NON-EXTENDABILITY OF THE FINITE HILBERT TRANSFORM

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Abstract. The finite Hilbert transform $T\colon X \to X$ acts continuously on every rearrangement invariant space X on $(-1,1)$ having non-trivial Boyd indices. It is proved that T cannot be further extended, whilst still taking its values in X, to any larger domain space. That is, $T\colon X \to X$ is already optimally defined.

1. Introduction and main result

The finite Hilbert transform $T(f)$ of $f \in L^1(-1,1)$ is the well known principal value integral

$$(T(f))(t) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right)\frac{f(x)}{x-t}\,dx,$$

which exists for a.e. $t \in (-1,1)$ and is a measurable function. It has important applications to aerodynamics and elasticity via the airfoil equation, [3], [16], [20], [23], [24], and to problems arising in image reconstruction; see, for example, [11], [21]. We refer to [7], [8], [9], [10], [15] where one-dimensional singular integral operators closely related to the finite Hilbert transform are studied in great detail

For each $1 < p < \infty$ the classical linear operator $f \mapsto T(f)$ maps $L^p(-1,1)$ continuously into itself; denote this operator by T_p. Tricomi showed that T_p is a Fredholm operator and exhibited inversion formulae, [23], except for the case when $p = 2$, [24, §4.3] (see also [12, Ch. 11], [19, Ch. 14.4-3] and the references therein). For T_2 the situation is significantly different, as already pointed out somewhat earlier in [22, p.44]. Partial operator theoretic results for T_2 on $L^2(-1,1)$ were obtained by Okada and Elliott, [17]; see also the references.

In [4] the finite Hilbert transform T was studied when acting on suitable rearrangement invariant (r.i., in short) spaces X on $(-1,1)$; see below for the relevant definitions. Actually, T acts continuously on X (denote this operator by T_X) precisely when the Boyd indices of X are non-trivial, that is, when $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$; see [13, pp.170–171]. This class of r.i. spaces is the largest and most adequate replacement for the L^p-spaces when undertaking a further study of the finite Hilbert transform T. This is due to two

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critical facts: that $T\colon X \to X$ is injective if and only if the function $1/\sqrt{1-x^2} \notin X$, and that $T\colon X \to X$ has non-dense range if and only if $1/\sqrt{1-x^2}$ belongs to the associate space X' of X (whenever X is separable). In terms of r.i. spaces the previous conditions can be phrased as follows: *$T\colon X \to X$ is injective if and only if $L^{2,\infty}(-1,1) \nsubseteq X$* and *$T\colon X \to X$ has a non-dense range if and only if $X \subseteq L^{2,1}(-1,1)$* (for X separable). Here $L^{2,1}(-1,1)$ and $L^{2,\infty}(-1,1)$ are the usual Lorentz spaces.

Various types of inversion results of Tricomi for the operator T_p (when $1 < p < 2$ and $2 < p < \infty$) have been extended to T_X whenever the Boyd indices of X satisfy the condition $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$; see [4, Theorems 3.2 and 3.3]. Moreover, T is necessarily a Fredholm operator in such r.i. spaces, [4, Remark 3.4]. Results of this kind admit the possibility for a refinement of the solution of the airfoil equation; see [4, Corollary 3.5]. Additional operator theoretic results concerning T_X in r.i. spaces X occur in the recent article [5] (e.g., compactness, order boundedness, integral representation, etc.).

An important problem is the possibility of extending the domain of T_p, with T_p still maintaining its values in $L^p(-1,1)$. It was shown in [18, Example 4.21], for all $1 < p < \infty$ with $p \neq 2$, that there is *no* larger Banach function space (B.f.s. in short) containing $L^p(-1,1)$ such that T_p has an $L^p(-1,1)$-valued continuous extension to this space. This result was generalized in [4, Theorem 4.7]. Namely, it is not possible to extend the finite Hilbert transform $T_X\colon X \to X$ for any r.i. space X satisfying

$$0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2 \quad \text{or} \quad 1/2 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1. \tag{1}$$

The arguments used in [4] for establishing the above result do not apply to T_X for r.i. spaces X which fail to satisfy (1). In particular, they do not apply to $T_2\colon L^2(-1,1) \to L^2(-1,1)$. However, in [4] it was also established, via a completely different approach, that at least T_2 does not have a continuous $L^2(-1,1)$-valued extension to any larger B.f.s., [4, Theorem 5.3].

Thus, the question of extendability of T_X remains unanswered for a large sub-family of r.i. spaces which have non-trivial Boyd indices. Indeed, with the exception of $X = L^2(-1,1)$, this is the case for all those r.i. spaces X satisfying $0 < \underline{\alpha}_X \leqslant 1/2 \leqslant \overline{\alpha}_X < 1$. In particular, this includes all the Lorentz spaces $L^{2,q}$ for $1 \leqslant q \leqslant \infty$ with $q \neq 2$. The proof given in [4] for T_2, based on the Hilbert space structure of $L^2(-1,1)$, is not applicable to other r.i. spaces of the kind just mentioned.

The aim of this note is to answer the above question for *all* r.i. spaces X on which T_X is continuous, via a new and unified proof.

Theorem. *Let X be a r.i. space on $(-1,1)$ with non-trivial Boyd indices. The finite Hilbert transform $T_X\colon X \to X$ has no continuous, X-valued extension to any genuinely larger B.f.s. containing X.*

2. Preliminaries

In this paper the relevant measure space is $(-1,1)$ equipped with its Borel σ-algebra $\mathcal{B}$ and Lebesgue measure m (restricted to $\mathcal{B}$). We denote by $L^0(-1,1) = L^0$ the space (of equivalence classes) of all $\mathbb{C}$-valued measurable functions, endowed with the topology of convergence in measure. The space $L^p(-1,1)$ is denoted simply by L^p, for $1 \leqslant p \leqslant \infty$.

A *Banach function space* (B.f.s.) X on $(-1,1)$ is a Banach space $X \subseteq L^0$ satisfying the ideal property, that is, $g \in X$ and $\|g\|_X \leqslant \|f\|_X$ whenever $f \in X$, $g \in L^0$ and $|g| \leqslant |f|$ a.e. The *associate space* X' of X consists of all $g \in L^0$ satisfying $\int_{-1}^1 |fg| < \infty$, for every $f \in X$, equipped with the norm $\|g\|_{X'} := \sup\{|\int_{-1}^1 fg| : \|f\|_X \leqslant 1\}$. The space X' is a closed subspace of the Banach space dual X^* of X. The space X satisfies the Fatou property if, whenever $\{f_n\}_{n=1}^\infty \subseteq X$ satisfies $0 \leqslant f_n \leqslant f_{n+1} \uparrow f$ a.e. with $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f_n\|_X \to \|f\|_X$. In this paper *all* B.f.s.' X are on $(-1,1)$ relative to Lebesgue measure and, as in [1], satisfy the Fatou property.

A *rearrangement invariant* (r.i.) space X on $(-1,1)$ is a B.f.s. such that if $g^* \leqslant f^*$ with $f \in X$, then $g \in X$ and $\|g\|_X \leqslant \|f\|_X$. Here $f^* \colon [0,2] \to [0,\infty]$ is the decreasing rearrangement of f, that is, the right continuous inverse of its distribution function: $\lambda \mapsto m(\{t \in (-1,1) : |f(t)| > \lambda\})$. The associate space X' of a r.i. space X is again a r.i. space. Every r.i. space X on $(-1,1)$ satisfies $L^\infty \subseteq X \subseteq L^1$. Moreover, if $f \in X$ and $g \in X'$, then $fg \in L^1$ and $\|fg\|_{L^1} \leqslant \|f\|_X \|g\|_{X'}$, i.e., Hölder's inequality is available.

The family of r.i. spaces includes many classical spaces appearing in analysis, in particular the Lorentz $L^{p,q}$ spaces, [1, Definition IV.4.1].

Given a r.i. space X on $(-1,1)$, due to the Luxemburg representation theorem there exists a r.i. space $\widetilde{X}$ on $(0,2)$ such that $\|f\|_X = \|f^*\|_{\widetilde{X}}$ for $f \in X$, [1, Theorem II.4.10]. The dilation operator E_t for $t > 0$ is defined, for each $f \in \widetilde{X}$, by $E_t(f)(s) := f(st)$ for $0 \leqslant s \leqslant \min\{2, 1/t\}$ and zero for $\min\{2,1/t\} < s \leqslant 2$. The operator $E_t \colon \widetilde{X} \to \widetilde{X}$ is bounded with $\|E_{1/t}\|_{\widetilde{X}\to\widetilde{X}} \leqslant \max\{t,1\}$. The *lower* and *upper Boyd indices* of X are defined, respectively, by

$$\underline{\alpha}_X := \sup_{0<t<1} \frac{\log \|E_{1/t}\|_{\widetilde{X}\to\widetilde{X}}}{\log t} \quad \text{and} \quad \overline{\alpha}_X := \inf_{1<t<\infty} \frac{\log \|E_{1/t}\|_{\widetilde{X}\to\widetilde{X}}}{\log t};$$

see [2] and also [1, Definition III.5.12]. They satisfy $0 \leqslant \underline{\alpha}_X \leqslant \overline{\alpha}_X \leqslant 1$. Note that $\underline{\alpha}_{L^p} = \overline{\alpha}_{L^p} = 1/p$.

For all of the above and further facts on r.i. spaces see [1], for example.

3. Proof of the Theorem

Given X, a r.i. space on $(-1,1)$ with non-trivial Boyd indices, consider the space

$$[T,X] := \big\{f \in L^1 : T(h) \in X,\ \forall |h| \leqslant |f|\big\},$$

which is a B.f.s. for the norm

$$\|f\|_{[T,X]} := \sup_{|h|\leqslant|f|} \|T(h)\|_X, \quad f \in [T,X].$$

The proof of this fact uses, in an essential way, a deep result of Talagrand concerning L^0-valued measures, [4, Proposition 4.5]. The space $[T,X]$ is the largest B.f.s. containing X to which $T_X \colon X \to X$ has a continuous, linear, X-valued extension, [4, Theorem 4.6].

In particular, $X \subseteq [T, X]$. Thus, in order to show that no genuine extension of T_X is possible it suffices to show that $[T, X] \subseteq X$; see Theorems 4.7 and 5.3 in [4].

Fix $N \in \mathbb{N}$. Given $a_1, \ldots, a_N \in \mathbb{C}$ and disjoint sets $A_1 \ldots, A_N$ in $\mathcal{B}$, define the simple function

$$\phi := \sum_{n=1}^{N} a_n \chi_{A_n}.$$

On $\Lambda := \{1, -1\}^N$ consider the probability measure $d\sigma$, which is the product measure of N copies of the uniform probability on $\{1, -1\}$. Define the bounded measurable function F on Λ by

$$\sigma = (\sigma_1, \ldots, \sigma_N) \in \Lambda \mapsto F(\sigma) := \left\| T_X\left(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \right) \right\|_X.$$

Observe, since the sets $A_1, \ldots, A_N$ are pairwise disjoint, that for every $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Lambda$ one has

$$\left| \sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \right| = \sum_{n=1}^{N} |a_n| \chi_{A_n} = |\phi|,$$

whence

$$\|F\|_{L^\infty(\Lambda)} = \sup_{\sigma \in \Lambda} \left\| T\left(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \right) \right\|_X \leqslant \sup_{|h| \leqslant |\phi|} \|T(h)\|_X = \|\phi\|_{[T,X]}. \tag{2}$$

On the other hand, an application of Fubini's theorem yields

$$\begin{aligned}
\|F\|_{L^\infty(\Lambda)} &\geqslant \|F\|_{L^1(\Lambda)} \\
&= \int_\Lambda |F(\sigma)| \, d\sigma \\
&= \int_\Lambda \left\| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right) \right\|_X d\sigma \\
&= \int_\Lambda \left(\sup_{\|g\|_{X'}=1} \int_{-1}^{1} |g(t)| \left| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| dt \right) d\sigma \\
&\geqslant \sup_{\|g\|_{X'}=1} \int_\Lambda \left(\int_{-1}^{1} |g(t)| \left| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| dt \right) d\sigma \\
&= \sup_{\|g\|_{X'}=1} \int_{-1}^{1} |g(t)| \left(\int_\Lambda \left| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| d\sigma \right) dt. \qquad (3)
\end{aligned}$$

Consider now the inner integral over Λ in the last term (3) of the previous expression. For $t \in (-1, 1)$ fixed, set

$$\beta_n := a_n T\left(\chi_{A_n}\right)(t), \quad n = 1 \ldots, N.$$

It is known that the coordinate projections

$$P_n : \sigma \in \Lambda \mapsto \sigma_n \in \{-1, 1\}, \quad n = 1, \dots, N,$$

form an orthonormal set, that is,

$$\int_\Lambda P_j P_k \, d\sigma = \int_\Lambda \sigma_j \sigma_k \, d\sigma = \delta_{j,k}, \quad j, k = 1, \dots, N.$$

Then, for the inner integral in (3), we have

$$\int_\Lambda \left| \sum_{n=1}^N \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| d\sigma = \int_\Lambda \left| \sum_{n=1}^N \beta_n P_n(\sigma) \right| d\sigma.$$

Apply the Khintchine inequality, [6, Inequality 1.10 and p.23], for $\{P_n\}_{n=1}^N$ yields

$$\int_\Lambda \left| \sum_{n=1}^N \beta_n P_n(\sigma) \right| d\sigma \geqslant \frac{1}{\sqrt{2}} \left(\sum_{n=1}^N |\beta_n|^2 \right)^{1/2}.$$

Accordingly,

$$\int_\Lambda \left| \sum_{n=1}^N \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| d\sigma \geqslant \frac{1}{\sqrt{2}} \left(\sum_{n=1}^N |a_n|^2 \left| T\left(\chi_{A_n}\right)(t) \right|^2 \right)^{1/2}. \tag{4}$$

Then, from (3) and (4), it follows that

$$\begin{aligned} \|F\|_{L^\infty(\Lambda)} &\geqslant \frac{1}{\sqrt{2}} \sup_{\|g\|_{X'}=1} \int_{-1}^1 |g(t)| \left(\sum_{n=1}^N |a_n|^2 \left| T\left(\chi_{A_n}\right)(t) \right|^2 \right)^{1/2} dt \\ &= \frac{1}{\sqrt{2}} \left\| \left(\sum_{n=1}^N |a_n|^2 \left| T\left(\chi_{A_n}\right) \right|^2 \right)^{1/2} \right\|_X . \end{aligned} \tag{5}$$

We recall the following consequence of the Stein-Weiss formula for the distribution function of the Hilbert transform H on $\mathbb{R}$ of a characteristic function, due to Laeng, [14, Theorem 1.2]. Namely, for $A \subseteq \mathbb{R}$ with $m(A) < \infty$ (where m also denotes Lebesgue measure in $\mathbb{R}$), we have

$$m(\{x \in A : |H(\chi_A)(x))| > \lambda\}) = \frac{2m(A)}{e^{\pi\lambda} + 1}, \quad \lambda > 0.$$

In particular, for any set $A \subseteq (-1, 1)$ it follows, for each $\lambda > 0$, that

$$m(\{x \in A : |T(\chi_A)(x)| > \lambda\}) = m(\{x \in A : |H(\chi_A)(x)| > \lambda\}) = \frac{2m(A)}{e^{\pi\lambda} + 1}.$$

That is,

$$m(\{x \in A : |T(\chi_A)(x)| > \lambda\}) = \frac{2m(A)}{e^{\pi\lambda} + 1}, \quad A \in \mathcal{B}, \quad \lambda > 0. \tag{6}$$

Set $\lambda = 1$ and $\delta := 2/(e^\pi + 1) < 1$. For each $n = 1, \dots, N$, define

$$A_n^1 := \{x \in A_n : |T(\chi_{A_n})(x)| > 1\}.$$

Then (6) implies that

$$m(A_n^1) = \frac{2m(A_n)}{e^\pi + 1} = \delta m(A_n), \quad n = 1, \dots, N. \tag{7}$$

Since the sets $A_1, \dots, A_N$ are pairwise disjoint, so are their subsets $A_1^1, \dots, A_N^1$. Note that $|T(\chi_{A_n})(x)| > 1$ for $x \in A_n^1$, for $n = 1, \dots, N$. Thus, on $(-1, 1)$ we have the pointwise estimates

$$\begin{aligned}\left(\sum_{n=1}^N |a_n|^2 |T(\chi_{A_n})|^2\right)^{1/2} &\geqslant \left(\sum_{n=1}^N |a_n|^2 |T(\chi_{A_n})|^2 \chi_{A_n}\right)^{1/2} \\ &= \sum_{n=1}^N |a_n| \, |T(\chi_{A_n})| \, \chi_{A_n} \\ &\geqslant \sum_{n=1}^N |a_n| \chi_{A_n^1}. \end{aligned}\tag{8}$$

Since $\|\cdot\|_X$ is a lattice norm, (8) yields

$$\left\| \left(\sum_{n=1}^N |a_n|^2 |T(\chi_{A_n})|^2\right)^{1/2} \right\|_X \geqslant \left\| \sum_{n=1}^N |a_n| \chi_{A_n^1} \right\|_X = \|\varphi\|_X, \tag{9}$$

where φ is the simple function

$$\varphi := \sum_{n=1}^N a_n \chi_{A_n^1}.$$

From (7) it follows that

$$m(\{x \in (-1,1) : |\varphi(x)| > \lambda\}) = \delta m(\{x \in (-1,1) : |\phi(x)| > \lambda\}), \quad \lambda > 0. \tag{10}$$

Consider the dilation operator $E_\delta \colon \widetilde{X} \to \widetilde{X}$ for $\delta < 1$ given above, that is, $E_\delta(f)(t) = f(\delta t)$ for $0 \leqslant s \leqslant \min\{2, 1/\delta\}$ and zero otherwise. For the decreasing rearrangements ϕ^* and φ^* of ϕ and φ, respectively, it follows from (10) that

$$\phi^* = E_\delta(\varphi^*).$$

Consequently, with $\|E_\delta\|$ denoting the operator norm of $E_\delta \colon \widetilde{X} \to \widetilde{X}$, we have

$$\|\phi\|_X = \|\phi^*\|_{\widetilde{X}} = \|E_\delta(\varphi^*)\|_{\widetilde{X}} \leqslant \|E_\delta\| \cdot \|\varphi^*\|_{\widetilde{X}} = \|E_\delta\| \cdot \|\varphi\|_X. \tag{11}$$

It follows, from (2), (5), (9) and (11) that

$$\begin{aligned}\|\phi\|_X &\leqslant \|E_\delta\| \cdot \|\varphi\|_X \\ &\leqslant \|E_\delta\| \cdot \left\| \left(\sum_{n=1}^N |a_n|^2 |T(\chi_{A_n})|^2\right)^{1/2} \right\|_X \\ &\leqslant \sqrt{2} \|E_\delta\| \cdot \|\phi\|_{[T,X]}. \end{aligned}$$

That is, there exists a constant $M > 0$, depending exclusively on X, such that

$$M\|\phi\|_X \leqslant \|\phi\|_{[T,X]}, \tag{12}$$

for *all* simple functions ϕ.

In order to extend (12) to all functions in $[T,X]$ fix $f \in [T,X]$. For every simple function ϕ satisfying $|\phi| \leqslant |f|$ it follows from (12) that

$$M\|\phi\|_X \leqslant \|\phi\|_{[T,X]} \leqslant \|f\|_{[T,X]}.$$

Taking the supremum with respect to all such ϕ yields, via the Fatou property of X, that $f \in X$ and

$$M\|f\|_X \leqslant \|f\|_{[T,X]}.$$

In particular, $[T,X] \subseteq X$. Consequently, $[T,X] = X$ with equivalent norms. Thus, no genuine X-valued extension of $T_X : X \to X$ is possible. □

The above Theorem has an immediate consequence. Namely, it extends, to *all* r.i. spaces with non-trivial Boyd indices, certain results known for those r.i. spaces X satisfying $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$, [4, Corollary 4.8], and for $X = L^2(-1,1)$, [4, Corollaries 5.4 and 5.5].

Corollary. *Let X be a r.i. space on $(-1,1)$ with non-trivial Boyd indices.*

(a) *There exists a constant $\beta > 0$ such that, for every $f \in X$, we have*

$$\frac{\beta}{4}\|f\|_X \leqslant \sup_{A\in\mathcal{B}} \big\|T_X(\chi_A f)\big\|_X \leqslant \sup_{|\theta|=1} \big\|T_X(\theta f)\big\|_X \leqslant \sup_{|h|\leqslant|f|} \big\|T_X(h)\big\|_X \leqslant \|f\|_X.$$

(b) *For a function $f \in L^1$ the following conditions are equivalent.*
   - (a) *$f \in X$.*
   - (b) *$T(f\chi_A) \in X$ for every $A \in \mathcal{B}$.*
   - (c) *$T(f\theta) \in X$ for every $\theta \in L^\infty$ with $|\theta| = 1$ a.e.*
   - (d) *$T(h) \in X$ for every $h \in L^0$ with $|h| \leqslant |f|$ a.e.*

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