The simultaneous onset and interaction of Taylor and Dean instabilities in a Couette geometry

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Abstract. The fluid flow between a pair of coaxial circular cylinders generated by the uniform rotation of the inner cylinder and an azimuthal pressure gradient is susceptible to both Taylor and Dean type instabilities. The flow can be characterised by two parameters: a measure of the relative magnitude of the rotation and pressure effects and a non-dimensional Taylor number. Neutral curves associated with each instability can be constructed but it has been suggested that these curves do not cross but rather posses 'kinks'. Our work is based in the small gap, large wavenumber limit and considers the simultaneous onset of Taylor and Dean instabilities. The two linear instabilities interact at exponentially small orders and a consistent, matched asymptotic solution is found across the whole annular domain, identifying five regions of interest: two boundary adjustment regions and three internal critical points. We construct necessary conditions for the concurrent onset of the linear Taylor and Dean instabilities and show that neutral curve crossing is possible.

1. Introduction

The flow of a viscous fluid in the annular gap between two coaxial circular cylinders has been extensively studied analytically, numerically and experimentally for over a century. In particular, the flow generated by the uniform rotation of the inner cylinder and its subsequent instability to Taylor cells is familiar to all students of fluid mechanics. A wide-ranging review of the Couette-Taylor problem, its study and background, is given by Tagg [1]. The imposition of an azimuthal pressure gradient within the Couette geometry, although hard to realise in practice, provides an excellent model for curved channel flow and can itself be susceptible to linear perturbations and the cellular Dean instability. In this paper, we study the simultaneous onset of Taylor and Dean instabilities in the small gap, large wavenumber limit.

The composite Taylor-Dean problem has been discussed extensively in the literature and the linear eigenvalue equations governing perturbations to the unperturbed azimuthal flow in the limit of a small annular gap are well known. They have been studied by, among others, Di Prima [2], Hall [3] and Kachoyan [4]. The eigenvalue problem is determined by three parameters: the axial wavenumber k of the perturbations; a measure of the relative dominance of the effects of rotation and pressure, β; and a Taylor number $\widehat{T}$. For each value of β there exists a critical Taylor number $\widehat{T}_c$; for $\widehat{T} > \widehat{T}_c$ the nature of the instability depends upon the values of β and k. When the principal destabilising effect is rotation the instability will be akin to an isolated Taylor instability close to the inner cylinder. Conversely, when the main destabilising effect is pressure, a Dean-like instability will be observed in the main flow and away from the boundaries. At sufficiently large β there exists a range of wavenumbers for which linear perturbations of neither pure Taylor nor Dean type exist but rather time-periodic solutions are the dominant instability.

Taylor-Dean flow in the annulus is not easy to construct physically due the difficulty in maintaining azimuthal symmetry. However, the two distinct driving mechanisms of the flow are found together in many practical applications; e.g. an electrogalvanizing line in the steel-making industry or rotating drum filters in paper- and board-making processes (see [5]). The Taylor-Dean configuration is usually emulated in the laboratory using a partially-filled annulus to produce a combination of Couette and curved Poiseuille flow. There has been considerable interest in the diverse patterns that can be formed from this apparatus, including the coexistence of different states, and Laure & Mutabazi [6] provide a brief background to this work.

Kachoyan [4] considered the eigenvalue problem for general values of β and determined neutral curves associated with both the Taylor-like and Dean-like instabilities for several values of β. Of particular interest is when $\beta > 1$ and the pressure gradient acts against rotation. Kachoyan observed that, for $\beta = \beta_0 \simeq 1.275$, the leading order critical Taylor numbers agree at the limit of very large wavenumbers. As β is increased beyond β_0 the neutral curves associated with the two types of instability appear to cross at finite values of k. But, upon closer numerical examination Kachoyan concluded that the neutral curves did not intersect but rather there was a “kink” in each neutral curve.

We are concerned with large wavenumber perturbations in small-gap Taylor-Dean flow and consider the concurrent onset of linear Taylor and Dean instabilities at the same critical Taylor number. Our objective is to explore the kinking behaviour suggested by Kachoyan and to place it upon a firm rational footing. The two possible modes of instability are not treated in isolation but are shown to interact in a subtle way, linked at exponentially small orders by a WKB analysis. A consistent, matched asymptotic solution to the perturbation equations for the velocities, Taylor number and inverse wavenumber is found across the whole annular domain. Internal critical points result in the appearance of oscillatory solutions. We find that it is possible for neutral curve crossing to occur at discrete values of β but the accuracy required in calculations to observe such a phenomenon is not realisable.

2. Problem formulation

Consider an incompressible fluid occupying the region between two coaxial cylinders of radii R_1 and R_2 ($R_2 = R_1 + d > R_1$). The outer cylinder is at rest while the inner rotates about its axis with angular velocity Ω. In addition a constant, azimuthal pressure gradient is applied. There exists a solution to the Navier-Stokes equations given by

$$p = -\rho\kappa\theta + \rho\int \widehat{u}_\theta^2 r^{-1}\mathrm{d}r, \qquad \widehat{\mathbf{u}} = \widehat{u}_\theta \mathbf{e}_\theta = \Big\{\mathcal{A}r^{-1} + \mathcal{B}r - (2\nu)^{-1}\kappa r\ln r\Big\}\mathbf{e}_\theta\,. \tag{1}$$

Here p, ρ, $\widehat{\mathbf{u}}$ are the pressure, density and velocity fields respectively, (r,θ) the usual polar co-ordinates, ν the kinematic viscosity, κ a measure of the pressure gradient and $\mathcal{A}$ and $\mathcal{B}$ are known constants. We non-dimensionalise lengths with respect to d, radial and axial velocities by $\nu/2d$, time by d^2/ν, and $\widehat{u}_\theta$ by $V_T + V_D$ (where $V_T = \Omega R_1/2$ and $V_D = \kappa d^2/12\nu$ are typical velocities associated with Taylor and Dean effects). Then, in the limit of small gap $d \ll 1$, the dimensionless base flow is given by $\mathbf{u} = V\mathbf{e}_\theta$ where

$$V = 2(1-\beta)(1-x) + 6\beta x(1-x), \tag{2}$$

$\beta = V_D/(V_T + V_D)$ and $x = \big(r - R_1\big)/d$ is a new, scaled radial variable. The value $\beta = 0$ corresponds to Taylor-Couette flow and $\beta = 1$ to classical Dean flow. The linearised governing equations for small axisymmetric perturbations, $\big(u(x), \mathrm{v}(x), w(x)\big)e^{ikz}$ at neutral stability are

$$\mathcal{L}^2 u = \varepsilon^2\widehat{T}V\mathrm{v}, \qquad \mathcal{L}\mathrm{v} = \tfrac{1}{2}\varepsilon^2 uV', \tag{3}$$

where $\varepsilon = k^{-1}$, $\mathcal{L} \equiv \varepsilon^2(\mathrm{d}^2/\mathrm{d}x^2) - 1$ and $\widehat{T} = 4d^3(V_T + V_D)^2/\nu^2 R_1$ is the Taylor number.

We consider high wavenumbers $\varepsilon \ll 1$ and appropriate scalings of (3) are $T = \varepsilon^4\widehat{T}$, $v = \varepsilon^{-2}\mathrm{v}$. This leads to

$$\mathcal{L}^2 u = TVv, \qquad \mathcal{L}v = \tfrac{1}{2}uV' \tag{4}$$

subject to boundary conditions $u = v = u' = 0$ on the walls $x = 0,\,1$.

The operator $\mathcal{L}$ possesses two implicit scales and suggests a WKB analysis. Critical points arise where $TVV'/2$ takes values 0 or -1 and these require separate consideration. The latter case corresponds to two minima of VV' where the Rayleigh criterion is most violated and the onset of instability is expected to occur. The critical points are

$$\begin{aligned} x_0 &= \frac{1}{6\beta}\left(4\beta - 1 + \frac{1+2\beta}{\sqrt{3}}\right) \quad (VV' \text{ minimum}), \qquad & x_1 &= \frac{4\beta - 1}{6\beta} \quad (V' = 0), \\ x_2 &= \frac{\beta - 1}{3\beta} \quad (V = 0), & x_3 &= 0 \quad (VV' \text{ local minimum}). \end{aligned} \tag{5}$$

It is known that the structure at x_0 corresponds to the onset of a Dean instability and that x_3 corresponds to a Taylor instability at the cylinder wall. At leading order the isolated Taylor numbers are given by

$$T_D = -2/VV'\Big|_{x_0} = \frac{9\sqrt{3}\beta}{(1+2\beta)^2}, \qquad T_T = -2/VV'\Big|_{x_3} = -\frac{1}{2(4\beta-1)(1-\beta)}. \tag{6}$$

Therefore there are five clearly defined regions (shown later in figure 2): one near the wall $x = 1$ where the boundary conditions must hold; a critical point x_0 where the Dean instability appears; two internal critical points x_1, x_2 where the WKB solution is no longer appropriate and the wall zone at $x = 0$ where the Taylor instability exists.

We are concerned with the simultaneous onset of Taylor and Dean-like instabilities and their interaction. The question of whether their neutral stability curves cross is equivalent to whether it is possible to construct a consistent solution across the domain for particular values of the parameters β, ε and Taylor number, T. Therefore, we require the Taylor numbers T_D, T_T correspond at leading order. From (6) we find that $\beta = \beta_0 = (5 + 3\sqrt{3})/8$ and $T_D = T_T = 4/9$. To match at higher orders we perturb β,

$$\beta = \beta_0 + \delta, \qquad \delta \ll 1, \tag{7}$$

and construct asymptotic series for the physical quantities of the system in terms of δ.

The critical points x_0, x_3 lead to layers of widths $\varepsilon^{1/2}$ and $\varepsilon^{2/3}$ respectively and the corresponding corrections to the Taylor numbers for the isolated modes of instability are $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^{2/3})$. Comparison of the asymptotic forms of T_T, T_D therefore suggest an expansion of ε given by

$$\varepsilon = \varepsilon_0\delta^{3/2} + \varepsilon_1\delta^2 + \dots. \tag{8}$$

Let us now consider the asymptotic series for the velocities u, v and Taylor number T. We shall see that the various critical points are associated with layers of width $\mathcal{O}(\delta^{3/4})$, $\mathcal{O}(\delta)$ with an embedded layer of $\mathcal{O}(\delta^{3/2})$, and $\mathcal{O}(\delta^{9/8})$ and thus we expect that velocities (u, v) can be written:

$$(u_0, v_0) + \delta^{1/2}(u_1, v_1) + \delta^{3/4}(u_2, v_2) + \delta(u_3, v_3) + \delta^{9/8}(u_4, v_4) + \delta^{5/4}(u_5, v_5) + \delta^{3/2}(u_6, v_6) + \dots. \tag{9}$$

Other orders of u are not forced by the dynamics of the problem and merely reproduce, by linearity, the main solution. We expand the Taylor number in a similar manner but for later clarity we present two expansions – one appropriate at x_0, the other at x_3:

$$\begin{aligned} T &= T_0 + \delta^{1/2} T_1 + \delta^{3/4} T_2 + \delta T_3 + \delta^{9/8} T_4 + \delta^{5/4} T_5 + \delta^{3/2} T_6 + \dots, \\ &= \tau_0 + \delta^{1/2} \tau_1 + \delta^{3/4} \tau_2 + \delta \tau_3 + \delta^{9/8} \tau_4 + \delta^{5/4} \tau_5 + \delta^{3/2} \tau_6 + \dots . \end{aligned} \tag{10}$$

Note that since the Taylor coefficients (and velocities) will be functions of β, T_i is not necessarily equal to τ_i. Finally we denote the values of V, V' at x_j by V_j, V'_j respectively.

3. A linked flow solution throughout the domain

3.1. The WKB solution in the main flow

Consider the flow away from the critical points of g and the boundary walls. There are two scales x and εx and we seek a solution of the form

$$u = \mathsf{u}E, \qquad v = \mathsf{v}E \qquad \text{where} \quad E \equiv \exp\left\{\varepsilon^{-1} \int^x g(s)\mathrm{d}s\right\}. \tag{11}$$

Neglecting terms of $\mathcal{O}(\varepsilon^3)$, equations (4) then yield the system

$$\begin{aligned} (g^2-1)^2\mathsf{u} + \varepsilon\big[4\mathsf{u}'g(g^2-1) + 2\mathsf{u}g'(3g^2-1)\big] + \varepsilon^2\big[2\mathsf{u}''(3g^2-1) + 12\mathsf{u}'g'g + \mathsf{u}g''(4g+3)\big] &= T\mathsf{v}V, \\ (g^2-1)\mathsf{v} + \varepsilon(2\mathsf{v}'g + \mathsf{v}g') + \varepsilon^2\mathsf{v}'' &= \tfrac{1}{2}\mathsf{u}V'. \end{aligned} \tag{12}$$

Substituting the asymptotic series for u, v and ε we can equate orders in δ. At $\mathcal{O}(1)$ we find

$$(g^2-1)^2\mathsf{u}_0 = T_0\mathsf{v}_0V, \qquad (g^2-1)\mathsf{v}_0 = \tfrac{1}{2}\mathsf{u}_0V' \qquad \Longrightarrow \quad (g^2-1)^3 = T_0VV'/2. \tag{13}$$

Therefore there are six roots which we label

$$g_{\pm j} = \pm\sqrt{1 + \omega_j \operatorname{sgn}\left(\tfrac{1}{2}T_0VV'\right)\left|\tfrac{1}{2}T_0VV'\right|^{1/3}} \quad \text{where} \quad \omega_j = e^{2\pi j\mathrm{i}/3}, \quad j = 0,1,2. \tag{14}$$

The solutions $g_{\pm 0}$ correspond to exponentially growing and decaying solutions whereas the complex solutions are oscillatory with exponentially growing or decaying amplitudes. Figure 1 shows a typical plot of $-\frac{1}{2}T_0VV'$ and identifies critical points of (13) where the WKB form of solution is no longer valid. At $x = x_1$, x_2, the function $\frac{1}{2}T_0VV'$ vanishes and both of the sets of three roots $\{g_{+j}\}$, $\{g_{-j}\}$ coalesce leaving only two remaining real roots $g = \pm 1$. At x_0 and x_3, $\frac{1}{2}T_0VV'$ takes the value -1 and the two real roots $g_{\pm 0}$ coalesce leaving the four remaining complex roots and $g = 0$. A sketch of the behaviour of the roots of g in the complex plane is included in figure 1.

Equating coefficients of $\delta^{1/2}$, $\delta^{3/4}$, δ, $\delta^{9/8}$ and $\delta^{5/4}$ we find that, for a consistent solution of (12), $T_1 = T_2 = T_3 = T_4 = T_5 = 0$. This is not surprising – at $x = x_0$, $\beta = \beta_0$ the correction to the Taylor number is $\mathcal{O}(\varepsilon)$. At $\mathcal{O}(\delta^{3/2})$ the dynamics are forced by u_0 and v_0:

$$\begin{aligned} (g^2-1)^2\mathsf{u}_6 + \varepsilon_0\big[4\mathsf{u}_0'g(g^2-1) + 2\mathsf{u}_0g'(3g^2-1)\big] &= T_0\mathsf{v}_6 + T_6\mathsf{v}_0V, \\ (g^2-1)\mathsf{v}_6 + \varepsilon_0(2\mathsf{v}_0'g + \mathsf{v}_0g') &= \tfrac{1}{2}\mathsf{u}_6V'. \end{aligned} \tag{15}$$

Using (13) and (15) a first order differential equation for u_0 can be determined, namely

$$\frac{\mathsf{u}_0'}{\mathsf{u}_0} + \frac{g'}{2g} + \frac{V''}{3V'} - \frac{T_6}{6\varepsilon_0T_0}\frac{g^2-1}{g} = 0. \tag{16}$$

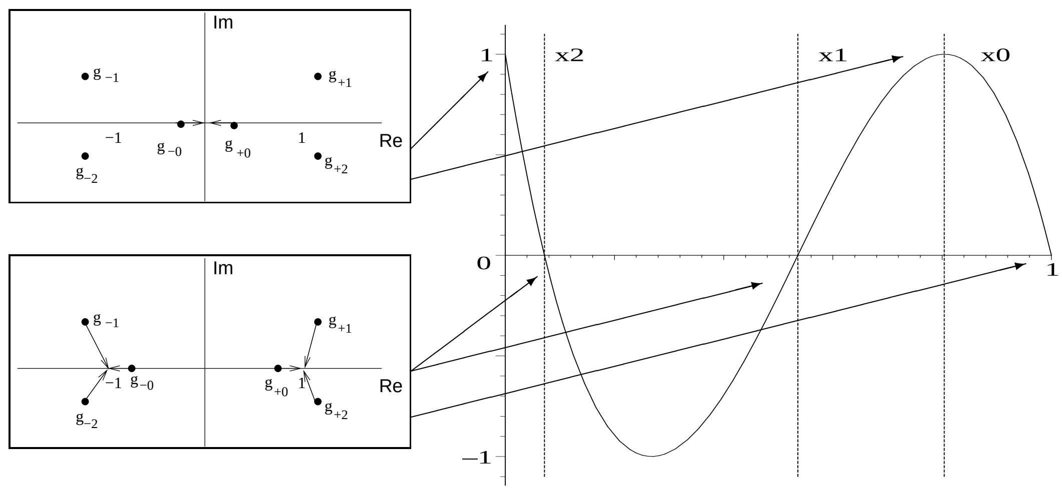

Figure 1. A typical plot of $-T_0VV'/2$ showing the critical points of the WKB solution and the behaviour of the roots of g there.

Solving (16) we find that

$$u_0 = \mathsf{u}_0 E \propto |g|^{-1/2}|V'|^{-1/3} \exp\left\{\frac{1}{\varepsilon}\int^x g(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int^x \frac{g(s)^2-1}{g(s)}\mathrm{d}s\right\},$$
$$v_0 = \mathsf{v}_0 E = \frac{\mathsf{u}_0 V' E}{2(g^2-1)} \propto \frac{|g|^{-1/2}|V'|^{2/3}}{2(g^2-1)} \exp\left\{\frac{1}{\varepsilon}\int^x g(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int^x \frac{g(s)^2-1}{g(s)}\mathrm{d}s\right\}. \quad (17)$$

The six roots of g give the six linearly independent leading order velocities outside the critical layers. We will link these solutions across the critical layers (where roots coalesce and separate asymptotic analysis is needed) and to the boundaries $x = 0, 1$. We assume that both the Taylor and Dean disturbances are $\mathcal{O}(1)$ and that they are spatially decaying, interacting at exponentially small orders via the WKB solutions, to produce necessary conditions for simultaneous onset.

3.2. The critical layer at x_0

Although as x_0 is approached the complex roots $g_{\pm j}$, $j = 1, 2$ remain distinct and their velocity contribution is unaffected by the layer, the real roots of g coalesce at 0. It is a second order critical point and a balance of scales shows that the layer is $\mathcal{O}(\delta^{3/4})$. Thus we introduce a scaled variable given by $\xi = (x - x_0)\delta^{-3/4}$. Expressing the differential operator $\mathcal{L}$ in terms of ξ and expanding the base velocity V about x_0 we can equate orders of δ within (4). At $\mathcal{O}(1)$ we have

$$u_0 = T_0 v_0 V_0, \quad -v_0 = \tfrac{1}{2}u_0 V_0' \quad \Longrightarrow \quad T_0 = -2/V_0V_0'. \quad (18)$$

Thus T_0 is determined for the neutral stability of Dean disturbances. The next few orders relate u_k and v_k, $k = 1 \ldots 5$, but at $\mathcal{O}(\delta^{3/2})$ we have a linear equation governing u_0:

$$\frac{d^2u_0}{d\xi^2} + \frac{u_0}{3\varepsilon_0^2}\left(\frac{T_6}{T_0} + \frac{3V''}{2V_0}\xi^2\right) = 0. \quad (19)$$

Equation (19) reduces to the parabolic cylinder equation

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}X^2} - u_0\left(a + \tfrac{1}{4}X^2\right) = 0 \quad \text{where } X = \left(\frac{-2V''}{\varepsilon_0^2 V_0}\right)^{1/4}\xi, \quad a = -\frac{T_6}{3\varepsilon_0 T_0}\left(\frac{V_0}{-2V''}\right)^{1/2}, \quad (20)$$

which possesses two linearly independent solutions $U(a, X)$, $V(a, X)$. The asymptotic forms of these solutions for $X \gg 1$ are given by Abramowitz & Stegun [7]. The solution $V(a, X)$ displays exponential growth as $X \to \infty$. But we are investigating (the concurrent) onset of instability and assume that any deviation from the isolated $\mathcal{O}(1)$ Taylor and Dean modes of instability is tiny. Thus we reject any contribution to u_0 from $V(a, X)$. By linearity we take $u_0 = U(a, X)$ whereby

$$
\begin{aligned}
u_0 &\sim X^{-a-\frac{1}{2}} e^{-X^2/4}, \qquad\qquad \text{as } X \to \infty, \\
u_0 &\sim \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+a\right)}(-X)^{a-\frac{1}{2}} e^{X^2/4} - \sin \pi a (-X)^{-a-\frac{1}{2}} e^{-X^2/4}, \quad \text{as } X \to -\infty.
\end{aligned}
\tag{21}
$$

Although u_0 seems to possess a term with exponential growth as $X \to -\infty$, at $a = -1/2$ its coefficient vanishes, corresponding to an isolated Dean mode of instability. We require that a be exponentially close to $-1/2$ and that any growth will become $\mathcal{O}(1)$ at $x = x_3$, corresponding to the interaction between the Taylor and Dean modes (see figure 2). This condition will relate T_6 and ε_0.

We match this layer onto the main flow for $x_1 < x < x_0$ using the method of intermediate scales. Introducing a lower limit, $l_0 = x_0 - \delta^{3/4}$ to accommodate the simple pole of the integrand, we find that the radial velocity for $x_1 < x < x_0$ is given by

$$
\begin{aligned}
u_0 = \mathcal{C}_+ |g_{+0}|^{-1/2} |V'|^{-1/3} \exp\left\{ \frac{1}{\varepsilon} \int_{x_0}^{x} g_{+0}(s) \mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0} \int_{l_0}^{x} \frac{g_{+0}(s)^2 - 1}{g_{+0}(s)} \mathrm{d}s \right\} \\
+ \mathcal{C}_- |g_{-0}|^{-1/2} |V'|^{-1/3} \exp\left\{ \frac{1}{\varepsilon} \int_{x_0}^{x} g_{-0}(s) \mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0} \int_{l_0}^{x} \frac{g_{-0}(s)^2 - 1}{g_{-0}(s)} \mathrm{d}s \right\},
\end{aligned}
\tag{22}
$$

where $\alpha = -2V_0/V''$ and

$$
\mathcal{C}_+ = -\sin \pi a \; \alpha^{-\frac{1}{4}} (-V_0')^{\frac{1}{3}} \delta^{\frac{3}{8}} \left(\frac{4}{\varepsilon_0^2 \alpha} \right)^{-\frac{1}{4}\left(a-\frac{1}{2}\right)}, \quad \mathcal{C}_- = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+a)} \alpha^{-\frac{1}{4}} (-V_0')^{\frac{1}{3}} \delta^{\frac{3}{8}} \left(\frac{4}{\varepsilon_0^2 \alpha} \right)^{\frac{1}{4}\left(a-\frac{1}{2}\right)}.
\tag{23}
$$

We note that the factor $\delta^{3/8}$ implies that the flow becomes more intense in the layer, as might be expected.

3.3. The critical layer about x_1

In the main flow the leading order velocities u_0, v_0 are proportional to $|V'|^{-1/3}$ and $|V'|^{2/3}(g^2 - 1)^{-1}$ respectively. As we approach $x = x_1$ we find that $V' \to 0$, $g \to \pm 1$ and $|V'|^{2/3}(g^2 - 1)^{-1} \to 0$. It is clear therefore that, near this critical point, the assumption that u_0, v_0 are of the same order is no longer valid. By careful consideration of the appropriate asymptotic scales and the governing equations it can be seen that, near x_1, $u \sim \mathcal{O}(\delta^{-3/8})$, $v \sim \mathcal{O}(\delta^{3/8})$ in a layer of width $\mathcal{O}(\delta^{9/8})$. Introducing the scaled variable $\xi = (x - x_1)\delta^{-9/8}$ we look for solutions of the form

$$
u = \delta^{-3/8} \widetilde{u}(\xi) \exp(\pm x/\varepsilon), \qquad v = \delta^{3/8} \widetilde{v}(\xi) \exp(\pm x/\varepsilon).
\tag{24}
$$

The velocity V and operator $\mathcal{L}$ can once again be expressed in terms of ξ and, equating at lowest orders of δ, the governing equations about x_1 become

$$
\widetilde{u}_0''' = \pm \frac{T_0 V_1 V''}{16\varepsilon_0^3} \xi \widetilde{u}_0, \qquad \widetilde{v}_0 = \frac{4\varepsilon_0^2}{T_0 V_1} \widetilde{u}_0''.
\tag{25}
$$

A change of variables simplifies the system within the layer to two third order differential equations corresponding to the two sets of roots, $\{g_{+j}\}$ and $\{g_{-j}\}$ in the main flow:

$$\frac{\mathrm{d}^3\widetilde{u}_0}{\mathrm{d}X_\pm^3} = -X_\pm \widetilde{u}_0, \quad \text{where} \quad X_+ = (2\varepsilon_0)^{-3/4}\gamma^{1/4}\xi = e^{-\pi \mathrm{i}/4}X_-, \quad \gamma = -\tfrac{1}{2}T_0V_1V''. \tag{26}$$

In the limit $|X_\pm| \gg 1$ there are three linearly independent solutions with asymptotic forms

$$f_{\pm j} = X_\pm^{-1/3}\exp\left\{-\tfrac{3}{4}\omega_j X_\pm^{4/3}\right\}, \qquad j = 0, 1, 2 \tag{27}$$

and our solution will match a linear combination of the $f_{\pm j}$. But, to obtain a consistent solution to (26) throughout the complex plane for $|X_\pm| \gg 1$, we require that, as the argument of $X_\pm$ varies (and we cross the layer), the coefficients of the $f_{\pm j}$ change as we cross Stokes lines. The changes are given by known Stokes multipliers. The asymptotic behaviour of equation (26) has been analysed by Heading [8] and as $\arg X_+$ goes from 0_+ to π_- the linear combination $c_0f_{+0}+c_1f_{+1}+c_2f_{+2}$ becomes $c_0f_{+0}+c_1f_{+1}+(e^{2\pi \mathrm{i}/3}c_0+e^{\pi \mathrm{i}/3}c_1+c_2)f_{+2}$. Similarly, as $\arg X_-$ goes from $\pi/4$ to $5\pi/4$, $c_0f_{-0}+c_1f_{-1}+c_2f_{-2}$ becomes $c_0f_{-0}+e^{2\pi \mathrm{i}/3}c_2f_{-1}+(e^{2\pi \mathrm{i}/3}c_0+e^{\pi \mathrm{i}/3}c_1+c_2)f_{-2}$. We can therefore trace the evolution of our solution (22) in $x_1 < x < x_0$ across the layer about x_1 to the region $x_2 < x < x_1$. It is found that the dominant solution associated with $(g_{-0}+1)$ – an increasing function about x_1 – remains dominant and the sub-dominant solution relating to $(g_{+0}-1)$ – a decreasing function about x_1 – becomes a pair of oscillating solutions in $[x_2, x_1]$:

$$\begin{aligned} u_0 = \mathcal{D}_+|g_{+1}|^{-1/2}|V'|^{-1/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_1}^{x} g_{+1}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0T_0}\int_{x_1}^{x}\frac{g_{+1}(s)^2-1}{g_{+1}(s)}\mathrm{d}s\right\} + \text{c.c.} \\ + \mathcal{D}_-|g_{-0}|^{-1/2}|V'|^{-1/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_1}^{x} g_{-0}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0T_0}\int_{x_1}^{x}\frac{g_{-0}(s)^2-1}{g_{-0}(s)}\mathrm{d}s\right\}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \mathcal{D}_+ &= \mathcal{C}_+e^{\pi \mathrm{i}/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_0}^{x_1} g_{+0}\mathrm{d}s + \frac{T_6}{6\varepsilon_0T_0}\int_{l_0}^{x_1}\frac{g_{+0}^2-1}{g_{+0}}\mathrm{d}s\right\}, \\ \mathcal{D}_- &= \mathcal{C}_-\exp\left\{\frac{1}{\varepsilon}\int_{x_0}^{x_1} g_{-0}\mathrm{d}s + \frac{T_6}{6\varepsilon_0T_0}\int_{l_0}^{x_1}\frac{g_{-0}^2-1}{g_{-0}}\mathrm{d}s\right\}. \end{aligned} \tag{29}$$

3.4. The critical layer about x_2

The layer about x_2 is analogous to that at x_1, except that now it is V that vanishes rather than V'. The appropriate scaling is $u_0 \sim \mathcal{O}(1)$ and, since $g \to \pm 1$, $v_0 \sim \mathcal{O}(\delta^{-3/8})$ within a layer of $\mathcal{O}(\delta^{9/8})$. We introduce the new variable $\xi = (x - x_2)\delta^{-9/8}$ and write $u = \widetilde{u}(\xi)e^{\pm x/\varepsilon}$, $v = \delta^{-3/8}\widetilde{v}(\xi)e^{\pm x/\varepsilon}$. We find that it is now the azimuthal velocity that satisfies the parabolic cylinder equation,

$$\frac{\mathrm{d}^3\widetilde{v}_0}{\mathrm{d}X_\pm^3} = -X_\pm \widetilde{v}_0, \quad \text{where} \quad X_- = (2\varepsilon_0)^{-3/4}\lambda^{1/4}\xi = e^{-\pi \mathrm{i}/4}X_+, \quad \lambda = -\tfrac{1}{2}T_0(V_2')^2, \tag{30}$$

and $\widetilde{u}_0 = 4\varepsilon_0\widetilde{v}_0'/V_2'$. Therefore the asymptotic solutions and Stokes multipliers remain unchanged. Near x_2, g_{-0} is decreasing about -1 and thus f_{-0} is the sub-dominant solution. Therefore when we match to the left this solution maps onto two oscillating solutions whereas the oscillating solutions associated with $g_{+1,2}$ are equally dominant and merge into a single solution. Matching the solution for $\widetilde{v}_0$ in the layer about x_2 to the left, it can be shown, using (17b), that in the region $x_3 < x < x_2$, the *radial* velocity is given by

$$\begin{aligned}u_0 = \mathcal{E}_+|g_{+0}|^{-1/2}|V'|^{-1/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_2}^{x} g_{+0}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_2}^{x}\frac{g_{+0}(s)^2-1}{g_{+0}(s)}\mathrm{d}s\right\}\\ + \mathcal{E}_-|g_{-1}|^{-1/2}|V'|^{-1/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_2}^{x} g_{-1}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_2}^{x}\frac{g_{-1}(s)^2-1}{g_{-1}(s)}\mathrm{d}s\right\} + \text{c.c.},\end{aligned} \quad (31)$$

where

$$\begin{aligned}\mathcal{E}_+ &= \mathcal{D}_+\exp\left\{\frac{1}{\varepsilon}\int_{x_1}^{x_2} g_{+1}\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_1}^{x_2}\frac{g_{+1}^2-1}{g_{+1}}\mathrm{d}s\right\} + \text{c.c.},\\ \mathcal{E}_- &= \mathcal{D}_-\exp\left\{\frac{1}{\varepsilon}\int_{x_1}^{x_2} g_{-0}\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_1}^{x_2}\frac{g_{-0}^2-1}{g_{-0}}\mathrm{d}s\right\}.\end{aligned} \quad (32)$$

3.5. The layer at $x_3 = 0$

We now study the final critical point $x_3 = 0$, a local minimum of $\frac{1}{2}T_0VV'$. It is not a stationary point of the function and hence the layer is of width $\mathcal{O}(\delta)$. We rescale with $\xi = x\delta^{-1}$ and expanding asymptotically about x_3 as usual. We can equate coefficients in δ but, as will become apparent, it is necessary to use the asymptotic expansion of T in terms of τ_j. We find at $\mathcal{O}(1)$ that

$$u_0 = \tau_0 v_0 V_3, \quad -v_0 = \tfrac{1}{2}u_0 V_3' \quad \Longrightarrow \quad \tau_0 = -1/2V_3V_3'. \quad (33)$$

At the next two orders the equations reduce to $\tau_1 = \tau_2 = 0$, but at $\mathcal{O}(\delta)$ the problem is forced so that

$$u_3 - 2\varepsilon_0^2 u_0'' = \tau_0 V_3 v_3 + \tau_0\xi V_3' v_0 + \tau_3 V_3 v_0, \qquad \varepsilon_0^2 v_0'' - v_3 = \tfrac{1}{2}\xi V'' u_0 + \tfrac{1}{2}V_3' u_3. \quad (34)$$

Note that T_0, τ_0 only agree up to $\mathcal{O}(\delta)$ and hence $\tau_3 \neq 0$. Using (33), system (34) reduces to

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}X^2} = X u_0 \ \text{ where } \ X = \lambda_0^{1/3}\xi - \lambda_1\lambda_0^{-2/3}, \quad \lambda_0 = -\frac{1}{3\varepsilon_0^2}\left(\frac{V''}{V_3'} + \frac{V_3'}{V_3}\right) > 0, \quad \lambda_1 = \frac{\tau_3}{3\tau_0\varepsilon_0^2}. \quad (35)$$

The Airy functions $\mathrm{Ai}(X)$ and $\mathrm{Bi}(X)$ solve (35) and their asymptotic behaviours for $X \gg 1$ are

$$\mathrm{Ai}(X) \sim 2^{-1}\pi^{-1/2}X^{-1/4}\exp\left(-\tfrac{2}{3}X^{3/2}\right), \qquad \mathrm{Bi}(X) \sim \pi^{-1/2}X^{-1/4}\exp\left(\tfrac{2}{3}X^{3/2}\right). \quad (36)$$

It is clear that Bi is exponentially increasing and cannot contribute at $\mathcal{O}(1)$ as this would not correspond to an isolated mode of instability. In fact, the coefficient of Bi must be exponentially small indicating a slight detuning of the isolated mode and link with the incoming WKB mode corresponding to g_{+0}. Thus we find the leading order contribution to Bi is

$$\frac{\mathcal{E}_+}{\sqrt{\pi\varepsilon_0}}\delta^{1/4}\lambda_0^{-1/6}(V_3')^{-1/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_2}^{0} g_{+0}\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_2}^{0}\frac{g_{+0}^2-1}{g_{+0}}\mathrm{d}s\right\}\mathrm{Bi}(X). \quad (37)$$

We require the coefficient of Ai to be $\mathcal{O}(1)$ as we wish the two instabilities, Taylor and Dean, to occur simultaneously and be of similar sizes. It is clear that as $X \to \infty$ this links to the velocity corresponding with the root g_{-0} in the region $x_3 < x < x_2$ which, as it encounters the critical point x_2 from the left, will become two oscillatory, complex conjugate solutions corresponding to $g_{-1,2}$. These will continue to decrease exponentially to the wall $x = 1$. Thus the coefficient

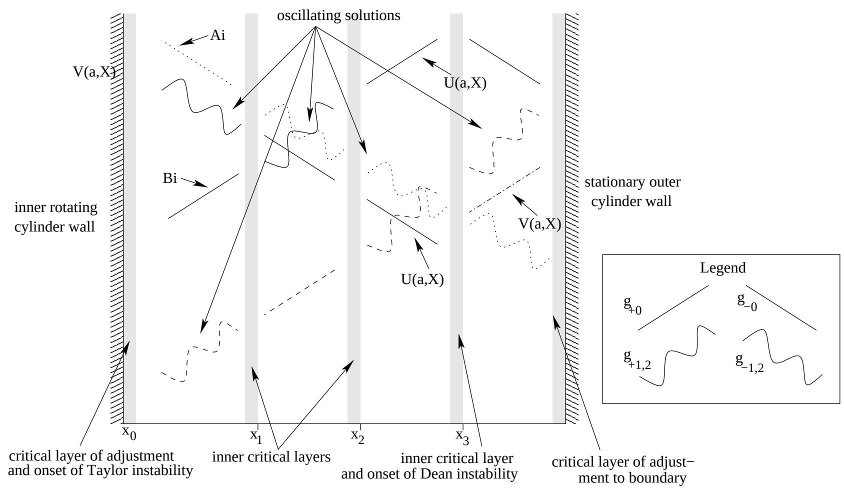

Figure 2. Diagram of the six linearly independent solutions across the flow domain.

of Ai is not known and we can only say at this stage that $u_0 = \mathcal{F}\mathrm{Ai}(X)$ where $\mathcal{F}$ is some $\mathcal{O}(1)$ constant.

We are investigating corrections to the Taylor number up to $\mathcal{O}(\delta^{3/2})$ and continue to equate at higher orders of δ. At $\mathcal{O}(\delta^{9/8})$, $\mathcal{O}(\delta^{5/4})$ we find that $\tau_4 = \tau_5 = 0$. But, at $\mathcal{O}(\delta^{3/2})$, we find

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}X^2} - Xu_1 = -\frac{\tau_6}{3\varepsilon_0^2\lambda_0^{2/3}}u_0 + \frac{2\varepsilon_1}{\varepsilon_0}\frac{\mathrm{d}^2 u_0}{\mathrm{d}X^2}, \tag{38}$$

with the same change of variables as (35). Equation (38) is solved in terms of Airy functions to give

$$u_1 = \mathcal{G}\mathrm{Ai}(X) - \left(\frac{\tau_6}{3\varepsilon_0^2\lambda_0^{2/3}} + \frac{2\varepsilon_1 X}{3\varepsilon_0}\right)\mathcal{F}\mathrm{Ai}'(X). \tag{39}$$

Before we can write down the final conditions governing the parameters of our linked leading order solutions we must consider the boundary conditions at the two cylinder walls $x = 0,\ 1$.

3.6. Boundary conditions at $x = 0$ and $x = 1$

The boundary conditions $u = v = w = 0$ on $x = 0,\ 1$ are equivalent to $u = u' = \mathcal{L}^2 u = 0$ on the walls. Near $x = 1$ the flow has decayed to be exponentially small but even here it is necessary to ensure adherence to the boundary conditions. We have found six linearly independent solutions across the flow domain but it is clear that at any position in the fluid not all these solutions can exist at leading order. The condition that the flow should only be $\mathcal{O}(1)$ at the two points of instability coupled with the boundary conditions allows us to anticipate the dominant flow solutions across the domain and, in particular, at the cylinder boundaries. Figure 2 shows diagrammatically the evolution of the six flow elements. The different styles of line illustrate the linking of solutions and the legend distinguishes the cartoons used to depict each of the six types of solution. An indication of the magnitude of a solution is given by its vertical position.

We begin by considering the wall at $x = 1$. Of the six solutions here, the two oscillatory decaying solutions associated with $g_{-1,2}$ link directly to the decay of the Airy solution at x_3. As such they have decayed throughout the whole fluid domain and at $x = 1$ we would expect them to be far smaller than the other flow components. Thus to satisfy the boundary conditions we have four remaining solutions at our disposal: one associated with the $\mathcal{O}(1)$ Dean instability whose coefficient is predetermined; two oscillating solutions which decay all the way to the wall $x = 0$; and the solution associated with g_{+0} which links with the solution $V(a, X)$ of the parabolic cylinder equation. Thus we are able to satisfy the boundary conditions at $x = 1$ wall by choosing the coefficients of the latter three, which will inevitably be be exponentially small at $x = 1$.

The boundary $x = 0$ requires more detailed consideration. Examining figure 2 we see that of the six solutions, the exponentially increasing one associated with Bi is linked to the Dean instability and hence is exponentially small. The oscillatory growing modes link with the exponentially small oscillating modes at the wall $x = 1$ and have further decayed across the domain. The only remaining, possibly $\mathcal{O}(1)$, modes are the oscillatory decaying solutions g_{-j} (also linking to the Dean instability $U(a, X)$) and the decreasing solution associated with Ai. However we have three homogeneous boundary conditions and the three dominant solutions will be unable to satisfy them non-trivially. Hence an inner adjustment layer is introduced to satisfy the boundary conditions.

3.7. The inner layer at $x = 0$

A balance of scales reveals an inner layer of width $\mathcal{O}(\delta^{3/2})$ at $x = 0$. Introducing $\xi \equiv x\delta^{-3/2}$, we solve the leading order governing equations and apply the boundary conditions at $\xi = 0$ to give

$$u_0 = \mathcal{H}\left[1 - (12)^{1/4}\xi/\varepsilon_0 + e^{-3^{1/4}\cos(\pi/12)\xi/\varepsilon_0}\left(\sqrt{3}\sin\left(3^{1/4}\sin(\pi/12)\xi/\varepsilon_0\right)\right.\right.$$
$$\left.\left. - \cos\left(3^{1/4}\sin(\pi/12)\xi/\varepsilon_0\right)\right)\right], \quad (40)$$

where $\mathcal{H}$ is a constant, $\sqrt{1-\omega_1} = 3^{1/4}e^{-\pi \mathrm{i}/12}$, $\sqrt{1-\omega_2} = 3^{1/4}e^{\pi \mathrm{i}/12}$, and we have discarded the exponentially increasing and growing oscillatory solutions at this order as required.

We require that (40) matches with the velocity in the outer layer given by the Airy solutions, $\mathcal{F}\mathrm{Ai}(X)$ and (39), together with the two oscillating, decaying WKB solutions from (31). To achieve this requires $\mathrm{Ai}(-\lambda_1\lambda_0^{-2/3}) = 0$ and $\mathcal{H} = \widetilde{\mathcal{H}}\delta^{1/2}$ with $\widetilde{\mathcal{H}} = \mathcal{O}(1)$; in particular

$$\lambda_1\lambda_0^{-2/3} = \frac{\tau_3}{3\tau_0\varepsilon_0^2}\left[-\frac{1}{3\varepsilon_0^2}\left(\frac{V''}{V_3'} + \frac{V_3'}{V_3}\right)\right]^{-2/3} = 2.3381,$$

$$\mathcal{F}\left(2.3381\frac{2\varepsilon_1}{3\varepsilon_0} - \frac{\tau_6}{3\varepsilon_0^2\lambda_0^{2/3}}\right)\mathrm{Ai}'(-2.3381) = \widetilde{\mathcal{H}}, \qquad \mathcal{F}\lambda_0^{1/3}\mathrm{Ai}'(-2.3381) = -(12)^{1/4}\widetilde{\mathcal{H}}/\varepsilon_0. \quad (41)$$

Matching the oscillatory, decaying solutions yields

$$\mathcal{E}_-(1-\omega_1)^{-1/4}(V_3')^{-1/3}\exp\left\{\frac{1}{\varepsilon}\int_{x_2}^{0} g_{-1}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_2}^{0}\frac{g_{-1}(s)^2 - 1}{g_{-1}(s)}\mathrm{d}s\right\} = \delta^{1/2}\widetilde{\mathcal{H}}e^{4\pi \mathrm{i}/3}. \quad (42)$$

We now have all the necessary equations. First, observe that the two Taylor number expansions (10) must be equivalent. We expand τ_0, T_0 as series in δ using (18), (33) and equate coefficients. At $\mathcal{O}(\delta)$ we find $\tau_3 = 1.6501$ and then (41a) is used to determine ε_0:

$$\varepsilon_0 = \left(\frac{\tau_3}{7.01432\tau_0}\right)^{3/2}\left[-\frac{1}{3\varepsilon_0^2}\left(\frac{V''}{V_3'} + \frac{V_3'}{V_3}\right)\right]^{-1} = 0.0688 + 0.1631\delta + \mathcal{O}(\delta^2). \quad (43)$$

From the arguments of §3.2 we know that a is exponentially close to $-1/2$ so that, from (20),

$$\tau_6 = T_6 = -\frac{3}{2}\varepsilon_0 T_0\left(\frac{-2V''}{V_0}\right)^{1/2} = 0.2421 + 0.4082\delta + \mathcal{O}(\delta^2). \quad (44)$$

It follows from (41c) and (41b) respectively that

$$\widetilde{\mathcal{H}}/\mathcal{F} = -0.2741 + 0.0018\delta + \mathcal{O}(\delta^2), \qquad \varepsilon_1 = -0.0105 - 0.0134\delta + \mathcal{O}(\delta^2). \tag{45}$$

Our final conditions are given by equation (42). Using (23), (29), (32) we find that

$$\mathcal{E}_- \sim 0.7439\left(\tfrac{1}{2} + a\right)\delta^{3/8} \exp\left\{\frac{1}{\varepsilon}\int_{x_0}^{x_2} g_{-0}\mathrm{d}s + 1.3195\int_{l_0}^{x_2} \frac{g_{-0}^2 - 1}{g_{-0}}\mathrm{d}s\right\} \in \Re. \tag{46}$$

Equating magnitudes and argument in (42) yields

$$\begin{aligned} \tfrac{1}{2} + a \sim 3.1091\widetilde{\mathcal{H}}\delta^{1/8} \exp\Bigg\{-\Re\left[\frac{1}{\varepsilon}\int_{x_2}^{0} g_{-1}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_2}^{0}\frac{g_{-1}(s)^2 - 1}{g_{-1}(s)}\mathrm{d}s\right] \\ -\frac{1}{\varepsilon}\int_{x_0}^{x_2} g_{-0}\mathrm{d}s - 1.3195s\Bigg\}, \\ 4\pi/3 = \pi/24 + \Im\left[\frac{1}{\varepsilon}\int_{x_2}^{0} g_{-1}(s)\mathrm{d}s + \frac{T_6}{6\varepsilon_0 T_0}\int_{x_2}^{0}\frac{g_{-1}(s)^2 - 1}{g_{-1}(s)}\mathrm{d}s\right] \quad \text{modulus } 2\pi. \end{aligned} \tag{47}$$

Equation (47b) will determine the discrete values of δ at which the solution has the correct phase to satisfy the boundary conditions. As $\delta \to 0$, the right hand side will increase unboundedly and there are will be multiple values of δ for which this expression is satisfied.

4. Conclusions

In the preceding sections we have demonstrated the manner in which a consistent solution of the governing equations for the simultaneous onset of exponentially linked Taylor and Dean instabilities is constructed. Our consideration of the leading order velocity has determined the leading coefficients in the asymptotic expansions of ε and the Taylor number T. In order to precisely determine the values of δ for which the phase matching condition (47) is satisfied we would require ε to be determined to $\mathcal{O}(\delta^3)$. The process is extremely involved but once completed would produce an implicit expression to determine δ. Having demonstrated the approach necessary we do not pursue these higher orders.

The values of δ which satisfy (47) will be discrete – sparse for moderate values of δ but becoming more closely gathered as zero is approached. Therefore we predict that neutral curve crossing is theoretically possible but will be almost impossible to observe numerically. The values of β at which numerical experimentation is performed will need to be precise – any deviation from the discrete theoretical values required will result in exponential divergence from the required solution phase.

In making comparisons with previous studies, and in particular that of [4], it must be remembered that our study concentrates upon high wavenumbers. However, as Kachoyan suggests, the asymptotic behaviours observed here might be used as a guide to interpret the behaviour of neutral curves for other values of β.

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