A Discontinuous Galerkin Method for Approximating the Stationary Distribution of Stochastic Fluid-Fluid Processes

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Abstract

The stochastic fluid-fluid model (SFFM) is a Markov process $\{(X_t, Y_t, \varphi_t), t \geq 0\}$, where $\{\varphi_t, t \geq 0\}$ is a continuous-time Markov chain, the first fluid, $\{X_t, t \geq 0\}$, is a classical stochastic fluid process driven by $\{\varphi_t, t \geq 0\}$, and the second fluid, $\{Y_t, t \geq 0\}$, is driven by the pair $\{(X_t, \varphi_t), t \geq 0\}$. Operator-analytic expressions for the stationary distribution of the SFFM, in terms of the infinitesimal generator of the process $\{(X_t, \varphi_t), t \geq 0\}$, are known. However, these operator-analytic expressions do not lend themselves to direct computation. In this paper the discontinuous Galerkin (DG) method is used to construct approximations to these operators, in the form of finite dimensional matrices, to enable computation. The DG approximations are used to construct approximations to the stationary distribution of the SFFM, and results are verified by simulation. The numerics demonstrate that the DG scheme can have a superior rate of convergence compared to other methods.

Keywords Stochastic fluid-fluid processes · Stationary distribution · Discontinuous Galerkin method

1 Introduction

An unbounded stochastic fluid process $\{(\widehat{X}_t, \varphi_t), t \geq 0\}$ is a Markov process where the phase $\{\varphi_t\}$ is a continuous-time Markov chain on a finite state space $\mathcal{S}$, and the fluid $\{\widehat{X}_t\}$ varies linearly at rate c_{φ_t}. A subset of Markov additive processes, stochastic fluids have been well-analysed in the past two decades. There have been recent generalisations of stochastic fluid

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processes to a higher dimension: Miyazawa and Zwart (2012) analysed discrete-time multidimensional Markov additive processes, and Bean and O'Reilly (2014) studied the so-called *stochastic fluid-fluid model*. The latter is the focus of this paper.

An unbounded stochastic fluid-fluid model (SFFM) is a Markov process $\{(\widehat{X}_t, \widehat{Y}_t, \varphi_t), t \geq 0\}$, where the phase $\{\varphi_t\}$ is a continuous-time Markov chain on a finite state space $\mathcal{S}$; $\{\widehat{X}_t\}$ is the first fluid, which varies linearly at rate c_{φ_t}

$$\widehat{X}_t := \widehat{X}_0 + \int_0^t c_{\varphi_s} \, ds;$$

and $\widehat{Y}_t$ is the second fluid, which varies at rate $r_{\varphi_t}(\widehat{X}_t)$:

$$\widehat{Y}_t := \widehat{Y}_0 + \int_0^t r_{\varphi_s}(\widehat{X}_s) \, ds.$$

Regulated boundaries may also be included for both fluids. To distinguish between unbounded and bounded processes, we use the notations $\widehat{X}_t$ and $\widehat{Y}_t$ to denote unbounded processes, and X_t and Y_t to denote fluid levels with a regulated lower boundary at 0.

As classic fluid processes, $\{(\widehat{X}_t, \varphi_t), t \geq 0\}$, or bounded analogues, are used extensively in many areas, such as insurance and environmental modelling, it is clear that stochastic fluid-fluid models have an even wider range of applicability.

An example of application for an SFFM is the modelling of growth and bleaching of coral reefs, as described in (Bean and O'Reilly 2014). In this process, we can model the density of symbiotic zooxanthellae at time t by X_t, with the positive rates c_i corresponding to the growth of the zooxanthellae, the negative rates to the bleaching. The density, X_t, determines the net rate at which the coral stores the lipids produced by the zooxanthellae. The amount of stored lipids is modelled by Y_t, and the coral dies when the stored lipids run out, that is, $Y_t = 0$.

Some specifications of stochastic fluid-fluid models have already been analysed; Bean and O'Reilly (2013) and the to-date unpublished work of Bean *et al.* (*Matrix-Analytic Methods for the analysis of Stochastic Fluid-Fluid Models*, 2020) study cases where $\{X_t\}$ and $\{Y_t\}$ are independent, given $\{\varphi_t\}$, and Latouche et al. (2013) and O'Reilly and Scheinhardt (2017) study cases where $\{Y_t\}$ depends on whether $\{X_t\}$ is above, or below, some specific threshold. Here, we derive approximations to the theoretical operators in (Bean and O'Reilly 2014), which covers a much wider class of models than the specific ones already studied, therefore this work applies to a much larger class of models.

While the analyses in (Bean and O'Reilly 2014; Miyazawa and Zwart 2012) are markedly different, both papers drew inspiration from Neuts' matrix-analytic approach (Neuts 1981; Latouche and Ramaswami 1999) to obtain the limiting behaviour of these processes, working with operators on function spaces instead of matrices. Thus, their closed-form expressions for the limiting distributions ([Theorem 2] Bean and O'Reilly (2014), [Theorem 4.1] Miyazawa and Zwart (2012)) are given in terms of operators acting on measures, which are not immediately amenable to numerical computations for real-life applications. Only in the simplest cases can the solutions to these operator equations be readily evaluated and, beyond the simplest cases, approximations are needed. One way to numerically handle operators on function spaces is to construct approximations of the operators. To this end, there exist numerical procedures such as finite difference, finite volume, finite element and discontinuous Galerkin (DG) methods (Cockburn 1999; Hesthaven and Warburton 2007). In the context of approximating fluid queues, Bean and O'Reilly (2013) derive an approximation to the process $\{(X_t, \varphi_t)\}$ which is a continuous-time Markov chain (specifically, a

quasi-birth-and-death process). It turns out that the finite-volume method with an upwind flux, the DG method with a single (constant) basis function in each cell and an upwind flux, and the Markov chain approximation of Bean and O'Reilly (2013) are all equivalent.

In our application to SFFMs, the operators we wish to approximate are acting on a function space of probability densities and therefore our approximation method must respect certain properties of probability densities, such as conservation of probability. This is the case in the DG method (Cockburn 1999).

In this paper, the DG method is used to approximate the operators appearing in (Bean and O'Reilly 2014), and ultimately, the joint stationary distribution of an SFFM. We numerically illustrate the effectiveness of the methodology using an on-off bandwidth-sharing system of two processors (Latouche et al. 2013). In this example, inputs into the processors, $\{X_t\}$ and $\{Y_t\}$, are turned on and off by a Markov chain, $\{\varphi_t\}$; the combined output capacity is fixed and allocated according to the workload of the first, high-priority, processor $\{X_t\}$. Latouche et al. (2013) evaluate the marginal limiting distribution of the first processor $\{X_t\}$, and provides bounds for the marginal limiting distribution of the workload of the second processor $\{Y_t\}$. We verify our DG approximations by comparing them against Monte Carlo simulations, against analytical results obtained, and against our intuitive understanding of the system dynamics. In all considered cases, we find the approximations to be accurate. Further, the numerics also demonstrate a superior rate of convergence over the method of Bean and O'Reilly (2013) whereby the first fluid is approximated by a continuous-time Markov chain.

The paper is organised as follows. In Sect. 2, we give relevant background to present the joint stationary distribution of a stochastic fluid-fluid model. We construct, in Sect. 3, a discontinuous Galerkin scheme to approximate the infinitesimal generator of a stochastic fluid process. Sect. 4 uses this approximation to the infinitesimal generator to construct approximations of the first-return operator, and stationary distribution, of an SFFM. Numerical experiments are reported in Sect. 5. In Appendix 1 we provide a proof that our DG approximations conserve probability. Lastly, since the paper is notationally heavy, we apply the DG method to a small toy example in Appendix 2.

2 Preliminaries

Consider a stochastic fluid-fluid model $\{(X_t, Y_t, \varphi_t), t \geq 0\}$. We assume that $X_t, Y_t \in [0, \infty)$ and that there is a regulated boundary at level 0 for both buffers:

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t := \max\{0, c_i\} \quad \text{if } X_t = 0 \text{ and } \varphi_t = i,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_t := \max\{0, r_i(x)\} \quad \text{if } Y_t = 0, X_t = x, \text{ and } \varphi_t = i,$$

for $i \in \mathcal{S} = \{1, ..., N_{\mathcal{S}}\}$. Let T be the irreducible generator for the finite-state Markov chain $\{\varphi_t\}$. We denote by $C := \mathrm{diag}(c_i)_{i \in \mathcal{S}}$ the diagonal fluid-rate matrix for $\{X_t\}$, and $R(x) := \mathrm{diag}(r_i(x))_{i \in \mathcal{S}}$ the diagonal fluid-rate matrix of functions for $\{Y_t\}$.

Remark 2.1 For future reference we require some notation regarding the elements of the model introduced above. We use the notation $\boldsymbol{u} = (u_h)_{h \in \mathcal{H}}$ to denote a row-vector, $\boldsymbol{u}$, defined by its elements, u_h, indexed by $h \in \mathcal{H}$, where $\mathcal{H}$ is some countable index set. Similarly, $\boldsymbol{u} = (\boldsymbol{u}_h)_{h \in \mathcal{H}}$, is a row-vector defined by a collection of row-vectors $\boldsymbol{u}_h$. The notation

$\boldsymbol{u}_m = (u_h)_{h \in \mathcal{H}_m}$ refers to the vector containing the subset of elements corresponding to $\mathcal{H}_m \subseteq \mathcal{H}$. When the index set is empty, the resulting vector $\boldsymbol{u}_m$ is a vector of dimension 0. In cases when the elements have two indices, we order the elements of the vector according to the first index, then the second; i.e. $\boldsymbol{u} = (u_g^h)_{g \in \mathcal{G}, h \in \mathcal{H}} = ((u_g^h)_{g \in \mathcal{G}})_{h \in \mathcal{H}}$. Here we use the convention that for a vector $\boldsymbol{u} = (u)_{h \in \mathcal{H}}$ where the elements u do not depend on the index h and H is some index set, then we repeat u h-times; i.e. $\boldsymbol{u} = (u)_{h \in \mathcal{H}} = \underbrace{(u, \dots, u)}_{h- \text{ times}}$. The nota-

tion $U = [u_{gh}]_{g \in \mathcal{G}, h \in \mathcal{H}}$ is used to denote a matrix defined by its elements, or sub-blocks, u_{gh}.

Let $\mathcal{S}_- = \{i \in \mathcal{S} \mid c_i < 0\}$, $\mathcal{S}_+ = \{i \in \mathcal{S} \mid c_i > 0\}$, $\mathcal{S}_0 = \{i \in \mathcal{S} \mid c_i = 0\}$, $\mathcal{S}_\nabla = \{i \in \mathcal{S} \mid c_i \leq 0\}$, $\mathcal{S}_\Delta = \{i \in \mathcal{S} \mid c_i \geq 0\}$. Define matrices $C_m := \mathrm{diag}(c_i)_{i \in \mathcal{S}_m}$, $m \in \{+, -, 0, \nabla, \Delta\}$, and define the sub-matrices of transition rates $T_{mn} = [T_{ij}]_{i \in \mathcal{S}_m, j \in \mathcal{S}_n}$, $m, n \in \{+, -, 0, \nabla, \Delta\}$.

For the remainder of this section, we summarise the findings of (Bean and O'Reilly 2014) on the joint stationary distribution of $\{(X_t, Y_t, \varphi_t), t \geq 0\}$. For each Markovian state $i \in \mathcal{S}$, we partition the state space of X_t, $[0, \infty)$, according to the rates of change $r_i(\cdot)$ for the second fluid $\{Y_t\}$: $[0, \infty) := \mathcal{F}_i^+ \cup \mathcal{F}_i^- \cup \mathcal{F}_i^0$, where

$$\mathcal{F}_i^+ := \{u \in \mathcal{F} : r_i(u) > 0\}, \quad \mathcal{F}_i^- := \{u \in \mathcal{F} : r_i(u) < 0\}, \quad \mathcal{F}_i^0 := \{u \in \mathcal{F} : r_i(u) = 0\}. \tag{1}$$

For all $i \in \mathcal{S}$, the functions $r_i(\cdot)$ are assumed to be sufficiently well-behaved that $\mathcal{F}_i^m$, $m \in \{+, -, 0\}$, is a finite union of intervals and isolated points.

We assume that the process $\{(X_t, Y_t, \varphi_t), t \geq 0\}$ is positive recurrent, in order to guarantee the existence of the joint stationary density. Define stationary operators

$$\overline{\pi}_i(y)(\mathcal{A}) := \lim_{t \to \infty} \frac{\partial}{\partial y} \mathbb{P}[X_t \in \mathcal{A}, Y_t \leq y, \varphi_t = i], \quad y > 0, \tag{2}$$

$$\overline{\mathbb{p}}_i(\mathcal{A}) := \lim_{t \to \infty} \mathbb{P}[X_t \in \mathcal{A}, Y_t = 0, \varphi_t = i], \tag{3}$$

where $\mathcal{A} \subset [0, \infty)$.

Then let $\overline{\pi}(y) = (\overline{\pi}_i(y))_{i \in \mathcal{S}}$ be a vector containing the joint stationary density operators and $\overline{\mathbb{p}} = (\overline{\mathbb{p}}_i)_{i \in \mathcal{S}}$ a vector containing the joint stationary mass operators.

The determination of $\overline{\pi}(y)$ involves two important matrices of operators, $\overline{\mathbb{B}}$ and $\overline{\psi}$. The operator $\overline{\mathbb{B}}$ is the *infinitesimal generator* of the process $\{(X_t, \varphi_t)\}$. The operator $\overline{\psi}$ is such that $\overline{\mu_{\psi}}(\mathcal{A})$ is the conditional probability of $\{Y_t\}$ returning to level zero and doing so when $X_t \in \mathcal{A}$, given that the initial distribution is $\overline{\mu}$.

2.1 Matrix $\overline{\mathbb{B}}$ of Operators

Since $\{(X_t, \varphi_t), t \geq 0\}$ is a Markov process, the evolution of probability can be described by a semigroup. Let $\mathcal{M}(\mathcal{S} \times \mathbb{R}_+)$ be the set of integrable complex-valued Borel measures on the Borel σ-algebra $\mathcal{B}_{\mathcal{S} \times \mathbb{R}_+}$. For $\overline{\mu} \in \mathcal{M}(\mathcal{S} \times \mathbb{R}_+)$, we can write $\overline{\mu} = (\overline{\mu}_i)_{i \in \mathcal{S}}$. The measures $\overline{\mu}_i(\cdot)$ represent an initial distribution, $\overline{\mu}_i(\cdot) = \mathbb{P}(X_0 \in \cdot, \varphi_0 = i)$. Let $\{\overline{\mathbb{V}}(t)\}_{t \geq 0}$, $\overline{\mathbb{V}}(t) : \mathcal{M}(\mathcal{S} \times \mathbb{R}_+) \mapsto \mathcal{M}(\mathcal{S} \times \mathbb{R}_+)$ be the semigroup describing the evolution of probability for $\{(X_t, \varphi_t), t \geq 0\}$ structured as a matrix of operators, $\left[\overline{\mathbb{V}}(t)\right]_{ij} = \overline{\mathbb{V}}_{ij}(t)$ where,

$$\overline{\mu}_i \overline{\mathbb{V}}_{ij}(t)(\mathcal{A}) = \int_{x\in[0,\infty)} d\overline{\mu}_i(x)\mathbb{P}(X_t \in \mathcal{A}, \varphi_t = j \mid X_0 = x, \varphi_0 = i).$$

Intuitively, the operator $\overline{\mathbb{V}}(t)$ maps an initial measure $\overline{\mu}$ on (X_0, φ_0) to the measure $\mathbb{P}(X_t \in \mathcal{A}, \varphi_t = j) =: \overline{\mu}_j(t)(\mathcal{A})$. The matrix of operators $\overline{\mathbb{B}} := [\overline{\mathbb{B}}_{ij}]_{i,j\in\mathcal{S}}$ is the *infinitesimal generator* of the semigroup $\{\overline{\mathbb{V}}(t)\}$ defined by

$$\overline{\mathbb{B}} = \left.\frac{d}{dt}\overline{\mathbb{V}}(t)\right|_{t=0},$$

with domain the set of measures for which this limit exists. Specifically, the domain of $\overline{\mathbb{B}}$ is the set of measures, $\overline{\mu} = (\overline{\mu}_i)_{i\in\mathcal{S}}$, for which each $\overline{\mu}_i$ admits an absolutely continuous density on $(0, \infty)$, and can have a point mass at 0 if $i \in \mathcal{S}_\nabla$; call this set of measures $\mathcal{M}_0$. That is, the measures $\overline{\mu}_i$, $i \in \mathcal{S}$, with $(\overline{\mu}_i)_{i\in\mathcal{S}} \in \mathcal{M}_0$ are absolutely continuous on $(0, \infty)$ and may have point masses at 0 if $i \in \mathcal{S}_\nabla$. The measure $\overline{\mu}_i$ cannot have a point mass at 0 if $i \notin \mathcal{S}_\nabla$. In the sequel we write $v_i(x)$, $x > 0$, as the density of $\overline{\mu}_i$ and q_i as the point mass of $\overline{\mu}_i$ at $x = 0$ (if such a point mass exists).

To use the operators $\{\overline{\mathbb{V}}(t)\}$ and $\overline{\mathbb{B}}$ to analyse the fluid-fluid model, Bean and O'Reilly (2014) explicitly track when $(X_t, \varphi_t) \in (\mathcal{F}_i^m, i)$ for $i \in \mathcal{S}$, $m \in \{+, -, 0\}$ by partitioning the operators $\overline{\mathbb{V}}(t)$ and $\overline{\mathbb{B}}$ into $\overline{\mathbb{V}}_{ij}^{mn}$ and $\overline{\mathbb{B}}_{ij}^{mn}$, for $i, j \in \mathcal{S}$, $m, n \in \{+, -, 0\}$, where

$$\overline{\mu}_i|_{\mathcal{F}_i^m} \overline{\mathbb{V}}_{ij}^{mn}(t)(\mathcal{A}) := \int_{x\in[0,\infty)} d\overline{\mu}_i|_{\mathcal{F}_i^m}(x)\mathbb{P}(X_t \in \mathcal{A} \cap \mathcal{F}_j^n, \varphi_t = j \mid X_0 = x, \varphi_0 = i),$$

and $\overline{\mu}_i|_E$ is the restriction of $\overline{\mu}_i$ to E. Similarly, for $\overline{\mathbb{B}}_{ij}^{mn}$, $i, j \in \mathcal{S}$, $m, n \in \{+, -, 0\}$.

We claim that numerical schemes are needed to approximate the analytic operator equations introduced in Bean and O'Reilly (2014). The DG scheme we choose to use here works by first partitioning the state space of the fluid level, $\{X_t\}$, into a collection of intervals, $\mathcal{D}_k = [x_k, x_{k+1}]$ then, on each interval, the operator $\overline{\mathbb{B}}$ is projected onto a basis of polynomials. So, to help elucidate the connection between the operators $\{\overline{\mathbb{V}}(t)\}$, $\overline{\mathbb{B}}$ and their DG approximation counterparts, we take a slightly different approach to partitioning these operators than that taken in Bean and O'Reilly (2014). Rather than partition according to the sets $\mathcal{F}_i^m$, $i \in \mathcal{S}$, $m \in \{+, -, 0\}$, we use the same partition as that in the construction of the DG scheme. By doing so, we can directly correspond elements of the partitioned operators to their approximation counterparts. Since the partition used to construct the DG scheme is finer, then we can always reconstruct the partition in terms of the sets $\mathcal{F}_i^m$, $i \in \mathcal{S}$, $m \in \{+, -, 0\}$.

Let us first partition the space $[0, \infty)$ into $\mathcal{D}_\nabla = \{0\}$, and non-trivial intervals $\mathcal{D}_k = [x_k, x_{k+1}] \setminus \{0\}$, with $x_1 = 0$, $x_k < x_{k+1}$, $k = 1, 2, \dots$. The symbol ∇ is used to refer to sets and quantities which are relevant to boundary at $x = 0$. For $\mu \in \mathcal{M}_0(\mathcal{S} \times \mathbb{R}_+)$ we write $\mu = (\mu_i^k)_{i\in\mathcal{S}, k\in\{\nabla, 1, 2, \dots\}}$, where $\mu_i^k(\cdot) = \mu_i(\cdot \cap \mathcal{D}_k)$, $k = \nabla, 1, 2, \dots$. We also have densities, $v_i^k(x)$, $x > 0$, associated with each measure, μ_i^k. For $i, j \in \mathcal{S}$, $k, \ell \in \{\nabla, 1, 2, \dots\}$ define the operators

$$\mu_i^k \mathbb{V}_{ij}^{k\ell}(t)(\mathcal{A}) := \int_{x\in\mathcal{D}_k} d\mu_i^k(x)\mathbb{P}(X_t \in \mathcal{A} \cap \mathcal{D}_\ell, \varphi_t = j \mid X_0 = x, \varphi_0 = i),$$

and the matrices of operators $\mathbb{V}^{k\ell}(t) := \left[\mathbb{V}_{ij}^{k\ell}(t)\right]_{i,j\in\mathcal{S}}$, $k, \ell \in \{\nabla, 1, 2, \dots\}$ and write

$$\mathbb{V}(t) = \begin{bmatrix} \mathbb{V}^{\nabla,\nabla}(t) & \mathbb{V}^{\nabla,1}(t) & \mathbb{V}^{\nabla,2}(t) & \dots \\ \mathbb{V}^{1,\nabla}(t) & \mathbb{V}^{1,1}(t) & \mathbb{V}^{1,2}(t) & \dots \\ \mathbb{V}^{2,\nabla}(t) & \mathbb{V}^{2,1}(t) & \mathbb{V}^{2,2}(t) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now define $\mathbb{B} = \left.\frac{d}{dt}\mathbb{V}(t)\right|_{t=0}$ as the infinitesimal generator of $\{\mathbb{V}(t)\}$, resulting in the tridiagonal matrix of operators

$$\mathbb{B}(t) = \begin{bmatrix} \mathbb{B}^{\nabla,\nabla}(t) & \mathbb{B}^{\nabla,1}(t) & & \\ \mathbb{B}^{1,\nabla}(t) & \mathbb{B}^{1,1}(t) & \mathbb{B}^{1,2}(t) & \\ & \mathbb{B}^{2,1}(t) & \mathbb{B}^{2,2}(t) & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

where the blocks $\mathbb{B}^{k\ell} := \left[\mathbb{B}^{k\ell}_{ij}(t)\right]_{i,j\in\mathcal{S}}$, $k,\ell \in \{\nabla,1,2,\dots\}$. The tridiagonal structure arises since, for $|k-\ell| \geq 2$ (where we take $\nabla = 0$ if it appears in the differences) it is impossible for $\{X_t\}$ to move from $\mathcal{D}_k$ to $\mathcal{D}_\ell$ in an infinitesimal amount of time.

Remark 2.2 We use a blackboard bold font with an overline above the character (e.g. $\overline{\overline{\mathbb{B}}}$ and $\overline{\mathbb{V}}(t)$) to represent theoretical operators derived in (Bean and O'Reilly 2014) which are constructed using the partition in Eq. (1). The operators denoted with an overline play a minor role in the introductory sections of this paper, but do not appear again. We use a blackboard font sans overline (e.g. $\mathbb{V}(t)$ and $\mathbb{B}$) to represent the same operators but which are constructed with the finer partition defined by $\mathcal{D}_k$, $k = \nabla, 1, 2, \dots$. We use the letters $i, j \in \mathcal{S}$ to represent states of the phase process, letters $m, n, \in \{+, -, 0\}$ to refer to the partition in terms of the sets in Eq. (1), and the letters $k, \ell \in \{\nabla, 1, 2, \dots\}$ to refer to the finer partition into sets $\{\mathcal{D}_k\}_k$. With a slight abuse of notation, whenever we use the dummy variables k, ℓ without qualification we imply $k, \ell \in \{\nabla, 1, 2, \dots\}$, the dummy variables m, n without qualification imply $m, n \in \{+, -, 0\}$ and the dummy variables i, j without qualification imply $i, j \in \mathcal{S}$. E.g. $\mathbb{B}^{k\ell}_{ij}$ means $\mathbb{B}^{k\ell}_{ij}$, $i, j \in \mathcal{S}, k, \ell \in \{\nabla, 1, 2, \dots\}$ and $\mathbb{B}^{mn}_{ij}$ means $\mathbb{B}^{mn}_{ij}$, $i, j \in \mathcal{S}, m, n \in \{+, -, 0\}$.

By an appropriate choice of the intervals $\{\mathcal{D}_k\}$, $k \in \{\nabla, 1, 2, \dots, \}$, the partition used in (Bean and O'Reilly 2014) can be recovered. Intuitively, we must ensure that each of the boundaries of $\mathcal{F}^m_i$, $i \in \mathcal{S}$, $m \in \{+, -, 0\}$, align with a boundary of a cell $\mathcal{D}_k = [x_k, x_{k+1}] \setminus \{0\}$. Then, each set $\mathcal{F}^m_i$, $i \in \mathcal{S}$, $m \in \{+, -, 0\}$, can be written as a union of cells, $\mathcal{D}_k$, $k = \nabla, 1, 2, \dots$, sans a collection of points which have measure zero for all measures in $\mathcal{M}_0$, and this collection of points is inconsequential for the purposes of the approximations presented here.

Formally, to recover the partition used in (Bean and O'Reilly 2014) we choose the intervals $\mathcal{D}_k$ such that $l(\mathcal{D}_k \cap \mathcal{F}^m_i) \in \{l(\mathcal{D}_k), 0\}$ for all $i \in \mathcal{S}$, $m \in \{+, -, 0\}$, $k \in \{\nabla, 1, 2, \dots\}$, for all measures $l \in \mathcal{M}_0$. That is, we choose $\mathcal{D}_k$ such that it is contained (up to sets of measure 0 with respect to measures in $\mathcal{M}_0$) within $\mathcal{F}^m_i$ for some $m \in \{+, -, 0\}$ and all $i \in \mathcal{S}$. We assume such a partition for the rest of the paper. For $i \in \mathcal{S}$, $m \in \{+, -, 0\}$, let $\mathcal{K}^m_i = \{k \in \{\nabla, 1, 2, \dots\} \mid l(\mathcal{D}_k \cap \mathcal{F}^m_i) = l(\mathcal{D}_k), l \in \mathcal{M}_0\}$, so that $\bigcup_{k \in \mathcal{K}^m_i} \mathcal{D}_k$ and $\mathcal{F}^m_i$ are equal up to a set of l-measure 0 for all $l \in \mathcal{M}_0$. Define $\mathcal{K}^m = \bigcup_{i \in \mathcal{S}} \mathcal{K}^m_i$, $m \in \{+, -, 0\}$.

To recover the partition defined by (1) we bundle together the elements of $\mathbb{V}(t)$ which correspond to $\mathcal{F}_i^m$ and $\mathcal{F}_j^n$. That is, for $m, n \in \{+, -, 0\}$, define $\mathbb{V}_{ij}^{mn}(t)$ as the matrix of operators

$$\mathbb{V}_{ij}^{mn}(t) = \left[\mathbb{V}_{ij}^{k\ell}(t) \right]_{k \in \mathcal{K}_i^m, \ell \in \mathcal{K}_j^n}.$$

Then, for $i, j \in \mathcal{S}$, $m, n \in \{+, -, 0\}$, we can write $\overline{\mathbb{V}}_{ij}^{mn}(t) = \mathbf{1}_{|\mathcal{K}_i^m|} \mathbb{V}_{ij}^{mn}(t) \mathbf{1}_{|\mathcal{K}_j^n|}^{\mathrm{T}}$ where $\mathbf{1}_{|\mathcal{K}_i^m|}$ and $\mathbf{1}_{|\mathcal{K}_j^n|}$ are row-vectors of 1's of length $|\mathcal{K}_i^m|$ and $|\mathcal{K}_j^n|$, respectively, and $^{\mathrm{T}}$ denotes the transpose. The same construction can be achieved with $\mathbb{B}$.

Let $\mathcal{S}_k^+ = \{i \in \mathcal{S} \mid r_i(x) > 0, \forall x \in \mathcal{D}_k\}$, $\mathcal{S}_k^0 = \{i \in \mathcal{S} \mid r_i(x) = 0, \forall x \in \mathcal{D}_k\}$, $\mathcal{S}_k^- = \{i \in \mathcal{S} \mid r_i(x) < 0, \forall x \in \mathcal{D}_k\}$ and $\mathcal{S}_k^{\bullet} = \{i \in \mathcal{S} \mid r_i(x) \neq 0, \forall x \in \mathcal{D}_k\}$ for $k \in \{\nabla, 1, \ldots\}$. For later reference, we need the following constructions. For $k, \ell \in \{\nabla, 1, 2, \ldots\}$

$$\mathbb{B}^{k\ell} = \left[\mathbb{B}_{ij}^{k\ell} \right]_{i,j \in \mathcal{S}}, \tag{4}$$

for $i, j \in \mathcal{S}$

$$\mathbb{B}_{ij} = \left[\mathbb{B}_{ij}^{k\ell} \right]_{k,\ell \in \{\nabla, 1, 2, \ldots\}}, \tag{5}$$

and for $m, n \in \{+, -, 0\}$

$$\mathbb{B}^{mn} = \left[\left[\mathbb{B}_{ij}^{k\ell} \right]_{i \in \mathcal{S}_k^m, j \in \mathcal{S}_\ell^n} \right]_{k \in \mathcal{K}^m, \ell \in \mathcal{K}^n}, \tag{6}$$

$$\mathbb{B}^{kn} = \left[\left[\mathbb{B}_{ij}^{k\ell} \right]_{i \in \mathcal{S}_k^m, j \in \mathcal{S}_\ell^n} \right]_{\ell \in \mathcal{K}^n} \quad \text{for } k \in \{\nabla, 1, 2, \ldots\}, \tag{7}$$

$$\mathbb{B}^{m\ell} = \left[\left[\mathbb{B}_{ij}^{k\ell} \right]_{i \in \mathcal{S}_k^m, j \in \mathcal{S}_\ell^n} \right]_{k \in \mathcal{K}^m} \quad \text{for } \ell \in \{\nabla, 1, 2, \ldots\}. \tag{8}$$

We persist with the partition $\mathcal{D}_k$, $k \in \nabla, 1, 2, \ldots$ throughout this paper, as this is consistent with the partition used in the DG method, and note that for all the operators defined with this partition, the partitioning used in (Bean and O'Reilly 2014) can always be recovered by the above construction.

We can write $\mu_i^k \mathbb{B}_{ij}^{k\ell}(\mathcal{A})$ in kernel form as $\int_{x \in \mathcal{D}_k, y \in \mathcal{A}} \mathrm{d}\mu_i^k(x) \mathbb{B}_{ij}^{k\ell}(x, \mathrm{d}y)$. It is known that

$$\mu_i^k \mathbb{B}_{ij}^{kk}(\mathrm{d}y) := \int_{x \in \mathcal{D}_k} \mathrm{d}\mu_i^k(x) \mathbb{B}_{ij}^{kk}(x, \mathrm{d}y) = \begin{cases} v_i^k(y) T_{ij} \, \mathrm{d}y, & i \neq j, \\ v_i^k(y) T_{ii} \, \mathrm{d}y - c_i \frac{\mathrm{d}}{\mathrm{d}y} v_i^k(y) \, \mathrm{d}y, & i = j, \end{cases}$$

on the interior of $\mathcal{D}_k$ (Karandikar and Kulkarni 1995). Intuitively, $v_i^k(y) T_{ij} \, \mathrm{d}y$ represents the instantaneous rate of transition from phase i to j in the infinitesimal interval $\mathrm{d}y$, $v_i^k(y) T_{ii} \, \mathrm{d}y$ represents no such transition occurring, and $-c_i \frac{\mathrm{d}}{\mathrm{d}y} v_i^k(y) \, \mathrm{d}y$ represents the drift across the interval $\mathrm{d}y$ when the phase is i.

Translating the results of Bean and O'Reilly ([2014]) to use the partition $\{\mathcal{D}_k\}$ we may state that, for all $i, j \in \mathcal{S}, k \in \{1, 2, \dots\}$,

$$\mu_i^k \mathbb{B}_{ij}^{kk}(\mathcal{D}_k) = \int_{x \in \mathcal{D}_k} v_i^k(x) T_{ij}\, dx - c_i v_i^k(x_{k+1}) \mathbb{1}_{(c_i > 0, i = j)} + c_i v_i^k(x_k) \mathbb{1}_{(c_i < 0, i = j)},$$

where $\mathbb{1}$ is the indicator function. Intuitively, the first term represents the instantaneous rate of the stochastic transitions of the phase process $\{\varphi_t\}$, the second term represents the flux out of the right-hand edge of $\mathcal{D}_k$ which occurs when $c_i > 0$ only, and the last term represents the flux out of the left-hand edge of $\mathcal{D}_k$ which occurs when $c_i < 0$ only.

The results of Bean and O'Reilly ([2014]) also imply that,

$$\mu_i^k \mathbb{B}_{ii}^{k,k+1}(\mathcal{D}_{k+1}) = c_i v_i^k(x_{k+1}) \mathbb{1}_{(c_i > 0)}, \text{ for all } i \in \mathcal{S}, k \in \{1, 2, \dots\},$$
$$\mu_i^k \mathbb{B}_{ii}^{k,k-1}(\mathcal{D}_{k-1}) = -c_i v_i^k(x_k) \mathbb{1}_{(c_i < 0)}, \text{ for all } i \in \mathcal{S}, k \in \{2, 3, \dots\}.$$

Intuitively, the first equation represents the flux from $\mathcal{D}_k$ to $\mathcal{D}_{k+1}$ across the shared boundary at x_{k+1} which occurs when $c_i > 0$ only. The second expression represents the flux from $\mathcal{D}_k$ to $\mathcal{D}_{k-1}$ across the shared boundary at x_k which occurs when $c_i < 0$ only.

Bean and O'Reilly ([2014]) also state that, at the boundary $x = 0$, for states $i \in \mathcal{S}$ with $c_i \leq 0$ such that a point mass at 0 is possible, we have

$$\mu_i^\nabla \mathbb{B}_{ii}^{\nabla, \nabla} = \mu_i^\nabla(\{0\}) T_{ii},$$
$$\mu_i^\nabla \mathbb{B}_{ij}^{\nabla, \nabla} = \mu_i^\nabla(\{0\}) T_{ij}, j \in \mathcal{S}, c_j \leq 0,$$
$$\mu_i^\nabla \mathbb{B}_{ij}^{\nabla, 1} = \mu_i^\nabla(\{0\}) T_{ij}, j \in \mathcal{S}, c_j > 0,$$
$$\mu_i^1 \mathbb{B}_{ii}^{1, \nabla} = -c_i v_i^1(0^+), j \in \mathcal{S}, c_j < 0,$$

where 0^+ is the right limit at 0. Otherwise $\mu_i^k \mathbb{B}_{ij}^{k\ell} = 0$, for $|k - \ell| \geq 2$, $i, j \in \mathcal{S}$ or $|k - \ell|$ $= 1$, $i, j \in \mathcal{S}, i \neq j$, where we take $\nabla = 0$ if it appears in the differences, capturing the facts that the process $\{X_t\}$ is continuous and that drift across boundaries occurs only when $\{\varphi_t\}$ remains in the same phase.

Note that we have not presented $\mathbb{B}$ in its full detail here and refer the reader to (Bean and O'Reilly [2014]) for the details. The main goal here is to show how $\mathbb{B}$ is used to construct the stationary distribution of the SFFM and to illustrate the link between the operator $\mathbb{B}$ and the DG approximation of the same object. As we shall see later, these expressions closely resemble the DG approximations to the same quantities.

2.2 Matrix $\mathbb{D}(s)$ of Operators

Let $b(t) := \int_0^t \left| r_{\varphi_z}(X_z) \right| dz$ be the total unregulated amount of fluid that has flowed into or out of the second buffer $\{Y_t\}$ during $[0, t]$, and let $\omega(y) := \inf\{t > 0 : b(t) = y\}$ be the first time this accumulated in-out amount hits level y. Note that at the stopping time $\omega(y)$ it must be that $(X_{\omega(y)}, \varphi_{\omega(y)}) \in (\mathcal{F}_i^m, i)$ for some $i \in \mathcal{S}$ and $m \in \{+, -\}$, i.e. $m \neq 0$. We define the operators $\mathbb{U}_{ij}^{k\ell}(y, s) : \mathcal{M}_0(\mathcal{D}_k \cap \mathcal{F}_i^m) \mapsto \mathcal{M}_0(\mathcal{D}_\ell \cap \mathcal{F}_j^n)$, for $k \in \mathcal{K}_i^+ \cup \mathcal{K}_i^-, \ell \in \mathcal{K}_j^+ \cup \mathcal{K}_j^-$, and $i \in \mathcal{S}_k, j \in \mathcal{S}_k$, by

$$\mu_i^k \mathbb{U}_{ij}^{k\ell}(y,s)(\mathcal{A}) := \int_{x \in \mathcal{D}_k} \mathrm{d}\mu_i^k(x) \mathbb{E}\left[e^{-s\omega(y)}1\{\varphi_{\omega(y)} = j, X_{\omega(y)} \in \mathcal{A} \cap \mathcal{D}_\ell\} \mid \varphi_0 = i, X_0 = x\right].$$

Then, construct the matrix of operators

$$\mathbb{U}(y,s) := \left[[\mathbb{U}_{ij}^{k\ell}(y,s)]_{i \in \mathcal{S}_k, j \in \mathcal{S}_\ell}\right]_{k,\ell \in \mathcal{K}^+ \cup \mathcal{K}^-}.$$

The matrix of operators $\mathbb{D}(s)$ is the infinitesimal generator of the semigroup $\{\mathbb{U}(y,s)\}_{y \geq 0}$ defined by

$$\mathbb{D}(s) = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{U}(y,s)|_{y=0},$$

whenever this limit exists.

Recalling the constructions in Eqs. (4)-(8) and using Lemma 4 of (Bean and O'Reilly 2014) gives the following expression for $\mathbb{D}(s)$.

Lemma 2.1 *For $y \geq 0$, $s \in \mathbb{C}$ with $Re(s) \geq 0$, $i,j \in \mathcal{S}$, $k \in \mathcal{K}_i^+ \cup \mathcal{K}_i^-$, $\ell \in \mathcal{K}_j^+ \cup \mathcal{K}_j^-$,*

$$\mathbb{D}_{ij}^{k\ell}(s) = [\mathbb{R}^k(\mathbb{B}^{k\ell} - s\mathbb{I} + \mathbb{B}^{k0}(s\mathbb{I} - \mathbb{B}^{00})^{-1}\mathbb{B}^{0\ell})]_{ij},$$

where $\mathbb{I}$ is the identity operator, and $\mathbb{R}^k := \mathrm{diag}(\mathbb{R}_i^k)_{i \in \mathcal{S}}$ is a diagonal matrix of operators $\mathbb{R}_i^k$ given by

$$\mu_i^k \mathbb{R}_i^k(\mathcal{A}) := \int_{x \in \mathcal{A} \cap \mathcal{D}_k} \frac{1}{r_i(x)} \mathrm{d}\mu_i^k(x), \quad k \in \mathcal{K}_i^+ \cup \mathcal{K}_i^-.$$

Also, construct the matrices of operators

$$\mathbb{D}^{mn} := \left[[\mathbb{D}_{ij}^{k\ell}]_{i \in \mathcal{S}_k^m, j \in \mathcal{S}_k^n}\right]_{k \in \mathcal{K}^m, \ell \in \mathcal{K}^n}.$$

2.3 Matrix $\psi(s)$ of Operators

We denote by $\psi(s)$ the matrix of operators with the same dimensions as $\mathbb{D}^{+-}$, recording the Laplace-Stieltjes transforms of the time for $\{Y_t\}$ to return, for the first time, to the initial level of zero as introduced in (Bean and O'Reilly 2014) but constructed with respect to the finer partition $\{\mathcal{D}_k\}$. Define the stopping time $\theta(y) := \inf\{t > 0 : Y_t = y\}$ to be the first time $\{Y_t\}$ hits level y, then each component $\psi_{ij}^{k\ell}(s) : \mathcal{M}_0(\mathcal{D}_k) \mapsto \mathcal{M}_0(\mathcal{D}_\ell), i,j \in \mathcal{S}, k \in \mathcal{K}_i^+$ and $\ell \in \mathcal{K}_j^-$, is given by

$$\mu_i^k \psi_{ij}^{k\ell}(s)(\mathcal{A}) := \int_{x \in \mathcal{D}_k} \mathrm{d}\mu_i^k(x) \mathbb{E}\left[e^{-s\theta(0)}1\{\varphi_{\theta(0)} = j, X_{\theta(0)} \in \mathcal{A} \cap \mathcal{D}_\ell\} \mid X_0 = x, Y_0 = 0, \varphi_0 = i\right].$$

Bean and O'Reilly [Theorem 1] Bean and O'Reilly (2014) give the following result which characterises $\psi(s)$.

Theorem 2.1 *For Re* $(s) \geq 0$, $\psi(s)$ *satisfies the equation*:

$$\mathbb{D}^{+-}(s) + \Psi(s)\mathbb{D}^{-+}(s)\Psi(s) + \mathbb{D}^{++}(s)\Psi(s) + \Psi(s)\mathbb{D}^{--}(s) = 0.$$

Furthermore, if s is real then $\psi(s)$ *is the minimal nonnegative solution.*

2.4 Stationary Distribution

Let $\psi := \psi(0)$. We define $\theta_n := \inf\{t \geq \theta_{n-1} : Y_t = 0\}$, for $n \geq 2$, to be the sequence of hitting times to level 0 of Y_t, with $\theta_1 := \theta(0)$. Consider a discrete-time Markov process $\{(X_{\theta_n}, \varphi_{\theta_n}), n \geq 1\}$, and for $i \in \mathcal{S}, k \in \mathcal{K}_i^-$ define the measures ξ_i^k as follows $\xi_i^k(\mathcal{A}) := \lim_{n \to \infty} \mathbb{P}\left[X_{\theta_n} \in \mathcal{A} \cap \mathcal{D}_k, \varphi_{\theta_n} = i\right]$. By (Bean and O'Reilly 2014), the vector of measures $\xi := (\xi_i^k)_{i \in \mathcal{S}_k^-, k \in \mathcal{K}^-}$ satisfies the following set of equations

$$\begin{bmatrix} \xi & 0 \end{bmatrix} \left(-\begin{bmatrix} \mathbb{B}^{--} & \mathbb{B}^{-0} \\ \mathbb{B}^{0-} & \mathbb{B}^{00} \end{bmatrix}\right)^{-1} \begin{bmatrix} \mathbb{B}^{-+} \\ \mathbb{B}^{0+} \end{bmatrix} \Psi = \xi, \tag{9}$$

$$\sum_{k \in \mathcal{K}^-} \sum_{i \in \mathcal{S}_k^-} \xi_i^k(\mathcal{F}_i^-) = 1. \tag{10}$$

We reproduce Theorem 2 of (Bean and O'Reilly 2014) below, which gives the joint stationary distribution of $\{(X_t, Y_t, \varphi_t)\}$. Recall that the joint stationary density operator $\pi(y) = (\pi_i(y))_{i \in \mathcal{S}}$ for $\{(X_t, Y_t, \varphi_t)\}$ and the joint stationary mass operator $\mathbb{p} = (\mathbb{p}_i)_{i \in \mathcal{S}}$ are defined by (2) and (3), respectively. We can partition π as follows

$$\pi(y) = \begin{bmatrix} \pi^+(y) & \pi^-(y) & \pi^0(y) \end{bmatrix}$$
$$= \left[\left(\pi_i^k(y)\right)_{i \in \mathcal{S}_k^+, k \in \mathcal{K}^+} \quad \left(\pi_i^k(y)\right)_{i \in \mathcal{S}_k^-, k \in \mathcal{K}^-} \quad \left(\pi_i^k(y)\right)_{i \in \mathcal{S}_k^0, k \in \mathcal{K}^0} \right],$$

where

$$\pi_i^k(y)(\mathcal{A}) = \pi_i(y)(\mathcal{A} \cap \mathcal{D}_k).$$

Similarly, we can write

$$\mathbb{p} = \begin{bmatrix} \mathbb{p}^- & \mathbb{p}^0 \end{bmatrix} = \left[\left(\mathbb{p}_i^k\right)_{i \in \mathcal{S}_k^-, k \in \mathcal{K}^-} \quad \left(\mathbb{p}_i^k\right)_{i \in \mathcal{S}_k^0, k \in \mathcal{K}^0} \right],$$

where $\mathbb{p}_i^k(\mathcal{A}) = \mathbb{p}_i(\mathcal{A} \cap \mathcal{D}_k)$.

Theorem 2.2 *The density* $\pi^m(y)$, *for* $m \in \{+, -, 0\}$ *and* $y > 0$, *and the probability mass* $\mathbb{p}^m$, *for* $m \in \{-, 0\}$, *satisfy the following set of equations*:

$$\mathbb{w}^0(y) = \begin{bmatrix} \mathbb{w}^+(y) & \mathbb{w}^-(y) \end{bmatrix} \begin{bmatrix} \mathbb{B}^{+0} \\ \mathbb{B}^{-0} \end{bmatrix} \left(-\mathbb{B}^{00}\right)^{-1}, \tag{11}$$

$$\begin{bmatrix} \mathbb{w}^+(y) & \mathbb{w}^-(y) \end{bmatrix} = \begin{bmatrix} \mathbb{p}^- & \mathbb{p}^0 \end{bmatrix} \begin{bmatrix} \mathbb{B}^{-+} \\ \mathbb{B}^{0+} \end{bmatrix} \begin{bmatrix} e^{\mathbb{K}y} & e^{\mathbb{K}y}\mathbb{\psi} \end{bmatrix} \begin{bmatrix} \mathbb{R}^+ & 0 \\ 0 & \mathbb{R}^- \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} \mathbb{p}^- & \mathbb{p}^0 \end{bmatrix} = z \begin{bmatrix} \mathbb{\xi} & \mathbf{0} \end{bmatrix} \left(-\begin{bmatrix} \mathbb{B}^{--} & \mathbb{B}^{-0} \\ \mathbb{B}^{0-} & \mathbb{B}^{00} \end{bmatrix}\right)^{-1}, \tag{13}$$

$$\sum_{m\in\{+,-,0\}} \sum_{i\in\mathcal{S}} \int_{y=0}^{\infty} \mathbb{w}_i^m(y)(\mathcal{F}_i^m)\,\mathrm{d}y + \sum_{m\in\{-,0\}} \sum_{i\in\mathcal{S}} \mathbb{p}_i^m(\mathcal{F}_i^m) = 1, \tag{14}$$

where $\mathbb{K} := \mathbb{D}^{++}(0) + \mathbb{\psi}\mathbb{D}^{(-+)}(0)$ *and z is a normalising constant.*

At this point we reiterate that Eqs. (11)-(14) are operator equations and are only amenable to numerical evaluation in the simplest of cases. Sources of this intractability come from, for example, the need to find the inverse operator $(-\mathbb{B}^{00})^{-1}$, and the need to find the solution, $\mathbb{\psi}(s)$, of the operator equation in Theorem 2.1. There is also the complexity of the partition of the operators defined by the sets $\mathcal{F}_i^m$, $i \in \mathcal{S}$, $m \in \{+, -, 0\}$ to consider. Therefore, there is the need for approximation schemes such as the DG scheme we introduce next.

3 Discontinuous Galerkin Approximation of a Stochastic Fluid Model: Approximating $\mathbb{B}$

Discontinuous Galerkin (DG) methods can be used to approximate the solutions to systems of partial differential equations (PDEs). For a more thorough description of these methods see (Hesthaven and Warburton 2007). The domain of approximation is partitioned into intervals, referred to individually as *cells* and collectively as a *mesh*. On each cell, we have a finite element approximation, which constructs a finite-dimensional smooth Sobolev space using piecewise-polynomial basis functions, and then projects the partial differential equations onto this space. This projection leads to a new system of equations, referred to as the *weak form* of the original system of PDEs. Next, we must approximate the *flux operator* which moves probability from one cell to another, in a manner similar to the underlying principle of a finite-volume approximation. This method conserves probability, and can handle discontinuities, such as jumps and point masses. Here we construct the DG approximation to the matrix of operators $\mathbb{B}$ which we use later to construct a DG approximation to $\mathbb{D}(s)$ then $\mathbb{\psi}(s)$, and ultimately the stationary distribution of an SFFM.

3.1 The Partial Differential Equation

We start by introducing the PDE from which we will extract the approximation to the generator $\mathbb{B}$.

Let $f_i(x, t)$ be the joint density of $\{(X_t, \varphi_t)\}$:

$$f_i(x, t) := \frac{\partial}{\partial x} \mathbb{P}\big[X_t \leq x, \varphi_t = i\big], \quad x > 0, \, i \in \mathcal{S},$$

which satisfies the system of partial differential equations

$$\frac{\partial}{\partial t} f_i(x, t) = \sum_{j \in \mathcal{S}} f_j(x, t) T_{ji} - c_i \frac{\partial}{\partial x} f_i(x, t), \quad x > 0, i \in \mathcal{S}$$

subject to suitable boundary conditions (Bean and O'Reilly 2014). In matrix form,

$$\frac{\partial}{\partial t} \boldsymbol{f}(x, t) = \boldsymbol{f}(x, t) T - \frac{\partial}{\partial x} \boldsymbol{f}(x, t) C, \tag{15}$$

where $\boldsymbol{f}(x, t) = \big(f_i(x, t)\big)_{i \in \mathcal{S}}$. This system of PDEs is closely related to the generator $\mathbb{B}$. For $\mathcal{A} \subset (0, \infty)$, and assuming $\boldsymbol{\mu}(t)$ admits a density,

$$\boldsymbol{\mu}(t)(\mathcal{A}) = \int_{x \in \mathcal{A}} \boldsymbol{f}(x, t) \, \mathrm{d}x.$$

That is, $f_i(x, t)$ is the density of $\mu_i(t)$. Then $\boldsymbol{\mu}(t)$ satisfies the operator differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\mu}(t)(\, \mathrm{d}x) = \boldsymbol{\mu}(t)\mathbb{B}(\, \mathrm{d}x) = \boldsymbol{f}(x, t) T \, \mathrm{d}x - \frac{\partial}{\partial x} \boldsymbol{f}(x, t) C \, \mathrm{d}x,$$

on the interior of the space $[0, \infty)$. Thus, by approximating the operator on the right-hand side of Eq. (15) we can approximate the infinitesimal operator $\mathbb{B}$. The DG method does exactly this, by approximating the operator with a matrix.

3.2 Cells, Test Functions, and Weak Formulation

To begin with, consider an unbounded first fluid level $\{\widehat{X}_t, t \geq 0\}$, $\widehat{X}_t \in (-\infty, \infty)$. We will eventually truncate this space so that we have a finite dimensional approximation; however, this requires a discussion on boundary conditions which we save for later. Let $\mathcal{D}_k = [x_k, x_{k+1}]$, $k \in \mathbb{Z}$ partition the domain $(-\infty, \infty)$. We call the $\mathcal{D}_k$ cells.

On each cell $\mathcal{D}_k$ we choose N_k linearly independent functions $\{\phi_r^k\}_{r=1}^{N_k}$, compactly supported on $\mathcal{D}_k$ (i.e. $\phi_r^k(x) = 0$ for $x \notin \mathcal{D}_k$) to form a basis for the space W_k, in which we formulate the approximation. Here, as is standard in DG methods (Hesthaven and Warburton 2007), we take $\{\phi_r^k\}_{r=1}^{N_k}$ to be the space of polynomials of degree $N_k - 1$. It is convenient in this work to take $\{\phi_r^k\}_{r=1}^{N_k}$ as a basis of Lagrange interpolating polynomials defined by the Gauss-Lobatto quadrature points, since our approximations inherit nice properties from this (Hesthaven and Warburton 2007). However, the constructions presented here are general, and any basis can be used. For the sake of illustration, the reader may think of $\{\phi_r^k\}_{r=1}^{N_k}$ as the Lagrange polynomials. On each cell $\mathcal{D}_k$ we approximate

$$f_i(x, t) \approx u_i^k(x, t) = \sum_{r=1}^{N_k} a_{i,r}^k(t) \phi_r^k(x),$$

where $a_{i,r}^k(t)$ are yet-to-be-determined time-dependent coefficients. We refer to u_i^k as the *local* approximation on $\mathcal{D}_k$, while the *global* approximation is given by $\sum_{k \in \mathbb{Z}} u_i^k$ on the whole domain. The whole approximation space is $\bigoplus_{k \in \mathbb{Z}} W_k$.

Let $\mathcal{N}_k := \{1, \ldots, N_k\}$, $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, define *local* row-vectors

$$\boldsymbol{\phi}^k(x) = (\phi_r^k(x))_{r \in \mathcal{N}_k}, \quad \boldsymbol{a}_i^k(x) = (a_{i,r}^k(x))_{r \in \mathcal{N}_k}, \ i \in \mathcal{S}.$$

Note that we will always use the letter r to index the basis function within each cell.

The DG method proceeds by considering the *weak-formulation* of the PDE, which is constructed from the *strong-form* of the PDE, Eq. (15). In general, to construct the weak-form we need a set of test functions, say W. Now, take the strong form of the PDE, multiply it by some test function $\psi(x) \in W$, integrate with respect to x, and apply integration by parts to the derivative with respect to x, to get

$$\int_{x \in \mathbb{R}} \frac{\partial}{\partial t} f_j(x,t) \psi(x) \, \mathrm{d}x = \int_{x \in \mathbb{R}} \sum_{i \in \mathcal{S}} f_i(x,t) T_{ij} \psi(x) \, \mathrm{d}x + \int_{x \in \mathbb{R}} f_j(x,t) c_j \frac{\mathrm{d}}{\mathrm{d}x} \psi(x) \, \mathrm{d}x$$
$$- [f_j(x,t) c_j \psi(x)]_{x=-\infty}^{x=\infty}, \tag{16}$$

for $j \in \mathcal{S}$. It is common to choose W such that $\psi(-\infty) = \psi(\infty) = 0$, in which case the last term on the right is zero. Requiring (16) to hold for every $\psi \in W$ gives the weak-formulation of the PDE. For a sufficiently rich set of test functions W the weak and strong forms of the PDE are equivalent. Solutions to (16) are known as *weak* solutions and generalise the concept of a solution of the PDE. For example, this may allow discontinuities with respect to x in the solution – something which is ill-defined for the strong form.

For the purpose of DG, we take the set of test functions to be $W = \bigoplus_{k \in \mathbb{Z}} W^k$, the same as the set of basis functions of our solution space. Proceeding as described above, the weak formulation is

$$\int_{x \in \mathcal{D}_k} \frac{\partial}{\partial t} f_j(x,t) \phi_r^k(x) \, \mathrm{d}x = \int_{x \in \mathcal{D}_k} \sum_{i \in \mathcal{S}} f_i(x,t) T_{ij} \phi_r^k(x) \, \mathrm{d}x + \int_{x \in \mathcal{D}_k} f_j(x,t) c_j \frac{\mathrm{d}}{\mathrm{d}x} \phi_r^k(x) \, \mathrm{d}x$$
$$- [f_j(x,t) c_j \phi_r^k(x)]_{x=x_k}^{x=x_{k+1}},$$

since ϕ_r^k is compactly supported on $\mathcal{D}_k$, for all $j \in \mathcal{S}$, $r \in \mathcal{N}_k$, $k \in \mathbb{Z}$. Now, note that any function $g(x)$ can be decomposed as $g(x) = g^W(x) + g^\perp(x)$ where $g^W \in W$ and $g^\perp \in W^\perp$, and $W^\perp$ is the orthogonal complement of W. Since $g^\perp$ is orthogonal to W, $\int_x g^\perp(x) \phi_r^k(x) \, \mathrm{d}x = 0$ for $r \in \mathcal{N}_k$, $k \in \mathbb{Z}$. Also, note that $\frac{\mathrm{d}}{\mathrm{d}x} \phi_r^k(x) \in W$. Using this, we can write

$$\int_{x \in \mathcal{D}_k} \frac{\partial}{\partial t} \left(f_j^W(x,t) + f_j^\perp(x,t) \right) \phi_r^k(x) \, \mathrm{d}x = \int_{x \in \mathcal{D}_k} \sum_{i \in \mathcal{S}} \left(f_i^W(x,t) + f_i^\perp(x,t) \right) T_{ij} \phi_r^k(x) \, \mathrm{d}x$$
$$+ \int_{x \in \mathcal{D}_k} \left(f_j^W(x,t) + f_j^\perp(x,t) \right) c_j \frac{\mathrm{d}}{\mathrm{d}x} \phi_r^k(x) \, \mathrm{d}x$$
$$- [f_j(x,t) c_j \phi_r^k(x)]_{x=x_k}^{x=x_{k+1}},$$

which is equivalent to

$$\int_{x \in \mathcal{D}_k} \frac{\partial}{\partial t} f_j^W(x,t) \phi_r^k(x) \, \mathrm{d}x = \int_{x \in \mathcal{D}_k} \sum_{i \in \mathcal{S}} f_i^W(x,t) T_{ij} \phi_r^k(x) \, \mathrm{d}x + \int_{x \in \mathcal{D}_k} f_j^W(x,t) c_j \frac{\mathrm{d}}{\mathrm{d}x} \phi_r^k(x) \, \mathrm{d}x$$
$$- [f_j(x,t) c_j \phi_r^k(x)]_{x=x_k}^{x=x_{k+1}}.$$

$$(17)$$

Now, $f_j^W(x,t) \in W$ so, on $\mathcal{D}_k$, it can be expressed as $\boldsymbol{a}_j^k(t) \boldsymbol{\phi}^k(x)^{\mathrm{T}}$, which we now substitute into (17) and repeat this for all test functions $\phi_r^k(x)$, $r = 1, ..., N_k$, to get the following system of equations,

$$\int_{x \in \mathcal{D}_k} \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}_j^k(t) \boldsymbol{\phi}^k(x)^{\mathrm{T}} \boldsymbol{\phi}^k(x) \, \mathrm{d}x = \int_{x \in \mathcal{D}_k} \sum_{i \in \mathcal{S}} \boldsymbol{a}_i^k(t) \boldsymbol{\phi}^k(x)^{\mathrm{T}} T_{ij} \boldsymbol{\phi}^k(x) \, \mathrm{d}x$$
$$+ \int_{x \in \mathcal{D}_k} \boldsymbol{a}_j^k(t) \boldsymbol{\phi}^k(x)^{\mathrm{T}} c_j \frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{\phi}^k(x) \, \mathrm{d}x$$
$$- c_j [f_j(x,t) \boldsymbol{\phi}^k(x)]_{x=x_k}^{x=x_{k+1}}, \quad k \in \mathbb{Z}.$$

$$(18)$$

3.3 Mass, Stiffness, and Flux

For $k \in \mathbb{Z}$, define local *mass* and *stiffness* matrices M_k and G_k by

$$M_k := \int_{x \in \mathcal{D}_k} \boldsymbol{\phi}^k(x)^{\mathrm{T}} \boldsymbol{\phi}^k(x) \, \mathrm{d}x, \quad G_k := \int_{x \in \mathcal{D}_k} \boldsymbol{\phi}^k(x)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{\phi}^k(x) \, \mathrm{d}x.$$

We can write (18) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}_j^k(t) M_k = \sum_{i \in \mathcal{S}} \boldsymbol{a}_i^k(t) M_k T_{ij} + c_j \boldsymbol{a}_j^k(t) G_k - c_j [f_j(x,t) \boldsymbol{\phi}^k(x)]_{x=x_k}^{x=x_{k+1}}. \quad (19)$$

It remains to approximate the *flux*, $f_j(x,t)$ at the cell edges x_k, $k \in \mathbb{Z}$, so that we may evaluate the terms $[f_j(x,t) \phi_r^k(x)]_{x=x_k}^{x=x_{k+1}}$, $r = 1, ..., N_k$, $k \in \mathbb{Z}$. This is the key for DG – it joins the local approximations on each cell $\mathcal{D}_k$, into a global approximation on the whole domain of approximation. The flux is the instantaneous rate (with respect to time) at which density moves across the boundaries x_k, $k \in \mathbb{Z}$. There are different choices for the flux, and we refer the reader to (Cockburn 1999; Hesthaven and Warburton 2007), and references therein, for some discussion of the topic. Here, we choose the *upwind* scheme, which, as we shall see, closely resembles the flux terms from the generator $\mathbb{B}$. The approximate flux, also known as the *numerical flux*, is given by

$$f_j^*(x,t) = sign(c_j) \lim_{\varepsilon \to 0^+} u_j(x - \varepsilon c_j, t),$$

at each $x = x_k$, $k \in \mathbb{Z}$. Intuitively, the upwind flux takes the value of the density immediately on the upwind side of each x_k.

Denote by x^- and x^+ the left and right limits at x, respectively. Assume first $c_j > 0$, then

$$-c_j[f_j(x,t)\phi_r^k(x)]_{x=x_k}^{x=x_{k+1}} \approx -c_j[f_j^*(x,t)\phi_r^k(x)]_{x=x_k}^{x=x_{k+1}}$$
$$= -c_j f_j^*(x_{k+1},t)\phi_r^k(x_{k+1}) + c_j f_j^*(x_k,t)\phi_r^k(x_k)$$
$$= -c_j u_j(x_{k+1}^-,t)\phi_r^k(x_{k+1}) + c_j u_j(x_k^-,t)\phi_r^k(x_k)$$
$$= -c_j u_j^k(x_{k+1}^-,t)\phi_r^k(x_{k+1}) + c_j u_j^{k-1}(x_k^-,t)\phi_r^k(x_k)$$
$$= -c_j \boldsymbol{a}_j^k(t)\boldsymbol{\phi}^k(x_{k+1}^-)^{\mathrm{T}}\phi_r^k(x_{k+1}) + c_j \boldsymbol{a}_j^{k-1}(t)\boldsymbol{\phi}^{k-1}(x_k^-)^{\mathrm{T}}\phi_r^k(x_k).$$

In matrix form,

$$-c_j[f_j(x,t)\boldsymbol{\phi}^k(x)]_{x=x_k}^{x=x_{k+1}} \approx -c_j[f_j^*(x,t)\boldsymbol{\phi}^k(x)]_{x=x_k}^{x=x_{k+1}}$$
$$= -c_j \boldsymbol{a}_j^k(t)\boldsymbol{\phi}^k(x_{k+1}^-)^{\mathrm{T}}\boldsymbol{\phi}^k(x_{k+1}) + c_j \boldsymbol{a}_j^{k-1}(t)\boldsymbol{\phi}^{k-1}(x_k^-)^{\mathrm{T}}\boldsymbol{\phi}^k(x_k)$$
$$= c_j \boldsymbol{a}_j^k(t)F_j^{k,k} + c_j \boldsymbol{a}_j^{k-1}(t)F_j^{k-1,k},$$

where, for $j \in \mathcal{S}$ with $c_j > 0$, we define $F_j^{k,k} := -\boldsymbol{\phi}^k(x_{k+1}^-)^{\mathrm{T}}\boldsymbol{\phi}^k(x_{k+1})$, $k \in \mathbb{Z}$ and $F_j^{k-1,k} := \boldsymbol{\phi}^{k-1}(x_k^-)^{\mathrm{T}}\boldsymbol{\phi}^k(x_k)$, $k \in \mathbb{Z}$.

Now proceed similarly for $c_j < 0$ to get the approximation

$$-c_j[f_j(x,t)\boldsymbol{\phi}^k(x)]_{x=x_k}^{x=x_{k+1}} \approx -c_j[f_j^*(x,t)\boldsymbol{\phi}^k(x)]_{x=x_k}^{x=x_{k+1}}$$
$$= -c_j \boldsymbol{a}_j^{k+1}(t)\boldsymbol{\phi}^{k+1}(x_{k+1}^+)^{\mathrm{T}}\boldsymbol{\phi}^k(x_{k+1}) + c_j \boldsymbol{a}_j^k(t)\boldsymbol{\phi}^k(x_k^+)^{\mathrm{T}}\boldsymbol{\phi}^k(x_k)$$
$$= c_j \boldsymbol{a}_j^{k+1}(t)F_j^{k+1,k} + c_j \boldsymbol{a}_j^k(t)F_j^{k,k},$$

where, for $j \in \mathcal{S}$ with $c_j < 0$, we define $F_j^{k+1,k} := -\boldsymbol{\phi}^{k+1}(x_{k+1}^+)^{\mathrm{T}}\boldsymbol{\phi}^k(x_{k+1})$, $k \in \mathbb{Z}$, and $F_j^{k,k} := \boldsymbol{\phi}^k(x_k^+)^{\mathrm{T}}\boldsymbol{\phi}^k(x_k)$, $k \in \mathbb{Z}$.

The matrices $F_j^{k-1,k}$, $F_j^{k,k}$, and $F_j^{k+1,k}$ are the local flux matrices. For convenience, we also define the matrices $F_j^{k,k+1} = 0$ for $c_j < 0$ and $F_j^{k,k-1} = 0$ for $c_j > 0$, $k \in \mathbb{Z}$.

To write this out as a *global* system, define the row-vectors

$$\boldsymbol{a}^k(t) = (\boldsymbol{a}_i^k(t))_{i \in \mathcal{S}}, \quad \boldsymbol{a}(t) = (\boldsymbol{a}^k(t))_{k \in \mathbb{Z}},$$

and the block-diagonal matrix

$$\widetilde{M} = \begin{bmatrix} \ddots & & \\ & I_{N_\mathcal{S}} \otimes M_k & \\ & & \ddots \end{bmatrix}, \text{where } N_\mathcal{S} = |\mathcal{S}|, \otimes \text{ is the Kronecker product,}$$

and the block-tridiagonal matrix $\widetilde{\mathfrak{B}}$

$$= \begin{bmatrix} \ddots & \ddots & \ddots & & \\ & \widetilde{\mathfrak{B}}^{k,k-1} & \widetilde{\mathfrak{B}}^{k,k} & \widetilde{\mathfrak{B}}^{k,k+1} & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where, for $k \in \mathbb{Z}$,

$$\widetilde{\mathfrak{B}}^{kk} = \begin{bmatrix} T_{11}M_k + c_1(F_1^{kk} + G_k) & T_{12}M_k & T_{1N_S}M_k \\ T_{21}M_k & & \\ \vdots & \ddots & \vdots \\ & & T_{N_S-1,N_S}M_k \\ T_{N_S1}M_k & T_{N_S,N_S-1}M_k & T_{N_S,N_S}M_k + c_{N_S}(F_{N_S}^{kk} + G_k) \end{bmatrix},$$

$$\widetilde{\mathfrak{B}}^{k,k+1} = \begin{bmatrix} c_1 F_1^{k,k+1} & & \\ & \ddots & \\ & & c_{N_S}F_{N_S}^{k,k+1} \end{bmatrix},$$

$$\widetilde{\mathfrak{B}}^{k,k-1} = \begin{bmatrix} c_1 F_1^{k,k-1} & & \\ & \ddots & \\ & & c_{N_S}F_{N_S}^{k,k-1} \end{bmatrix}.$$

The matrices $\widetilde{\mathfrak{B}}^{k\ell}$ are defined by sub-blocks; denote these sub-blocks by $\widetilde{\mathfrak{B}}_{ij}^{k\ell}$:

$$\widetilde{\mathfrak{B}}_{ij}^{kk} = \begin{cases} T_{ij}M_k + c_i(F_i^{kk} + G_k) & i = j, \\ T_{ij}M_k & i \neq j, \end{cases}$$

$$\widetilde{\mathfrak{B}}_{ij}^{k\ell} = \begin{cases} c_i F_i^{k\ell} & i = j, \\ 0 & i \neq j, \end{cases} \quad \ell \in \{k-1, k+1\}.$$

The global system of equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}(t)\widetilde{M} = \boldsymbol{a}(t)\widetilde{\mathfrak{B}}. \tag{20}$$

3.4 Boundary conditions

To enable computation, this numerical approximation has to take place on a finite interval, which means we must consider a bounded domain and specify boundary conditions. Recall from Sect. 2 that we wish to impose a regulated boundary at $x = 0$. To apply the DG method, we must truncate the state space of the first fluid at some finite interval upper bound, $[0, \mathcal{I}]$, for some $\mathcal{I} < \infty$, and specify the boundary behaviour at $x = \mathcal{I}$. Here we consider $\mathcal{I}$ to be a regulated boundary. Let us denote the doubly-bounded fluid level by $\overline{X}_t$. Ultimately, we wish to approximate a fluid-fluid queue where the first fluid level, X_t, is bounded below at 0, only. Thus, the first step in the approximation scheme is to approximate X_t by $\overline{X}_t$. The truncation of X_t to $\overline{X}_t$ will result in an artificial point mass at the upper bound, which we have to address properly. It is important to choose an $\mathcal{I}$ sufficiently large to control the error induced by the artificial upper bound, however, with larger $\mathcal{I}$ there comes increased computational burden. We shall further comment on this in Sect. 5, where we report our numerical experiments.

Let $[0, \mathcal{I}]$ be the domain of the approximation, where $\mathcal{I} < \infty$, and assume there is a regulated boundary for $\{\overline{X}_t\}$ at $x = \mathcal{I}$. We partition the space $[0, \mathcal{I}]$ into $\mathcal{D}_\nabla = \{0\}$, $\mathcal{D}_\Delta = \{\mathcal{I}\}$, and K non-trivial intervals, $\mathcal{D}_k = [x_k, x_{k+1}] \setminus \{\{0\} \cup \{\mathcal{I}\}\}$, $x_k < x_{k+1}$, $k = 1, ..., K$, $x_1 = 0$, $x_{K+1} = \mathcal{I}$ and define $h_k := x_{k+1} - x_k$. The notation Δ refers to quantities and sets which are relevant to the boundary at $\mathcal{I}$.

For states with $c_i \leq 0$, there is the possibility of point mass accumulating at the boundary at 0. Denote these point masses by $q_{\nabla,i}(t)$ for $i \in \mathcal{S}_\nabla$. For states with $c_i > 0$ there is

no possibility of a point mass at 0. Similarly, for $c_i \geq 0$ there is the possibility of a point mass at $\mathcal{I}$. Denote these point masses by $q_{\Delta,i}(t)$, for $i \in \mathcal{S}_\Delta$. For states with $c_i < 0$ there is no possibility of a point mass at $\mathcal{I}$. Let $\boldsymbol{q}_\nabla(t) = (q_{\nabla,i}(t))_{i \in \mathcal{S}_\nabla}$ and $\boldsymbol{q}_\Delta(t) = (q_{\Delta,i}(t))_{i \in \mathcal{S}_\Delta}$ and $\boldsymbol{f}_m(x, t) = (f_i(x, t))_{i \in \mathcal{S}_m}$, $m \in \{+, -, 0\}$.

Let us first consider the boundary at $\overline{X}_t = 0$. Bean and O'Reilly (2014) show the following boundary conditions describe the evolution of probability/density of a stochastic fluid model with a regulated boundary at 0;

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}_\nabla(t) = \boldsymbol{q}_\nabla(t)T_{\nabla\nabla} - \boldsymbol{f}_\nabla(0, t)C_\nabla, \tag{21}$$

$$\boldsymbol{q}_\nabla(t)T_{\nabla+} = \boldsymbol{f}_+(0, t)C_+. \tag{22}$$

Equation (21) states that point mass moves between phases according to the sub-generator matrix $T_{\nabla\nabla}$, and that the flux of probability density into the point masses is $-\boldsymbol{f}_\nabla(0, t)C_\nabla$. Substituting the DG approximation for $\boldsymbol{f}_\nabla(0, t)$ into Eq. (21) gives, for $j \in \mathcal{S}_\nabla$,

$$\frac{\mathrm{d}}{\mathrm{d}t}q_{\nabla,j}(t) = \sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t)T_{ij} - \boldsymbol{a}_j^1(t)\boldsymbol{\phi}^1(0)^\mathrm{T} c_j.$$

Equation (22) describes the flux of probability mass to density upon a transition from a phase in $\mathcal{S}_\nabla$ to a phase in $\mathcal{S}_+$. Thus, the flux into the left-hand edge of $\mathcal{D}_1$ in phase $j \in \mathcal{S}_+$ is, $\sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t)T_{ij}$. Therefore, we can now evaluate

$$\begin{aligned}
-c_j[f_j(x, t)\boldsymbol{\phi}^1(x)]_{x=0}^{x=x_1} &= -c_j f_j(x_1, t)\boldsymbol{\phi}^1(x_1) + c_j f_j(0, t)\boldsymbol{\phi}^1(0) \\
&\approx -c_j(f_j^*(x_1, t)\boldsymbol{\phi}^1(x_1) + \sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t)T_{ij}\boldsymbol{\phi}^1(0) \\
&= c_j \boldsymbol{a}_j^1(t)F_j^{1,1} + \sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t)T_{ij}\boldsymbol{\phi}^1(0),
\end{aligned}$$

for $j \in \mathcal{S}_+$. Thus, the DG approximation of the flux into point masses $q_{\nabla,j}(t)$ is $-\boldsymbol{a}_j^1(t)\boldsymbol{\phi}^1(0)^\mathrm{T} c_j$, $j \in \mathcal{S}_-$, the rate of transition of point mass within $\boldsymbol{q}_\nabla(t)$ is $T_{\nabla\nabla}$, and the DG approximation of the transition of point mass to density is $\sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t)T_{ij}\boldsymbol{\phi}^1(0)$, $j \in \mathcal{S}_+$.

Similarly, for the boundary at $\overline{X}_t = \mathcal{I}$ the boundary conditions are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}_\Delta(t) = \boldsymbol{q}_\Delta(t)T_{\Delta\Delta} + \boldsymbol{f}_\Delta(\mathcal{I}, t)C_\Delta,$$

$$\boldsymbol{q}_\Delta(t)T_{\Delta-} = -\boldsymbol{f}_-(\mathcal{I}, t)C_-.$$

Using the same arguments as above,

$$\frac{\mathrm{d}}{\mathrm{d}t}q_{\Delta,j}(t) = \sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(t)T_{ij} + \boldsymbol{a}_j^K(t)\boldsymbol{\phi}^K(\mathcal{I})^\mathrm{T} c_j,$$

$$-c_j[f_j(x, t)\boldsymbol{\phi}^K(x)]_{x=x_K}^{x=\mathcal{I}} \approx c_j \boldsymbol{a}_j^K(t)F_j^{K,K} + \sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(t)T_{ij}\boldsymbol{\phi}^K(\mathcal{I}),$$

for $j \in \mathcal{S}_\Delta$. Thus, the DG approximation of the flux into the point mass $q_{\Delta,j}(t)$ is $\boldsymbol{a}_j^K(t)\boldsymbol{\phi}^K(\mathcal{I})^\mathrm{T} c_j$, $j \in \mathcal{S}_+$, the rate of transition of point mass within $\boldsymbol{q}_\Delta(t)$ is $T_{\Delta\Delta}$, and the DG approximation of the transition of point mass to density is $\sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(t)T_{ij}\boldsymbol{\phi}^K(\mathcal{I})$, $j \in \mathcal{S}_-$.

To include this behaviour in the DG generator we truncate the doubly-infinite matrix $\widetilde{\mathfrak{B}}$ at $k = 1$ and $k = K$, then append $|\mathcal{S}_\nabla|$ rows and columns to the top and left, and $|\mathcal{S}_\Delta|$ rows and columns to the bottom and right. These represent the point masses $\boldsymbol{q}_\nabla(t)$ and $\boldsymbol{q}_\Delta(t)$, respectively. Given the discussion above, the truncated matrix is

$$\widehat{\mathfrak{B}} = \begin{bmatrix} T_{\nabla\nabla} & \widetilde{\mathfrak{B}}^{\nabla 1} & & & & & \\ \widetilde{\mathfrak{B}}^{1\nabla} & \widetilde{\mathfrak{B}}^{11} & \widetilde{\mathfrak{B}}^{12} & & & & \\ & \widetilde{\mathfrak{B}}^{21} & \widetilde{\mathfrak{B}}^{22} & & \widetilde{\mathfrak{B}}^{23} & & \\ & & \ddots & \ddots & & \ddots & \\ & & \widetilde{\mathfrak{B}}^{K-1,K-2} & \widetilde{\mathfrak{B}}^{K-1,K-1} & \widetilde{\mathfrak{B}}^{K-1,K} & & \\ & & & \widetilde{\mathfrak{B}}^{K,K-1} & \widetilde{\mathfrak{B}}^{K,K} & \widetilde{\mathfrak{B}}^{K,\Delta} & \\ & & & & \widetilde{\mathfrak{B}}^{\Delta,K} & T_{\Delta\Delta} \end{bmatrix},$$

where $\widetilde{\mathfrak{B}}^{\nabla 1} := T_{\nabla +} \otimes \boldsymbol{\phi}^1(0)$, $\widetilde{\mathfrak{B}}^{1\nabla} := -\mathrm{diag}(c_i \mathbb{1}_{(c_i < 0)})_{i \in \mathcal{S}} \otimes \boldsymbol{\phi}^1(0)^{\mathrm{T}}$, $\widetilde{\mathfrak{B}}^{\Delta K} := T_{\Delta -} \otimes \boldsymbol{\phi}^K(\mathcal{I})$ and $\widetilde{\mathfrak{B}}^{K,\Delta} := \mathrm{diag}(c_i \mathbb{1}_{(c_i > 0)})_{i \in \mathcal{S}} \otimes \boldsymbol{\phi}^K(\mathcal{I})^{\mathrm{T}}$, and $\otimes$ is the Kronecker product. Where we have used the same sub-block notation as we have for $\widetilde{\mathfrak{B}}$.

After the addition of the boundary conditions, the system of ODEs (20) can now be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \boldsymbol{q}_\nabla(t) & \boldsymbol{a}(t) & \boldsymbol{q}_\Delta(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_\nabla(t) & \boldsymbol{a}(t) & \boldsymbol{q}_\Delta(t) \end{bmatrix} \widehat{\mathfrak{B}} \widehat{M}^{-1}, \tag{23}$$

where $\widehat{M} = \begin{bmatrix} I_{|\mathcal{S}_\nabla|} & & & & \\ & I_{N_S} \otimes M_1 & & & \\ & & \ddots & & \\ & & & I_{N_S} \otimes M_K & \\ & & & & I_{|\mathcal{S}_\Delta|} \end{bmatrix}$. Define $B = \widehat{\mathfrak{B}} \widehat{M}^{-1}$, with the sub-block

as we used for $\widetilde{\mathfrak{B}}$.

Regarding our notational convention, the matrices in fraktur fonts (e.g. $\mathfrak{B}$) are intermediary constructions that are not directly referred to for the rest of the paper (but do appear again in the appendix). We use regular mathematics fonts to represent

DG approximations to operators, i.e. B is a DG approximation to $\mathbb{B}$ and Ψ is an approximation to ψ.

We prove the following result in Appendix 1.

Corollary 3.1 *The approximate generator B conserves probability. That is, for all $t \geq 0$,*

$$\sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t) + \sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(t) + \sum_{i \in \mathcal{S}} \int_{x \in [0, \mathcal{I}]} u_i(x, t)\, \mathrm{d}x$$

$$= \sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(0) + \sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(0) + \sum_{i \in \mathcal{S}} \int_{x \in [0, \mathcal{I}]} u_i(x, 0)\, \mathrm{d}x.$$

3.5 Putting It All Together

Recall that the ultimate goal for our DG approximation is to approximate the operator $\mathbb{B}$. We have that $B^{k\ell}$ is an approximation to $\mathbb{B}^{k\ell}$, $k, \ell \in \{\nabla, 1, ..., K, \Delta\}$.

Given we have now truncated the space and added boundaries, let us define $\mathcal{M}_{0,\mathcal{I}}$ as the set of measures, μ_i, which admit an absolutely continuous density on $(0,\mathcal{I})$, may have a point mass at $x = 0$ if $i \in \mathcal{S}_\nabla$, and another at $x = \mathcal{I}$ if $i \in \mathcal{S}_\Delta$. The set $\mathcal{M}_{0,\mathcal{I}}$ is the domain of the operator $\mathbb{B}$ truncated to the interval $[0, \mathcal{I}]$ with regulated boundaries at $x = 0$ and $x = \mathcal{I}$. Also, redefine $\mathcal{K}_i^m = \{k \in \{\nabla, 1, ..., K, \Delta\} \mid l(\mathcal{D}_k \cap \mathcal{F}_i^m) = l(\mathcal{D}_k)\}$ for $i \in \mathcal{S}, m \in \{+, -, 0\}$ for all $l \in \mathcal{M}_{0,\mathcal{I}}$.

Approximations B_{ij}^{mn}, B_{ij}, and B^{mn} to $\mathbb{B}_{ij}^{mn}$, $\mathbb{B}_{ij}$, and $\mathbb{B}^{mn}$, $i, j \in \mathcal{S}$, $m, n \in \{+, -, 0\}$, are constructed from the block-matrices $B_{ij}^{k\ell}, i, j \in \mathcal{S}, k, \ell \in \{\nabla, 1, \ldots, K, \Delta\}$, as

$$B_{ij}^{mn} = \left[B_{ij}^{k\ell} \right]_{k \in \mathcal{K}_i^m, \ell \in \mathcal{K}_j^n}, \quad i, j \in \mathcal{S}, m, n \in \{+, -, 0\},$$

$$B_{ij} = \left[B_{ij}^{k\ell} \right]_{k, \ell \in \{\nabla, 1, \ldots, K, \Delta\}}, \quad i, j \in \mathcal{S},$$

$$B^{mn} = \left[\left[B_{ij}^{k\ell} \right]_{i \in \mathcal{S}_k^m, j \in \mathcal{S}_\ell^n} \right]_{k \in \mathcal{K}^m, \ell \in \mathcal{K}^n}, \quad m, n \in \{+, -, 0\}.$$

4 Application to an SFFM

Given our approximation B to the generator $\mathbb{B}$ we now follow the recipe from (Bean and O'Reilly 2014), replacing the actual generator $\mathbb{B}$ with its approximation B, to construct approximations, $\boldsymbol{\pi}$ and $\boldsymbol{p}$, to the stationary operators,$_\mathbb{\pi}$ and $\mathbb{p}$.

It may be convenient to think of our approximations in terms of approximations of kernels. Recall that the operators in (Bean and O'Reilly 2014) can be thought of in terms of kernels. That is, for some function $\boldsymbol{g} = (g_i(x))_{i \in \mathcal{S}}$, we can write

$$\boldsymbol{\mu} \mathbb{B} \boldsymbol{g}^\mathrm{T} = \sum_{k, \ell \in \{\nabla, 1, \ldots, K, \Delta\}} \sum_{i, j \in \mathcal{S}} \int_{x,y} \mathrm{d}\mu_i(x) \mathbb{B}_{ij}^{k\ell}(x, \mathrm{d}y) g_j(y),$$

where $\mathbb{B}_{ij}^{k\ell}(x, \mathrm{d}y)$ is the kernel of the operator $\mathbb{B}_{ij}^{k\ell}$.

Let $\boldsymbol{a}^\nabla(t) := \boldsymbol{q}_\nabla(t)$ and $\boldsymbol{a}^\Delta(t) := \boldsymbol{q}_\Delta(t)$, and define basis functions $\boldsymbol{\phi}^\nabla(x) = \phi_1^\nabla(x) = \delta(x)$ and $\boldsymbol{\phi}^\Delta(x) = \phi_1^\Delta(x) = \delta(x - \mathcal{I})$, where δ is the Dirac delta function, $N_\nabla = N_\Delta = 1$, and $\mathcal{N}_\nabla = \mathcal{N}_\Delta = \{1\}$. Also define $\widehat{M}_\nabla = I_{|\mathcal{S}_\nabla|}$ and $\widehat{M}_\Delta = I_{|\mathcal{S}_\Delta|}$ and row-vectors

$$\boldsymbol{\phi}(x) = (\boldsymbol{\phi}^k(x))_{k \in \{\nabla, 1, \ldots, K, \Delta\}}, \quad \boldsymbol{a}_i(t) = (a_i^k(t))_{k \in \{\nabla, 1, \ldots, K, \Delta\}}, i \in \mathcal{S}.$$

To pose the approximation $\mathbb{B}$ in kernel form let $\boldsymbol{a}_i \boldsymbol{\phi}(x)^\mathrm{T} \in W, i \in \mathcal{S}$ be the initial density of the process, and $\boldsymbol{\phi}(x) \boldsymbol{b}_i^\mathrm{T} \in W, i \in \mathcal{S}$ be a test function. Observe that, from our DG construction earlier and the definition of $\widehat{M}$,

$$\sum_{i,j \in \mathcal{S}} \int_{x,y \in [0,\mathcal{I}]} \boldsymbol{a}_i \boldsymbol{\phi}(x)^\mathrm{T} \boldsymbol{\phi}(x) \, \mathrm{d}x \widehat{M}^{-1} B_{ij} \boldsymbol{\phi}(y)^\mathrm{T} \boldsymbol{\phi}(y) \boldsymbol{b}_j \, \mathrm{d}y = \sum_{i,j \in \mathcal{S}} \boldsymbol{a}_i B_{ij} \widehat{M} \boldsymbol{b}_j.$$

Thus, we can think of

$$\boldsymbol{\phi}(x) \widehat{M}^{-1} B_{ij} \boldsymbol{\phi}(y)^\mathrm{T} \, \mathrm{d}y,$$

as an approximation to the kernel $\mathbb{B}_{ij}(x, \mathrm{d}y)$. This concept can be extended to all the approximations of operators considered in this work.

4.1 Approximating the Operator $\mathbb{R}$

Recall the operator $\mathbb{R}^k$ from Lemma 2.1. Essentially, the operator $\mathbb{R}^k$ takes an initial measure μ^k and multiplies each element by $1/|r_i(x)|$ on cells $\mathcal{D}_k$ where $r_i(x) \neq 0$. In the context of DG the initial distribution is given by $a_i\phi(x)^{\mathrm{T}} \in W$, $i \in \mathcal{S}$. Thus, for $k \in \{\nabla, 1, ..., K, \Delta\}$ such that $r_i(x) \neq 0$ on $\mathcal{D}_k$, we have

$$a_i^k \phi^k(x)^{\mathrm{T}} \mathbb{R}_i^k = \frac{a_i^k \phi^k(x)^{\mathrm{T}}}{|r_i(x)|}.$$

Decompose the right-hand side into a component which lies in W and another orthogonal to W:

$$\frac{a_i^k \phi^k(x)^{\mathrm{T}}}{|r_i(x)|} = \rho_i^k \phi^k(x)^{\mathrm{T}} + g_i^{\perp}(x),$$

where $\rho_i^k \phi^k(x)^{\mathrm{T}} \in W$, $g_i^{\perp} \in W^{\perp}$. Now, multiply by test functions $\{\phi_r^k(x)\}_{r=1}^{N_k}$ and integrate over $[0, \mathcal{I}]$:

$$a_i^k \int_{x \in [0,\mathcal{I}]} \frac{\phi^k(x)^{\mathrm{T}} \phi^k(x)}{|r_i(x)|}\, \mathrm{d}x = \rho_i^k \int_{x \in [0,\mathcal{I}]} \phi^k(x)^{\mathrm{T}} \phi^k(x)\, \mathrm{d}x + \int_{x \in [0,\mathcal{I}]} g_i^{\perp}(x)\phi^k(x)\, \mathrm{d}x$$

$$= \rho_i^k \int_{x \in [0,\mathcal{I}]} \phi^k(x)^{\mathrm{T}} \phi^k(x)\, \mathrm{d}x = \rho_i^k M_k,$$

since $g_i(x)^{\perp} \in W^{\perp}$. Define the matrix $M_k^r := \int_{x \in [0,\mathcal{I}]} \frac{\phi^k(x)^{\mathrm{T}} \phi^k(x)}{|r_i(x)|}\, \mathrm{d}x$, then $a_i^k M_k^r = \rho_i^k M_k$, which implies $\rho_i^k = a_i^k M_k^r M_k^{-1}$. Thus, we have the approximation

$$a_i^k \phi^k(x)^{\mathrm{T}} \mathbb{R}_i^k = \frac{a_i^k \phi^k(x)^{\mathrm{T}}}{|r_i(x)|} \approx a_i^k M_k^r M_k^{-1} \phi^k(x)^{\mathrm{T}}.$$

Since a_i^k is arbitrary, we see that we approximate $\mathbb{R}_i^k$ by $R_i^k = M_k^r M_k^{-1}$, and $\mathbb{R}^k$ by $R^k = \mathrm{diag}(R_i^k)_{i \in \mathcal{S}_k}$.

In practice, we implement a Gauss-Lobatto quadrature approximation to compute the elements of M_k^r.

4.2 Approximating the Operator *D* and the DG Riccati Equation

Recalling Lemma 2.1 and replacing the operators $\mathbb{R}^k$ and $\mathbb{B}^{\ell m}$, by their approximations we have the following approximation to $\mathbb{D}^{mn}(s)$

$$D^{mn}(s) = \left[R^m \left(B^{mn} - sI + B^{m0} \left(B^{00} - sI \right)^{-1} B^{0n} \right) \right], \quad m, n \in \{+, -\}.$$

Let $\boldsymbol{\phi}^k(x)M_k^{-1}\Psi_{ij}^{k\ell}(s)\boldsymbol{\phi}^\ell(y)^{\mathrm{T}}\,\mathrm{d}y$, $i,j\in\mathcal{S}$, $k\in\mathcal{K}_i^+,\ell\in\mathcal{K}_j^-$ be a finite-dimensional approximation of the operator kernel $\psi_{ij}^{k\ell}(s)(x,\,\mathrm{d}y)$, where $\Psi_{ij}^{k\ell}(s)$ is a matrix of constants for a given s. Construct an approximation to $\psi(s)(x,\,\mathrm{d}y)$ by

$$\boldsymbol{\phi}^+(x)\widehat{M}_+^{-1}\Psi(s)\boldsymbol{\phi}^-(y)^{\mathrm{T}}\,\mathrm{d}y=\left[\left[\boldsymbol{\phi}^k(x)M_k^{-1}\Psi_{ij}^{k\ell}(s)\boldsymbol{\phi}^\ell(y)^{\mathrm{T}}\,\mathrm{d}y\right]_{i\in\mathcal{S}_k^+,j\in\mathcal{S}_\ell^-}\right]_{k\in\mathcal{K}^+,\ell\in\mathcal{K}^-},$$

where $\boldsymbol{\phi}^+(x)=(\boldsymbol{\phi}^k(x))_{i\in\mathcal{S}_k^+,k\in\mathcal{K}^+}$ and $\boldsymbol{\phi}^-(y)=(\boldsymbol{\phi}^k(y))_{i\in\mathcal{S}_k^-,k\in\mathcal{K}^-}$ are row-vectors, $\Psi(s)$ is a matrix of constants for a given s with the same size as D^{+-}, and $\widehat{M}_m$, $m\in\{+,-,0\}$ is a block diagonal matrix $\widehat{M}_m=\mathrm{diag}\left((M_k)_{i\in\mathcal{S}_k^m}\right)_{k\in\mathcal{K}^m}$, $m\in\{+,-,0\}$. Now replace the theoretical kernels in Theorem 2.1 by their DG approximations to get

$$\boldsymbol{\phi}^+(x)\widehat{M}_+^{-1}D^{+-}(s)\boldsymbol{\phi}^-(y)^{\mathrm{T}}\,\mathrm{d}y$$
$$+\int_{z_1,z_2}\boldsymbol{\phi}^+(x)\widehat{M}_+^{-1}\Psi(s)\boldsymbol{\phi}^-(z_1)^{\mathrm{T}}\,\boldsymbol{\phi}^-(z_1)\widehat{M}_-^{-1}D^{-+}(s)\boldsymbol{\phi}^+(z_2)\boldsymbol{\phi}^+(z_2)\widehat{M}_+^{-1}\Psi(s)\boldsymbol{\phi}^-(y)^{\mathrm{T}}\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}y$$
$$+\int_{z_1}\boldsymbol{\phi}^+(x)\widehat{M}_+^{-1}D^{++}(s)\boldsymbol{\phi}^+(z_1)^{\mathrm{T}}\,\boldsymbol{\phi}^+(z_1)\widehat{M}_+^{-1}\Psi(s)\boldsymbol{\phi}^-(y)^{\mathrm{T}}\,\mathrm{d}z_1\,\mathrm{d}y$$
$$+\int_{z_1}\boldsymbol{\phi}^+(x)\widehat{M}_+^{-1}\Psi(s)\boldsymbol{\phi}^-(z_1)^{\mathrm{T}}\,\boldsymbol{\phi}^-(z_1)\widehat{M}_-^{-1}D^{--}(s)\boldsymbol{\phi}^-(y)^{\mathrm{T}}\,\mathrm{d}z_1\,\mathrm{d}y=0.$$

Multiplying on the left by $\boldsymbol{\phi}^+(x)^{\mathrm{T}}$ and on the right by $\boldsymbol{\phi}^-(y)$, integrating over both x and y, then post-multiplying by $\widehat{M}_-^{-1}$ gives the following matrix Riccati equation

$$D^{+-}(s)+\Psi(s)D^{-+}(s)\Psi(s)+D^{++}(s)\Psi(s)+\Psi(s)D^{--}(s)=0. \tag{24}$$

Thus, we may find $\Psi(s)$ by solving Eq. (24), using one of the methods in (Bean et al. 2009). Here, we use the Newtons method.

Given the stochastic interpretation of $\psi(0)$ we know that $\nu\psi(0)([0,\infty))=1$ for every vector of measures ν such that $\nu([0,\infty)\mathbf{1}=1$, when an SFFM is recurrent. It appears that this result carries over to the matrix $\Psi(0)$ giving the property that $\int_{y\in[0,\mathcal{I}]}\Psi(0)\boldsymbol{\phi}^-(y)^{\mathrm{T}}\,\mathrm{d}y=\mathbf{1}$. However, we have only observed this numerically and have no proof of this property.

Table 1 Notation for the approximation of the stationary operators of an SFFM. The first column contains the operators which we are approximating, the second column contains indices for which the operators are defined, the third column defines the notation we use for the coefficients of the approximation, and the last column defines how the coefficients and basis functions are used to approximate the operators

Exact operator	Operator indices	Approximation notation	Approximations
ξ_i^k	$i\in\mathcal{S}_k^-,k\in\mathcal{K}^-$	$\boldsymbol{\xi}_i^k:=(\xi_{i,r}^k)_{r\in\mathcal{N}_k}$	$\xi_i^k(\mathrm{d}x)\approx\boldsymbol{\xi}_i^k\boldsymbol{\phi}^k(x)^{\mathrm{T}}\,\mathrm{d}x,$
p_i^k	$i\in\mathcal{S}_k^-\cup\mathcal{S}_k^0,$ $k\in\bigcup_{m\in\{-,0\}}\mathcal{K}_m$	$\boldsymbol{p}_i^k:=(p_{i,r}^k)_{r\in\mathcal{N}_k}$	$p_i^k(\mathrm{d}x)\approx\boldsymbol{p}_i^k\boldsymbol{\phi}^k(x)^{\mathrm{T}}\,\mathrm{d}x$
$\pi_i^k(y)$	$i\in\mathcal{S},$ $k\in\{\nabla,1,\ldots,K,\Delta\}$	$\boldsymbol{\pi}_i^k(y):=(\pi_{i,r}^k(y))_{r\in\mathcal{N}_k}$	$\pi_i^k(y)(\mathrm{d}x)\approx\boldsymbol{\pi}_i^k(y)\boldsymbol{\phi}^k(x)^{\mathrm{T}}\,\mathrm{d}x$

4.3 Putting It All Together: Constructing an Approximation to the Stationary Distribution

We find an approximation to the stationary distribution by replacing the theoretical operators in Theorem 2.2 with their approximations. Table 1 defines the notation we use for the DG approximations to stationary operators.

With the notation in Table 1 define row-vectors

$$
\begin{aligned}
\boldsymbol{\xi}^k &:= (\xi_i^k)_{i \in \mathcal{S}_k^-}, \quad k \in \mathcal{K}_i^-, \\
\boldsymbol{\xi} &:= (\boldsymbol{\xi}^k)_{k \in \mathcal{K}^-}, \\
\boldsymbol{p}^{k,m} &:= (p_i^k)_{i \in \mathcal{S}_k^m}, \quad k \in \mathcal{K}^m, m \in \{-, 0\}, \\
\boldsymbol{p}^m &:= (\boldsymbol{p}^{k,m})_{k \in \mathcal{K}^m}, \quad m \in \{-, 0\}, \\
\boldsymbol{p} &:= (\boldsymbol{p}^m)_{m \in \{-, 0\}}, \\
\boldsymbol{\pi}^{k,m}(y) &:= (\pi_i^k(y))_{i \in \mathcal{S}_k^m}, \quad k \in \{\nabla, 1, \ldots, K, \Delta\}, m \in \{+, -, 0\}, \\
\boldsymbol{\pi}^m(y) &:= (\boldsymbol{\pi}^{k,m}(y))_{k \in \mathcal{K}^m}, \quad m \in \{+, -, 0\}, \\
\boldsymbol{\pi}(y) &:= (\boldsymbol{\pi}^m(y))_{m \in \{+, -, 0\}}.
\end{aligned}
$$

Proceeding similarly to the derivation of the Riccati Eq. (24), we can argue that the coefficients $\boldsymbol{\xi}$ are the solution to the matrix system

$$
\begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} \left(- \begin{bmatrix} B^{--} & B^{-0} \\ B^{0-} & B^{00} \end{bmatrix} \right)^{-1} \begin{bmatrix} B^{-+} \\ B^{0+} \end{bmatrix} \Psi(0) = \boldsymbol{\xi},
$$

$$
\int_{x \in [0, \mathcal{I}]} \boldsymbol{\xi} \begin{bmatrix} \boldsymbol{\phi}^-(x)^{\mathrm{T}} \\ \boldsymbol{\phi}^0(x)^{\mathrm{T}} \end{bmatrix} \mathrm{d}x \mathbf{1} = 1.
$$

Essentially, we replace the theoretical operators in Eqs. (9) and (10) with their DG counterparts.

Similarly, the coefficients $\boldsymbol{p}$ are given by

$$
\begin{bmatrix} \boldsymbol{p}^- & \boldsymbol{p}^0 \end{bmatrix} = z \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} \left(- \begin{bmatrix} B^{--} & B^{-0} \\ B^{0-} & B^{00} \end{bmatrix} \right)^{-1}, \tag{25}
$$

where z is a normalising constant. The coefficients $\boldsymbol{\pi}(y)$ are given by

$$
\boldsymbol{\pi}^0(y) = \begin{bmatrix} \boldsymbol{\pi}^+(y) & \boldsymbol{\pi}^-(y) \end{bmatrix} \begin{bmatrix} B^{+0} \\ B^{-0} \end{bmatrix} (-B^{00})^{-1}, \tag{26}
$$

$$
\begin{bmatrix} \boldsymbol{\pi}^+(y) & \boldsymbol{\pi}^-(y) \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}^- & \boldsymbol{p}^0 \end{bmatrix} \begin{bmatrix} B^{-+} \\ B^{0+} \end{bmatrix} \begin{bmatrix} e^{Ky} & e^{Ky}\Psi(0) \end{bmatrix} \begin{bmatrix} R^+ & 0 \\ 0 & R^- \end{bmatrix}, \tag{27}
$$

$$
\sum_{i \in \mathcal{S}} \sum_{k \in \{\nabla, 1, \ldots, K, \Delta\}} \int_{y=0}^{\infty} \int_{x \in [0, \mathcal{I}]} \pi_i^k(y) \boldsymbol{\phi}^k(x)^{\mathrm{T}} \, \mathrm{d}x \, \mathrm{d}y \tag{28}
$$

$$+ \sum_{i \in \mathcal{S}} \sum_{\ell \in \{-,0\}} \sum_{k \in \mathcal{K}_i^\ell} \int_{x \in [0,\mathcal{I}]} \boldsymbol{p}_i^k \boldsymbol{\phi}^k(x)^{\mathrm{T}} \, \mathrm{d}x = 1, \tag{29}$$

where $K := D^{++}(0) + \Psi(0)D^{(-+)}(0)$, and z is a normalising constant.

To assist the reader in understanding these constructions and the notation we provide an explicitly worked toy-example in Appendix 2.

5 Numerical Experiments

To illustrate the validity of our discontinuous Galerkin approximation, we perform numerical experiments on a stochastic fluid-fluid model, in a three-pronged approach.

First, we run Monte Carlo simulations, in order to compare the simulated joint density of $\{(X_t, \varphi_t)\}$ evaluated at the time $\{Y_t\}$ first returns to the initial level 0 against that which is obtained via the return-probability matrix $\Psi(0)$. This numerically verifies the accuracy of our proposed approximation for the operator matrix $\Psi(0)$. Second, we evaluate approximations to the limiting marginal joint density of $\{(X_t, \varphi_t)\}$ by first approximating the limiting distribution of $\{(\overline{X}_t, Y_t, \varphi_t)\}_{t \geq 0}$ and then integrating over Y_t. We compare this against the analytical density. Third, we vary the parameters of the second fluid $\{Y_t\}$ to confirm that the approximating joint density for $\{(\overline{X}_t, \varphi_t)\}$ does not change, while the marginal limiting distribution for $\{Y_t\}$ does, both of which are consistent with our intuitive understanding of the chosen example. In all three procedures, we find the approximations to be accurate.

We also analyse different choices for the level of spatial discretisation and the degree of polynomial basis functions, with respect to the order of convergence in relevant error terms.

5.1 An On-Off Bandwidth-Sharing Model

The following example is a modification of the example presented in (Latouche et al. 2013) where it was first analysed. The modification of the example is necessary since, in (Latouche et al. 2013), the model is level dependent; in particular, the rate of change for the process $\{X_t\}$ at time t depends on whether X_t is above, or below, some threshold, x^*.

The modified example we consider here is as follows. Consider a stochastic fluid-fluid $\{(X_t, Y_t, \varphi_t), t \geq 0\}$, where $\{X_t\}$ and $\{Y_t\}$ represent the workloads in Buffers 1 and 2 at time $t \geq 0$, respectively, both driven by the phase $\{\varphi_t\}$, which is a Markov chain on the state space $\mathcal{S} = \{11, 10, 01, 00\}$. Both $\{X_t\}$ and $\{Y_t\}$ have a regulated boundary at 0. Here, the state 11 indicates inputs to both buffers being ON, the state 00 indicates both being OFF, the state 10 is when only the first input is ON, and the state 01 is when only the second is ON. The input of Buffer k is switched from ON to OFF with rate γ_k, and from OFF to ON with rate β_k, for $k = 1, 2$. Thus, the infinitesimal generator T for φ_t is given by

$$T = \begin{bmatrix} -(\gamma_1 + \gamma_2) & \gamma_2 & \gamma_1 & 0 \\ \beta_2 & -(\gamma_1 + \beta_2) & 0 & \gamma_1 \\ \beta_1 & 0 & -(\gamma_2 + \beta_1) & \gamma_2 \\ 0 & \beta_1 & \beta_2 & -(\beta_1 + \beta_2) \end{bmatrix}.$$

The net rates of change for X_t, denoted c_i, are given by

$$(c_{11}, c_{10}, c_{01}, c_{00}) = (\lambda_1 - \theta_1, \ \lambda_1 - \theta_1, \ -\theta_1, \ -\theta_1),$$

and the net rates of change for Y_t, denoted r_i, are as follows

$$(r_{11}, r_{10}, r_{01}, r_{00}) = \begin{cases} (\lambda_2 - \kappa, & -\kappa, & \lambda_2 - \kappa, & -\kappa) & \text{if } X_t = 0, \\[2mm] (\lambda_2 - \theta_2, & -\theta_2, & \lambda_2 - \theta_2, & -\theta_2) & \text{if } X_t \in (0, x^*), \\[2mm] (\ \ \lambda_2, & 0, & \lambda_2, & 0) & \text{if } X_t \geq x^*. \end{cases}$$

For our numerical experiments, we use the parameter choices given in (Latouche et al. 2013):

$$\gamma_1 = 11, \quad \beta_1 = 1, \quad \lambda_1 = 12.48, \quad \theta_1 = 1.6, \quad \kappa = 2.6, \tag{30}$$

$$\gamma_2 = 22, \quad \beta_2 = 1, \quad \lambda_2 = 16.25, \quad \theta_2 = 1.0, \quad x^* = 1.6. \tag{31}$$

While the true problem has an unbounded domain $[0, \infty)$, the discontinuous Garlekin method requires the domain of approximation to be a finite interval. Here we choose an upper bound of $\mathcal{I} = 48$ and approximate the process on $[0, \mathcal{I}]$ with regulated boundaries at 0 and $\mathcal{I}$. The effect of this truncation can be partly quantified by evaluating $\lim_{t \to \infty} \mathbb{P}[X_t > \mathcal{I}, \varphi_t = i], i \in \mathcal{S}$. We observe that our approximations have an error of at least this amount.

5.2 Numerical Rxperiment for Ψ

Here we construct an approximation to Ψ using our DG methodology and compare the results to estimates obtained from Monte Carlo simulation which we treat as the *ground truth*. We choose the initial distribution of $\{(X_t, Y_t, \varphi_t)\}$ to be a point mass at $(X_0 = 5, Y_0 = 0, \varphi_0 = 01)$ and zero elsewhere. Given the initial distribution and the
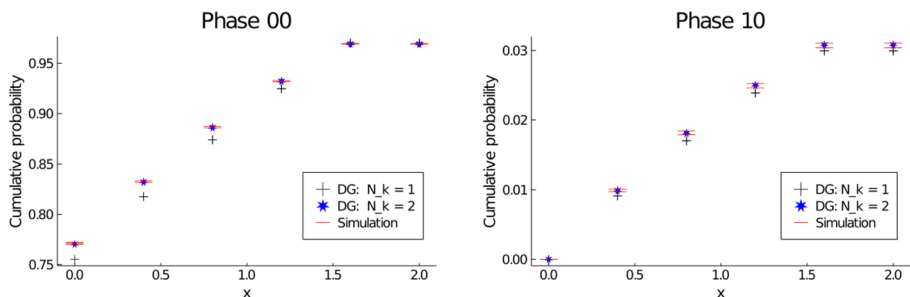
Fig. 1 Approximations to the cumulative probabilities $\mathbb{P}[X_{\theta(0)} \leq x_k, \varphi_{\theta(0)} = i]$, $i \in \{00, 10\}$ and $x_k = 0, 0.4, 0.8, ..., 2$, obtained by Monte Carlo (red bars), DG with one basis function on each cell (black crosses) and DG with two basis functions on each cell (blue stars). For the Monte Carlo estimate we plot the 95% confidence interval. The cumulative probabilities, $\mathbb{P}[X_{\theta(0)} \leq x_k, \varphi_{\theta(0)} = i]$ are constant for $i \in \{00, 10\}$ and $x_k > 1.6$, and are zero for $\varphi \in \{11, 01\}$, as it is impossible for $\theta(0)$ to occur whilst the process is in these phases; hence, we do not show them here

model parameters defined above, we simulate 10^6 sample paths until the first return time $\theta(0)$ and record $(X_{\theta(0)}, \varphi_{\theta(0)})$. We then use the samples to estimate the probabilities $\mathbb{P}[X_{\theta(0)} \leq x_k, \varphi_{\theta(0)} = i]$, $x_k = 0, 0.4, 0.8, ..., 48$, $i \in \mathcal{S}$ by finding the proportion of samples that lie in these sets. To account for Monte Carlo error we use the bootstrap (10^4 times) to estimate the 95% confidence interval for these estimates. In Fig. 1 the lower and upper red horizontal bars at each point represent these confidence intervals.

We then use the DG methodology with a constant cell width of $h_k = 0.4$, $k = 1, ..., 120$, and $N_k = 1$ or $N_k = 2$ basis functions on each cell to estimate the same probabilities. Results are plotted in Fig. 1.

As shown in Fig. 1, the piecewise-constant DG approximation with one basis function on each cell provides a reasonable approximation, while the piecewise-linear DG approximation with two basis functions on each cell gives an approximation which is almost indistinguishable from the Monte Carlo estimate.

5.3 The Marginal Stationary Distribution of *X*

Since Buffer 1, $\{X_t\}$, is conditionally independent of Buffer 2, $\{Y_t\}$, given $\{\varphi_t\}$, we can use results from the existing literature on stochastic fluid flows (da Silva Soares 2005) to obtain the marginal limiting density $\chi(x) = (\chi_i(x))_{i \in \mathcal{S}}$ of $\{X_t\}$:

$$\chi_i(x) := \frac{\partial}{\partial x} \lim_{t \to \infty} \mathbb{P}[X_t \leq x, \varphi_t = i], \quad i \in \{11, 10, 01, 00\}.$$

On the other hand, using the methodology of Sect. 3, we can approximate the joint stationary operators

$$\pi_i(y)(x) \, dx \approx \frac{\partial}{\partial y} \lim_{t \to \infty} \mathbb{P}[X_t \in \mathcal{A}, Y_t \leq y, \varphi_t = i],$$

$$p_i(x) \, dx \approx \lim_{t \to \infty} \mathbb{P}[X_t \in \mathcal{A}, Y_t = 0, \varphi_t = i],$$

for $i \in \{11, 10, 01, 00\}$. Then, we marginalise over y to approximate the marginal stationary distribution $\chi(x)$

$$\int_{y \in [0, \infty)} \pi_i(y)(x) \, dy + p_i(x) \approx \chi_i(x).$$

Let two vectors χ^1 and χ^2 denote respectively the piecewise-constant and piecewise-linear DG approximations of χ. We use a mesh with a constant cell width $h_k = 0.4$ for the DG approximation. Figure 2 presents the analytical densities $\chi_i(x)$, $i \in \{11, 10, 01, 00\}$, the piecewise-constant DG approximation, and the piecewise-linear DG approximation.

The piecewise-linear approximation performs well and is almost indistinguishable from the analytic density, while the piecewise-constant approximation does not perform as well. The piecewise-constant approximation underestimates the point masses and the density at lower values of x, and redistributes this mass in the tails.

5.4 Sensitivity to the Dynamics of $\{Y_t\}$

To further confirm that the discontinuous Galerkin approximation $\Psi(0)$ of the operator $\psi(0)$ accurately captures the dynamics of $\{Y_t\}$, we vary the rate at which the input to this buffer
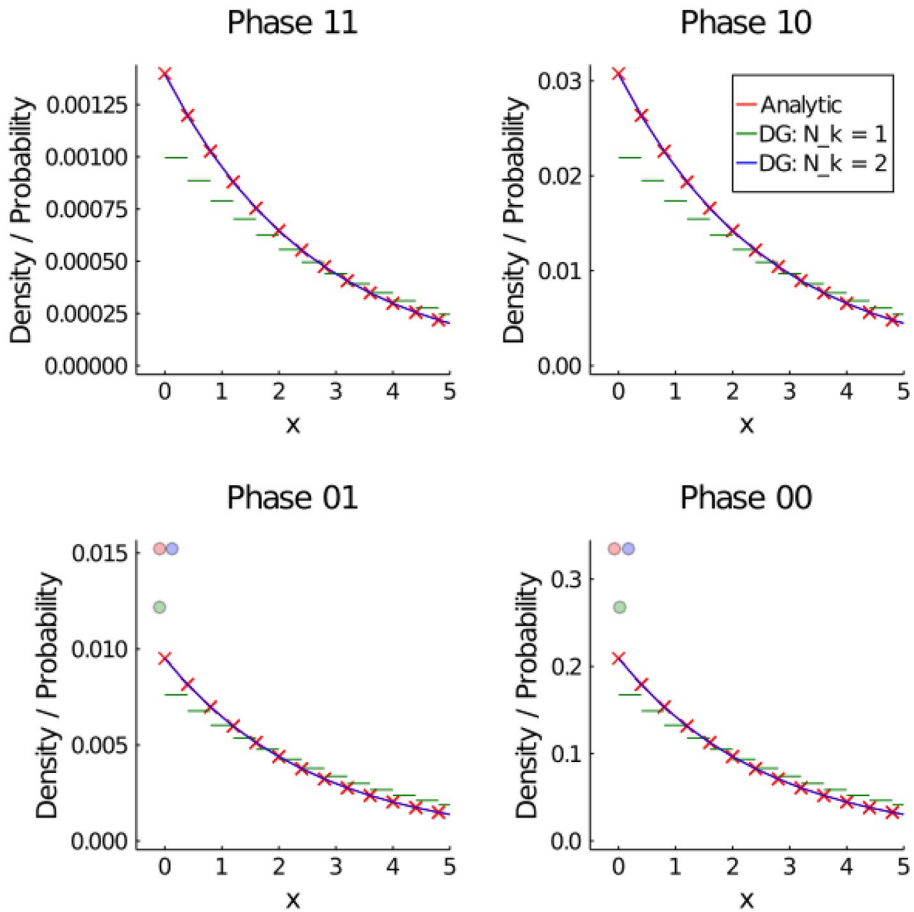
Fig. 2 A plot of the analytic density functions, $\chi_i(x)$, and the approximations to the density functions $\chi_i^1(x)$, $\chi_i^2(x)$, for $i \in \{11, 10, 01, 00\}$ and $x \in [0, 5]$. The analytic density, $\chi_i(x)$, is the solid red line with red crosses, $\chi_i^1(x)$ the solid green lines and $\chi_i^2(x)$ the solid blue line. The analytic density and DG approximation with linear basis functions are indistinguishable. The value of the point masses at $X = 0$ are represented by the height of the circles of the same colour as the corresponding density and have been *jittered* so that they do not lie on top of each other

switches OFF (denoted by γ_2). As we modify this rate, we expect to see a change in the distribution of probability between

$$p^0 := \lim_{t \to \infty} \mathbb{P}[Y_t = 0],$$

$$p^+ := \mathbb{P}[Y_t > 0].$$

Table 2 The probabilities p^0 and p^+ while varying γ_2

	$\gamma_2 = 11, \ \beta_2 = 1$	$\gamma_2 = 16, \ \beta_2 = 1$	$\gamma_2 = 22, \ \beta_2 = 1$
p^0	≈ 0.0	≈ 0.138	≈ 0.275
p^+	≈ 1.0	≈ 0.862	≈ 0.725
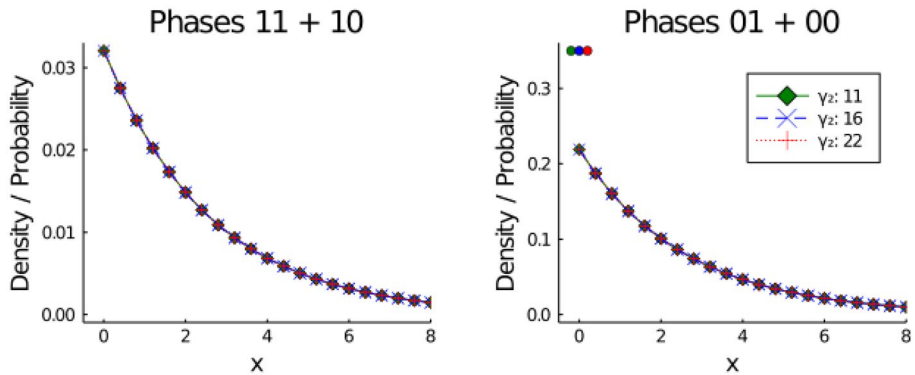
Fig. 3 Approximations to the density functions $\chi_{11}(x) + \chi_{10}(x)$ (left) and $\chi_{01} + \chi_{00}$ (right) on $x \in [0, 8]$ for $\gamma_2 = 11$ (green with diamonds), $\gamma_2 = 16$ (blue with crosses) and $\gamma_2 = 22$ (red with plusses). The coloured circles at $x = 0$ on the right-hand plot represent the point masses and are coloured according to $\gamma_2 = 11$ (green), $\gamma_2 = 16$ (blue) and $\gamma_2 = 22$ (red)

The net input rates for $\{X_t\}$, $c_{11} = c_{10}$ and $c_{01} = c_{00}$, and the proportion of time that the phase process spends in the sets $\phi_t \in \{11, 10\}$ and $\phi_t \in \{01, 00\}$ remains unchanged as γ_2 changes. Also, as γ_2 increases, the phase process spends proportionally more time in states $\{10, 00\}$ than in states $\{11, 01\}$, and thus more time in phases with $r_i(x) \leq 0$. Hence, as γ_2 increases, we expect that Y_t will spend more time at $Y_t = 0$, on average, yet the sum of the densities $\chi_{11}(x) + \chi_{10}(x)$ and $\chi_{01}(x) + \chi_{00}(x)$ should remain unchanged.

Regarding our approximations, we keep the mesh ($h_k = 0.4$) and basis functions ($N_k = 2$) fixed, and vary γ_2 and record the probabilities p^0 and p^+, shown in Table 2. We also plot the densities $\chi_{11}^2(x) + \chi_{10}^2(x)$ and $\chi_{01}^2(x) + \chi_{00}^2(x)$ in Fig. 3. As γ_2 increases, the amount of probability in p^0 increases, while the densities $\chi_{11}^2(x) + \chi_{10}^2(x)$ and $\chi_{01}^2(x) + \chi_{00}^2(x)$ remain unchanged, as expected.

5.5 Numerical Errors

It has been shown that operators such as $\mathbb{B}$ (defined in Sect. 2.1) under a DG approximation can have an error which converges at the order of $\mathcal{O}(h_k^{N+2})$, where h_k is the discretisation and $N = N_k$ is the number of basis function on each cell (Hesthaven and Warburton 2007). However, this result cannot be easily translated across to the operator Ψ. The DG approximation Ψ is constructed by taking the DG approximation, B, of the operators $\mathbb{B}$, constructing an approximation, D, to $\mathbb{D}$ and then solving the Riccati Eq. (24) with D. We then use the approximation, Ψ, of ψ to derive an approximation for the limiting density π. With such a construction, it is not known how the error propagates through the process of solving the Riccati equation, and then through further calculations to determine π. Determining bounds for the approximation errors of Ψ and π, as functions of the discretisation and basis selection, is a topic for future research.

As a preliminary step in this direction, we empirically investigate how the approximation error of the marginal limiting density of the fluid $\{X_t\}$ (see Sect. 5.3) changes with respect to the choice of basis functions and the level of discretisation. Let $[0, \mathcal{I}]$ be the interval on which we approximate the solution, then both the approximations and the analytical
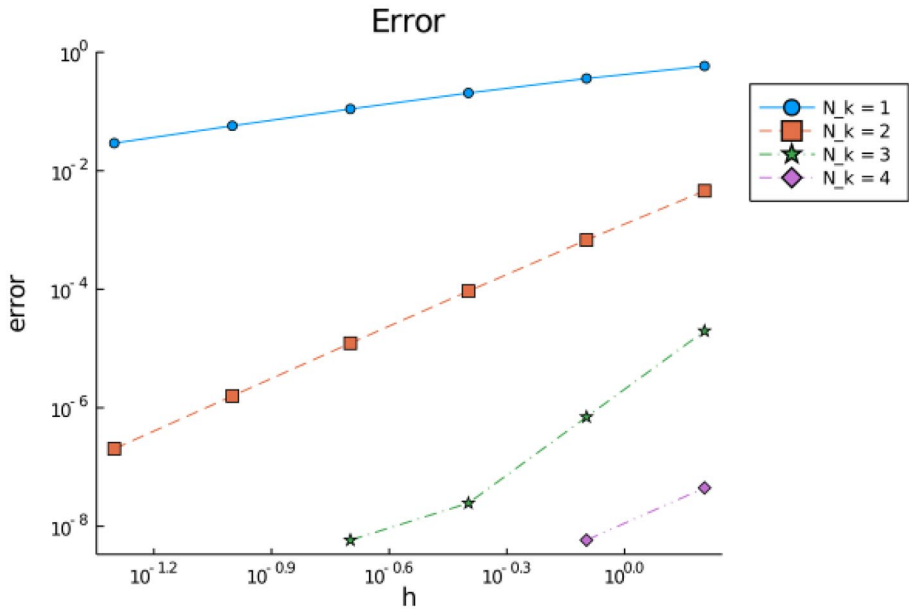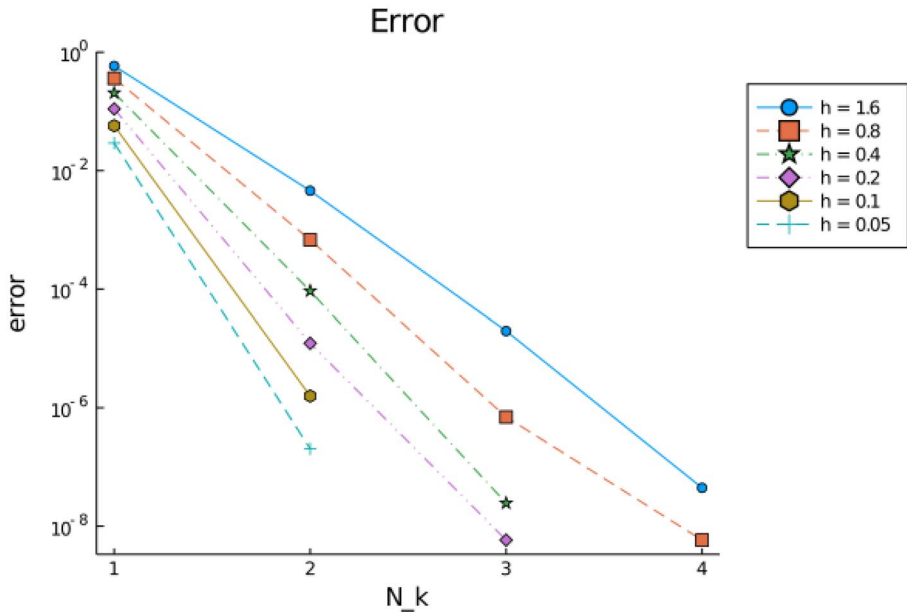
Fig. 4 A log-log plot of errors of approximation for the marginal stationary distribution of $\{X_i, \varphi_t\}$ for different mesh sizes, h, and number of basis functions, N_k. The error due to truncation at $\mathcal{I} = 48$ is $\approx 5.83 \times 10^{-9}$ contributes the majority of the error in the approximations with $N_k = 3$ bases and a cell width of $h = 0.2$ and $N_k = 4$ bases and a cell width of $h = 0.8$

solution belong to the space $\mathcal{S} \times \mathcal{C}^{-1}([0, \mathcal{I}])$, where $\mathcal{C}^{-1}([0, \mathcal{I}])$ is the set of functions with countably many discontinuities. We compute the error of the approximation as

$$\sum_{i \in \mathcal{S}_-} \left| q_{\nabla, i} - \lim_{t \to \infty} \mathbb{P}[X_t = 0, \varphi_t = i] \right| + \sum_{i \in \mathcal{S}_+} |q_{\Delta, i}|$$

$$+ \sum_{i \in \mathcal{S}} \sum_{k=1}^{K} \left| \int_{\mathcal{D}_k} u_i^k(x) \, dx - \lim_{t \to \infty} \mathbb{P}[X_t \in \mathcal{D}_k, \varphi_t = i] \right|,$$

where $u_i^k(x)$, $q_{\nabla, i}$ and $q_{\Delta, i}$, represent the DG approximation to the marginal stationary density of $\{(X_t, \varphi_t)\}$, and the marginal stationary point masses at 0 and $\mathcal{I}$, respectively.

A log-log plot of the errors of the approximations versus cell width, $h = h_k$, are shown in Fig. 4 for piecewise-constant ($N_k = 1$), piecewise-linear ($N_k = 2$), piecewise-quadratic ($N_k = 3$) and piecewise-cubic ($N_k = 4$) polynomial bases. As log-mesh size decreases the log-error of the approximation decreases linearly. For $N_k = 3$ and $h = 0.2$ and for $N_k = 4$ and $h = 0.8$, a significant amount of the errors of the approximations are from other sources of error such as truncation. Recall that we truncated the problem to the interval $[0, \mathcal{I}]$ with $\mathcal{I} = 48$, from which we compute $\lim_{t \to \infty} \mathbb{P}[X_t > 48] \approx 5.83 \times 10^{-9}$ as the error due to truncation. Ignoring the approximations where the truncation error is significant, we use least-squares to estimate the slopes of the lines for $N_k = 1$, $N_k = 2$ and $N_k = 3$ to be 0.871, 2.90, and 4.82, respectively.

Fig. 5 Log-errors of approximation for the marginal stationary distribution of $\{X_i, \varphi_t\}$ for different mesh sizes, h, and number of basis functions, N_k. The error due to truncation at $\mathcal{I} = 48$ is $\approx 5.83 \times 10^{-9}$ and contributes the majority of the error in the approximations with $N_k = 3$ bases and a cell width of $h = 0.2$ and $N_k = 4$ bases and a cell width of $h = 0.8$

In Fig. 5 we plot log-error against N_k, the number of basis functions in each cell. This shows that log-error decreases linearly as the number of basis functions on each cell, N_k, increases. Using least-squares we estimate the slopes of the lines for $h = 1.6, 0.8, 0.4, 0.2, 0.1,$ and 0.05 to be $-5.46, -6.07, -7.97, -8.38, -10.5$ and -11.9, respectively.

To help understand the trade-offs between error, computation time, memory and the overall *size* of the computational problem, we report, in Table 3, computation statistics for the data in Figs. 4-5. The *size* of the approximation scheme is quantified by the total number of basis functions required, denoted n_ϕ. Computation time and memory requirements are approximately the same for a given problem size, n_ϕ; however, the error of the approximation is greatly reduced if a larger number of basis functions is used as opposed to a smaller cell width. For example, using a piecewise-constant approximation, $N_k = 1$, with cell width $h = 0.4$, then $n_\phi = 480$ basis functions are required and the error of this approximation is ≈ 0.205. Compare this to the piecewise-cubic approximation, $N_k = 4$, with a cell width of $h = 1.6$ which also requires $n_\phi = 480$ basis function, but achieves an error of $\approx 4.44 \times 10^{-8}$ – an improvement of over 7 orders of magnitude for approximately the same computation effort as measured by run-time or memory.

The approximation scheme of Bean and O'Reilly (2013) is equivalent to the DG scheme introduce here when $N_k = 1$. The width of the cell, h, corresponds to the parameter 'Δx' in Bean and O'Reilly (2013). Further, the finite-volume method with an *upwind flux* also results in an equivalent approximation. Thus, examining the first column of Table 3 we can observe the numerical performance of these related methods.

Table 3 Computation statistics for DG approximations to the marginal stationary density of $\{X_t, \varphi_t\}$. n_ϕ is the total number of basis functions required for the approximation. Memory is the total memory (in megabytes) allocated during the construction of the operators B, R, D, Ψ, and the approximation to the stationary density, π. The code was implemented in Julia using scientific libraries and was performed on a 2.3 Ghz Dual-Core Intel Core i5 processor with 16GB of RAM running macOS Catalina version 10.15.6

h		Number of basis functions, N_k			
		1	2	3	4
1.6	error	5.85×10^{-1}	4.58×10^{-3}	1.97×10^{-5}	4.44×10^{-8}
	time (sec)	0.0276	0.0871	0.205	0.302
	memory (MB)	5.81	14.0	28.4	48.8
	n_ϕ	120	240	360	480
0.8	error	3.61×10^{-1}	6.79×10^{-4}	6.98×10^{-7}	5.83×10^{-9}
	time (sec)	0.113	0.315	0.646	1.3
	memory (MB)	15.6	47.9	103.0	181.0
	n_ϕ	240	480	720	960
0.4	error	2.05×10^{-1}	9.30×10^{-5}	2.46×10^{-8}	-
	time (sec)	0.352	1.22	2.85	
	memory (MB)	50.4	179.0	394.0	
	n_ϕ	480	960	1440	
0.2	error	1.10×10^{-1}	1.23×10^{-5}	5.83×10^{-9}	-
	time (sec)	1.38	5.62	13.9	
	memory (MB)	188.0	695.0	1540.0	
	n_ϕ	960	1920	2880	
0.1	error	5.73×10^{-2}	1.58×10^{-6}	-	-
	time (sec)	7.69	31.4		
	memory (MB)	727.0	2740.0		
	n_ϕ	1920	3840		
0.05	error	2.92×10^{-2}	2.03×10^{-7}	-	-
	time (sec)	43.6	230.0		
	memory (MB)	2860.0	10900.0		
	n_ϕ	3840	7680		

Comparing the first column of Table 3 to the first row of Table 3 suggests that the DG scheme can converge much faster than these other schemes.

Unfortunately, larger scale analysis of the errors of approximation for Ψ is computationally prohibitive since the *ground truth* must be evaluated via Monte Carlo simulation.

6 Conclusions

We proposed the application of the discontinuous Galerkin method to approximate operators associated with stochastic fluid-queues and stochastic fluid-fluid models. Analysis of SFFMs using these operators requires us to partition the operators into regions corresponding to when the second fluid level is increasing, decreasing or constant. The DG method is a natural candidate to approximate these operators due to the cell-based discretisation. By correctly utilising the discretisation of the state space, we used the DG method to

approximate the infinitesimal generator of a stochastic fluid model $\{(X_t, \varphi_t)\}$, ensuring that the necessary partition can be recovered. Furthermore, DG methods are also appealing as they can maintain mass conservation, and high-order schemes can be constructed while maintaining the necessary spatial-discretisation.

Here, we have detailed how one constructs approximations to various operators which arise in the analysis of stochastic fluid-fluid models. We demonstrated this with an application to approximate all the operators needed to construct the joint stationary distribution of a stochastic fluid-fluid models. The DG method also enabled us to obtain other performance measures of stochastic fluid-fluid models that are also analytically presented by operators, such as Ψ.

The numerical results showed that DG approximations of the operator Ψ, as well as the stationary distribution of a SFFM, are accurate and effective. We also verified that the operators and their dynamics were captured accurately. Furthermore, in our example, we observed that adding more regularity in the basis functions was more effective in reducing the error of the approximation than decreasing cell width, for approximately the same computational effort. The numerical illustration also demonstrates that the DG scheme can converge rapidly compared to other methods.

Future work includes determining theoretical error bounds for approximations of the operator Ψ, as well as of the stationary distribution. Another interesting topic for future research is to determine conditions under which the approximation to the stationary distribution is a positive function. It may also be possible to extend the methods presented here to processes with a level-dependent first fluid, such as that in (Latouche et al. 2013).

Properties of *B*

Recall that the coefficients $\boldsymbol{a}_i^k(t)$ can be used to construct an approximate solution to a differential equation at time t as $u_i^k(x, t) = \boldsymbol{a}_i^k(t)\boldsymbol{\phi}^k(x)^{\mathrm{T}}$. For $i \in \mathcal{S}$, $k \in \{\nabla, 1, ..., K, \Delta\}$, $r \in \mathcal{N}_k$, define $\alpha_{i,r}^k(t) := a_{i,r}^k(t) \int_{x \in \mathcal{D}_k} \phi_r^k(x)\,\mathrm{d}x$, and row-vectors $\boldsymbol{\alpha}_i^k(t) = (\alpha_{i,r}^k(t))_{r \in \mathcal{N}_k}$. Motivated by the fact that we may be interested in approximations of the probabilities $\mathbb{P}(X(t) \in \mathcal{D}_k, \varphi(t) = i)$ rather than the function u_i^k itself, we can pose the problem equivalently in terms of the integrals

$$\mathbb{P}\left[X(t) \in \mathcal{D}_k, \varphi(t) = i\right] \approx \boldsymbol{a}_i^k(t) \int_{x \in \mathcal{D}_k} \boldsymbol{\phi}^k(x)^{\mathrm{T}}\,\mathrm{d}x = \boldsymbol{\alpha}_i^k(t)\mathbf{1}.$$

Define

$$\boldsymbol{\alpha}^k(t) = (\boldsymbol{\alpha}_i^k(t))_{i \in \mathcal{S}}, \text{ and } \boldsymbol{\alpha}(t) = (\boldsymbol{\alpha}^k(t)))_{k \in \{\nabla, 1, ..., K, \Delta\}},$$

and matrices

$$P_k = \mathrm{diag}\left(\int_{x \in \mathcal{D}_k} \phi_r^k(x)\,\mathrm{d}x\right)_{r \in \mathcal{N}_k}, \ k \in \{1, ..., K\},$$

$$P = \begin{bmatrix} I_{N_{|\mathcal{S}|}} \otimes P_1 & & \\ & \ddots & \\ & & I_{N_{|\mathcal{S}|}} \otimes P_K \end{bmatrix}.$$

By choosing the basis $\{\phi_r^k\}_{r\in\mathcal{N}_k,k\in\{1,\dots,K\}}$ such that $\int_{x\in\mathcal{D}_k}\phi_r^k(x)\,\mathrm{d}x\neq 0$ for all r,k, then P is invertible. This is the case for the Lagrange polynomials, but not, for example, for the Legendre polynomials. We can (loosely) interpret the new coefficients $\alpha_{i,r}^k(t)$ as representing the amount of probability captured by the basis function $\phi_r^k(x)$ in phase i.

The differential Eq. (23) can be equivalently expressed as $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}(t)=\boldsymbol{\alpha}(t)\mathfrak{B}$, where

$$\mathfrak{B}=\begin{bmatrix} I_{|\mathcal{S}_\nabla|} & & \\ & P^{-1} & \\ & & I_{|\mathcal{S}_\Delta|} \end{bmatrix} B \begin{bmatrix} I_{|\mathcal{S}_\nabla|} & & \\ & P & \\ & & I_{|\mathcal{S}_\Delta|} \end{bmatrix}.$$

Let

$$\mathfrak{B}^{\nabla 1} := T_{\nabla+}\otimes\left(\boldsymbol{\phi}^1(0)M_1^{-1}P_1\right),$$

$$\mathfrak{B}^{1\nabla} := -\mathrm{diag}(c_i\mathbb{1}_{(c_i<0)})_{i\in\mathcal{S}}\otimes P_1^{-1}\boldsymbol{\phi}^1(0)^{\mathrm{T}},$$

$$\mathfrak{B}^{\Delta K} := T_{\Delta-}\otimes\left(\boldsymbol{\phi}^K(\mathcal{I})M_K^{-1}P_K\right)$$

$$\mathfrak{B}^{K,\Delta} := \mathrm{diag}(c_i\mathbb{1}_{(c_i>0)})_{i\in\mathcal{S}}\otimes P_K^{-1}\boldsymbol{\phi}^K(\mathcal{I})^{\mathrm{T}},\ \mathfrak{B}_{ij}^{kk}$$

$$\mathfrak{B}_{ij}^{\mathfrak{k},\mathfrak{k}} := \begin{cases} T_{ii}I_{N_k}+c_iP_k^{-1}(F_i^{kk}+G_k)M_k^{-1}P_k & i=j, \\ T_{ij}I_{N_k} & i\neq j, \end{cases}\quad\text{for } k=1,\dots K,$$

$$\mathfrak{B}_{ij}^{k,k+1} := \begin{cases} c_iP_k^{-1}F_i^{k,k+1}M_{k+1}^{-1}P_{k+1} & i=j, \\ 0_{N_k} & i\neq j, \end{cases}\quad\text{for } k=1,\dots K-1,$$

$$\mathfrak{B}_{ij}^{k-1,k} := \begin{cases} c_iP_k^{-1}F_i^{k,k-1}M_{k-1}^{-1}P_{k-1} & i=j, \\ 0_{N_k} & i\neq j, \end{cases}\quad\text{for } k=2,\dots K,$$

$$\mathfrak{B}^{kk} =: \begin{bmatrix} \mathfrak{B}_{11}^{kk} & \cdots & \mathfrak{B}_{1N_S}^{kk} \\ \vdots & \ddots & \vdots \\ \mathfrak{B}_{N_S1}^{kk} & \cdots & \mathfrak{B}_{N_S,N_S}^{kk} \end{bmatrix},\quad\text{for } k=1,\dots K,$$

$$\mathfrak{B}^{k,k+1} =: \begin{bmatrix} \mathfrak{B}_{1,1}^{k,k+1} & \cdots & \mathfrak{B}_{1,N_S}^{k,k+1} \\ \vdots & \ddots & \vdots \\ \mathfrak{B}_{N_S,1}^{k,k+1} & \cdots & \mathfrak{B}_{N_S,N_S}^{k,k+1} \end{bmatrix},\quad\text{for } k=1,\dots K-1,$$

$$\mathfrak{B}^{k,k-1} =: \begin{bmatrix} \mathfrak{B}_{1,1}^{k,k-1} & \cdots & \mathfrak{B}_{1,N_S}^{k,k-1} \\ \vdots & \ddots & \vdots \\ \mathfrak{B}_{N_S,1}^{k,k-1} & \cdots & \mathfrak{B}_{N_S,N_S}^{k,k-1} \end{bmatrix},\quad\text{for } k=2,\dots K.$$

Then

$$\mathfrak{B}=\begin{bmatrix} T_{\nabla\nabla} & \mathfrak{B}^{\nabla 1} & & & & & \\ \mathfrak{B}^{1\nabla} & \mathfrak{B}^{11} & \mathfrak{B}^{12} & & & & \\ & \mathfrak{B}^{21} & \mathfrak{B}^{22} & & \mathfrak{B}^{23} & & \\ & & \ddots & \ddots & & \ddots & \\ & & \mathfrak{B}^{K-1,K-2} & \mathfrak{B}^{K-1,K-1} & \mathfrak{B}^{K-1,K} & \\ & & & \mathfrak{B}^{K,K-1} & \mathfrak{B}^{K,K} & \mathfrak{B}^{K,\Delta} \\ & & & & \mathfrak{B}^{\Delta,K} & T_{\Delta\Delta} \end{bmatrix}.$$

Remark 7.1 One may recognise the structure of $\mathfrak{B}$ as the structure of a quasi-birth-and-death process (QBD), with levels $k = 1, ..., K$. This raises the question of whether $\mathfrak{B}$ is indeed a representation of the generator matrix of a QBD, or QBD-like process. In the case of a constant basis function on each cell, i.e. $N_k = 1$ and $\phi_1^k(x) \propto 1$, $k = 1, ..., K$, then $\mathfrak{B}$ is the generator of a QBD: it has zero row-sums, negative diagonal entries, and non-negative off-diagonal entries, the QBD-phase variable is $\{\varphi_t\}$ and the level is $k = \nabla, 1, ..., K, \Delta$. In fact, if h_k is the same for every $k = 1, ..., K$, then this is the same QBD discretisation of a stochastic fluid process analysed by Bean and O'Reilly (2013). However, for higher-degree polynomials $\mathfrak{B}$ is not necessarily the generator of a QBD process. We conjecture that, using polynomial basis functions, then $N_k = 1$ and $\phi_1^k(x) \propto 1$, $k = 1, ..., K$ is the only DG approximation which has an interpretation as a QBD-like process – not even as a QBD-RAP (Bean and Nielsen 2010).

In the following lemma, we use the following properties of the Lagrange interpolating polynomials defined by the Gauss-Lobatto quadrature nodes. **Property 1** $\sum_{s=1}^{N_k} \phi_s^k(x) = \begin{cases} 1 & x \in \mathcal{D}_k, \\ 0 & x \notin \mathcal{D}_k. \end{cases}$

For $k \in \{1, ..., K\}$, let e_n^k be a row-vector of length N_k with a 1 in the nth position and zeros elsewhere. **Property 2** At the cell edges, $\boldsymbol{\phi}^k(x_k) = e_1^k$ and $\boldsymbol{\phi}^k(x_{k+1}) = e_{N_k}^k$, $k = 1, ..., K$.

Lemma 7.1 *If* $\{\phi_r^k(x)\}_{r \in \mathcal{N}_k}$, *are chosen as the Lagrange interpolating polynomials on* $\mathcal{D}_k$, $k \in \{1, ..., K\}$, *then the matrix* $\mathfrak{B}$ *has zero row-sums.*

Proof Let **1** and **0** be column vectors of ones and zeros, respectively, with an appropriate length depending on the context. Using Property 1, observe that

$$M_k \mathbf{1} = \left(\sum_{s=1}^{N_k} \int_{x \in \mathcal{D}_k} \phi_r^k(x) \phi_s^k(x) \, dx \right)_{r \in \mathcal{N}_k}^{\mathrm{T}}$$

$$= \left(\int_{x \in \mathcal{D}_k} \phi_r^k(x) \sum_{s=1}^{N_k} \phi_s^k(x) \, dx \right)_{r \in \mathcal{N}_k}^{\mathrm{T}}$$

$$= \left(\int_{x \in \mathcal{D}_k} \phi_r^k(x) \, dx \right)_{r \in \mathcal{N}_k}^{\mathrm{T}}$$

$$= P_k \mathbf{1},$$

hence $M_k^{-1} P_k \mathbf{1} = \mathbf{1}$. Also

$$G_k \mathbf{1} = \left(\sum_{s=1}^{N_k} \int_{x \in \mathcal{D}_k} \phi_r^k(x) \frac{\mathrm{d}}{\mathrm{d}x} \phi_s^k(x) \, dx \right)_{r \in \mathcal{N}_k}^{\mathrm{T}}$$

$$= \left(\int_{x \in \mathcal{D}_k} \phi_r^k(x) \frac{\mathrm{d}}{\mathrm{d}x} \sum_{s=1}^{N_k} \phi_s^k(x) \, dx \right)_{r \in \mathcal{N}_k}^{\mathrm{T}}$$

$$= \left(\int_{x \in \mathcal{D}_k} \phi_r^k(x) \frac{\mathrm{d}}{\mathrm{d}x} 1 \, dx \right)_{r \in \mathcal{N}_k}^{\mathrm{T}}$$

$$= \mathbf{0},$$

where we have again used Property 1.

Consider first $c_i > 0$. Let $\boldsymbol{b}$ and $\boldsymbol{d}$ be arbitrary row-vectors of length N_k and N_{k+1}, respectively. By Property 2, for $k = 1, ..., K - 1$,

$$
\begin{aligned}
F_i^{kk}\boldsymbol{b} &= -\boldsymbol{\phi}^k(x_{k+1})^{\mathrm{T}}\, \boldsymbol{\phi}^k(x_{k+1})\boldsymbol{b} \\
&= -(\boldsymbol{e}_{N_k}^k)^{\mathrm{T}}\, \boldsymbol{e}_{N_k}^k \boldsymbol{b} \\
&= -b_{N_k}(\boldsymbol{e}_{N_k}^k)^{\mathrm{T}}, \\
F_i^{k,k+1}\boldsymbol{d} &= \boldsymbol{\phi}^k(x_{k+1})^{\mathrm{T}}\, \boldsymbol{\phi}^{k+1}(x_{k+1})\boldsymbol{d} \\
&= (\boldsymbol{e}_{N_k}^k)^{\mathrm{T}}\, \boldsymbol{e}_1^{k+1}\boldsymbol{d} \\
&= d_1(\boldsymbol{e}_{N_k}^k)^{\mathrm{T}}.
\end{aligned}
$$

Therefore, for $c_i > 0$, we claim

$$
\sum_{j \in \mathcal{S}} T_{ij}I_{N_k}\boldsymbol{1} + c_i P_k^{-1}(F_i^{kk} + G_k)M_k^{-1}P_k\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k+1}M_k^{-1}P_k\boldsymbol{1} = \boldsymbol{0}.
$$

The first sum is zero since T is a generator of a continuous-time Markov chain. This leaves the other two terms, which, using our previous observations, we get

$$
\begin{aligned}
&c_i P_k^{-1}(F_i^{kk} + G_k)M_k^{-1}P_k\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k+1}M_{k+1}^{-1}P_{k+1}\boldsymbol{1} \\
&= c_i P_k^{-1}(F_i^{kk} + G_k)\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k+1}\boldsymbol{1} \\
&= c_i P_k^{-1}F_i^{kk}\boldsymbol{1} + c_i P_k^{-1}G_k\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k+1}\boldsymbol{1} \\
&= c_i P_k^{-1}(-\boldsymbol{e}_{N_k}^k)^{\mathrm{T}} + \boldsymbol{0} + c_i P_k^{-1}(\boldsymbol{e}_{N_k}^k)^{\mathrm{T}} \\
&= \boldsymbol{0}.
\end{aligned}
$$

Similarly, for $c_i < 0$, and row-vectors $\boldsymbol{b}$ and $\boldsymbol{d}$ of length N_k and N_{k-1}, respectively,

$$
\begin{aligned}
F_i^{kk}\boldsymbol{b} &= \boldsymbol{\phi}^k(x_k)^{\mathrm{T}}\, \boldsymbol{\phi}^k(x_k)\boldsymbol{b} \\
&= (\boldsymbol{e}_1^k)^{\mathrm{T}}\, \boldsymbol{e}_1^k \boldsymbol{b} \\
&= b_1(\boldsymbol{e}_1^k)^{\mathrm{T}} \\
F_i^{k,k-1}\boldsymbol{d} &= -\boldsymbol{\phi}^k(x_k)^{\mathrm{T}}\, \boldsymbol{\phi}^{k-1}(x_k)\boldsymbol{d} \\
&= -(\boldsymbol{e}_1^k)^{\mathrm{T}}\, \boldsymbol{e}_{N_{k-1}}^{k-1}\boldsymbol{d} \\
&= -d_{N_{k-1}}(\boldsymbol{e}_1^k)^{\mathrm{T}}.
\end{aligned}
$$

Therefore, for $c_i < 0$ and $k = 2, ..., K$, using the same arguments as before we have

$$
\begin{aligned}
&\sum_{j \in \mathcal{S}} T_{ij}I_{N_k}\boldsymbol{1} + c_i P_k^{-1}(F_i^{kk} + G_k)M_k^{-1}P_k\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k-1}M_k^{-1}P_k\boldsymbol{1} \\
&= \boldsymbol{0} + c_i P_k^{-1}(F_i^{kk} + G_k)M_k^{-1}P_k\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k+1}M_{k-1}^{-1}P_{k-1}\boldsymbol{1} \\
&= c_i P_k^{-1}(F_i^{kk} + G_k)\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k-1}\boldsymbol{1} \\
&= c_i P_k^{-1}F_i^{kk}\boldsymbol{1} + c_i P_k^{-1}G_k\boldsymbol{1} + c_i P_k^{-1}F_i^{k,k-1}\boldsymbol{1} \\
&= c_i P_k^{-1}(\boldsymbol{e}_1^k)^{\mathrm{T}} + \boldsymbol{0} + c_i P_k^{-1}(-\boldsymbol{e}_1^k)^{\mathrm{T}} \\
&= \boldsymbol{0}.
\end{aligned}
$$

For the lower boundary,

$$T_{\nabla\nabla}\mathbf{1} + \mathfrak{B}^{\nabla 1}\mathbf{1} = T_{\nabla\nabla}\mathbf{1} + \left[T_{\nabla+} \otimes \left(\boldsymbol{\phi}^1(0)M_1^{-1}P_1\right)\right]$$

1.Swapping the order of summation and recalling

$$M_k^{-1}P_k\mathbf{1} = \mathbf{1} \text{ then this is equal to } T_{\nabla\nabla}\mathbf{1} + \left[T_{\nabla+} \otimes \left(\boldsymbol{\phi}^1(0)M_1^{-1}P_1\right)\mathbf{1}\right]\mathbf{1}$$

$$= T_{\nabla\nabla}\mathbf{1} + \left[T_{\nabla+} \otimes e_1^1\mathbf{1}\right]\mathbf{1}$$

$$= T_{\nabla\nabla}\mathbf{1} + \left[T_{\nabla+} \otimes \mathbf{1}\right]\mathbf{1}$$

$$= T_{\nabla\nabla}\mathbf{1} + T_{\nabla+}\mathbf{1}$$

$$= \mathbf{0}.$$

Also, for $c_i < 0$,

$$-c_iP_1^{-1}\boldsymbol{\phi}^1(0)^{\mathrm{T}} + c_iP_1^{-1}(F_i^{1,1} + G_k)M_1^{-1}P_1\mathbf{1} = -c_iP_1^{-1}\boldsymbol{\phi}^1(0)^{\mathrm{T}} + c_iP_1^{-1}(F_i^{1,1} + G_k)\mathbf{1}$$

$$= -c_iP_1^{-1}\boldsymbol{\phi}^1(0)^{\mathrm{T}} + c_iP_1^{-1}F_i^{1,1}\mathbf{1}$$

$$= -c_iP_1^{-1}(e_1^1)^{\mathrm{T}} + c_iP_1^{-1}(e_1^1)^{\mathrm{T}}$$

$$= \mathbf{0}.$$

For the upper boundary,

$$T_{\Delta\Delta}\mathbf{1} + \mathfrak{B}^{\Delta K}\mathbf{1} = T_{\Delta\Delta}\mathbf{1} + [T_{\Delta-} \otimes (\boldsymbol{\phi}^K(\mathcal{I})M_K^{-1}P_K)]$$

1.Swapping the order of summation and recalling

$$M_k^{-1}P_k\mathbf{1} = \mathbf{1} \text{ then this is equal to } T_{\Delta\Delta}\mathbf{1} + [T_{\Delta-} \otimes (\boldsymbol{\phi}^K(\mathcal{I})M_K^{-1}P_K)\mathbf{1}]\mathbf{1}$$

$$= T_{\Delta\Delta}\mathbf{1} + [T_{\Delta-} \otimes e_{N_K}^K\mathbf{1}]\mathbf{1}$$

$$= T_{\Delta\Delta}\mathbf{1} + [T_{\Delta-} \otimes e_{N_K}^K]\mathbf{1}$$

$$= T_{\Delta\Delta}\mathbf{1} + T_{\Delta-}\mathbf{1}$$

$$= \mathbf{0}.$$

Also, for $c_i > 0$,

$$c_iP_K^{-1}\boldsymbol{\phi}^K(\mathcal{I})^{\mathrm{T}} + c_iP_K^{-1}(F_i^{K,K} + G_K)M_K^{-1}P_K\mathbf{1} = c_iP_K^{-1}\boldsymbol{\phi}^K(\mathcal{I})^{\mathrm{T}} + c_iP_K^{-1}(F_i^{K,K} + G_K)\mathbf{1}$$

$$= c_iP_K^{-1}\boldsymbol{\phi}^K(\mathcal{I})^{\mathrm{T}} + c_iP_K^{-1}F_i^{K,K}\mathbf{1}$$

$$= c_iP_K^{-1}(e_{N_K}^K)^{\mathrm{T}} + c_iP_K^{-1}(-e_{N_K}^K)^{\mathrm{T}}$$

$$= \mathbf{0}.$$

Combining all of the above we have shown that the row sums of $\mathfrak{B}$ are zero.

Corollary 7.1 *The DG approximation to the generator B conserves probability. That is, for all $t \geq 0$,*

$$\sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(t) + \sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(t) + \sum_{i \in \mathcal{S}} \int_{x \in [0,\mathcal{I}]} u_i(x,t)\,\mathrm{d}x$$

$$= \sum_{i \in \mathcal{S}_\nabla} q_{\nabla,i}(0) + \sum_{i \in \mathcal{S}_\Delta} q_{\Delta,i}(0) + \sum_{i \in \mathcal{S}} \int_{x \in [0,\mathcal{I}]} u_i(x,0)\,\mathrm{d}x.$$

Proof Let $\{\psi_r^k(x)\}_{r \in \mathcal{N}_k, k \in \{1,...,K\}}$, be a basis for $span(\phi_r^k(x), r \in \mathcal{N}_k, k \in \{1,...,K\})$, where $\{\phi_r^k(x)\}_{r \in \mathcal{N}_k, k \in \{1,...,K\}}$ are the Lagrange polynomials. Also define $\psi_1^\nabla(x) = \delta(x)$ and $\psi_1^\Delta(x) = \delta(x - \mathcal{I})$ to capture the point masses at the boundaries. Let us use the same vector notation for the basis $\psi_r^k(x)$ as we do for $\phi_r^k(x)$. For $k \in \{1,...,K\}$, since $\{\psi_r^k(x)\}_{r \in \mathcal{N}_k}$ and $\{\phi_r^k(x)\}_{r \in \mathcal{N}_k}$ have the same span, then there is a matrix V^k such that $\boldsymbol{\psi}^k(x)^\mathrm{T} = V^k \boldsymbol{\phi}^k(x)^\mathrm{T}$. Trivially, this also holds for $k = \nabla, \Delta$.

Let

$$W = \begin{bmatrix} I_{|\mathcal{S}_\nabla|} & & \\ & P & \\ & & I_{|\mathcal{S}_\Delta|} \end{bmatrix} \text{ and } V = \begin{bmatrix} I_{|\mathcal{S}_\nabla|} & & & & \\ & V^1 & & & \\ & & \ddots & & \\ & & & V^K & \\ & & & & I_{|\mathcal{S}_\Delta|} \end{bmatrix}.$$

For a DG approximation, B, constructed from $\{\psi_r^k\}_{r \in \mathcal{N}_k, k \in \{\nabla,1,...,K,\Delta\}}$, it can be shown that B is similar to $\mathfrak{B}$ with similarity matrix, VW, such that $B_{ij} = VW\mathfrak{B}_{ij}W^{-1}V^{-1}$, $i, j \in \mathcal{S}$. Therefore,

$$\int_{x \in [0,\mathcal{I}]} B_{ij} \boldsymbol{\psi}(x)^\mathrm{T} \, \mathrm{d}x = VW\mathfrak{B}_{ij}W^{-1}V^{-1} \int_{x \in [0,\mathcal{I}]} V\boldsymbol{\phi}(x)^\mathrm{T} \, \mathrm{d}x$$

$$= VW\mathfrak{B}_{ij}W^{-1}W\mathbf{1}$$

$$= VW\mathfrak{B}_{ij}\mathbf{1},$$

since $\int_{x \in [0,\mathcal{I}]} \boldsymbol{\phi}(x)^\mathrm{T} \, \mathrm{d}x = W\mathbf{1}$. The row sums of $\mathfrak{B}$ are 0, hence

$$\int_{x \in [0,\mathcal{I}]} \sum_{j \in \mathcal{S}} B_{ij} \boldsymbol{\psi}(x)^\mathrm{T} \, \mathrm{d}x = VW \sum_{j \in \mathcal{S}} \mathfrak{B}_{ij}\mathbf{1} \tag{32}$$

$$= VW\mathbf{0} \tag{33}$$

$$= \mathbf{0}. \tag{34}$$

Let $^\Psi\boldsymbol{a}_i(t)$, $i \in \mathcal{S}$ denote the coefficients related to the DG approximation constructed with the basis $\{\psi_r^k\}_{r \in \mathcal{N}_k, k \in \{\nabla,1,...,K,\Delta\}}$ (to distinguish them from $\boldsymbol{a}$ and $\boldsymbol{\alpha}$ used above). The DE constructed by the DG method is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(^\Psi\boldsymbol{a}_j(t)\right)\boldsymbol{\psi}(x) = \sum_{i \in \mathcal{S}} \left(^\Psi\boldsymbol{a}_i(t)\right)B_{ij}\boldsymbol{\psi}(x).$$

Integrating over $x \in [0, \mathcal{I}]$ and summing over $j \in \mathcal{S}$ we get

$$\int_{x \in [0,\mathcal{I}]} \sum_{j \in \mathcal{S}} \frac{\mathrm{d}}{\mathrm{d}t}\left(^\Psi\boldsymbol{a}_j(t)\right)\boldsymbol{\psi}(x) \, \mathrm{d}x = \int_{x \in [0,\mathcal{I}]} \sum_{j \in \mathcal{S}} \sum_{i \in \mathcal{S}} \left(^\Psi\boldsymbol{a}_i(t)\right)B_{ij}\boldsymbol{\psi}(x).$$

Exchanging the order of operations gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j \in \mathcal{S}} \left(^\Psi\boldsymbol{a}_j(t)\right) \int_{x \in [0,\mathcal{I}]} \boldsymbol{\psi}(x) \, \mathrm{d}x = \sum_{i \in \mathcal{S}} \left(^\Psi\boldsymbol{a}_i(t)\right) \int_{x \in [0,\mathcal{I}]} \sum_{j \in \mathcal{S}} B_{ij}\boldsymbol{\psi}(x) \, \mathrm{d}x = 0, \tag{35}$$

where the right-hand side is 0 due to Eq. (32). This holds for all $t \geq 0$. The left-hand side of (35) is the rate of change (with respect to time) of the total mass of the system. Since this is 0 for all $t \geq 0$, there is no change in the total mass of the system and thus probability is conserved.

A Toy Example

Here we include a small toy example to show how we construct a DG approximation and to help clarify the notation.

Consider a process $\{(\overline{X}_t, Y_t, \varphi_t)\}_{t \geq 0}$ with two phases, $\varphi_t \in S = \{1, 2\}$ and generator matrix T. Let $\mathcal{I} = 1.8$, and partition into two intervals $\mathcal{D}_1 = [0, 1]$ and $\mathcal{D}_2 = [1, 1.8]$, hence $x_1 = 0$, $x_2 = 1$, $x_3 = 1.8$. We choose a basis of Lagrange polynomials of order 1 to define our approximation space. That is,

$$\phi_1^1(x) = 1 - x, \qquad \phi_2^1(x) = x, \qquad x \in \mathcal{D}_1,$$
$$\phi_1^2(x) = \frac{1.8 - x}{0.8}, \qquad \phi_2^2(x) = \frac{x - 1}{0.8}, \qquad x \in \mathcal{D}_2.$$

The mesh and basis functions are shown in Fig. 6.

We can verify that the matrices M and G are given by

$$M = \begin{bmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/6 & 1/3 & 0 & 0 \\ 0 & 0 & 4/15 & 4/30 \\ 0 & 0 & 4/30 & 4/15 \end{bmatrix}, \quad G = \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}.$$
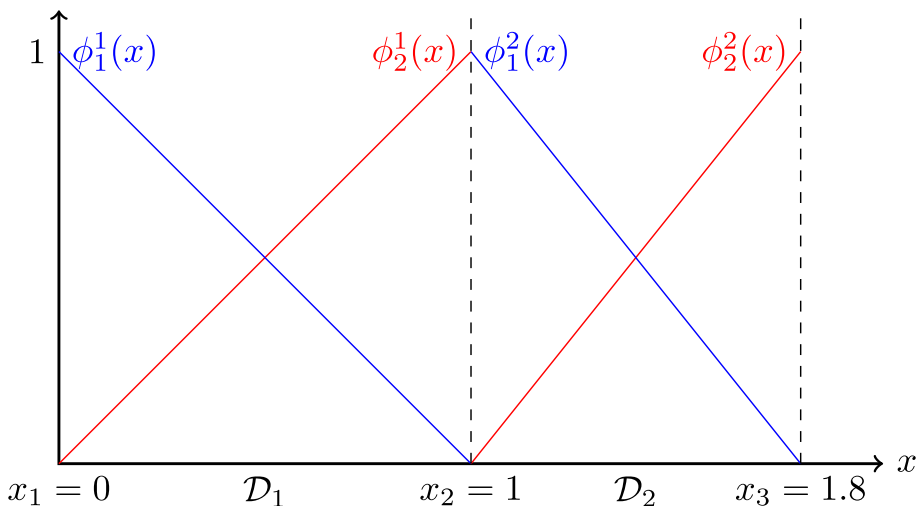
Fig. 6 A mesh with nodes $x_1 = 0$, $x_2 = 1$ and $x_3 = 1.8$ and cells $\mathcal{D}_1 = [0, 1]$, $\mathcal{D}_2 = [1, 1.8]$. There are two basis functions on each cell. Point masses are located at $x_1 = 0$ and $x_3 = 1.8$

The matrix P is given by

$$P = \left[\begin{array}{cc|cc} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 2/5 \end{array}\right].$$

Let $c_1 = 1$ and $c_2 = -2$. Then the flux matrices are given by

$$F_1 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right], \quad F_2 = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Suppose that $r_1(x) > 0$ on $\mathcal{D}_1 = \mathcal{F}_1^+$ and $r_1(x) < 0$ on $\mathcal{D}_2 \cup \{1\} = \mathcal{F}_1^-$, and further, that $r_2(x) < 0$ on $\{0\} \cup \mathcal{D}_1 = \mathcal{F}_2^-$ and $r_2(x) > 0$ on $\mathcal{D}_2 = \mathcal{F}_2^+$. Specifically, let

$$r_1(x) = \begin{cases} 1 & x \in [0,1], \\ -1 & x \in [1, 1.8], \end{cases} \qquad r_2(x) = \begin{cases} -1 & x = 0, \\ -2 & x \in (0,1], \\ 1 & x \in [1, 1.8]. \end{cases}$$

Then, constructing B we get

	$\mathcal{D}_1$				$\mathcal{D}_2$				∇
$\mathcal{F}_2^-$	$\mathcal{F}_1^+$		$\mathcal{F}_2^-$		$\mathcal{F}_1^-$		$\mathcal{F}_2^+$		$\mathcal{F}_1^-$
$q_{\nabla,1}$	$a_{1,1}^1$	$a_{1,2}^1$	$a_{2,1}^1$	$a_{2,2}^1$	$a_{1,1}^2$	$a_{1,2}^2$	$a_{2,1}^2$	$a_{2,2}^2$	$q_{\Delta,1}$
T_{22}	$4T_{21}$	$-2T_{21}$	0	0	0	0	0	0	0
0	$T_{11} - 3$	3	T_{12}	0	0	0	0	0	0
0	-1	$T_{11} - 1$	0	T_{12}	5	-2.5	0	0	0
2	T_{21}	0	$T_{22} - 2$	-2	0	0	0	0	0
0	0	T_{21}	6	$T_{22} - 6$	0	0	0	0	0
0	0	0	0	0	$T_{11} - \frac{15}{4}$	$\frac{15}{4}$	T_{12}	0	0
0	0	0	0	0	$-\frac{5}{4}$	$T_{11} - \frac{5}{4}$	0	T_{12}	1
0	0	0	-4	8	T_{21}	0	$T_{22} - \frac{5}{2}$	$-\frac{5}{2}$	0
0	0	0	0	0	0	T_{21}	$\frac{15}{2}$	$T_{22} - \frac{15}{2}$	0
0	0	0	0	0	0	0	$-2T_{12}$	$4T_{12}$	T_{11}

We also have sub-matrices

$$B_{11}^{++} = \begin{bmatrix} T_{11} - 3 & 3 \\ -1 & T_{11} - 1 \end{bmatrix}, \ B_{11}^{+-} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 5 & -2.5 & 0 \end{array} \right], \ B_{11}^{--} = \left[\begin{array}{cc|c} T_{11} - \frac{15}{2} & \frac{15}{2} & 0 \\ -\frac{5}{2} & T_{11} - \frac{5}{2} & 1 \\ \hline 0 & 0 & T_{11} \end{array} \right],$$

$$B_{12}^{+-} = \left[\begin{array}{c|cc} 0 & T_{12} & 0 \\ 0 & 0 & T_{12} \end{array} \right], \ B_{12}^{-+} = \left[\begin{array}{cc} T_{12} & 0 \\ 0 & T_{12} \\ \hline -2T_{12} & 4T_{12} \end{array} \right],$$

$$B_{21}^{+-} = \left[\begin{array}{cc|c} T_{21} & 0 & 0 \\ 0 & T_{21} & 0 \end{array} \right], \ B_{21}^{-+} = \left[\begin{array}{cc} 4T_{21} & -2T_{21} \\ T_{21} & 0 \\ 0 & T_{21} \end{array} \right],$$

$$B_{22}^{++} = \begin{bmatrix} T_{22} - \frac{5}{2} & -\frac{5}{2} \\ \frac{15}{2} & T_{22} - \frac{15}{2} \end{bmatrix}, \ B_{22}^{+-} = \left[\begin{array}{c|cc} 0 & -4 & 8 \\ 0 & 0 & 0 \end{array} \right], \ B_{22}^{--} = \left[\begin{array}{c|cc} T_{22} & 0 & 0 \\ \hline 2 & T_{22} - 2 & -2 \\ 0 & 6 & T_{22} - 6 \end{array} \right],$$

and $B_{11}^{-+} = 0_{3 \times 2}$, $B_{12}^{++} = 0_{2 \times 2}$, $B_{12}^{--} = 0_{2 \times 3}$, $B_{21}^{++} = 0_{2 \times 2}$, $B_{21}^{--} = 0_{3 \times 3}$, $B_{22}^{-+} = 0_{3 \times 2}$, where $0_{n \times m}$ denotes a $n \times m$ matrix of zeros. Furthermore,

$$B^{++} = \left[\begin{array}{cc|cc} T_{11} - 3 & 3 & 0 & 0 \\ -1 & T_{11} - 1 & 0 & 0 \\ \hline 0 & 0 & T_{22} - \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{15}{2} & -\frac{15}{2} \end{array} \right],$$

$$B^{+-} = \left[\begin{array}{c|cc|cc|c} 0 & T_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & T_{12} & 5 & -2.5 & 0 \\ \hline 0 & -4 & 8 & T_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & T_{21} & 0 \end{array} \right],$$

$$B^{-+} = \left[\begin{array}{cc|cc} 4T_{21} & -2T_{21} & 0 & 0 \\ \hline T_{21} & 0 & 0 & 0 \\ 0 & T_{21} & 0 & 0 \\ \hline 0 & 0 & T_{12} & 0 \\ 0 & 0 & 0 & T_{12} \\ \hline 0 & 0 & -2T_{12} & 4T_{12} \end{array} \right],$$

$$B^{--} = \left[\begin{array}{c|cc|cc|c} T_{22} & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & T_{22} - 2 & -2 & 0 & 0 & 0 \\ 0 & 6 & T_{22} - 6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & T_{11} - \frac{15}{4} & -\frac{15}{4} & 0 \\ 0 & 0 & 0 & -\frac{5}{4} & T_{11} - \frac{5}{4} & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & T_{11} \end{array} \right].$$

Since $r_1(x)$ and $r_2(x)$ are constant on each cell then R^+ and R^- take a particularly simple form. We have

$$R^+ = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad R^- = \left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The DG approximations $D^{mn}(s)$, $m, n \in \{+, -\}$ can now be constructed as

$$D^{mn}(s) = \begin{cases} R^m(B^{mm} - sI) & n = m, \\ R^m B^{mn} & n \neq m. \end{cases}$$

For a given value of s, we construct and solve the matrix Riccati equation,

$$D^{+-}(s) + \Psi(s)D^{-+}(s)\Psi(s) + D^{++}(s)\Psi(s) + \Psi(s)D^{--}(s) = 0,$$

for the matrix $\Psi(s)$ using, for example, Newtons method (Bean et al. 2009). To obtain the stationary distribution we require $\Psi(0)$.

Now, to find $\boldsymbol{\xi}$, we solve the linear system in Eqs. (9)-(10). The result is a vector which we denote,

$$\boldsymbol{\xi} = \left[\, \xi_2^\nabla \,\middle|\, \xi_{2,1}^1 \;\; \xi_{2,2}^1 \,\middle|\, \xi_{1,1}^2 \;\; \xi_{1,2}^2 \,\middle|\, \xi_1^\Delta \,\right],$$

where ξ_2^∇ is an approximation to $\lim_{n\to\infty} \mathbb{P}\left[\overline{X}_{\theta_n} = 0, \varphi_{\theta_n} = 2\right]$ and ξ_1^Δ is an approximation to the artificial point mass $\lim_{n\to\infty} \mathbb{P}\left[\overline{X}_{\theta_n} = 1.8, \varphi_{\theta_n} = 1\right]$. For $x \in \mathcal{D}_1$ an approximation to the density of $\lim_{n\to\infty} \mathbb{P}\left[\overline{X}_{\theta_n} \in \mathrm{d}x, \varphi_{\theta_n} = 2\right]$, is constructed as $\xi_{2,1}^1 \phi_1^1(x) + \xi_{2,2}^1 \phi_2^1(x)$. For $x \in \mathcal{D}_2$ an approximation to the density of $\lim_{n\to\infty} \mathbb{P}\left[\overline{X}_{\theta_n} \in \mathrm{d}x, \varphi_{\theta_n} = 1\right]$, is constructed as $\xi_{1,1}^2 \phi_1^2(x) + \xi_{1,2}^2 \phi_2^2(x)$.

Next, given a value of y, we solve the system (25)-(29) to find $\boldsymbol{p} = \boldsymbol{p}^-$ and $\boldsymbol{\pi}(y)$.

For the point masses we have

$$\boldsymbol{p}^- = \left[\, p_2^\nabla \,\middle|\, p_{2,1}^1 \;\; p_{2,2}^1 \,\middle|\, p_{1,1}^2 \;\; p_{1,2}^2 \,\middle|\, p_1^\Delta \,\right],$$

where p_2^∇ is an approximation to $\lim_{t\to\infty} \mathbb{P}\left[\overline{X}_t = 0, Y_t = 0, \varphi_t = 2\right]$ and p_1^Δ is an approximation to the artificial point mass $\lim_{t\to\infty} \mathbb{P}\left[\overline{X}_t = 1.8, Y_t = 0, \varphi_t = 1\right]$. For $x \in \mathcal{D}_1$, an approximation to the density of $\lim_{t\to\infty} \mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t = 0, \varphi_t = 2\right]$, is constructed as $p_{2,1}^1 \phi_1^1(x) + p_{2,2}^1 \phi_2^1(x)$. For $x \in \mathcal{D}_2$, an approximation to the density of $\lim_{t\to\infty} \mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t = 0, \varphi_t = 1\right]$, is constructed as $p_{1,1}^2 \phi_1^2(x) + p_{1,2}^2 \phi_2^2(x)$.

Similarly, for $\boldsymbol{\pi}^-(y)$, we have

$$\boldsymbol{\pi}^-(y) = \left[\, \pi_2^\nabla(y) \,\middle|\, \pi_{2,1}^1(y) \;\; \pi_{2,2}^1(y) \,\middle|\, \pi_{1,1}^2(y) \;\; \pi_{1,2}^2(y) \,\middle|\, \pi_1^\Delta(y) \,\right],$$

where $\pi_2^\nabla(y)$ is an approximation to $\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t = 0, Y_t \in \mathrm{d}y, \varphi_t = 2\right]$ and $\pi_1^\Delta(y)$ is an approximation to the artificial point mass $\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t = 1.8, Y_t \in \mathrm{d}y, \varphi_t = 1\right]$. For $x \in \mathcal{D}_1$ an approximation to the density of $\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t \in \mathrm{d}y, \varphi_t = 2\right]$, is constructed as $\pi_{2,1}^1(y)\phi_1^1(x) + \pi_{2,2}^1(y)\phi_2^1(x)$. For $x \in \mathcal{D}_2$ an approximation to the density of $\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t \in \mathrm{d}y, \varphi_t = 1\right]$, is constructed as $\pi_{1,1}^2(y)\phi_1^2(x) + \pi_{1,2}^2(y)\phi_2^2(x)$.

For $\boldsymbol{\pi}^+(y)$, we have

$$\boldsymbol{\pi}^+(y) = \left[\begin{array}{cc|cc} \pi_{1,1}^1(y) & \pi_{1,2}^1(y) & \pi_{2,1}^2(y) & \pi_{2,2}^2(y) \end{array}\right].$$

For $x \in \mathcal{D}_1$ an approximation to the joint density of $\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t \in \mathrm{d}y, \varphi_t = 1\right]$ is constructed as $\pi_{1,1}^1(y)\phi_1^1(x) + \pi_{1,2}^1(y)\phi_2^1(x)$. For $x \in \mathcal{D}_2$ an approximation to the density of $\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t \in \mathrm{d}y, \varphi_t = 2\right]$ is constructed as $\pi_{2,1}^2(y)\phi_1^2(x) + \pi_{2,2}^2(y)\phi_2^2(x)$.

In summary, for $i \in \mathcal{S}$, a global approximation of the joint stationary distribution is

$$\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t \in \mathrm{d}y, \varphi_t = i\right] \approx \sum_{r\in\{1,2\},k\in\{1,2\}} \pi_{i,r}^k(y)\phi_r^k(x)\,\mathrm{d}x\,\mathrm{d}y, \quad x \in (0,1.8),\ y > 0,$$

$$\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t \in \mathrm{d}x, Y_t = 0, \varphi_t = i\right] \approx \sum_{r\in\{1,2\},k\in\{1,2\}} p_{i,r}^k\phi_r^k(x)\,\mathrm{d}x, \quad x \in (0,1.8),$$

$$\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t = 0, Y_t \in \mathrm{d}y, \varphi_t = i\right] \approx \pi_i^\nabla(y)\,\mathrm{d}y, \quad y > 0,$$

$$\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t = 0, Y_t = 0, \varphi_t = i\right] \approx p_i^\nabla,$$

$$\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t = 1.8, Y_t \in \mathrm{d}y, \varphi_t = i\right] \approx \pi_i^\Delta(y)\,\mathrm{d}y, \quad y > 0,$$

$$\lim_{t\to\infty}\mathbb{P}\left[\overline{X}_t = 1.8, Y_t = 0, \varphi_t = i\right] \approx p_i^\Delta.$$

Code Availability Code available here: https://github.com/angus-lewis/SFFM.

Declarations

Ethics Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication All authors consent to the submission of this article to this journal.

Conflicts of Interest None.

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