SOME EXACT AND ASYMPTOTIC RESULTS FOR BEST UNIFORM APPROXIMATION

by

BINH LAM (née NGUYEN THI BINH)

B.Sc. (Hons)

Submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy

UNIVERSITY OF TASMANIA

HOBART

(September, 1972)

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and that, to the best of my knowledge and belief, this thesis contains no copy or paraphrase previously published or written by another person, except when due reference is made in the text.

(Binh Lam)

ACKNOWLEDGEMENT

I wish to express my appreciation to my supervisor, Professor D. Elliott, who provided not only guidance but also enthusiasm throughout my three years working under him. To Miss Y.L. Watson goes my thanks for her superb typing.

Finally, I wish to thank the University of Tasmania for the award of a University Research Scholarship for the years 1970-1972.

Chapter VII has been published in SIAM (see [16]). Some results from chapters IV and V are to appear in the same journal (see [8]). Chapter III has been submitted for publication (see [17]).

CONTENTS

CHAPTER I

INTRODUCTION

1.1. Introduction.

Let C be a normed linear space over the field of real or complex numbers and M be a finite dimensional linear subspace of C. Given an element $f \in C$, let us define a quantity $\rho_M(f)$ by

$$(1.1.1) \qquad \rho_M(f) = \inf_{g \in M} \|f-g\| .$$

The general linear approximation problem can be formulated as one of determining the existence, uniqueness and characterization of the set of elements $h \in M$ satisfying

$$(1.1.2) \qquad \rho_M(f) = \|f-h\| .$$

Such functions h are called the best approximations to f with respect to M and their existence is guaranteed by the following theorem.

Theorem 1.1.

For any given element $f \in C$, there exists an element $h \in M$ satisfying equation (1.1.2).

Proof. see Buck [4]; Achieser [1], p.10.

In this thesis, we shall be concerned with the problem of uniform (or Chebyshev) approximation, in both linear and non-linear cases. In the case of linear uniform approximation, let A be a compact space and $C[A]$ be the space of a continuous real- or complex-valued functions defined on A. Then $C[A]$ is a normed linear space with respect to the uniform (or Chebyshev) norm which is defined by

$$(1.1.3) \qquad \|f\| = \sup_{x \in A} |f(x)| \quad ,$$

(see, for example, Simmons [32], p.55).

Given a function $f \in C[A]$, the uniqueness of the best uniform approximation to f with respect to a finite dimensional subspace M of $C[A]$ can be established provided M satisfies the so-called "Haar condition".

Definition 1.1.

A linear subspace M of $C[A]$ of finite dimension n is said to fulfill the "Haar condition" if it possesses the property that every function in M which is not identically equal to zero vanishes at no more than $(n-1)$ points of A.

This property gives rise to a sufficient condition for the uniqueness of the best uniform approximation.

Theorem 1.2.

If the linear subspace M of $C[A]$ satisfies the Haar condition, then for every $f \in C[A]$, there exists a unique best uniform approximation to f with respect to M.

Proof. see Meinardus [21], p.16; Achieser [1], p.68.

In the case of non-linear uniform approximation, let B be a given set of parameters and N denote a set of functions $F(b;x)$ in $C[A]$, which depend on the parameters $b \in B$. For a given function $f \in C[A]$, we define the minimal deviation $\rho_N(f)$ by

$$(1.1.4) \qquad \rho_N(f) = \inf_{b \in B} \|f(x) - F(b;x)\| \quad .$$

A function $F(c;x) \in N$, $c \in B$, is called a best uniform approximation to f with respect to N if it satisfies the following equation

$$(1.1.5) \qquad \rho_N(f) = \|f(x) - F(c;x)\| .$$

The general theory of non-linear uniform approximation has been investigated only in the last few years, although some results for special cases were given much earlier. For example, the case of rational approximation was considered by Chebyshev in 1899 and de la Vallée Poussin in 1919. Unlike the linear case, for a given function $f \in C[\Lambda]$, a best approximation to f with respect to N does not always exist (see Rice [28]). However, the uniqueness of the best approximation, when it exists, may be established by an appropriate generalization of the Haar condition (see Meinardus [21], p.142).

In this thesis, we shall consider the problems of approximation of continuous functions of one variable by polynomials and rational functions. These problems have been of interest to mathematicians for more than a century; in particular, for the last two decades when the development of electronic computers made the computation of best uniform approximations possible. Chebyshev was the first mathematician to study the characterization of the best uniform approximations. The problem was later investigated by

Haar, Kolmogroff and de la Vallée Poussin (see Achieser [1], chapter 2; Meinardus [21], chapters 3 and 8). From their results, Bernstein [2] was able to obtain analytic expressions for the best uniform approximations to a few particular functions.

We shall restrict ourselves to the normed linear space of all continuous real-valued functions of one variable defined on a finite closed interval $[a,b]$, which will be denoted by $C[a,b]$. In the following two sections, we shall take a closer look at the existence, uniqueness and characterization of the best uniform approximations for both polynomial and rational approximations in the space $C[a,b]$.

1.2. On best uniform polynomial approximation.

Without loss of generality, we may consider the uniform polynomial approximation in the space $C[-1,1]$. Let P_n denote the space of all polynomials of degree at most n, then P_n is a finite dimensional subspace of $C[-1,1]$.

Definition 1.2.

Let $f(x) \in C[-1,1]$. For any non-negative integer n, the quantity $E_n(f)$ is defined by

$$E_n(f) = \inf_{p \in P_n} \|f-p\| \quad . \tag{1.2.1}$$

Definition 1.3.

Let $f(x) \in C[-1,1]$. For any non-negative integer n, a polynomial $p_n(x)$ is called a best uniform polynomial approximation of degree n to $f(x)$ if it satisfies the following equation

$$E_n(f) = \|f-p_n\| \quad . \tag{1.2.2}$$

Theorem 1.1 assures us that, for a given function $f(x) \in C[-1,1]$, there exists a best uniform polynomial approximation $p_n(x)$ to $f(x)$. The quantity $E_n(f)$ is the maximum error occuring in the approximation. Since $P_n \subset P_{n+1}$, $E_{n+1}(f) \leq E_n(f)$, for all n. Furthermore, Weierstrass' theorem (see below) implies that as $n \to \infty$, $E_n(f) \to 0$.

Theorem 1.3. (Weierstrass).

If $f(x) \in C[-1,1]$, then for every $\varepsilon > 0$, there exists a polynomial $q_n(x)$ of degree $n = n(\varepsilon)$ such that

$$|f(x) - q_n(x)| \leq \varepsilon \quad , \tag{1.2.3}$$

for all $x \in [-1,1]$.

Proof. see Achieser [1], p.30; Rivlin [31], p.12.

Jackson, Bernstein and several other authors have investigated the order of convergence of the quantity $E_n(f)$ and its dependency on the structure of the function f and its derivatives (see, for example, Meinardus [21], chapter 5). In addition, various lower and upper bounds for $E_n(f)$ have been obtained by de la Vallée Poussin, Bernstein, Blum & Curtis and others (see Rivlin [29]). These results enable us to determine how closely a given function with some known properties can be approximated by polynomials. They do not, however, help us to obtain a best uniform polynomial approximation. To this end, we need the following characterization theorem.

Theorem 1.4. (Chebyshev).

Let $f(x) \in C[-1,1]$. Then $p_n(x)$ is a best uniform polynomial approximation to f out of P_n if and only if the difference $f(x) - p_n(x)$ attains its extreme values alternately in at least $(n+2)$ points of $[-1,1]$.

Proof. see Achieser [1], p.55; Rivlin [31], p.26.

This property of $p_n(x)$ gives rise to its uniqueness.

Theorem 1.5.

If $f(x) \in C[-1,1]$ and $p_n(x)$ is a best uniform polynomial approximation to f out of P_n, then

(1.2.4) $$\|f - p_n\| < \|f-p\| \quad ,$$

for all $p(x) \in P_n$, other than $p_n(x)$.

Proof. see Rivlin [31], p.28.

To sum up, for a given function $f(x) \in C[-1,1]$, there exists a unique best uniform polynomial approximation $p_n(x)$ of degree n to f. The polynomial $p_n(x)$ is characterized by theorem 1.4 and the maximum error $E_n(f)$ occuring in the approximation is given by equation (1.2.2). We shall now obtain similar results for the case of uniform rational approximation.

1.3. On best uniform rational approximation.

Let $V_{n,m}$ denote the set of all irreducible rational functions $r(n,m;x)$ of the form

(1.3.1) $$r(n,m;x) = \frac{p(x)}{q(x)} \quad ,$$

where $p(x) \in P_n$, $q(x) \in P_m$ and $-1 \leq x \leq 1$. The rational function $r(n,m;x)$ is said to be irreducible if the polynomials $p(x)$ and $q(x)$ have no common factors.

Definition 1.4.

Let $f(x) \in C[-1,1]$. For non-negative integers n and m, the quantity $E_{n,m}(f)$ is defined by

$$(1.3.2) \qquad E_{n,m}(f) = \inf_{r \in V_{n,m}} \|f(x) - r(n,m;x)\| .$$

Definition 1.5.

Let $f(x) \in C[-1,1]$. For non-negative integers n and m, a rational function $R(n,m;x)$ of the form (1.3.1) is called a best uniform rational approximation to f out of $V_{n,m}$ if it satisfies the following equation

$$(1.3.3) \qquad E_{n,m}(f) = \|f(x) - R(n,m;x)\| .$$

For a given function $f(x) \in C[-1,1]$, it is considerably more difficult to establish the existence of a best uniform rational approximation to f out of $V_{n,m}$.

Theorem 1.6.

If $f(x) \in C[-1,1]$, then there exists a rational function $R(n,m;x)$ such that

$$\|f(x) - R(n,m;x)\| \leq \|f(x) - r(n,m;x)\| ,$$

for all $r(n,m;x) \in V_{n,m}$.

Proof. see Achieser [1], p.53; Rivlin [31], p.121.

In analogy with theorem 1.4, we have the following characterization property of $R(n,m;x)$.

Theorem 1.7. (Chebyshev).

If $f(x) \in C[-1,1]$, then $R(n,m;x) = p(x)/q(x)$ is a best uniform rational approximation to f out of $V_{n,m}$ if and only if the difference $f(x) - R(n,m;x)$ attains its extreme values in at least N points of $[-1,1]$, where

$$(1.3.4) \qquad N = 2 + \max\,(n + \overset{\partial q}{\cancel{\partial p}},\ m + \overset{\partial p}{\cancel{\partial q}}) ,$$

∂p and ∂q being the degrees of the polynomials $p(x)$ and $q(x)$ respectively.

Proof. see Achieser [1], p.55; Rivlin [31], p.123.

Let us now denote by $\bar{r}(n,m;x)$ a rational function of the form (1.3.1), where the coefficients of x^n and x^m of $p(x)$ and $q(x)$, respectively, do not vanish. We derive from theorem 1.7 the following result which will be used later on.

Lemma 1.1.

$\bar{r}(n,m;x)$ is the best uniform rational approximation to $f(x)$ out of $V_{n+u,m+v}$ if the difference $f(x) - \bar{r}(n,m;x)$ attains its extreme values alternately in at least $(n+m+2+t)$ points of $[-1,1]$, where ~~$0 \leq u,v \leq t$~~. $t = \max(u,v)$.

Proof. This is a direct consequence of theorem 1.7.

As in the case of polynomial approximation, the best uniform rational approximation characterized by theorem 1.7 is in fact unique.

Theorem 1.8.

If $f(x) \in C[-1,1]$ and $R(n,m;x)$ is a best uniform rational approximation to f out of $V_{n,m}$, then

$$(1.3.5) \qquad \|f - R(n,m;x)\| < \|f - r(n,m;x)\| \quad ,$$

for all $r(n,m;x) \in V_{n,m}$, other than $R(n,m;x)$.

Proof. see Rivlin [31], p.125.

Thus, let $f(x)$ be a given function in $C[-1,1]$, and let n,m be any two non-negative integers. Then there exists a unique best uniform rational approximation $R(n,m;x)$ to f out of $V_{n,m}$. $R(n,m;x)$ is characterized by theorem 1.7 and the maximum error $E_{n,m}(f)$ involved in the approximation is given by (1.3.3).

It is worth noting that the case of uniform polynomial approximation considered in the previous section is in fact a special case of uniform rational approximation. When $m = 0$, $E_{n,m}(f) = E_n(f)$ and $R(n,0;x) = p_n(x)$.

Unlike polynomials, rational functions do not have the pleasant property of depending linearly on their coefficients. The theory of uniform rational approximation is, therefore, far less well developed than that of uniform polynomial approximation. Fewer exact and asymptotic results are to be found in the literature. Furthermore, the question of how closely it is possible to approximate a given function by rational functions has not yet been fully investigated. A result by Newman [24], however, illustrates that for the function $|x|$, rational approximations are far more effective than polynomial approximations. Similar results along this line have been obtained by Turán [36], Freud and Szabados [9].

We shall now introduce a system of orthogonal polynomials which plays an important role in the theory of uniform approximation.

<u>Definition 1.6.</u>

For any non-negative integer n, the Chebyshev polynomial of the first kind $T_n(x)$ of degree n is defined by

$$(1.3.6) \qquad T_n(x) = \cos n\theta ,$$

where $x = \cos\theta$ and $-1 \le x \le 1$.

$\{T_n(x)\}_{n=0}^{\infty}$ is a system of orthogonal polynomials on [-1,1]. We have

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_r(x) T_s(x) dx = \begin{cases} \pi, \text{ if } r = s = 0 \\ \frac{1}{2}\pi, \text{ if } r = s \ne 0, \\ 0 , \text{ if } r \ne s , \end{cases}$$

(see Clenshaw [5]).

Given a function f(x), defined on [-1,1], we can express f(x) in terms of a series expansion of these polynomials.

Definition 1.7.

The Chebyshev series expansion of f(x) defined on [-1,1] is defined as

$$(1.3.7) \qquad f(x) \simeq \sum_{k=0}^{\infty}{}' a_k T_k(x) \quad ,$$

where $\sum'$ denotes a sum whose first term is halved and the Chebyshev coefficient a_k, $k = 0,1,2,\ldots$, are given by

$$(1.3.8) \qquad a_k = \frac{2}{\pi}\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} f(x) T_k(x) dx \quad .$$

Definition 1.8.

For any non-negative integer n, the truncated Chebyshev series expansion $s_n(x)$ of f(x) of degree n is defined by

$$(1.3.9) \qquad s_n(x) = \sum_{k=0}^{n}{}' a_k T_k(x) \quad .$$

From (1.3.6), we find that $s_n(x)$ is a polynomial of degree at most n. For some functions, $s_n(x)$ differs very little from the best uniform polynomial $p_n(x)$, and is therefore often used as an approximation to $f(x)$. The maximum error S_n occuring in the approximation is defined as follows.

Definition 1.9.

For any non-negative integer n, the quantity S_n is defined by

$$S_n = \|f - s_n\| . \tag{1.3.10}$$

Let ~~If~~ $f(x) \in C[-1,1]$, and suppose there exists a Chebyshev series expansion of the form (1.3.7), which is uniformly convergent for all $x \in [-1,1]$ (see Clenshaw [5]). For many functions, we have analytic expressions for a_k. The values of these coefficients have also been extensively tabulated (see, for example, Clenshaw [5], Luke [19]). Only for a few functions, however, do we have analytic expressions for the quantities $E_n(f)$ or $E_{n,m}(f)$ and the best uniform polynomial or rational approximations. In this thesis, we shall add to the list of these known functions, some special class of continuous functions whose best uniform rational approximations and the quantity $E_{n,m}(f)$ can be given explicitly. In addition, we shall show how asymptotic estimates of the quantities $E_n(f)$ and $E_{n,m}(f)$, for large n, may be readily computed from the knowledge of the Chebyshev coefficients of $f(x)$.

From this result, we shall obtain the best uniform polynomial and rational approximations to certain functions $f(x) \in C[-1,1]$. Apart from quoted results, chapters III, IV, V, VI and VII are original.

In chapter II, we shall give a survey of known results where the best uniform polynomial and rational approximations are given either exactly or asymptotically. We shall consider in detail those results which are of some relevance to the later chapters of the thesis, other results will only be briefly mentioned.

In chapter III, we shall obtain some explicit results for the best uniform rational approximation. We shall define a class of rational functions which will be extensively used throughout the thesis. These functions can be looked upon as generalizations of the Chebyshev polynomials of the first kind. From this class of functions, we shall construct a special class of continuous functions whose best uniform rational approximations will be obtained explicitly. From these general results, we can recover some exact results for the best uniform polynomial and rational approximations previously given by Bernstein [2], Boehm [3], and Rivlin [30].

In chapters IV, V and VI, we shall obtain some asymptotic results. An algorithm for finding an asymptotic estimate for the quantity $E_{n,m}(f)$, for large n, where $f(x) \in C[-1,1]$ and $m = O(1)n$, will be given in chapter IV. We shall prove that when f is a polynomial of degree (n+r+1), n and r being two non-negative integers, then an asymptotic estimate for $E_{n,m}(f)$, for large n, is given by the eigenvalue of $(m+1)^{st}$ largest modulus of a certain symmetric matrix of order (m+r+1). The coefficients of this matrix are in fact the Chebyshev coefficients of f. This result will then be generalized to certain functions which are continuous on [-1,1].

In chapter V, we shall consider in detail the case of uniform polynomial approximation. An asymptotic estimate for $E_n(f)$, for large n, will be obtained as a special case of chapter IV. From this, we shall then find the polynomial which is asymptotic to the best uniform polynomial approximation. Some numerical examples will be given in the last section. These results, which are obtained with much less effort, are nevertheless comparable to those given by other authors (see Hastings [13], Murnaghan and Wrench [23]).

ChapterVI will deal with the case of uniform rational approximation. The analysis for determining the rational function which is asymptotic to the best uniform rational approximation is not as satisfactory as in the case of polynomial approximation. But unfortunately, we have not been able to improve the analysis any further.

Finally, in chapterVII, we shall use some results from chapter V to prove a conjecture of Clenshaw [6]. This conjecture concerns with the maximum value of the quantity S_n/E_n, taken over all polynomials of degree (n+r+1), n and r being two non-negative integers. Although this chapter bears no direct relation to the main theme of the thesis, it was this problem, in fact, that led us to other problems considered in chapters III, IV, V and VI.

CHAPTER II

A SURVEY OF KNOWN RESULTS

In section 2.1, we shall give a survey of the known results where the best uniform polynomial approximations can be obtained either explicitly or asymptotically. Section 2.2 will be devoted to the best uniform rational approximations. We shall consider in detail those results which are of some relevance to the later chapters of the thesis; other results will only be briefly mentioned.

2.1. On best uniform polynomial approximation.

2.1.1. Exact results.

Whereas many explicit analytical results for particular functions may be found in literature for least-square approximation, (see, for example, Rivlin [31], Chapter 2); this is not so for uniform polynomial approximation. There are a very few examples where the best uniform polynomial approximation $p_n(x)$ of degree n to f(x) and the maximum error $E_n(f)$ can be obtained explicitly.

2.1.1.1. The polynomial of degree n which deviates least from zero.
(The monic polynomial)

The problem of finding the polynomial of degree n which deviates least from zero on the interval [-1,1], is the same as that of finding the best uniform polynomial approximation of degree (n-1) over [-1,1] to the function $f(x) = x^n$. Let us consider the Chebyshev polynomial of degree n defined by (1.3.6). From this definition, we find that the function x^n can be expressed in terms of Chebyshev polynomials $T_k(x)$, $k = 0(1)n$, as follows.

$$(2.1.1) \qquad x^n = \frac{1}{2^{n-1}} T_n(x) + \alpha_{n-1}T_{n-1}(x) + \dots + \alpha_0 T_0(x),$$

where α_i, $i = 0(1)(n-1)$, are real numbers (see, for example, Clenshaw [5]). Now, let

$$(2.1.2) \qquad p_{n-1}(x) = \alpha_{n-1}T_{n-1}(x) + \dots + \alpha_0 T_0(x) ,$$

then the function $f(x) - p_{n-1}(x)$ takes its extreme values $\pm \frac{1}{2^{n-1}}$ alternately at (n+1) points x_k of [-1,1], where

$$(2.1.3) \qquad x_k = \cos \frac{k\pi}{n} , \qquad \text{for } k = 0(1)n .$$

Thus, by theorem 1.4, $p_{n-1}(x)$ is the best polynomial uniform approximation to $f(x)$ of degree (n-1) and $\frac{1}{2^{n-1}} T_n(x)$ is the polynomial of degree n which deviates least from zero, on [-1,1] , (see Achieser [1], p.57).

2.1.1.2. <u>The Weierstrass function</u>.

Let a be a positive number less than 1 and b be an odd integer greater than 1, for $\theta \in [0,2\pi]$, the Weierstrass function is defined by

$$f(\theta) = \sum_{k=0}^{\infty} a^k \cos b^k\theta \quad , \tag{2.1.4}$$

(see [1], p.66).

Since $|\cos b^k\theta| \leq 1$ for all $\theta \in [0,2\pi]$ and $k = 0,1,2,\ldots,$ the infinite series converges uniformly for all $\theta \in [0,2\pi]$ and the function $f(\theta)$ is continuous on $[0,2\pi]$. For a given non-negative integer n, let j be such that

$$b^j \leq n < b^{j+1} \,, \tag{2.1.5}$$

and define the function $S_n(\theta)$ by

$$S_n(\theta) = \sum_{k=0}^{j} a^k \cos b^k\theta \quad , \tag{2.1.6}$$

then $S_n(\theta)$ is a trigonometric polynomial of degree $\leq n$.

Now, let us put

$$L = \max_{\theta \in [0,2\pi]} |f(\theta) - S_n(\theta)| \,. \tag{2.1.7}$$

Then,

$$L \leq \sum_{k=j+1}^{\infty} a^k = \frac{a^{j+1}}{1-a} .$$

We observe that this bound is attained by the function $f(\theta) - S_n(\theta)$ at $(2b^{j+1} + 1)$ points θ_m of $[0,2\pi]$, where

(2.1.8) $$\theta_m = \frac{m\pi}{b^{j+1}} , \quad m = 0(1)(2b^{j+1}) .$$

Thus, the function $f(\theta) - S_n(\theta)$ takes its extreme values $\pm L$, where $L = \frac{a^{j+1}}{1-a}$, alternately at $(2b^{j+1} + 1)$ points θ_m of $[0,2\pi]$. But, from (2.1.5), $2b^{j+1} + 1 \geq 2n + 2$; it follows from theorem 1.4 that $S_n(\theta)$ is the best uniform polynomial approximation to $f(\theta)$ of degree n, with $E_n(f) = \frac{a^{j+1}}{1-a}$.

We shall now consider a problem which can be looked upon as a generalization of the problem of Chebyshev discussed in section 2.1.1.1.

2.1.1.3. Zolotareff's problem.

Zolotareff posed the problem of determining the best uniform polynomial approximation of degree n over $[-1,1]$ to the polynomial $f(x)$ of degree $(n+2)$ given by

(2.1.9) $$f(x) = x^{n+2} - \sigma x^{n+1} ,$$

where σ is a given real number (see Meinardus [21], p.41; Achieser [1], p.281). Without loss of generality, we may assume that $\sigma \geq 0$. If $p_n(x)$ is the required best approximation, the polynomial $Z_{n+2}(x)$ of degree (n+2), where

$$(2.1.10) \qquad Z_{n+2}(x) = x^{n+2} - \sigma x^{n+1} - p_n(x) ,$$

is the so-called Zolotareff polynomial. For

$$(2.1.11) \qquad 0 \leq \sigma \leq (n+2) \tan^2 \frac{\pi}{2(n+2)} ,$$

Zolotareff has obtained

$$(2.1.12a) \qquad Z_{n+2}(x) = 2^{-n-1} \left(1 + \frac{\sigma}{n+2}\right)^{n+2} T_{n+2}\left(\frac{x - \frac{\sigma}{n+2}}{1 + \frac{\sigma}{n+2}}\right) ,$$

with

$$(2.1.12b) \qquad E_n(f) = 2^{-n-1} \left(1 + \frac{\sigma}{n+2}\right)^{n+2} .$$

From definition 1.6, we observe that (2.1.12a) can be written in the form

$$(2.1.13) \qquad Z_{n+2}(x) = x^{n+2} - \sigma x^{n+1} + q_n(x) ,$$

where $q_n(x)$ is a polynomial of degree n. Furthermore, the function $T_{n+2}(v)$, where

$$v = \frac{x - \frac{\sigma}{n+2}}{1 + \frac{\sigma}{n+2}} , \tag{2.1.14}$$

has (n+3) extreme values in the interval $-1 \le v \le 1$. From theorem 1.4, we require that at least (n+2) of these extreme points must lie in the interval $-1 \le x \le 1$ so that the polynomial $q_n(x)$ satisfies the characterization property of the best approximation. Hence, in the extreme case, we must have

$$\frac{1 - \frac{\sigma}{n+2}}{1 + \frac{\sigma}{n+2}} = \cos \frac{\pi}{n+2} ,$$

or

$$\sigma = (n+2)\tan^2 \frac{\pi}{2(n+2)} .$$

Thus, the solution (2.1.12a) is only true for those values of σ satisfying the condition (2.1.11). The Zolotareff polynomial can also be obtained in terms of elliptic functions for $\sigma > (n+2)\tan^2 \frac{\pi}{2(n+2)}$, on using a function-theoretical method, (see Achieser [1], p.282-285), but we shall not go into detail here.

2.1.1.4. A problem of Bernstein and Achieser.

Bernstein and Achieser considered a special problem of Chebyshev (or uniform) approximation (see Achieser [1], p.249-251;

Meinardus [21], p.36-41). From the results obtained, they derived explicit expressions for the quantity $E_n(f)$ for certain rational functions.

Let $p_m(x)$ be a real, positive polynomial of degree m in the interval [-1,1], defined by

(2.1.15) $$p_m(x) = \prod_{\nu=1}^{m} \left(1 - \frac{x}{a_\nu}\right) ,$$

where the numbers a_ν satisfy $|a_\nu| > 1$, for $\nu = 1(1)m$.

Let V be finite space of dimension n whose basis consists of the functions of the form

$$\frac{x^\nu}{p_m(x)} , \qquad \nu = 0(1)(n-1) .$$

We want to approximate the function

(2.1.16) $$f(x) = \frac{x^n}{p_m(x)} ,$$

by functions out of the space V. Let us firstly define the numbers c_ν, where $\nu = 1(1)m$, by

(2.1.17) $$c_\nu^2 - 2c_\nu a_\nu + 1 = 0 .$$

Then,

(2.1.18) $$a_\nu = \frac{1}{2}\left(c_\nu + \frac{1}{c_\nu}\right) .$$

We choose c_ν such that $|c_\nu| < 1$, for $\nu = 1(1)m$. Now, let the real variable x, where $|x| \le 1$, be related to the complex value v by the equation

$$(2.1.19) \qquad x = \frac{1}{2}\left(v + \frac{1}{v}\right), \quad |v| = 1 \text{ and } \mathrm{Im}(v) \ge 0,$$

and set

$$(2.1.20) \qquad H_m(v) = \prod_{\nu=1}^{m} (v - c_\nu) \quad .$$

For $n \ge m$, we define the function $T_n(x,p_m)$ by

$$(2.1.21) \qquad T_n(x,p_m) = \frac{K_n}{2}\left\{v^{n-2m}\frac{H_m(v)}{H_m(\frac{1}{v})} + v^{2m-n}\frac{H_m(\frac{1}{v})}{H_m(v)}\right\} p_m(x) \ ,$$

where

$$(2.1.22) \qquad K_n = \begin{cases} \dfrac{2^{1-m}\prod_{\nu=1}^{m}(1+c_\nu^{\,2})}{1+\prod_{\nu=1}^{m} c_\nu^{\,2}} \ , & \text{for } n = m, \\[2ex] 2^{1-n}\prod_{\nu=1}^{m}(1+c_\nu^{\,2}) \ , & \text{for } n > m. \end{cases}$$

After some algebra, we find that $T_n(x,p_m)$ is a polynomial of degree n whose coefficient of x^n is unity. We have the following result.

Theorem 2.1.

If $g(x)$ is a best uniform approximation to $f(x)$ out of V, then for $n \geq m$,

(2.1.23) $$f(x) - g(x) = \frac{T_n(x,p_m)}{p_m(x)} ,$$

and

(2.1.24) $$\|f-g\| = K_n .$$

Proof see Meinardus [21], p.38.

By choosing appropriate polynomial $p_m(x)$ and applying this result, we can obtain explicit expressions for $E_n(f)$ for certain rational functions. As an example, let us put

(2.1.25) $$p_2(x) = (1 - \frac{x^2}{a^2}) ,$$

where a is a real number such that $a > 1$. Then,

(2.1.26) $$H_2(v) = v^2 - \alpha^2 ,$$

where $\alpha = a - \sqrt{a^2-1}$. From (2.1.21), we can show that, for $r \geq 1$, $T_{2r}(x,p_2)$ is a polynomial of degree r in x^2, and

$$\frac{T_{2r}(x,p_2)}{p_2(x)} = \frac{C_r}{x^2-a^2} + (\text{a polynomial of degree } 2r-2),$$

where $\qquad C_r = -2K_{2r}\alpha^{2-2r}a^2(a^2-1).$

Thus, the maximum error occuring when we approximate the function $\frac{1}{x^2-a^2}$ on the interval $[-1,1]$ by polynomials of degree $\leq 2r$ or $\leq 2r+1$ is given by

$$(2.1.27) \qquad E_{2r}\left(\frac{1}{x^2-a^2}\right) = E_{2r+1}\left(\frac{1}{x^2-a^2}\right) = \frac{(a-\sqrt{a^2-1})^{2r}}{2a^2(a^2-1)}$$

It is worth noting that the function $T_n(x,p_m)$ defined by (2.1.21) will be introduced again in chapters 3 and 4 under a different notation. Furthermore, a detailed study of its properties will be given.

We shall now consider some further explicit results given by Bernstein [2] and Rivlin [30].

2.1.1.5. Some further results due to Bernstein and Rivlin.

In one of his papers, Bernstein [2] considered the convergence of the series expansion of a given function $f(x)$. Suppose that we can write $f(x)$ in the form

$$(2.1.28) \qquad f(x) = \sum_{k=0}^{\infty} y_k \quad ,$$

where y_k is some polynomial of degree k. The series $\sum_{k=0}^{\infty} y_k$ is called "the most economical polynomial polynomial expansion" of f(x) if

$$(2.1.29) \qquad E_n(f) = \|f(x) - \sum_{k=0}^{n} y_k\| ,$$

for each n, where $E_n(f)$ is defined as in equation (1.2.2); in other words, if the n^{th} partial sum is the best uniform polynomial approximation of degree n to f(x), for each n. In the following theorem, he showed that for certain class of functions f(x), the expansion of f(x) in a series of Chebyshev polynomials is the most economical one.

Theorem 2.2. (S.N. Bernstein).

The expansion

$$(2.1.30) \qquad f(x) \doteq \sum_{k=0}^{\infty}{}' \alpha_k T_k(x) ,$$

where $\alpha_k \geq 0$, is the most economical one of f(x) if and only if the ratio k_{i+1}/k_i of the indices of two sucessive non-vanishing coefficients $\alpha_{k_{i+1}}$ and α_{k_i} is an odd integer, for every i. (see [10], p.16~~3~~4).

Proof. We first suppose that the condition is satisfied, then for every n such that $k_i \le n < k_{i+1}$, for some i, the n^{th} partial sum $S_n(x)$ is defined by

$$(2.1.31) \qquad S_n(x) = \sum_{k=0}^{k_i}{}' \alpha_k T_k(x) .$$

Since $T_{k_{i+1}}(x)$, $T_{k_{i+2}}(x), \ldots,$ all have the values ± 1 alternately at $(k_{i+1}+1)$ points x_ℓ, where

$$x_\ell = \cos\left(\frac{\ell\pi}{k_{i+1}}\right) , \qquad \ell = 0(1)(k_{i+1}) ,$$

the function $f(x) - S_n(x)$ takes the value $\sum_{k=k_{i+1}}^{\infty} \alpha_k$ alternately at these points. As $k_{i+1} + 1 \ge n + 2$, it follows from theorem 1.4 that $S_n(x)$ is the best uniform polynomial approximation to $f(x)$.

Conversely, if $S_n(x)$ is the best uniform polynomial approximation of degree n to $f(x)$, where $k_i \le n \le k_{i+1}$. Then

$$\|f(x) - S_n(x)\| = \left|\sum_{k=k_{i+1}}^{\infty} \alpha_k T_k(1)\right| = \sum_{k=k_{i+1}}^{\infty} \alpha_k .$$

Again, using theorem 1.4, this maximum must be obtained in at least (n+2) points t_ℓ, $\ell = 0(1)(n+1)$, of [-1,1]. It follows that, for each ℓ, $T_{k_{i+1}}(t_\ell) = T_{k_{i+2}}(t_\ell) = \ldots = \pm 1$, the sign is independent of k_j, $j \geq i+1$. This implies that the ratio k_{i+1}/k_i is an odd integer, for each i, and the theorem is proved.

We note that the Weirstrass function (2.1.4) is a special case of this theorem, where $\alpha_i = a^i$ and $k_i = b^i$, for $i = 0,1,2,\ldots$, a being a positive number less than 1, and b being an odd integer greater than 1.

We shall now consider another class of functions whose truncated Chebyshev expansions are also the best uniform polynomial approximations if the coefficients of the highest remaining terms are suitably adjusted.

<u>Theorem 2.3</u> (T.J. Rivlin)

Let a,b be non-negative integers, a > 0 and

$$f(x) = \sum_{j=0}^{\infty} t^j \, T_{aj+b}(x) , \tag{2.1.32}$$

where $|t| < 1$. Then

$$f(x) = \frac{T_b(x) - t\, T_{|b-a|}(x)}{1 + t^2 - 2t\, T_a(x)} . \tag{2.1.33}$$

If $ak + b \leq n < a(k+1) + b$ and if we put

$$(2.1.34) \qquad q(x) = \sum_{j=0}^{k} t^j T_{aj+b}(x) + \frac{t^{k+2}}{1-t^2} T_{ak+b}(x) ,$$

then

$$(2.1.35) \qquad p_n(x) = q(x) ,$$

and

$$(2.1.36) \qquad E_n(f) = \frac{|t|^{k+1}}{1-t^2} .$$

Proof. (2.1.33) follows directly from (2.1.32) if we write f(x) in the form

$$(2.1.37) \qquad f(x) = \mathrm{Re}[e^{ib\theta} \sum_{j=0}^{\infty} (te^{ia\theta})^j] ,$$

and sum the infinite series. Now, let us put

$$\varepsilon(x) = f(x) - q(x) .$$

Then

$$\varepsilon(x) = \mathrm{Re}\left[e^{ib\theta} \left(\frac{t^{k+1} e^{ia(k+1)\theta}}{1 - te^{ia\theta}} - \frac{t^{k+2}}{1-t^2} e^{iak\theta} \right) \right].$$

After some algebra, we obtain

$$\varepsilon(x) = \frac{t^{k+1}}{1-t^2} \frac{A(\theta)}{B(\theta)} , \tag{2.1.38}$$

where $A(\theta)$ and $B(\theta)$ are trigonometric polynomials given by

$$A(\theta) = \cos[(k+1)a+b]\theta - 2t\cos(ka+b)\theta + t^2\cos[(k-1)a+b]\theta ,$$

and

$$B(\theta) = 1 + t^2 - 2t\cos a\theta .$$

If we now define an angle ψ by

$$\cos\psi = \frac{-2t + (1+t^2)\cos a\theta}{B(\theta)} \tag{2.1.39}$$

and

$$\sin\psi = \frac{(1-t^2)\sin a\theta}{B(\theta)} . \tag{2.1.40}$$

Then,

$$\varepsilon(x) = \frac{t^{k+1}}{1-t^2}\cos[(ak+b)\theta + \psi] \ . \tag{2.1.41}$$

As θ varies from 0 to π, it follows from (2.1.39) and (2.1.40) that ψ varies from 0 to $a\pi$, and $[(ak+b)\theta + \psi]$ varies continuously from 0 to $[a(k+1)+b]\pi$. Thus, $\varepsilon(x)$ takes the extreme values $\pm \frac{|t|^{k+1}}{1-t^2}$ alternately at $[a(k+1)+b+1]$ points. Since $n+2 \leq a(k+1)+b+1$, the result follows from theorem 1.4.

Corollary 2.1.

The result of this theorem can be readily extended to the case of the function $\alpha f(x) + \beta$, where α and β are two arbitrary (real or complex) numbers (see Rivlin [30]).

With suitable choice of the parameters a,b α,β, and t, Rivlin then recovered some results previously given by Bernstein ([2], p.120), Talbot [35] and Hornecker [14].

(i) If $t = \lambda - (\lambda^2-1)^{\frac{1}{2}}$, where $\lambda > 1$, then $|t| < 1$. By choosing $a = 1$, $b = 0$, $\alpha = 4t/(t^2-1)$ and $\beta = -2t/(t^2-1)$, we obtain the function

$$f(x) = \frac{1}{x-\lambda} \ , \qquad \lambda > 1 \text{ and } -1 \leq x \leq 1 \ . \tag{2.1.42}$$

The best uniform polynomial approximation of degree n to f(x) is given by

$$p_n(x) = \frac{-2t}{t^2-1} + \frac{4t}{t^2-1} \sum_{j=0}^{n-1} t^j T_j(x) - \frac{4t^{n+1}}{(1-t^2)^2} T_n(x) \,, \tag{2.1.43a}$$

and

$$E_n(f) = \frac{4t^{n+2}}{(1-t^2)^2} \,. \tag{2.1.43b}$$

This is precisely the result given by Bernstein ([2], p.120) and Talbot [35].

(ii) Let $t = -(1+2c^2) + 2c(1+c^2)^{\frac{1}{2}}$, where c is any real number, then $-1 < t < 0$. By choosing $a = 2$, $b = 0$, $\beta = \frac{-4t}{1-t^2}$ and $\alpha = 8t/(1-t^2)$, we find

$$f(x) = \frac{1}{c^2+x^2} \,, \quad \text{where } -1 \le x \le 1. \tag{2.1.44}$$

If $2k \le n < 2(k+1)$, then

$$p_n(x) = \frac{8t}{1-t^2}\left[\sum_{j=0}^{k-1} t^j T_{2j}(x) + \frac{t^k}{1-t^2} T_{2k}(x)\right] \,, \tag{2.1.45}$$

and

$$E_n(f) = \frac{8|t|^{k+2}}{(1-t^2)^2} , \tag{2.1.46}$$

as given by Hornecker [14].

So far, we have considered several explicit results for the best uniform polynomial approximations. In the following section, we shall list a few examples where $p_n(x)$ and E_n were given asymptotically, for large n.

2.1.2. Asymptotic results.

The function f(z) is said to belong to the space A[-1,1] if there exist ellipses with foci at -1 and +1, such that f(z) is holomorphic in their interiors. Let ξ_ρ denote an ellipse whose semi-sum of the axes is ρ, and let $q = q(f)$ be the supremum of all numbers ρ such that f(z) is holomorphic within ξ_ρ. The ellipse ξ_q is called the "regularity ellipse" of f.

The behaviour of the quantity $E_n(f)$, for $f \in A[-1,1]$, as $n \to \infty$, was first investigated in a series of papers by Bernstein (see [2]). He gave several theorems which led to asymptotic estimates for $E_n(f)$, for large n, for certain classes of functions. In particular, for entire functions, he obtained the following results.

Theorem 2.4.

Let f(z) be an entire transcendental function which is real for real z, and let a_ν, $\nu = 0,1,2,\ldots$, be the coefficients in the Chebyshev series expansion of f(z). Then there exists a sequence of integers n_μ such that

$$\lim_{\mu\to\infty} \frac{E_{n_\mu}(f)}{|a_{n_\mu+1}|} = 1 , \tag{2.1.47}$$

provided that

(i) $a_{n_\mu+1} \neq 0$, for $\mu = 1,2,\ldots$, and

(ii) $\sum_{\nu=n_\mu+2}^{\infty} |a_\nu| = O(|a_{n_\mu+1}|)$ as $\mu\to\infty$.

Proof. see Meinardus [21], p.95.

From the result of this theorem, we can obtain an asymptotic estimate for $E_n(f)$, for large n, where f is an entire function. For example, let us put

$$f(z) = e^{tz} , \tag{2.1.48}$$

where t is a real number. Then,

$$(2.1.49) \qquad f(z) = \sum_{\nu=0}^{\infty}{}' \, I_\nu(t)\, T_\nu(z) ,$$

where

$$(2.1.50) \qquad I_\nu(t) = \sum_{\mu=0}^{\infty} \frac{(t/2)^{2\mu+\nu}}{\mu!(\nu+\mu)!} ,$$

is the modified Bessel function of order ν with purely imaginary argument. We can put $n_\mu = \mu$, for all μ, and obtain from (2.1.47)

$$(2.1.51) \qquad E_n(e^{tz}) = \frac{|t|^{n+1}}{2^n(n+1)!} (1+O(1)) ,$$

as $n \to \infty$.

From the exact results for rational functions with single poles (see section 2.1.1.5), Bernstein also derived the following asymptotic formulas:

$$(2.1.52) \qquad E_n(\log(\lambda-x)) \sim \frac{[\lambda-(\lambda^2-1)^{\frac{1}{2}}]^n}{n(\lambda^2-1)^{\frac{1}{2}}} ,$$

and

$$(2.1.53) \qquad E_n((\lambda-x)^{-s}) \sim \frac{n^{s-1}}{|\Gamma(s)|} \frac{[\lambda-(\lambda^2-1)\ ]^n}{(\lambda^2-1)^{(1+s)/2}} ,$$

for n large, where λ is a real numbers such that $\lambda > 1$ and s is an arbitrary number. The proofs of these results can be found in [2], p.121-123.

In one of his paper, Hornecker [14] also obtained asymptotic expressions for $E_n(f)$ and $p_n(x)$, for large n, in terms of Chebyshev coefficients of f, by using the characterisation of the best approximation (see theorem 1.4). Since the results are rather lengthy, we shall not reproduce them here.

So far, we have given a survey of exact and asymptotic results found in literature for best uniform polynomial approximation. In the following section, a similar survey will be given for the case of rational approximation.

2.2. On best uniform rational approximation.

2.2.1. Exact results.

The determination of a best uniform rational approximation is, in general, considerably more difficult than that of a best uniform polynomial approximation. It is not surprising, therefore, to find that there are even fewer examples where the best uniform rational approximation $R(n,m;x)$ and the quantity $E_{n,m}(f)$ are given explicitly.

2.1.1.1. A result due to Chebyshev and Talbot.

Let $f(x)$ be a polynomial of degree N defined by

$$(2.2.1) \qquad f(x) = \sum_{\nu=0}^{N}{}' c_{N-\nu} T_\nu(x) ,$$

where c_j, $j = 0(1)N$, are given real numbers. For a non-negative integer n satisfying $n < N$, we define a Hankel matrix $C = (c_{i,j})$ of order $(n+1)$ by

$$(2.2.2) \qquad c_{i,j} = \begin{cases} c_{n-i-j+2} , & \text{for } i+j \le n+2 , \\ 0 , & \text{for } i+j > n+2 , \end{cases}$$

where $i,j = 1(1)(n+1)$.

Then we have the following result.

Lemma 2.1.

For non-negative integers N and n, where $n < N$,

$$(2.2.3) \qquad E_{N-1,n}(f) = \frac{|\lambda|}{2^{N-1}} ,$$

where λ is the eigenvalue of smallest modulus of Hankel matrix C.

On using the characterisation theorem (see theorem 1.7), Chebyshev obtained this result by a complex variable method, (see Achieser [1], p.278-280). Later on, Talbot [35] re-derived this result as an application of what he called "the surd factorisation theorem". He also gave an explicit form for the rational function $R(N-1,m;x)$ of best uniform approximation. The analysis is rather lengthy and we shall not go into detail here. It is worth noting that Meinardus ([21], p.166) posed the question "whether the function $f(x) - R(N-1,n;x)$ plays a role in the theory of rational approximation similar to the role of Tchebycheff polynomials in the theory of polynomial approximation". We shall attempt to answer this question later on in the thesis (see Chapters 3,6).

2.2.1.2. A result due to Boehm.

In one of his papers, Boehm [3] gave complete characterisations for some classes of functions whose best uniform rational approximations are polynomials. He then constructed a family of functions from certain series of Chebyshev polynomials and obtained their best approximations explicitly.

Let k be a positive integer and q be an odd integer greater than unity. For any summable sequence $\alpha = \{a_i\}$ of non-negative real number a_i, we define the function $f(\alpha,k,q)$ on $[-1,1]$ by

$$(2.2.4) \qquad f(\alpha,k,q) = \sum_{i=1}^{\infty} a_i \, T_{k_q i-1}(x) \quad ,$$

where $T_j(x)$ is the Chebyshev polynomial of degree j (see definition 1.6).

For any non-negative integer p, let us put

(2.2.5) $$f_p(\alpha,k,q) = \sum_{i=1}^{p} a_i \, T_{k_q^{i-1}}(x) ,$$

then we have the following result.

Lemma 2.2.

The polynomial $f_p(\alpha,k,q)$ of degree kq^{p-1} is the best uniform rational approximation $R(kq^{p-1} + t,u;x)$, (see definition 1.5), to $f(\alpha,k,q)$ on $[-1,1]$, with maximum error $\sum_{i=p+1}^{\infty} a_i$, for all integers t,u such that

(2.2.6) $$0 \le t \le k(q^p - q^{p-1}) - 1 \quad \text{and} \quad 0 \le u \le k(q^p - q^{p-1}) - 1.$$

Proof. We first observe that since $|T_{k_q^{i-1}}| \le 1$ for all $i = 1,2,3,\ldots$ and $x \in [-1,1]$, the series (2.2.4) converges uniformly for all $x \in [-1,1]$, and the function $f(\alpha,k,q)$ is continuous on this interval. Now, let

(2.2.7) $$\varepsilon(x) = f(\alpha,k,q) - f_p(\alpha,k,q) = \sum_{i=p+1}^{\infty} a_i \, T_{k_q^{i-1}}(x) .$$

For each $i \geq p+1$, the Chebyshev polynomial $T_{kq^{i-1}}(x)$ assumes its maximum value ± 1 alternately at (kq^p+1) points x_ν of $[-1,1]$, where

$$(2.2.8) \qquad x_\nu = \cos \frac{\nu\pi}{kq^p} , \qquad \nu = 0(1)(kq^p) .$$

Furthermore, the sign is independent of i, since q is an odd integer. Thus, $\varepsilon(x)$ assumes its maximum value $\sum_{i=p+1}^{\infty} a_i$ alternately at these (kq^p+1) points. The result now follows from lemma 1.1.

We note that the Weierstrass function defined in (2.1.4) is a special case of the functions $f(\alpha,k,q)$ if we put $\alpha = \{a^i\}$, $k = 1$ and $q = b$ in equation (2.2.4). The functions $f(\alpha,k,q)$ can be generalised further so that the result of lemma 2.2 still holds. The only condition we have to impose on the series (2.2.4) is that the degree of each Chebyshev polynomial in the series being an odd multiple (greater than unity) of the degree of the previous term. The series obtained in this way were originally investigated by Bernstein for the best uniform polynomial approximations, (see section 2.1.1.5).

We shall now consider some particular functions where the quantity $E_{n,m}(f)$ is given asymptotically, for large n.

2.2.2. Asymptotic results.

Although much work has been done to determine the order of convergence of the quantity $E_{n,m}(f)$ and its dependency on the structure of the function $f(x)$ or the modulus of continuity of f and its derivatives (see [2],[13],[21]), one can not find many particular functions for which $E_{n,m}(f)$ is obtained asymptotically for large n.

2.2.2.1. The exponential function

Meinardus (see [21], p.168) represented the exponential e^x in the form (2.1.51). On constructing a special rational function of the form $\bar{r}(n,1;x)$, he then proved that, as $n\to\infty$,

$$E_{n,1}(e^x) = \frac{1}{2^{n+1}(n+1)(n+2)!}\,(1+O(1)). \tag{2.2.9}$$

Comparing this result with (2.1.51), we see that asymptotically, for large n, the function e^x can be approximated more closely by rational functions in $V_{n,1}$ than by polynomials in P_n. By using the same method, asymptotic values for $E_{n,m}(f)$, where $m > 1$, for large n, may be obtained, and he conjectured that

(2.2.10) $$E_{n,m}(e^x) = \frac{n!m!}{2^{n+m}(n+m)!(n+m+1)!}(1+0(1)),$$

as $n+m \to \infty$.

2.2.2.2. The function $|x|$.

For some classes of functions, the orders of convergence of the quantities $E_n(f)$ and $E_{n,m}(f)$ in the limit as $n\to\infty$ are the same. In other words, for such functions, the uniform polynomial approximation is equally effective as the uniform rational approximation, (see, for example, Lorentz [18], chapter 4; Goncar [11]). This is not always true, however, as we have seen in the previous section for the case of an entire function. To emphasize this fact, Newman [24] has given a remarkable example where the quantity $E_{n,n}(f)$ converges to zero as $n\to\infty$ at a much faster rate than $E_n(f)$. For the function $f(x) = |x|$, where $-1 \le x \le 1$, he constructed a particular rational function which is of the form $\bar{r}(n,n;x)$ if n is even and of the form $\bar{r}(n+1,n-1;x)$ if n is odd. He then obtained the maximum error occuring in approximating this rational function to $|x|$ and derived the following result.

Theorem 2.5. (Newman).

For non-negative integer $n > 4$,

(2.2.11) $$E_{n,n}(|x|) \le 3e^{-n^{\frac{1}{2}}} \quad ;$$

if n is even, and

(2.2.12) $$E_{(n+1),(n-1)}(|x|) \le 3e^{-n^{\frac{1}{2}}} \quad ,$$

if n is odd.

Proof. see [24].

The above theorem provides an upper bound for $E_{n,n}(|x|)$, he also obtained its lower bound, by using a different method.

Theorem 2.6 (Newman).

If $r(x) \in V_{k,\ell}$, where $k,\ell \le n$, then

(2.2.13) $$\max \big| |x| - r(x) \big| \ge \frac{1}{2} e^{-9n^{\frac{1}{2}}} \quad .$$

Proof. see [24].

From this result, we find that the rational function constructed by Newman is not too far from the best rational approximation. Furthermore, the order of convergence of $E_{n,n}(|x|)$ is $e^{-c\sqrt{n}}$, where c is a positive constant, which is far better than the order of convergence k/n of $E_n(|x|)$, where k is also a constant.

Similar result can also be obtained for a wider class of functions.

2.2.2.3. The function x^α, where α is a positive real number.

Using a different technique, Freud and Szabados [9] constructed a rational function of the form $\bar{r}(n,n;x)$ and prove the following result.

Theorem 2.7.

Let α be an arbitrary positive real number. Then there exists a rational function $r_n(x,\alpha)$ of the form $\bar{r}(n,n;x)$ such that

$$(2.2.14) \qquad |x^\alpha - r_n(x;\alpha)| \le 19 \exp\{-0.78\alpha. \frac{n^{1/3}}{(1+3\alpha)^{2/3}}\}$$

for all x in [0,1] and for sufficiently large n.

Proof. see [9].

Thus, $E_{nn}(x^\alpha)$ converges to zero at a rate of at least $e^{-cn^{1/3}}$, whereas $E_n(x^\alpha)$ converges to zero at a rate of k/x^α, where c,k are some constants.

Further results along this line can be found in Szabados [34] and Turán [36].

In the next chapter, we shall generalize some exact results given in this chapter.

CHAPTER III

SOME EXPLICIT RESULTS FOR THE BEST UNIFORM RATIONAL APPROXIMATION TO CERTAIN CONTINUOUS FUNCTIONS

In section 3.1, we shall define a class of rational functions which will be extensively used later on. We shall derive some of its properties, and then introduce a special class of continuous functions. The best uniform approximations to these continuous functions will be given explicitly in sections 3.2 and 3.3. From these general results, we recover some particular results for the best uniform polynomial and rational approximations previously given by Bernstein [8], Boehm [3] and Rivlin [30] (see also sections 2.1.1.5 and 2.2.1.2).

3.1. A class of rational functions $F_{k,\ell}(x)$ and its properties.

We shall begin with some definitions.

Definition 3.1.

Let α be any integer and t be any real or complex number such that $|t| < 1$. For $\theta \in [0,\pi]$, we define a function $\delta(\alpha,t;\theta)$ by

$$(3.1.1) \qquad \delta(\alpha,t;\theta) = i \log \left[\frac{te^{i\alpha\theta}-1}{t - e^{i\alpha\theta}} \right],$$

where we choose that branch of the logarithmic function so that $\delta(\alpha,t;0) = 0$ and $\delta(\alpha,t;\pi) = \alpha\pi$.

The function $\delta(\alpha,t;\theta)$ has the following properties.

Lemma 3.1.

(i) $\delta(\alpha,t;\theta)$ is a continuous function of θ in $[0,\pi]$.

(ii) $\delta(\alpha,t;\theta) = \delta(\alpha,\bar{t};\theta)$, where the bar denotes the complex conjugate.

(iii) δ is real for all θ in $[0,\pi]$ when t is real.

Proof. Since $|t| < 1$, $\delta(\alpha,t;\theta)$ has no singularities in $[0,\pi]$, hence it is a continuous function. (ii) and (iii) can be obtained directly from (3.1.1).

Definition 3.2.

Let T_ℓ be a set of ℓ real or complex numbers $\{t_1,t_2,\ldots,t_\ell\}$ satisfying the conditions

(i) $|t_s| < 1$, for $s = 1(1)\ell$;

(ii) if $t_s \in T_\ell$ and $\mathrm{Im}(t_s) \neq 0$, then $\bar{t}_s \in T_\ell$.

To each t_s in T_ℓ, we associate a number ε_s, where ε_s is either +1 or -1. We let Σ_ℓ denote the set $\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_\ell\}$ with the restriction that if t_s is complex, then we shall associate the same value ε_s to t_s and $\bar{t}_s$. When $\ell = 0$, T_ℓ and Σ_ℓ are taken to be null sets.

To each t_s in T_ℓ, we also associate a set of integers $\{\alpha_{j,s}\}$ which may be finite or infinite. If t_s is complex, then we shall impose the condition that the set of integers $\{\alpha_{j,s}\}$ is the same for t_s and $\bar{t}_s$. We shall denote by A the set of integers $\{\alpha_{j,s}\}$ taken over all j and all s.

Definition 3.3.

For $\theta \in [0,\pi]$, we define the function $\Delta_j(A,T_\ell,\Sigma_\ell;\theta)$ by

$$(3.1.2) \qquad \Delta_j(A,T_\ell,\Sigma_\ell;\theta) = \sum_{s=1}^{\ell} \varepsilon_s \delta(\alpha_{j,s},t_s;\theta) .$$

When there is no possibility of confusion over the sets A, T_ℓ and Σ_ℓ, we shall abbreviate $\Delta_j(A,T_\ell,\Sigma_\ell;\theta)$ by $\Delta_{\ell,j}(\theta)$. The function $\Delta_{\ell,j}(\theta)$ has the following properties.

Lemma 3.2.

(i) $\Delta_{\ell,j}(\theta)$ is a real continuous function of θ on $[0,\pi]$.

(ii) $\Delta_{\ell,j}(0) = 0$ and $\Delta_{\ell,j}(\pi) = \sum_{s=1}^{\ell} \varepsilon_s \alpha_{j,s} \pi$.

Proof. follows directly from lemma 3.1.

Definition 3.4.

For any integer k, we define a real function $F_k(A,T_\ell,\Sigma_\ell;x)$ on $[-1,1]$ by

(3.1.3) $$F_k(A,T_\ell,\Sigma_\ell;x) = \cos[k\theta + \Delta_j(A,T_\ell,\Sigma_\ell;\theta)] ,$$

where $x = \cos\theta$.

Again, when there is no possibility of confusion over the sets A, T_ℓ and Σ_ℓ, we shall abbreviate $F_k(A,T_\ell,\Sigma_\ell;x)$ by $F_{k,\ell}(x)$.

These functions can be looked upon as generalizations of the Chebyshev polynomials of the first kind. For when $\ell = 0$, $F_k(A,T_0,\Sigma_0;x) = T_k(x)$. It is worth noting that although we have introduced $F_k(A,T_\ell,\Sigma_\ell;x)$ as in(3.1.3), this class of functions has also been used by other authors under different notations, (see, for example, section 2.1.1.4 and Meinardus [21], p.38). We shall now obtain some properties of $F_{k,\ell}(x)$.

Lemma 3.3.

$F_{k,\ell}(x)$ attains its extreme values of ± 1 alternately in at least $(1 + |k + \sum_{s=1}^{\ell} \varepsilon_s \alpha_{j,s}|)$ points of $[-1,1]$.

Proof. As θ varies from 0 to π, $[k\theta + \Delta_{\ell,j}(\theta)]$ takes all values in $[0,(k + \sum_{s=1}^{\ell} \varepsilon_s \alpha_{j,s})\pi]$, since $\Delta_{\ell,j}(\theta)$ is a real continuous function of θ in $[0,\pi]$. The result follows immediately.

Lemma 3.4.

$F_{k,\ell}(x)$ is a quotient of two polynomials of degree $(|k| + \sum_{s=1}^{\ell} |\alpha_{j,s}|)$ and $(\sum_{s=1}^{\ell} |\alpha_{j,s}|)$ respectively.

Proof. We can assume without loss of generality that $\varepsilon_s = +1$, for $s = 1(1)\ell_1$, and -1 for $s = (\ell_1+1)(1)(\ell)$. Then $F_{k,\ell}(x)$ can be written in the form

$$F_{k,\ell}(x) = \mathrm{Re}\left\{ e^{ik\theta} \prod_{s=1}^{\ell_1} \left[\frac{t_s e^{i\alpha_{j,s}\theta} - 1}{t_s - e^{i\alpha_{j,s}\theta}} \right] \prod_{s=\ell_1+1}^{\ell} \left[\frac{t_s - e^{i\alpha_{j,s}\theta}}{t_s e^{i\alpha_{j,s}\theta} - 1} \right] \right\}.$$

By multiplying both the numerator and denominator by the complex conjugate of the latter, we observe that the denominator is a polynomial of degree $(\sum_{s=1}^{\ell} |\alpha_{j,s}|)$ in $\cos\theta$, and the numerator is a polynomial of degree $(|k| + \sum_{s=1}^{\ell} |\alpha_{j,s}|)$, also in $\cos\theta$. The result now follows immediately.

The class of rational functions $F_{k,\ell}(x)$ possesses some further important properties as we shall see in chapter 4. We shall now introduce a special class of continuous functions which will be used in the next two sections.

Let $\{k_j\}_{j=0}^{\infty}$ be a subsequence of non-negative integers and $\lambda = \{\lambda_j\}_{j=0}^{\infty}$ be any sequence of arbitrary (real or complex) numbers such that $\sum_{j=0}^{\infty} |\lambda_j|$ is finite. We define a function $f(A,T_\ell,\Sigma_\ell,\lambda;x)$ by

$$(3.1.4) \qquad f(A,T_\ell,\Sigma_\ell,\lambda;x) = \sum_{j=0}^{\infty} \lambda_j \, F_{k_j}(A,T_\ell,\Sigma_\ell;x) \quad .$$

When there is no possibility of confusion over the sets A,T_ℓ,Σ_ℓ and the sequence λ, we shall denote $f(A,T_\ell,\Sigma_\ell,\lambda;x)$ by $f(x)$. The function $f(x)$ has the following property.

<u>Lemma 3.5.</u>

$f(x)$ is continuous for all $x \in [-1,1]$.

Proof. Since $|F_{k_j,\ell}(x)| \leq 1$ for every $x \in [-1,1]$, and $\sum_{j=0}^{\infty} |\lambda_j|$ is finite, the series $\sum_{j=0}^{\infty} \lambda_j F_{k_j,\ell}(x)$ converges uniformly on $[-1,1]$. Hence, $f(x)$ is a continuous function of x on $[-1,1]$, (see, for example, Simmons [32], p.84).

We shall now obtain an extension of a result by Rivlin [30], using appropriate choice of the sets Λ, Σ_ℓ and the sequences $\{\lambda_j\}$, $\{k_j\}$.

3.2. A generalization of a result due to Rivlin.

Let $\alpha_{j,s} = \alpha_s$, for $s = 1(1)\ell$ and all j, so that A is a finite set of ℓ integers. Also, let $\varepsilon_s = +1$, for $s = 1(1)\ell$, and $\lambda_j = \gamma^j$, where γ is a real number satisfying $|\gamma| < 1$. If we choose $k_j = aj+b$, where a,b are positive integers, $a \geq 1$ and $b \geq 0$. Then the function $f(x)$ can be expressed as follows.

Lemma 3.6.

$$(3.2.1) \qquad f(x) = \frac{F_{b,\ell}(x) - \gamma F_{b-a,\ell}(x)}{1 + \gamma^2 - 2\gamma\, T_a(x)} .$$

Proof. From (3.1.4), we have

$$f(x) = \sum_{j=0}^{\infty} \gamma^j F_{aj+b,\ell}(x) ,$$

which can be re-written as

$$f(x) = \mathrm{Re}\{e^{i[b\theta+\Delta_{\ell,j}]} \sum_{j=0}^{\infty} (\gamma e^{ia\theta})^j\} .$$

Identity (3.2.1) is now obtained by summing the infinite series.

For any non-negative integer p, we define the function $f_p(A,T_\ell,\Sigma_\ell,\lambda;x)$ by

$$(3.2.2) \quad f_p(A,T_\ell,\Sigma_\ell,\lambda;x) = \sum_{j=0}^{p} \gamma^j F_{aj+b,\ell}(x) + \frac{\gamma^{p+2}}{1-\gamma^2} F_{ap+b,\ell}(x) .$$

We shall denote the function $f_p(A,T_\ell,\Sigma_\ell,\lambda;x)$ by $f_p(x)$ and obtain the following theorem.

Theorem 3.1.

Let α_s be positive integers such that $\sum_{s=1}^{\ell} \alpha_s \le a-1$, and let m be any integer satisfying

$$(3.2.3) \qquad \sum_{s=1}^{\ell} \alpha_s \le m \le a-1 .$$

Let n be any non-negative integer, and suppose p is such that

$$(3.2.4) \qquad ap+b + \sum_{s=1}^{\ell} \alpha_s \le n \le a(p+1) + b - 1 \quad .$$

Then $f_p(x)$ is the best uniform rational approximation $R(n,m;x)$ to $f(x)$ and

$$(3.2.5) \qquad E_{n,m}(f) = \frac{|\gamma|^{p+1}}{1-\gamma^2} \quad .$$

Proof. Let us write

$$\varepsilon(x) = f(x) - f_p(x) .$$

Then,

$$\varepsilon(x) = \mathrm{Re}\left\{ e^{i[b\theta+\Delta_{\ell,j}]} \left[\sum_{j=p+1}^{\infty} (\gamma e^{ia\theta})^j - \frac{\gamma^{p+2}}{1-\gamma^2} e^{iap\theta} \right] \right\} ,$$

$$= \mathrm{Re}\left\{ \frac{\gamma^{p+1}}{1-\gamma^2} e^{i[b\theta+\Delta_{\ell,j}]} e^{iap\theta} \left[\frac{e^{ia\theta}-\gamma}{1-\gamma e^{ia\theta}} \right] \right\} ,$$

on summing the series. Hence,

$$(3.2.6) \qquad \varepsilon(x) = \frac{\gamma^{p+1}}{1-\gamma^2} F_{ap+b}(A, T_{\ell+1}, \Sigma_{\ell+1}; x) \quad ,$$

where $\varepsilon_{\ell+1} = +1$, $t_{\ell+1} = \gamma$ and $\alpha_{\ell+1} = a$.

From lemma 3.3, the function $F_{ap+b}(A,T_{\ell+1},\Sigma_{\ell+1};x)$ attains its extreme values ± 1 alternately in at least

$(ap + b + \sum_{s=1}^{\ell} \alpha_s + a + 1)$ points. Also, from lemma 3.4, $f_p(x)$ is a rational function of the form $\bar{r}(ap + b + \sum_{s=1}^{\ell} \alpha_s, \sum_{s=1}^{\ell} \alpha_s;x)$.

Thus, by lemma 1.1, $f_p(x)$ is the best uniform rational approximation $R(ap + b + \sum_{s=1}^{\ell} \alpha_s + u, \sum_{s=1}^{\ell} \alpha_s + v;x)$ to $f(x)$, where $0 \le u,v \le a - \sum_{s=1}^{\ell} \alpha_s - 1$. By putting $n = ap + b + \sum_{s=1}^{\ell} \alpha_s + u$ and $m = \sum_{s=1}^{\ell} \alpha_s + v$, the conditions (3.2.3) and (3.2.4) follow.

We have

$$E_{n,m}(f) = \max_{x\in[-1,1]} |\varepsilon(x)| = \frac{|\gamma|^{p+1}}{1 - \gamma^2} .$$

<u>Corollary 3.1</u>.

The results in theorem 3.1 can be readily extended to the case of the function $(\alpha+\beta f)$, where α and β are two arbitrary (real or complex) constants.

Comment.

If $\ell = 0$, a,b are two integers such that $a \geq 1$, $b \geq 0$ and γ is a real number satisfying $|\gamma| < 1$, then

$$(3.2.7) \qquad f(x) = \frac{T_b(x) - \gamma T_{|b-a|}(x)}{1 + \gamma^2 - 2\gamma T_a(x)} .$$

Let m be an integer satisfying $0 \leq m \leq a-1$ and n be any given non-negative integer, then the best uniform rational approximation $R(n,m;x)$ to $f(x)$ out of $V_{n,m}$ is a polynomial of degree $(ap+b)$, where p is such that $ap+b \leq n \leq a(p+1)+b-1$. We have

$$(3.2.8) \qquad R(n,m;x) = \sum_{j=0}^{p} \gamma^j \, T_{aj+b} + \frac{\gamma^{p+2}}{1-\gamma^2} \, T_{ap+b} ,$$

and

$$(3.2.9) \qquad E_{n,m}(f) = \frac{|\gamma|^{p+1}}{1 - \gamma^2} .$$

In particular, when $m = 0$, $R(n,0;x) = p_n(x)$, $E_{n,0}(f) = E_n(f)$, and we obtain precisely Rivlin's results [30] for the case of polynomial approximation. (see section 2.1.1.5).

3.3. A generalisation of a result due to Bernstein.

Let ℓ be an even integer, $\varepsilon_s = +1$ for $s = 1(1)(\ell/2)$ and $\varepsilon_s = -1$ for $s = (\ell/2+1)(1)(\ell)$. Let $\{k_j\}_{j=0}^{\infty}$ be such that the ratios k_{j+1}/k_j, $j = 0,1,2,\ldots$, are odd integers greater than $(2\ell+1)$. Furthermore, we choose $\alpha_{j,s} = k_j$, for $s = 1(1)\ell$ and let $\{\lambda_j\}_{j=0}^{\infty}$ be a sequence of non-negative real numbers such that $\sum_{j=0}^{\infty} \lambda_j$ is finite. We define the function $f_p(A,T_\ell,\Sigma_\ell,\lambda;x)$ by

$$f_p(A,T_\ell,\Sigma_\ell,\lambda;x) = \sum_{j=0}^{p} \lambda_j F_{k_j}(A,T_\ell,\Sigma_\ell;x). \tag{3.3.1}$$

We shall denote the function $f_p(A,T_\ell,\Sigma_\ell,\lambda;x)$ by $f_p(x)$ and obtain the following result.

Theorem 3.2.

Given two non-negative integers n,m, and suppose p is such that

$$(\ell+1)k_p \le n \le k_{p+1} - \ell k_p - 1,$$
$$\text{and} \qquad \ell k_p \le m \le k_{p+1} - (\ell+1)k_p - 1. \tag{3.3.2}$$

Then $f_p(x)$ is the best uniform rational approximation $R(n,m;x)$ to $f(x)$ and

(3.3.3) $$E_{n,m}(f) = \sum_{j=p+1}^{\infty} \lambda_j$$

Proof. Let us write

(3.3.4) $$\begin{aligned}\varepsilon(x) &= f(x) - f_p(x),\\ &= \sum_{j=p+1}^{\infty} \lambda_j F_{k_j}(A, T_\ell, \Sigma_\ell; x).\end{aligned}$$

We want to consider the values of $\varepsilon(x)$ at $(k_{p+1} + 1)$ points x_q of $[-1,1]$, where

$$x_q = \cos\theta_q = \cos \frac{q\pi}{k_{p+1}}, \quad q = 0(1)(k_{p+1}).$$

Then,

$$\varepsilon(x_q) = \sum_{j=p+1}^{\infty} \lambda_j \cos[k_j\theta_q + \sum_{s=1}^{\ell} \varepsilon_s \delta(k_j, t_s; \theta_q)] .$$

But from definition (3.1), we have

$$\delta(k_j, t_s; \theta_q) = \delta\left(\frac{qk_j}{k_{p+1}}, t_s; \pi\right) = \frac{qk_j\pi}{k_{p+1}} .$$

This is independent of s, so that

$$\sum_{s=1}^{\ell} \varepsilon_s \delta(k_j, t_s; \theta_q) = \frac{qk_j\pi}{k_{p+1}} \sum_{s=1}^{\ell} \varepsilon_s = 0 ,$$

and

$$\varepsilon(x_q) = \sum_{j=k+1}^{\infty} \lambda_j \cos\left(\frac{qk_j}{k_{p+1}}\pi\right) = (-1)^q \sum_{j=p+1}^{\infty} \lambda_j ,$$

as k_j/k_{p+1} is an odd integer whenever $j > (p+1)$. Thus, $\varepsilon(x)$ attains its extreme values $\sum_{j=p+1}^{\infty} \lambda_j$ in at least $(k_{p+1} + 1)$ points in $[-1,1]$. From lemma 3.4, we find that $f_p(x)$ is a rational function of the form $\bar{r}((\ell+1)k_p, \ell k_p;x)$. By using the result of lemma 1.1, f_p is the best uniform rational approximation $R((\ell+1)k_p + u, \ell k_p + v; x)$ to $f(x)$, where $0 \leq u,v \leq k_{p+1}-(2\ell+1)k_p-1$. We now put $n = (\ell+1)k_p+u$ and $m = \ell k_p+v$ and obtain the conditions (3.3.2) for p. Theorem 3.2 is then proved.

Comments.

(i) When $\ell = 0$, the Chebyshev series

$$(3.3.5) \qquad f(A,T_0,\Sigma_0,\lambda;x) = \sum_{j=0}^{\infty} \lambda_j T_{k_j}(x)$$

has its truncated series $f_p(A,T_0,\Sigma_0,\lambda;x)$ of degree p as its best uniform rational approximation $R(n,m;x)$ if p satisfies $k_p \leq n \leq k_{p+1}-1$ and $0 \leq m \leq k_{p+1}-1$. In particular, when $m = 0$, we obtain a result for the best uniform polynomial approximation given by Bernstein (see section 2.1.1.5 and Golomb [10], p.163).

(ii) As a special case of theorem 3.2, we choose $k_j = ab^j$, where a is a positive integer and be is an odd integer $> (2\ell+1)$. We obtain a result which can be looked upon as an extension of a result due to Boehm [3], who considered essentially the case $\ell = 0$, (see also section 2.2.1.2).

(iii) We now choose $k_j = a^j$, where a is an odd integer greater than $(2\ell+1)$, and $\lambda_j = \gamma^j$, where γ is a real number satisfying $0 < \gamma < 1$. When $\ell = 0$, $f(A,T_0,\Sigma_0,\lambda;x)$ is then the well-known Weierstrass function

$$(3.3.6) \qquad f(A,T_0,\Sigma_0,\lambda;x) = \sum_{j=0}^{\infty} \gamma_j \, T_{a^j}(x) ,$$

(see Achieser [1], p.66). Putting $m = 0$, we obtain its best uniform polynomial approximation (see section 2.1.1.2).

We have obtained some explicit results for the best uniform rational approximation. In the next three chapters, we shall obtain some asymptotic results. An algorithm for finding an asymptotic estimate for $E_{n,m}(f)$, for large n, where f is a continuous function on $[-1,1]$ and $m = O(1)n$, will be given in chapter IV. Special cases of this result will be discussed in chapters V and VI.

CHAPTER IV

AN ALGORITHM FOR ESTIMATING $E_{n,m}(f)$, WHERE $m = O(1)n$

Let $f(x) \in C[-1,1]$. The Chebyshev coefficients a_k of $f(x)$, defined by (1.3.8) exist for all $k = 0,1,2,\ldots$ For many functions, we have analytic expressions for a_k, which have also been extensively tabulated (see, for example, Clenshaw [5], Luke [19]). As we have seen in chapters II and III, however, only for a few functions do we have analytic expressions for the quantity $E_{n,m}(f)$ and the rational function of best uniform approximation $R(n,m;x)$. In all other cases, we have to evaluate them numerically using, for example, Maehly's algorithm (see [12]). The purpose of this chapter is to show how an estimate of $E_{n,m}(f)$, for large n, may be readily computed from the knowledge of the Chebyshev coefficients of $f(x)$. In section 4.1, we shall investigate some further important properties of the class of rational functions $F_{k,\ell}(x)$ which was defined in the last chapter. In sections 4.2 and 4.3, we shall prove the main result, which states that when $f(x)$ is a polynomial of degree $(n+r+1)$, then an estimate $E^*_{n,m}(f)$

of $E_{n,m}(f)$, valid for large n, is given by the modulus of the eigenvalue of $(m+1)^{st}$ largest modulus of a certain matrix of order $(m+r+1)$. In section 4.4, we shall consider the generalisation of this result to certain functions that are continuous on $[-1,1]$.

4.1. Some further properties of $F_{k,\ell}(x)$.

Throughout the rest of the thesis, we shall denote by $F_{k,\ell}(x)$ the function $F_k(A,T_\ell,\Sigma_\ell;x)$, where A is a finite set of ℓ integers taking the value of unity, so that $\alpha_{j,s} = \alpha_s = 1$, for all j and $s = 1(1)\ell$. The functions $\delta(\alpha_{j,s},t_s;\theta)$ and $\Delta_{\ell,j}(\theta)$ are therefore independent of j and we may denote them by $\delta_s(\theta)$ and $\Delta_\ell(\theta)$ respectively. We also impose another condition on the set T_ℓ, in addition to the conditions (i) and (ii) in definition 3.2,

(iii) $t_s \neq 0$, for $s = 1(1)\ell$.

We shall now obtain some important properties of $F_{k,\ell}(x)$ and of some related functions. ~~which~~ These properties will be extensively used on. These results are rather lengthy; furthermore, they only play a minor role in the analysis discussed in other sections of this thesis. We, therefore, suggest the reader to go on to the next section and refer back to this section whenever necessary.

Lemma 4.1.

For $\ell = 1,2,3,\ldots$ and any integer k, $F_{k,\ell}(x)$ satisfies the following recurrence relations:

$$(4.1.1) \qquad 2[\tfrac{1}{2}(t_\ell + \frac{1}{t_\ell}) - x]F_{k,\ell} = t_\ell^{-\varepsilon_\ell} F_{k+1,\ell-1} - 2F_{k,\ell-1} + t_\ell^{\varepsilon_\ell} F_{k-1,\ell-1} ,$$

and

$$(4.1.2) \qquad F_{k,\ell} = \varepsilon_\ell (t_\ell - \frac{1}{t_\ell}) \sum_{j=0}^{k-1}{}' \; t_\ell^{-\varepsilon_\ell(k-j)} F_{j,\ell-1} - t_\ell^{-\varepsilon_\ell} F_{k,\ell-1}$$

$$+ \frac{1}{2} \varepsilon_\ell \, t_\ell^{-\varepsilon_\ell k} (t_\ell - \frac{1}{t_\ell}) \left\{ F_{0,\ell-1} + \frac{F_{-1,\ell-1} - t_\ell^{-\varepsilon_\ell} F_{0,\ell-1}}{\frac{1}{2}(t_\ell + \frac{1}{t_\ell}) - x} \right\} .$$

Proof. (4.1.1) follows in a straightforward manner by writing

$$(4.1.3) \qquad F_{k,\ell} = \frac{1}{2} \{\exp[i(k\theta + \Delta_\ell)] + \exp[-i(k\theta + \Delta_\ell)]\},$$

and observing from definition 3.1 that

$$(4.1.4) \qquad e^{i\varepsilon_\ell \delta_\ell} = \frac{t_\ell^{\varepsilon_\ell} - e^{i\theta}}{t_\ell^{\varepsilon_\ell} e^{i\theta} - 1} .$$

By using the same identity (4.1.3), if we now write

$$e^{i\varepsilon_\ell \delta_\ell} = (e^{-i\theta} - t_\ell^{-\varepsilon_\ell})(1 - t_\ell^{-\varepsilon_\ell} e^{-i\theta})^{-1} ,$$

and

$$e^{-i\varepsilon_\ell \delta_\ell} = (e^{i\theta} - t_\ell^{-\varepsilon_\ell})(1 - t_\ell^{-\varepsilon_\ell} e^{i\theta})^{-1} ,$$

then expand these equations in terms of the infinite series of $e^{-i\theta}$ and $e^{i\theta}$ respectively, (4.1.2) will be obtained.

We shall now introduce the so-called "elementary symmetric functions" of the quantities $t_s^{-\varepsilon_s}$, $s = 1(1)\ell$.

<u>Definition 4.1</u>.

For $\ell = 1,2,3,\ldots$, and $j = 1(1)\ell$, let $P_{j,\ell}$ denote the sum of the products of $t_s^{-\varepsilon_s}$, $s = 1(1)\ell$, taken j at a time. We also define

$$P_{0,\ell} = 1 \qquad \text{for } \ell = 1,2,3,\ldots$$

and

$$P_{j,\ell} = 0 \qquad \text{for } j > \ell \text{ and } j < 0 .$$

As an immediate consequence of this definition, we have the following results.

Lemma 4.2.

For $\ell = 1,2,3,\ldots$ and $j = 1(1)\ell$,

(4.1.5) $$P_{j,\ell} = P_{j,\ell-1} + t_\ell^{-\varepsilon_\ell} P_{j-1,\ell-1} .$$

Proof. follows readily from definition 4.1.

Lemma 4.3.

The quantities $t_s^{-\varepsilon_s}$, $s = 1(1)\ell$, are the zeros of the polynomial $\Psi_\ell(z)$ of degree ℓ defined by

(4.1.6) $$\Psi_\ell(z) = \sum_{j=0}^{\ell} (-1)^j P_{j,\ell} z^{\ell-j} .$$

Proof. This follows immediately from the definition of $P_{j,\ell}$, and since $P_{0,\ell} = 1$, the polynomial $\Psi_\ell(z)$ is exactly of degree ℓ.

In addition to the quantities $P_{j,\ell}$, we shall also require the "homogeneous product sums of weight j" of $t_s^{-\varepsilon_s}$, $s = 1(1)\ell$, which we shall denote by $Q_{j,\ell}$.

Definition 4.2.

For $j,\ell = 1,2,3,\ldots$, the quantities $Q_{j,\ell}$ are defined by

$$(4.1.7) \qquad \sum_{s=0}^{j} (-1)^s P_{s,\ell}\, Q_{j-s,\ell} = 0 ,$$

where $Q_{0,\ell} = 1$.

As an immediate consequence of this definition, we have the following result.

Lemma 4.4.

For $j,\ell = 1,2,3,\ldots$,

$$(4.1.8) \qquad Q_{j,\ell} = \sum_{s=1}^{j} (-1)^{s+1} P_{s,\ell}\, Q_{j-s,\ell} ,$$

where $Q_{0,\ell} = 1$ and $P_{j,\ell} = 0$ for $j > \ell$.

Proof. follows readily from the definition of $Q_{j,\ell}$.

Equation (4.1.5) relates the quantities $P_{j,\ell}$ and $P_{j,\ell-1}$, we shall now derive a similar result for $Q_{j,\ell}$.

Lemma 4.5.

For $j,\ell = 1,2,3,\ldots,$

$$Q_{j,\ell} = \sum_{s=0}^{j} t_\ell^{-\varepsilon_\ell s} Q_{j-s,\ell-1} . \tag{4.1.9}$$

Proof. This follows by induction on ℓ, using lemma 4.4.

Definition 4.3.

For $\ell = 1,2,3,\ldots$, we define the polynomial $\Omega_\ell(x)$ of degree ℓ by

$$\Omega_\ell(x) = \prod_{s=1}^{\ell} [\tfrac{1}{2}(t_s + \frac{1}{t_s}) - x] . \tag{4.1.10}$$

We want to express its Chebyshev coefficients in terms of the quantities $P_{j,\ell}$ and $Q_{j,\ell}$.

Lemma 4.6.

For $\ell = 1,2,3,\ldots,$

$$\Omega_\ell(x) = {\sum_{m=0}^{\ell}}' b_{\ell-m} T_m(x) , \tag{4.1.11}$$

where

$$(4.1.12) \qquad b_m = \frac{(-1)^{\ell+m}}{2^{(\ell-1)} P_{\ell,\ell}} \left(\sum_{s=0}^{m} P_{s,\ell} \, P_{\ell-m+s,\ell} \right) ,$$

for $m = 0(1)\ell$.

Proof. Let us write

$$(4.1.13) \qquad g(x) = \prod_{s=1}^{\ell} (t_s^{-\varepsilon_s} - x) = \sum_{s=0}^{\ell} (-1)^s P_{\ell-s,\ell} \, x^s .$$

The second equality follows from lemma 4.3. Then,

$$(4.1.14) \qquad x^{\ell} g(\tfrac{1}{x}) = (-1)^{\ell} P_{\ell,\ell} \sum_{s=1}^{\ell} (t_s^{\varepsilon_s} - x) = \sum_{s=0}^{\ell} (-1)^{\ell-s} P_{s,\ell} x^s .$$

We have

$$(4.1.15) \qquad \begin{aligned} g(x)\, g(\tfrac{1}{x}) &= (-1)^{\ell} P_{\ell,\ell} \prod_{s=1}^{\ell} [(x+\tfrac{1}{x}) - (t_s + \tfrac{1}{t_s})] , \\ &= 2^{\ell} P_{\ell,\ell} \, \Omega_{\ell}(z) , \end{aligned}$$

where $z = \frac{1}{2} (x + \frac{1}{x})$.

Also,

$$g(x)\ g(\tfrac{1}{x}) = \frac{1}{x^{\ell}}\Big(\sum_{s=0}^{\ell} (-1)^s P_{\ell-s,\ell}x^s\Big)\Big(\sum_{s=0}^{\ell} (-1)^{\ell-s} P_{s,\ell}x^s\Big)$$

(4.1.16)

$$= \frac{1}{x^{\ell}}\Big(\sum_{m=0}^{2\ell} C_m\, x^m\Big) ,$$

where

(4.1.17) $$C_m = (-1)^{\ell+m} \sum_{s=0}^{m} P_{s,\ell}P_{\ell-m+s,\ell} , \quad \text{for } m = 0(1)2\ell ,$$

But, for $m = 0(1)\ell$,

$$C_{2\ell-m} = C_m ,$$

since $P_{j,\ell} = 0$ for $j > \ell$ and $j < 0$. Hence, we can re-write (4.1.14) as

(4.1.18) $$g(x)\ g(\tfrac{1}{x}) = 2 \sum_{m=1}^{\ell} C_{\ell-m}\, T_m(z) + C_{\ell} ,$$

using the identity $2T_p(z) = x^p + x^{-p}$, for $p = 0,1,2,\ldots$

Lemma 4.6 then follows by comparing (4.1.15) and (4.1.18).

Lemma 4.7.

For $k,\ell = 1,2,3,\ldots,$

(4.1.19) $$2^{\ell}\Omega_{\ell}(x)\ F_{k,\ell}(x) = \sum_{j=0}^{2\ell} B_j^{(\ell)}\ T_{|k+j-\ell|}(x) ,$$

where

$$(4.1.20)\qquad B_j^{(\ell)} = \begin{cases} \dfrac{(-1)^j}{P_{\ell,\ell}} \displaystyle\sum_{m=0}^{j} P_{j-m,\ell}\, P_{m,\ell}\ , \text{ for } \quad j = 0(1)\ell\ . \\[2ex] \dfrac{(-1)^j}{P_{\ell,\ell}} \displaystyle\sum_{m=0}^{2\ell-j} P_{\ell-m,\ell}\, P_{m+j-\ell,\ell}\ , \text{ for } j = (\ell+1)(1)(2\ell). \end{cases}$$

Proof. By applying the recurrence relation (4.1.1) ℓ times, we find that $2^{\ell}\Omega_{\ell}(x)\, F_{k,\ell}(x)$ may be expressed as a sum of Chebyshev polynomials $T_{|k+j-\ell|}(x)$, where $j = 0(1)2\ell$. Furthermore, the coefficients of $T_{|k+j-\ell|}(x)$ only depend on j and ℓ, and we shall denote them by $B_j^{(\ell)}$. From (4.1.1), we can deduce that $B_j^{(\ell)}$ satisfy the following recurrence relation

$$(4.1.21)\qquad B_j^{(\ell)} = t_{\ell}^{-\varepsilon_{\ell}}\, B_{j-2}^{(\ell-1)} - 2B_{j-1}^{(\ell-1)} + t_{\ell}^{\varepsilon_{\ell}} B_j^{(\ell-1)},$$

$$j = 2(1)(2\ell-2),$$

with conditions

$$(4.1.22)\qquad \begin{cases} B_0^{(\ell)} = \dfrac{1}{P_{\ell,\ell}}\ , \qquad B_1^{(\ell)} = \dfrac{-2P_{1,\ell}}{P_{\ell,\ell}} \\[2ex] B_{2\ell-1}^{(\ell)} = -2P_{\ell-1,\ell}\ , \quad B_{2\ell}^{(\ell)} = P_{\ell,\ell}\ . \end{cases}$$

First, we observe that these conditions are satisfied by the expressions (4.1.20) for $B_j^{(\ell)}$. For $j = 2(1)\ell$, we substitute (4.1.20) in the left hand side of (4.1.21) and obtain

$$B_j^{(\ell)} = \frac{(-1)^j t_\ell^{\varepsilon_\ell}}{P_{\ell-1,\ell-1}} \{ \sum_{m=0}^{j-2} t_\ell^{-2\varepsilon_\ell} P_{j-m,\ell-1} P_{m,\ell-1} +$$

$$\sum_{m=0}^{j-1} 2t_\ell^{-\varepsilon_\ell} P_{j-1-m,\ell-1} P_{m,\ell-1} + \sum_{m=0}^{j} P_{j-m,\ell-1} P_{m,\ell-1} \}$$

On using (4.1.5), this equation becomes

$$B_j^{(\ell)} = \frac{(-1)^j}{P_{\ell,\ell}} \sum_{m=0}^{j} P_{j-m,\ell} P_{m,\ell} .$$

Thus, for $j = 0(1)\ell$, the expressions (4.1.20) for $B_j^{(\ell)}$ satisfy the linear recurrence relation (4.1.21) and the conditions (4.1.22). Similar results may be obtained for $j = (\ell+1)(1)(2\ell)$, and the lemma is proved.

We have found the Chebyshev coefficients for two polynomials $\Omega_\ell(x)$ and $2^\ell\Omega_\ell(x)\ F_{k,\ell}(x)$, we shall now derive a similar result for the polynomial part of the rational function $F_{k,\ell}(x)$. By continued application of (4.1.2), we note that $F_{k,\ell}(x)$ is a polynomial of degree k plus terms involving products of factors like $t_\ell^{-\varepsilon_\ell k}/[\frac{1}{2}(t_\ell + \frac{1}{t_\ell}) - x]$. Let us denote the polynomial part of $F_{k,\ell}(x)$ by $G_{k,\ell}(x)$ and the remainder by $M_{k,\ell}(x)$. Then,

(4.1.23) $$F_{k,\ell}(x) = G_{k,\ell}(x) + M_{k,\ell}(x) .$$

Lemma 4.8.

For $k = 0,1,2,\ldots$ and $\ell = 1,2,3,\ldots$, $G_{k,\ell}$ satisfies the recurrence relation

$$(4.1.24)\quad G_{k,\ell} = \varepsilon_\ell\left(t_\ell - \frac{1}{t_\ell}\right) \sum_{j=0}^{k-1}{}' \; t_\ell^{-\varepsilon_\ell(k-j)} G_{j,\ell-1} - t_\ell^{-\varepsilon_\ell} G_{k,\ell-1},$$

with $G_{k,0}(x) = T_k(x)$ and $G_{k,-1}(x) = 0$.

Proof. follows directly from lemma 4.1.

We now let $A_j(G_{k,\ell})$ denote the coefficient of $T_j(x)$ in the Chebyshev series expansion of $G_{k,\ell}(x)$ and obtain the following property.

Lemma 4.9.

For $k,\ell = 0,1,2,\ldots$ and $j = 0(1)k$,

$$(4.1.25)\qquad A_j(G_{k,\ell}) = A_{j+1}(G_{k+1,\ell}) .$$

Proof. We note first that for all values of ℓ, $G_{k,\ell}(x)$ is a polynomial of degree k. In (4.1.24), the coefficients of $G_{j,\ell-1}$ are independent of x. And since for a given ℓ, these coefficients depend only upon the value of $(k-j)$, the result follows.

We shall now find an explicit expression for $A_j(G_{k,\ell})$ in terms of the quantities $P_{j,\ell}$ and $Q_{j,\ell}$.

Lemma 4.10.

For $j = 0(1)k$, and $k,\ell = 0,1,2,\ldots,$

$$(4.1.26) \qquad A_j(G_{k,\ell}) = (-1)^{k-j+\ell} \sum_{s=0}^{k-j} (-1)^s Q_{s,\ell} P_{\ell+s-k+j,\ell} .$$

Proof. From the result of lemma 4.9, we observe that (4.1.26) is true if we can prove that

$$(4.1.27) \qquad A_1(G_{k,\ell}) = (-1)^{k+\ell-1} \sum_{s=0}^{k-1} (-1)^s Q_{s,\ell} P_{\ell+s-k+1,\ell} .$$

For convenience, let us now denote $A_1(G_{k,\ell})$ by $g_{k,\ell}$. Then, from lemma 4.8, $g_{k,\ell}$ must satisfy the recurrence relation

$$(4.1.28) \qquad g_{k,\ell} = \varepsilon_\ell\left(t_\ell - \frac{1}{t_\ell}\right) \sum_{j=1}^{k-1} t_\ell^{-\varepsilon_\ell(k-j)} g_{j,\ell-1} - t_\ell^{-\varepsilon_\ell} g_{k,\ell-1} ,$$

for $\ell = 1,2,3,\ldots,$ together with the conditions

$$(4.1.29) \qquad \begin{cases} g_{1,\ell} = (-1)^{\ell} P_{\ell,\ell} \ , & \text{for } \ell = 1,2,3,\ldots \\ g_{k,0} = \delta_{k,1} \ , & \text{for } k = 0,1,2,\ldots \end{cases}$$

First we note that these boundary conditions are satisfied by the expression (4.1.27) of $g_{k,\ell}$. Next, let us denote the right hand side of (4.1.28) by $h_{k,\ell}$. On using (4.1.27), we have

$$h_{k,\ell} = \varepsilon_\ell (t_\ell - \frac{1}{t_\ell}) \sum_{j=1}^{k-1} \sum_{s=0}^{j-1} (-1)^{j+\ell+s} t_\ell^{-\varepsilon_\ell (k-j)} Q_{s,\ell-1} P_{\ell+s-j,\ell-1} - t_\ell^{-\varepsilon_\ell} g_{k,\ell-1} \ ,$$

$$= \sum_{s=0}^{k-2} Q_{s,\ell-1} (-1)^{\ell+s} \left\{ \sum_{j=s+1}^{k-1} (-1)^j t_\ell^{-\varepsilon_\ell (k-j)} \varepsilon_\ell (t_\ell - \frac{1}{t_\ell}) P_{\ell+s-j,\ell-1} \right\} - t_\ell^{-\varepsilon_\ell} g_{k,\ell-1} \ ,$$

On inverting the order of summation in the double sum. With judicious application of lemma 4.2, the term inside the brackets can be written as

$$\sum_{j=s+1}^{k} (-1)^{j-1} t_\ell^{-\varepsilon_\ell (k-j)} P_{\ell+s-j+1,\ell} - (-1)^{k+1} t_\ell^{-\varepsilon_\ell} P_{\ell+s-k,\ell-1} \ .$$

Thus,

$$h_{k,\ell} = \sum_{s=0}^{k-2} Q_{s,\ell-1} (-1)^{\ell+s} \Big(\sum_{j=s+1}^{k} (-1)^{j-1} t_\ell^{-\varepsilon_\ell(k-j)} P_{\ell+s-j+1,\ell}$$

$$- (-1)^{k+1} t_\ell^{-\varepsilon_\ell} P_{\ell+s-k,\ell-1} \Big) - t_\ell^{-\varepsilon_\ell} \sum_{s=0}^{k-1} (-1)^s Q_{s,\ell-1} P_{\ell+s-k,\ell-1}$$

Applying lemma 4.4 to the last term, we find after a little algebra that

$$h_{k,\ell} = \sum_{s=0}^{k-1} Q_{s,\ell-1} (-1)^{\ell+s} \sum_{j=s+1}^{k} (-1)^{j-1} t_\ell^{-\varepsilon_\ell(k-j)} P_{\ell+s+1-j,\ell} .$$

If in the second sum, we write $s+1-j = -u$ and invert the orders of summation, we obtain

$$h_{k,\ell} = (-1)^\ell \sum_{u=0}^{k-1} (-1)^u P_{\ell-u,\ell} \sum_{s=0}^{k-(1+u)} t_\ell^{-\varepsilon_\ell(\ell-1-u-s)} Q_{s,\ell-1} .$$

On using lemma 4.5, we find that $h_{k,\ell} = g_{k,\ell}$ as given by (4.1.27). Thus the expression for $g_{k,\ell}$ as given by equation (4.1.27) satisfies the linear recurrence relation (4.1.28) and the boundary condition (4.1.29) and the lemma is proved.

We have obtained an explicit expression for the Chebyshev coefficients of the polynomial part of $F_{k,\ell}(x)$, we shall now find an expression for the remainder $M_{k,\ell}(x)$.

Lemma 4.11.

For $k,\ell = 0,1,2,\ldots,$

$$(4.1.30) \qquad M_{k,\ell}(x) = \sum_{s=1}^{\ell} H_{s,\ell}(x)\, t_s^{-\varepsilon_s k},$$

where $H_{s,\ell}(x)$ is independent of k.

Proof. From (4.1.2) and (4.1.23), it follows that $M_{k,\ell}(x)$ satisfies the recurrence relation

$$(4.1.31) \quad M_{k,\ell}(x) = (t_\ell - \frac{1}{t_\ell}) \left\{ \sum_{j=0}^{k-1}{}' \, t_\ell^{-\varepsilon_\ell(k-j)} M_{j,\ell-1}(x) - \left(\frac{t_\ell^{-2\varepsilon_\ell}}{1-t_\ell^{-2\varepsilon_\ell}}\right) M_{k,\ell-1}(x) + \phi(t_1,t_2,\ldots,t_\ell;x)\, t_\ell^{-\varepsilon_\ell k} \right\},$$

with $M_{k,0}(x) = 0$, for all k. The function ϕ is given by

$$\phi(t_1,t_2,\ldots,t_\ell;x) = \frac{1}{2} \left\{ F_{0,\ell-1}(x) + \frac{F_{-1,\ell-1}(x) - t_\ell^{-\varepsilon_\ell} F_{0,\ell-1}(x)}{\frac{1}{2}(t_\ell + \frac{1}{t_\ell}) - x} \right\}.$$

The proof of the lemma follows by induction on ℓ. We shall assume that $t_1,t_2,\ldots,t_\ell$ are distinct, the modifications that have to be made when this is not the case are straightforward. First, from (4.1.31),

$$M_{k,1}(x) = \varepsilon_1(t_1 - \frac{1}{t_1})\,\phi(t_1;x)\,t_1^{-\varepsilon_1 k} ,$$

for all k, so that the lemma is true when $\ell = 1$. Next, let us suppose that the lemma is true for $\ell = 1(1)(L-1)$ and all k. That is, we shall assume that $H_{s,\ell}(x)$ is independent of k for $s,\ell = 1(1)(L-1)$. With $\ell = L-1$, on substituting (2.1.30) into (2.1.31), we obtain $M_{k,L}(x)$ in the required form provided we define

$$H_{s,L}(x) = \frac{t_s^{-\varepsilon_s} t_L^{-\varepsilon_{L-1}}}{t_L^{-\varepsilon_L} - t_s^{-\varepsilon_s}} H_{s,L-1}(x), \quad \text{for } s = 1(1)(L-1),$$

and

$$H_{L,L}(x) = \varepsilon_L(t_L - \frac{1}{t_L}) \left\{ \frac{1}{2} \sum_{s=1}^{L-1} \frac{t_L^{-\varepsilon_L} + t_s^{-\varepsilon_s}}{t_L^{-\varepsilon_L} - t_s^{-\varepsilon_s}} H_{s,L-1}(x) \right.$$

$$\left. + \phi(t_1,t_2,\ldots,t_L;x) \right\} .$$

Thus, $H_{s,L}(x)$ is independent of k for $s = 1(1)L$, and the lemma follows by induction principle.

We shall now consider a generalisation of lemmas 4.10 and 4.11. By applying the recurrence relation (4.1.2) u times, where $0 \leq u \leq \ell$, we find that $F_{k,\ell}(x)$ can be written as the sum of two parts. The first part $G_{k,\ell,u}(x)$ is the sum of terms involving $F_{j,\ell-u}(x)$, $j = 0(1)k$, whose coefficients will be denoted by $C_{j,u}^{(k,\ell)}$. The second part which we shall call the remainder $M_{k,\ell,u}(x)$ consists of products of factors like $t_\ell^{-\varepsilon_\ell k}/[\frac{1}{2}(t_\ell + \frac{1}{t_\ell}) - x]$. Thus,

$$F_{k,\ell}(x) = G_{k,\ell,u}(x) + M_{k,\ell,u}(x) , \tag{4.1.32}$$

where

$$G_{k,\ell,u}(x) = \sum_{j=0}^{k}{}' C_{j,u}^{(k,\ell)} F_{j,\ell-u}(x) . \tag{4.1.33}$$

Lemma 4.12.

For $k,\ell = 0,1,2,\ldots$, $u = 0(1)\ell$ and $j = 0(1)k$,

$$C_{j,u}^{(k,\ell)} = (-1)^{k-j+u} \sum_{s=0}^{k-j} (-1)^s Q_{s,u} P_{u+s-k+j,u} , \tag{4.1.34}$$

where $P_{s,u}$ and $Q_{s,u}$ are symmetric functions of $t_s^{-\varepsilon_s}$, $s = (\ell-u+1)(1)(\ell)$, as defined in definitions 4.1 and 4.2.

Proof. We note that when $u = \ell$, $C_{j,\ell}^{(k,\ell)} = A_j(G_{k,\ell})$, for $j = 0(1)k$, and $G_{k,\ell,\ell}(x) = G_{k,\ell}(x)$. The proof follows in exactly the same way as that of lemma 4.10 if we replace Chebyshev polynomials $T_j(x)$, $j = 0(1)k$, by functions $F_{j,\ell-u}(x)$, and the symmetric functions $P_{s,\ell}$, $Q_{s,\ell}$ by $P_{s,u}$, $Q_{s,u}$ respectively.

In analogy with lemma 4.11, we have the following result.

Lemma 4.13

For $k,\ell = 0,1,2,\ldots$ and $u = 0(1)\ell$,

$$(4.1.35) \quad M_{k,\ell,u}(x) = \sum_{s=\ell-u+1}^{\ell} H_{s,\ell,u}(x)\, t_s^{-\varepsilon_s k} ,$$

where $H_{s,\ell,u}(x)$, for $s = (\ell-u+1)(1)\ell$, are independent of k.

Proof. With appropriate modifications, the proof may be obtained in a way similar to that of lemma 4.11 if we define

$$H_{s,L,u}(x) = \frac{t_s^{-\varepsilon_s} t_L^{-\varepsilon_L} - 1}{t_L^{-\varepsilon_L} - t_s^{-\varepsilon_s}} H_{s,L-1,u}(x) , \quad \text{for } s = (L-u+1)(1)(L-1),$$

and

$$H_{L,L,u}(x) = \varepsilon_L\left(t_L - \frac{1}{t_L}\right) \left\{ \frac{1}{2} \sum_{s=L-u}^{L-1} \frac{t_L^{-\varepsilon_L} + t_s^{-\varepsilon_s}}{t_L^{-\varepsilon_L} - t_s^{-\varepsilon_s}} H_{s,L-1,u}(x) + \phi(t_1, t_2, \ldots t_L; x) \right\} .$$

We can now state sufficient conditions under which $M_{k,\ell,u}(x)$ will be negligible for all $x \in [-1,1]$.

Lemma 4.14.

If $|t_s^{-\varepsilon_s}| < 1$, for $s = (\ell-u+1)(1)\ell$, then

(4.1.36) $$\lim_{k\to\infty} M_{k,\ell,u}(x) = 0, \quad \text{for all } x \in [-1,1] .$$

Proof. This follows from lemma 4.13, since the right hand side of (4.1.35) is a finite sum and the functions $H_{s,\ell,u}(x)$, for $s = (\ell-u+1)(1)\ell$, are independent of k.

We have obtained several important properties of the class of rational functions $F_{k,\ell}(x)$ and some related functions. In the following section, we shall apply these results to show how an asymptotic estimate $E^*_{n,m}(f)$ of the quantity $E_{n,m}(f)$ may be readily obtained from the Chebyshev coefficients of f(x).

4.2. An expression for $E^*_{n,m}(f_{n+r+1})$.

In this and the next section, we shall assume that f(x) is a polynomial of degree (n+r+1), where n,r are two non-negative integers. We shall denote it by $f_{n+r+1}(x)$. If R(n,m;x) denotes the rational function of best uniform approximation to $f_{n+r+1}(x)$ out of $V_{n,m}$, where m = 0(1)n, then we have

$$(4.2.1) \qquad f_{n+r+1}(x) - R(n,m;x) = E_{n,m}\, \Phi(x) .$$

In this equation (and as we shall do subsequently) we have written $E_{n,m}$ for $E_{n,m}(f_{n+r+1})$. Assuming that $f_{n+r+1}(x)$ is normal for (n,m), (see Rivlin [31], p.126), the error function $\Phi(x)$ has the property that it take its extreme values of ± 1 alternately in at least (n+m+2) points of [-1,1]. For an arbitrary polynomial $f_{n+r+1}(x)$, it does not appear possible to find analytic expressions for $E_{n,m}$ and $\phi(x)$. Instead, following Bernstein [2] and Clenshaw [6], we shall first of all consider a representation of $f_{n+r+1}(x)$ in terms of rational function $F_{k,\ell}(x)$, with suitable values of k and ℓ.

Let $F_{n+r+1,m+r}(x)$ denote the rational function $F_{n+r+1}(A,T_{m+r},\Sigma_{m+r};x)$, where A is a finite set of (m+r) integers taking the value of +1, so that it possesses all properties given in the previous section. Furthermore, we choose the set Σ_{m+r} such that $\varepsilon_s = +1$, for s = 1(1)m and $\varepsilon_s = -1$, for s = (m+1)(1)(m+r).

Applying the recurrence relation (4.1.2) r times, we find that $F_{n+r+1,m+r}(x)$ can be written in the form

$$(4.2.2) \quad F_{n+r+1,m+r}(x) = G_{n+r+1,m+r,r}(x) + M_{n+r+1,m+r,r}(x) ,$$

where

$$(4.2.3) \quad G_{n+r+1,m+r,r}(x) = \sum_{j=0}^{n+r+1}{}' \; C_{j,r}^{(n+r+1,m+r)} F_{j,m}(x) ,$$

as in equation (4.1.32) and (4.1.33). From lemma 4.14, $F_{n+r+1,m+r}(x)$ is asymptotically equal to $G_{n+r+1,m+r,r}(x)$, for n sufficiently large, provided $|t_s| < 1$, for $s = (m+1)(1)(m+r)$. From (4.2.3) and lemma 3.4, we see that $G_{n+r+1,m+r,r}(x)$ may be decomposed into a quotient of two polynomials, both of degree m, and a polynomial of degree (n+r+1). We observe that this polynomial is the polynomial part $G_{n+r+1,m+r}(x)$ of $F_{n+r+1,m+r}(x)$ as defined in (4.1.23), and whose Chebyshev coefficients are given in lemma 4.10. Alternatively, we can write

$$G_{n+r+1,m+r,r} = \frac{\text{a polynomial of degree } n}{\text{a polynomial of degree } m} + (\text{a polynomial of degree } (n+r+1) \text{ whose lowest degree of } T_k(x) \text{ is } (n+1-m)),$$

as $0 \le m \le n$.

Given a polynomial $f_{n+r+1}(x)$ of degree $(n+r+1)$, let us write it as

$$(4.2.4)\quad f_{n+r+1}(x) = \frac{p^*_{n;r}(x)}{q^*_{m;r}(x)} \pm E^*_{n,m;r}\, G_{n+r+1,m+r,r}(x) ,$$

where $E^*_{n,m;r}$ is chosen to be a positive constant, $p^*_{n;r}(x)$ and $q^*_{m;r}(x)$ are two polynomials of degree $\le n$ and $\le m$ respectively. It is easily seen that it is plausible to write $f_{n+r+1}(x)$ in this way if we choose $q^*_{m;r}(x)$ to be $\Omega_m(x)$ (see definition 4.3). By multiplying both sides of (4.2.4) by $\Omega_m(x)$ and comparing the coefficients $T_j(x)$, $j = 0(1)(m+n+r+1)$, we obtain a set of $(n+m+r+2)$ equations with $(n+m+r+2)$ unknowns. These unknowns are the $(n+1)$ coefficients of $p^*_{n;r}(x)$, the constant $E^*_{n,m;r}$ and the constants t_s, $s = 1(1)(m+r)$, which determine $G_{n+r+1,m+r,r}(x)$ and $\Omega_m(x)$.

For n sufficiently large and if $|t_s| < 1$, for $s = (m+1)(1)(m+r)$, $M_{m+r+1,m+r,r}(x)$ will be negligible for all $x \in [-1,1]$ by lemma 4.14 and the behaviour of $G_{n+r+1,m+r,r}(x)$ will closely resemble that of the rational function $F_{n+r+1,m+r}(x)$. In particular, from lemma 3.3, $G_{n+r+1,m+r,r}(x)$ will oscillate between its extreme values of ± 1, taking these values alternately in at least $(m+n+2)$ points of $[-1,1]$. This is precisely the behaviour of the error function $\Phi(x)$ in equation (4.2.1), and we may compare

$E^*_{n,m;r}$ with $E_{n,m}$, and the rational function $p^*_{n;r}(x)/q^*_{m;r}(x)$ with $R(n,m;x)$.

To sum up, for a given polynomial $f_{n+r+1}(x)$, if we can find the constants t_s, $s = 1(1)(m+r)$, such that $|t_s| < 1$, for $s = (m+1)(1)(m+r)$ and (4.2.4) is valid, then for large enough n, $E^*_{n,m;r}$ will be asymptotic to $E_{n,m}$ and $p^*_{n;r}(x)/q^*_{n;r}(x)$ will be asymptotic to $R(n,m;x)$. We are now in a position to obtain an expression for the quantity $E^*_{n,m;r}$, using the equation (4.2.4). Before doing so, it is convenient to define a matrix and a column vector.

Definition 4.4.

For non-negative integers n,r, and $m = 0(1)n$, let $A_{n,m+r} = (a_{i,j})$ be the symmetric matrix of order $(m+r+1)$ defined by

$$(4.2.5) \qquad a_{i,j} = \begin{cases} a_{n+i+j-m-1}, & i+j \le m+r+2, \\ 0 \qquad , & i+j > m+r+2, \end{cases}$$

for $i,j = 1(1)(m+r+1)$, where a_k, $k = 0(1)(n+r+1)$, is the coefficient of $T_k(x)$ in the Chebyshev series expansion of $f_{n+r+1}(x)$, (see definition 1.6).

Definition 4.5.

For non-negative integers m,r, let $\underline{P}_{m+r}$ be the column vector of dimension (m+r+1) defined by

$$(4.2.6)\qquad \underline{P_{m+r}}^T = (P_{0,m+r},-P_{1,m+r},\ldots,(-1)^{m+r}P_{m+r,m+r}),$$

where $P_{s,m+r}$, s = 0(1)(m+r) are the elementary symmetric functions of $t_j^{-\varepsilon_j}$, j = 1(1)(m+r), (see definition 4.1).

We shall now state and prove two theorems which provide the basis for the algorithm from which the quantity $E^*_{n,m;r}$ may be computed.

Theorem 4.1.

Suppose there exist $E^*_{n,m;r}$ and a set T_{m+r} such that equation (4.2.4) is valid. Then, either $+E^*_{n,m;r}$ or $-E^*_{n,m;r}$ is an eigenvalue of the matrix $A_{n,m+r}$ with $\underline{P}_{m+r}$ being the corresponding eigenvector.

Proof. By comparing coefficients of $T_j(x)$, for j = (n+1-m)(1)(n+r+1), on each side of equation (4.2.4), we have

$$(4.2.7) \qquad a_j = \pm E^*_{n,m;r} \, A_j(G_{n+r+1,m+r,r}).$$

We observe that, for $j = (n+1-m)(1)(n+r+1)$, $A_j(G_{n+r+1,m+r,r})$ are precisely the same as $A_j(G_{n+r+1,m+r})$ whose explicit expressions are given in lemma 4.10. Thus,

$$(4.2.8) \quad a_j = \pm E^*_{n,m;r} \sum_{s=m-n+1+j}^{m+r} (-1)^s \, P_{s,m+r} \, Q_{s-m+n+1-j,m+r} \, ,$$

for $j = (n+1-m)(1)(n+r+1)$. Or,

$$(4.2.9) \quad a_{n+k+1-m} = \pm E^*_{n,m;r} \sum_{s=k}^{m+r} (-1)^s \, P_{s,m+r} \, Q_{s-k,m+r},$$

for $k = 0(1)(m+r)$. Now, writing $k = \ell+j$, multiplying both sides by this equation by $(-1)^j \, P_{j,m+r}$, and summing over j from 0 to $(m+r-\ell)$, we find

$$\sum_{j=0}^{m+r-\ell} a_{n+\ell+1-m+j}(-1)^j \, P_{j,m+r} = \pm E^*_{n,m;r} \sum_{j=0}^{m+r-\ell} \sum_{s=\ell+j}^{m+r} \Big\{(-1)^s \, P_{s,m+r} \, Q_{s-\ell-j,m+r}(-1)^j \, P_{j,m+r}\Big\}$$

$$= \pm E^*_{n,m;r} \sum_{s=\ell}^{m+r} (-1)^s \, P_{s,m+r} \sum_{j=0}^{s-\ell} \Big\{(-1)^j \, P_{j,m+r} \, Q_{s-\ell-j,m+r}\Big\},$$

on interchanging the order of summation in the double sum.

From definition 4.2, we have

$$\sum_{j=0}^{s-\ell} (-1)^j P_{j,m+r} Q_{s-\ell-j,m+r} = \begin{cases} 0, & \text{for } s \neq \ell, \\ 1, & \text{for } s = \ell. \end{cases}$$

Hence,

$$(4.2.10) \qquad \sum_{j=0}^{m+r-\ell} a_{n+\ell+1-m+j}(-1)^j P_{j,m+r} = \pm E^*_{n,m;r}(-1)^\ell P_{\ell,m+r},$$

for $\ell = 0(1)(m+r)$. We can write this sytem of equations in matrix form, as

$$(4.2.11) \qquad A_{n,m+r} \underline{P_{m+r}} = \pm E^*_{n,m;r} \underline{P_{m+r}},$$

and the theorem is proved.

This theorem is proved under the assumption that the polynomial $f_{n+r+1}(x)$ can be written in the form given by equation (4.2.4). This will not always be true, and whether $f_{n+r+1}(x)$ can be written in this way, will not be known a priori. Thus, we would like to state sufficient conditions under which $f_{n+r+1}(x)$ can be written in the form of equation (4.2.4), together with the condition that $M_{n+r+1,m+r,r}(x)$ is negligible for all $x \in [-1,1]$.

Theorem 4.2.

Given a polynomial $f_{n+r+1}(x)$ with $a_{n+r+1} \neq 0$. Let λ be a non-zero eigenvalue of the matrix $A_{n,m+r}$ and $\underline{x} = (x_0, x_1, \ldots, x_{m+r})$ be the corresponding eigenvector. Suppose $\psi_{m+r}(z)$ denotes the polynomial $\sum_{s=0}^{m+r} (-1)^s x_s z^{m+r-s}$, and let $\{\beta_1, \beta_2, \ldots, \beta_{m+r}\}$ be the zeros of $\psi_{m+r}(z)$. If

(a) $x_0 \neq 0$, and

(b) $|\beta_s| > 1$, for $s = (0)(1)(m)$, and $|\beta_s| < 1$, for $s = (m+1)(1)(m+r)$;

then

(i) $\{\beta_1, \beta_2, \ldots, \beta_{m+r}\} = T_{m+r}$,

(ii) $|\lambda| = E^*_{n,m;r}$, $\underline{P_{m+r}} = \underline{x}/x_0$,

(iii) equation (4.2.4) is valid,

(iv) $\lim_{n\to\infty} M_{n+r+1,m+r,r}(x) = 0$, for all $x \in [-1,1]$.

Proof. Since $A_{n,m+r}$ is a real, symmetric matrix, λ and $\underline{x}$ are real. The polynomial $\psi_r(z)$ has real coefficients and therefore its zeros occur as complex conjugate pairs. From

$$(4.2.12) \qquad A_{n,m+r}\underline{x} = \lambda \underline{x} ,$$

we find in particular that $a_{n+r+1}\, x_0 = \lambda x_{m+r}$. Since none of λ, x_0 and a_{n+r+1} is zero, $x_{m+r} \neq 0$. Consequently, none of the zeros of $\psi_{m+r}(z)$ can be zero. Again, since $x_0 \neq 0$, $\psi_{m+r}(z)$ is a polynomial of degree $(m+r)$, so that it has $(m+r)$ zeros (with due regard to multiplicity). These results, together with (b), show that the zeros $\{\beta_1, \beta_2, \ldots, \beta_{m+r}\}$ form a set T_{m+r}, and (i) is proved.

Since $x_0 \neq 0$, we can normalize the vector $\underline{x}$ so that $x_0 = 1$. The polynomial $\psi_{m+r}(z)$ can be identified with $\Psi_{m+r}(z)$ (see equation (4.1.6)), and the normalized vector $\underline{x}$ with $\underline{P_{m+r}}$ (see definition 4.5). From (4.2.12), we have

$$(4.2.13) \qquad \sum_{j=0}^{m+r-\ell} a_{n+\ell+1-m+j}(-1)^j\, P_{j,m+r} = \lambda(-1)^\ell \, P_{\ell,m+r} ,$$

for $\ell = 0(1)(m+r)$. On reversing the steps in the proof of theorem 4.1, we find after some algebra that

$$(4.2.14) \qquad a_j = \lambda A_j\,(G_{n+r+1,m+r,r}) ,$$

for $j = (n+1-m)(1)(n+r+1)$. Comparing this with equation (4.2.7), we obtain $\lambda = \pm E^*_{n,r}$. Thus, (ii) and (iii) are proved.

Finally, (iv) follows from (b), on using lemma 4.14.

We have shown that either $+E^*_{n,m;r}$ or $-E^*_{n,m;r}$ is an eigenvalue of the matrix $A_{n,m;r}$, the sign being chosen so that $E^*_{n,m;r}$ will be positive. There remains to be determined, however, the criterion that must be used for choosing $E^*_{n,m;r}$ from among the $(m+r+1)$ eigenvalues of $A_{n,m+r}$ so that, in particular, condition (b) of theorem 4.2 is satisfied. We shall consider this problem in the next section.

4.3. Characterisation of $E^*_{n,m;r}$.

Throughout this section, we shall assume that n,m and r are fixed non-negative integers, where $0 \le m \le n$. For convenience, we shall write $N = m+r$. In the last section, we have insisted that $E^*_{n,m;r}$ is positive, hence there is no guarantee that $E^*_{n,m;r}$ is an eigenvalue of $A_{n,N}$. We shall therefore introduce another matrix C_{2N} which has the property that both $+E^*_{n,m;r}$ and $-E^*_{n,m;r}$ are its eigenvalues, one of these being an eigenvalue of $A_{n,N}$.

Definition 4.6.

For $k = 0(1)N$, the square matrix $B_k = (b_{i,j})$ of order $(k+1)$ is defined by

$$(4.3.1)\qquad b_{i,j} = \begin{cases} a_{n+r+1+(i-j)}, & i \le j, \\ 0, & i > j, \end{cases}$$

where $i,j = 1(1)(k+1)$.

Definition 4.7.

For $k = 0(1)N$, the symmetric matrix C_{2k} of order $(2k+2)$ is given by

$$(4.3.2)\qquad C_{2k} = \left(\begin{array}{c|c} 0 & B_k \\ \hline B_k^T & 0 \end{array}\right) .$$

We shall obtain some properties of C_{2k}.

Lemma 4.15.

All eigenvalues of C_{2k} are real, and if μ is an eigenvalue, so is $-\mu$. Furthermore, μ^2 is an eigenvalue of $B_k^T B_k$.

Proof. Since C_{2k} is symmetric, all its eigenvalues are real. If μ is an eigenvalue of C_{2k}, then we shall have

$$\det\left\{\left(\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} -\mu I & B_k \\ \hline B^T_k & -\mu I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right)\right\} = 0$$

On multiplying out the matrices, we have

$$\det \left(\begin{array}{c|c} \mu I & B_k \\ \hline B^T_k & \mu I \end{array}\right) = 0 ,$$

so that $-\mu$ is also an eigenvalue of C_{2k}. Finally, this last equation can be written as $\det(B_k{}^T B_k - \mu^2 I) = 0$, thus, μ^2 is an eigenvalue of the matrix $B_k{}^T B_k$.

Definition 4.8.

For $k = 0(1)N$, let the eigenvalues of C_{2N} be denoted by $\pm\mu_{j,k}$, for $j = 0(1)k$, and suppose they are ordered so that

$$\mu_{0,k} \geq \mu_{1,k} \geq \dots \geq \mu_{k,k} \geq 0 .$$

The importance of the matrix C_{2N} is given by the following lemma.

Lemma 4.16.

If λ is an eigenvalue of $A_{n,N}$, then both $+\lambda$ and $-\lambda$ are eigenvalues of C_{2N}.

Proof. Let $\underline{x}$ be an eigenvector of $A_{n,N}$ corresponding to the eigenvalue λ, where $\underline{x}^T = (x_0,x_1,\ldots,x_N)$. Let $\underline{y}$ denote a column vector defined by

$$\underline{y}^T = (x_0,x_1,\ldots,x_N,x_N,\ldots,x_1,x_0) .$$

It is easily seen that the equation

$$C_{2N}\,\underline{y} = \lambda\underline{y} \,, \tag{4.3.3}$$

is no more than the equation $A_{n,N}\,\underline{x} = \lambda\underline{x}$ written down twice. Since λ is an eigenvalue of C_{2N}, so is $-\lambda$ by lemma 4.15, and the result follows.

From theorem 4.2, $E^*_{n,m;r}$ is the modulus of one of the eigenvalues of $A_{n,N}$, it follows that $E^*_{n,m;r}$ must be chosen from among the positive eigenvalues $\mu_{j,N}$, $j = 0(1)N$, of the matrix C_{2N}. Our criterion for choosing $E^*_{n,m;r}$ is that r zeros of the polynomial constructed from the corresponding eigenvector, must lie within the unit circle. In order to apply this criterion, we shall make use of the Schur-Cohn theorem (see theorem 4.3).

Let $\mu_{j,N}$, $j = 0(1)N$, be any positive eigenvalue of C_{2N}, and let $\underline{y_{j,N}}$ be the corresponding eigenvector. Now, let us write

$$(4.3.4) \qquad \underline{y^T_{j,N}} = (x_0, x_1, \ldots, x_N, x_N, \ldots, x_0) ,$$

where for ease of notation, we have not used the suffices j,N on the elements of the vector $\underline{y_{j,N}}$.

Definition 4.9.

Let $\psi_N(z)$ denote a polynomial of degree $\leq N$ defined by

$$(4.3.5) \qquad \psi_N(z) = \sum_{s=0}^{N} (-1)^s x_s z^{N-s} .$$

Definition 4.10.

For $k = 0(1)N$, let $M_k = (m_{i,j})$ be the square matrix of order $(k+1)$, defined by

$$(4.3.6) \qquad m_{i,j} = \begin{cases} x_{N+i-j}, & i \le j, \\ 0, & i > j, \end{cases}$$

for $i,j = 1(1)(k+1)$.

Definition 4.11.

For $k = 0(1)N$, let $M^*_k = (m^*_{i,j})$ be the square matrix of order $(k+1)$, defined by

$$(4.3.7) \qquad m^*_{i,j} = \begin{cases} x_{j-i}, & i \le j, \\ 0, & i > j, \end{cases}$$

for $i,j = 1(1)(k+1)$.

Theorem 4.3. (Schur-Cohn).

For $k = 0(1)N$, let

$$(4.3.8) \qquad \Delta_k = \det \left(\begin{array}{c|c} M_k^T & M^*_k \\ \hline M^{*T}_k & M_k \end{array} \right) .$$

If $\Delta_k \neq 0$, for $k = 0(1)(N-1)$, then $\psi_N(z)$ has no zeros on $|z| = 1$. If there are p variations of sign in the sequence $1, \Delta_0, \Delta_1, \ldots, \Delta_{N-1}$, then $\psi_N(z)$ has p zeros inside the unit circle $|z| = 1$.

Proof. see Marden [20], chapter 10.

From the structure of the matrix C_{2N}, it can be readily seen that the equation

$$(4.3.9) \qquad C_{2N} \underline{y_{j,N}} = \mu_{j,N} \underline{y_{j,N}} ,$$

is equivalent to the system of equations

$$(4.3.10) \qquad M_k^{*T} B_k^T = \mu_{j,N} M_k^T ,$$

for $k = 0(1)N$.

In order to apply the Schur-Cohn theorem, we need to determine sgn Δ_k, for $k = 0(1)N$.

Lemma 4.17.

If $\mu_{j,N} \neq 0$ and $\det(M_k^*) \neq 0$ for $k = 0(1)N$, then

$$\text{(4.3.11)} \quad \operatorname{sgn} \Delta_k = (-1)^{k+1} \operatorname{sgn} \prod_{\ell=0}^{k} (\mu^2_{j,N} - \mu^2_{\ell,k}) ,$$

for $k = 0(1)N$.

Proof. From equation (4.3.8), since the matrices M_k^T and M_k^{*T} commute, we have

$$\Delta_k = \det(M_k^T M_k - M_k^{*T} M^*_k).$$

On using (4.3.10), we can write this as

$$\Delta_k = \det\left\{M_k^* \left[\frac{1}{\mu^2_{j,N}} (B_k^T B_k) - I \right] M_k^* \right\} ,$$

since $\mu_{j,N} \neq 0$. Since $\det(M_k^*) \neq 0$, we obtain

$$\operatorname{sgn} \Delta_k = \operatorname{sgn} \det(B_k^T B_k - \mu^2_{j,N} I) .$$

The result now follows immediately since the eigenvalues of $B_k^T B_k$ are given by $\mu^2_{\ell,k}$, for $\ell = 0(1)k$. (see lemma 4.15).

Thus, we have shown that, for $k = 0(1)N$, the signs of Δ_k vary with the choice of $\mu_{j,N}$ from among (N+1) positive eigenvalues of C_{2N}. We need the following result.

Lemma 4.18.

For $k = 0(1)N$ and $j = 0(1)k$,

$$\mu^2_{j+1,k} \le \mu^2_{j,k-1} \le \mu^2_{j,k} \quad . \tag{4.3.12}$$

Proof. This is merely a statement of the separation theorem for the eigenvalues of a real symmetric matrix and those of its leading principal minor as applied to the sequence of matrices $B_k^T B_k$ and $B^T_{k-1} B_{k-1}$. For a proof, see Wilkinson [38], pp.103-104.

We are now in a position to state sufficient conditions to enable us to obtain $E^*_{n,m;r}$ from the moduli of the eigenvalues of $A_{n,N}$.

Theorem 4.4.

Given $A_{n,N}$ with $a_{n+r+1} \neq 0$. Let λ be an eigenvalue of $(m+1)^{st}$ largest modulus of $A_{n,N}$. Suppose λ satisfies the following conditions

(a) $x_0 \neq 0$,

(b) neither $+\lambda$ nor $-\lambda$ is an eigenvalue of $A_{n+1,N-1}$,

(c) $|\lambda|$ is greater than the modulus of the eigenvalue of largest modulus of $A_{n+m+1,N-m-1}$.

Then,

(i) λ is the only eigenvalue of $(m+1)^{st}$ largest modulus of $A_{n,N}$,

(ii) $E^{*}_{n,m;r} = |\lambda|$.

Proof. We note that in the special case when $m = 0$, conditions (b) and (c) are equivalent.

(i) Since λ is an eigenvalue of $(m+1)^{st}$ largest modulus of $A_{n,N}$, $\mu_{m,N} = |\lambda|$ is a positive eigenvalue of C_{2N}. From (b), $\mu_{m,N}$ cannot be an eigenvalue of $C_{2(N-1)}$, so that we must have

$$\mu_{m,N-1} < \mu_{m,N} < \mu_{m-1,N-1} \quad , \tag{4.3.13}$$

by lemma 4.18. Again, from lemma 4.18, we have $\mu_{m-1,N-1} \leq \mu_{m-1,N}$ and $\mu_{m+1,N} \leq \mu_{m,N-1}$. Thus, $\mu_{m+1,N} < \mu_{m,N} < \mu_{m-1,N}$ and $\mu_{m,N}$ is a simple eigenvalue of C_{2N}. This implies that λ is the only eigenvalue of $(m+1)^{st}$ largest modulus of $A_{n,N}$.

(ii) The result follows from theorem 4.2 if conditions (b) of that theorem is satisfied. We want the polynomial $\psi_N(z)$ to have r zeros inside the unit circle. From Schur-Cohn theorem, the sequence $\{1,\Delta_0,\ldots,\Delta_{N-1}\}$ must therefore have r variations in sign. From (c), we have

(4.3.14) $$\mu_{m,N} > \mu_{0,N-m-1} \quad .$$

Thus, $\operatorname{sgn} \prod_{\ell=0}^{k} (\mu^2_{m,N} - \mu^2_{\ell,k})$ is strictly positive for $k = 0(1)(N-m-1)$, so that the sequence $\{1,\Delta_0,\ldots,\Delta_{N-m-1}\}$ has $N-m=r$ variations in sign, by lemma 4.17. From (4.3.12), (4.3.13) and (4.3.14), we find

(4.3.15) $$\mu_{m-k+1,N-k} < \mu_{m,N} < \mu_{m-k,N-k} ,$$

for $k = 1(1)m$. Thus, by lemma 4.18,

(4.3.16) $$\operatorname{sgn} \Delta_{N-k} = (-1)^{N-k+1}(-1)^{m-k+1} = (-1)^{N+m} ,$$

for $k = 1(1)m$, which is independent of k so that there is no variation in sign in the sequence $\{\Delta_{N-m},\Delta_{N-m+1},\ldots,\Delta_{N-1}\}$. Furthermore, since $\operatorname{sgn} \Delta_{N-m-1} = \operatorname{sgn} \Delta_{N-m} = (-1)^{N+m}$, the sequence $\{1,\Delta_0,\Delta_1,\ldots,\Delta_{N-1}\}$ has r variations in sign, and the polynomial $\psi_N(z)$ has r zeros inside the unit circle. Finally, it is readily seen that under the conditions of this theorem, λ is the only eigenvalue of $A_{n,N}$ for which the corresponding polynomial $\psi_N(z)$ has r zeros inside the unit circle. For any eigenvalue $\mu_{j,N}$ of

C_{2N}, other than $\mu_{m,N}$, the sequence $\{1,\Delta_0,\Delta_1,\ldots,\Delta_{N-1}\}$ will have less or greater than r variations in signs. Hence, (ii) is proved.

We shall now state an alternative condition to condition (a) of theorems 4.2 and 4.3.

<u>Lemma 4.19</u>.

Given $A_{n,N}$ with $a_{n+r+1} \neq 0$. Let λ be a non-zero eigenvalue of $A_{n,N}$. If neither $+\lambda$ nor $-\lambda$ is an eigenvalue of $A_{n+2,N-2}$, then $x_0 \neq 0$.

<u>Proof</u>. If neither $+\lambda$ nor $-\lambda$ is an eigenvalue of $A_{n+2,N-2}$, then both $+\lambda$ and $-\lambda$ are not eigenvalues of $C_{2(N-2)}$. Since λ is an eigenvalue of $A_{n,N}$, both $+\lambda$ and $-\lambda$ are eigenvalues of C_{2N}. Hence,

$$(4.3.17) \qquad C_{2N}\,\underline{y} = \pm\lambda\underline{y},$$

where $\underline{y}^T = (x_0,x_1,\ldots,x_{N-1},x_N,x_N,x_{N-1},\ldots,x_1,x_0)$. Let us now suppose that $x_0 = 0$, then from (4.3.17), we have, in particular, (4.3.18) $$a_{n+r+1}\, x_0 \;=\; \pm\lambda x_N \,.$$

Since $a_{n+r+1} \neq 0$ and $\lambda \neq 0$, this implies that $x_N = 0$. The equation (4.3.17) is then reduced to

(4.3.19) $$C_{2(N-2)}\underline{Y} = \pm\lambda\, \underline{Y} \;,$$

where $\underline{Y}^T = (x_1,\ldots,x_{N-1},x_{N-1},\ldots,x_1)$. Hence $\pm\lambda$ are eigenvalues of $C_{2(N-2)}$. This is a contradiction. Thus $x_0 \neq 0$ and the lemma is proved.

To sum up, let $f_{n+r+1}(x)$ be a polynomial of degree $(n+r+1)$, where n,r are two non-negative integers. We have shown that, under certain conditions, an asymptotic estimate $E^*_{n,m;r}$ for the quantity $E_{n,m}(f)$, for $m = 0(1)n$, and n sufficiently large, can be obtained directly from the Chebyshev coefficients of $f_{n+r+1}(x)$. $E^*_{n,m;r}$ is given by the modulus of the eigenvalue of $(m+1)^{st}$ largest modulus of a symmetric matrix of order $(m+r+1)$ defined by (4.2.5). In the following section, we shall consider the generalisation of this result to certain functions that are continuous on the interval $[-1,1]$.

4.4. The limiting case as $r \to \infty$.

In the last two sections, we have assumed that the function $f(x)$ is a polynomial of degree $(n+r+1)$, we would like to extend these results to the case when $f(x)$ is a continuous function. Let us suppose that $f(x)$ is a continuous function such that we can write

$$(4.4.1) \qquad f(x) = \lim_{r\to\infty} \sum_{k=0}^{n+r+1}{}' a_k T_k(x) \quad ,$$

the sum converging to $f(x)$ for all $x \in [-1,1]$. Having chosen n,m, where $0 \le m \le n$, we shall firstly approximate to $f(x)$ by a polynomial of degree $(n+r+1)$ and use the above analysis to obtain the quantity $E^*_{n,m;r}$. An estimate of $E_{n,m}(f)$ is then found by considering the limit of the sequence $\{E^*_{n,m;r}\}$ as $r \to \infty$. The question immediately arises as to under what conditions does $\lim_{r\to\infty} E^*_{n,m;r}$ exists, and to partially answer this, we have the following theorem.

Theorem 4.5.

Suppose $f(x) = \lim_{r\to\infty} f_{n+r+1}(x)$, for all $x \in [-1,1]$. For given n and m, where $0 \le m \le n$, the sequence $\{E^*_{n,m;r}\}$ converges as $r \to \infty$, provided $|a_k| \le A/k^{\phi}$, where $\phi > 1$ and $A > 0$.

<u>Proof</u>. As we have seen in theorem 4.4, the quantity $E^*_{n,m;r}$ is the modulus of an eigenvalue of the matrix $A_{n,N}$. Consider the Gerschgorin circles of this matrix, we obtain

$$E^*_{n,m;r} \le \sum_{k=0}^{m+r} |a_{n+k-m+1}| \quad . \tag{4.4.2}$$

If $|a_k| \le A/k^{\phi}$, where $A > 0$ and $\phi > 1$, then this series converges in the limit as $r \to \infty$. (This proof is not complete) .

The condition $|a_k| \le A/k^{\phi}$, $\phi > 1$, of theorem 4.5 is not a very restrictive one in practice. Indeed, it is also a sufficient condition to guarantee that $\lim_{r\to\infty} f_{n+r+1}(x) = f(x)$, uniformly for all $x \in [-1,1]$. ~~This~~

In this chapter, we have given an algorithm for estimating the quantity $E_{n,m}(f)$, where $m = O(1)n$, and $f(x)$ is a certain continuous function on $[-1,1]$. We have neither, however, investigated the ratio $E_{n,m} / E^*_{n,m;r}$, for a given value of r, nor the asymptotic rational function of best uniform approximation $p^*_{n;r}(x)/q^*_{m;r}(x)$ of $f(x)$. These problems will be dealt with in the next two chapters. In chapter V, we shall consider the case of best uniform polynomial approximation. We shall obtain the asymptotic polynomial of best uniform approximation $p^*_{n;r}(x)$ and various bounds for the ratio $E_n/E^*_{n;r}$. In chapter VI, an expression for the asymptotic rational function of best uniform approximation $p^*_{n;r}(x)/q^*_{m;r}(x)$ will be given.

CHAPTER V

THE CASE OF BEST UNIFORM POLYNOMIAL APPROXIMATION

In section 5.1, we shall obtain the asymptotic polynomial of best uniform approximation $p^*_{n;r}(x)$. Various bounds for the ratio $E_n/E^*_{n;r}$, where $E^*_{n;r}$ is an asymptotic estimate of E_n, for large n will be given in section 5.2. Finally, in section 5.3, we shall deal with some numerical examples which will be compared with the results given by other authors.

5.1. The asymptotic polynomial of best uniform approximation $p^*_{n;r}(x)$.

As a special case of section 4.2, we let n,r be two given non-negative integers and m = 0. Suppose $f_{n+r+1}(x)$ is a polynomial of degree (n+r+1), from equation (4.2.4), we may write

$$(5.1.1) \quad f_{n+r+1}(x) = p^*_{n;r}(x) \pm E^*_{n;r}\, G_{n+r+1,r}(x) ,$$

where $E^*_{n;r}$ is a positive constant and $p^*_{n;r}$ is a polynomial of degree $\le n$. The function $G_{n+r+1,r}(x)$ is the polynomial part of the rational function $F_{n+r+1,r}(x)$, (see equation 4.1.23). We note here that $F_{n+r+1,r}(x)$ denotes the function $F_{n+r+1}(A,T_r,\Sigma_r;x)$, defined as in (3.1.3), where A is a finite set of r integers taking the value of +1 and Σ_r is such that $\varepsilon_s = -1$, for $s = 1(1)r$. From lemma 3.3, we find that $F_{n+r+1,r}(x)$ attains its extreme values ± 1 in at least (n+2) points of $[-1,1]$, and therefore satisfies the characterisation property of the error function of the best uniform polynomial approximation. Also, from lemma 4.14, for n sufficiently large and if $|t_s| < 1$, for $s = 1(1)r$, the behaviour of $G_{n+r+1,r}(x)$ will closely resemble that of $F_{n+r+1,r}(x)$. Thus, for a given polynomial $f_{n+r+1}(x)$, if we can find the constants t_s, $s = 1(1)r$, such that $|t_s| < 1$ and (5.1.1) is valid, then for n sufficiently large, $E^*_{n;r}$ will be asymptotic to E_n and $p^*_{n;r}(x)$ will be asymptotic to $p_n(x)$. We have the following result.

<u>Theorem **5.1**</u>

Given $A_{n,r}$ (see definition 4.4) with $a_{n+r+1} \ne 0$. Let λ be an eigenvalue of largest modulus of $A_{n,r}$. Suppose λ satisfies the following conditions

(a) the corresponding eigenvector of λ has a non-zero first component,

(b) neither $+\lambda$ nor $-\lambda$ is an eigenvalue of $A_{n+1,r-1}$.

Then,

(i) λ is the only eigenvalue of largest modulus of $A_{n,r}$,

(ii) $E^*_{n;r} = |\lambda|$.

Proof. This can be obtained as a special case of theorem 4.4, on putting m = 0, and observing that the conditions (b) and (c) of that theorem are equivalent in this case.

Suppose now that the polynomial $p^*_{n;r}(x)$ is written as

$$p^*_{n;r}(x) = \sum_{k=0}^{n} a^*_k \, T_k(x) \quad , \tag{5.1.2}$$

then we have the following result.

Theorem 5.2.

For k = 0(1)n,

$$a^*_k = a_k \mp E^*_{n;r} \sum_{s=0}^{r} (-1)^s \, P_{s,r} \, Q_{n+1-k+s,r} \, , \tag{5.1.3}$$

where $P_{s,r}$ and $Q_{s,r}$ are symmetric functions of the quantities t_j, $j = 1(1)r$, (see definitions 4.1 and 4.2).

Proof. We note that under the conditions of theorem 5.1, we can write $f_{n+r+1}(x)$ in the form (5.1.1), (see theorems 4.2 and 4.4). The result now follows by comparing the coefficients of $T_k(x)$, for $k = 0(1)n$, of both sides of equation (5.1.1), on using lemma 4.10.

Thus, we see that in order to evaluate the polynomial $p^*_{n;r}(x)$, we must know the eigenvector $\underline{P_r}$ of $A_{n,r}$ which corresponds to the eigenvalue λ. The vector $\underline{P_r}$ is of dimension $(r+1)$ and is defined by

(5.1.4) $$\underline{P_r}^T = (P_{0,r}, -P_{1,r}, \ldots, (-1)^r P_{r,r}),$$

(see definition 4.5). Once the quantities $(-1)^s P_{s,r}$ are known, we may compute $Q_{s,r}$, for $s = 1(1)(n+r+1)$, recursively from the relation

(5.1.5) $$Q_{s,r} = \sum_{j=1}^{s} (-1)^{j+1} P_{j,r} Q_{s-j,r} \quad ,$$

where $Q_{0,r} = 1$ and $P_{j,r} = 0$ for $j > r$, (see lemma 4.4). Finally, we note that the result (i) of theorem 5.1, together with the condition $P_{0,r} = 1$ enables the eigenvector $\underline{P_r}$ to be determined uniquely. This enables us to determine the polynomial $p^*_{n;r}(x)$.

5.2. Bounds for $E_n/E^*_{n;r}$.

As we have mentioned in the last section, for a fixed value of r, $E^*_{n;r}$ is given as the modulus of the eigenvalue of the largest modulus of the matrix $A_{n;r}$. We want to obtain an upper bound for the quantity $E_n/E^*_{n;r}$.

Lemma 5.1.

For non-negative integers n,r.

$$\frac{E_n}{E^*_{n,r}} \leq \max_{x\in[-1,1]} |G_{n+r+1,r}(x)| , \tag{5.2.1}$$

or

$$\frac{E_n}{E^*_{n,r}} \leq 1 + \max_{x\in[-1,1]} |M_{n+r+1,r}(x)| . \tag{5.2.2}$$

Proof. By definition 1.1, we have

$$E_n \leq \max_{x\in[-1,1]} |f_{n+r+1}(x) - p^*_{n;r}(x)| .$$

Thus,

$$(5.2.3) \qquad E_n \le \max_{x\in[-1,1]} |E^*_{n,r} G_{n+r+1,r}(x)| ,$$

on using equation (5.1.1). (5.2.1) immediately follows. The second inequality (5.2.2) is obtained from (5.2.3), on using equation (4.1.23) and the fact that $\max_{x\in[-1,1]} |F_{n+r+1,r}(x)| = 1$.

We note that the first upper bound is useful in practice. Once the quantities $(-1)^s P_{s,r}$, $s = 1(1)r$, are known, the Chebyshev coefficients of $G_{n+r+1,r}(x)$ may be computed, using lemma 4.10. However, for analytical purpose, as we are interested in the limit of $E_n/E^*_{n;r}$ as $r \to \infty$, (5.2.2) is to be preferred. We shall now obtain a lower bound for $E_n/E^*_{n;r}$.

Lemma 5.2.

For non-negative integers n,r,

$$(5.2.4) \qquad \frac{E_n}{E^*_{n;r}} \ge \frac{\pi}{4} \left| \sum_{s=0}^{r} (-1)^s P_{s,r} Q_{s,r} \right| .$$

Proof. From equation (5.1.1), we have

$$
(5.2.5) \qquad
\begin{aligned}
a_{n+1} &= \pm E^{*}_{n,r} \, A_{n+1} \, (G_{n+r+1,r}) \, , \\
&= \pm E^{*}_{n,r} \sum_{s=0}^{r} (-1)^{s} \, P_{s,r} \, Q_{s,r} \, ,
\end{aligned}
$$

on using lemma 4.10. Since

$$
(5.2.6) \qquad E_n \ \geq \ \frac{\pi}{4} \, |a_{n+1}| \, ,
$$

(see Rivlin [26]), (5.2.4) readily follows.

This result may be further improved in the case when all the Chebyshev coefficients a_k of $f_{n+r+1}(x)$ are non-negative.

Lemma 5.3.

For non-negative integers n, r, if a_k are non-negative, for $k = 0(1)(n+r+1)$, then

$$
(5.2.7) \qquad \frac{E_n}{E^{*}_{n,r}} \ \geq \ \Big| \sum_{s=0}^{r} (-1)^{s} \, P_{s,r} \, Q_{s,r} \Big| \, .
$$

Proof. When a_k are non-negative, we have

(5.2.8) $$E_n \geq a_{n+1} ,$$

(see Rivlin [29]). The result immediately follows from (5.2.5).

The lower bounds (5.2.4) and (5.2.7) although very useful for numerical purposes, do not have the advantage of the upper bound (5.2.2), since they do not provide us with any information about their behaviour when $r \to \infty$. Unfortunately we have not been able to obtain any bound better suited to that purpose.

We have seen in section 4.4 that if $f(x) = \lim_{r\to\infty} f_{n+r+1}(x)$, for all $x \in [-1,1]$, then the sequence $\{E^*_{n;r}\}$ converges as $r \to \infty$, provided $|a_k| \leq \frac{A}{k^\phi}$, where $\phi > 1$ and $A > 0$. We shall now obtain a result from the case where $0 < \phi < 1$.

Lemma 5.4.

If $a_k = A/k^\phi$, where $0 < \phi < 1$ and $A > 0$, then the sequence $\{E^*_{n;r}\}$ diverges in the limit as $r \to \infty$.

Proof. Since $E^*_{n;r}$ is the largest eigenvalue of the matrix C_{2r}, we have

(5.2.9) $$E^*_{n;r} \geq \frac{\underline{u}^T C_{2r} \underline{u}}{\underline{u}^T \underline{u}} ,$$

where $\underline{u}$ is any real column vector of dimension $(2r+2)$. From definition 4.7, we can write this as

$$E^*_{n;r} \geq \frac{\underline{v}^T(B_r + B_r^T)\underline{v}}{2\underline{v}^T\underline{v}} \quad ,$$

where $\underline{v}$ is now any column vector of dimension $(r+1)$. In particular, if we choose $\underline{v}$ so that each of its element is 1, then

$$E^*_{n;r} \geq \frac{1}{r+1} \sum_{k=1}^{r+1} ka_{n+k} \geq \frac{A(r+2)}{2(n+r+1)^{\phi}} \quad ,$$

as $\phi > 0$. Also, since $\phi < 1$, this lower bound diverges in the limit as $r \to \infty$.

In the following section, we shall obtain some numerical examples and compare them with the results given by Murnaghan and Wrench [23].

5.3 Some numerical examples.

We shall consider two functions $f(x) = \arctan x$ and $f(x) = \log\left(\frac{a+x}{a-x}\right)$, where $a = \frac{10^{\frac{1}{2}}+1}{10^{\frac{1}{2}}-1}$, $-1 \leq x \leq 1$, which have been discussed by Murnaghan and Wrench [23]. For a fixed value of n, we shall compute the quantity $E^*_{n;r}$ for various values of r to obtain E_n which is the limit of the sequence $\{E^*_{n;r}\}$ as $r \to \infty$. In practice, the convergence is quite rapid as we shall see below. Once the value E_n is obtained at a certain value of r, we may compute $Q_{s,r}$, $s = 1(1)(n+r+1)$, recursively from equation (5.1.5), where $(-1)^s P_{s,r}$, $s = 0(1)r$, are the components of the eigenvector corresponding to the eigenvalue $+E^*_{n;r}$ or $-E^*_{n,r}$. The coefficients of the asymptotic polynomial of best uniform approximation will then be given by (5.1.3).

(i) $f(x) = \arctan x$, where $-1 \le x \le 1$.

The Chebyshev coefficients of arctan x are given by

$$(5.3.1) \qquad a_{2k+1} = \frac{(-1)^k p^{2k+1}}{2k+1} ,$$

for $k = 0,1,2,\ldots$, where $p = 2^{\frac{1}{2}} - 1 = 0.414213562$, (see [23]). For $n = 6$, the rapid convergence of $E^*_{6;r}$ to E_6 is well illustrated in table 5.1.

In tables 5.2 and 5.3, we exhibit the values of the quantities $P_{s,8}$, for $s = 0(2)8$, and of the quantities $Q_{s,8}$, for $s = 0(2)14$. Since $f(x) = \arctan x$ is an odd function, all the values of $P_{s,8}$ and $Q_{s,8}$, where s is an odd integer, vanish.

We note that in all tables, $0.(k)d_{k+1}d_{k+2}\ldots$ denotes that there are k zeros immediately following the decimal point.

VALUES OF $E^*_{6;r}(f)$, $r = 4(4)16$, $f(x)$ = artan x.

r	$E^*_{6;r}$
4	0.(3)89925
8	0.(3)60859
12	0.(3)60859
16	0.(3)60859

TABLE 5.1

VALUES OF $P_{s,8}$, $s = 0(2)8$, FOR $f(x)$ = artan x.

s	$P_{s,8}$
0	1.0
2	-0.1335 7106
4	0.0187 6247
6	-0.(2)27 2406
8	0.(3)3 9717

TABLE 5.2

VALUES OF $Q_{s,8}$, s = 0(2)14, FOR f(x) = artan x.

s	$Q_{s,8}$
0	1.0
2	0.1335 7106
4	-0.(3)9 2124
6	0.(4) 9489
8	-0.(5) 336
10	-0.(4) 5779
12	-0.(5) 703
14	0.(6) 10

TABLE 5.3

In table 5.4, we give the polynomial $p^*_{6,8}(x)$ for artan x, and compare it with $p_6(x)$ as computed by Hastings [13], Murnaghan and Wrench [23].

POLYNOMIAL APPROXIMATION TO artan x, $-1 \le x \le 1$

	k	HASTING'S RESULT	M. & W.'S RESULT	$P^*_{6,8}(x)$
COEFFICIENTS OF x^{2k+1}	0	0.9953 54	0.9953 580	0.9953 5796
	1	-0.2886 79	-0.2886 902	-0.2886 9024
	2	0.0793 31	0.0793 390	0.0793 3904
E_6		0.(3)6 086	0.(3)6 086	0.(3)6 0859

TABLE 5.4

(ii) $f(x) = \log\left(\frac{a+x}{a-x}\right)$, where $a = \frac{10^{\frac{1}{2}}+1}{10^{\frac{1}{2}}-1}$, and $-1 \le x \le 1$.

The Chebyshev coefficients of f(x) are given by

$$a_{2k+1} = \frac{4Mp^{2k+1}}{2k+1}, \tag{5.3.2}$$

for $k = 0,1,2,\ldots$, where $M = \log e = 0.4342\ 9448$ and $p = a - (a^2-1)^{\frac{1}{2}} = 0.28013$, (see [23]).

The values of $E^*_{4;r}$, $r = 4(4)16$, are given in table 5.5. In table 5.6 and 5.7, we exhibit the values of $P_{s,8}$, for $s = 0(2)8$ and of $Q_{s,8}$, for $s = 0(2)12$. As in the previous example, $P_{s,8}$

and $Q_{s,8}$, where s is an odd integer, vanish. In table 5.8, we give the polynomial $p^*_{4;8}(x)$ for $\log\left(\frac{a+x}{a-x}\right)$, and compare it with $p_4(x)$ as computed by Murnaghan and Wrench [23].

VALUES OF $E^*_{4;r}(f)$, $r = 4(4)16$, $f(x) = \log \frac{a+x}{a-x}$.

r	$E^*_{4;r}$
4	0.(3)89732
8	0.(3)60123
12	0.(3)60123
16	0.(3)60123

TABLE 5.5

VALUES OF $P_{s,8}$, $s = 0(2)8$, FOR $f(x) = \log \frac{a+x}{a-x}$.

s	$P_{s,8}$
0	1.0
2	0.0560 6772
4	0.(2)34 2267
6	0.(3)2 1978
8	0.(4) 1454

TABLE 5.6

VALUES OF $Q_{s,8}$, $s = 0(2)12$, FOR $f(x) = \log \frac{a+x}{a-x}$.

s	$Q_{s,8}$
0	1.0
2	-0.0560 6772
4	-0(3)2 7908
6	-0.(4) 1223
8	-0.(6) 58
10	0.(6) 95
12	-0.(7) 4

TABLE 5.7

POLYNOMIAL APPROXIMATION TO $\log \frac{a+x}{a-x}$, $a = \frac{10^{\frac{1}{2}}+1}{10^{\frac{1}{2}}-1}$, $-1 \leq x \leq 1$.

	k	M. & W.'s RESULT	$p^*_{4,8}(x)$
COEFFICIENTS OF x^{2k+1}	0	0.4483470	0.4483470
	1	0.0510518	0.05105176
E_4		0.(3)6012	0.(3)60123

TABLE 5.8

We note here that our results are comparable to those given by Hastings, Murnaghan and Wrench. Furthermore, these results were obtained with much less effort since the algorithm involves only the computation of the eigenvalue of largest modulus of a symmetric matrix and its corresponding eigenvectors.

We have, in this chapter, considered in detail the evaluation of $E_n(f)$ and $p_n(x)$ for the best polynomial approximation. In the next chapter, we shall discuss the problem of finding the asymptotic rational function of best uniform approximation.

CHAPTER VI

THE ASYMPTOTIC RATIONAL FUNCTION OF BEST UNIFORM APPROXIMATION $p^*_{n;r}(x)/q^*_{m;r}(x)$.

In section 6.1, we shall discuss the special case when f(x) is a polynomial of degree (n+1). All the results for this case will be obtained exactly, and not asymptotically. In section 6.2, an expression for the rational function $p^*_{n;r}/q^*_{m;r}$ for the general case when f(x) is a polynomial of degree (n+r+1), r being a non-negative integer, will be given.

6.1. The case when f(x) is a polynomial of degree (n+1).

We shall denote f(x) by $f_{n+1}(x)$ throughout this section. We want to approximate to $f_{n+1}(x)$ by rational function from the set $V_{n,m}$, where $0 \le m \le n$. This problem has been previously discussed by other authors (see Achieser [1], p.278; Meinardus [21], p.166; Talbot [35]; see also section 2.2.1.1). An exact expression for $E_{n,m}(f_{n+1})$ has been given by all authors; Talbot has also given an expression for the corresponding best rational approximation $R(n,m;x)$. However, no author has determined explicitly the error function

$$(6.1.1) \qquad f_{n+1}(x) - R(n,m;x)$$

or discussed its properties in any detail. We propose to do so in this section.

We first observe that this problem is a special case of that analysed in section 4.2 where we have put $r = 0$. From equation (4.2.4), we may write

$$(6.1.2) \qquad f_{n+1}(x) = \frac{p^*_{n;o}(x)}{q^*_{m;0}(x)} \pm E^*_{n,m;0} F_{n+1,m}(x) ,$$

where $E^*_{n,m;0}$ is a positive constant, and $p^*_{n;0}(x)$ and $q^*_{m;o}(x)$ are two polynomials of degree $\leq n$ and $\leq m$ respectively. The right hand side of (6.1.2) follows from the fact that $G_{n+1,m,0}(x) = F_{n+1,m}(x)$ and $M_{n+1,m,0}(x) = 0$, (see equation (4.2.2)). Again, we note that $F_{n+1,m}(x)$ denotes the rational function $F_{n+1}(A,T_m,\Sigma_m;x)$, defined as in (3.1.3), where A is a finite set of m integers taking the value +1, and Σ_m is such that $\varepsilon_s = +1$, for $s = 1(1)m$. From lemma 3.4, $F_{n+1,m}(x)$ is a rational function of the form $\bar{r}(n+m+1,m;x)$. As we have mentioned in section 4.2, it is easily seen that it is plausible to write $f_{n+1}(x)$ in this way if we choose $q^*_{m;0}(x)$ to be $\Omega_m(x)$, (see definition 4.3).

From lemma 3.3, $F_{n+1,m}(x)$ attains its extreme values of ± 1 alternately in at least $(n+m+2)$ points of $[-1,1]$. This is precisely the required property which characterizes the best uniform rational approximation. Thus, equation (6.1.2) provides us exactly with the rational function of best uniform approximation $p_n(x)/q_m(x)$ and the maximum error $E_{n,m}(f_{n+1})$. We can rewrite (6.1.2) as

$$(6.1.3) \qquad f_{n+1}(x) = \frac{p_n(x)}{q_m(x)} \pm E_{n,m}\, F_{n+1,m}(x) \quad ,$$

and obtain the following result.

Theorem 6.1.

Given $A_{n,m}$ (see definition 4.4) with $a_{n+1} \neq 0$. Let λ be an eigenvalue of smallest modulus of $A_{n,m}$. Suppose λ satisfies the following conditions

(a) the corresponding eigenvector of λ has a non-zero first component,

(b) neither $+\lambda$ nor $-\lambda$ is an eigenvalue of $A_{n+1,m-1}$.

Then,

(i) λ is the only eigenvalue of smallest modulus of $A_{n,m}$,

(ii) $E_{n,m} = |\lambda|$.

Proof. This can be obtained as a special case of theorem 4.4, on putting $r = 0$. The condition (c) of theorem 4.4 is no longer necessary in this case since the matrix $A_{n+m+1,-1}$ has no element.

We note that under the conditions of theorem 6.1, the eigenvector $\underline{P_m}$ corresponding to λ is given by

$$(6.1.4) \qquad \underline{P_m}^T = (P_{0,m}, -P_{1,m}, \ldots, (-1)^m P_{m,m}) ,$$

where $P_{0,m} \neq 0$ and $P_{m,m} \neq 0$. Furthermore, equation (6.1.3) is valid, (see theorems 4.2 and 4.4). We shall now find explicit expressions for $p_n(x)/q_m(x)$. First, let us write

$$(6.1.5) \qquad q_m(x) = \sum_{j=0}^{m}{}' \; b_{m-j} \, T_j(x) ,$$

then we have the following result.

Lemma 6.1.

For $j = 0(1)m$,

(6.1.6) $$b_j = \frac{(-1)^{j+m}}{2^{(m-1)}p_{m,m}} \left(\sum_{s=0}^{j} P_{s,m} P_{m-j+s,m} \right) .$$

Proof. This follows directly from lemma 4.6, since we have chosen $q_m(x)$ to be $\Omega_m(x)$.

Now let,

(6.1.7) $$f_{n+1}(x)\, q_m(x) = \sum_{j=0}^{n+m+1}{}' \; d_j \, T_j(x) \quad ,$$

then we have the following lemma.

Lemma 6.2.

(6.1.8) $$d_0 = \frac{1}{2} \sum_{i=0}^{m}{}' \; a_i \, b_{m-i} \quad ,$$

and

(6.1.9) $$d_j = \frac{1}{2} \sum_{i=0}^{m}{}' \; b_{m-i} \left(a_{j+i} + a_{|j-i|} \right) \quad ,$$

for $j = 1(1)(n+m+1)$, where $a_k = 0$ for $k > (n+1)$.

Proof. This can be readily obtained from (6.1.7), on using a property of Chebyshev polynomials of the first kind,

$$2T_k(x)\,T_j(x) = T_{k+j}(x) + T_{|k-j|}(x) \quad , \tag{6.1.10}$$

for $k,j = 0,1,2,\ldots$.

Finally, let us write $p_n(x)$ as

$$p_n(x) = \sum_{j=0}^{n}{}' \; e_j T_j(x) \quad , \tag{6.1.11}$$

and obtain explicit expressions for **its** coefficients.

Lemma 6.3.

$$e_j = d_j, \qquad \text{for } j = 0(1)(n-m) \ , \tag{6.1.12}$$

and

$$e_j = d_j \mp E_{n,m}\, B^{(m)}_{j+m-n-1}, \quad \text{for } j = (n+1-m)(1)(n), \tag{6.1.13}$$

where

$$B_k^{(m)} = \frac{(-1)^k}{2^m P_{m,m}} \sum_{s=0}^{k} P_{k-s,m}\, P_{s,m}\,, \quad k = 0(1)m. \tag{6.1.14}$$

Proof. By multiplying both sides of equation (6.1.2) by $q_m(x)$ and comparing the Chebyshev coefficients, the result is obtained on using (6.1.7), (6.1.11) and lemma 4.7.

Thus, we have obtained an exact expression for the best uniform rational approximation $R(n,m;x)$ of $f_{n+1}(x)$. It is worth mentioning here that Meinardus, (see [21], p.166), posed the question "whether the function $f_{n+1}(x) - R(n,m;x)$ plays a role in the theory of rational approximation similar to the role of Tchebycheff polynomials in the theory of polynomial approximation". As we have seen in this section, the function of interest is no more than the rational function $F_{n+1,m}(x)$, which is a generalisation of a Chebyshev polynomial of the first kind. When $m = 0$, $F_{n+1,0}(x) = T_{n+1}(x)$, and we recover a result for polynomial approximation, (see section 2.1.1.1). Furthermore, a more general form of $F_{n+1,m}(x)$, which we have dealt with in chapter III. has led us to several explicit results in the best uniform rational approximation. From these results, we recovered some particular results for best uniform polynomial and rational approximations previously given by Bernstein [2], Boehm [3] and Rivlin [30], where

Chebyshev polynomials of the first kind were used. Thus, the answer to Meinardus' question is no doubt in the affirmative. However, we do not know how important is the role of rational functions $F_{k,\ell}(x)$ in the theory of uniform rational approximation. This problem still remains to be investigated.

We have obtained the exact results for the special case where f(x) is a polynomial of degree (n+1). In the following section, we shall consider the asymptotic results for the general case where f(x) is a polynomial of degree (n+r+1), r being a positive integer.

6.2. An expression for $p^*_{n;r}(x)/q^*_{m;r}(x)$.

As we have discussed in chapter IV, given a continuous function f(x), we first approximate to it by a polynomial of degree (n+r+1). We then compute the quantity $E^*_{n,m;r}$, for $0 < m \leq n$, which is given as the modulus of the eigenvalue of $(m+1)^{st}$ largest modulus of the symmetric matrix $A_{n,m+r}$ (see definition 4.4 and theorem 4.4). An estimate of $E_{n,m}(f)$ is obtained as the limit of $E^*_{n,m;r}$ as $r \to \infty$. The rapid convergence of $E^*_{n,m;r}$ to $E_{n,m}$ for some functions is well illustrated in tables 6.1 and 6.2, where we have computed $E^*_{n,n;r}$ for two functions $f(x) = \log x$, $\frac{1}{2} \leq x \leq 1$, and $f(x) = e^x$, $-1 \leq x \leq 1$. The results are comparable with those given by Curtis and Osborne [7], and Ralston [26], who have obtained $10^5 E_{2,2}(\log x) = 0.1714628$ and $10^4 E_{2,2}(e^x) = 0.8689996$ respectively.

VALUES OF $10^5 E^*_{2,2;r}(f)$, $r = 1(2)9$, $f(x) = \log x$, $\frac{1}{2} \leq x \leq 1$.

r	$10^5 E^*_{2,2;r}(f)$
1	5.6319 9627
3	0.2257 7531
5	0.1714 7154
7	0.1714 7145
9	0.1714 7145

TABLE 6.1

VALUES OF $10^4 E^*_{2,2;r}(f)$, $r = 1(2)5$, $f(x) = e^x$, $-1 \leq x \leq 1$.

r	$10^4 E^*_{2,2;r}(f)$
1	4.5474 1131
3	0.8663 2432
5	0.8690 4548

TABLE 6.2

Once the value for $E_{n,m}(f)$ is obtained as the quantity $E^*_{n,m;r}(f)$, for a certain value of r, we may compute the asymptotic rational function of the best uniform approximation $p^*_{n;r}(x)/q^*_{m;r}(x)$, using the analysis which will be discussed below.

Let $\underline{P_{m+r}}$ (see definition 4.5) be the corresponding eigenvector of either $+E^*_{n,m;r}$ or $-E^*_{n,m;r}$. Under the conditions of theorem 4.4, the polynomial $\Psi_{m+r}(z)$ (see equation 4.1.6) has exactly m zeros $t_j^{-\varepsilon_j}$, $j = 1(1)m$, outside and r zeros $t_j^{-\varepsilon_j}$, $j = (m+1)(1)(m+r)$, inside the unit circle. Now, let $P_{s,m}^{(1)}$, $s = 0(1)m$, and $P_{s,r}^{(2)}$, $s = 0(1)r$, denote the "elementary symmetric functions" of the quantities $t_j^{-\varepsilon_j}$, $j = 1(1)m$, and of the quantities $t_j^{-\varepsilon_j}$, $j = (m+1)(1)(m+r)$, respectively. Similarly, let $Q_{s,m}^{(1)}$ and $Q_{s,r}^{(2)}$, for $s = 0,1,2,\ldots$, be the corresponding "homogeneous product sums of weight s", (see definitions 4.1 and 4.2). Furthermore, let us write the polynomial $q^*_{m;r}(x)$ as follows

$$(6.2.1) \qquad q^*_{m;r}(x) = \sum_{j=0}^{m}{}' \; b^*_{m-j}\, T_j(x) .$$

We want to express the coefficients b^*_{m-j}, $j = 0(1)m$, in terms of the quantities $P_{s,m}^{(1)}$, $s = 0(1)m$.

Lemma 6.4.

For $j = 0(1)m$,

$$(6.2.2) \qquad b^*_j = \frac{(-1)^{j+m}}{2^{(m-1)} P_{m,m}^{(1)}} \left(\sum_{s=0}^{j} P_{s,m}^{(1)} \, P^{(1)}_{m-j+s,m} \right).$$

Proof. This follows directly from lemma 4.6, since we have chosen $q^*_{m;r}$ to be $\Omega_m(x)$ (see section 4.2). We also note that $P_{m,m}^{(1)} \neq 0$ since $|t_j^{-\varepsilon_j}| > 1$, for $j = 1(1)m$.

Now, let

$$(6.2.3) \qquad f_{n+r+1}(x)\, q^*_{m;r}(x) = \sum_{j=0}^{n+m+r+1}{}' \, d_j^* \, T_j(x) ,$$

then we have the following lemma.

Lemma 6.5.

$$(6.2.4) \qquad d^*_0 = \frac{1}{2} \sum_{i=0}^{m}{}' \, a_i \, b^*_{m-i} ,$$

and

$$(6.2.5) \qquad d^*_j = \frac{1}{2} \sum_{i=0}^{m}{}' \, b^*_{m-i} \, (a_{j+i} + a_{|j-i|}) ,$$

for $j = 1(1)(n+m+r+1)$, where $a_k = 0$ for $k > (n+r+1)$.

Proof. The result readily follows by using (6.1.10).

By using equation (4.1.33), we can re-write (4.2.4) as

$$(6.2.6) \qquad f_{n+r+1}(x) = \frac{p^*_{n;r}(x)}{q^*_{m;r}(x)} \pm E^*_{n,m;r} \sum_{j=0}^{n+r+1}{}' C_{j,r}^{(n+r+1,m+r)} F_{j,m}(x) ,$$

where the coefficients $C_{j,r}^{(n+r+1,m+r)}$, $j = 0(1)(n+r+1)$, are given by

$$(6.2.7) \qquad C_{j,r}^{(n+r+1,m+r)} = (-1)^{n-j+1} \sum_{s=0}^{n+r+1-j} (-1)^s Q_{s,r}^{(2)} P^{(2)}_{s-n+j-1,r} ,$$

on applying lemma 4.12. Finally, let us write the polynomial $p^*_{n;r}(x)$ as

$$(6.2.8) \qquad p^*_{n;r}(x) = \sum_{j=0}^{n}{}' e_j^* T_j(x) ,$$

and obtain explicit expressions for its coefficients.

Lemma 6.6.

For $k = 0(1)n$,

$$(6.2.9)\qquad e_k^{\,*} = d_k^{\,*} \mp E^*_{n,m;r} \sum_{j=0}^{n+r+1}{}' \, C_{j,r}^{(n+r+1,m+r)} \, B^{(m)}_{k+m-j} \, ,$$

where

$$(6.2.10)\quad B_k^{(m)} = \begin{cases} \dfrac{(-1)^k}{2^m P_{m,m}^{(1)}} \displaystyle\sum_{s=0}^{k} P_{k-s,m}^{(1)} \, P_{s,m}^{(1)} \, , & \text{for } k = 0(1)m, \\[2ex] \dfrac{(-1)^k}{2^m P_{m,m}^{(1)}} \displaystyle\sum_{s=0}^{2m-k} P_{m-2,m}^{(1)} \, P_{s+k-m,m}^{(1)}, & \text{for } k = (m+1)(1)(2m), \end{cases}$$

amd $B_k^{(m)} = 0$ for $k < 0$ and $k > 2m$.

Proof. By multiplying both sides of equation (6.2.6) by $q^*_{m;r}(x)$ and comparing the Chebyshev coefficients, the result follows from (6.2.3), (6.2.8) and lemma 4.7.

To sum up, we have obtained explicit expressions for $p^*_{n;r}(x)$ and $q^*_{m;r}(x)$. Unfortunately, this analysis has one disadvantage in practice. In the case of uniform polynomial approximation (see Chapter V), we only require the components $(-1)^s P_{s,r}$, $s = 0(1)r$, of the eigenvector corresponding to the chosen eigenvalue, in order to compute the best approximation. In this general case, however, once the quantities $(-1)^s P_{s,m+r}$, $s = 0(1)(m+r)$, are obtained, we require the knowledge of the quantities $P_{s,m}^{(1)}$, $s = 0(1)m$ and $P_{s,r}^{(2)}$, $s = 0(1)r$.

$P_{s,m}^{(1)}$, $s = 0(1)m$, are the elementary symmetric functions of the m roots outside the unit circle of the polynomial $\Psi_{m+r}(z)$, whereas $P_{s,r}^{(2)}$, $s = 0(1)r$, are the elementary symmetric functions of the r roots inside the unit circle. Thus, the computation of $p^*_{n;r}(x)$ and $q^*_{m;r}(x)$ involves an extra step of finding the zeros of $\Psi_{m+r}(z)$, from which the values of $P_{s,m}^{(1)}$, $s = 0(1)m$, and $P_{s,r}^{(2)}$, $s = 0(1)r$, may be obtained. This disadvantage would be overcome, however, if a relationship between the quantities $P_{s,m+r}$, $P_{s,m}^{(1)}$ and $P_{s,r}^{(2)}$, which enables us to compute $P_{s,m}^{(1)}$ and $P_{s,r}^{(2)}$ directly from the knowledge of $P_{s,m+r}$, were found. Unfortunately, we have not been able to obtain such a relationship, and therefore have not been able to improve the analysis any further.

CHAPTER VII

ON A CONJECTURE OF C.W. CLENSHAW

~~If~~ Let $f(x) \in C[-1,1]$, and suppose there exists a Chebyshev series expansion of the form (1.3.7), which is uniformly convergent for all $x \in [-1,1]$ (see Clenshaw [5]). As we have mentioned in chapter IV, while the evaluation of the best uniform polynomial approximation $p_n(x)$ is somewhat tedious; the truncated Chebyshev series expansion $s_n(x)$ of degree n (see definition 1.7), on the other hand, is easier to compute. For many functions, we have analytic expressions for the Chebyshev coefficients a_k, which have also been extensively tabulated (see Clenshaw [5], Luke [19]). It is useful, therefore, to ask whether it is worthwhile to compute $p_n(x)$, or to use the truncated Chebyshev series expansion $s_n(x)$ to approximate the best uniform polynomial approximation to a given function. Some work on the comparison of the effectiveness of $s_n(x)$ and $p_n(x)$ has been carried out by many authors, and a summary of these results has been given by Rivlin [29]. They provide us with various bounds for the quantity S_n/E_n, which are very useful in the case of rapidly convergent Chebyshev series. In one of his paper, Clenshaw [6] has considered the same problem under less restrictive conditions.

For given non-negative integers n and r, Clenshaw has dealt with the problem of finding the maximum value of S_n/E_n taken over all polynomials $f_{n+r+1}(x)$ of degree $(n+r+1)$. He considered in great detail the cases when r = 1,2, and 3, and from these results, he was led to a conclusion concerning the maximum value of S_n/E_n for all values of r. He made the following three assumptions:

(i) n is large,

(ii) for all r, S_n/E_n has its overall maximum when all the coefficients of $f_{n+r+1}(x) - s_n(x)$ are either of the same, or of strictly alternating sign,

(iii) a quantity $P_{j,r}$ (see definition 4.1) is equal to $\binom{-\frac{1}{2}}{j}$ for all $j = 0(1)r$.

The third assumption was a conjecture obtained from a study of the cases when r = 1,2, and 3, assuming that (i) and

(ii) are valid. In this chapter, we shall show that under the assumptions (i) and (ii), the necessary condition for S_n/E_n to be a maximum over all polynomials of degree $(n+r+1)$ is satisfied if we choose $P_{j,r} = \binom{-\frac{1}{2}}{j}$ for $j = 0(1)r$ and $r = 0,1,2,\ldots$ In section 7.1, we shall find an explicit expression for the asymptotic estimate, for large n, of the quantity S_n/E_n, which we shall denote by $|N_{r,n}|$. An extreme value of this quantity will be given in section 7.2.

7.1. An explicit expression for $N_{r,n}$.

Suppose $f_{n+r+1}(x)$ is a polynomial of degree $(n+r+1)$, where n,r are non-negative integers. We can write $f_{n+r+1}(x)$ in a series of Chebyshev polynomials, as

$$(7.1.1) \qquad f_{n+r+1}(x) = \sum_{k=0}^{n+r+1}{}' a_k T_k(x) \ .$$

From equation (1.3.10), we have

$$(7.1.2) \qquad S_n = \max_{-1\le x\le 1} \left| \sum_{k=n+1}^{n+r+1} a_k T_k(x) \right| \ ,$$

where the coefficients a_k are given by

$$(7.1.3)\quad a_k = \frac{2}{\pi}\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} [f_{n+r+1}(t) - p_n(t)]\, T_k(t)dt.$$

On using (1.3.8), this last result follows from the fact that $T_k(t)$, for $k = (n+1)(1)(n+r+1)$, is orthogonal to all polynomials of degree n in the interval $[-1,1]$. If we write $x = \cos\psi$ and $t = \cos\theta$, (7.1.2.) becomes

$$S_n = \max_{0\le\psi\le\pi} \left|\sum_{k=n+1}^{n+r+1} \cos k\psi \left(\frac{2}{\pi}\int_0^{\pi} [f_{n+r+1}(\cos\theta) - p_n(\cos\theta)]\cos k\theta d\theta\right)\right|.$$

On assuming that the coefficients a_k, $k = (n+1)(1)(n+r+1)$, are of the same, or strictly alternating, sign, we can re-write this equation as

$$(7.1.4)\quad S_n = \left|\sum_{k=n+1}^{n+r+1} \frac{2}{\pi}\int_0^{\pi} [f_{n+r+1}(\cos\theta) - p_n(\cos\theta)]\cos k\theta d\theta\right|.$$

We now need an expression for $f_{n+r+1}(\cos\theta) - p_n(\cos\theta)$. As in chapter V, we let $E^*_{n;r}$ denote an asymptotic estimate of E_n, and $p^*_{n;r}(x)$ be asymptotic to the best uniform polynomial approximation $p_n(x)$, for large n. Then,

$$(7.1.5)\quad f_{n+r+1}(\cos\theta) - p^*_{n;r}(\cos\theta) = E^*_{n;r}\, G_{n+r+1,r}(\cos\theta)\ ,$$

where $G_{n+r+1,r}(\cos\theta)$ is the polynomial part of the rational function $F_{n+r+1,r}(\cos\theta)$. Again, we note that $F_{n+r+1,r}(\cos\theta)$

denotes the function $F_{n+r+1}(\Lambda,T_r,\Sigma_r;\cos\theta)$, where Λ is a finite set of r integers taking the value of unity and Σ_r is such that $\varepsilon_s = -1$, for $s = 1(1)r$; so that $F_{n+r+1,r}(\cos\theta)$ satisfies the required characterisation of the error function of the best uniform polynomial approximation. We also note that (7.1.5) was obtained under the assumption that the elements t_s, for $s = 1(1)r$, of the set T_r, satisfy the condition $|t_s| < 1$.

From equation (7.1.4) and (7.1.5), on replacing $p_n(\cos\theta)$ by $p^*_{n;r}(\cos\theta)$, we find

$$\frac{S_n}{E^*_{n;r}} = \left|\sum_{k=n+1}^{n+r+1} \frac{2}{\pi}\int_0^\pi G_{n+r+1,r}(\cos\theta)\cos k\theta d\theta\right| .$$

Thus,

$$\frac{S_n}{E^*_{n;r}} = \left|\sum_{k=0}^{r} A_{n+k+1}\,(G_{n+r+1,r})\right| , \tag{7.1.6}$$

where $A_k(G_{n+r+1,r})$ denotes the k^{th} Chebyshev coefficient of $G_{n+r+1,r}$. For convenience, we shall denote the finite sum in the right hand side of (7.1.6) by $N_{r,n}$. We obtain an explicit expression for $N_{r,n}$ in terms of the symmetric functions $P_{j,r}$ and $Q_{j,r}$, $j = 0(1)r$, of the quantities t_s, $s = 1(1)r$, (see definition 4.1 and 4.2).

Lemma 7.1.

For non-negative integers n and r,

$$(7.1.7) \qquad N_{r,n} = \sum_{k=0}^{r} (-1)^k P_{k,r} \sum_{\ell=0}^{k} Q_{\ell,r} .$$

Proof. From lemma 4.10,

$$N_{r,n} = \sum_{\ell=1}^{r+1} \sum_{s=0}^{\ell-1} (-1)^{r+\ell+s+1} Q_{s,r} P_{r+s+1-\ell,r} .$$

If, in place of summation over s, we sum over k, where $r+s+1-\ell = k$ and then interchange the orders of summation, we find

$$N_{r,n} = \sum_{k=0}^{r} \sum_{\ell=r+1-k}^{r+1} (-1)^k Q_{k+\ell-r-1,r} P_{k,r} .$$

Equation (7.1.7) readily follows.

7.2. An extreme value of $N_{r,n}$.

Lemma 7.1 has given us an explicit expression for $N_{r,n}$

as a function of the quantities $P_{j,r}$, $j = 0(1)r$. We shall now prove that in the particular case when $P_{j,r}$ is taken equal to $\binom{-\frac{1}{2}}{j}$ for $j = 1(1)r$ and all r, then $\dfrac{\partial N_{r,n}}{\partial P_{j,r}} = 0.$

First, we observe from lemma 4.2 that, for a given integer ℓ such that $1 \le \ell \le r$, the quantity $Q_{\ell,r}$ is a function of $P_{j,r}$, where $j = 1(1)\ell$, only. Thus, we have trivially.

<u>Lemma 7.2</u>.

(7.2.1.) $$\frac{\partial Q_{\ell,r}}{\partial P_{j,r}} = 0, \quad \text{for } j > \ell \ge 1.$$

<u>Lemma 7.3</u>.

For $s \ge j$ and $j = 1(1)(r-1)$,

(7.2.2) $$\frac{\partial Q_{s+1,r}}{\partial P_{j+1,r}} = - \frac{\partial Q_{s,r}}{\partial P_{j,r}} .$$

<u>Proof</u>. From definition 4.2, we can write for $k = 1(1)r$,

$$\sum_{s=0}^{k} (-1)^s P_{k-s,r} Q_{s,r} = 0 .$$

For any $j = 1(1)(r-1)$, we have on differentiating partially with respect to $P_{j,r}$,

$$(-1)^{k-j}\, Q_{k-j,r} + \sum_{s=j}^{k} (-1)^s\, P_{k-s,r} \frac{\partial Q_{s,r}}{\partial P_{j,r}} = 0.$$

On replacing j,k by (j+1) and (k+1) respectively, we obtain

$$(-1)^{k-j}\, Q_{k-j,r} + \sum_{s=j+1}^{k+1} (-1)^s\, P_{k+1-s,r} \frac{\partial Q_{s,r}}{\partial P_{j+1,r}} = 0 .$$

Subtracting these two equations, we find that

$$\sum_{s=j}^{k} (-1)^s\, P_{k-s,r} \left[\frac{\partial Q_{s,r}}{\partial P_{j,r}} + \frac{\partial Q_{s+1,r}}{\partial P_{j+1,r}} \right] = 0 ,$$

from which (7.2.2) follows.

Lemma 7.4.

For $r = 1,2,3,\ldots,$ and $j = 1(1)r$,

$$(7.2.3)\quad \frac{\partial N_{r,n}}{\partial P_{j,r}} = (-1)^j \sum_{\ell=0}^{j} Q_{\ell,r} + \sum_{\ell=j}^{r} (-1)^{j-1} \frac{\partial Q_{\ell-j+1,r}}{\partial P_{1,r}} \sum_{k=\ell}^{r} (-1)^k P_{k,r} .$$

Proof. This result follows from differentiating (7.1.7) partially with respect to $P_{j,r}$, using lemma 7.2 and making repeated use of

lemma 7.3.

We now want to consider the behaviour of $\frac{\partial N_{r,n}}{\partial P_{j,r}}$ for $j = 1(1)r$, when $P_{j,r}$ takes the particular value of $\binom{-\frac{1}{2}}{j}$ as suggested by Clenshaw. To this end, we need some further results.

Lemma 7.5.

If for $j = 0(1)r$, $P_{j,r} = \binom{-\frac{1}{2}}{j}$, then

$$Q_{j,r} = (-1)^j \binom{\frac{1}{2}}{j} \quad . \tag{7.2.4}$$

Proof. The result is trivially true when $j = 0$. Suppose now that it is true for $j = 0(1)J$. Then, from lemma 4.4,

$$\begin{aligned} Q_{J+1,r} &= (-1)^J \sum_{s=1}^{J+1} \binom{-\frac{1}{2}}{j} \binom{\frac{1}{2}}{J+1-s} \quad , \\ &= (-1)^J \left\{ \sum_{s=0}^{J+1} \binom{-\frac{1}{2}}{s} \binom{\frac{1}{2}}{J+1-s} - \binom{\frac{1}{2}}{J+1} \right\} \quad . \end{aligned}$$

The sum vanishes by Vandermonde's theorem (see [22]), and the result follows.

Lemma 7.6.

If for $j = 0(1)r$, $P_{j,r} = \binom{-\frac{1}{2}}{j}$, then

$$\frac{\partial Q_{1,r}}{\partial P_{1,r}} + \frac{\partial Q_{2,r}}{\partial P_{1,r}} = 0 , \tag{7.2.5}$$

and

$$\frac{\partial Q_{j,r}}{\partial P_{1,r}} = 0 , \quad \text{for } j = 3(1)r. \tag{7.2.6}$$

Proof. From lemma 4.4, $Q_{1,r} = P_{1,r}$ and $Q_{2,r} = P_{1,r}^{\,2} - P_{2,r}$, so that (7.2.5) readily follows. Using lemmas 4.4 and 7.5, we can prove (7.2.6) by induction.

We now come to the main result.

Theorem 7.1.

With $P_{j,r} = \binom{-\frac{1}{2}}{j}$, for $j = 1(1)r$,

$$\frac{\partial N_{r,n}}{\partial P_{j,r}} = 0 . \tag{7.2.7}$$

Proof. From lemmas 7.4 and 7.6, we have

$$\frac{\partial N_{r,n}}{\partial P_{j,r}} = (-1)^j \sum_{\ell=0}^{j} Q_{\ell,r} + \sum_{\ell=j}^{r} (-1)^{j-1} \frac{\partial Q_{\ell-j+1,r}}{\partial P_{1,r}} \sum_{k=\ell}^{r} (-1)^k P_{k,r} ,$$

$$= (-1)^j \sum_{\ell=0}^{j} Q_{\ell,r} - P_{j,r} .$$

Using lemma 7.5, it follows that the right hand side vanishes and the theorem is proved.

Thus, we have shown that when $P_{j,r}$ takes the value $\binom{-\frac{1}{2}}{j}$, for $j = 1(1)r$, we have satisfied necessary conditions for $N_{r,n}$, and therefore S_n/E_n to have a local maximum. The conditions

$$\frac{\partial N_{r,n}}{\partial P_{j,r}} = 0, \quad \text{for } j = 1(1)r,$$

however, are not alone sufficient to guarantee that $N_{r,n}$ has a local maximum. Unfortunately, we have not been able to show that the sufficient conditions are satisfied for all r. From examination of particular cases when r = 1,2,3 and 4, however, it can be shown that these conditions are satisfied.

Finally, we obtain an explicit expression for the maximum value of $N_{r,n}$, denoted by $R^{(r)}$.

Theorem 7.2.

For $r = 0,1,2,\ldots,$

$$R^{(r)} = \sum_{j=0}^{r} \binom{-\frac{1}{2}}{j}^2 . \tag{7.2.8}$$

Proof. The result follows directly from lemmas 7.1 and 7.5.

From (7.2.8), Clenshaw concluded that

$$\frac{S_n}{E_n} \le \frac{1}{\pi} \left(3.35 + \log r + O\left(\frac{1}{r}\right)\right) , \tag{7.2.9}$$

for large r. He also noted in particular that this bound is less than 2 if $r \le 18$, so that "in the great majority of practical applications, the extra labour involved in calculating a best approximation may not be worth-while". As we have mentioned earlier, the evaluation of S_n/E_n for various functions has also been considered by many authors, and a summary of such results has been given by Rivlin [29]. More recently, Powell [25] has shown that if $f(x)$ is continuous in $[-1,1]$, then

$$\frac{S_n}{E_n} \le 1 + \frac{4}{\pi^2} \log n, \text{ for large } n. \tag{7.2.10}$$

Now, the polynomials of degree $(n+r+1)$ are certainly continuous functions in $[-1,1]$, so that (7.2.10) is true for all values of r. On comparing the inequalities (7.2.9) and (7.2.10), we see that for $r > n$, Powell's result gives a better upper bound than Clenshaw's. However, for $1 \leq r < n$, (7.2.9) is to be preferred and indeed it was for r in this range that Clenshaw did his numerical experiments, which led to his conjecture for the values of $P_{j,r}$, $j = 0(1)r$.

APPENDIX

LIST OF SYMBOLS

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