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Cognitive Functioning in Mathematical Problem Solving During Early Adolescence

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Problem-solving in school mathematics has traditionally been considered as belonging only to the concrete symbolic mode of thinking, the mode which is concerned with making logical, analytical deductions. Little attention has been given to the place of the intuitive processes of the ikonic mode. The present study was designed to explore the interface between logical and intuitive processes in the context of mathematical problem solving. Sixteen Year 9 and 10 students from advanced mathematics classes were individually assessed while they solved five mathematics problems. Each student's problem-solving path, for each problem, was mapped according to the type of strategies used. Strategies were broadly classified into Ikonic (IK) or Concrete Symbolic (CS) categories. Students were given two types of problems to solve: (i) those most likely to attract a concrete symbolic approach; and (ii) problems with a significant imaging or intuitive component. Students were also assessed as to the vividness and controllability of their imaging ability, and their creativity. Results indicated that the nature of the problem is a basic factor in determining the type of strategy used for its solution. Students consistently applied CS strategies to CS problems, and IK strategies to IK problems. In addition, students tended to change modes significantly more often when solving CS-type problems than when solving IK-type problems. A switch to IK functioning appeared to be particularly helpful in breaking an unproductive set when solving a CS-type problem. Individual differences in strategy use were also found, with students high on vividness of imagery using IK strategies more frequently than students who were low on vividness. No relationship was found between IK strategy use and either students' degree of controllability of imagery or their level of creativity. The instructional implications of the results are discussed.

Problem solving in school mathematics has traditionally been considered as belonging only to the concrete symbolic mode of thinking: the mode which is concerned with making logical connections between data and a mathematical model, teasing out the relationship between the variable(s) in the model and then relating the result back to the concrete data. Little, if any, attention has been given to the place of the intuitive processes of the ikonic mode at this level despite the well-known use of these processes in the thinking of researchers at a high level of science and mathematics (Hadamard, 1954). Moreover, research in out-of-school thinking (Resnick, 1987) during the past decade points to the predominance of intuitive mode strategies in everyday problem solving especially in mathematical situations with unschooled individuals.

In the past decade, educators have begun to realise the limitations of the traditional approach, and mathematics curricula which are appearing throughout the world as a result of national enquiries are focusing on achieving a balance between the two modes of functioning (Cockcroft, 1982; National Council of Teachers of Mathematics, 1989). The problem is that there is little relevant research which deals with learning or its assessment from this point of view (there is of course an enormous amount of literature dealing with assessment in the concrete symbolic mode). The result is that curriculum, teaching and assessment decisions are being made on the basis of what is recognised to be good exemplary practice at this stage (e.g., California State Department of Education, 1989; de Lange, 1987; *Graded Assessment*, 1988). A recent analysis (Collis & Romberg, 1991) of some assessment items from one of the projects (California State Department of Education, 1989) shows clearly how deficient and inefficient this procedure is likely to be even in the relatively short term.

A search of the literature reveals why there has been a reliance upon good practice and teachers' intuitions as the model for assessment in this area of problem solving. Most of the research has concentrated on school-based problem solving during the very early elementary years (e.g., Carpenter, 1985; Carpenter & Moser, 1983; Swing, Stoiber & Peterson, 1988) or on teasing out variables associated with the various processes in the psychological domain that appear relevant (e.g., Greeno, 1983; Kintsch & Greeno, 1985). In addition a series of studies describing the nature of "outside school" or "work place" mathematical problem solving have been reported in the past decade (e.g., Carraher, 1989; Resnick, 1987; Schliemann & Acioly, 1989; Scribner, 1986). It is fair to say that, although many of these studies have figured in the intuitions of mathematics educators when designing recent problem solving assessment procedures and items, little effort has been made to look at the problem scientifically with assessment specifically in mind. The study reported here is an attempt to redress this balance and to begin to look at the problem solving processes of the students in high school with a view to devising a rational means of assessing their competence.

Theoretical Orientation of the Study

The study reported here is grounded in the Biggs and Collis formulation of the development of cognition as updated in 1991 (Biggs & Collis, 1991) from its original 1982 version (Biggs & Collis, 1982). In addition the work of Collis and Romberg (1991) on the assessment of open-ended items in mathematical problem solving has been utilised in the basic framework for gathering the empirical data. Let us examine briefly these two contributions to the background of this study.

Biggs and Collis (1991) and Collis and Biggs (1991) pointed out that much of our thinking in the area of problem solving is multimodal. In particular, students at the early secondaryl level, when presented with a novel mathematical problem, have available three modes of intellectual functioning which they can bring to bear to seek a solution: sensori motor, ikonic and concrete symbolic. The first of these is usually not of great significance in school-based problem solving and will not be discussed in this paper. The second is highly developed by the time students reach the age level of interest here and has reached this high level largely without school help. The third is very dependent on school-based teaching, and the subsequent learning, of the concrete symbolic systems of reading, writing and mathematics. In this paper it is the development of the concrete symbolic system of mathematics which is the major concern.

Biggs and Collis (1991) argued, following Piaget and others, that the ikonic mode of functioning begins in early childhood (i.e., around 18 months) with the beginning of true thought in the form of the internalisation of action (Piaget, 1950), a form of imaging which Bruner (1964) later referred to as an "ikon." From a very rudimentary form externalised in the one word "sentence" of infants, it generalises, with the aid of language throughout the pre-school years , to the richly imaged world of the early school child. Throughout its development, ikonic thought not only draws on imaging but very often has heavy overtones of the child's affective life. By adopting this mode for encoding the real world, young children find much satisfaction in interpreting many of the interactions they see in real life in terms of myths and stories (Egan, 1984).

This then is the beginning of the ikonic mode but it continues to develop until well into adulthood where ikonic thought clearly goes beyond primitive explanations to the intuitive thinking displayed in such areas as aesthetics, mathematics and science. The latter two domains, in particular, appear to make regular use of this mode in problem solving; Hadamard (1954) supplies many examples from mathematics, and Kekule's use of this mode in solving the problem associated with the structure of the organic ring is well known. The ikonic mode is clearly not simply to be associated with the presymbolic thought of early childhood but continues to grow in both power and complexity into adulthood. Moreover, as the later modes develop, there appears to be an interaction between modes which enables typical ikonic-mode strategies to be applied to symbolic ways of representing reality (Collis & Biggs, 1991).

The move to the concrete symbolic mode marks a significant shift from a direct imaging of reality to a written, higher order symbolisation of reality. These second order symbols form a system with direct referents in the experienced world. The symbols also have a logic and an order among themselves which are represented directly in reality and give us one of our most powerful tools for acting on the environment. These symbol systems include written language, mathematics, maps, musical notations and so on, which are by no means the result of "natural" learning in the way ikonic or sensori-motor achievements may be regarded as "natural." They are the products of deliberate, carefully organised instruction. Indeed, the mastering of these systems and their application to the real world activities of the individual are most often considered as the primary task of schooling during the compulsory years as they form the basis for much of the everyday reasoning and record keeping in our modern adult society.

From this starting point, Collis and Romberg (1991) analysed a sample of the kinds of open-ended problem-solving items used in a range of recent assessment projects listed above. The results of the analysis of students' responses and, in some cases, instructors' expectations (e.g., California State Department of Education Project, 1989) led to the hypotheses that students had some common approaches to school-based mathematical problem solving and that, in general, the instructors' expectations were not compatible with these. The former is of major concern here.

Collis and Romberg (1991) found that all of the projects that they analysed in their study made use of open questions. These consisted of items in which the students had to construct their own responses. A wide variety of this item type is currently being tested throughout the world, and although the items vary on many dimensions, they all set out to test the higher order aims of mathematical problem solving and to reveal the student's reasoning as he or she moves towards a solution. The variations between the items include: (a) time allowed for solution, from traditional examination type conditions to projects which might take a semester to complete; (b) degree of cooperation and outside help allowed, from none to unlimited; (c) amount of mathematisation of the data required; and (d) amount of writing required, from virtually none to persuasive essays. As would be expected, each has its own particular advantages and disadvantages and most projects reported problems with achieving objectivity.

Although there was a variety of techniques used to set up the problem situation, the basic format for this type of item came down to one in which the context was set by a series of propositional statements followed by questions to which the student was expected to construct a response. The ways in which the context set for the problem could be varied and the open-ended nature of the construction required of the response, made different demands upon both student and assessor from the traditional form of test item. Let us consider a student's task when faced with a typical item of this type.

As was described above, the problem is placed in context by a series of propositional statements, followed by a question which seeks a response. The student needs to take the given propositions and decide on a course of action which might be schematised as shown in Figure 1 (adapted from Collis & Romberg, 1991).

After absorbing the data, an initial decision is made and the student proceeds down column L or R. At row C, Column L splits and leads in the case of C(i) to an irrelevant conclusion (in the given context), or in the case of C(ii) to an intuitive relevant solution. The route shown on Column R is associated with traditional mathematical problem solving. It is rare, however, that a student will stay purely on one track (L(ii) or R) if the problem is novel. In this case, there is likely to be movement both ways at either rows B or C or both; this will apply whether the subject is basically following route L or route R. The work-place mathematics literature shows that minimally educated adults solving problems in the ikonic mode utilise any number skills that they have (Carraher, 1988), while the use mathematically competent individuals make of the ikonic mode is well known (Hadamard, 1954). It would appear that the task for the instructor is to find ways of assessing what the individual is doing in rows B and C.

Testing at C_L is concerned with the ability to work cooperatively, to handle a particular context, to use outside resources, to define data to be used, to judge the degree of precision required, to assess attitude and intuitive ability in problem solving, and to judge flexibility of reasoning. Testing at C_R is focused on knowledge and understanding of mathematical procedures, mathematical concepts, communicative ability with mathematical language, and problem solving with mathematical models.

C_L and C_R together seem to encompass recent ideas of what is required of students in mathematical problem solving. Under the traditional assessment procedures the focus was on a very narrow range of C_R abilities. Tests for the new generation of problem-solving skills will need to be devised so that the assessor can look at *both* C_L and C_R *and* their interaction.

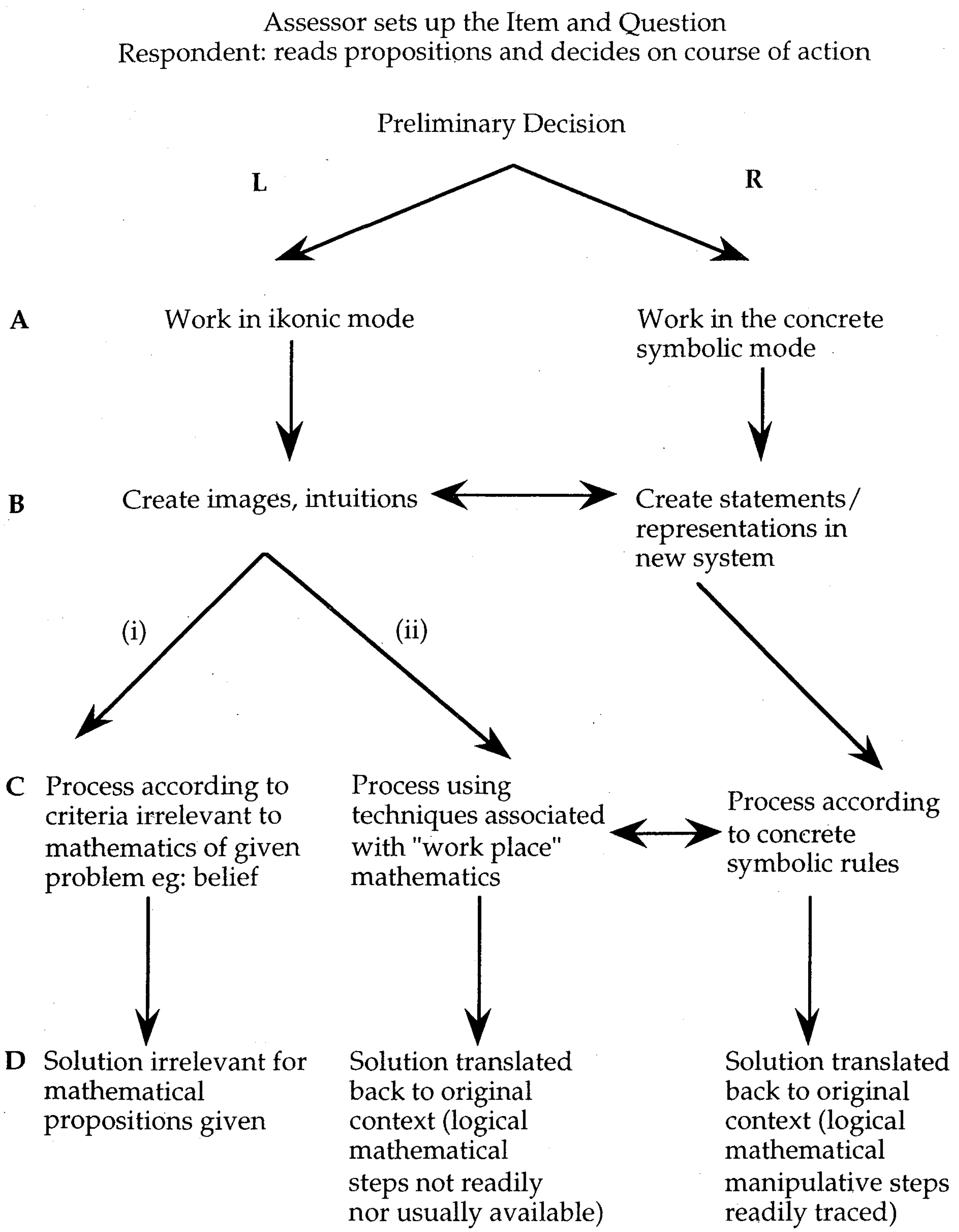

Figure 1. The Problem Solving Path
(adapted from Collis & Romberg, 1991)

This project was set up as the first step in devising guidelines for designing assessment techniques for testing students' skills in handling this new generation of problem types. Before guidelines were formulated, it was necessary to examine, empirically, the hypotheses developed from theory which formed the basis of the earlier analytical study by Collis and Romberg (1991) summarised above.

In the context of the data gathering proposed this is most easily considered in relation to the following questions:

Question 1. Do students have the same primary strategy by which to approach different classes of problems? If not, then do students have the same primary strategy by which to approach particular types of problems?

Question 2. Overall, are the strategies used dependent upon the obvious characteristics of the item?

Question 3. In the process of solving each type of problem, what typical path do students follow, if any?

Question 4. Do specific individual characteristics, as assessed by standard measures of imagining and creativity, relate to the nature of each student's strategies?

The Study

Crucial to examining the questions were (a) the selection of items which would form the main instrument for gathering the data; (b) devising suitable interview procedures to obtain relevant data; and (c) determining the appropriate form of analysis which would be suitable to answer the questions using the data obtained, while taking into account the necessarily small sample of students which the investigators could afford to involve. These considerations will be dealt with in turn.

The Items

It was necessary to select problems that could possibly be solved by techniques associated with the different courses of action a student might decide to take (see Figure 1). For example, the following problem can be solved in several ways.

Hungry Men Problem

Three tired and hungry men had a bag of apples. When they were asleep one of them awoke, ate 1/3 of the apples and went back to sleep. Later a second man awoke, ate 1/3 of the remaining apples, and went back to sleep. Finally, the third man awoke and ate 1/3 of the remaining apples, leaving 8 apples in the bag. How many apples were in the bag originally? Explain your solution.

As Watson (1988) has shown, the most likely methods of solution involve symbols and diagrams. These methods are primarily school-taught, and are based on the concrete symbolism traditionally associated with mathematical problem solving in the school context. This item was adopted as a model for items for which the students were likely to be attracted towards a concrete symbolic approach.

An example of a second type of problem, where a different set of solution methods would be expected, is the following:

Cube Painting Problem

> A cube that is 3cm by 3cm by 3cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Without drawing the cube, explain how you would find out how many of each kind of smaller cube there are.

This problem would appear to involve a significant visual or imaging, intuitive component. This is so even if the individual attempts to draw a diagram or picture during the course of seeking a solution—it certainly does not lend itself readily to concrete symbolic, school-based approach. This item was adopted as the model for later items which were meant to suggest a use of the ikonic mode as the primary strategy.

In summary, two categories of problems were identified according to the expected initial-solution strategy—those which would suggest the use of concrete-symbolic processes (CS) and those which would suggest ikonic processing (IK). On this basis, two sets of five problems (see Appendix) were prepared for the interviews with both categories of problem present in each set.

Data Collection

Sample. Sixteen students from two Advanced Mathematics classes (Years 9 and 10) in a Tasmanian High School were selected as the sample.

Imaging Tests. At the first session three group tests were administered, two to assess aspects of each student's ability to image situations and one to obtain a measure of each student's creativity. All thirty-eight students completed this battery of tests which consisted of the following individual tests: the Betts Questionnaire on Mental Imagery (QMI) (Sheehan, 1967); the Gordon Test of Visual Imagery Control (Gordon, 1949); and the well-known "Uses of a Brick" Creativity Test (Guilford, 1967). This set of tests provided data to assist in answering Question 4.

The Interviews. The sixteen students selected were interviewed individually on their problem solving strategies in relation to the item types described above; eight students were allocated to each problem set.

For each interview, the interviewer sat next to the student and tape-recorded the interview. Problems, or parts of problems, were presented to the student one at a time on white cards. Students were asked to tell the interviewer everything they thought about while reading the problem (or a part of the problem) and while solving it. If necessary, the interviewer prompted the student by asking if they "saw" any aspects of the story, or if they were anticipating what the problem was about, or what information they thought they still needed to solve the problem. Students could start solving the problem whenever they were ready, and had a pen and paper, ruler and calculator available if required. Half of the students received the problems in one order and the other half were given the same problems but in reverse order.

Plan of Analysis

If the students' interview responses were to be analysed in relation to the strategies being sought, a way of recording what the students were saying in terms of the mode (CS or IK) which each individual statement apparently represented was required. A variation of Haylock's Think-Board (Haylock, 1984) seemed to offer promise for exploring the responses in this way. Haylock developed the Think-Board, not as a research tool, but as a device to encourage the problem solver to consider different representations for a problem in attempting to find a solution. It was decided to adapt the Think-Board technique so that it could be used to record the problem solver's actual "moves" during the process.

The adapted Think-Board appears in Figure 2. The broken line divides the board into two regions, the upper region represents Concrete Symbolic (CS) responses while the lower half represents Ikonic (IK) responses. Each of these regions is divided into three smaller regions by diagonals; these smaller regions represent subsets of the CS and IK regions of which they form a part.

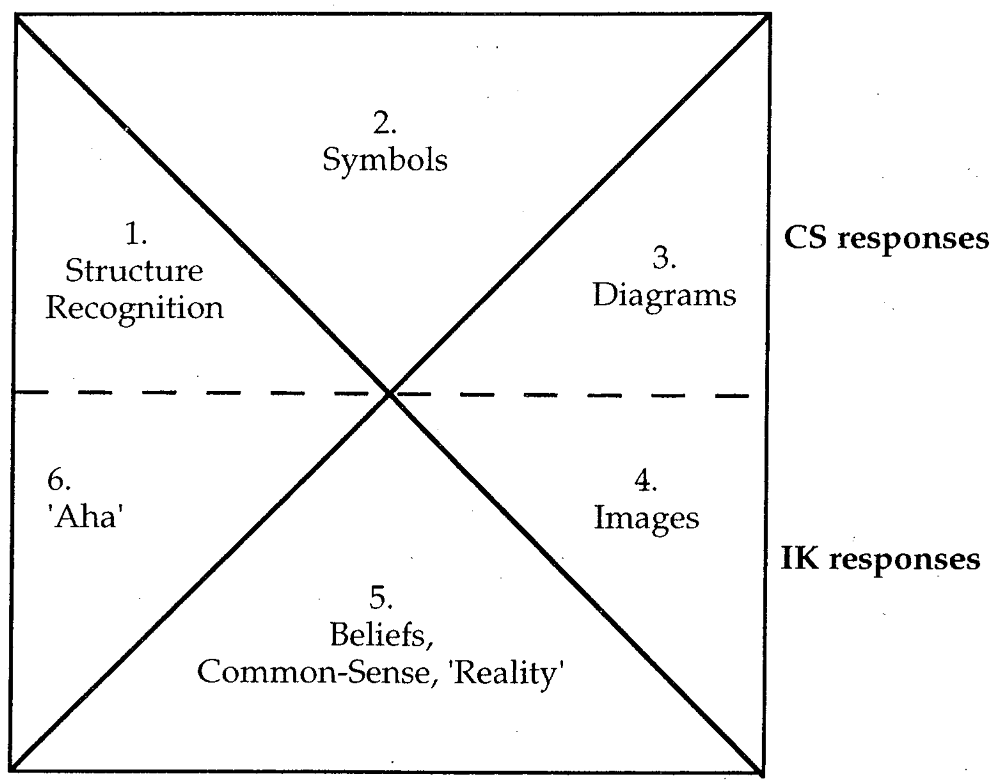

Figure 2. The adapted Think-Board

Starting at the top left, the following definitions have been made for each of the smaller regions.

Structure Recognition: The recognition of a CS structure in a problem as similar to a previous example, e.g., "This problem will be solved using the same mathematical ideas that I used in the last one."

Symbols: The recognition of an appropriate CS procedure to solve the problem, e.g. "This should be able to be solved using simultaneous equations."

Diagrams: The use of a CS diagram, e.g., Venn diagram, graph, etc. to solve the problem.

Images: Reporting visual images related to the problem, e.g., "I can see the pieces of the cube with one side painted."

Reality, beliefs etc.: The use of real world experience which appears to have some practical relationship to the problem, e.g., "Everyone knows that you can't run a car without petrol!"

"Aha" Experience: A sudden, apparently unbidden, "insight," into the structure of the problem, usually visual, e.g., "Oh I see what I have to do" or "I see what it's getting at."

Interviews were transcribed and each student's responses to each question were mapped onto a Think-Board. The mapping of each response was by agreement among the three investigators. These maps were then transformed to paths in which the sections of the Think-Board were denoted as follows: Structure Recognition (S_1), Symbols (S_2), Diagrams (S_3), Images (I_1), Beliefs, Common-Sense, Reality (I_2), and "Aha" (I_3).

The number of responses falling in the upper (CS) half of the board and the number falling in the lower (IK) half of the board for each question were determined and used as the basis for the statistical analyses. Where a response was seen to be a combination of more than one section of the Think Board it was decided that it should contribute to the score for each of those sections. As an illustration of the technique, the protocol for one student's interview on the Cube Painting Problem (see above) reads as follows:

"I am imagining a whole cube that isn't cut up. (I_1)

I can see the cube being cut up. (I_1)

Now I can see the cube in slices coming away from each other and I can see how many have 2, and how many have 3. (I_1)

It is like a Rubik cube. (I_2)

[From this image the student worked out: 3 sides (I_1), no sides (I_1), and 2 sides (I_1).]

Then I added the numbers together and found that there were 6 left so there are 6 with one side painted. (S_2)

For this student the path was I_1, I_1, I_1, I_2, I_1, I_1, I_1, S_2, and the resulting scores were IK 7, CS 1.

Statistical Procedures Adopted

To examine the questions set out above it was necessary to use techniques that would give some assurance that any differences which were detected were unlikely to be the result of a chance association. In addition, the procedures available were narrowed by the small numbers in the sample and the nature of the data gathered. A summary of the statistical techniques used to shed light on each question follows.

Question 1. The interest here was whether the individual student did in fact adapt his/her strategy (basically CS or basically IK) when faced with items which were designed by the experimenters to favour one approach over the other. This was done in two steps. First, the data were examined overall to see if there was a consistent approach to all the problems regardless of type. Second, the data on each type of item were analysed separately to see if there was a consistent approach within item type. Given this type of data and size of the sample, the Cochran *Q* test (Siegel, 1956) was decided on as the most appropriate form for statistical analysis. It is a method of testing whether the probabilities of a particular type of response are different in various conditions. The entries required for the tables were obtained by determining the primary strategy each student used for solving each problem. This was done by counting the two types of response, if there were more IK responses than CS responses, a score of "1" was assigned; if not, the score assigned was "0." These data were then tabulated and Cochran *Q* tests were carried out.

Question 2. This question was not concerned with the individual student strategies as such but used the pooled data to examine the more general notion that the students' approaches to a problem in general would be determined by the obvious characteristics of the item. The total number of both CS and IK responses for all subjects was separately calculated and then averaged according to the question set-type in order to give equal weighting to CS and IK responses across all subjects. This was necessary since Problem Set 1 contained three CS and two IK question types compared to the two CS and three IK question types in Problem Set 2. These averaged totals were then entered as appropriate against the two categories of problem. Comparison of the number of CS and IK responses for each problem type was achieved using dependent t-tests.

Question 3. This question sought to check the students' movement between the modes as they proceeded towards a solution. A count was made of the number of times that each student changed from one mode to the other during the specific question types. A total count of the number of times each student changed modes for both CS and IK questions was obtained for each student, and then these values were adjusted for set-type. A dependent t -test was performed on the these means to determine if the mean number of times students changed modes when answering CS-type problems was significantly different from the mean number of times students changed modes when answering IK-type problems.

Question 4. The final question was asked in order to assess the effect of students' ability to create images on the nature of their strategies. The standard measures of imagery and creativity mentioned earlier were scored, producing four variables for each student. The total score on Betts QMI yielded a vividness of imagery (Vividness) rating as well as a visual subscale which was included as a separate variable (VisVivid). A control of visual imagery score (Control) was provided by the Gordon Test of Visual Imagery Control. Finally, the fluency, flexibility and uniqueness scores on the Uses of a Brick Creativity test were added to produce a measure of Creativity. The students interviewed were divided into High and Low groups on all four variables based upon a mean split. The problem-solving strategies of the High and Low groups on each of four variables were then compared by taking the difference in the average numbers of CS and IK responses per question and performing a Mann-Whitney *U* test on the ranked values.

Results

The results of data analyses are presented separately for each question.

Question 1 addressed the issue of whether students tended to use the same primary strategy (CS or IK) regardless of the type of problem they were set. Close inspection of the data, using all ten problems, revealed that the primary strategies used by the students were not consistent across the two different types of problems (CS & IK). This general finding also held for each set of 5 problems considered separately (Cochran Q test values: $Q = 26.52, p<.001$; $Q = 27.30, p<.001$). Further analysis to check whether primarily IK strategies were applied in response to CS problems produced non-significant results. Inspection revealed that the strategies applied in these cases were primarily CS. Likewise it was found that students did not consistently apply CS strategies to IK problems, rather IK strategies were primarily used to solve IK problems. These results seem to indicate that over and above any individual differences in the use of ikonic or concrete symbolic strategies, the nature of the problem itself is highly significant in determining which type of strategy will be used.

Question 2 supplemented Question 1 analysis, and sought to check whether the strategies used seemed to be dependent upon the obvious characteristics of the item. The dependent *t*-test performed on the pooled CS and pooled IK responses of the whole sample for all items supported the view that *a priori* CS problems elicit more CS responses than *a priori* IK problems ($t(15) = 5.98, p<.001$). There was, however, no difference between the number of IK responses elicited from IK problems or CS problems ($t(15) = 0.03, p>.05$). In other words, it appears that CS strategies are more affected by the problem characteristics than IK strategies. In general, significantly more CS responses than IK responses were given to CS problem types ($t(15) = 2.40, p<.05$), and significantly more IK responses than CS responses were given to IK problems ($t(15) = -3.42, p<.02$). Furthermore, there was a significantly greater number of CS responses given to CS problems compared to the number of IK responses given to IK problems ($t(15) = 4.47, p<.001$). These results support the conclusions at least for the CS problem types from Question 1 that it is the apparent nature of the problem itself that determines the strategy to be used.

Table 1

Problem Category and Type of Response: Total Number of Responses of 16 Subjects adjusted for Set Type

	CS-type Problems	IK-Type Problems
CS Response	89.2	26.0
IK Response	56.4	55.6

Question 3 sought information on the typical path which students seemed to follow during the process of solving each type of problem. The total number of times that students changed modes per type of question was 35.8 for CS-type problems and 22.6 for the IK type problems, giving means per student of 2.2 (SD=1.1) and 1.4 (SD=1.4), respectively. The dependent *t*-test on the means of the number of modal changes in each type of question for each student produced a significant result (t (15) = 3.10, $p<.02$), the direction of the differences indicating that students tended to change modes significantly more often when answering CS-type problems than when answering IK-type problems. A dependent *t*-test on the pooled CS and IK responses (see Table 1) showed significantly more IK responses to be given to CS problem types than CS responses given to IK problems ($t(15) = 3.97$, $p<.001$).

This may suggest that students are more likely to utilise IK strategies if they find some degree of difficulty in solving CS problems and are unable to use CS strategies effectively. In the case of IK problems, however, students are unlikely to use CS strategies to assist in their problem solving pathway. In summary, closer inspection of the data summarised in Table 1 indicates that when students are working on a CS approach and find some difficulty in proceeding, they are more inclined to use an IK approach to make progress, whereas when they use an IK approach as the initial step, they either solve the problem or discontinue their attempt, i.e., if they saw any chance of using a CS approach they would have used it in the first place.

Question 4 assessed how the specific individual characteristics of the students, as assessed by standard measures of imaging and creativity, might relate to the nature of their strategies. The results from the Mann-Whitney *U* tests performed on the pooled CS and IK responses for CS and IK type questions indicate that the degree of overall Vividness may be important in determining the type of strategy adopted by the student during a problem solving task. This effect was found to be approaching significance even with the small sample tested (U = 17.5, p = .06). Since Vividness is a general measure of vividness of imagery which includes such components as kinaesthetic, tactile and aural imagery, among others, the visual subscale (VisVivid) was treated as a separate variable. Further analyses using the Mann-Whitney *U* test indicated that it is this visual component of imagery that strongly influences the student's selection of the most effective strategy to be used in a problem solving task (U = 6.5, $p<.001$). It appears that students with high VisVivid scores are more likely to produce IK responses than their counterparts with low VisVivid scores. Finally, measures of both Controllability and Creativity were found not to contribute significantly to the type of strategy adopted by the student during specific problem-solving tasks ($U = 22.5$, $p>.05$ and $U = 27.5$, $p>.05$, respectively). In other words, students with high Control scores in relation to visual imagery, or with high Creativity scores are not significantly different from students with low Control or low Creativity scores.

Conclusion

Within the limitations set by the data and the small sample, this study highlights some important factors which must be taken into account in the teaching, learning and assessing of school mathematics especially as it relates to problem solving in novel situations.

1. There seem to be two basic approaches to problem solving at this level, one based on CS, school taught procedures, the other related to IK mode processing. The latter, in this study, means appealing to common sense, everyday life and visualisation, and the use of intuitive reasoning. It is interesting to note that students, who see what they believe to be a solution using ikonic approaches, see little point in backing up their solution by an appeal to mathematics.

2. In the course of solving a problem, students will move from mode to mode but are more likely to move from CS to IK when a school-based method fails to satisfy rather than vice versa. They do, however, move both ways. In the case of IK to CS procedures, the move is most often made only after the problem is really solved, as illustrated by the student response to the Cube Painting Problem.

3. The characteristics of the problem and its context appear to be the basic factors in determining the initial strategy, particularly when CS type problems are presented.

4. Students who show heightened overall Vividness and have the ability to obtain clear visual representations of a problem in developing their ideas tend to prefer using IK strategies during problem-solving tasks.

These findings, tentative as they must be because of the limitations of the study, have two important implications for mathematics teachers in particular. First, those responsible for assessing mathematical problem solving at this level, should take into account the fact that the obvious problem characteristics may determine the basic strategy of the students. Second, all teachers should be aware of the significance of the IK mode in problem solving and adjust their mathematics teaching and assessing accordingly. Presmeg (1986) has reported that the "stars" in senior secondary mathematics classes are almost invariably non-visualisers who prefer verbal-logical as opposed to visual thinking. This is in sharp contrast to the cognitive preferences of creative adult mathematicians, many of whom are visual thinkers (Hadamard, 1954; Kuyk, 1982). One reason for this discrepancy postulated by Presmeg is the reinforcement which the traditional school mathematics curriculum may give to nonvisual methods. The present study, which uses a different means of defining visualisers from that of Presmeg, indicates that students do spontaneously use the ikonic mode when problem solving. The study also indicates that this is particularly useful when integrated with the use of CS strategy such that the student can alternate flexibly between the two, thereby breaking an unproductive set towards the problem. Teachers may, however, be unaware of this use of the ikonic mode, and not model it, encourage it or assess it. Presmeg (1986) found that most teachers did not investigate their students' use of visual imagery, nor did they feel they had the time to teach such methods. This omission means that the visualiser may not learn how to transcend the concreteness of an image or diagram, while the nonvisualiser may not acquire the richness and flexibility inherent in ikonic mode approaches to problem solving.

The findings described above should highlight the importance of further research into the cognitive mechanisms behind the variables involved in students' problem-solving techniques. Without a concerted effort in this direction, the ideas behind the reports, mentioned in the introduction, which have been widely acclaimed as pointing the way forward into the 1990s will very likely share the fate of the many other promising innovatory proposals of the last fifty years.

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Appendix A: Problem Set One

Hungry Men Problem

Three tired and hungry men had a bag of apples. When they were asleep one of them awoke, ate 1/4 of the apples and went back to sleep. Later a second man awoke, ate 1/2 of the remaining apples and went back to sleep. Finally, the third man awoke and ate 2/3 of the remaining apples, leaving 2 apples in the bag. How many apples were there originally?

Doctors and Dentists Problem

A journalist looking for a story came across the following two survey results. The Dentists' Association found that half of the population is nervous about going to the dentist. Independently the Medical Association found that half of the population is nervous about going to the doctor. Michael read the article and thought everyone must be afraid going to either a dentist or a doctor. Is Michael right? Why or why not?

Cube Painting Problem

A cube that is 3cm by 3cm by 3cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Without drawing the cube, explain how you would find out how many of each kind of smaller cube there are.

Waitresses Problem

A. Three waitresses, Julie, Alice, and Tina, put all their tips in one jar.
B. Julie went home first and took 1/3 of the money as her share.
C. Alice, not knowing that Julie had taken her share, took 1/3 of what was there as her share.
D. Tina, unaware that the others had already taken what they thought were their shares, took 1/3 of the remaining money.
E. There was $8.00 left in the jar. How much did the waitresses have in tips in the beginning?

Fuel-savers Problem

A. Three companies have designed fuel-savers for car engines;
B. Company X's fuel-saver will reduce fuel consumption by 20%;
C. Company Y's fuel-saver will save 30%;
D. and Company Z's saves 50%;
E. Caroline thinks that if she uses all three fuel-savers on the engine of her car she won't need to buy any more petrol. Is Caroline right? Why or why not?

APPENDIX B: Problem Set Two

Envelopes Problem (Wason & Johnson-Laird, 1972)

Subject presented with set of 4 envelopes:
Unsealed , Sealed , 30c stamp $\boxed{\;\#}$, 43c stamp $\boxed{\;\#}$

Subjects told that there is a general postal regulation:

If a letter is sealed, then it has a 43c stamp on it.

Subject is to select those letters that would definitely need to be turned over to find out whether the regulation was being violated.

Chickens and Goats Problem

A farmer is counting the chickens and goats in his yard. He counts a total of 50 heads and 140 feet. How many chickens and how many goats does the farmer have?

Trip Problem (Wason & Johnson-Laird, 1972)

Subject presented with set of 4 cards: $\boxed{\text{Bus}}$, $\boxed{\text{Car}}$, $\boxed{\text{Oatlands}}$, $\boxed{\text{Launceston}}$
Told each card has a town on one side and a mode of transport on the other. Given rule:

If I go to Launceston, then I travel by bus

Subject is to select those cards that would definitely need to be turned over to find out whether the rule was true.

Garage Sale Problem

A. A lady buys 20 plates at a garage sale. Some are large plates and some are small.
B. The small plates have a pattern with two roses on them.
C. The pattern on the large plates has four roses.
D. She counts 56 roses altogether on her plates.
E. How many large plates and how many small plates does the lady have?

Wason's 4-card Problem (Wason & Johnson-Laird, 1972)

Subject told each card has a letter on one side and a number on the other. Given rule:

If a card has a vowel on one side, then it has an even number on the other side.

and cards: $\boxed{\text{E}}$, $\boxed{\text{K}}$, $\boxed{\text{4}}$, $\boxed{\text{7}}$
Subject is to select those cards that would definitely need to be turned over to find out whether the rule was true.