Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Quadratic Conformal Supersymmetry

To cite this article: L A Yates 2019 *J. Phys.: Conf. Ser.* **1194** 012115

Quadratic Conformal Supersymmetry

L A Yates

School of Physical Sciences, University of Tasmania, Hobart, Tasmania, Australia

E-mail: `luke.yates@utas.edu.au`

Abstract. The non-appearance of predicted superparticles in high-energy particle experiments has placed severe constraints on candidate models of supersymmetry; in particular on the masses of the superpartners of known particles. Drawing on our earlier work investigating quadratic deformations of Lie superalgebras we present in this paper our recent findings that for certain extensions of space-time supersymmetries, namely the conformal superalgebra, there are representations without any superpartners (see J. Phys. A: Math. Theor. 51 (2018) 145203). This possibility arises due to a remarkable coincidence of the allowable quadratic extensions of the algebra, where one demands that a generalisation of the PBW theorem holds, and the minimal polynomial identity satisfied by the even subalgebra.

1. Introduction

As an extended symmetry principle, supersymmetry places within a single algebraic framework a model of particles and their interactions, enlarging the standard model to include the prediction of superparticles. In its simplest form, each particle of the standard model would be accompanied by a single superpartner. The appeal of supersymmetry is the improved behaviour of supersymmetric quantum field theories with respect to infinities and renormalization. However, in spite of ever increasing energy thresholds in experimental particle physics, up to and including the first run of the Large Hadron Collider, predicted superparticles of the known particles in the standard model have not appeared, pushing upwards the bounds on superparticle masses and significantly constraining candidate models [1].

At the technical level, the application of symmetry principles in theoretical physics goes hand-in-hand with the investigation of related algebraic structures. Within these structures, the symmetry itself is realised via the specific (non-commutative) multiplicative relations between algebraic elements and the study of invariants, representations and topological properties provides important (physical) information such as prediction of the spectrum of particle states. In this context, the investigation of physics beyond the standard model, or indeed beyond standard supersymmetric models, necessarily involves variations, extensions or deformations of the corresponding algebraic structures.

Above sufficiently high (unification) energy thresholds, candidate symmetry groups are unbroken and therefore restricted by established no-go theorems, namely the Coleman-Mandula Theorem [2] and its supersymmetric extension, the Haag−Lopuszański−Sohnius (HLS) theorem [3]. At lower energy scales, however, symmetry breaking is imperative, both in the standard model as, for example, a mechanism for mass generation, and in the context of supersymmetry to lift the mass degeneracy for pairs of superpartners. In this regime, non-linear algebraic structures such as quantum groups, $\mathcal{W}$-algebras and other deformations of the classical groups are candidate algebraic frameworks in which to investigate symmetry principles beyond the standard model.

It is in this context that we investigate a class of quadratic deformations of the universal enveloping algebra of ordinary Lie superalgebras, termed *quadratic superalgberas*. Originally introduced by Jarvis et. al [4], these algebras are generated by an underlying $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$. The defining relations satisfy non-linear graded commutation relations of the form

$$[L_{\bar{0}}, L_{\bar{0}}] \subset L_{\bar{0}} \qquad [L_{\bar{0}}, L_{\bar{1}}] \subset L_{\bar{1}} \qquad \{L_{\bar{1}}, L_{\bar{1}}\} \subset (L_{\bar{0}} \otimes L_{\bar{0}}) + L_{\bar{0}} + \mathbb{C}, \tag{1}$$

in addition to fulfilling an analogue of the usual (graded) Jacobi identities. In this paper we review key results concerning quadratic superalgebras, in particular the existence of so-called zero-step modules. We apply these results in the context of spacetime conformal supersymmetry exploiting the occurrence of zero-step modules to provide new insights into the predicted particle spectrum. The paper concludes with a short appendix containing a brief overview of the theory of quadratic algebras.

2. Quadratic Superalgebras

Let $L = L_{\bar{0}} + L_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded complex vector space. Take the even and odd subspaces $L_{\bar{0}}$ and $L_{\bar{1}}$ to be spanned by basis elements x_1, $i = 1, 2, ..., n$, and y_r, $r = 1, 2, ..., m$, respectively. The tensor algebra $T(L) = \bigoplus_{n=0}^{\infty} \otimes^n (L) \cong \mathbb{C} \oplus L \oplus L \otimes L \oplus \cdots$ inherits the $\mathbb{Z}_2$-grading in the natural way.

Imposing the multiplicative conditions (1) we have in terms of basis elements

$$[x_i, x_j] = c_{ij}{}^k x_k \qquad [x_i, y_p] = \bar{c}_{ip}{}^q y_q \qquad \{y_p, y_q\} = d_{pq}{}^{kl} x_k \otimes x_l + b_{pq}{}^k x_k + a_{pq}, \tag{2}$$

where the arrays of complex numbers $c_{ij}{}^k$, $\bar{c}_{ip}{}^q$, $b_{pq}{}^k$ and $d_{pq}{}^{kl}$ take the role of generalised structure constants. Demanding that $[\ ,\ ]$ and $\{\ ,\ \}$ fulfill the standard (graded) commutation relations we obtain a set of quadratic relations $I \subset L \otimes L + L + \mathbb{C}$ spanned by the set

$$\begin{aligned} &x_i \otimes x_j - x_j \otimes x_i - c_{ij}{}^k x_k; \\ &x_i \otimes y_p - y_p \otimes x_i - \bar{c}_{ip}{}^q y_q; \\ &y_p \otimes y_q - y_q \otimes y_p - d_{pq}{}^{kl} x_k \otimes x_l - b_{pq}{}^k x_k - a_{pq}. \end{aligned} \tag{3}$$

Let us demand that the relations (2) satisfy the ordinary graded Jacobi identities. Let $\{w_i, i = 1, \cdots, m+n\}$ be a fixed homogeneous basis for L such that $|i| = 0$ and $|i| = 1$ denote the even and odd grading respectively. In this basis the quadratic relations (3) take the form

$$w_i w_j - (-1)^{|i||j|} w_j w_i - d_{ij}{}^{lm} w_l w_m - c_{ij}^k w_k - a_{ij}, \tag{4}$$

where $d_{ij}{}^{lm}$ and a_{ij} are non-vanishing only when both $|i| = |j| = 1$ and $|l| = |m| = 0$. The graded Jacobi identities are

$$[w_i, [w_j, w_k]] = [[w_i, w_j], w_k] + (-1)^{|i|.|j|} [w_j, [w_i, w_k]], \tag{5}$$

where nested brackets of the form $[\{y_p, y_q\}, w_i]$, for which the inner bracket contains quadratic terms, are defined by

$$[\{y_p, y_q\}, w_i] \equiv d_{pq}{}^{lo} \left(x_l [x_o, w_i] + [x_l, w_i] x_o \right) + c_{pq}^k [x_k, w_i].$$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012115 doi:10.1088/1742-6596/1194/1/012115

In terms of the generalised structure constants the Jacobi identities (5) take the form

$$
\begin{aligned}
&c_{ij}{}^{l}c_{lk}{}^{o} = c_{ik}{}^{l}c_{lj}{}^{o} + c_{jk}{}^{l}c_{li}{}^{o};\\
&c_{ij}{}^{l}\bar{c}_{lp}{}^{q} = \bar{c}_{ir}{}^{q}\bar{c}_{jp}{}^{r} - \bar{c}_{jr}{}^{q}\bar{c}_{ip}{}^{r};\\
&c_{in}{}^{k}d_{pq}{}^{nl} + c_{in}{}^{l}d_{pq}{}^{kn} = \bar{c}_{ip}{}^{s}d_{sq}{}^{kl} + \bar{c}_{ip}{}^{s}d_{sq}{}^{kl},\\
&b_{pq}{}^{o}c_{io}{}^{n} = \bar{c}_{ip}{}^{s}b_{sq}{}^{n} + \bar{c}_{iq}{}^{s}b_{ps}{}^{n},\\
&\bar{c}_{op}{}^{s}b_{qr}{}^{o} + \bar{c}_{oq}{}^{s}b_{rp}{}^{o} + \bar{c}_{or}{}^{s}b_{pq}{}^{o} = 0,\\
&\bar{c}_{op}{}^{s}d_{qr}{}^{ol} + \bar{c}_{oq}{}^{s}d_{rp}{}^{ol} + \bar{c}_{or}{}^{s}d_{pq}{}^{ol} = 0.
\end{aligned}
\tag{6}
$$

Definition 2.1 (Quadratic Superalgebra)**.** Let $L = L_{\bar{0}} + L_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded complex vector space satisfying the multiplicative relations (2) and the Jacobi identities (6) above. Let (I) be the two-sided ideal in the tensor algebra $T(L)$ generated by the set of qudratic relations I as in (3). The subalgebra of the tensor algebra defined by $U(L) = T(L)/(I)$ is called the quadratic superalgebra associated with I.

Remark. $L_{\bar{0}}$ is a Lie algebra and $L_{\bar{1}}$ in an $L_{\bar{0}}$-module. When $d_{pq}{}^{kl} = b_{pq} = 0$ then $U(L) = T(L)/(I)$ is the universal enveloping algebra of the ordinary Lie superalgebra $L = L_{\bar{0}} + L_{\bar{1}}$. The imposition of the Jacobi identities turns out to have a key role in determining the existence of a Poincaré-Birkhoff-Witt (PBW) basis. If the odd module $L_{\bar{1}}$ is a real representation of the even part then we call $U(L)$ a balanced quadratic superalgebra. These may be further categorised into type I$'$ and type II where the former admit a decomposition $L_{\bar{1}} = L_+ + L_-$, where L_+ is an irreducible L_0 representation and L_- is its contragredient, with $\{L_\pm, L_\pm\} = 0$ imposed to be consistent with type I Lie superalgebras.

Theorem 2.2 (PBW basis theorem)**.** *As before let $\{w_i, i = 1, \cdots, m+n\}$ be a fixed homogeneous basis for L, and consider the defining relations* (4) *which generate the ideal (I). Then the quadratic superalgebra $U(L) = T(L)/(I)$ has a basis of ordered monomials if an index ordering exists such that only those $d_{ij}{}^{lo}$ are nonvanishing for which both l and o precede i and j.*

Proof. We outline two methods of proof.

Method 1. This method involves the explicit construction of the ordered monomial elements followed by a proof of their linear independence via induction on the monomial degree. The linear independence depends explicitly on satisfaction of both the Jacobi identity (6) and the ordering condition. This method mirrors classical proofs of the PBW theorem for Lie algebras, see Lemma 2.9 [5] for the complete proof.

Method 2. This method employs techniques from the theory of quadratic algebras (see the appendix for a brief overview). These include a generalised PBW theorem and a special case of the so-called diamond lemma which is used verify the linear independence of the derived (Grobner) basis, see Lemma 2.8 [6] for full details.

Theorem 2.3 (Structure of $gl_2(n/1)$)**.** *Take $L_{\bar{0}} = gl(n)$ with (Gel'fand) basis $E^a{}_b$ $a, b = 1, ..., n$ satisfying $[E^a{}_b, E^c{}_d] = \delta^c{}_b E^a{}_d - \delta^a{}_d E^c{}_b$. Take $L_{\bar{1}} = L_{-1} + L_{+1} \cong \{1\} + \{\bar{1}\}$ where $\{1\}$ and $\{\bar{1}\}$ are the fundamental and fundamental contragredient $L_{\bar{0}}$-representations with bases $\overline{Q}^a$ and Q_a respectively. These satisfy*

$$[E^a{}_b, \overline{Q}^c] = \delta^c{}_b \overline{Q}^a \qquad [E^a{}_b, Q_c] = -\delta^a{}_c Q_b.$$

The most general solution of the anticommutator $\{\overline{Q}^a, Q_b\} \subset L_{\bar{0}} \otimes L_{\bar{0}} + L_{\bar{0}} + \mathbb{C}$ such that $gl_2(n/1) \equiv L_{\bar{0}} + L_{\bar{1}}$ is a quadratic superalgebra is (up to an overall scaling)

$$\{\overline{Q}^a, Q_b\} = (E^2)^a{}_b - E^a{}_b(\langle E\rangle - \alpha) - \frac{1}{2}\delta^a{}_b\left(\langle E^2\rangle - \langle E\rangle^2 + (n-1+2\alpha)\langle E\rangle\right) + \delta^a{}_b\mathsf{c}, \tag{7}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012115 doi:10.1088/1742-6596/1194/1/012115

where α is a free parameter, c is a central charge, and $\langle E \rangle$ and $\langle E^2 \rangle$ are the linear and quadratic Casimir operators respectively.

Proof. See section 4 [4] and section 4.1 [6].

The close relationship between $gl_2(n/1)$ and Lie superalgebra $sl(n/1)$ is revealed by a simple re-parametrisation. Let $gl_2(n/1)^{\lambda,\mathsf{c}}$ denote the quadratic superalgebra resulting from an overall rescaling of (7) by $\lambda \equiv \frac{1}{\alpha}$, explicitly

$$\{\overline{Q}^a, Q_b\} = E^a{}_b - \delta^a{}_b\langle E\rangle + \lambda\big[(E^2)^a{}_b - E^a{}_b\langle E\rangle - \tfrac{1}{2}\delta^a{}_b\big(\langle E^2\rangle - \langle E\rangle^2 + (n-1)\langle E\rangle - 2\mathsf{c}\big)\big].$$

Lemma 2.4 (Contraction limit of $gl_2(n/1)^{\lambda,\mathsf{c}}$([6], Lemma 2.2))**.** *The ordinary Lie superalgebra $sl(n/1)$ is obtained from $gl_2(n/1)^{\lambda,\mathsf{c}}$ in the contraction limit $\lambda \to 0$.*

Since the even part is an ordinary Lie Algebra, the representation theory of quadratic superalgebras can be developed in a manner analogous to that of Lie superalgberas. This includes the method of induced Kac-modules for which there exists a class of truncated irreducible modules called atypical modules. Atypicality arises in Lie superalgebras due to the fact that for certain highest weight $L_{\overline{0}}$-modules $V_0(\lambda)$, there exists $v \in V_0(\lambda)$ such that

$$T_+T_-.v = 0, \tag{8}$$

where $T_\pm \equiv \prod_{y_i \in L_\pm} y_i$. As a consequence of (8), the induced module, $V(\lambda)$, contains non-trivial submodules which must be factored out to obtain the corresponding irreducible L-module with highest weight λ [7]. An alternative method, due to Gould [8], takes a more constructive approach, employing the theory of characteristic identities and the related theory of tensor projection operators [9,10] to determine the set of even modules which comprise a given L-module of highest weight λ. One of the unique features of quadratic superalgebras is the existence of completely degenerate atypical modules; these are *zero-step* atypical modules which comprise a single irreducible $L_{\overline{0}}$-module, that is $V(\lambda) = V_0(\lambda)$. This occurs due to the remarkable possibility that for certain representations $V_0(\lambda)$, the right-hand-side of the anticommutator coincides with a quadratic characteristic identity and is therefore identically zero. We illustrate some finite-dimensional zero-step modules below; a class of infinite-dimensional examples appears in the subsequent section.

Example 2.5 (Finite-dimensional zero-step modules)**.** The (minimal) degree of the polynomial identity satisfied by the array of Gel'fand generators E is determined, in the case of finite-dimensional representations $V_0(\lambda)$, by the number of distinct dominant integral weights in the sequence, $\lambda+\varphi_1, \lambda+\varphi_2, ..., \lambda+\varphi_n$, where $\varphi_i = -\varepsilon_i$ comprise the weight space of the fundamental contragredient representation of $gl(n)$ [11,12]. In order to illustrate a class of zero-step modules, we fix

$$\lambda = (k^r, 0^{n-r}) \cong \overbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \; & & & & & & & \\ \hline \; & & & & & & & \\ \hline \; & & & & & & & \\ \hline \end{array}}^{k \text{ columns}} \left.\vphantom{\begin{array}{c} a\\a\\a\\a \end{array}}\right\} r \text{ rows},$$

for which the Casimir operators, evaluated on $V_0(\lambda)$, take the eigenvalues

$$\langle E\rangle = rk \qquad \langle E^2\rangle = rk(k+n-r).$$

Given the requirement that $\lambda + \varphi_i$ be dominant integral, it is evident the only rows from which a single box may be removed are the r^{th} and n^{th}. These correspond to the addition of φ_{n-r+1}

and φ_1 respectively. The resulting quadratic $gl(n)$-polynomial identity, whose roots are given by $a_i = \lambda_i + n - 1$, is

$$E(E - (k + n - r)) = 0. \tag{9}$$

Zero-step modules will occur whenever (9) coincides with the right-hand side of the anticommuator (7). This leads to the following constraints on the central charge for $n \geq 3$,

$$c = \frac{1}{2} r(r-1)k(k+1). \tag{10}$$

The following table is easily computed and gives concrete instances for the restricted case $\alpha = 0$.

n	3	4	5	5	6	7	7	7	8	9	9	9	10	10
r	2	2	2	3	2	2	3	4	2	2	3	5	2	4
k	1	2	3	1	4	5	2	1	6	7	3	1	8	2
c	2	6	12	6	20	30	18	12	42	56	36	20	72	36

Zero-step atypicals ($\alpha = 0$): $gl_2(n/1)$ Kac modules $V(k^r, 0^{n-r})$ for $n = 3, 4, ..., 10$

3. Quadratic Conformal Supersymmetry

Our aim in this section is to illustrate how the existence of zero-step modules provides for new possibilities in extended models of supersymmetry. In particular, we show that for a certain quadratic deformation of the algebra of $N = 1$ space-time conformal supersymmetry, the zero-step modules are precisely the class of massless representations of the even part $su(2, 2)$. Due to the degeneracy of these supermultiplets, the corresponding particles of the standard model, those descended from massless multiplets at unification energies, are not accompanied by superpartners.

We begin by introducing $L = su_2(2, 2/1)$, a real from of the quadratic superalgebra $gl_2(4/1)$. The even part is $L_{\bar{0}} \cong u(2, 2) \cong su(2, 2)+gl(1)$ and the odd part is the L_0-module $L_{\bar{1}} \cong \{\bar{1}\}+\{1\}$. We use the same basis as $gl_2(4/1)$ (see Theorem 2.3), however, in this non-compact case, the requirement of having unitary representations imposes the hermiticity conditions,

$$(E^a{}_b)^\dagger = \eta^b{}_{b'} E^{b'}{}_{a'} \eta^{a'}{}_a, \qquad (Q_a)^\dagger = \eta^a{}_{b'} \overline{Q}^{b'},$$

where η =diag$(-1, -1, 1, 1)$ [1]. Due to Lemma 2.4, $su(2, 2/1)$ is obtained in the appropriate contraction limit of $su_2(2, 2/1)$.

Representations of the conformal superalgebra are induced from representations of the even subalgebra. In the case of massless representations, these supermultiplets are *shortened* to remove the even submodules which are themselves not massless [13]. The (positive energy) unitary irreducible representations of the conformal group have been classified by Mack [15] and are characterised by their energy, mass and spin/helicity. These are infinite-dimensional modules possessing a lowest weight $\lambda = (d; -j_1, -j_2)$ where d, j_1 and j_2 relate to eigenvalues of the Cartan elements,

$$\begin{aligned} H_0 &\equiv \tfrac{1}{2}(E^1{}_1 + E^2{}_2 - E^3{}_3 - E^4{}_4) \\ H_1 &\equiv \tfrac{1}{2}(E^1{}_1 - E^2{}_2) \\ H_2 &\equiv \tfrac{1}{2}(E^3{}_3 - E^4{}_4). \end{aligned} \tag{11}$$

[1] This basis may be brought into correspondence with that of [13, 14]. Set $L^a{}_b = E^a{}_b - \frac{1}{4}\delta^a{}_b \langle E \rangle$ and define $T^a{}_b = \eta^a{}_{a'} L^{a'}_{b'} \eta^{b'}{}_b$ satisfying $[T^a{}_b, T^c{}_d] = \eta^a{}_d T^c{}_b - \eta^c{}_b T^a{}_d$.

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012115 doi:10.1088/1742-6596/1194/1/012115

These action of these on the lowest weight vector $|\lambda\rangle$ is

$$H_0|\lambda\rangle = d|\lambda\rangle, \qquad H_1|\lambda\rangle = -j_1|\lambda\rangle, \qquad H_2|\lambda\rangle = -j_2|\lambda\rangle.$$

The unitary irreducible representations comprise the following five classes [15]:

$$\begin{array}{rlll}
(\textit{Trivial}) & (1) & d = j_1 = j_2 = 0. & \\
(\textit{Massive}) & (2) & j_1 \neq 0,\, j_2 \neq 0,\, d > j_1 + j_2 + 2 & s = |j_1 - j_2| ... j_1 + j_2 \\
 & (3) & j_1 j_2 = 0,\, d > j_1 + j_2 + 1 & s = j_1 + j_2 \\
 & (4) & j_1 \neq 0, j_2 \neq 0,\, d = j_1 + j_2 + 2 & s = j_1 + j_2 \\
(\textit{Massless}) & (5) & j_1 j_2 = 0,\, d = j_1 + j_2 + 1 & \text{helicity } = j_1 - j_2.
\end{array}$$

In the present (non-compact) case, the determination of the minimal polynomial identity satisfied by E is a difficult problem since the unitary representations are infinite-dimensional and the fundamental contragredient representation is non-unitary. This problem has been solved in principle through the analysis of Kostant [11] and Gould [16], although in practice it still requires a subtle understanding of the structure of each module. Drawing on the detailed analyis of the representation theory of $su(2,2)$ due to Yao [17, 18], we have shown in [6] that E satisfies the following quadratic minimal identity for representations $V_0(\lambda)$ belonging to the class of massless representations satisfing (class (5) above),

$$E(E - (2j_1 - 2j_2 + 1) = 0. \tag{13}$$

In particular, this result depends on the so-called *maximal degeneracy* of the massless representations which, in this context, means that the entire module can be decomposed into a sum of finite-dimensional modules of the maximal compact subgroup where each submodule occurs with unit multiplicity (see Lemma 4.1 [6] for details). This leads to the main result.

Lemma 3.1 (Massless multiplets are zero-step modules (Lemma 4.2 [6])). *The anticommutator for* $su_2(2, 2/1)$ *may be brought into correspondence with a quadratic minimal polynomial identity for* massless *representations of the even subalgebra. The coincidence occurs for the parameter value* $\alpha = -3$ *and for central charge* $\mathsf{c} = 0$.

Proof. We begin by noting that for $\alpha = -3$ and $\mathsf{c} = 0$ the anticommutator (7) may be expressed in the form

$$\{\overline{Q}^a, Q_b\} = p(E)^a{}_b + \delta^a{}_b \mathsf{Tr}[p(E)],$$

where $p(E) = E(E-(\langle E\rangle+3))$. It remains to show that $p(E)$ coincides with a quadratic identity of the massless representations. Using (11), the eigenvalue of the linear Casimir $\langle E\rangle$ is easily determined to be $2j_1 - 2j_2 - 2$. The result is obtained by comparison with (13). □

4. Conclusion

In this work we identify a class of non-linear algebras, natural extensions of standard supersymmetric algebras, in which supersymmetry principles may be imposed without the need for each particle of the standard model to be accompanied by a superpartner. Non-linear algebraic structures such as these have a natural and established place in theoretical physics. Quantum groups and related non-linear structures such as Yangians and $\mathcal{W}$-algebras have found numerous applications since their advent in the early 1980's [19, 20] and they arise as symmetries in very elementary systems [21]. This work builds upon initial investigations of 'supersymmetry without superpartners' [22] where similar results were obtained in the restricted setting of oscillator representations of the even subalgebra. While all oscillator representations of $su(2,2)$ are known to be massless representations [23, 24], in this work we are at pains to provide an algebraic foundation for the results, and as such provide a robust framework for detailed model implementations. Future work includes the investigation of both a coproduct for quadratic superalgebras as well as the potential relationship between these algebras and instances of finite $\mathcal{W}$-(super)algebras.

5. Acknowledgments

This paper is based on a contributed talk at Group 32, Prague, July 2018 and draws from joint research with Peter Jarvis, School of Physical Sciences, University of Tasmania. The work is partially supported by an Australian Commonwealth APA postgraduate scholarship.

References

[1] Bechtle P, Plehn T and Sander C 2015 *The Large Hadron Collider: Harvest of Run 1* ed Schörner-Sadenius T (Springer International Publishing) pp 421–462
[2] Coleman S and Mandula J 1967 *Physical Review* **159** 1251
[3] Haag R, Łopuszański J T and Sohnius M 1975 *Nuclear Physics B* **88** 257–274
[4] Jarvis P and Rudolph G 2003 *Journal of Physics A: Mathematical and General* **36** 5531
[5] Jarvis P D, Rudolph G and Yates L A 2011 *Journal of Physics A: Mathematical and Theoretical* **44** 235205
[6] Yates L A and Jarvis P D 2017 *Journal of Physics A: Mathematical and Theoretical* **51** 145203
[7] Kac V 1977 *Advances in Mathematics* **26** 8 – 96
[8] Gould M 1989 *Journal of Physics A: Mathematical and General* **22** 1209
[9] Bracken A J and Green H S 1971 *Journal of Mathematical Physics* **12** 2099–2106
[10] O'Brien D, Cant A and Carey A L 1977 *Annales de l'IHP Physique théorique* vol 26 pp 405–429
[11] Kostant B 1975 *Journal of Functional Analysis* **20** 257 – 285
[12] Gould M 1985 *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics* **26**(03) 257–283
[13] Binegar B 1986 *Phys. Rev. D* **34**(2) 525–532
[14] Flato M and Fronsdal C 1984 *Letters in Mathematical Physics* **8** 159–162
[15] Mack G 1977 *Communications in Mathematical Physics* **55** 1–28
[16] Gould M 1984 *Journal of Physics A: Mathematical and General* **17** 1
[17] Yao T 1967 *Journal of Mathematical Physics* **8** 1931–1954
[18] Yao T 1968 *Journal of Mathematical Physics* **9** 1615–1626
[19] Takhtadzhan L A and Faddeev L D 1979 *Russian Mathematical Surveys* **34** 11–68
[20] Bonatsos D and Daskaloyannis C 1999 *Progress in particle and nuclear physics* **43** 537–618
[21] de Boer J, Harmsze F and Tjin T 1996 *Physics Reports* **272** 139–214
[22] Jarvis P 2012 *J.Phys.* **A45** 322001
[23] Mack G and Todorov I 1969 *Journal of Mathematical Physics* **10** 2078–2085
[24] Kurşunoğlu B 1967 *Journal of Mathematical Physics* **8** 1694–1699
[25] Polishchuk A and Positselski L 2005 *American Mathematical Society, Providence, RI* **8**
[26] Braverman A and Gaitsgory D 1996 *Journal of Algebra* **181** 315 – 328
[27] Priddy S B 1970 *Transactions of the American Mathematical Society* **152** 39–60

Appendix A. Quadratic Algebras

The class of quadratic superalgebras may be framed within the much larger class of *quadratic algebras*. In this setting there exists a generalisation of the classical PBW theorem to which there corresponds a set of generalised Jacobi identities, and in certain cases, an algorithm to derive a basis of ordered monomials. This appendix is a short excerpt from our earlier work (see §2 [6]); it provides a brief account of the main definitions and theorems, drawing substantially from the text due to Polishchuk and Positselski [25] and references therein including Braverman and Gaitsgory [26] and Priddy [27].

Let X be a finite-dimensional vector space of dimension n and let $T(X)$ be the tensor algebra generated by X. We fix a set of *non-homogeneous quadratic relations* $I \subset (X \otimes X) \oplus X \oplus \mathbb{C}$ to which there corresponds a set of *homogeneous relations* $I_2 \subset T \otimes T$ which are the projection of I onto $X \otimes X$. We denote by (I) and (I_2) the ideal generated in $T(X)$ by I and I_2 respectively.

Definition A1 (Quadratic Algebra)**.** Let X and I be defined as above. The algebras

$$U = T(X)/(I) \qquad \text{and} \qquad A = T(X)/(I_2).$$

are called the **inhomogeneous quadratic algebra** and the **homogenous quadratic algebra** respectively, generated by X and I.

Note that the direct sum decomposition $I \subset X \otimes X + X + \mathbb{C}$ enables maps $\alpha : I_2 \to X$ and $\beta : I_2 \to \mathbb{C}$ to be defined such that

$$I = \{x - \alpha(x) - \beta(x) | x \in I_2\}.$$

Within the class of quadratic algebras there exists an important subclass possessing certain cohomological properties which permit a generalisation of the classical PBW theorem. This class contains only homogenous quadratic algebras and is defined by the notion of Koszulness for which there are many equivalent definitions. We give the following definition in terms of the distributivity of certain vector subspaces in the tensor algebra.

Definition A2 (Koszul Algebra([25], chapter 2, theorem 4.1))**.**
A homogeneous quadratic algebra $A = T(X)/(I_2)$ is Koszul iff for all $n \geq 0$ the collection of subspaces

$$X^{\otimes i-1} \otimes I_2 \otimes X^{\otimes n-i-1} \subset X^{\otimes n}, \quad i = 1, ..., n-1$$

is distributive.

Associated with the tensor algebra is the filtration defined by $T_n = \sum_{k=0}^{n} T^k$ in such a way that $\mathbb{C} \cong T_0 \subset T_1 \subset T_2 \subset \cdots$, that is, $T_n \subset T_{n+1}$. U inherits this filtration in the natural way so that we have $\mathbb{C} \cong U_0 \subset U_1 \subset U_2 \subset \cdots$ and we define the *associated graded algebra* as the direct sum

$$grU \equiv \bigoplus_{n}^{\infty} U_n/U_{n-1}.$$

A, the homogeneous version of U, is generated by first homogenising, that is truncating, each term in the generating relations and then factoring the tensor algebra by the resulting ideal. The construction of grU, on the other hand, is obtained by initially retaining the full set of non-homogeneous relations and instead truncating the terms appearing in the corresponding ideal. The generalisation of the PBW theorem is a statement of the conditions under which these two graded algebras coincide.

Theorem A3 (Generalised PBW theorem([25], chapter 5, theorem 2.1))**.**
When A is Koszul, and the following conditions are satisfied:

(J1) $(\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)|_{I_2 \otimes L \cap L \otimes I_2} \subset I_2$;
(J2) $\alpha \circ (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)|_{I_2 \otimes L \cap L \otimes I_2} = -(\beta \otimes \mathsf{id} - \mathsf{id} \otimes \beta)|_{I_2 \otimes L \cap L \otimes I_2}$
(J3) $\beta \circ (\alpha \otimes \mathsf{id} - \mathsf{id} \otimes \alpha)|_{I_2 \otimes L \cap L \otimes I_2} = 0$

we have the isomorphism

$$gr(U) \cong A.$$

The relations (J1)-(J3) are called the *generalised Jacobi identities*. Proving Koszulness is in general a difficult undertaking and one may select from a variety of methods (for example [25], chapter 2). Here instead we investigate when the homogeneous algebra A of theorem A3 satisfies the stronger condition of being a *PBW algebra*, that is, admitting an ordered basis of monomials in the generators $x_i \in X$ (see definition A4). A theorem due to Priddy ([27], theorem 5.3) states that every homogeneous PBW algebra is Koszul.

Before giving a formal definition of a PBW algebra we must make precise the notion of an ordered set of monomials with respect to a set of quadratic relations. Following [27] let x_i, $i \in S_1 := \{1, .., n\}$ be a basis for X. S_1 defines an ordering on X such $x_i < x_j$ when $i < j$. S_1 also defines a lexicographic ordering on monomials belonging to $T(X)$. I_2 comprises expressions of the form $\sum c^{kl} x_k x_l$. Each such expression contains a unique *leading monomial* which is the highest quadratic term, with respect to the lexicographic ordering, in the sum. We define

$$S_2 := \{(i, j) | i, j \in S_1,\ x_i x_j \text{ is not a leading monomial}\}.$$

We also define

$$S_i := \{(i_1, i_2, ..., i_n) | (i_j, i_{j+1}) \in S_2, j = 1, 2, ..., n-1\}.$$

Consider now the grading of A inherited from $T(X)$. Since I_2 is by construction homogeneous, each graded subspace A_i contains only homogeneous elements of degree i. It follows that

$$\begin{aligned} &A_0 \cong \mathbb{C}, \\ &A_1 \cong X \text{ has a basis } \{x_i | x_i \in X,\ i \in S_1\}, \\ &A_2 \text{ has a basis } \{x_i x_j | x_i, x_j \in X,\ (i, j) \in S_2\}. \end{aligned}$$

Definition A4 (PBW Algebra)**.** The homogenous quadratic algebra A is a *PBW-algebra* if the monomials $\{x_{i_1} x_{i_2} ... x_{i_n},\ (i_1, i_2, ..., i_n) \in S_n\}$ are a basis for A. In this case the monomials are called a *PBW-basis* of A.

Note that these monomials always span A, thus the task of establishing the PBW property is to determine their linear independence. In order to proceed with this task we define the mapping $\pi : X \otimes X \to X \otimes X$

$$\pi(x_i x_j) = \begin{cases} x_i x_j & (i, j) \in S_2 \\ \sum_{(k,l) \in S_2} c^{kl} x_k x_l & (i, j) \notin S_2 \end{cases}$$

which extends linearly to $X \otimes X$ and essentially replaces leading monomials with a unique sum of non-leading terms as determined by I_2. Finally we define

$$\pi^{12} = \pi \otimes I : T_3 \to T_3. \qquad \pi^{23} = I \otimes \pi : T_3 \to T_3.$$

Lemma A5 (Thm 2.1 p.82 [25] - Diamond Lemma)**.**
A is a PBW-algebra iff the cubic monomials ($x_i x_j x_k$, $(i, j, k) \in S_3$) are linearly independent in A_3. Equivalently A is a PBW-algebra iff the following equation holds:

$$\cdots \pi^{12} \pi^{23} \pi^{12} \pi^{23} \pi^{12} = \cdots \pi^{23} \pi^{12} \pi^{23} \pi^{12} \pi^{23}. \tag{A-1}$$

Remarks. The infinite composition is well defined since π decreases the order. To establish (A-1) we need only consider basis elements $x_i x_j x_k \in T_3$ such that both $(i, j), (j, k) \notin S_2$. For if one of these belonged to S_2 then one of either π^{12} or π^{23} will act trivially on the starting term $x_i x_j x_k$ and (A-1) follows immediately. The PBW property depends on the choice of ordering given to the generators of X. That is, a fixed homogeneous quadratic algebra may be a PBW algebra given one ordering but not for another.