On congruences of automata defined by directed graphs

A.V. Kelarev [a,1] O.V. Sokratova [b,1,2]

[a] *Faculty of Science and Engineering, University of Tasmania GPO Box 252-37, Hobart, Tasmania 7001, Australia*

[b] *Institute of Computer Science, Tartu University J.Liivi 2, 50409 Tartu, Estonia*

Abstract

Graphs and various objects derived from them are basic essential tools that have been actively used in various branches of modern theoretical computer science. In particular, graph grammars and graph transformations have been very well explored in the literature. We consider finite state automata defined by directed graphs, characterize all their congruences, and give a complete description of all automata of this type satisfying three properties for congruences introduced and considered in the literature by analogy with classical semisimplicity conditions that play important roles in structure theory.

Key words: automata, congruences, directed graphs

1 Introduction

Graphs and various objects derived from them are basic essential tools that have been actively used in various branches of modern theoretical computer science. In particular, graph grammars and graph transformations have been very well explored in the literature (see, for example, [3]). It makes sense to consider new ways of defining classical finite state acceptors using graph labelings, and determine how properties of the acceptors depend on the properties

Email addresses: `Andrei.Kelarev@utas.edu.au` (A.V. Kelarev), `olga@cs.ut.ee` (O.V. Sokratova).

1 The authors were supported by Australian Research Council, grant K10367.

2 The author was supported by Estonian Science Foundation, grant 4912, and by Fulbright Fellow Award.

of the original graph labeling. Our paper contributes to this direction and continues the investigation initiated in [5], where the concept of a graph algebra has been used in order to define finite state automata. In the present paper we study congruences of these automata.

Congruences of automata are important tools for the investigation of the structure of automata, efficiency of their applications, optimization algorithms, and issues that arise in implementing automata. In this paper we consider three properties of automata congruences that have been known and used in the study of other types of objects for a long time. These properties are natural analogues of classical semisimplicity conditions playing key roles in structure theory as crucial concepts for describing several classes of objects. The research devoted to structural descriptions using methods relying on these properties has developed into a large area involving general considerations in the framework of category theory (see, in particular, [4]). In the case of finite state automata these concepts have been first addressed in [9]. Instead of trying to be complete in discussing achievements and branches of this direction, we refer the reader to [10] and [11] for the history of this topic, detailed explanations of roles of these semisimplicity conditions, known facts obtained earlier, and a few introductory references to relevant papers of other authors. The exact definitions of these properties (O1), (O2), and (O3) are given in Section 2.

We use standard concepts of automata and languages theory, following [7] and [14]. Let X be an alphabet. A *language* over X is a subset of the free monoid X^* generated by X. Throughout the word *graph* means a finite directed graph without multiple edges but possibly with loops. In this paper we consider graphs and their algebras as means to define language recognizers.

Graph algebras have been investigated by several authors (see, for example, [2], [6], [8], [12], and [13]) in relation to various problems of discrete mathematics and computer science. The *graph algebra* $\mathrm{Alg}(D)$ of a graph $D = (V, E)$ is the set $V \cup \{0\}$ equipped with multiplication given, for all $x, y \in V$, by the rule

$$xy = \begin{cases} x \text{ if } (x, y) \in E, \\ 0 \text{ otherwise.} \end{cases}$$

Let $\mathrm{Alg}(D)^1$ be the graph algebra with identity 1 adjoined, T a subset of $\mathrm{Alg}(D)^1$, and let $f\colon X \to \mathrm{Alg}(D)^1$ be any mapping. Consider the *graph algebra automaton* $\mathrm{Atm}(D, T)$, where

(A1) the set of states is $\mathrm{Alg}(D)^1$;
(A2) 1 is the initial state;
(A3) T is the set of terminal states;
(A4) the next-state function is defined by left multiplications of elements of the

graph algebra, i.e., $a \cdot x = f(x)a$, for $a \in \mathrm{Alg}(D)^1$, $x \in X$.

To illustrate the definition we include an easy example (see Figure 1).

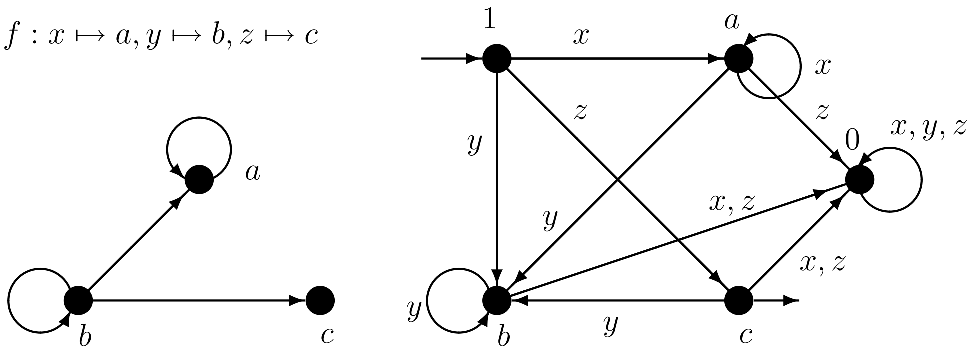

Fig. 1. Directed graph and its automaton

The *language recognized* or *accepted* by $\mathrm{Atm}(D,T)$ is $\{u \in X^* \mid 1 \cdot u \in T\}$. To illustrate let us mention that if D is a null graph, then all products in $\mathrm{Alg}(D)$ are zero, and in this case the language recognized by $\mathrm{Atm}(D,T)$ is $f^{-1}(1)^* f^{-1}(T)$ if $0 \notin T$, and $X^* f^{-1}(T) \cup X^*(X - f^{-1}(1))X^*(X - f^{-1}(1))X^*$ if $0 \in T$.

Let D' be the subgraph induced in D by the set $V' = V \cap f(X)$ of vertices. Consider the automaton $\mathrm{Atm}(D',T)$, defined by the graph D' and the same function f. It is easily seen that $\mathrm{Atm}(D',T)$ recognizes the same language as the original automaton $\mathrm{Atm}(D,T)$. Thus throughout we may assume that $V \subseteq f(X)$.

All languages recognized by the graph algebra automata have been described in terms of regular expressions and combinatorial properties in [5]. This description has answered several natural questions concerning the class $\mathcal{G}$ of languages recognized by graph algebra automata. For example, it has been shown that this class contains certain fairly large subclasses, and that it is a proper subclass of the class of regular languages. Besides, it has been proven that the whole class $\mathcal{G}$ is closed under the Kleene $*$-operation and complement. Although $\mathcal{G}$ is not closed for union, intersection, and product, it can be represented as a union of two classes one of which is closed under intersection and left derivative, and another is closed under union and right derivative.

Our main theorem describes all graph algebra automata satisfying the conditions (O1), (O2), and (O3) for congruences, and shows that in this case all three properties are equivalent (see Theorem 1). Two of our technical lemmas used in the proof are of independent interest. For each automaton $\mathrm{Atm}(D,T)$, they describe all congruences (Lemma 2) and the Nerode equivalence (Lemma 3).

2 The Main Theorem

A few definitions are required for the main theorem. An equivalence relation ϱ on the set of states compatible with the next-state function defines the quotient automaton in a standard fashion. Since the initial state 1 is adjoined to every graph externally, in order to make sure that the quotient automaton is also a graph algebra automaton, we have to restrict our attention to equivalence relations such that the class containing 1 is a singleton. In this case we say that ϱ is an *equivalence relation of the automaton* $\mathrm{Atm}(D,T)$.

The quotient automaton recognizes the same language if and only if the relation *saturates* the set T of terminal states, i.e., T is the union of some classes of the relation. Equivalence relations of this sort will be called *congruences* on the automaton $\mathrm{Atm}(D,T)$. This ensures that the quotient automaton is also defined by a graph. More formally, an equivalence relation ρ on the set of states is called a *congruence* of $\mathrm{Atm}(D,T)$ if and only if it satisfies the following three conditions:

(C1) $(a,b)\in\rho$ implies $(a\cdot x, b\cdot x)\in\rho$, for all $a,b\in \mathrm{Alg}(D), x\in X$;
(C2) if $(a,b)\in\rho$ and $a\in T$, then $b\in T$;
(C3) the class containing 1 is a singleton.

Denote by $\mathrm{Con}(D,T)$ the set of all congruences on $\mathrm{Atm}(D,T)$. Given congruences ρ,δ on $\mathrm{Atm}(D,T)$, the meet $\rho\bigwedge\delta$ and join $\rho\bigvee\delta$ denote their intersection and the transitive closure of their union, respectively. It is well known and easy to verify that the set of all congruences on any automaton forms a lattice with respect to $\bigwedge$ and $\bigvee$. Therefore $\mathrm{Con}(D,T)$ is a sublattice of the lattice of all equivalence relations on $\mathrm{Alg}(D)^1$ with 1 in a separate class. The largest congruence on $\mathrm{Atm}(D,T)$ is the *Nerode congruence* σ_T described by

$$\begin{aligned}\sigma_T = \{&(a,b)\in \mathrm{Alg}(D)\times \mathrm{Alg}(D) \mid a\cdot u\in T \text{ iff } b\cdot u\in T\\ &\text{for all } u\in X^*\}\cup\{(1,1)\}\end{aligned}\tag{1}$$

(see, e.g., [1]). Denote the equality relation on $\mathrm{Atm}(D,T)$ by ι. A congruence is said to be *proper* if it is distinct from ι and σ_T. A congruence ρ on $\mathrm{Atm}(D,T)$ is said to be *essential* if, for every $\delta\in \mathrm{Con}(D,T)$, the equality $\rho\bigwedge\delta=\iota$ implies $\delta=\iota$. A congruence ρ in $\mathrm{Atm}(D,T)$ is called a *direct summand* if there exists $\delta\in \mathrm{Con}(D,T)$ such that $\rho\bigwedge\delta=\iota$ and $\rho\bigvee\delta=\sigma_T$. In this case δ is called a *complement* of ρ, and we write $\rho\perp\delta$.

For any subset S of $\mathrm{Alg}(D)^1$, denote by $\overline{S}$ the set $\mathrm{Alg}(D)\setminus S$. Note that

$\overline{S}$ never contains 1. Clearly, an equivalence relation on $\mathrm{Alg}(D)^1$, with 1 in a separate class, saturates a subset $S \subseteq \mathrm{Alg}(D)$ if and only if it saturates $\overline{S}$. In order to consider the cases where $0 \in T$ and $0 \notin T$ simultaneously, we define the following set:

$$T_0 = \begin{cases} T \setminus \{1\} & \text{if } 0 \in T, \\ \overline{T} & \text{otherwise.} \end{cases}$$

The *in-neighbourhood* and *out-neighbourhood* of a vertex v of $D = (V, E)$ are the sets $\mathrm{In}(v) = \{w \in V \mid (w, v) \in E\}$ and $\mathrm{Out}(v) = \{w \in V \mid (v, w) \in E\}$. We say that a subset $S \subseteq \mathrm{Alg}(D)$ is *in-closed* if $\mathrm{In}(S) \subseteq S$, where $\mathrm{In}(S) = \bigcup\limits_{s \in S} \mathrm{In}(s)$. Putting $\mathrm{In}(0) = \emptyset$ we see that $\{0\}$ is in-closed.

Let C_0 be the set of all elements $c \in T_0$ such that there does not exist any vertex $v \in \overline{T}_0$ with a directed path from v to c. Obviously, C_0 is the largest in-closed subset of T_0, and it always contains 0. The main result of this paper to be proven in Section 4 is the following

Theorem 1 *Let $D = (V, E)$ be a finite directed graph, and let $T \subseteq V \cup \{0, 1\}$. Then the following conditions are equivalent:*

(O1) *the automaton $\mathrm{Atm}(D, T)$ has no proper essential congruences;*
(O2) *the Nerode congruence σ_T on $\mathrm{Atm}(D, T)$ is a join of minimal congruences;*
(O3) *every proper congruence on $\mathrm{Atm}(D, T)$ is a direct summand;*
(O4) *there exists an in-closed subset M of $C_0 \setminus \{0\}$ satisfying the following four properties:*
  (i) *all connected components of the subgraph induced by M in D are strongly connected;*
  (ii) *all vertices in M with nonzero in-degrees have pairwise distinct in-neighbourhoods;*
  (iii) *for each $a \in C_0 \setminus \{0\}$, there exists $a' \in M$ such that $\mathrm{In}(a) = \mathrm{In}(a')$;*
  (iv) *for all $a, b \in T_0 \setminus C_0$ or $a, b \in \overline{T}_0$, the equality $\mathrm{In}(a) \cap \overline{C}_0 = \mathrm{In}(b) \cap \overline{C}_0$ implies $\mathrm{In}(a) = \mathrm{In}(b)$.*

3 Technical Lemmas

For any subset S of T_0, consider three auxiliary relations

$$\begin{aligned} \mu_{S,S} &= (S \cup \{0\}) \times (S \cup \{0\}), \\ \mu_S^{T_0} &= \{(a, b) \mid \mathrm{In}(a) \cap \overline{S} = \mathrm{In}(b) \cap \overline{S} \text{ and } a, b \in T_0 \setminus (S \cup \{0\})\}, \\ \mu_S^{\overline{T}_0} &= \{(a, b) \mid \mathrm{In}(a) \cap \overline{S} = \mathrm{In}(b) \cap \overline{S} \text{ and } a, b \in \overline{T}_0\}. \end{aligned}$$

We introduce the relation

$$\mu_S = \{(1,1)\} \cup \mu_{S,S} \cup \mu_S^{T_0} \cup \mu_S^{\overline{T}_0}. \tag{2}$$

Clearly, μ_S is an equivalence relation on $\mathrm{Atm}(D,T)$, and $\mu_S = \mu_{S\cup\{0\}} = \mu_{S\setminus\{0\}}$. It may happen that $S_1 \subseteq S_2$, but $\mu_{S_1} \not\subseteq \mu_{S_2}$, for example, if D has isolated vertices. The following lemma describes all congruences in $\mathrm{Con}(D,T)$.

Lemma 2 *Let ρ be an equivalence relation on $\mathrm{Atm}(D,T)$. Denote by S the class of ρ containing 0. Then ρ is a congruence on $\mathrm{Atm}(D,T)$ if and only if S is an in-closed subset of C_0 and $\rho \subseteq \mu_S$. In particular, for every in-closed subset $S \subseteq C_0$, the relation μ_S is a congruence on $\mathrm{Atm}(D,T)$.*

PROOF. The 'if' part: Suppose that $\rho \subseteq \mu_S$ and S is an in-closed subset of C_0. Since μ_S satisfies conditions (C2) and (C3), it follows that the same can be said of ρ. In order to verify (C1) for ρ, consider any pair $(a,b) \in \rho$ and $f(x) = c$ where $c \notin \{0,1\}$, i.e. $c \in V$.

First, if $c \in \mathrm{In}(a) \cap \mathrm{In}(b)$, then $(a \cdot x, b \cdot x) = (ca, cb) = (c,c) \in \rho$.

Second, if $c \notin \mathrm{In}(a) \cup \mathrm{In}(b)$, then $(a \cdot x, b \cdot x) = (0,0) \in \rho$, too.

Third, suppose that $c \in \mathrm{In}(a) \setminus \mathrm{In}(b)$. We claim that $c \in S$. Indeed, if $a \in S$, then $c \in \mathrm{In}(a) \subseteq \mathrm{In}(S) \subseteq S$, because S is in-closed. If, however, $a \notin S$, then $\rho \subseteq \mu_S$ implies that $\mathrm{In}(a) \cap \overline{S} = \mathrm{In}(b) \cap \overline{S}$, and $c \in S$ again. It follows that $(a \cdot c, b \cdot c) = (ca, cb) = (c, 0) \in S \times S \subseteq \rho$.

The case where $c \in \mathrm{In}(b) \setminus \mathrm{In}(a)$ is similar, and so we have proved that (C1) holds. Thus ρ is a congruence on $\mathrm{Atm}(D,T)$.

The 'only if' part: Suppose that ρ is a congruence on the automaton $\mathrm{Atm}(D,T)$. Clearly, $S \subseteq T_0$, because ρ saturates T. To prove that S is in-closed, take any vertex $a \in S$. Condition (C1) implies that $(b,0) = (ba, b0) \in \rho$, for every $b \in \mathrm{In}(a)$. Therefore $\mathrm{In}(S) \subseteq S$.

In order to show that $\rho \subseteq \mu_S$, pick any pair $(a,b) \in \rho$. If $a, b \in S$, then $(a,b) \in \mu_S$, because $S \cup \{0\}$ is an equivalence class of μ_S.

Furthermore, assume that $a, b \notin S$. Condition (C2) shows that ρ saturates T, and so $a, b \in T_0 \setminus S$ or $a, b \in \overline{T}_0$. If there exists $c \in \mathrm{In}(a) \setminus \mathrm{In}(b)$, then (C1) implies $(c, 0) = (ca, cb) \in \rho$, and we get $c \in S$. Hence $\mathrm{In}(a) \setminus \mathrm{In}(b) \subseteq S$. Similarly, $\mathrm{In}(b) \setminus \mathrm{In}(a) \subseteq S$, and so $\mathrm{In}(a) \cap \overline{S} = \mathrm{In}(b) \cap \overline{S}$. By the definition of μ_S we see that $(a,b) \in \mu_S$. Thus $\rho \subseteq \mu_S$. $\square$

Lemma 3 *The Nerode congruence σ_T coincides with μ_{C_0}.*

PROOF. By Lemma 2, the class $0/\sigma_T$ of the Nerode congruence containing 0 is in-closed. Therefore it is contained in the largest in-closed subset C_0 of T_0.

To prove the reversed inclusion, consider any vertex $c \in C_0$ and use the equality (1). Since C_0 is in-closed, we see that $c \cdot w \in T_0$ for all $w \in X^*$. Obviously, $0 \cdot w \in T_0$ for all $w \in X^*$, too. It follows from (1) that $(c, 0) \in \sigma_T$. Thus $C_0 = 0/\sigma_T$.

Lemma 2 tells us that $\sigma_T \subseteq \mu_{C_0}$. However, σ_T is the largest congruence. Therefore $\sigma_T = \mu_{C_0}$. □

For a subset A of $\mathrm{Alg}(D)$, denote by $\Theta(A)$ the equivalence relation $\iota \cup (A \times A)$ on $\mathrm{Atm}(D, T)$.

Lemma 4 *For any $A \subseteq \mathrm{Alg}(D)$, the equivalence relation $\Theta(A)$ is a congruence if and only if one of the following conditions holds:*

(i) *all vertices of A have the same in-neighbourhoods, and either $A \subseteq \overline{T}_0$ or $A \subseteq T_0 \setminus \{0\}$;*
(ii) *A is an in-closed subset of T_0 and $0 \in A$.*

PROOF. The 'if' part: If (i) holds, then $\Theta(A) \subseteq \mu_{\{0\}}$. If (ii) holds, then $\Theta(A) \subseteq \mu_A$. In both cases $\Theta(A)$ is a congruence by Lemma 2.

The 'only if' part: Suppose that $\Theta(A)$ is a congruence. If $0 \notin A$, then $0/\Theta(A) = \{0\}$, and so $\Theta(A) \subseteq \mu_{\{0\}}$ by Lemma 2. The definition of $\mu_{\{0\}}$ shows that (i) holds. If $0 \in A$, then $0/\Theta(A) = A$, and so A is an in-closed subset of T_0 by Lemma 2, i.e., condition (ii) is satisfied. □

A graph is said to be *strongly connected* if, for every vertices $u \neq v$, there exists a directed path from u to v.

Lemma 5 *Let M be an in-closed subset of the set V of vertices. Then the subgraph of $D = (V, E)$ induced by M is strongly connected if and only if M is a minimal nonempty in-closed subset of V.*

PROOF. If M is a singleton, then the assertion is trivial, and so we assume that $|M| > 1$. For any vertex $v \in V$, denote by $\mathrm{In}^*(v)$ the set of all vertices $u \in V$ such that there exists a directed path from u to v. Obviously, $\mathrm{In}^*(v)$ is the smallest in-closed set containing v. It is easily seen that, for every in-closed subset M of vertices, the subgraph of D induced by M is strongly connected if and only if $M = \mathrm{In}^*(v)$, for every $v \in M$. Let us verify that the latter fact is equivalent to M being a minimal in-closed subset of V.

To prove one implication, suppose to the contrary that $M = \mathrm{In}^*(v)$, for every $v \in M$, but there exists an in-closed proper subset N of M. Then, for each vertex $v \in N$, the set $\mathrm{In}^*(v)$ is contained in the in-closed set N, contradicting $M = \mathrm{In}^*(v)$.

For the converse implication, suppose that M is a minimal in-closed subset of V. Take any $v \in M$. Obviously, $\mathrm{In}^*(v) \subseteq M$ since M is in-closed. Since the set $\mathrm{In}^*(v)$ is in-closed, by the minimality of M we get $M = \mathrm{In}^*(v)$. This completes our proof. $\Box$

4 Proof of the Main Theorem

PROOF of Theorem 1. (O2)$\Rightarrow$(O1): Suppose that σ_T is a join of minimal congruences α_i on $\mathrm{Atm}(D,T)$, where $i \in I$. Consider a proper congruence ρ on $\mathrm{Atm}(D,T)$. For every $i \in I$, either $\alpha_i \subseteq \rho$, or $\alpha_i \bigwedge \rho = \iota$. If all the α_i are contained in ρ, then $\rho = \sigma_T$, a contradiction with ρ being proper. Therefore there exists $i \in I$ such that $\alpha_i \bigwedge \rho = \iota$. Thus ρ is not essential, and so $\mathrm{Atm}(D,T)$ has no proper essential congruences.

(O3)$\Rightarrow$(O2): Denote by α_i, where $i \in I$, all minimal congruences on $\mathrm{Atm}(D,T)$. Suppose to the contrary that $\alpha = \bigvee\limits_{i\in I} \alpha_i$ is properly contained in σ_T. By (O3) there exists a congruence $\delta \neq \iota$ such that $\alpha \perp \delta$. Choose a minimal proper congruence $\delta' \subseteq \delta$. Then $\delta' \subseteq \alpha$, and so $\delta' = \alpha \bigwedge \delta' \subseteq \alpha \bigwedge \delta = \iota$. This contradiction shows that $\sigma_T = \bigvee\limits_{i\in I} \alpha_i$.

(O4)$\Rightarrow$(O3): Assume that there exists an in-closed subset M of $C_0 \setminus \{0\}$ satisfying properties (i) to (iv). Take any proper congruence ρ on the automaton $\mathrm{Atm}(D,T)$.

First, consider the easy case where $M = \emptyset$. Condition (iii) implies that $C_0 = \{0\}$. Lemma 2 shows that an equivalence relation is a congruence if and only if it is contained in $\mu_{\{0\}}$. By Lemma 3, $\sigma_T = \mu_{\{0\}}$. Denote by K_i, where $i \in I$, all $\mu_{\{0\}}$-classes not equal to $\{0\}$. Evidently, every class K_i is a disjoint union of some ρ-classes: $K_i = \bigcup\limits_{j\in J_i} K_{ij}$, for some J_i. In this notation all classes of the congruence ρ are $\{0\}$ and the K_{ij}, where $i \in I, j \in J_i$. For each $i \in I$, pick one element a_{ij} in each K_{ij}. The equivalence relation $\tau = \bigcup\limits_{i\in I} \Theta(\{a_{ij} \mid j \in J_i\})$ is a congruence, because $\tau \subseteq \mu_{\{0\}}$. It is easily seen that $\rho \perp \tau$. Thus condition (O3) holds.

Second, consider the case where $M \neq \emptyset$. If a vertex v of T_0 has in-degree 0 and does not belong to M, then it is an isolated vertex of the subgraph induced in D by $M \cup \{v\}$, and so we can adjoin it to M. Therefore without loss of generality we may assume that M contains all vertices of T_0 with in-degree zero. We are going to define a congruence δ which is a direct complement to ρ. To this end we have to introduce notation for certain subsets in C_0 and classify the classes of ρ in relation to these sets.

Denote by $M_1, \ldots, M_n$ the sets of vertices of all connected components of the subgraph induced by M in D. By (i) all these components are strongly connected. Lemma 5 says that $M_1, \ldots, M_n$ are minimal in-closed subsets of $C_0 \setminus \{0\}$. Take any minimal in-closed subset M' of $C_0 \setminus \{0\}$, which does not consist of a single vertex with in-degree 0. Since the subgraph induced by M' is strongly connected by Lemma 5, we get $\mathrm{In}(a) \neq \emptyset$, for any vertex $a \in M'$. Therefore (iii) implies that $\mathrm{In}(a) = \mathrm{In}(b)$, for some $b \in M_i$, where $1 \leq i \leq n$. Since both M' and M_i are in-closed, we get $\mathrm{In}(a) \subseteq M' \cap M_i$, and so $M' = M_i$, because they are minimal. Thus M is a disjoint union of all minimal in-closed subsets of $C_0 \setminus \{0\}$.

If M_i consists of a single vertex with in-degree zero, then put $N_i = M_i$. Otherwise, denote by N_i the set of all vertices a in C_0 such that $\mathrm{In}(a) = \mathrm{In}(a')$, for some $a' \in M_i$. Condition (iii) implies that $C_0 \setminus \{0\}$ is a disjoint union of the N_i, $i = 1, \ldots, n$.

For any i, put $P_i = N_i \setminus M_i$. Note that $\mathrm{Out}(a) \cap C_0 = \emptyset$, for every vertex a of $\bigcup_{i=1}^{n} P_i$. Indeed, if $(a, b) \in E$, for some $b \in C_0$, then by (iii) there exists $b' \in M$, such that $\mathrm{In}(b) = \mathrm{In}(b')$, and so $(a, b') \in E$, which is impossible, because M is in-closed.

Denote by S the class $0/\rho$. By Lemma 2, S is in-closed, and so $S \cap M$ is in-closed, too. Moreover, since M is a disjoint union of all minimal in-closed subsets of C_0, we see that both the sets $M \setminus S$ and $M \cap S$ are disjoint unions of some minimal in-closed subsets M_i of M. Assume that $M \cap S = \bigcup_{i=1}^{k} M_i$ and $M \setminus S = \bigcup_{i=k+1}^{n} M_i$, and that every set $M_1, \ldots, M_s, M_{k+1}, \ldots, M_t$ consists of a single vertex with in-degree zero, where $0 \leq s \leq k \leq t \leq n$. For $i = s+1, \ldots, k$, let $P'_i = P_i \cap S$. The set C_0 and its subsets are illustrated in Figure 2. Note that C_0 is a disjoint union of some ρ-classes. The class S is the disjoint union $\{0\} \cup (\bigcup_{i=1}^{k} M_i) \cup \bigcup_{i=s+1}^{k} P'_i$.

We claim that each ρ-class $K \subseteq C_0$ such that $K \neq S$ has one of the following four types:

Type 1: $K \subseteq P_i$, where $i = t+1, \ldots, n$, and all vertices of K have the same in-neighbourhoods.

<table>
<tr><td>N_n</td><td>M_n</td><td colspan="2">P_n</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td colspan="2">$\vdots$</td></tr>
<tr><td>N_{t+1}</td><td>M_{t+1}</td><td colspan="2">P_{t+1}</td></tr>
<tr><td>N_t</td><td>M_t</td><td></td><td></td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td></td><td></td></tr>
<tr><td>N_{k+1}</td><td>M_{k+1}</td><td></td><td></td></tr>
<tr><td>N_k</td><td>$\mathbf{M_k}$</td><td>$\mathbf{P'_k}$</td><td>$P_k \setminus P'_k$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td></tr>
<tr><td>N_{s+1}</td><td>$\mathbf{M_{s+1}}$</td><td>$\mathbf{P'_{s+1}}$</td><td>$P_{s+1} \setminus P'_{s+1}$</td></tr>
<tr><td>N_s</td><td>$\mathbf{M_s}$</td><td></td><td></td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td></td><td></td></tr>
<tr><td>N_1</td><td>$\mathbf{M_1}$</td><td></td><td></td></tr>
<tr><td></td><td>$\mathbf{0}$</td><td></td><td></td></tr>
</table>

Fig. 2. C_0

Type 2: $K \subseteq M_i \cup P_i$, where $i = t+1, \dots, n$, $|K \cap M_i| = 1$ and all vertices of K have the same in-neighbourhoods.

Type 3: $K \subseteq (P_{s+1} \setminus P'_{s+1}) \cup \dots \cup (P_k \setminus P'_k) \cup M_{k+1} \cup \dots \cup M_t$ and $K \cap M \neq \emptyset$.

Type 4: $K \subseteq (P_{s+1} \setminus P'_{s+1}) \cup \dots \cup (P_k \setminus P'_k)$.

Indeed, first suppose that K contains a vertex a of N_i, where $i \in \{t+1, \dots, n\}$. Take any b in K. By the definition of N_i we get $\mathrm{In}(a) = \mathrm{In}(a')$, for some $a' \in M_i$. Since the in-neighbourhoods of all vertices in M_i are nonempty, we see that $\emptyset \neq \mathrm{In}(a) \subseteq M_i$. By Lemma 2, $\mathrm{In}(b) \cap \overline{S} = \mathrm{In}(a)$. If $b \in N_j$, then $\mathrm{In}(b) \subseteq M_j$, and so $j = i$. Hence $b \in N_i$. It follows that $\mathrm{In}(b) = \mathrm{In}(a)$. Thus $K \subseteq N_i$, and the in-neighbourhoods of all vertices of K are equal to $\mathrm{In}(a)$. Now, if $K \cap M_i = \emptyset$, then K is of Type 1. Otherwise, if K contains a vertex of M_i, then (ii) shows that $|K \cap M_i| = 1$, and so K is of Type 2.

Suppose now that K contains a vertex a in the union

$$(P_{s+1} \setminus P'_{s+1}) \cup \dots \cup (P_k \setminus P'_k) \cup M_{k+1} \cup \dots \cup M_t.$$

Take any $b \in K$. Lemma 2 shows that $\mathrm{In}(b) \cap \overline{S} = \mathrm{In}(a) \cap \overline{S}$. We have $\mathrm{In}(a) \subseteq M_{s+1} \cup \dots \cup M_k \subseteq S$, and so $\mathrm{In}(b) \subseteq S$. Observe that $b \notin \bigcup\limits_{i=t+1}^{n} N_i$, because otherwise $\emptyset \neq \mathrm{In}(b) \subseteq \bigcup\limits_{i=t+1}^{n} M_i$, which is impossible, because $S \cap \bigcup\limits_{t+1}^{n} M_i = \emptyset$.

Therefore $b \in \bigcup_{i=1}^{t} N_i$. Since $S \cap K = \emptyset$, we get $K \subseteq (P_{s+1} \setminus P'_{s+1}) \cup \ldots \cup (P_k \setminus P'_k) \cup M_{k+1} \cup \ldots \cup M_t$. Now, if $K \cap (M_{k+1} \cup \ldots \cup M_t) \neq \emptyset$, then K is of Type 3. Otherwise, K is of Type 4.

Let us mention that using Lemma 2 it is possible to show that every partition of C_0 such that one class is a disjoint union $\{0\} \cup (\bigcup_{i=1}^{k} M_i) \cup \bigcup_{i=s+1}^{k} P'_i$, and other classes are arbitrary subsets of Types 1, 2, 3, or 4, is the restriction on C_0 of some congruence on $\mathrm{Atm}(D, T)$.

In order to define the required δ, we first construct its class containing 0. Let Q_1 be the union $M_{t+1} \cup \ldots \cup M_n$. In each ρ-class of Type 1 pick a vertex v belonging to some set P_i, where $t + 1 \leq i \leq n$, and denote the set of chosen vertices by Q_2. Choose one vertex with in-degree zero in each ρ-class of Type 3, and denote the set of these vertices by Q_3. We claim that the set $Q = Q_1 \cup Q_2 \cup Q_3$ is in-closed. Indeed, take a vertex $a \in Q$. If $a \in Q_1$, then $\mathrm{In}(a) \subseteq Q_1$, because Q_1 is a union of in-closed sets. If $a \in Q_2$, then a belongs to a set P_i, for some $i \in \{t + 1, \ldots, n\}$, and there exists a vertex $a' \in M_i$ such that $\mathrm{In}(a) = \mathrm{In}(a') \subseteq M_i \subseteq Q_1$. If $a \in Q_3$, then $\mathrm{In}(a) = \emptyset$. Thus $\mathrm{In}(a) \subseteq Q$, in all cases. Therefore

$$\delta_1 = \Theta(Q \cup \{0\})$$

is a congruence by Lemma 4.

Next, in each ρ-class of Type 4 we pick a vertex u, which belongs to some $P_i \setminus P'_i$, where $s + 1 \leq i \leq k$, and denote the set of these vertices u by U. For each $u \in U$, by (iii) there exists a vertex $u' \in M_i$ such that $\mathrm{In}(u) = \mathrm{In}(u')$. Hence $\Theta(u, u')$ is a congruence by Lemma 4. Denote by U' a minimal set of all vertices $u' \in M$ such that, for each $u \in U$ there exists $u' \in U'$ with $\mathrm{In}(u) = \mathrm{In}(u')$. Define a congruence

$$\delta_2 = \bigvee_{u \in U} \Theta(u, u').$$

It is routine to verify that the family of all classes of δ_2 consists of all classes

$$H_{u'} = \{u'\} \cup \{u \in U \mid \mathrm{In}(u) = \mathrm{In}(u')\}, \tag{3}$$

where u' runs over U'.

Furthermore, let K_i, $i \in I$, be all σ_T-classes not equal to C_0. Since σ_T is the largest congruence, every class K_i is a disjoint union of some ρ-classes: $K_i = \bigcup_{j \in J_i} K_{ij}$ for some J_i. Therefore all classes of ρ lying outside of C_0 are K_{ij}, where $i \in I, j \in J_i$. For $i \in I$, pick one element a_{ij} in each K_{ij} and consider the equivalence relation $\tau_i = \Theta(\{a_{ij} \mid j \in J_i\})$. For each $i \in I$, all the elements a_{ij}, $j \in J_i$, belong to the same σ_T-class K_i. Hence $\mathrm{In}(a_{ij}) \cap \overline{C}_0$ is the same for all $j \in J_i$ in view of Lemma 3. Condition (iv) implies that all the elements a_{ij}, where $j \in J_i$, have the same in-neighbourhood. Therefore τ_i is a congruence by Lemma 4. Put

$$\delta_3 = \bigcup_{i \in I} \tau_i.$$

For any $i_1, i_2 \in I$, $i_1 \neq i_2$, we see that the only possibly non-singleton class $\{a_{i_1 j} \mid j \in J_{i_1}\}$ of τ_{i_1} does not intersect the class $\{a_{i_2 j} \mid j \in J_{i_2}\}$ of τ_{i_2}. It follows that δ_3 is an equivalence relation. Since all the τ_i are congruences, it follows that δ_3 is a congruence, too.

Finally, define

$$\delta = \delta_1 \bigcup \delta_2 \bigcup \delta_3.$$

Since $(U \cup U') \cap Q = \emptyset$, we see that the relation $\delta_1 \cup \delta_2$ is transitive, and therefore it is a congruence. By construction, all non-singleton classes of $\delta_1 \cup \delta_2$ lie in C_0, while all non-singleton classes of δ_3 lie outside of C_0. It follows that δ is a transitive relation and, moreover, a congruence.

Now, we are going to verify that $\rho \perp \delta$. Take any pair $(c, d) \in \rho \bigwedge \delta$. First, assume that $(c, d) \in \delta_1$. By the definition of δ_1 and because $Q \cap S = \emptyset$, we get either $c = d$ or $c, d \in Q$. In the latter case we see that $\mathrm{In}(c)$ and $\mathrm{In}(d)$ are nonempty since $Q \cap S = \emptyset$. Now $(c, d) \in \rho$ implies $\mathrm{In}(c) = \mathrm{In}(d)$, because Q is in-closed. Therefore $c = d$ by (ii). Second, suppose that $(c, d) \in \delta_2$. Then $c = d$, or $c, d \in H_{u'}$, for some $u' \in U'$. The definition (3) of the class $H_{u'}$ shows that in the latter case $\mathrm{In}(c)$ and $\mathrm{In}(d)$ are nonempty and by (ii) $c = d$, too. If $(c, d) \in \delta_3$, then $(c, d) \in \tau_i$, for some $i \in I$, and the definition of τ_i yields $c = d$. Therefore $\rho \bigwedge \delta = \iota$.

In order to prove that $\rho \bigvee \delta = \sigma_T$, take any pair $(c, d) \in \sigma_T$. If $c, d \in T_0 \setminus C_0$ or $c, d \in \overline{T}_0$, then c, d lie in some σ_T-class K_i. Let $c \in K_{ij_1}$ and $d \in K_{ij_2}$, for some $j_1, j_2 \in J_i$, and let $c' \in K_{ij_1}$ and $d' \in K_{ij_2}$ be the elements which were chosen in these ρ-classes when we defined τ_i. Then $(c, c'), (d, d') \in \rho$ and $(c', d') \in \tau_i$, which yields $(c, d) \in \rho \circ \tau_i \circ \rho \subseteq \rho \bigvee \delta$. Finally, assume that $c, d \in C_0$. It is enough to verify that $(c, 0) \in \rho \bigvee \delta$. To this end, consider the ρ-class K that contains c. Note that the set Q contains one element in each ρ-class of Types 1, 2, and 3 contained in C_0. Therefore if K is of Type 1, 2, or 3, then $(c, 0) \in \rho \circ \delta_1$. If K is of Type 4, then there exist $u \in K \cap U$ and the corresponding $u' \in M_{s+1} \cup \ldots \cup M_k$ such that $(u, u') \in \delta_2$. Hence $(c, 0) \in \rho \circ \delta_2 \circ \rho$. Thus in all these cases $(c, 0) \in \rho \bigvee \delta$.

Therefore $\sigma_T = \rho \bigvee \delta$. We have proved that ρ is a direct summand.

(O1)$\Rightarrow$(O4): Assume that the automaton $\mathrm{Atm}(D, T)$ has no proper essential congruences. If $C_0 = \{0\}$, then conditions (i) to (iv) are vacuously true for $M = \emptyset$. Thus we may assume that $C_0 \neq \{0\}$. In particular, $\sigma_T \neq \iota$ by Lemma 3.

Let $M_1, \ldots, M_n$ be all nonempty minimal in-closed subsets in $C_0 \setminus \{0\}$. Clearly, they are pairwise disjoint, and $n \geq 1$. The union $M = M_1 \cup \ldots \cup M_n$ is in-closed. Besides, it contains all vertices of C_0 with in-degree zero. We are going to verify that M satisfies (i) to (iv).

The connected components of the subgraph induced by M in D are the subgraphs induced by $M_1, \ldots, M_n$. By Lemma 5, each subgraph of this sort is strongly connected. Thus (i) is satisfied.

Suppose that (ii) does not hold, i.e., there exist two distinct vertices a, b of M with nonzero in-degree such that $\mathrm{In}(a) = \mathrm{In}(b) \neq \emptyset$. Since all the M_i are in-closed, it follows that a and b lie in the same set M_i.

Denote by $\mathrm{Out}^*(M_i)$ the set of all vertices $u \in V$ such that there exists a directed path from some vertex of M_i to u. Consider the set

$$H = C_0 \setminus \mathrm{Out}^*(M_i).$$

Since $M \setminus M_i$ is in-closed, there do not exist any directed paths from vertices of M_i to vertices of $M \setminus M_i$. Therefore

$$M \setminus M_i \subseteq H.$$

We are going to prove that μ_H is a proper essential congruence.

In order to verify that μ_H is a congruence, by Lemma 2 it suffices to check that the set H is in-closed. To this end, take any $u \in H$, and suppose to the contrary that there exists $v \in \mathrm{In}(u) \setminus H$. Since C_0 is in-closed, we get $v \in C_0$. Then $v \in \mathrm{Out}^*(M_i)$, and so there exists a directed path from some vertex $x \in M_i$ to v. Completing this path by the edge (v, u) we get a directed path from x to u; whence $u \in \mathrm{Out}^*(M_i)$. This contradicts the choice of u and establishes that μ_H is a congruence.

By (2) and the definition of μ_H, we get $(a, b) \in \mu_H$. Hence $\mu_H \neq \iota$. Besides, $\mu_H \neq \sigma_T$, because $\sigma_T = \mu_{C_0} \supseteq \{0\} \times M_i$ and $\mu_H \cap (\{0\} \times M_i) = \emptyset$. Thus μ_H is a proper congruence.

It remains to show that μ_H is essential. Take an arbitrary congruence $\delta \neq \iota$. We need to verify that $\mu_H \bigwedge \delta \neq \iota$. Denote by S the equivalence class of δ containing 0. By Lemma 2, the set S is in-closed.

First, consider the case where $S \neq \{0\}$. Choose a minimal in-closed subset S' in $S \setminus \{0\}$. We have $S' \subseteq M$. If $S' = M_i$, then $(a, b) \in \mu_H \bigwedge \delta$. If $S' \neq M_i$, then $S' \subseteq H$, and therefore $\Theta(S' \cup \{0\}) \subseteq \mu_H \bigwedge \delta$. Hence $\mu_H \bigwedge \delta \neq \iota$ in this case.

Second, suppose that $S = \{0\}$. Pick any pair $(c, d) \in \delta$ with $c \neq d$. Lemma 2 tells us that $(c, d) \in \mu_{\{0\}}$. We get $\mathrm{In}(c) = \mathrm{In}(d)$.

If $\operatorname{In}(c) = \emptyset$, then either $c, d \in \overline{T}_0$ and $(c, d) \in \mu_H$, or $c, d \in M$. Note that neither c nor d belong to M_i, because every vertex of in-degree zero forms a minimal (singleton) in-closed subset in M and $|M_i| \geq 2$. Hence $c, d \in M \setminus M_i \subseteq H$, and so $(c, d) \in \mu_H$.

Further, assume that $\operatorname{In}(c) \neq \emptyset$. It can be easily seen that $c \in \operatorname{Out}^*(M_i)$ if and only if $d \in \operatorname{Out}^*(M_i)$ because $\operatorname{In}(c) = \operatorname{In}(d)$. Now, suppose that one of the vertices, say c, belongs to H. Then $\operatorname{In}(c) \subseteq C_0$, because C_0 is in-closed. Since $(c, d) \in \mu_{\{0\}}$, we get $d \in T_0$. Assume that $d \in T_0 \setminus C_0$. Since C_0 is the largest in-closed subset of T_0, there exists $x \in \operatorname{In}(d) \setminus C_0$, which is impossible because $\operatorname{In}(d) = \operatorname{In}(c) \subseteq C_0$. Hence $c \in H$ if and only if $d \in H$. Therefore $(c, d) \in \mu_H$, again.

Thus, in the second case, we get $\delta \subseteq \mu_H$, and so $(\mu_H) \bigwedge \delta = \delta \neq \iota$, as required. Hence the congruence μ_H is essential. This contradicts (O1) and shows that (ii) holds.

(Note that in the special case where $H = \emptyset$ our proof above remains valid and can be simplified.)

Next, let us verify (iii). If M_i consists of a single vertex with in-degree zero, then put $N_i = M_i$. Otherwise, denote by N_i the set of all vertices a in C_0 such that $\operatorname{In}(a) = \operatorname{In}(a')$, for some $a' \in M_i$. Since the M_i are in-closed, we see that the sets $N_1, \ldots, N_n$ are pairwise disjoint. The set $N = N_1 \cup \ldots \cup N_n$ is in-closed, because $\operatorname{In}(a) \subseteq M \subseteq N$, for every vertex $a \in N$.

We claim that $\mu_{\{0\}} \subseteq \mu_N$. Indeed, take any pair $(a, b) \in \mu_{\{0\}}$. Since the definition of $\mu_{\{0\}}$ is more demanding then the definition of μ_N are the same on $\overline{T}_0$, we only have to consider the case where $a, b \in T_0 \setminus \{0\}$. The definition of $\mu_{\{0\}}$ implies $\operatorname{In}(a) = \operatorname{In}(b)$. If $a, b \in T_0 \setminus N$, then $(a, b) \in \mu_N$. Further, we may assume that one of these vertices, say a, belongs to N_i. First, suppose that a has in-degree zero, and so is the only vertex of some $N_i = M_i$. Then b is also a vertex with in-degree zero, hence $b \in C_0$, and so $b \in M \subseteq N$, too. This means that $(a, b) \in \mu_N$, as required. Second, suppose that $|\operatorname{In}(a)| > 0$. By the definition of N_i, there exists $a' \in M_i$ such that $\operatorname{In}(a) = \operatorname{In}(a')$. Then $\operatorname{In}(b) = \operatorname{In}(a') \subseteq C_0$, hence $b \in C_0$, and so $b \in N_i \subseteq N$, again. Thus in both cases $a, b \in N$, and hence $(a, b) \in \mu_N$, by the definition of μ_N. Therefore $\mu_{\{0\}} \subseteq \mu_N$, as claimed.

Suppose that $N \neq C_0 \setminus \{0\}$. Then μ_N is a proper congruence. By (O1), μ_N is not essential. Hence there exists a congruence $\delta \neq \iota$ such that $\mu_N \bigwedge \delta = \iota$. Denote by K the class of δ containing 0. If $K = \{0\}$, then $\delta \subseteq \mu_{\{0\}} \subseteq \mu_N$. This contradiction shows that $K \neq \{0\}$. By Lemma 2, K is an in-closed subset of C_0, and we can choose a minimal in-closed subset M_{n+1} in $K \setminus \{0\}$ that does not intersect N. This contradicts the definition of M and shows that $N = C_0 \setminus \{0\}$. Thus (iii) holds.

It remains to prove (iv). Suppose to the contrary that there exist two elements a, b, which both belong either to the set $T_0 \setminus C_0$ or to the set $\overline{T}_0$, and $\mathrm{In}(a) \cap \overline{C}_0 = \mathrm{In}(b) \cap \overline{C}_0$, but $\mathrm{In}(a) \neq \mathrm{In}(b)$. Consider the equivalence relation ρ on $\mathrm{Atm}(D, T)$ which has the same classes as σ_T with the only exception: we divide a/σ_T into two new classes,

$$\{c \in a/\sigma_T \mid \mathrm{In}(c) = \mathrm{In}(a)\} \text{ and } \{c \in a/\sigma_T \mid \mathrm{In}(c) \neq \mathrm{In}(a)\}.$$

Since $a \notin C_0$, it follows from Lemma 2 that ρ is a congruence on $\mathrm{Atm}(D, T)$.

We prove that ρ is essential. Assume that $\rho \bigwedge \delta = \iota$ for some congruence δ. Then $0/\delta = \{0\}$, because $0/\rho = C_0$. Take any $(c, d) \in \delta$. If $c \notin a/\sigma_T$, then $c/\rho = c/\sigma_T \supseteq c/\delta$; whence $(c, d) \in \rho$. If $d \notin a/\sigma_T$, then $(c, d) \in \rho$ and so further we assume that $c, d \in a/\sigma_T$. Since $0/\delta = \{0\}$, Lemma 2 yields $\mathrm{In}(c) = \mathrm{In}(d)$. If $\mathrm{In}(c) = \mathrm{In}(a)$, then both c and d lie in a/ρ by the definition of ρ, and therefore $(c, d) \in \rho$. Similarly, if $\mathrm{In}(c) \neq \mathrm{In}(a)$, then c, d lie in b/ρ, and $(c, d) \in \rho$, again. Thus $(c, d) \in \iota$ in all cases. Thus ρ is essential. This contradiction shows that (iv) is satisfied, which completes our proof. □

The authors are grateful to two referees for several corrections to our proof and improvements to the text of this paper.

References

[1] J. Berstel, Finite automata and rational languages: an introduction, in: Formal Properties of Finite Automata and Applications, Lect. Notes Comp. Sci. 386, Springer, New York, 1989, 2–14.

[2] K. Baker, G.F. McNulty, H. Werner, Finitely based varieties of graph algebras, Acta Sci. Math. 51 (1987), 3–15.

[3] H. Ehrig, G. Engels, H.-J. Kreowski and G. Rozenberg (Eds.), Handbook of Graph Grammars and Computing by Graph Transformation, Vol. 1, 2, World Sci. Publishing, River Edge, NJ, 1999.

[4] B.J. Gardner, Radical Theory, Longman, Pitman, 1989.

[5] A.V. Kelarev, O.V. Sokratova, Languages recognized by a class of finite automata, Acta Cybernetica 15 (2001), 45-52.

[6] E.W. Kiss, R. Pöschel, P. Pröhle, Subvarieties of varieties generated by graph algebras, Acta Sci. Math. 54 (1990), 57–75.

[7] G. Lallement, Semigroups and Combinatorial Applications, Wiley, New York, 1979.

[8] S. Oates-Williams, Graphs and universal algebra, Lect. Notes Math. 884 (1981), 351–354.

[9] R.H. Oehmke, On semisimplicity for automata, in: Words, Languages and Combinatorics (Kyoto), World Scientific, Singapore, 1990, 373–384.

[10] R.H. Oehmke, Automata with complemented lattices of congruences, in: Words, Languages and Combinatorics (Kyoto), World Scientific, Singapore, 1992, 294–300.

[11] R.H. Oehmke, The direct product of right congruences, Korean J. Comput. Appl. Math. 5 (1998), no. 3, 475–480.

[12] R. Pöschel, The equational logic for graph algebras, Z. Math. Logik Grundlag Math. 35 (1989), no. 3, 273–282.

[13] R. Pöschel, Graph algebras and graph varieties, Algebra Universalis 27 (1990), no. 4, 559–577.

[14] G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, Vol. 1, 2, 3, Springer, New York, 1997.