Received: 23 August 2020 | Revised: 12 December 2021 | Accepted: 25 April 2022

DOI: 10.1002/for.2871

RESEARCH ARTICLE

WILEY

Are internally consistent forecasts rational?

Jing Tian[1] | Firmin Doko Tchatoka[2] | Thomas Goodwin[3]

[1]Tasmanian School of Business and Economics, University of Tasmania, Hobart, Tasmania, Australia

[2]School of Economics and Public Policy, University of Adelaide, Adelaide, South Australia, Australia

[3]Hillwood Berries, Hillwood, Tasmania, Australia

Correspondence
Jing Tian, Tasmanian School of Business and Economics, University of Tasmania, Private Bag 84, Hobart 7001, TAS, Australia.
Email: jing.tian@utas.edu.au

Abstract

Economic forecasts, such as the Survey of Professional Forecasters (SPF), are revised multiple times before realization of the target. This paper studies the sources of forecast revisions. By decomposing the fixed-event forecast error into a rational component due to unanticipated future shocks and an irrational component due to measurement error in acquired information or forecasters' reactions, we derive the conditions under which fixed-event forecasts that contain an irrational forecast error can still possess the second moment properties of rational forecasts. We show that internal consistency of fixed-event forecasts depends on the magnitude of the rational and irrational components in the revisions. As such, fixed-event forecasts subject to irrational error may still be internally consistent, although they are not rational, as evidence in many empirical studies. We illustrate our methodology with the SPF inflation forecasts data. Our results show evidence of a sizeable and heterogeneous irrational forecast error component across forecast horizons and a high irrational-to-news ratio for most forecasters. This finding also provides insight into why forecast revision effort is not always fully compensated by revision reward.

KEYWORDS

fixed-event forecast, bi-error structure, irrational forecast, internal consistency, SPF forecasts

1 | INTRODUCTION

The rationality of multi-horizon fixed-event forecasts is formed on the rational expectations hypothesis that rational forecasters efficiently utilize existing information when forming and updating their expectations. Several econometric frameworks have been suggested in the literature to investigate whether information is incorporated fully and immediately into recently released forecasts. For instance, internal consistency, which describes that rational forecasts of the same targeted event that are published at different forecast horizons should be internally consistent with each other, is often used to assess the "term structure" of fixed-event forecasts across horizons. Patton and Timmermann (2012) provide a suite of second moment bounds implied by a sequence of rational forecasts. Specifically, as the forecast horizon grows, rational forecasts are expected to have (i) weakly decreasing variance; (ii) weakly increasing mean squared errors (MSFE); and (iii) weakly increasing mean squared (cumulative) revisions (MSFR).

The empirical evidence on these conditions is often mixed in the sense that only a restricted set of them is observed in the data. For example, Figure 1 illustrates the

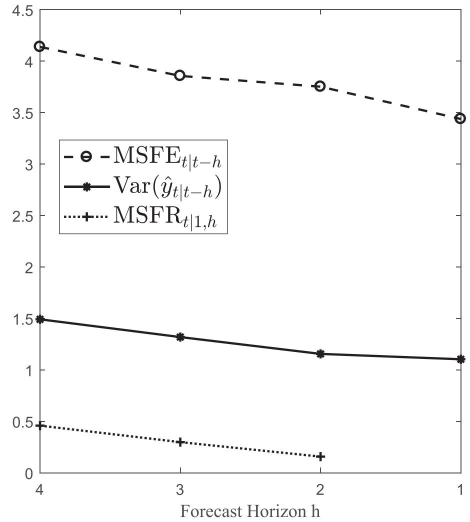

FIGURE 1 Mean squared forecast error, forecast variances, and mean squared forecast revision for the U.S. headline CPI inflation (%)

sequences of MSFE, variance of the forecasts, and MSFR from the mean responses of the Survey of Professional Forecasters (SPF) for the quarter-over-quarter headline CPI inflation from 1981:Q4 to 2018:Q1.[1] Although the MSFE and MSFR decline as forecast time approaches the target date (thus satisfying forecast rationality), the variance of forecasts decreases contradicting forecast rationality. Such a mixed outcome is also reported in Patton and Timmermann (2012),[2] where the Federal Reserve Greenbook forecasts for three macroeconomic variables from 1981:Q2 to 2004:Q4 were evaluated. Similar to the SPF mean forecasts of inflation (Figure 1), Patton and Timmermann (2012) find that both the MSFE and variance of forecasts of GDP deflator decrease as the forecast horizon shrinks, but the GDP growth forecasts do not present decreasing MSFE despite the fact that the variance increases as the horizon decreases.

This raises the question of whether a violation of any internal consistency conditions in Patton and Timmermann (2012) implies irrational forecast revisions.[3] In other words, given that the forecasts are "irrational," can they still be internally consistent, and if so, under which conditions?

To investigate these questions, we consider a bi-error framework where a fixed-event forecast is decomposed as the sum of rational and irrational components (similar to Chang et al., 2013; Davies & Lahiri, 1995) and document why mixed outcomes of Patton and Timmermann's (2012) second moment bounds may be observed in real data. Specifically, we hypothesize that an unbiased fixed-event forecast is subject to two types of forecast errors: a rational forecast error that occurs due to unanticipated information over forecast horizons; and an irrational forecast error that occurs due to either measurement error in news or forecasters' "irrational" behavior to news. Unlike the existing literature, we allow for heterogeneous variances across horizons for both irrational error and the newly arrived information in revised forecasts. This assumption aligns with the uneven information flow discussed in Isiklar and Lahiri (2007), Lahiri and Sheng (2010), and Lahiri (2012) and is strongly supported by the empirical analysis of the SPF CPI inflation forecasts (see Section 4.2).

We derive the conditions under which fixed-event forecasts containing an irrational forecast error can still possess the second moment properties of rational forecasts. We show that the internal consistency conditions are determined by the variances and correlations of irrational error and newly arrived information and that the requirements among the suite of internal consistency properties are not always the same. Therefore, it is possible to observe only a subset of monotonic patterns in the second moments of fixed-event forecasts in empirical data. In essence, our finding suggests that fixed-event forecasts may be internally consistent but not rational.

In addition, we use the bi-error model to show that overreaction to news results in a sequence of forecast revisions that follow a first-order autoregressive (AR) process with a non-positive coefficient. Therefore, a Nordhaus-type test for forecast rationality, as suggested by Clements and Taylor (2001), is likely to be rejected in the presence of irrational forecast error. We show that the magnitude of the irrational-to-news ratio (i.e., the standard deviation of irrational error relative to the standard deviation of news) plays a key role in this test outcome. Although a rejection of the Nordhaus-type test implies that forecasts are not rational, a subset of internal consistency conditions may still hold. In the empirical application of the SPF inflation forecast data (see Section 4.2.2), a Nordaus-type test rejects rationality for most forecasters due to large irrational-to-news ratios. However, Patton and Timmermann's (2012) second moment monotonic patterns are observed for many of these forecasters (see Table 2).

Literature close to our study includes Isiklar and Lahiri (2007) and Lahiri (2012), who compare revision effort and revision reward to assess whether rational forecasters react to news in an optimal way. By allowing for

[1]Historical SPF forecasts of CPI inflation rate can be downloaded at https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files/cpi, The Federal Reserve Bank of Philadelphia (2020)

[2]See Figures 1 and 2 of Patton and Timmermann (2012).

[3]Irrational forecast revisions refer to the revisions that do not efficiently incorporate newly available information.

the presence of an irrational component in the multi-horizon fixed-event forecasts and a potential non-zero correlation between the irrational component and the newly arrived information, this study generalizes Isiklar and Lahiri (2007). We show that if the revised forecasts contain an irrational error that is uncorrelated with the target or is due to overreaction to news, the revision effort measured by MSFR cannot be fully remunerated by the improvement in MSFE, meaning that forecast revisions are suboptimal. We demonstrate that eliminating irrational errors in revisions is not the only reason for revision effort to be fully rewarded. Remuneration can also be achieved when forecasters smooth news in a way that the irrational-to-news ratio does not exceed the correlation between the irrational error and news. Interestingly, this finding implies that the equality between the MSFR and the reduction in the MSFE does not necessarily suggest an efficient use of newly arrived information or a rational forecast revision.

The remainder of this paper is organized as follows. Section 2 discusses fixed-event forecast decomposition. We derive the conditions of internal consistency using the bi-error model in Section 3. Section 4 illustrates our methodology with a panel of the SPF inflation forecasts. Finally, Section 5 concludes. Additional proofs are included in the appendices.

2 | SETUP

Let y_t denote a target event and $\hat{y}_{t|t-h}$ be a fixed-event h-period ahead forecast of y_t before the target date t (i.e., $h<t$), where $h=1,2,...,H$. The initial forecast, $\hat{y}_{t|t-H}$, is made at the longest horizon $h=H$ and is then updated as the forecast horizon shrinks. The long, medium, and short horizons will be denoted by l, m, and s, respectively, hereafter, and we assume that $l>m>s$. We can decompose y_t (see Davies & Lahiri, 1995) as

$$y_t=\hat{y}^*_{t|t-h}+\lambda_{th}, \tag{2.1}$$

where $\hat{y}^*_{t|t-h}$ is a rational forecast at horizon h (i.e., the unobserved value y_t would take on for time t if no shocks occurred from horizon h until t). The term λ_{th} represents the cumulative effect of unanticipated shocks that occurs from horizon h until t, that is,

$$\lambda_{th}=\sum_{i=0}^{h-1}\omega_{i,t}, \tag{2.2}$$

where $\omega_{i,t}$ are independent across i and t, each with a zero mean and a finite variance $\sigma^2_{\omega_i}$ (i.e., $0<\sigma^2_{\omega_i}<\infty$), and are uncorrelated with $\hat{y}^*_{t|t-h}$. In this representation, we refer to λ_{th} as the rational forecast error (Davies & Lahiri, 1995). Figure 2 illustrates the timing of rational forecast $\hat{y}^*_{t|t-h}$ relative to aggregate shocks that occur after the forecast is made. Rational forecasters update the previously made forecast $\hat{y}^*_{t|t-h-1}$ by incorporating newly arrived information $\omega_{h,t}$ but do not anticipate future shocks $\omega_{h-1,t},...,\omega_{0,t}$. Therefore, λ_{th} is orthogonal to the rational forecast component $\hat{y}^*_{t|t-h}$ by construction, that is, $\mathrm{Cov}(\lambda_{th},\hat{y}^*_{t|t-h})=0$, and as a result, we have $\mathrm{Cov}(\lambda_{th},y_t)=\sum_{i=0}^{h-1}\sigma^2_{\omega_i}>0$.

Now, consider the following decomposition of the forecast $\hat{y}_{t|t-h}$ (similar to Chang et al., 2013; Davies & Lahiri, 1995)[4]:

$$\hat{y}_{t|t-h}=\hat{y}^*_{t|t-h}-\xi_{th}, \tag{2.3}$$

where $-\xi_{th}$ is the irrational component added to the rational forecast to form a fixed-event forecast. The irrational component ($-\xi_{th}$) is orthogonal to rational forecast errors that are unanticipated future shocks.

Irrational component of fixed-event forecasts may exist for various reasons. First, noise or measurement error embedded in the information received by forecasters is uncorrelated with the true effect of shocks to the target, thus rendering forecasts irrational. For instance, Jeong and Maddala (1991) identify that errors-in-variable lead to rejection of the rational expectations hypothesis. Lovell (1986) characterizes the type of error uncorrelated with the realization as an "implicit" forecast error on the basis of the implicit expectation hypothesis by Mills (1957). Second, the irrational forecast component may be correlated with the target if forecasters have incentives to ignore or partially update newly arrived information. Schotese (1994) models forecasts based on the objectives of both accuracy and reputation and argues that forecasters who attempt to cultivate a reputation for producing stable forecasts intentionally underutilize the currently available information in order to smooth forecast revisions. The evidence of information rigidity in professional macroeconomic and financial forecast revisions has been empirically documented; see Abarbanell and Bernard (1992), Coibion and Gorodnichenko (2015), and Andrade and Le Bihan (2013). Because smoothing aims to reduce the impact of available information, a negative correlation between the irrational forecast

[4]Chang et al. (2013) decompose forecasts to econometric model-based forecasts and an expert's intuition for investigating relationship between current revisions and lagged revisions. Their econometric model-based forecasts are defined as conditional expectations of the target based on the data generating process and hence are essentially the same as our rational forecast component.

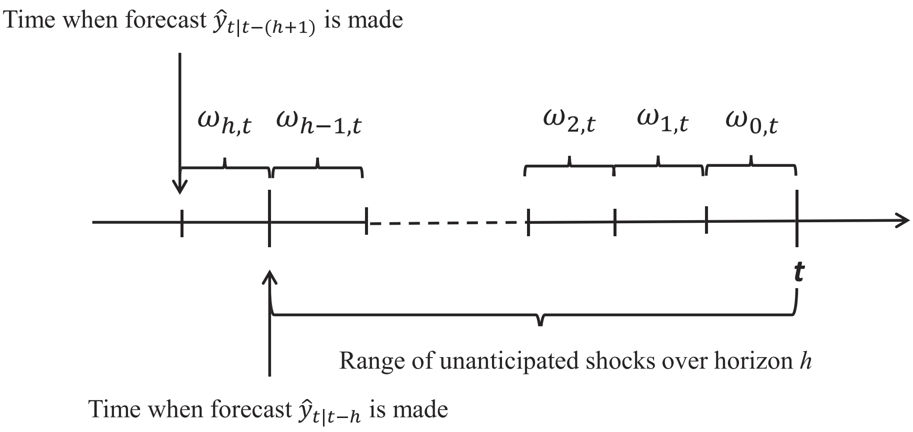

FIGURE 2 Illustration of the timing of forecasts and the range of aggregate unanticipated shocks

component and acquired information of the target captures such an "irrational" behavior of forecasters. Third, forecasters may overreact to new information. For instance, stock market investors may overlook the impact of news on the fundamentals when forming their expectations on stock prices, which could explain why the stock price movement is much more volatile than that of the counterpart macro fundamentals (see Bordalo et al., 2018; Shiller, 1981). As such, overreaction to news requires the irrational forecast component to be positively correlated with newly available information.

To account for all these various settings, we consider the following assumption on the irrational forecast error ξ_{th} and the unanticipated shocks $\omega_{h,t}$.

> **Assumption 2.1.** $(a)\xi_{th}$ are independent across t and h have zero mean and variance $\sigma_\xi^2(h) \equiv \sigma_{\xi_h}^2$ for $h = 1,...,H$; $(b)\mathrm{Cov}(\xi_{th}, \omega_{l,t}) = 0$ if $l \neq h$ for any $h = 1,...,H$.

Assumption 2.1a implies that the usual monotonic pattern of the rational forecast component is no longer guaranteed, as this variance may increase or decrease with h. Assumption 2.1a also characterizes the irrational forecast component as being uncorrelated across horizons (as opposed to the rational forecast error) and non-autocorrelated over time. Assumption 2.1b indicates that $\mathrm{Cov}(\xi_{th}, \omega_{l,t}) \neq 0$ only when $l = h$. This assumption allows the irrational forecast component to be correlated only with the newly arrived information, not with any earlier news that have already been incorporated in previously released forecasts.

Under Assumption 2.1, forecasters *smooth newly revealed information* when the irrational forecast component, $-\xi_{th}$, is negatively correlated with the unanticipated shock $\omega_{h,t}$, that is, $\mathrm{Cov}(-\xi_{th}, \omega_{h,t}) < 0$, and they *overreact to newly revealed information* when $-\xi_{th}$ is positively correlated with $\omega_{h,t}$, that is $\mathrm{Cov}(-\xi_{t,h}, \omega_{h,t}) > 0$. Because we have $\mathrm{Cov}(\xi_{th}, \omega_{h,t}) = \theta_h \sigma_{\omega_h} \sigma_{\xi_h}$, where θ_h is the common correlation coefficient between ξ_{th} and $\omega_{h,t}$ for all t, the condition for smoothing news is equivalent to $0 < \theta_h \leq 1$ whereas the condition for overreacting to news amounts to $-1 \leq \theta_h < 0$, given any forecast horizon $h = 1,...,H$. Clearly, θ_h measures the reaction of forecasters when new information arrives. In this perspective, *implicit forecast error* occurs when $\mathrm{Cov}(\hat{y}_{t|t-h}, \xi_{th}) \neq 0$ and $\mathrm{Cov}(y_t, \xi_{th}) = 0$ for all t given any h.

From (2.1) to (2.3), it is easy to see that the forecast error made at each horizon, $e_{th} = y_t - \hat{y}_{t|t-h}$, takes the form

$$e_{th} = \sum_{i=0}^{h-1} \omega_{i,t} + \xi_{th}, \tag{2.4}$$

where the summation term is the rational forecast error due to unanticipated information that occurs after the forecasts were made and ξ_{th} is the irrational forecast error.[5] Equation (2.4) characterizes a *bi-structure* of the forecast error e_{th}. A similar expression was given earlier in Davies and Lahiri (1999) under the assumption that $\omega_{i,t}$ are independent across i and t. Equation (2.4) shows that forecast revisions are made due to two reasons. First, forecasters update their forecasts due to the amount of new information aggregated after the forecasts were made. Second, forecasters update their forecasts to adjust the irrational forecast error components. Specifically, note that we can express the difference between short-horizon and long-horizon forecasts (for the same target y_t), under (2.4), as:

[5] It is important to remember that the rational forecast error made at horizon h is by definition uncorrelated with the irrational forecast error component.

$$d_{t,sl} := \hat{y}_{t|t-s} - \hat{y}_{t|t-l} = \sum\nolimits_{i=s}^{l-1} \omega_{i,t} + \xi_{tl} - \xi_{ts}. \tag{2.5}$$

The first term $\sum_{i=s}^{l-1} \omega_{i,t}$ is the aggregated amount of new information relevant to the target y_t that is incorporated in the updated short horizon forecast. This revision component is the reduction in rational forecast error. The last two terms indicate an adjustment to irrational forecast components. Note that the irrational component of the short-horizon forecast (i.e., $-\xi_{ts}$) may be correlated with $\omega_{s,t}$ but is uncorrelated with $\omega_{s+1,t}, \ldots, \omega_{l-1,t}$. The next section characterizes the conditions for internal consistency of fixed-event forecasts under the *bi-error structure* (2.4).

3 | INTERNAL CONSISTENCY

A simple but powerful testing framework for fixed-event forecast rationality is to investigate the monotonicity properties of second moment bounds across multiple horizons. In this section, we discuss the internal consistency of fixed-event forecasts subject to both rational and irrational forecast errors. Patton and Timmermann (2012) characterize *internal consistency* of fixed-event forecasts by second moment bounds of forecast errors, revisions, and the target. In this study, we extend the concept to include a comparison between forecast reward and forecast effort, which are both measured by the second moments of fixed-event forecasts.

It is well-known that rational forecasts are internally consistent across horizons (see, e.g., Lahiri, 2012; Patton & Timmermann, 2012). However, because the bi-error decomposition (2.4) incorporates an irrational component, an understanding of the extent to which this model may still deliver internal consistency is of paramount importance for both forecasters and policy makers. In this section, we use the framework of Section 2 to establish the necessary and sufficient conditions under which fixed-event forecasts are still internally consistent under the bi-error model. These conditions are for (a) *monotonicity in the variances* of forecasts, forecast errors, and forecast revisions; (b) *monotonicity in the covariances* related to forecasts, forecast errors, and the target; (c) *monotonicity and bounds on covariances* related to forecast revisions; and (d) *remuneration* of revision effort by the revision reward. The details of these conditions are included in **Panels A–D** in Table A1 of Appendix A. The table also summarizes the implication of these conditions on the bi-error model parameters, which generally take the form of inequality constraints between variances and covariances. A step-by-step derivation of these conditions is presented in Appendix B.

Our main goal in the following subsections is to show that in the bi-error model, fixed-event forecasts may still be internally consistent even when they contain a significant amount of the irrational component.

3.1 | Monotonicity in the variances, covariances, and bounds on covariances

We first study the condition for monotonicity in the variances of forecasts, forecast errors, and forecast revisions. As discussed in Section 2, a rational forecast error is information yet to be available to forecasters at a given horizon. Hence, the irrational forecast error made at the same horizon is independent to the rational error component. Consequently, the MSFEs are characterized by the variances of the two errors, that is,

$$\mathrm{MSFE}_{t|t-h} := \mathbb{E}(e_{th}^2) = \sum_{i=0}^{h-1} \sigma_{\omega_i}^2 + \sigma_{\xi_h}^2,\ h = 1, \ldots, H. \tag{3.1}$$

Comparing the values of MSFE between $h = s$ (short horizon) and $h = l$ (long horizon), we get

$$\begin{aligned} \Delta\mathrm{MSFE}_{t,sl} &= \mathrm{MSFE}_{t|t-s} - \mathrm{MSFE}_{t|t-l} \\ &= -\sum_{i=s}^{l-1} \sigma_{\omega_i}^2 + \sigma_{\xi_s}^2 - \sigma_{\xi_l}^2; \end{aligned} \tag{3.2}$$

that is, the changes in MSFE are due to either a reduction in the rational forecast error $\left(-\sum_{i=s}^{l-1} \sigma_{\omega_i}^2\right)$ or a correction of the irrational forecast error $\left(\sigma_{\xi_s}^2 - \sigma_{\xi_l}^2\right)$. The relative size of these two sources determines whether MSFE can be improved by forecast revisions. Clearly, we see from (3.2) that

$$\mathbb{E}(e_{ts}^2) \le \mathbb{E}(e_{tl}^2) \Leftrightarrow \sum_{i=s}^{l-1} \sigma_{\omega_i}^2 \ge \sigma_{\xi_s}^2 - \sigma_{\xi_l}^2, \tag{3.3}$$

which suggests that nondecreasing MSFE with horizon h can be observed under the following two scenarios. First, when revisions lead to a smaller size of irrational forecast error, we have $\sigma_{\xi_s}^2 - \sigma_{\xi_l}^2 < 0$, and hence, $\mathrm{MSFE}_{t|t-s} < \mathrm{MSFE}_{t|t-l}$, and revisions improve forecast accuracy. Second, when revisions incur a larger size of irrational forecast error, as long as the information accumulation during two updating points of time (i.e., rational component of forecast revision) outweighs the increment in $\sigma_{\xi_h}^2$, revisions are still able to improve forecasting accuracy.

We can also express the cumulative mean square forecast revision (MSFR) made at the short horizon s for revising forecasts made previously at a longer horizon $h > s$ in a similar way as the MSFE, that is,

$$\mathrm{MSFR}_{t,sh} = E\left(d_{t,sh}^2\right) = \sum_{i=s}^{h-1} \sigma_{\omega_i}^2 + \sigma_{\xi_s}^2 + \sigma_{\xi_h}^2 - 2\mathrm{Cov}(\omega_{s,t}, \xi_{ts}). \tag{3.4}$$

Because the covariance term in (3.4) does not depend on the forecast horizon h, it is straightforward to see that

$$\mathrm{MSFR}_{t,sm} \le \mathrm{MSFR}_{t,sl} \Leftrightarrow \sum_{i=m}^{l-1} \sigma_{\omega_i}^2 \ge \sigma_{\xi_m}^2 - \sigma_{\xi_l}^2. \tag{3.5}$$

That is, $\mathrm{MSFR}_{t,sh}$ increases with horizon h (for $h > s$) when (i) the variance of irrational forecast error decreases as the horizon shrinks or (ii) the variance of irrational forecast error increases as the horizon shrinks but the increment is less than information accumulation. This result explains why, more often in empirical studies, MSFE and MSFR share common patterns across forecast horizons.

We now focus on the variance of the forecasts. From the decomposition $e_{th} = y_t - \hat{y}_{t|t-h}$, where e_{th} is given in (2.4), it is straightforward to see that $\mathrm{Var}(\hat{y}_{t|t-h}) = \mathrm{Var}(y_t) - \sum_{i=0}^{h-1} \sigma_{\omega_i}^2 + \sigma_{\xi_h}^2 - 2\,\mathrm{Cov}(\omega_{h,t}, \xi_{th})$. Therefore, for forecasts made at the short horizon ($h = s$) and at the long horizon ($h = l$), it is the case that

$$\begin{aligned} \mathbb{E}(\hat{y}_{t|t-s}^2) \; &\ge \mathbb{E}(\hat{y}_{t|t-l}^2) \Leftrightarrow \sum_{i=s}^{l-1} \sigma_{\omega_i}^2 \ge \sigma_{\xi_l}^2 - \sigma_{\xi_s}^2 \\ &+ 2\mathrm{Cov}(\omega_{l,t}, \xi_{tl}) - 2\mathrm{Cov}(\omega_{s,t}, \xi_{ts}), \end{aligned} \tag{3.6}$$

which is stronger than the conditions in (3.3) and (3.5) for the MSFE and MSFR, respectively. In particular, to observe monotonic patterns in forecast variances, the rational forecast revision component, $\sum_{i=s}^{l-1} \sigma_{\omega_i}^2$, must now be greater than the difference in the variances of irrational components between a long and a short horizons plus twice the difference in the covariances between the irrational errors and the associated unanticipated shocks.

If the irrational forecast error is only a noise irrelevant to the target (i.e., $\mathrm{Cov}(\omega_{h,t}, \xi_{th}) = 0$), and if forecast revisions were made to reduce the noise (i.e., $\sigma_{\xi_l}^2 > \sigma_{\xi_m}^2 > \sigma_{\xi_s}^2$), then incorporating newly available information (i.e., $\sum_{i=s}^{l-1} \sigma_{\omega_i}^2$) leads us to observe a monotonic pattern in MSFE and MSFR for rational fixed-event forecasts, even if the noise in the revised forecasts outweighs the amount of newly revealed information; thus, forecasts are subrational.

Under Assumption 2.1, along with (2.1)–(2.4), we can also derive the conditions that guarantee monotonicity in the covariances related to forecasts, forecast errors, and the target. Indeed, it is straightforward to see under those conditions that for any forecast horizon h, we have:

$$\mathrm{Cov}(y_t, \hat{y}_{t|t-h}) = \mathrm{Var}(y_t) - \sum_{i=0}^{h-1} \sigma_{\omega_i}^2 - \mathrm{Cov}(\omega_{h,t}, \xi_{th}), \tag{3.7}$$

$$\mathrm{Cov}(y_t, e_{th}) = \sum_{i=0}^{h-1} \sigma_{\omega_i}^2 + \mathrm{Cov}(\omega_{h,t}, \xi_{th}). \tag{3.8}$$

Therefore, from (2.4), the covariance between the short- and any longer horizon forecasts ($h > s$) is given by

$$\mathrm{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-h}) = \mathrm{Var}(y_t) - \sum_{i=0}^{h-1} \sigma_{\omega_i}^2 - \mathrm{Cov}(\omega_{h,t}, \xi_{th}). \tag{3.9}$$

The following two equivalences hold under (3.9):

$$\begin{aligned} \mathrm{Cov}(y_t, \hat{y}_{t|t-s}) \; &\ge \mathrm{Cov}(y_t, \hat{y}_{t|t-l}) \wedge \mathrm{Cov}(y_t, e_{ts}) \\ \le \mathrm{Cov}(y_t, e_{tl}) \; &\Leftrightarrow \sum_{i=s}^{l-1} \sigma_{\omega_i}^2 \ge \mathrm{Cov}(\omega_{s,t}, \xi_{ts}) - \\ &\mathrm{Cov}(\omega_{l,t}, \xi_{tl}), \\ \mathrm{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-m}) \; &\ge \mathrm{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-l}) \\ \Leftrightarrow \sum_{i=m}^{l-1} \sigma_{\omega_i}^2 \; &\ge \mathrm{Cov}(\omega_{m,t}, \xi_{tm}) - \mathrm{Cov}(\omega_{l,t}, \xi_{tl}). \end{aligned} \tag{3.10}$$

Under (3.10), monotonic patterns in covariances between forecasts, forecast errors, and the target are observed. We see that for the three covariances to exhibit monotonic patterns, the accumulated variance of new information between two updating points must be greater than the difference in the covariances between irrational error and contemporaneous information at the two horizons. In empirical applications where the irrational forecast error is simply noise or a measurement error, the covariance of irrational forecast error and contemporaneous information is zero. Therefore, the above conditions are always met, and monotonic patterns of covariance are observed over forecast horizons, even if multi-horizon forecasts contain a substantial amount of forecast error irrelevant to the target. However, irrational forecast errors may occur due to smoothing newly available information (i.e., $\mathrm{Cov}(\omega_{h,t}, \xi_{th}) > 0$). In this case, when the forecaster's smoothing ability is weakened as the forecast horizon approaches to the target date, the accumulated variance of information over the two updating points always prevails. As a result, forecasts and associated errors covariate with the target in the same monotonic patterns as rational forecasts. In a scenario where forecasters always overreact to news (i.e., $\mathrm{Cov}(\omega_{h,t},\xi_{th})<0$), the strengthening impact of overreaction as horizon shortens leads to the accumulated variance of information over the two updating points to prevail, so that the covariances in relation to such subrational forecasts evolves across horizons in the same pattern as rational forecasts.

We can also establish bounds on the covariances related to forecast revisions that are made due to both information adoption and adjustment to the irrational forecast component. In **Panel C** of Table A1 in Appendix A, we show that $\mathrm{Cov}(\hat{y}_{t|t-s},d_{t,sm})\leq\mathrm{Cov}(\hat{y}_{t|t-s},d_{t,sl})$ if and only if $\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\mathrm{Cov}(\omega_{m,t},\xi_{tm})-\mathrm{Cov}(\omega_{l,t},\xi_{tl})$, while $\mathrm{Cov}(e_{tm},d_{t,sm})\leq\mathrm{Cov}(e_{tl},d_{t,sl})$ if and only if $\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_m}-\sigma^2_{\xi_l}$ for any forecast horizons $l>m>s$. These conditions are obviously the same as those established previously. In the absence of an irrational error, forecasts are revised only for updating the information that has occurred over the period since a previous forecast was made. Intuitively, the longer it takes to release an updated forecast, the more information it is expected to encapsulate, thereby making the newly released forecasts more relevant to the revisions. However, in the presence of irrational error, the relevance between revised forecasts and revisions does not always increase with the forecast horizon, as it depends on how irrational forecast error and newly acquired information change over forecast horizons. The covariance between the forecast error made at a longer h horizon and the revision occurring later at a short horizon s, however, follows a nondecreasing pattern in h under a requirement of information adoption compared with the difference in the variance of irrational forecast error only. The type and the size of irrational forecast error do not affect this monotonic property.

Patton and Timmermann (2012) propose that the variance of rational forecast revisions is bounded by its relevance to the target; that is, we must have $\mathrm{Var}(d_{t,sl})\leq 2\,\mathrm{Cov}(y_t,d_{t,sl})$ and $\mathrm{Var}(d_{t,ml})\leq 2\,\mathrm{Cov}\ (\hat{y}_{t|t-s},d_{t,ml})$ for any $l>m>s$. In our bi-error model, these two conditions are equivalent to $\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_s}+\sigma^2_{\xi_l}-2\,\mathrm{Cov}(\omega_{l,t},\xi_{tl})$ and $\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_m}+\sigma^2_{\xi_l}-2\,\mathrm{Cov}(\omega_{l,t},\xi_{tl})$, respectively. Again, the conditions are similar to the ones previously discussed. These bounds clearly require the accumulated unanticipated shocks to be larger in the case of overreaction than in the case of underreaction. This is because the variance of bi-error forecast revisions is higher when the irrational component is made due to overreacting to contemporaneous news than when it is made due to smoothing contemporaneous news; see Equation (3.4). The condition $\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_m}+\sigma^2_{\xi_l}-2\,\mathrm{Cov}(\omega_{l,t},\xi_{tl})$ suits the scenario of a target event being unobserved. The upper bound is then the covariance value of the revision and the shortest-horizon forecast.

3.2 | Remuneration of revision effort by revision reward

In this section, we are interested in the condition under which the forecast revision effort is compensated by the revision reward. Isiklar and Lahiri (2007) and Lahiri (2012) advocate an alternative method for testing rational fixed-event forecasts, which is also based on the second moments of forecasts. They compare improvement in forecast accuracy, measured by a reduction in the MSFE, with the amount of effort that forecasters make in the revision process, measured by the MSFR. When revisions efficiently incorporate newly available information, revision effort can be fully rewarded by an improvement in the forecast. Thus, they suggest that the difference between the two measures may indicate how forecasters react to new information. Here, we characterize the necessary and sufficient conditions in the bi-error model (that incorporates irrational forecast error) under which the forecast revision effort is fully rewarded by forecast improvement.

Consider forecasts made at a long horizon h and subsequently revised forecasts made at a short horizon $s<h$. From (3.2) to (3.4), the difference between $\mathrm{MSFR}_{t,sh}$ and $-\Delta\mathrm{MSFE}_{t,sh}$ is

$$\mathrm{MSFR}_{t,sh}-(-\Delta\mathrm{MSFE}_{t,sh})=2\sigma^2_{\xi_s}-2\mathrm{Cov}(\omega_{s,t},\xi_{ts}).\quad(3.11)$$

We see that this difference depends on the size and property of the irrational forecast error in the revised forecasts. If the irrational forecast error exists in the revised forecast but is simply a noise uncorrelated with the target event, (i.e., $\sigma_{\xi_s}\neq 0$ and $\mathrm{Cov}(\omega_{s,t},\xi_{ts})=0$), then the left-hand side of (3.11) is always greater than zero. Therefore, the revision effort cannot be fully compensated. The same conclusion can be drawn if the irrational error of revised forecasts is made due to forecasters' overreaction to contemporaneous news (i.e., if $\sigma_{\xi_s}\neq 0$ and $\mathrm{Cov}(\omega_{s,t},\xi_{ts})<0$). This latter implication is consistent with Lahiri (2012). However, it is possible to observe that revision award exceeds revision effort. If forecasters smooth new information when making revised forecasts, the irrational component offsets the true amount of news that should be incorporated into the revision. This may result in the MSFR being reduced due to the smoothing behavior in forecast revisions. Because the forecast improvement measure, $-\Delta\mathrm{MSFE}_{t,sh}$, is not affected by this type of irrational forecast error ξ_{ts}, it is possible that news-smoothing leads to $\text{MSFR}_{t,sh} \leq -\Delta\text{MSFE}_{t,sh}$.

Isiklar and Lahiri (2007) find that MSFR is significantly less than $-\Delta$MSFE in private sector forecast data for real GDP growth of multiple countries. Our bi-error model provides a framework within which one can characterize the condition under which this occurs. By noting that $\text{Cov}(\omega_{s,t}, \xi_{ts}) = \theta_s \sigma_{\omega_s} \sigma_{\xi_s}$, where θ_s is the correlation between $\omega_{s,t}$ and ξ_{ts} (with $\theta_s > 0$ for news-smoothing), along with the fact that $\sigma_{\xi_s} > 0$, it is clear from (3.11) that

$$\text{MSFR}_{t,sl} \leq -\Delta\text{MSFE}_{t,sl} \Longleftrightarrow \theta_s \geq \frac{\sigma_{\xi_s}}{\sigma_{\omega_s}}. \tag{3.12}$$

This means that for forecast improvement to at least compensate revision effort, the irrational-to-news ratio, that is, $\frac{\sigma_{\xi_s}}{\sigma_{\omega_s}}$, must not exceed the correlation between the irrational forecast error and contemporaneous news. Intuitively, a small value of θ_s may indicate that smoothing to contemporaneous news $\omega_{s,t}$ is a less dominant attribute than the irrational forecast error ξ_{ts}, and a larger amount of variation in ξ_{ts} is noise that is irrelevant to the target. As discussed earlier, if the irrational forecast error is purely a random noise, the forecast revision effort is less than the revision reward. Thus, in the case of smoothing contemporaneous news, the noise component of ξ_{ts} needs to be much smaller than the news in order to possibly lead to forecast improvement to be at least as large as forecast effort. Clearly, the equality between $-\Delta\text{MSFE}_{t,sl}$ and $\text{MSFR}_{t,sl}$ is a necessary condition for rational forecast revisions. Rejection of this equality will suggest the existence of an irrational error component in revised forecasts that may be caused by either overreaction to news, underreaction to news, or noise irrelevant to news. However, retaining the equality does not necessarily guarantee rational revisions; it can be a result of smoothing newly arrived news when $\frac{\sigma_{\xi_s}}{\sigma_{\omega_s}} = \theta_s$.

Remark 1. (**The Nordhaus-type test in the bi-error model**). Clements and Taylor (2001) provide a theoretical implication of smoothing news in the Nordhaus-type test. Let

$$d_{t,h} = \hat{y}_{t|t-h} - \hat{y}_{t|t-(h+1)}, \tag{3.13}$$

where $h = 1, \ldots, H-1$ denote the difference between the forecast of y_t made at horizon h and the forecast of y_t made at a previous horizon $h+1$. Thus, $d_{t,h}$ is the revision made between two adjacent updating points $t-h$ and $t-h-1$. In the absence of irrational forecast error ξ_{th} in (2.3), Clements and Taylor (2001) show that forecast rationality implies that the sequence of revisions $\{d_{t,h}\}, h = 1, \ldots, H-1$ is a white-noise. Therefore, the test for weak efficiency of forecasts can be formulated as testing for the null hypothesis of $\beta_h = 0$ in the regression

$$d_{t,h} = \beta_h d_{t,h+1} + \zeta_{t,h} \tag{3.14}$$

with $\zeta_{t,h}$ being a white-noise error term. In our bi-error model (i.e., when ξ_{th} is present in 2.3), it is easy to see from (2.5) and (3.13) that

$$d_{t,h} = \omega_{h,t} + \xi_{t(h+1)} - \xi_{th}, \tag{3.15}$$

$$d_{t,h+1} = \omega_{h+1,t} + \xi_{t(h+2)} - \xi_{t(h+1)}. \tag{3.16}$$

Therefore, we have $\text{Cov}(d_{t,h}, d_{t,h+1}) = \theta_{h+1}\sigma_{\omega_h} +1\sigma_{\xi_{h+1}} - \sigma^2_{\xi_{h+1}}$, where the first term on the right hand side is the covariance between the newly arrived information at the time when $(h+1)$-ahead forecast is made, $\omega_{h+1,t}$, and the irrational forecast error of the $(h+1)$-ahead forecast, $\xi_{t(h+1)}$. The second term $\sigma^2_{\xi_{h+1}}$ is the variance of the irrational forecast error $\xi_{t(h+1)}$.

Two implications regarding the process of revision can be drawn from our bi-error model. The first is that the sequence of revisions $\{d_{t,h}\}$ follows a negative $AR(1)$ process if $\theta_{h+1} < \frac{\sigma_{\xi_{h+1}}}{\sigma_{\omega_{h+1}}}$. This inequality must be satisfied if forecasters overreact to news ($\theta_{h+1} < 0$). Yet, other scenarios can also validate the inequality and thus result in a negative β_h in Equation (3.14). For example, when irrational forecast error is uncorrelated with the newly arrived information ($\theta_{h+1} = 0$) or when forecasters smooth news but $0 < \theta_{h+1} < \frac{\sigma_{\xi_{h+1}}}{\sigma_{\omega_{h+1}}}$. Second, the absence of irrational forecast error is not the only reason for observing revisions that follow a white-noise process. If forecasters smooth news ($\theta_{h+1} > 0$) in a way that $\theta_{h+1} = \frac{\sigma_{\xi_{h+1}}}{\sigma_{\omega_{h+1}}}$, then $\text{Cov}(d_{t,h}, d_{t,h+1}) = 0$ so that $\beta_h = 0$. Consequently, the estimated value of β_h in the regression (3.14) will be close to zero. Therefore, when testing whether $H_0 : \beta_h = 0$ in (3.14), retaining H_0 does not simply suggest rational forecasts.

3.3 | Conditions for internal consistency

Let $\text{Cov}(\omega_{k,t}, \xi_{tk}) = \theta_k \sigma_{\omega_k} \sigma_{\xi_k}$ for any $k \in \{s, l, m\}$. Theorem 3.1 gives the necessary and sufficient conditions for internal consistency of fixed-event forecasts in the bi-error model in which forecasts may contain a substantial amount of the irrational component.

Theorem 3.1. *Suppose that (2.1)–(2.3) and Assumption 2.1 are satisfied. Then for $l > m > s$, the necessary and sufficient conditions for internal consistency of fixed-event forecasts are*

$$(a)\ \sum_{i=s}^{l-1}\sigma_{\omega_i}^2 \geq \max\{\sigma_{\xi_s}^2-\sigma_{\xi_l}^2,\ -(\sigma_{\xi_s}^2-\sigma_{\xi_l}^2)+2\theta_l\sigma_{\omega_l}\sigma_{\xi_l} -2\theta_s\sigma_{\omega_s}\sigma_{\xi_s},\ \theta_s\sigma_{\omega_s}\sigma_{\xi_s}-\theta_l\sigma_{\omega_l}\sigma_{\xi_l},\ \sigma_{\xi_s}^2+\sigma_{\xi_l}^2 -2\theta_l\sigma_{\omega_l}\sigma_{\xi_l}\} \wedge \sum_{i=m}^{l-1}\sigma_{\omega_i}^2 \geq \max \left\{\sigma_{\xi_m}^2-\sigma_{\xi_l}^2, \theta_m\sigma_{\omega_m}\sigma_{\xi_m}-\theta_l\sigma_{\omega_l}\sigma_{\xi_l},\ \sigma_{\xi_m}^2+\sigma_{\xi_l}^2-2\theta_l\sigma_{\omega_l}\sigma_{\xi_l}\right\}$$

$$(b)\ 0<\theta_s\leq 1 \ \wedge\ \theta_s \geq \frac{\sigma_{\xi_s}}{\sigma_{\omega_s}}.$$

The proof of Theorem 3.1 follows from the derivations in Appendix B; therefore, it is omitted. The theorem states that internal consistency may still hold in multi-horizon fixed-event forecasts that incorporate a substantial amount of the irrational component. The practical implication of this result is that forecasts that fail to reject the rationality test based on the monotonicity of the second moments and the equality between revision effort and revision reward may not be rational, even if they are internally consistent. Because the conditions required for observing monotonicity can be different for various second moments, it is highly likely that mixed outcomes will be observed in empirical applications of fixed-event forecasts.

4 | EMPIRICAL APPLICATION

We explore the forecast error characteristics and internal consistency properties of a panel of SPF forecasts of the inflation rate. In Section 4.1, we discuss the data set and the strategy employed to estimate the variances, as well as the correlation over forecast horizons, of new forecast information and irrational forecast errors. We then present the main results on the internal consistency conditions and discuss their implications in Section 4.2.

4.1 | Data and methodology

The sample of SPF inflation forecasts consists of 146 target quarters from 1981Q4 to 2018Q1, but most of the forecasters do not report their forecasts for all target quarters. The initial forecast for each target quarter is made 4 quarters ahead, and a sequence of three forecasts is updated every quarter, with the shortest horizon forecast at 1 quarter before the target quarter, that is, $h=1,2,3,4$. The forecast values are for annualized quarter-over-quarter U.S. headline CPI inflation,[6] and we compute the realized values of CPI inflation based on the historical CPI published in the vintage of 2018Q2.[7]

Davies and Lahiri (1995) propose a methodology to estimate the rational and individual specific idiosyncratic errors and their variances for a panel of multiple horizon forecasts. However, they assume that the process that governs new information arriving between consecutive horizons (i.e., the news) is homoskedastic (constant variance across forecast horizons) and that the idiosyncratic error may be subject to individual heterogeneity but has a constant variance over horizons. Our bi-error model allows us to relax these strong assumptions by assuming different variances across forecast horizons. We thus modify the estimation strategy of Davies and Lahiri (1995) to allow for horizon specific $\sigma_{\omega_h}^2$ and $\sigma_{\xi_h}^2$.

As the methodology developed in this paper applies to unbiased forecasts, and it is well documented that the forecasters in SPF data may be subject to individual specific bias, we first estimate the individual bias and then remove it from the forecast error made for a target time t at horizon h, so that the remaining forecast errors have a bi-error structure. Specifically, let ϕ_j denote individual $j\in\{1,2,\ldots,N\}$, where N is the total number of forecasters. Averaging individual forecast errors over horizons and target time yields an estimated individual bias:

$$\hat{\phi}_j=\frac{1}{TH}\sum_{t=1}^{T}\sum_{h=1}^{H}\left(y_t-\hat{y}_{j,t|t-h}\right). \tag{4.1}$$

The bias-adjusted forecast error made at horizon h for a target time t by forecaster j is thus $e_{j,th}=y_t-\hat{y}_{j,t|t-h}-\widehat{\phi_j}$, and the rational forecast error is estimated by averaging $e_{j,th}$ over j:

$$\hat{\lambda}_{th}=\frac{1}{N}\sum_{j=1}^{N}e_{j,th}. \tag{4.2}$$

Given Equation (2.2), the newly arrived unanticipated information for forecasts made at horizon $h, \omega_{h,t}$, is estimated as

[6]The individual forecasts can be downloaded at https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files/cpi, The Federal Reserve Bank of Philadelphia (2021a).

[7]Real-time CPI can be download at https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files/cpi, The Federal Reserve Bank of Philadelphia (2021b). Quarterly CPI values are firstly computed by averaging monthly CPI within the quarter and are then transformed to quarter-over-quarter inflation, expressed in annualized percentage points.

$$\hat{\omega}_{h,t} = \hat{\lambda}_{t(h+1)} - \hat{\lambda}_{th} \text{ for } h = 1,..,H-1. \tag{4.3}$$

Following Davies and Lahiri (1995), the irrational forecast errors for individual forecaster j at horizon h and target time t, $\xi_{j,th}$, are estimated as

$$\hat{\xi}_{j,th} = e_{j,th} - \hat{\lambda}_{th}. \tag{4.4}$$

Under the assumption that $E\left(\xi^2_{j,th}\right) = \sigma^2_{\xi_{j,h}}$ (i.e., the variance of irrational forecast error is allowed to vary across forecast horizons), consistent estimates of $\sigma^2_{\xi_{j,h}}$ can be obtained by regressing $\hat{\xi}^2_{j,th}$ on a constant for each horizon h. Similarly, as $E(\omega^2_{h,t}) = \sigma^2_{\omega_h}$ for $h = 1,...,H-1$, the variance of newly arrived information for forecasts made at horizon $h, \sigma^2_{\omega_h}$, can be consistently estimated by regressing $\hat{\omega}^2_{h,t}$ on a constant.[8] From Theorem 3.1, the internal consistency conditions depend on the variances of newly arrived information and irrational forecast errors, as well as their correlation $\theta_{j,h}$. We estimate $\theta_{j,h}$ directly as $\hat{\theta}_{j,h} = \hat{\sigma}^{-1}_{\xi_h} \hat{\sigma}^{-1}_{\omega_h} \mathrm{Cov}(\hat{\xi}_{j,th}, \hat{\omega}_{h,t})$, which varies across forecast horizons and is specific to each forecaster.

We select 22 forecasters who report at least one forecast for no less than 50% of the targeted quarters. A total of 12,848 forecasts are expected if each of the 22 forecasters reports all 4 forecasts for every targeted quarter within the sample. However, we only have 6725 forecasts, and about half of the forecasts are missing. Following Davies and Lahiri (1995), when estimating various types of forecast errors by taking averages, such as in Equations (4.1) and (4.2), we replace the missing values with zeros and divide the summation by the number of non-missing values. To obtain the estimated values of variances and correlations, we compress the data by removing the targeted quarters that are associated with any missing forecasts. More specifically, the newly arrived information, $\hat{\omega}_{h,t}$, estimated using Equation (4.3) has a dimension of 146×3 with no missing values. However, as individual forecasters report forecasts for a subset of targeted quarters, the estimated irrational error for each forecaster, $\hat{\xi}_{j,th}$, computed as in (4.4), has many rows that contain missing values. To ensure that the variances ($\sigma^2_{\omega_h}$ and $\sigma^2_{\xi_{j,h}}$) and the correlations ($\theta_{j,h}$) are estimated from the same sample of targeted quarters, we compress the sample errors, $\hat{\xi}_{j,th}$, by removing the rows that contain any missing value and construct a matrix of newly arrived information $\hat{\omega}_{j,ht}$ where the targeted quarters match those in the compressed $\hat{\xi}_{j,th}$ for each forecaster j.[9]

4.2 | Main results

4.2.1 | Heterogeneity of irrational forecast error across horizons

Table 1 reports the estimated variances of the irrational error ($\hat{\xi}_{j,h}$) and newly arrived information ($\hat{\omega}_{j,h}$) across forecast horizons h, as well as their correlations for the 22 forecasters. The first column of the table reports the forecasters' individual identifier (ID), and the other columns contain the estimated variances and correlations. The numbers in parentheses are the standard errors associated with the reported estimates.

First, the table shows that the variance of the irrational error is relatively large across horizons, indicating that the reported forecasts contain sizeable irrational forecast errors for all 22 forecasters, irrespective of the forecast horizon. As seen, there is evidence of a strong heterogeneity in the irrational forecast error across horizons as its estimated variance varies substantially with forecast horizon h. This finding contrasts with Davies and Lahiri (1995), who assume that the irrational forecast error is homoskedastic across horizons. The key assumption of our bi-error model, that is the variance of the irrational forecast error may depend on forecast horizons, is sustained in the data. Considering columns 2–5 of Table 1 and moving from the left to the right, we see that for many forecasters, the estimated variance of the irrational error does not follow a decreasing pattern as horizon shrinks. For instance, consider forecaster 518. The estimated variance of the irrational error for this forecaster when $h = 1$ (1 quarter-ahead forecast) is twice the size of the estimated variance when $h = 2$ (2 quarters-ahead forecast), and more than 3 times that of the estimated variance when $h = 4$ (4 quarters-ahead forecast). This increasing pattern of the variance of the irrational error with h decreasing hinges on the possibility that some of the internal consistency conditions discussed in Section 3 may not hold, especially if the information accumulation during the forecast updating process (rational forecast revisions) is less than the increment in the variance of irrational forecast errors.

Despite the existence of sizeable irrational forecast errors in the CPI forecasts across horizons, it is also evident from Table 1 that the revised forecasts contain small but significant newly arrived information, as shown by

[8]Davies and Lahiri (1995) assume that the variance of disaggregated news is constant across horizons. This implies that $E(\lambda^2_{th}) = h\sigma^2_{\omega}$. As such, they estimate σ^2_{ω} by regressing the pooled TH-dimensional vector $\hat{\lambda}^2_{th}$ on the pooled vector of horizon indices $h = 1,...,H$.

[9]The number of target quarters in $\hat{\omega}_{j,ht}$ and $\hat{\xi}_{j,th}$ for each forecaster j is reported in Table C1, Appendix C.

TABLE 1 Estimation results of forecast errors for individual forecasters

ID	$h=4$ $\hat{\sigma}^2_{\xi j,4}$	$h=3$ $\hat{\sigma}^2_{\xi j,3}$	$h=2$ $\hat{\sigma}^2_{\xi j,2}$	$h=1$ $\hat{\sigma}^2_{\xi j,1}$	$h=3$ $\hat{\sigma}^2_{\omega j,3}$	$h=2$ $\hat{\sigma}^2_{\omega j,2}$	$h=1$ $\hat{\sigma}^2_{\omega j,1}$	$h=3$ $\hat{\theta}_{j,3}$	$h=2$ $\hat{\theta}_{j,2}$	$h=1$ $\hat{\theta}_{j,1}$
20	1.049	0.937	0.982	0.931	0.050	0.068	0.088	−0.439**	−0.420**	−0.422**
	(0.248)	(0.195)	(0.201)	(0.178)	(0.014)	(0.015)	(0.018)			
40	0.293	0.328	0.373	0.272	0.295	0.424	0.290	0.172	0.167	0.319
	(0.066)	(0.089)	(0.079)	(0.061)	(0.134)	(0.217)	(0.117)			
65	0.385	0.370	0.317	0.353	0.087	0.076	0.097	−0.263**	−0.236**	−0.296**
	(0.078)	(0.088)	(0.062)	(0.069)	(0.023)	(0.018)	(0.024)			
84	0.224	0.207	0.265	0.222	0.224	0.172	0.244	0.504**	0.145	−0.437**
	(0.060)	(0.041)	(0.102)	(0.046)	(0.053)	(0.043)	(0.108)			
407	0.277	0.309	0.249	0.244	0.070	0.063	0.121	0.270**	0.196	0.091
	(0.083)	(0.085)	(0.053)	(0.042)	(0.014)	(0.014)	(0.072)			
411	0.329	0.220	0.209	0.532	0.082	0.067	0.175	−0.074	0.090	−0.091
	(0.080)	(0.038)	(0.053)	(0.124)	(0.020)	(0.016)	(0.088)			
420	0.282	0.226	0.456	0.251	0.063	0.039	0.072	0.042	0.412**	0.098
	(0.071)	(0.061)	(0.281)	(0.093)	(0.022)	(0.011)	(0.016)			
421	0.261	0.218	0.134	0.156	0.064	0.064	0.121	0.203	0.014	−0.080
	(0.033)	(0.027)	(0.021)	(0.029)	(0.013)	(0.012)	(0.061)			
426	0.217	0.305	0.187	0.347	0.068	0.047	0.108	0.010	0.044	0.008
	(0.034)	(0.063)	(0.029)	(0.079)	(0.013)	(0.010)	(0.052)			
428	0.210	0.265	0.291	0.253	0.053	0.057	0.070	0.264	0.112	0.150
	(0.050)	(0.063)	(0.061)	(0.058)	(0.012)	(0.013)	(0.013)			
429	0.281	0.223	0.355	0.253	0.052	0.080	0.084	0.241	−0.127	−0.160
	(0.058)	(0.047)	(0.196)	(0.061)	(0.012)	(0.016)	(0.019)			
433	0.178	0.153	0.167	0.169	0.051	0.057	0.063	0.181	0.147	0.162
	(0.047)	(0.026)	(0.025)	(0.037)	(0.009)	(0.011)	(0.010)			
446	0.120	0.099	0.135	0.218	0.050	0.041	0.133	0.147	0.354**	0.613**
	(0.024)	(0.015)	(0.030)	(0.091)	(0.010)	(0.009)	(0.068)			
456	0.416	0.339	0.259	0.279	0.047	0.036	0.056	0.163	0.325**	−0.014
	(0.092)	(0.066)	(0.039)	(0.058)	(0.011)	(0.009)	(0.014)			
463	0.185	0.173	0.188	0.332	0.059	0.048	0.135	0.188	0.268**	0.284**
	(0.060)	(0.059)	(0.039)	(0.090)	(0.014)	(0.010)	(0.068)			
472	0.185	0.241	0.310	0.861	0.066	0.050	0.075	0.274	−0.025	−0.271
	(0.036)	(0.073)	(0.066)	(0.267)	(0.018)	(0.012)	(0.019)			
484	0.119	0.146	0.256	0.474	0.050	0.053	0.154	0.007	−0.139	0.465**
	(0.024)	(0.028)	(0.070)	(0.204)	(0.011)	(0.012)	(0.080)			
504	2.480	2.264	2.122	3.471	0.061	0.036	0.149	−0.151	0.147	0.122
	(0.715)	(0.437)	(0.650)	(1.204)	(0.018)	(0.011)	(0.092)			
507	0.262	0.296	0.276	0.304	0.044	0.045	0.080	0.287	0.188	0.227
	(0.056)	(0.063)	(0.066)	(0.054)	(0.010)	(0.011)	(0.020)			
508	0.276	0.173	0.371	0.482	0.026	0.013	0.103	0.026	−0.468	−40.417
	(0.141)	(0.066)	(0.161)	(0.249)	(0.010)	(0.004)	(0.048)			
510	0.743	0.384	0.669	0.742	0.058	0.040	0.138	−0.132	−0.003	0.192
	(0.250)	(0.115)	(0.166)	(0.228)	(0.015)	(0.010)	(0.074)			

(Continues)

TABLE 1 (Continued)

ID	$h=4$ $\hat{\sigma}^2_{\xi_{j,4}}$	$h=3$ $\hat{\sigma}^2_{\xi_{j,3}}$	$h=2$ $\hat{\sigma}^2_{\xi_{j,2}}$	$h=1$ $\hat{\sigma}^2_{\xi_{j,1}}$	$h=3$ $\hat{\sigma}^2_{\omega_{j,3}}$	$h=2$ $\hat{\sigma}^2_{\omega_{j,2}}$	$h=1$ $\hat{\sigma}^2_{\omega_{j,1}}$	$h=3$ $\hat{\theta}_{j,3}$	$h=2$ $\hat{\theta}_{j,2}$	$h=1$ $\hat{\theta}_{j,1}$
518	0.164	0.212	0.253	0.509	0.056	0.037	0.060	0.059	−0.102	−0.088
	(0.033)	(0.056)	(0.063)	(0.138)	(0.017)	(0.010)	(0.013)			

Note: Column 1 reports the ID of forecasters. Columns 2–5 report the estimated variance of irrational forecaster error across horizons $h=4,3,2$, and 1 quarters, and Columns 6–8 report the estimated variance of newly arrived information for the revised forecasts made at horizons $h=3,2$ and 1 quarters. Estimated standard errors of the variance estimates are in parentheses. The last three columns report the estimated correlations between irrational forecast error and newly arrived information for the revised forecasts. Numbers with double asterisks indicate that the correlation estimate is significant at 5% level of significance (the p values are computed using the MATLAB *corrcoef* function).

the variance estimates of news, $\hat{\sigma}^2_{\omega_{j,h}}$ for $h=1,2$ and 3. In addition, comparing $\hat{\sigma}^2_{\omega_{j,h}}$ across horizons, we see that the news incorporated into the forecasts made at the shortest horizon seems more volatile than those that have gone into the previously revised forecasts. However, for most forecasters, the estimated variance of newly arrived information is much lower than that of irrational forecast error at the same forecast horizon ($\hat{\sigma}^2_{\omega_{j,h}} < \hat{\sigma}^2_{\xi_{j,h}}$). The only exceptions are forecasters 40 and 84, who have at least one of the irrational-to-news ratios, $\frac{\hat{\sigma}_{\xi_h}}{\hat{\sigma}_{\omega_h}}$, less than 1. From (3.12), we know that values of the irrational-to-news ratio above 1 imply that the revision effort (measured by MSFR) cannot be remunerated by the revision reward (measured by improvement in MSFE). The estimation results suggest that for most SPF inflation forecasters in the sample, their revision effort cannot be remunerated by the revision reward. Furthermore, as discussed in Remark 1, the slope coefficient in the Nordhaus-type regression for forecast revisions is negative if the irrational-to-news ratio is greater than 1. Therefore, a Nordhaus-type test of forecast rationality is likely to be rejected. We will detail these results further in Section 4.2.2.

Now, focusing on the last three columns of Table 1, we see that the estimated correlations between irrational forecast error and newly arrived information are relatively high for many forecasters. Again, this indicates a presence of irrational forecast error in the CPI forecasts. However, the irrational forecast error for some forecasters can be simply considered as a noise uncorrelated with newly arrived information. For example, the estimated correlation is very small across horizons for forecasters 420, 426 and 518. We also observe that although some forecasters (for example 20 and 65) consistently overreact to news over horizons (with significantly negative $\hat{\theta}_h$ over h), the irrational forecast behavior of others may change over horizons. For example, the estimated correlation between irrational forecast error and new information for forecaster 84 has changed from a large positive value when $h=3$ to a large negative value as the forecast horizon shrinks to $h=1$, thus suggesting that forecaster 84 smooths news when revising forecasts 3 quarters ahead but overreacts to news when revising them at just 1 quarter before the target. One of the main contributions of our bi-error model with horizon specific variances and correlations is that the model allows us to capture and quantify how the irrational behavior of forecasters evolves across forecast horizons.

4.2.2 | Internal consistency in the inflation forecasts

In this section, we first investigate whether the internal consistency conditions may still be satisfied for these inflation forecasts that consist of a substantial amount of irrational forecast error. These conditions are discussed in Equations (3.3), (3.5), (3.6), and (3.12). Table 2 reports the estimated difference between the left and the right hand sides of each of the four equations. The values of the differences are calculated based on the variance and correlation estimates of the irrational error and new information reported in Table 1. Intuitively, large positive values indicate that internal consistency conditions are likely to be satisfied, whereas large negative values signal their rejection. As seen in Table 2, some forecasters meet at least a subset of internal consistency conditions; see, for example, forecasters 40 and 84. Moving to the last three columns for the remuneration conditions, we see that $\hat{\theta}_s - \frac{\hat{\sigma}_{\xi_s}}{\hat{\sigma}_{\omega_s}} < 0$ for all forecasters. This result implies that for all forecasters, the improvement in forecast accuracy is not sufficient enough to compensate the revision effort. As discussed in Section 4.2.1, these large negative values were expected as some forecasters overreact to news (e.g., forecasters 20 and 65 with $\hat{\theta}_s < 0$), and most forecasters consistently have an irrational-to-news ratio higher than 1 across forecast horizons.

In Remark 1 of Section 3.2, we have discussed that overreaction to news, large irrational-to-news ratios, and irrational errors with zero correlation to news can all lead to a negative slope in the Nordhaus-type regression model and thus a rejection of forecast rationality. To

TABLE 2 Examining internal consistency conditions for individual forecasters

ID	For MSFE $\sum_{i=1}^{l-1}\hat{\sigma}^2_{\omega_i}-\left(\hat{\sigma}^2_{\xi_1}-\hat{\sigma}^2_{\xi_l}\right)$ $l=2$	$l=3$	$l=4$	For MSFR $\sum_{i=m}^{l-1}\hat{\sigma}^2_{\omega_i}-\left(\hat{\sigma}^2_{\xi_m}-\hat{\sigma}^2_{\xi_l}\right)$ $m=2$ $l=3$	$m=2$ $l=4$	$m=3$ $l=4$	For variances $\sum_{i=1}^{l-1}\hat{\sigma}^2_{\omega_i}-\left(\hat{\sigma}^2_{\xi_l}-\hat{\sigma}^2_{\xi_1}\right)$ $-2\mathrm{Cov}(\hat{\omega}_{l,t},\hat{\xi}_{tl})$ $+2\mathrm{Cov}(\hat{\omega}_{1,t},\hat{\xi}_{t1})$ $l=2$	$l=3$	For effort vs. reward $\hat{\theta}_s-\frac{\hat{\sigma}_{\xi_s}}{\hat{\sigma}_{\omega s}}$ $s=1$	$s=2$	$s=3$
20	0.139	0.161	0.323	0.022	0.184	0.162	0.012	0.099	−3.681	−4.233	−4.759
40	0.392	0.771	1.031	0.379	0.639	0.260	0.236	0.732	−0.649	−0.771	−0.883
65	0.060	0.190	0.291	0.129	0.231	0.102	0.097	0.141	−2.207	−2.277	−2.330
84	0.287	0.401	0.642	0.115	0.356	0.241	−0.064	0.011	−1.391	−1.094	−0.456
407	0.126	0.249	0.288	0.123	0.162	0.038	0.098	0.071	−1.329	−1.790	−1.826
411	−0.148	−0.069	0.122	0.078	0.270	0.191	0.422	0.518	−1.832	−1.677	−1.711
420	0.278	0.086	0.205	−0.191	−0.072	0.119	−0.216	0.152	−1.767	−3.018	−1.849
421	0.099	0.248	0.354	0.148	0.255	0.107	0.119	0.053	−1.213	−1.431	−1.649
426	−0.051	0.113	0.093	0.165	0.145	−0.020	0.262	0.196	−1.783	−1.960	−2.104
428	0.108	0.139	0.138	0.031	0.030	−0.002	0.042	0.090	−1.756	−2.149	−1.965
429	0.187	0.134	0.244	−0.052	0.058	0.110	−0.022	0.095	−1.890	−2.237	−1.822
433	0.061	0.104	0.181	0.043	0.119	0.076	0.069	0.137	−1.477	−1.558	−1.553
446	0.049	0.054	0.125	0.005	0.076	0.070	0.373	0.480	−0.669	−1.468	−1.268
456	0.036	0.151	0.276	0.115	0.240	0.124	0.010	−0.012	−2.247	−2.354	−2.529
463	−0.008	0.024	0.095	0.033	0.103	0.070	0.348	0.425	−1.282	−1.712	−1.529
472	−0.476	−0.496	−0.486	−0.019	−0.009	0.010	0.495	0.538	−3.666	−2.523	−1.636
484	−0.064	−0.122	−0.098	−0.058	−0.035	0.023	0.655	0.785	−1.289	−2.345	−1.699
504	−1.199	−1.021	−0.744	0.178	0.455	0.277	1.592	1.679	−4.697	−7.496	−6.246
507	0.052	0.117	0.126	0.065	0.074	0.009	0.138	0.138	−1.720	−2.304	−2.318
508	−0.007	−0.193	−0.063	−0.186	−0.056	0.130	0.093	0.236	−2.577	−5.825	−2.530
510	0.064	−0.181	0.236	−0.245	0.172	0.417	0.336	0.698	−2.128	−4.105	−2.701
518	−0.196	−0.200	−0.193	−0.004	0.004	0.008	0.305	0.350	−2.998	−2.732	−1.883

Note: Column 1 reports forecaster's ID. Columns 2–4 report the estimated difference between the left and right hand sides of Equation (3.3). Columns 5–7 report the estimated difference between the left and right hand sides of Equation (3.5). The estimated difference between the left and right hand sides of Equation (3.6) is reported in columns 8 and 9. Finally, the last three columns refer to the condition of remuneration of revision effort by revision reward that is captured by Equation (3.12). Abbreviations: MSFE, mean squared forecast errors; MSFR, mean squared forecast revisions.

implement this Nordhaus test formally with the SPF inflation forecasts data, we consider the regression

$$d_{t,h}=\beta_0+\beta_h d_{t,h+1}+\zeta_{t,h},\ h=1,2, \tag{4.5}$$

where $d_{t,h}\equiv\hat{y}_{t|t-h}-\hat{y}_{t|t-(h+1)}$ is the revision made between consecutive horizons. Table 3 reports the estimated slope coefficient β_h $(h=1,2)$ from this regression. The associated standard errors are reported in parentheses, and the numbers with an asterisk indicate the estimated values are significant at a 10% level of significance. As seen in the table, most estimated slope coefficients are negative. These results are consistent with our finding that individual SPF inflation forecasts are characterized by irrational-to-news ratios above 1. These high irrational-to-news ratios, along with the fact that $\hat{\theta}_{h+1}$ is negative or very close to zero for many forecasters, yield $\hat{\theta}_{h+1}<\frac{\hat{\sigma}_{\xi_{h+1}}}{\hat{\sigma}_{\omega_{h+1}}}$ for all $h=1,2$; that is, a negative estimated slope coefficient is expected for all $h=1,2$. As such, the forecast rationality test with the null hypothesis of $\beta_h=0$ against the one-sided alternative of $\beta_h<0$ is rejected at least once at a 10% level for 19 out of 22 forecasters.

Overall, this application illustrates that the SPF inflation forecasts are subject to a substantial amount of the irrational forecast error that is heterogeneous

TABLE 3 Nordhaus-type test for individual forecasters

ID	$h=2$ $\hat{\beta}_2$	$h=1$ $\hat{\beta}_1$	ID	$h=2$ $\hat{\beta}_2$	$h=1$ $\hat{\beta}_1$	ID	$h=2$ $\hat{\beta}_2$	$h=1$ $\hat{\beta}_1$
20	−0.218*	−0.261*	426	−0.509*	−0.199	484	−0.041	−0.096
	(0.161)	(0.169)		(0.119)	(0.160)		(0.203)	(0.167)
40	0.167	−0.241*	428	−0.170	−0.326*	504	−0.224*	0.179
	(0.208)	(0.169)		(0.150)	(0.156)		(0.110)	(0.198)
65	−0.323*	−0.120	429	−0.300	−0.260*	507	−0.490*	−0.412*
	(0.132)	(0.125)		(0.328)	(0.161)		(0.189)	(0.165)
84	−0.007	0.098	433	−0.325*	−0.395*	508	−0.517*	−0.459*
	(0.166)	(0.187)		(0.104)	(0.129)		(0.237)	(0.317)
407	−0.305*	−0.216*	446	−0.357*	−0.331*	510	−0.499*	−0.515*
	(0.138)	(0.138)		(0.162)	(0.117)		(0.108)	(0.142)
411	−0.347*	0.383	456	0.252	0.130	518	−0.548*	−0.348*
	(0.131)	(0.344)		(0.210)	(0.216)		(0.165)	(0.170)
420	0.590	−0.343*	463	−0.198*	−0.542*			
	(0.315)	(0.142)		(0.118)	(0.195)			
421	−0.261*	0.194	472	−0.370*	−0.512*			
	(0.123)	(0.188)		(0.149)	(0.321)			

Note: The table reports the estimated slope coefficient β_h in the regression $d_{t,h}=\beta_0+\beta_h d_{t,h+1}+\zeta_{t,h}$, where $d_{t,h}=\hat{y}_{t|t-h}-\hat{y}_{t|t-(h+1)}$ and $h=1,2$. The associated standard errors of these estimates are included in the parentheses. The asterisk indicates significance at a 10% level.

across forecast horizons and forecasters. The correlation between irrational forecast error and newly arrived information also varies across horizons and forecasters, and the irrational-to-news ratio is consistently above 1 for most forecasters across forecast horizons. The sign of the correlation may change with horizons, suggesting that forecasters do not always react to newly arrived information in the same way over horizons. While some of the internal consistency conditions are satisfied despite the presence of an irrational forecast error, the high value of irrational-to-news ratio has led to the improvement in forecast accuracy insufficient to compensate the revision effort. The high irrational-to-news ratio has also resulted in the rejection of forecast rationality for most forecasters when the Nordhaus-type test is applied.

5 | CONCLUSION

This paper explores the extent to which multi-horizon fixed-event forecasts may still be internally consistent when subject to both rational and irrational forecast errors. We propose a bi-error model under which multi-horizon forecasts $\hat{y}_{t|t-h}$ can be decomposed as the sum of rational forecasts $(\hat{y}^*_{t|t-h})$ and an irrational component $(-\xi_{th})$. Focusing on both the monotonicity of the second moments (see Pattong & Timmermann, 2012) and the property of fully compensated forecast revisions (see Isiklar & Lahiri, 2007) within this bi-error framework, we derive the necessary and sufficient conditions under which revised forecasts that contain an irrational component can still possess the internal consistency property of rational forecasts. These conditions generally require that the accumulation of unanticipated information between two forecast updating points be larger than some threshold values that depend on the sizes of the irrational forecast error variance and its covariance with the most recent news. The correlation between irrational forecast error and the most recent news is also required to be no less than the irrational-to-news ratio for revision effort being remunerated by revision reward.

We illustrate our methodology with a panel of SPF inflation forecasts data. Our results show that a subset of the conditions required for the second moment properties featured by multi-horizon rational forecasts is satisfied, while a sizeable and heterogeneous irrational forecast error is observed across forecast horizons and forecasters. There is also evidence of high irrational-to-news ratios and overreaction to news, which provides insights into why the revision effort of SPF inflation forecasters is not remunerated by the improvement in the accuracy of inflation forecasts.

ACKNOWLEDGMENT

The authors would like to thank an anonymous referee for useful comments and suggestions on earlier versions of this manuscript. Open access publishing facilitated by University of Tasmania, as part of the Wiley - University of Tasmania agreement via the Council of Australian University Librarians.

DATA AVAILABILITY STATEMENT

Historical SPF forecasts of inflation rate and the realized values used in Section 1 and Section 4 are openly available at https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files/cpi; see The Federal Reserve Bank of Philadelphia (2020, 2021a,b).

ORCID

Jing Tian https://orcid.org/0000-0002-0186-4412
Firmin Doko Tchatoka https://orcid.org/0000-0003-1876-0633

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AUTHOR BIOGRAPHIES

Jing Tian is a senior lecturer at the Tasmanian School of Business and Economics, University of Tasmania. Her research interests include econometric forecasting and macro econometrics.

Firmin Doko Tchatoka is a professor at the School of Economics and Public Policy (University of Adelaide, Australia). His research interests include identification problems in structural models, model selection and optimal inference in GMM setting, identification of macroeconomic shocks, and identification of treatment effect in high-dimensional models.

Thomas Goodwin is a professional economist within the food and agribusiness sector. He deploys business process models within mobile apps to drive digital transformation.

How to cite this article: Tian, J., Doko Tchatoka, F., & Goodwin, T. (2022). Are internally consistent forecasts rational? *Journal of Forecasting*, 1–18. https://doi.org/10.1002/for.2871

APPENDIX A: CONDITIONS FOR INTERNAL CONSISTENCY FOR BI-ERROR FIXED-EVENT FORECASTS

This appendix summarizes the necessary and sufficient conditions under a bi-error structure that are discussed in Section 3.

The first column of Table A1 lists the second moment properties for internally consistent fixed-event forecasts, and the second column lists the necessary and sufficient conditions that we have derived for each property.

APPENDIX B: PROOFS FOR INTERNAL CONSISTENCY

This appendix provides a formal derivation for the second moment bounds of multi-horizon fixed-event forecasts that are subject to both rational and irrational forecast errors. We list the variables defined by our models, which will be used in the proofs in the following subsections, in Table B1.

B.1 | Variances of the forecasts, forecast errors, and revisions

Using the expression of the forecast errors in Table B1, the variances of forecasting errors $\mathrm{Var}(e_{th})=\sum_{i=0}^{i=h-1}\sigma^2_{\omega_i}+\sigma^2_{\xi_h}$ since irrational forecast error at horizon h do not correlate to the future unanticipated information. For two forecasts made at horizons $h=s$ and $h=l, \mathrm{Var}(e_{ts})-\mathrm{Var}(e_{tl})=-\sum_{i=s}^{l-1}\sigma^2_{\omega_i}+\sigma^2_{\xi_s}-\sigma^2_{\xi_l}$. Therefore, if and only if $\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_s}-\sigma^2_{\xi_l}$, then $\mathrm{Var}(e_{ts})\leq\mathrm{Var}(e_{tl})$ and $\mathrm{MSFE}_{ts}\leq\mathrm{MSFE}_{tl}$.

Using the expressions of the bi-error forecast revisions in Table B1 and the assumption that irrational forecast error ξ_{th} may only correlate with the contemporaneous news, $\mathrm{Var}(d_{t,sm})=\sum_{i=s}^{m-1}\sigma^2_{\omega_i}+\sigma^2_{\xi_m}+\sigma^2_{\xi_s}-2\mathrm{Cov}(\omega_{s,t},\xi_{ts})$, and $\mathrm{Var}(d_{t,sl})=\sum_{i=s}^{l-1}\sigma^2_{\omega_i}+\sigma^2_{\xi_l}+\sigma^2_{\xi_s}-2\mathrm{Cov}(\omega_{s,t},\xi_{ts})$. The differences in two variances $\mathrm{Var}(d_{t,sm})-\mathrm{Var}(d_{t,sl})=-\sum_{i=m}^{l-1}\sigma^2_{\omega_i}+\sigma^2_{\xi_m}-\sigma^2_{\xi_l}$. Therefore, if and only if

Internal consistency in second moment	Necessary and sufficient condition under bi-error structure
Panel A: Variances of forecasts, forecast errors and revisions	
$Var(\hat{y}_{t\|t-s})\geq Var(\hat{y}_{t\|t-l})$	$\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_l}-\sigma^2_{\xi_s}+2[Cov(\omega_{l,t},\xi_{tl})-Cov(\omega_{s,t},\xi_{ts}))]$
$Var(e_{ts})\leq Var(e_{tl})$	$\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_s}-\sigma^2_{\xi_l}$
$Var(d_{t,sm})\leq Var(d_{t,sl})$	$\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_m}-\sigma^2_{\xi_l}$
Panel B: Covariances between forecasts, forecast errors, and the target	
$Cov(\hat{y}_{t\|t-s},\hat{y}_{t\|t-m})\geq Cov(\hat{y}_{t\|t-s},\hat{y}_{t\|t-l})$	$\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq Cov(\omega_{m,t},\xi_{tm})-Cov(\omega_{l,t},\xi_{tl})$
$Cov(y_t,e_{ts})\leq Cov(y_t,e_{tl})$	$\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq Cov(\omega_{s,t},\xi_{ts})-Cov(\omega_{l,t},\xi_{tl})$
$Cov(y_t,\hat{y}_{t\|t-s})\geq Cov(y_t,\hat{y}_{t\|t-l})$	$\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq Cov(\omega_{s,t},\xi_{ts})-Cov(\omega_{l,t},\xi_{tl})$
Panel C: Covariances related to forecast revision	
$Cov(e_{tm},d_{t,sm})\leq Cov(e_{tl},d_{t,sl})$	$\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_m}-\sigma^2_{\xi_l}$
$Cov(\hat{y}_{t\|t-s},d_{t,sm})\leq Cov(\hat{y}_{t\|t-s},d_{t,sl})$	$\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq Cov(\omega_{m,t},\xi_{tm})-Cov(\omega_{l,t},\xi_{tl})$
$Var(d_{t,sl})\leq 2Cov(y_t,d_{t,sl})$	$\sum_{i=s}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_s}+\sigma^2_{\xi_l}-2Cov(\omega_{l,t},\xi_{tl})$
$Var(d_{t,ml})\leq 2Cov(\hat{y}_{t\|t-s},d_{t,ml})$	$\sum_{i=m}^{l-1}\sigma^2_{\omega_i}\geq\sigma^2_{\xi_m}+\sigma^2_{\xi_l}-2Cov(\omega_{l,t},\xi_{tl})$
Panel D: Forecast effort v.s. forecast reward	
$\mathrm{MSFR}_{t,sl}\leq\mathrm{MSFE}_{t\|t-l}-\mathrm{MSFE}_{t\|t-s}$	Forecasters smooth contemporaneous news and $\frac{\sigma_{\xi_s}}{\sigma_{\omega_s}}\leq\theta_s$

Note: h represents a generic forecast horizon and $h=H,H-1,...,l,l-1,...,m,m-1,...,s,s-1,...,0$, where H is the longest forecast horizon and $l\geq m\geq s$. The first ten second moment bounds listed in column 1 are the properties of internal consistency possessed by rational fixed-event forecasts (see Patton & Timmermann, 2012). The last second moment bound is from Lahiri (2012). Column 2 shows the necessary and sufficient condition for each moment bound be satisfied in our bi-error structure model.

TABLE A1 Conditions for internal consistency under bi-error structure

TABLE B1 Expression of the fixed-event forecasts, forecast errors, and revisions

Variable	Expression
Target	$y_t = \hat{y}^*_{t\|t-h} + \sum_{i=0}^{h-1} \omega_{i,t}$
Bi-error forecasts	$\hat{y}_{t\|t-h} = y_t - \sum_{i=0}^{h-1} \omega_{i,t} - \xi_{th}$
Bi-error forecast errors	$e_{th} = \sum_{i=0}^{h-1} \omega_{i,t} + \xi_{th}$
Bi-error forecast revisions between $h = l$ and $h = m$	$d_{t,ml} = \sum_{i=m}^{l-1} \omega_{i,t} + \xi_{tl} - \xi_{tm}$
Bi-error forecast revisions between $h = m$ and $h = s$	$d_{t,sm} = \sum_{i=s}^{m-1} \omega_{i,t} + \xi_{tm} - \xi_{ts}$
Bi-error forecast revisions between $h = l$ and $h = s$	$d_{t,sl} = \sum_{i=s}^{l-1} \omega_{i,t} + \xi_{tl} - \xi_{ts}$

Note: We use h to represent a generic forecast horizon and $h = H, H-1, ..., l, l-1, ..., m, m-1, ...s, s-1, ..., 0$, where H is the longest forecast horizon, and horizons $l \geq m \geq s$. The rational forecast error is $\sum_{i=0}^{h-1} \omega_{i,t}$, where $\omega_{i,t}$ have mean 0 and variance $\sigma^2_{\omega_i}$ and independent across i and t. The irrational forecast error ξ_{th} have mean 0 and variance $\sigma^2_{\xi_h}$ and are independent across target time t and forecast horizon h.

$\sum_{i=m}^{l-1} \sigma^2_{\omega_i} \geq \sigma^2_{\xi_m} - \sigma^2_{\xi_l}$, then $\text{Var}(d_{t,sm}) \leq \text{Var}(d_{t,sl})$, and $\text{MSFR}_{t|t-s} \leq \text{MSFR}_{t|t-l}$.

The variance of forecasts is given by $\text{Var}(\hat{y}_{t|t-h}) = \text{Var}(y_t) + \sum_{i=0}^{h-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_h} - 2\,\text{Cov}(y_t, \lambda_{th}) - 2\,\text{Cov}(\omega_{h,t}, \xi_{th}) = \text{Var}(y_t) - \sum_{i=0}^{h-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_h} - 2\,\text{Cov}(\omega_{h,t}, \xi_{th})$. Therefore, $\text{Var}(\hat{y}_{t|t-s}) - \text{Var}(\hat{y}_{t|t-l}) = \sum_{i=s}^{l-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_s} - \sigma^2_{\xi_l} - 2\,\text{Cov}(\omega_{s,t}, \xi_{ts}) + 2\,\text{Cov}(\omega_{l,t}, \xi_{tl})$. If and only if $\sum_{i=s}^{l-1} \sigma^2_{\omega_i} \geq \sigma^2_{\zeta_l} - \sigma^2_{\zeta_s} + 2\,\text{Cov}(\omega_{s,t}, \xi_{ts}) - 2\,\text{Cov}(\omega_{l,t}, \xi_{tl})$, we have $\text{Var}(\hat{y}_{t|t-s}) \geq \text{Var}(\hat{y}_{t|t-l})$, and $\text{MSF}_{ts} \geq \text{MSF}_{tl}$.

B.2 | Covariances between forecasts, forecast errors, and the target

Using the expression of the target and bi-error forecasts, the covariances between the forecasts $\hat{y}_{t|t-h}$ and the target y_t are given by $\text{Cov}(y_t, \hat{y}_{t|t-h}) = \text{Var}(y_t) - \sum_{i=0}^{h-1} \sigma^{\omega_i} - \text{Cov}(\omega_{h,t}, \xi_{th})$. Then $\text{Cov}(y_t, \hat{y}_{t|t-s}) - \text{Cov}(y_t, \hat{y}_{t|t-l}) = \sum_{i=s}^{l-1} \sigma^2_{\omega_i} - \text{Cov}(\omega_{s,t}, \xi_{ts}) + \text{Cov}(\omega\quad, \xi_{tl})$. If and only if $\sum_{i=s}^{l-1} \omega_{i,t} \geq \text{Cov}(\omega_{s,t}, \xi_{ts}) - \text{Cov}(\omega_{l,t}, \xi_{tl})$, then we have $\text{Cov}(y_t, \hat{y}_{t|t-s}) \geq \text{Cov}(y_t, \hat{y}_{t|t-l})$.

The covariances between the target and forecasting errors $\text{Cov}(y_t, e_{th}) = \sum_{i=0}^{h-1} \sigma^2_{\omega_i} + \text{Cov}(\omega_{h,t}, \xi_{th})$. Therefore, $\text{Cov}(y_t, e_{ts}) - \text{Cov}(y_t, e_{tl}) = -\sum_{i=s}^{l-1} \sigma^2_{\omega_i} - \text{Cov}(\omega_{l,t}, \xi_{ts}) + \text{Cov}(\omega_{s,t}, \xi_{ts})$, and if and only if $\sum_{i=s}^{l-1} \sigma^2_{\omega_i} \geq \text{Cov}(\omega_{s,t}, \xi_{ts}) - \text{Cov}(\omega_{l,t}, \xi_{ts})$, then $\text{Cov}(y_t, e_{ts}) \leq \text{Cov}(y_t, e_{tl})$.

The covariances between two forecasts $\text{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-m}) = \text{Cov}(y_t - \sum_{i=0}^{s-1} \omega_{i,t} - \xi_{ts}, y_t - \sum_{i=0}^{m-1} \omega_{i,t} - \xi_{tm}) = \text{Var}(y_t) - \sum_{i=0}^{m-1} \sigma^2_{\omega_i} - \text{Cov}(\omega\quad, \xi_{tm})$. Similarly, $\text{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-l}) = \text{Var}(y_t) - \sum_{i=0}^{l-1} \sigma^2_{\omega_i} - \text{Cov}(\omega_{l,t}, \xi_{tl})$. Thus, we have $\text{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-m}) - \text{Cov}(y_{t|t-s}, \hat{y}_{t|t-l}) = \sum_{i=m}^{l-1} \sigma^2_{\omega_i} - \text{Cov}(\omega_{m,t}, \xi_{tm}) + \text{Cov}(\omega_{l,t}, \xi_{tl})$. Therefore, $\text{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-m}) \geq \text{Cov}(\hat{y}_{t|t-s}, \hat{y}_{t|t-l})$ if and only if $\sum_{i=m}^{l-1} \sigma^2_{\omega_i} \geq \text{Cov}(\omega_{m,t}, \xi_{tm}) - \text{Cov}(\omega_{l,t}, \xi_{tl})$.

B.3 | Covariances related to forecast revisions

We first derive $\text{Cov}(\hat{y}_{t|t-s}, d_{t,sm})$ using the expression of $\hat{y}_{t|t-m}$ and $d_{t,sm}$ in Table B1. $\text{Cov}(\hat{y}_{t|t-s}, d_{t,sm}) = \text{Cov}(y_t - \sum_{i=0}^{s-1} \omega_{i,t} - \xi_{ts}, \sum_{i=s}^{m-1} \omega_{i,t} + \xi_{tm} - \xi_{ts}) = \sum_{i=s}^{m-1} \sigma^2_{\omega_i} + {}+\text{Cov}(\omega_{m,t}, \xi_{tm}) - 2\text{Cov}(\omega_{s,t}, \xi_{ts}) + \sigma^2_{\xi_s}$.

Similarly, $\text{Cov}(\hat{y}_{t|t-s}, d_{t,sl}) = \sum_{i=s}^{l-1} \sigma^2_{\omega_i} + \text{Cov}(\omega_{l,t}, \xi_{tl}) - 2\text{Cov}(\omega_{s,t}, \xi_{ts}) + \sigma^2_{\xi_s}$. Therefore, $\text{Cov}(\hat{y}_{t|t-s}, d_{t,sm}) - \text{Cov}(\hat{y}_{t|t-s}, d_{t,sl}) = -\sum^{i=m} \sigma^2_{\omega_i} + \text{Cov}(\omega_{m,t}, \xi_{tm}) - \text{Cov}(\omega_{l,t}, \xi_{tl})$.

If and only if $\sum_{i=m}^{l-1} \sigma^2_{\omega_i} \geq \text{Cov}(\omega_{m,t}, \xi_{tm}) - \text{Cov}(\omega_{l,t}, \xi_{tl})$, then we have $\text{Cov}(\hat{y}_{t|t-s}, d_{t,sm}) \leq \text{Cov}(\hat{y}_{t|t-s}, d_{t,sl})$.

When $h = m$, the covariances between forecast errors and forecasting revisions $\text{Cov}(e_{tm}, d_{t,sm}) = \text{Cov}(\sum_{i=0}^{m-1} \omega_{i,t} + \xi_{tm}, \sum_{i=s}^{m-1} \omega_{i,t} + \xi_{tm} - \xi_{ts}) = \sum_{i=s}^{m-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_m} - \text{Cov}(\omega_{s,t}, \xi_{ts})$. Similarly, $\text{Cov}(e_{tl}, d_{t,sl}) = \sum_{i=s}^{l-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_l} - \text{Cov}(\omega_{s,t}, \xi_{ts})$. Therefore, $\text{Cov}(e_{tm}, d_{t,sm}) - \text{Cov}(e_{tl}, d_{t,sl}) = -\sum_{i=m}^{l-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_m} - \sigma^2_{\xi_l}$. If and only if $\sum_{i=m}^{l-1} \sigma^2_{\omega_i} \geq \sigma^2_{\xi_m} - \sigma^2_{\xi_l}$, then we have $\text{Cov}(e_{tm}, d_{t,sm}) \leq \text{Cov}(e_{tl}, d_{t,sl})$.

Lastly, we work on the variance bounds for forecast revisions. Using the expression of $d_{t,sl}$ in Table B1, we have $\text{Var}(d_{t,sl}) = \sum_{i=s}^{l-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_l} + \sigma^2_{\xi_s} - 2\text{Cov}(\omega_{s,t}, \xi_{ts})$. The covariance between the target y_t and $d_{t,sl}$ is $\text{Cov}(y_t, d_{t,sl}) = \text{Cov}(\hat{y} + \sum_{i=0}^{l-1} \omega_{i,t} + \xi_{tl}, \sum_{i=s}^{l-1} \omega_{i,t} + \xi_{tl} - \xi_{ts}) = \text{Cov}(\omega_{l,t}, \xi_{tl}) + \sum_{i=s}^{l-1} \sigma^2_{\omega_i} - \text{Cov}(\omega_{s,t}, \xi_{ts})$. Then we have $\text{Var}(d_{t,sl}) - 2\text{Cov}(y_t, d_{t,sl}) = -\sum_{i=s}^{l-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_l} + \sigma^2_{\xi_s} - 2\text{Cov}(\omega_{l,t}, \xi_{tl})$. To have the variance bound condition satisfied by the bi-error forecast revisions, that is, $\text{Var}(d_{t,sl}) \leq 2\text{Cov}(y_t, d_{t,sl})$, we need $\sum_{i=s}^{l-1} \sigma^2_{\omega_i} \geq \sigma^2_{\xi_s} + \sigma^2_{\xi_l} - 2\text{Cov}(\omega_{l,t}, \xi_{tl})$.

Similarly, for the revision made at the medium horizon, $d_{t|m,l}$, we derive the variance $\text{Var}(d_{t,ml}) = \sum_{i=m}^{l-1} \sigma^2_{\omega_i} + \sigma^2_{\xi_m} + \sigma^2_{\xi_l} - 2\text{Cov}(\omega_{m,t}, \xi_{tm})$, and the covariance with the short horizon forecast $\text{Cov}(\hat{y}_{t|t-s}, d_{t,ml}) = \text{Cov}(\omega_{l,t}, \xi_{tl}) + \sum_{i=m}^{l-1} \omega^2_i - \text{Cov}(\omega_{m,t}, \xi\quad)$. It is then clear that $\text{Var}(d_{t,ml}) \leq 2\text{Cov}(\hat{y}_{t|t-s}, d_{t,ml})$ if and only if $\sum_{i=m}^{l-1} \sigma^2_{\omega_i} \geq \sigma^2_{\xi_m} + \sigma^2_{\xi_l} - 2\text{Cov}(\omega_{l,t}, \xi_{tl})$.

TABLE C1 The number of targeted quarters with all 4 forecasts being reported by individual forecasters

ID	No. of targets	ID	No. of targets	ID	No. of targets
20	41	426	79	484	52
40	40	428	51	504	45
65	59	429	29	507	42
84	39	433	78	508	15
407	57	446	61	510	56
411	47	456	40	518	46
420	34	463	61		
421	67	472	44		

Note: This table reports the number of targeted quarters used for estimating the variances and correlations of the irrational error and newly arrived information for individual forecasters.

APPENDIX C: INDIVIDUAL SPF CPI INFLATION FORECASTS

This appendix provides the number of targeted quarters used for estimating the variance and correlations of irrational error and newly arrived information in Section 4. Although the sample of panel data consists of 146 targeted quarters, the number of quarters for which a forecaster reports all 4 forecasts from horizon $h=4$ to horizon $h=1$ is much less than 146.