The effect of a normal electric field on wave propagation on a fluid film

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Long-wavelength, small-amplitude disturbances on the surface of a fluid layer subject to a normal electric field are considered. In our model, a dielectric medium lies above a layer of perfectly conducting fluid, and the electric field is produced by parallel plate electrodes. The Reynolds number of the fluid flow is taken to be large, with viscous effects restricted to a thin boundary layer on the lower plate. The effects of surface tension and electric field enter the governing equation through an inverse Bond number and an electrical Weber number, respectively. The thickness of the lower fluid layer is assumed to be much smaller than the disturbance wavelength, and a unified analysis is presented allowing for the full range of scalings for the thickness of the upper dielectric medium. A variety of different forms of modified Korteweg-de Vries equation are derived, involving Hilbert transforms, convolution terms, higher order spatial derivatives, and fractional derivatives. Critical values are identified for the inverse Bond number and electrical Weber number at which the qualitative nature of the disturbances changes. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4862975]

I. INTRODUCTION

In this paper, we consider disturbances on the interface between a fluid film and a dielectric medium lying between parallel plate electrodes. Our particular focus will be with the separation of the electrodes relative to the wavelength of the disturbances. The effect of electric fields on a thin fluid layer is important in a number of contexts,[1] including, for example, processes such as the electrostatic liquid film radiator.[2]

For an inviscid horizontal fluid layer in the absence of any electric field, a weakly nonlinear analysis for small-amplitude, long-wavelength disturbances gives rise to the Korteweg-de Vries (KdV) equation,[3] which admits the familiar sech-squared solitary solutions. The relative importance of surface tension forces compared to gravity enters through an inverse Bond number, which is defined precisely below. However, when the value of this parameter is close to a particular critical value the coefficient of the third derivative term in the KdV equation becomes small and wavelength shortening means that a different scaling between amplitude and wavelength must be considered. This gives rise to an additional fifth derivative term.[3] In each case, the effect of surface tension enters through a normal stress term at the fluid surface. The application of an electric field modifies these governing equations by introducing an additional stress at the surface.

Previous investigations into the effect of electric fields, detailed below, have involved solving the hydrodynamic flow alongside the calculation of the electric field for the specific physical problem. More recently, an equation for the evolution of surface disturbances was presented for an arbitrary

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surface stress.[4] This allows the investigation of the effect of electric field in a unified approach including perturbation terms omitted from earlier results. This is the motivation for the present paper.

We consider the case when the electric field is generated by parallel electrodes, so that in the undisturbed state the electric field is normal to the interface. Two horizontal parallel plates are separated by distance $h + d$ with a layer of inviscid fluid, of undisturbed depth h, lying on the lower impermeable plate. Between the fluid layer and the upper plate, lies a second fluid with different electric properties and this fluid is taken to be hydrodynamically passive. A potential difference V_0 is then applied between the two plates. Previous analyses of this problem have focused either on the case when the lower fluid is a perfect conductor, or on the case when both fluids are perfect dielectrics. In the first instance, the tangential component of electric field at the interface is zero, while in the second situation the surface charge density at the interface is zero. In both cases, the tangential force at the interface due to the electric field is zero, which simplifies the analysis somewhat since there is no need to consider a hydrodynamic boundary layer at the fluid surface.[4] Here, we examine the case when the lower fluid is assumed to be a perfect conductor, though the analysis can be readily modified to the case of two perfect dielectrics. We demonstrate how the magnitude of the imposed potential difference and the relative depth of the two layers affects the evolution of surface disturbances, allowing comparison with previous studies which have largely focused on the cases when the thickness of the upper medium is either much greater than,[5,6] or much less than[7−10] the disturbance wavelength.

When the separation distance of the electrodes is large compared with the disturbance wavelength, corresponding to the normal electric field tending to a constant far from the surface, a Korteweg-de Vries Benjamin-Ono type equation is obtained for an inviscid fluid.[5,6] The case when the electrode separation distance is comparable to the disturbance wavelength was briefly considered as part of a larger study on the existence of solitary waves subject to electric fields.[11] Then the effect of electric field enters through a convolution term and it was demonstrated that the results for large electrode separation distance[5,6] emerge as a natural limit.

More attention has been focused on when the upper medium is of comparable thickness to the depth of the fluid, and hence much less than the disturbance wavelength.[7−10] A KdV equation is obtained with coefficients involving the electric field parameters. However, the actual value of these coefficients do not agree in these different treatments, and some inconsistencies in the results are clear. One by-product of the present paper is the opportunity to revisit some of the results of these earlier works, and thereby eliminate the inconsistencies in their results. We are then able to demonstrate how the corrected results are consistent with more general theory.

The structure of the paper is as follows. In Sec. II, the setup of the problem is discussed in more detail, key scalings are introduced, and the equation for an arbitrary normal stress is discussed. In Sec. III, the normal stress at the surface due to the imposed electric field is evaluated as a function of electrode separation, and compared with other work for different scalings of this separation distance. The novelty of the present treatment lies in the concise derivations based on recent general theory[4] which allows additional terms, such as higher order derivatives and viscous effects, to be included in a rational manner. Numerical results are presented in Sec. IV for travelling waves and waves which evolve in time. In Sec. V, results are summarised with key scalings of parameters identified, with comparisons to experimental measurements made in Sec. VI. Finally, some minor corrections to the results of earlier papers[7,8] are noted in the Appendix.

II. GOVERNING EQUATION

Suppose that an impermeable electrode is located at $y_* = -h$ with a fluid layer occupying the region $-h < y_* < \eta_*(x_*, t_*)$ above the electrode, where we take x_*, y_* to be the horizontal and vertical dimensional coordinates and t_* the dimensional time. The flow in the perturbed fluid layer is taken to be irrotational and so the velocity field is given by $\mathbf{u^*} = \nabla^* \phi^*$ where $\phi^*(x_*, y_*)$ is the dimensional velocity potential and satisfies Laplace's equation. The electrode is maintained at potential $V^* = 0$ and we consider the case when the fluid is a perfect conductor and hence the electric field in the fluid layer is zero. Above this layer we have a dielectric medium assumed to be hydrodynamically
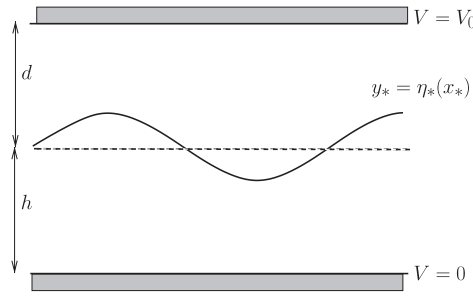
FIG. 1. Fluid layer occupying the region $-h < y_* < \eta_*(x_*)$ with upper electrode located at $y^* = d$ and maintained at potential V_0.

passive, but which supports an electric field $\mathbf{E}^* = \nabla V^*$. A second horizontal electrode is located at $y_* = d$ and maintained at potential $V^* = V_0$. In this case V^*, the dimensional electric potential, satisfies Laplace's equation in the region $\eta_*(x_*) < y_* < d$. The setup is illustrated in Figure 1.

At the interface the fluid velocity satisfies the kinematic condition, and since the lower fluid is a perfect conductor the tangential component of the electric field is zero. The normal component of the electric field at the interface gives rise to a normal stress on the surface, which together with the effect of surface tension lead to additional terms in the Bernoulli condition at the interface.

The set of equations governing the fluid flow, the electric field, and the interface conditions is non-dimensionalised using the undisturbed depth h, the gravitational acceleration g, and the potential difference V_0. The shallow water linear wave speed is $\sqrt{gh}$ and so we change to a frame moving at speed $c\sqrt{gh}$ with $c = O(1)$. Assuming that the amplitude of the perturbation of the surface is $O(\delta h)$ and that the wavelength is $O(\varepsilon^{-1}h)$, we write

$$\eta = \frac{1}{\delta h}\eta_*, \quad x = \frac{\varepsilon}{h}\left(x_* - c\sqrt{gh}t_*\right), \quad y = \frac{1}{h}y_*, \quad t = \delta\varepsilon\sqrt{\frac{g}{h}}t_*,$$

and in the small-amplitude long-wavelength limit we have $\delta, \varepsilon \ll 1$. The scaling chosen for the time is so that time derivative terms enter at the same order as the nonlinear terms. For an arbitrary stress applied at the surface of the fluid, a governing equation derived in the large Reynolds number limit is available.[4] In the present case, the tangential stress is zero, as discussed earlier, and the governing equation takes the form

$$2\eta_t + 3c\eta\eta_x = \frac{c\varepsilon^2}{\delta}\left(-\tfrac{1}{3}\eta_{xxx} - \frac{1}{45}\varepsilon^2\eta^{(v)}\right) - \frac{1-c^2}{c\delta}\eta_x + \frac{c^{\frac{1}{2}}\delta_b}{\delta}\mathcal{S}[\eta_x] + \frac{1}{c\delta^2}\frac{dT_n}{dx}. \tag{1}$$

Previously, it was assumed that the effect of the normal stress is $O(1)$ or smaller,[4] in which case $c = 1$. Here, we choose to retain greater generality, in order to allow for a greater range of scales for the electrical forcing.

The first three terms on the right hand side of (1) are hydrodynamic terms and the reason why the $O(\varepsilon^4)$ term is retained along with the $O(\varepsilon^2)$ terms, will become clear later in the exposition. The fourth term is due to dissipation in the viscous boundary layer on the lower electrode, with

$$\mathcal{S}[f] = \frac{1}{\pi}\int_0^\infty \frac{f(x+s,t)}{\sqrt{s}}\,ds, \quad \delta_b = \left(\frac{\mu}{\varepsilon\rho h\sqrt{gh}}\right)^{\frac{1}{2}}, \tag{2}$$

where $\delta_b h$ is the thickness of the viscous boundary layer at the base of the fluid and $\mathcal{S}(f)$ can be considered to be a fractional derivative. In the final term, T_n is the non-dimensional normal stress which here consists of contributions from the surface tension and the electric field. In dimensional terms,

$$T_n^* = \sigma \nabla^*.\mathbf{n} + \tfrac{1}{2}\epsilon_0(E_n^*)^2 = \frac{\sigma\eta_{x^*x^*}^*}{\left(1+\eta_{x^*}^{*2}\right)^{\frac{3}{2}}} + \frac{\epsilon_0(V_{y^*}^* - \eta_{x^*}^* V_{x^*}^*)^2}{2\left(1+\eta_{x^*}^{*2}\right)},$$

where σ is the surface tension parameter and ϵ_0 is the dielectric constant of the upper medium. Non-dimensionalising and using the scaling defined above gives

$$T_n = \left(\varepsilon^2 \delta \tau \eta_{xx} + \tfrac{1}{2} W_e D^3 Q^2\right)\left(1 + O(\delta^2 \varepsilon^2)\right), \qquad Q = V_y - \varepsilon^2 \delta \eta_x V_x,$$

where V is the electric potential, non-dimensionalised by V_0, $D = d/h$ and the non-dimensional parameters characterising the effects of the electric field and the surface tension are

$$W_e = \frac{\epsilon_0 V_0^2}{\rho g d^3}, \qquad \tau = \frac{\sigma}{\rho g h^2}, \tag{3}$$

an electric Weber number and an inverse Bond number, respectively. The governing equation then takes the form

$$2\eta_t + 3c\eta\eta_x = \frac{\varepsilon^2}{c\delta}\left((\tau - \tfrac{1}{3}c^2)\eta_{xxx} - \frac{1}{45}c^2\varepsilon^2\eta^{(v)} - \frac{1-c^2}{\varepsilon^2}\eta_x + E_1\right) + \frac{c^{\frac{1}{2}}\delta_B}{\delta}\,\mathcal{S}[\eta_x], \tag{4}$$

where

$$E_1 = \frac{W_e D^3}{2\varepsilon^2\delta}\frac{d(Q^2)}{dx}. \tag{5}$$

We now see why the fifth derivative term was retained in the asymptotic series, since if τ is close to its critical value of $\tfrac{1}{3}c^2$, then the fifth derivative term becomes comparable in magnitude with the third derivative term. At this stage it is more usual to set $c = 1$ to eliminate what appears to be the largest term on the right hand side of (4). However, we choose to retain added generality, which proves necessary when considering different forms of electric field. The exact form of E_1 depends on the imposed electric field and we now consider how the magnitude of the potential difference and the relative thickness of the two layers affects the governing equation.

III. ELECTRIC FIELD

In considering the effect of the separation of the two electrodes, we focus on two main cases; first, when the thickness of the upper (dielectric) medium is comparable to the disturbance wavelength and, second, when the thickness of the dielectric medium is comparable to the thickness of the lower layer (and hence much less than the wavelength of the disturbance). In terms of the dimensionless parameters introduced, the two cases correspond to $\varepsilon D = O(1)$ and $D = O(1)$, respectively. In each case, the problem reduces to solving

$$\varepsilon^2 V_{xx} + V_{yy} = 0, \qquad V(x, D) = 1, \qquad V(x, \delta\eta) = 0,$$

in order to express the electric term E_1, defined in Eq. (5), in terms of η.

A. Case $\varepsilon D = O(1)$

In this case, we write $\Delta = \varepsilon D$, $Y = \varepsilon y$, and $V(x, Y)$ then satisfies Laplace's equation. The boundary conditions on V are simplified if we write $V = (Y + \varepsilon\delta\phi)/\Delta$. Linearising the boundary condition at the interface $Y = \varepsilon\delta\eta$ to $Y = 0$ then gives

$$\phi_{xx} + \phi_{YY} = 0, \qquad \phi(x, 0) = -\eta, \qquad \phi(x, \Delta) = 0, \tag{6}$$

with Q and E_1 given by

$$Q = \frac{1}{D}\left(1 + \varepsilon\delta\phi_Y + o(\varepsilon^2\delta)\right), \qquad E_1 = \frac{W_e D}{\varepsilon}\phi_{xY}(x, 0).$$

The system (6) is readily solved by taking Fourier transforms with respect to x giving $\phi(x, Y)$ in the form of a convolution

$$\phi(x, Y) = \eta * f, \qquad f(x, Y) = \mathcal{F}^{-1}(F(k, Y)), \qquad F(k, Y) = -\frac{\sinh(k(\Delta - Y))}{\sinh(k\Delta)},$$

and hence

$$E_1 = \frac{W_e \Delta}{\varepsilon^2} \mathcal{G}[\eta_x], \qquad \mathcal{G}[\eta] \equiv \eta * g, \qquad g(x) = \mathcal{F}^{-1}\left(k\coth(k\Delta)\right). \tag{7}$$

Substituting into (4) we see that the wave speed is given by $c = 1$ and the governing equation becomes

$$2\eta_t + 3\eta\eta_x = \frac{\varepsilon^2}{\delta}\left((\tau - \tfrac{1}{3})\eta_{xxx} - \frac{1}{45}\varepsilon^2\eta^{(v)} + \frac{W_e\Delta}{\varepsilon^2}\mathcal{G}[\eta_x]\right) + \frac{\delta_B}{\delta}\mathcal{S}[\eta_x], \tag{8}$$

where $\eta^{(v)}$ denotes the fifth derivative of η with respect to x. In the inviscid limit, this agrees with earlier travelling wave analysis.[11]

If $\Delta \to \infty$, corresponding to separation much greater than the disturbance wavelength, then since ϕ satisfies Laplace's equation with $\phi \to 0$ as $Y \to \infty$ it can be readily shown that the first partial derivatives throughout $Y > 0$ are related by $\phi_Y = -\mathcal{H}[\phi_x]$, and $\phi_x = \mathcal{H}[\phi_Y]$, where $\mathcal{H}$ denotes the Hilbert transform with respect to x, defined as the Cauchy principal value of a convolution integral,[12]

$$\mathcal{H}[f] = PV\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s,t)}{x-s}\,ds\right).$$

Hence, the electrical forcing term in the governing equation becomes $E = \varepsilon^{-1}DW_e\mathcal{H}[\eta_{xx}]$. This can also be derived by taking the limit of (7) as $\Delta \to \infty$ and noting that $\mathcal{F}[\mathcal{H}(f)] = -i\,\text{sgn}(k)\mathcal{F}(f)$, to give

$$2\eta_t + 3\eta\eta_x = \frac{\varepsilon^2}{\delta}\left((\tau - \tfrac{1}{3})\eta_{xxx} - \frac{1}{45}\varepsilon^2\eta^{(v)} + \frac{W_e\Delta}{\varepsilon^2}\mathcal{H}[\eta_{xx}]\right) + \frac{\delta_B}{\delta}\mathcal{S}[\eta_x]. \tag{9}$$

Setting $\delta_b = 0$ this is in agreement with previous results[6,11] (observing that in their notation $E_b = DW_e$).

Considering instead the case $\Delta \ll 1$, then

$$k\coth(k\Delta) \sim \frac{1}{\Delta}\left(1 + \tfrac{1}{3}k^2\Delta^2 - \tfrac{1}{45}k^4\Delta^4 + O(\Delta^6)\right)$$

and $\mathcal{F}^{-1}(k\coth(k\Delta))$ is written as a sum of generalised functions to give

$$E_1 = \frac{W_e}{\varepsilon^2}\eta_x - \tfrac{1}{3}D^2W_e\eta_{xxx} - \tfrac{1}{45}\varepsilon^2 D^4W_e\eta^{(v)} + O(\varepsilon^4).$$

The first term corresponds to a shift in the wavespeed c and the second term modifies the dispersive term. Substituting into (4) we see that $c = (1 - W_e)^{\frac{1}{2}}$ and

$$2\eta_t + 3c\eta\eta_x = \frac{\varepsilon^2}{c\delta}\left[\left(\tau - \tfrac{1}{3}(c^2 + D^2W_e)\right)\eta_{xxx} - \frac{1}{45}\varepsilon^2(c^2 + D^4W_e)\eta^{(v)}\right] + \frac{c^{\frac{1}{2}}\delta_B}{\delta}\mathcal{S}[\eta_x]. \tag{10}$$

We now compare the limit $\varepsilon D \to 0$ with the $D = O(1)$ case.

B. Case $D = O(1)$

We set $z = y/D$ so the interface is located at $z = \gamma\eta$, where $\gamma = \delta h/d$. Writing $V = z + \gamma\phi$ gives

$$(\varepsilon D)^2\phi_{zz} + \phi_{xx} = 0, \qquad \phi(x, \gamma\eta) = -\eta, \qquad \phi(x, 1) = 0,$$

with solution

$$\phi(x, z) = -(1-z)r + \frac{1}{6}(\varepsilon D)^2(1-z)\left((1-z)^2 - (1-\gamma\eta)^2\right)r_{xx} + (\varepsilon D)^4 q(z, \eta)r^{(iv)} + O(\varepsilon^6),$$

where $r = \eta/(1 - \gamma\eta)$ and $q(z, \eta)$ can be readily calculated, but is not included here in the interests of conciseness.

Assuming that the disturbance amplitude is small compared with the depth of the dielectric medium, $\gamma \ll 1$, and hence

$$\phi_z(x, \gamma\eta) \sim \eta + \gamma\eta^2 - \frac{1}{3}\varepsilon^2 D^2 \eta_{xx} - \frac{1}{45}\varepsilon^4 D^4 \eta^{(iv)} + O(\gamma^2, \gamma\varepsilon^2, \varepsilon^4),$$

$$E_1 = \frac{W_e}{\varepsilon^2}(\eta_x + 3\gamma\eta\eta_x) - \frac{W_e D^2}{3}\eta_{xxx} - \frac{W_e \varepsilon^2 D^4}{45}\eta^{(iv)}.$$

Finally, substituting into Eq. (4), we see that the wave speed is given by $c = (1 - W_e)^{\frac{1}{2}}$ and

$$2\eta_t + 3c\left(1 - \frac{W_e}{Dc^2}\right)\eta\eta_x = \frac{\varepsilon^2}{c\delta}\left[\left(\tau - \frac{1}{3}(c^2 + D^2 W_e)\right)\eta_{xxx} - \frac{1}{45}\varepsilon^2(c^2 + D^4 W_e)\eta^{(v)}\right] + \frac{c^{\frac{1}{2}}\delta_B}{\delta}\mathcal{S}[\eta_x].$$

(11)

This agrees with (10), the small εD limit of (8), in the matching region $1 \ll D \ll \varepsilon^{-1}$. Finally, (11) can be re-written more compactly as

$$2\eta_t + 3A\eta\eta_x = B\eta_{xxx} + C\eta^{(v)} + \frac{c^{\frac{1}{2}}\delta_B}{\delta}\mathcal{S}[\eta_x],$$

$$A = \left(1 - \frac{W_e}{(1 - W_e)D}\right)c, \quad B = \frac{\varepsilon^2}{c\delta}\left(\tau - \frac{1}{3} - \frac{1}{3}W_e(D^2 - 1)\right), \quad C = -\frac{\varepsilon^4}{c\delta}\frac{1 + W_e(D^4 - 1)}{45}.$$

(12)

When the distinguished scaling $\delta = \varepsilon^2$ is taken and the viscous dissipation term is dropped this agrees with the (corrected) results of Easwaran[7] and Gonzalez and Castellanos,[8] discussed in the Appendix.

IV. NUMERICAL SOLUTIONS

Governing equations have been obtained for long-wavelength, small-amplitude disturbances when the separation of the electrodes is comparable with the wavelength (8) and short compared with the wavelength (11). In each case, the effect of viscous dissipation in the base boundary layer was included. The analysis for when the electrode separation is very small was presented largely to allow comparison with earlier work, and hence in this section we focus on numerical solutions of (8) which we re-write as

$$2\eta_t + 3\eta\eta_x = a\eta_{xxx} - b\eta^{(v)} + p\,\mathcal{G}[\eta_x] + q\,\mathcal{S}[\eta_x],$$

(13)

where $a = \operatorname{sgn}(\tau - \frac{1}{3})$,

$$b = \frac{\varepsilon^2}{45|\tau - \frac{1}{3}|}, \qquad p = \frac{W_e \Delta}{\varepsilon^2 |\tau - \frac{1}{3}|}, \qquad q = \frac{\delta_B}{\delta},$$

and we have set $\varepsilon^2 = |\tau - \frac{1}{3}|\delta$ so that the third derivative term enters at the same order as the quadratic nonlinearity. The transform terms $\mathcal{G}[\eta_x]$ and $\mathcal{S}[\eta_x]$ are defined in (2) and (7), respectively. In the absence of viscous dissipation, travelling wave solutions may exist, so the numerical approach we take is to search for such solutions when $q = 0$ and then look at how these solutions evolve when $q > 0$.

Noting that $\mathcal{G}[e^{ikx}] = \widehat{g}(k)e^{ikx}$ and $\mathcal{S}[e^{ikx}] = \widehat{s}(k)e^{ikx}$, where

$$\widehat{g}(k) = k\cosh(k\Delta), \qquad \widehat{s}(k) = \begin{cases} \dfrac{e^{\frac{i\pi}{4}}}{\sqrt{|k|}}, & k > 0, \\[2mm] \dfrac{e^{-\frac{i\pi}{4}}}{\sqrt{|k|}}, & k < 0, \end{cases}$$

this suggests that for both travelling and evolving solutions, a numerical scheme should be used in which spatial derivatives and transform terms are evaluated in spectral space.

A. Travelling wave solutions

We consider solutions of (13) travelling at speed C in the form $\eta = 2CN(Z)$, with $Z = \sqrt{2|C|}$ $(x - Ct)$, choosing the new variables so that in the case $b = p = q = 0$, the travelling wave solution is given by $N(Z) = \text{sech}^2 Z$. Recalling that (13) is in a frame moving with non-dimensional velocity $c = 1$, the non-dimensional speed of the travelling wave is $1 + C\delta$, so the Froude number F is given by $F = 1 + C\delta$, and the disturbance amplitude characterised by $\delta|C| = |F - 1|$. We then write $N(Z)$ as a Fourier series on $[-M\pi, M\pi]$,

$$N(Z) = \sum_{r=-n}^{n} c_r e^{iRZ}, \qquad R = \frac{r}{M},$$

to give the set of nonlinear equations

$$c_r - \frac{3}{2} \sum_s c_{r-s} c_s = \gamma \left(aR^2 - BR^4 + PR\cosh(\beta R) \right) c_r, \qquad r = -n \ldots n, \tag{14}$$

with unknowns c_r. Here, the parameters a, B, P, β are given in terms of the Froude number F, the inverse Bond number τ, the electric Weber number W_e, and the relative separation of the electrodes $D = d/h$ by, $a = \text{sgn}(\tau - \frac{1}{3})$, $\gamma = \text{sgn}(F - 1)$ and

$$B = \frac{2|F - 1|}{45|\tau - \frac{1}{3}|^2}, \qquad P = \frac{W_e D}{|2(F - 1)(\tau - \frac{1}{3})|^{1/2}}, \qquad \beta = \left| \frac{2(F - 1)}{\tau - \frac{1}{3}} \right|^{\frac{1}{2}} D. \tag{15}$$

Note that from the definition of the electric Weber number (3), keeping P constant and varying D corresponds to varying the depth, d, of the upper dielectric medium while keeping the average electric field in this region V_0/d constant. The set of equations (14) is then solved using Newton's method. In the large-β limit, this formulation agrees with earlier analysis[11] where the (B, P) parameter space was investigated and conditions for the existence of travelling waves determined.

We now illustrate the travelling wave solutions for a range of parameter values. We stress that at this stage the parameters are chosen to illustrate the changes in behaviour as parameters vary, rather than modelling a specific physical situation. Physical values of the parameters are discussed in Sec. VI.

In the large-D limit, the case $\tau > \frac{1}{3}$ has the richest mathematical behaviour with the existence of travelling waves depending on the set of parameters F, τ, and W_e and for this reason we present results for the case when $F < 1$ and $\tau > \frac{1}{3}$, so $\gamma = -1$ and $a = 1$, and then set $B = 0.5$ to allow comparison with earlier results[11] valid when $\beta \gg 1$. Choosing $\beta = 5.0$ gives results very similar to those of the $\beta \to \infty$ limit. In Figure 2(a), it is seen that as the magnitude of the electric field increases (i.e., P increases), the maximum disturbance amplitude decreases, but oscillations in the decaying tail appear and grow in amplitude. For P above a critical value $P_c \approx 2.29$, the numerical method does not converge to a solution, and it is reasonable to conclude that travelling wave solutions do not exist for $P > P_c$. Choosing a smaller separation distance, so $\beta = 2.5$, similar behaviour is seen with no solutions for $P > P_c \approx 2.1$. Decreasing β to 2.0, quite different behaviour occurs. In Figure 2(b), it is seen that the maximum disturbance amplitude decreases as P increases, but in this case no oscillations appear in the decaying tail. Numerically, no solutions are found for $P > P_c \approx 1.8$ which coincides with the maximum amplitude going to zero.

B. Effect of viscosity on evolving wave

Pseudospectral schemes are well established for solving Burgers equation and the KdV equation (see, for example, Fornberg[13]). In a pseudospectral formulation, all derivatives and spatial transforms linear in the dependent variable are evaluated in spectral space, as described above, but due to the presence of nonlinear terms the solution is most easily advanced forward in time in physical space. We take this approach to investigate the effect of viscous dissipation due to the lower boundary and also to validate the travelling wave solutions obtained above. Writing $\eta = 2CN(X, T)$ with $X = \sqrt{2|C|}x$ and $T = \sqrt{2|C|}t$, we take as a sample case, $a = 1$, $\gamma = -1$, $B = 0.5$, $\beta = 2.5$,
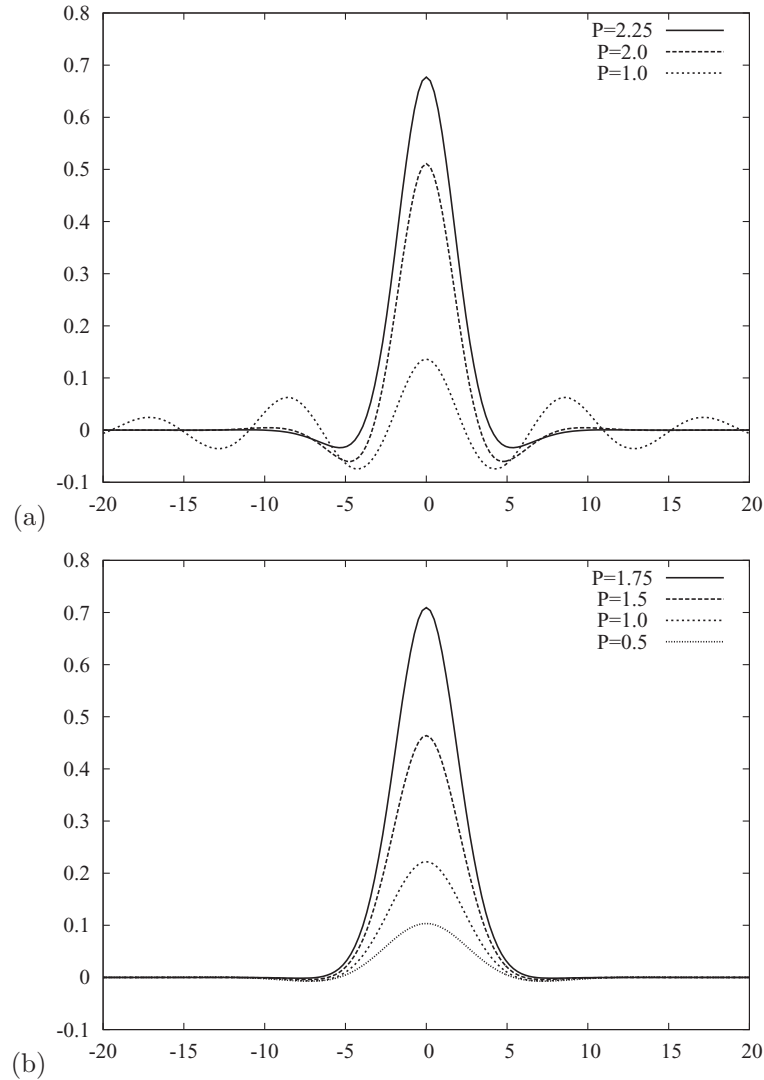
FIG. 2. Travelling wave form when $a = 1$, $\gamma = -1$, $B = 0.5$, with (a) $\beta = 5.0$ and $P = 1.0, 2.0, 2.25$; and (b) $\beta = 2.0$ and $P = 0.5, 1.0, 1.5, 1.75$.

$P = 1.0$. Since $\gamma = -1$, the Froude number is less than one and so the perturbation wave propagation speed is negative. For the numerical solutions, we choose $C = -\frac{1}{2}$. As the initial condition, we take the travelling wave solution illustrated in Figure 2(b) and then solve

$$2N_T + 3NN_X = \mathcal{R}[N],$$

where $\mathcal{R}[N]$, written in terms of Fourier components, is given by

$$\mathcal{R}[N] = -\gamma(iR)\left(aR^2 - BR^4 + P\cosh(\beta R)R + Qie^{\pm\frac{i\pi}{4}}\sqrt{|R|}\right)c_r, \qquad r = -n \dots n.$$

Setting $Q = 0$, the numerical solution propagates to the left at speed $\frac{1}{2}$, unchanged in form, which demonstrates the validity of the travelling wave form obtained earlier. Results are not illustrated for this case. Setting $Q = 0.05$ it is seen in Figure 3 that the solution propagates to the left, with speed close to $\frac{1}{2}$, but decays in amplitude, as is to be expected since wave energy is dissipated by the viscous boundary layer. For these parameter values, the disturbance is a depression wave which follows from the fact that the inverse Bond number exceeds $\frac{1}{3}$. We return to discuss the physical relevance of this case and in particular the effect of viscosity at the end of this paper.
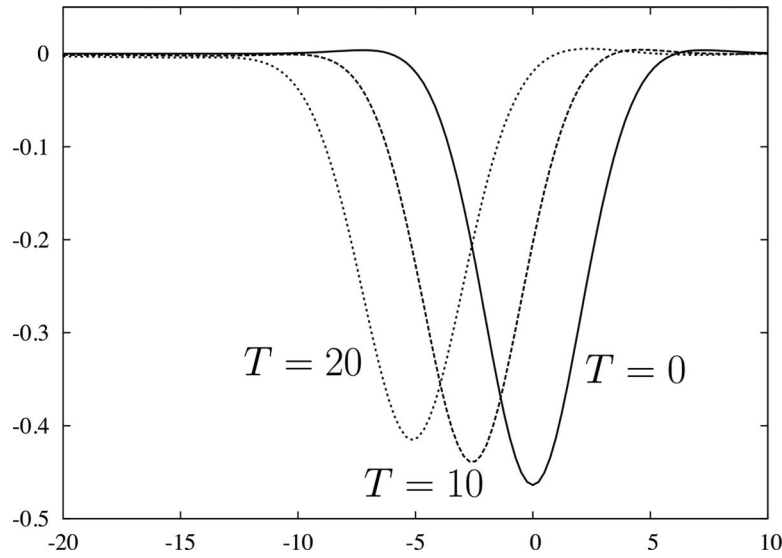
FIG. 3. Effect of viscous dissipation on the wave form $\eta = -N(X, T)$ when $a = 1$, $\gamma = -1$, $B = 0.5$, $\beta = 2.0$, $P = 1.0$, $C = -0.5$, with $Q = 0.05$ for times $T = 0, 10, 20$.

V. OVERVIEW OF GOVERNING EQUATIONS AND KEY SCALINGS

In Sec. III, governing equations have been obtained for the propagation of disturbances on the surface of a fluid layer subject to surface tension and a normal electric field due to parallel plate electrodes. When the thickness of the upper dielectric medium is comparable to the disturbance wavelength, ($\Delta = \varepsilon D = O(1)$), the disturbances are governed by (8). Unless the inverse Bond number τ is close to the critical value of $\frac{1}{3}$, the effect of the electric field is comparable to that of surface tension when the disturbance wavelength is such that $\delta = \varepsilon^2$ and the Weber number is $O(\varepsilon^2)$. Writing $W_e = \varepsilon^2 \widehat{W}_e$ gives

$$2\eta_t + 3\eta\eta_x = (\tau - \tfrac{1}{3})\eta_{xxx} + \Delta\widehat{W}_e\,\mathcal{G}[\eta_x] + \frac{\delta_B}{\delta}\,\mathcal{S}[\eta_x], \tag{16}$$

with wave speed $c = 1$. However, if τ is close to $\frac{1}{3}$, such that $\tau = \frac{1}{3} + \varepsilon^2\tau_1$, then the wavelength adjusts such that $\delta = \varepsilon^4$ and electric effects are comparable to the diffusive terms when $W_e = \varepsilon^4\widetilde{W}_e$, in which case

$$2\eta_t + 3\eta\eta_x = \tau_1\eta_{xxx} - \tfrac{1}{45}\eta^{(v)} + \Delta\widetilde{W}_e\,\mathcal{G}[\eta_x] + \frac{\delta_B}{\delta}\,\mathcal{S}[\eta_x], \tag{17}$$

again with $c = 1$. In the inviscid case and taking the limit $\Delta \gg 1$, all these conclusions agree with earlier results.[6, 11]

When $D = O(1)$, we have wave speed $c = (1 - W_e)^{\frac{1}{2}}$ and hence the analysis is only valid when $W_e < 1$. When W_e is close to one, the wave speed approaches zero and a new set of scalings is required. Other special cases arise when τ and W_e are close to critical values, making the coefficients of the nonlinear term or the third derivative term close to zero. These critical values are given by

$$W_{ec} = \frac{D}{D+1}, \qquad \tau_c = \tfrac{1}{3}(1 + W_e(D^2 - 1)),$$

and it is seen that the effect of the electric field can either increase or decrease the critical value of τ from $\frac{1}{3}$. All of these particular special cases can be considered in a unified way by defining a new timescale, $T = |A|t$, where A is defined in (12). The governing equation then becomes

$$2\eta_T + 3\eta\eta_x = \widehat{B}\eta_{xxx} + \widehat{C}\eta^{(v)} + \widehat{P}\,\mathcal{S}[\eta_x],$$

where

$$\widehat{B} = \left(\frac{\varepsilon^2 (\tau - \tau_c)}{\delta |W_e - W_{ec}|} \right) \frac{D}{D+1}, \qquad \widehat{C} = - \left(\frac{\varepsilon^4}{\delta |W_e - W_{ec}|} \right) \frac{D\big(1 + W_e(D^4 - 1)\big)}{45(D+1)},$$

and the coefficient of the viscous damping term is

$$\widehat{P} = \frac{(1 - W_e)^{\frac{3}{4}} \delta_b}{|W_e - W_{ec}|\delta} \frac{D}{D+1}.$$

Unless τ is close to τ_c, we have the distinguished scaling $\varepsilon^2 = \delta |W_e - W_{ec}|$ and hence

$$2\eta_T + 3\eta\eta_x = \frac{D(\tau - \tau_c)}{D+1}\eta_{xxx} + \widehat{P}\,\mathcal{S}[\eta_x], \tag{18}$$

whereas if $\tau = \tau_c + \varepsilon^2 \tau_2$ then the amplitude and wavelength are related by $\varepsilon^4 = \delta |W_e - W_{ec}|$ and the governing equation becomes

$$2\eta_T + 3\eta\eta_x = \left(\frac{D\tau_2}{D+1} \right) \eta_{xxx} - \left(\frac{D\big(1 + W_e(D^4 - 1)\big)}{45(D+1)} \right) \eta^{(v)} + \widehat{P}\,\mathcal{S}[\eta_x]. \tag{19}$$

Taken together, Eqs. (16)–(19), together with the scalings defined, provide a set of model equations for the weakly nonlinear evolution of small-amplitude, long-wavelength disturbances influenced by an electric field produced by a pair of parallel electrodes.

VI. SUMMARY

Governing equations have been obtained for the propagation of disturbances on the surface of a fluid layer subject to surface tension and a normal electric field due to parallel plate electrodes. Three non-dimensional parameters, τ, W_e, D enter the analysis, together with a parameter characterising viscous dissipation due to the thin boundary layer at the base of the fluid layer. The key parameter ranges have been identified and the corresponding equations (16)–(19) summarised in Sec. V.

In Sec. IV, numerical solutions were presented illustrating the forms of travelling waves possible in the regime where the disturbance length scale is comparable to the thickness of the dielectric layer. Those results illustrate the modification of the waveform due to the higher derivative term and the electric field via the hyperbolic cotangent transform term. The key qualitative observation is that as the electrodes are moved closer, oscillations in the tail of the soliton appear to be suppressed. In addition, the parameter ranges over which travelling wave solutions exist in the inviscid limit are modified by the exact form of the electric field imposed.

Finally, we consider how relevant these model equations are to physical situations of interest. We consider the case of a horizontal layer of either mercury or water with air of dielectric constant

$$\epsilon_A = 8.8 \times 10^{-12}\,\mathrm{F\,m^{-1}},$$

above this layer and below the upper electrode. Mercury and water are considered as they have been the subject of experimental investigation of solitary wave propagation,[14,15] though in the absence of any electric field. It is reasonable to approximate both impure water and mercury as perfect conductors and the relevant material parameters are then

$$\rho_W = 1 \times 10^3\,\mathrm{kg\,m^{-3}}, \quad \mu_W = 1 \times 10^{-3}\,\mathrm{N\,s\,m^{-2}}, \quad \sigma_W = 72 \times 10^{-3}\,\mathrm{N\,m^{-1}}$$
$$\rho_M = 13.5 \times 10^3\,\mathrm{kg\,m^{-3}}, \quad \mu_M = 1.5 \times 10^{-3}\,\mathrm{N\,s\,m^{-2}}, \quad \sigma_M = 484 \times 10^{-3}\,\mathrm{N\,m^{-1}},$$

where the subscripts W and M refer to water and mercury, respectively. For these material parameters, the critical value of the inverse Bond number, $\tau = \frac{1}{3}$, corresponds to fluid depths $h_W \approx 4.7$ mm and $h_M \approx 3.3$ mm for water and mercury, respectively. Experiments on water[14] focused on layer depths of 5 cm, appreciably larger than h_W in which case $\tau < \frac{1}{3}$ and elevation waves exist. For mercury, experiments were conducted[15] on layers of depth ranging from 2.12 mm to 8.5 mm and so covering the cases $\tau < \frac{1}{3}$ (when elevation solitary waves were measured) and $\tau > \frac{1}{3}$ (when depression waves were observed). The experiments on both water and mercury were conducted in the absence of

TABLE I. Key parameter values for water with air above.

h (m)	δ	ϵ	τ	b	q	E_c (V m^{-1})	λ^* (m)
0.5	0.1	0.55	2.9×10^{-5}	0.020	0.013	1.0×10^7	0.91
0.2	0.1	0.55	1.8×10^{-4}	0.020	0.025	6.4×10^6	0.36
0.05	0.1	0.55	0.0029	0.020	0.071	3.2×10^6	0.091
0.01	0.1	0.62	0.073	0.033	0.23	1.3×10^6	0.016
0.01	0.01	0.19	0.073	0.0031	4.1	7.5×10^5	0.053
0.005	0.1	1.6	0.29	1.4	0.23	5.9×10^5	0.0031
0.002	0.1	0.26	1.8	0.0010	1.2	9.4×10^5	0.0077

electric fields and the wave amplitude was approximately 10% of the fluid depth (corresponding to $\delta = 0.1$ in the notation of the present paper). We now consider how the effect of electric field is relevant to these parameter ranges.

When the depth of the air layer is comparable to the disturbance wavelength, the evolution of the wave is given by (13). Once the material parameters are fixed, the non-dimensional coefficients a, b, q are functions of h, the depth of the lower layer, and δ the relative amplitude of the disturbances. The parameter characterising effect of the electric field can be re-written as

$$p = \frac{\epsilon_A}{\rho g h} \sqrt{\frac{1}{\delta|\tau - \frac{1}{3}|}} E^2,$$

where $E = V_0/d$, the average electric field across the air layer. In order to assess the importance of the electric field to the evolution of the solitary wave, we define E_c to be the value of E such that $p = 1$ and the effect of the electric field is comparable to the hydrodynamic dispersion term. In Tables I and II, we illustrate the values of ϵ, τ, b, q, E_c and the dimensional wavelength $\lambda^* = h/\epsilon$, as a function of h and δ for water and mercury, respectively.

From these tables a number of conclusions can be drawn. First, the viscous dissipation is smaller in the case of mercury due to the smaller kinematic viscosity. For water the viscous terms are significant for water depths close to the critical depth of 4.5 mm and hence depression solitary waves are unlikely to be observable on water due to rapid damping, whereas for mercury the amplitude decay rate is much lower, allowing depression waves to be observed. Indeed, this was the primary reason for experimentalists using mercury when seeking depression solitary waves. Looking now at the magnitude of electric fields at which electric effects significantly modify the form of the solitary wave, we see that strong fields are required in both cases, though slightly lower for the case of air above water rather than mercury. For air, electric breakdown occurs when the field strength is approximately 3×10^6 V/m, while the breakdown for water is approximately 70×10^6 V/m. Thus, we see that when the thickness of the air layer is comparable with the disturbance wavelength, the

TABLE II. Key parameter values for mercury with air above.

h (m)	δ	ϵ	τ	b	q	E_c (V m^{-1})	λ^* (m)
0.5	0.1	0.55	1.4×10^{-5}	0.020	0.0041	3.9×10^7	0.91
0.2	0.1	0.55	8.8×10^{-5}	0.020	0.0082	2.4×10^7	0.36
0.05	0.1	0.55	0.0014	0.020	0.023	1.2×10^7	0.091
0.01	0.1	0.57	0.035	0.024	0.078	5.2×10^6	0.018
0.01	0.01	0.18	0.035	0.0024	1.4	2.9×10^6	0.056
0.005	0.1	0.73	0.14	0.062	0.12	3.3×10^6	0.0068
0.002	0.1	0.42	0.88	0.0073	0.30	2.7×10^6	0.0048

electric fields of strength less than the air breakdown threshold do have a significant effect on the form of solitary waves for thinner layers.

Turning now to the case when the thickness of the air layer is comparable to the thickness of the lower fluid layer, we see that when $D = 1$, the critical electrical Weber number is $\frac{1}{2}$. For a 1 cm layer of water, this requires an electric field of 2.4×10^6 V/m, while a layer of mercury of equal thickness requires a slightly higher field strength due to the higher density.

In conclusion, it has been demonstrated that the model governing equations summarised in Sec. V are relevant to physical problems involving thin fluid layers. It should be noted at this point that the derivation of the model equations is rigorous, but based on assumptions of the length scales present in the problem. However, this is not to say that these equations are uniformly valid as leading order approximations of the full nonlinear equations. For the simpler case when no electric field is present and the fluid layer is taken to be inviscid, analysis of the fully nonlinear case was undertaken using a numerical scheme based on an integrodifferential-equation formulation.[3] The numerical results show that in this case the Korteweg-de Vries equation does not provide a wholly accurate description of periodic gravity-capillary waves for $\tau < \frac{1}{3}$ due to the presence of short wavelength ripples in the tail of solitary-type waves, which invalidates the scaling argument used in the derivation of the KdV equation. The present treatment suggests an alternative critical value for the inverse Bond number, which may be less, or greater, than $\frac{1}{3}$ and a fully nonlinear numerical study is necessary in order to determine when Eqs. (16)–(19) are valid approximations of the full system. However, such an investigation using boundary integral methods is beyond the scope of the current paper. Despite this proviso, a significant result of the present work is that the key scalings of the full problem have been identified for further study.

APPENDIX: COMPARISON WITH EARLIER RESULTS

Here, we focus on the results of Easwaran[7] and Gonzalez and Castellanos.[8] Written in our notation, Easwaran[7] considered the inviscid case with $\delta = \varepsilon^2$ and electric Weber number $W_e = O(1)$, to obtain

$$2\eta_t + 3A\eta\eta_X = B\eta_{XXX},$$

where X is the coordinate in the frame moving at speed $c\sqrt{gh}, c = \sqrt{1 - W_e}$ and A, B are coefficients involving W_e and the relative depth of the two layers, $D = d/h$. Gonzalez and Castellanos[8] obtained a similar equation but with different expressions for the coefficients A and B.

In Easwaran,[7] calculations are carried out in dimensional form, but non-dimensionalising and using the scalings described in Sec. II, gives

$$A = \left(1 - \frac{W_e}{3(1 - W_e)D}\left(3 + 2d + d^2\right)\right)c, \qquad B = \frac{1}{c}\left(\tau - 1 - \frac{1}{2}W_e(3D^2 - 2)\right).$$

Two inconsistencies are immediately apparent. First, the coefficient A is not dimensionless due to the $3 + 2d + d^2$ multiplier. Second, in the absence of an electric field (i.e., $W_e = 0$), the equation becomes

$$2\eta_t + 3\eta\eta_X = (\tau - 1)\eta_{XXX}$$

and the coefficient of the third spatial derivative term does not agree with the well established result for disturbances on a thin fluid layer[3] which has $(\tau - \frac{1}{3})$ as the coefficient of the diffusive term.

In Gonzalez and Castellanos,[8] the effects of viscosity are also included and the governing equation for the surface elevation obtained using the Fredholm alternative. However, taking the inviscid limit gives (in the notation of Sec. II),

$$A = \left(1 - \frac{W_e}{(1 - W_e)D}\right)c, \qquad B = -\frac{1}{c}\left(2\tau + \frac{1}{3} + \frac{1}{3}W_e(D^2 - 1)\right).$$

While this is dimensionally correct, it suffers the same problem as the Easwaran[7] result in that setting $W_e = 0$ the standard $(\tau - \frac{1}{3})$ multiplier of the third derivative is not recovered.

Attempting to identify the source of the inconsistency using the Fredholm alternative[8] is considerably more involved than reworking the perturbation analysis of Easwaran.[7] It should be noted that the latter approach is subtly different from that taken in Sec. II of the present paper, in that all quantities are written as perturbation series, including the position of the interface. While this method[7] is equally valid, errors appear in the solution of the electric potential. Solving the system

$$\varepsilon \Phi_{\mu\mu} + \Phi_{yy} = 0, \quad \Phi(\mu, \eta) = 0, \quad \Phi(\mu, H) = \phi_0,$$

where $H = h + b$ and $\eta = h + \varepsilon \eta_1 + \epsilon^2 \eta_2$, as a perturbation series gives

$$\Phi = \frac{\phi_0 (y - h)}{b} + \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + O(\varepsilon^3),$$

where

$$\Phi_1 = \frac{\phi_0 \eta_1 (y - H)}{b^2}, \qquad \Phi_2 = -\frac{\phi_0 \eta_1''(y - H)^3}{6b^2} + \frac{\phi_0}{b} \left(\frac{\eta_1'' b}{6} + \frac{\eta_1^2}{b^2} + \frac{\eta_2}{b} \right)(y - H),$$

rather than the results given as Eqs. (10)–(12) in Easwaran.[7] Using these results for the electric potential, and correcting a minor error in the solution in the fluid layer, eventually yields

$$2\eta_t + 3A\eta\eta_X = B\eta_{XXX}, \qquad c = \sqrt{1 - W_e},$$

with

$$A = \left(1 - \frac{W_e}{(1 - W_e)D} \right) c, \qquad B = \frac{1}{c} \left(\tau - \tfrac{1}{3} - \tfrac{1}{3} W_e (D^2 - 1) \right).$$

This is clearly dimensionally correct and gives the expected result in the limit as $W_e \to 0$. Moreover, we see that the error in the results of Gonzalez and Castellanos[8] is solely in the coefficient of the inverse Bond number τ.

ACKNOWLEDGMENTS

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