DIRICHLET'S PROBLEM,

CONFORMAL MAPPING,

and

COMPLETE SETS IN HILBERT SPACE

by

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Outline of the Thesis

First Part:

One of the proofs of Poisson's formula is analysed. This leads readily to a method of solving Dirichlet's problem explicitly in some new cases.

The solution of Dirichlet's problem is equivalent to the conformal mapping of some given simply connected region on the interior of a circle. The new method for the solution of Dirichlet's problem is tested by the conformal mapping of an ellipse on a circle. Thus a result previously found by a different method by Szego is confirmed.

Second Part:

Some special types of regions are considered; the method of solving Dirichlet's problem discussed in the first part is applied to these regions, and thus a new problem arises. This method consists essentially in approximating a given function by a given set of functions. The meaning of "approximation" can be generalised by introducing the notion of function spaces. A new metric is introduced that is easier to handle. The problem is now a problem in "Hilbert space".

An important step in the discussion consists in considering a "truncated function", i.e., a function vanishing identically in some interval, as being the "projection" of a function. After this step the problem can be restated in finite or infinite dimensional Hilbert space.

The remaining part of the thesis consists mainly of a discussion of the notion of "completeness" and its connection with projections. A theorem by Dalzell and one by Vitali are found to be special cases of a general situation. The problem as such remains unsolved but some steps are made towards its solution. It is found that the notion of "completeness" is a complex one and that several grades of completeness have to be discriminated.

Introduction

Poissons Formula.

Let r, θ be the polar coordinates of a point in the plane, and let $z = r\,e^{i\theta}$. Let $f(z)$ be analytic in a region G properly including the circle $|z| \leqq R$, and let $u(r,\theta)$ be its real part. Then there exist complex numbers a_ν $(\nu = 0,1,\ldots)$ such that

$$f(z) = \sum_0^\infty a_\nu z^\nu \qquad |z| \leqq R.$$

Let $\qquad a_\nu = \alpha_\nu + i\,\beta_\nu$. Then

$$(1) \qquad u(r,\theta) = \sum_0^\infty (\alpha_\nu \cos \nu\theta - \beta_\nu \sin \nu\theta)\, r^\nu, \quad 0 \leqq r \leqq R.$$

Since $\sum_0^\infty a_\nu z^\nu$ is uniformly convergent at all points of the circle $|z| \leqq R$, the series in (1) converges for fixed r uniformly w.r.t. θ. Also the functions 1, $\cos \nu\theta$, $\sin \nu\theta$, $(\nu = 1,2,\ldots)$ are continuous, hence we may multiply both sides of (1) by $\cos n\theta$ or $\sin n\theta$ and integrate term by term. We get

$$\alpha_n R^n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta) \cos n\theta \, d\theta$$
$$\beta_n R^n = -\frac{1}{\pi}\int_0^{2\pi} u(R,\theta) \sin n\theta \, d\theta \qquad (n = 1,2,\ldots)$$
$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\, d\theta$$

Putting these values in (1), we get

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\varphi)\, d\varphi + \frac{1}{\pi}\sum_1^\infty \frac{r^\nu}{R^\nu}\int_0^{2\pi} u(R,\varphi)\,(\cos \nu\theta \cos \nu\varphi +$$
$$+ \sin \nu\theta \sin \nu\varphi)\, d\varphi\,.$$

But $\qquad \left|\frac{r^\nu}{R^\nu}\, u(R,\varphi) \cos \nu(\theta - \varphi)\right| \leqq \max |u(R,\varphi)|\, \frac{r^\nu}{R^\nu}$

$\sum_1^\infty \frac{r^\nu}{R^\nu}$ converges for $0 \leqq r < R$, hence the series

$\sum_1^\infty \frac{r^\nu}{R^\nu} U(R,\varphi)\cos \nu(\theta-\varphi)$ is uniformly convergent for $0 \leqq r < R$,
and since the terms are continuous we get for $0 \leqq r < R$,

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} u(r,\varphi)\left(\frac{1}{2} + \sum_1^\infty \cos n(\theta-\varphi)\frac{r^n}{R^n}\right)d\varphi$$

Put $z = r/R\, e^{i(\theta-\varphi)}$, then

$$\begin{aligned}\sum_1^\infty \frac{r^n}{R^n}\cos n(\theta-\varphi) &= \sum_1^\infty \mathcal{R}(z^\nu) = \mathcal{R}\sum_1^\infty z^\nu = \mathcal{R}z/(1-z)\\ &= \mathcal{R}z(1-\bar{z})/|1-z|^2 = (\mathcal{R}z - |z|^2)/|1-z|^2\\ &= \frac{r/R\cos(\theta-\varphi) - r^2/R^2}{1 - r/R\cos(\theta-\varphi) + r^2/R^2\sin^2(\theta-\varphi)}\\ &= \frac{1}{2}\,\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta-\varphi)} - \frac{1}{2}\end{aligned}$$

Hence for $0 \leqq r < R$,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2 - r^2)\,u(R,\varphi)\,d\varphi}{R^2 - 2rR\cos(\theta-\varphi) + r^2}$$

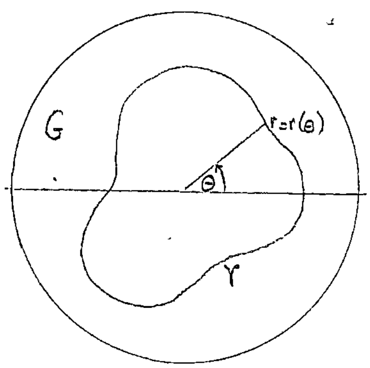
and this is known as Poisson's integral formula. $u(r,\theta)$ is harmonic inside the circle $|z| \leqq R$ and is uniquely determined by its values on the boundary of the circle. However these boundary values must consist of the reductions on $|z| = R$ to a function of θ of the real part of a function analytic in G.

Dirichlet's Problem.

The problem of determining a function harmonic in a given region and having as limiting function on the boundary a given continuous function is known as Dirichlet's problem. Poisson's formula has been derived by assuming that $u(r,\theta)$ is the real part of a function analytic in a region properly including the circle $|z| \leq R$. Because of this restriction the formula does not yet provethe existence of a solution of Dirichlet's prblem for the circle. However, the formula can be obtained by other methods without requiring the above restriction and thus solves Dirichlet's problem for the circle.[1]

Generalization of Poisson's Formula.

Suppose G is a region bounded by a circle. Let γ be a simple closed curve properly included in G such that if O, the centre of G, is the origin of the polar coordinates r, θ, then O is inside but not on γ and any straight line through O has only two points in common with γ.



Then in particular, if $f(z)$ is analytic in G and $f(z) = \sum_0^\infty a_\nu z^\nu$ inside G, then $\sum_0^\infty a_\nu z^\nu$ converges uniformly along γ, i.e. converges uniformly in the set of points comprising γ.

Let $r = r(\theta)$ along γ, this is a single valued function, $r(\theta) = r(\theta + 2\pi)$. Let $a_\nu = \alpha_\nu + i\beta_\nu$. Then the series $\sum_0^\infty(\alpha_\nu \cos \nu\theta - \beta_\nu \sin \nu\theta)\, r(\theta)^\nu$ converges uniformly w.r.t. θ to $u(r(\theta),\theta)$. Thus when the above conditions on G and γ are satisfied, the functions 1, $r(\theta)^\nu \cos \nu\theta$, $r(\theta)^\nu \sin \nu\theta$ ($\nu = 1,2,\ldots$) can be used to uniformly approximate any function which can be expressed as the reduction on γ to a function of θ of the real part of an analytic fubction in G. Hence it is possible, as the following theorem shows, to solve the boundary value (restricted to the class of continuous functions defined above) problem for G by the same method as we have used for the circle.

Harnack's Theorem.[2)]

Let G be any closed region in the plane, and let $\{U_n\}$ be an infinite sequence of functions harmonic in G. If the sequence converges uniformly on the boundary γ of G it converges uniformly throughout G, and the limit U isa harmonic in G.

Proof. $U_{n+p} - U_n$ is harmonic in G, hence by the maximum modulus theorem[3)], is either constant or attains its maximum value on γ.

Hence for $P \in G$, $P' \in \gamma$,

$$|U_{n+p}(P) - U_n(P)| \leqq |U_{n+p}(P') - U_n(P')|$$

But for arbitrary $\varepsilon > 0$, there exists a positive integer N such that

$$|U_{n+p}(P') - U_n(P')| < \varepsilon \qquad \text{for } n > N \text{ and } P' \in \gamma$$

and hence

$$|U_{n+p}(P) - U_n(P)| < \varepsilon \qquad " \quad " \quad " \quad P \in G.$$

i.e. $\{U_n\}$ converges uniformly in G.

Also, a uniformly convergent sequence of continuous functions has a continuous function as limit and hence the limit U of the sequence is continuous in G and on γ.

Let P be an arbitrary point of G. Take a circle K which lies in G and contains P in its interior. Using the centre of K as origin, let the polar coordinates of P be r, θ, and R be the radius of K. Then by Poisson's formula,

$$U_n(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, U_n(R,\varphi)}{R^2 - 2rR\cos(\theta - \varphi) + r^2}\, d\varphi$$

Let $$f(r,\theta,\varphi) = \frac{1}{2\pi}\,\frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2}$$

Then

$$\left| \int_0^{2\pi} f(r,\theta,\varphi)\,(U_n(R,\varphi) - U(R,\varphi))\, d\varphi \right| < \varepsilon \int_0^{2\pi} \left| f(r,\theta,\varphi) \right| d\varphi \text{ for } n > N$$

and arbitrary $\varepsilon > 0$. Since $\int_0^{2\pi} | f(r,\theta,\varphi) | \, d\varphi$ is finite,

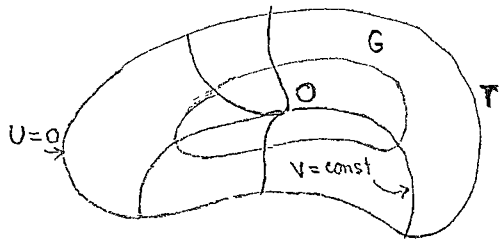
$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, U(R,\varphi)}{R^2 - 2rR\cos(\theta - \varphi) + r^2}\, d\varphi$$

Thus U is represented in the interior of K by a Poisson integral over the continuous boundary values of U, so that U is a harmonic function inside K and hence is harmonic in the neighbourhood of P. But since P is an arbitrary point in G, U is harmonic throughout G.

Conformal Mapping of Simply Connected Domains.

The solution of the boundary value problem for harmonic functions leads readily to the conformal mapping of simply connected domains on the interior of a circle.

Suppose G is a simple closed region with boundary γ, and O is a point inside G. Let (x,y) be the Cartesian coordinates and r, θ the corresponding polar coordinates of a point in G w.r.t. O.



Suppose there exists a function $w(x,y)$, harmonic in G with boundary values $-\log r$ on Υ. Let

$$U = \log r + w(x,y)$$

Then $U = 0$ on Υ and U is harmonic in G except at O. If V is the harmonic function conjugate to U,

$$U + i\,V = F(x + i\,y) = F(z)$$

is analytic as a function of the complex variable $z = x + i\,y$ in $G - \{O\}$ and has a singularity at O. The curve Υ belongs to the family of curves U = constant and the lines V = constant are the orthogonal trajectories of the family. Let C be a smooth closed curve in $G - \{O\}$ containing O. In passing around C once in the positive sense, V increases by a period k which is independent of the particular choice of C.

$$k = \int_C dV = \int_C \frac{dV}{ds}\,ds$$

where s denotes arc length on C. Let $x = x(s)$, $y = y(s)$ on C.

Then $$k = -\int_C \left(\frac{\partial V}{\partial x}\frac{dx}{ds} + \frac{\partial V}{\partial y}\frac{dy}{ds}\right) ds$$

By the Cauchy-Riemann equations,

$$k = -\int_C \left(\frac{\partial U}{\partial y}\frac{dx}{ds} - \frac{\partial U}{\partial x}\frac{dy}{ds}\right) ds$$

$$k = \int_C \frac{\partial U}{\partial n}\, ds \quad \text{where } \frac{\partial}{\partial n} \text{ denotes the normal derivative}$$

$$= \int_C \frac{\partial}{\partial n} \log r \, ds + \int_C \frac{\partial}{\partial n} w(x,y)\, ds$$

$$= \int_0^{2\pi} \frac{1}{r} r \, d\theta + \int_C dw' \quad \text{where } w' \text{ is the harmonic conjugate of } w$$

$$= 2\pi.$$

Put $\zeta = f(z) = e^{F(z)} = e^{U} e^{iV}$

On γ, $f(z) = e^{iV}$, $|\zeta| = 1.$

In one complete revolution on γ, $f(z)$ changes from e^{iV} to $e^{i(V+2\pi)} = e^{iV}.$

Hence γ is mapped onto the circumference of the unit circle $|\zeta| = 1.$ Thus the problem of the conformal mapping of simply connected domains on a circle can be subordinated to Dirichlet's Principle.

<u>Conformal Mapping of an Ellipse on a Circle.</u>

Let G be an ellipse of eccentricity ε and major axis 2b with boundary γ. Let O be a focus and r, θ be polar coordinates defined w.r.t. O and the major axis of the ellipse. Then on γ

$$r = \frac{b(1-\varepsilon^2)}{1 + \varepsilon \cos \theta}$$

Obviously the functions $1, r^\nu \cos \nu\theta, r^\nu \sin \nu\theta$ $(\nu = 1, 2, ..)$ are not orthogonal on γ, and hence the task of approximating the function $-\log r$ on γ by these functions would be tedious. However if we shift the origin O to the centre of the ellipse, define rectangular axes of x, y as continuations of the major and minor axes of the ellipse respectively, and use as the basic set of analytic functions in G, $\left\{ a \cosh \nu z;\ \nu = 0, 1, .. \right\}$, $(z = x + iy)$ we find that the harmonic functions consisting of the real and imaginary parts of the above, reduce on γ to the trigonometric functions which are orthogonal on γ. From this point the solution of the problem follows as previously outlined.

Let $\quad w = a \cosh z = u + i\,v$

Then $\qquad u = a \cosh x \cos y,$

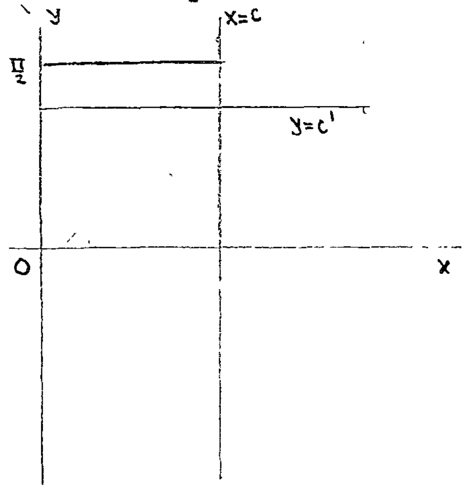
$\qquad\qquad v = a \sinh x \sin y,$

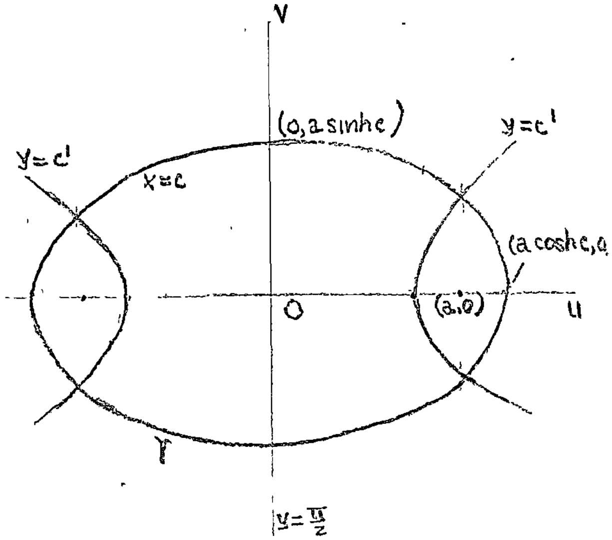
and it follows that

$$\frac{u^2}{a^2 \cosh^2 x} + \frac{v^2}{a^2 \sinh^2 x} = 1$$

$$\frac{u^2}{a^2 \cos^2 y} - \frac{v^2}{a^2 \sin^2 y} = 1$$

Hence the lines x = constant, y = constant, on the z-plane are mapped into confocal ellipses, hyperbolas, respectively, in the w-plane.





Let $u_\nu + i\, v_\nu = w_\nu = a \cosh \nu z$, $u_0 = a$, $(\nu = 1, 2 \ldots)$

Consider the ellipse $x = c$ $(c > 0)$ in the w-plane, with boundary γ.

On γ

$$u_\nu = \alpha_\nu \cos \nu y, \quad v_\nu = \beta_\nu \sin \nu y,$$

where

$$\alpha_\nu = a \cosh \nu c, \quad \beta_\nu = a \sinh \nu c$$

and

$$\log \sqrt{u^2 + v^2} = \log a + \tfrac{1}{2} \log (\cos^2 y + \sinh^2 c) = g(y) \quad \text{(say)}$$

$g(y)$ is continuous, has period 2π, and has[4] only a finite number of maxima and minima for $0 \leqq y \leqq 2\pi$. Hence its Fourier series converges to $g(y)$ for $0 \leqq y \leqq 2\pi$, and we may write

$$g(y) = A_0 + \sum_1^\infty (A_\nu \cos \nu y + B_\nu \sin \nu y), \quad 0 \leqq y \leqq 2\pi,$$

where

$$A_0 = -\frac{1}{4\pi} \int_0^{2\pi} \log (\cos^2 t + \sinh^2 c)\, dt - \log a,$$

$$A_\nu = -\frac{1}{2\pi}\int_0^{2\pi}\log(\cos^2 t + \sinh^2 c)\cos\nu t\,dt$$

$$B_\nu = -\frac{1}{2\pi}\int_0^{2\pi}\log(\cos^2 t + \sinh^2 c)\sin\nu t\,dt$$

The function

$$\log\sqrt{u^2+v^2} + A_0 + \sum_1^\infty\left(\frac{A_\nu}{\alpha_\nu}u_\nu + \frac{B_\nu}{\beta_\nu}v_\nu\right)$$

$$= \log|w| + A_0 + \sum_1^\infty\left(\frac{A_\nu}{\alpha_\nu}\mathcal{R}w_\nu - \frac{B_\nu}{\beta_\nu}\mathcal{R}iw\right)$$

$$= \mathcal{R}\left(\log w + A_0 + \sum_1^\infty w_\nu\left(\frac{A_\nu}{\alpha_\nu} - i\frac{B_\nu}{\beta_\nu}\right)\right)$$

is zero on γ. Hence the function

$$F(z) = \log\cosh z + A_0 + \sum_1^\infty\left(\frac{A_\nu}{\alpha_\nu} - i\frac{B_\nu}{\beta_\nu}\right)\cosh\nu z$$

is analytic except at $z = i\frac{\pi}{2}$ and $\mathcal{R}(F(z))\Big|_{x=c} = 0$.

The rectangle $0 \leqq x \leqq c$, $0 \leqq y \leqq \frac{\pi}{2}$ in the z-plane is mapped conformally onto the ellipse $(x = c)$ in the w-plane. Each z in the rectangle is mapped into 4 points, one in each quadrant of the ellipse.

Define $f(w)$ inside the ellipse by putting

(1) $$f(w) = F(\operatorname{arcosh}\frac{w}{a})$$

Then $f(w)$ is analytic except at $w = 0$ and $\mathcal{R}(f(w))$ is zero on γ. Hence $\zeta = e^{f(w)}$ maps the interior of the ellipse conformally onto the interior of the unit circle $|\zeta| = 1$.

We shall now evaluate our constants.

By partial integration we find that

$$A_\nu = -\frac{1}{\pi\nu}\int_0^{2\pi}\frac{\sin 2t\,\sin\nu t}{\cos 2t + \cosh 2c}\,dt$$

$$= 0 \quad \text{if } \nu \text{ is odd.}$$

and

$$B_\nu = \frac{1}{\pi\nu}\int_0^{2\pi}\frac{\sin 2t\cos\nu t}{\cos 2t+\cosh 2c}\,dt$$

$$= 0 \text{ if } \nu \text{ is odd.}$$

The above integrals are evaluated in the following by the calculus of residues.

Let $z = e^{it}$, then $dz = i\,z\,dt$ and

$$\frac{\sin 2t\sin 2\nu t}{\cos 2t+\cosh 2c} = \left(\frac{1}{2i}\right)^2\frac{(z^2-z^{-2})(z^{2\nu}-z^{-2\nu})}{\frac{1}{2}(z^2+z^{-2})+\cosh 2c}$$

$$= -\frac{1}{2}\,\frac{(z^4-1)(z^{4\nu}-1)}{z^{2\nu}(z^4+2\cosh 2c\,z^2+1)}$$

Thus we get the complex integral around the unit circle $|z| = 1$ in the z-plane:

$$A_{2\nu} = \frac{1}{4\pi\nu i}\int_{|z|=1}\frac{(z^4-1)(z^{4\nu}-1)\,dz}{z^{2\nu+1}(z^4+2z^2\cosh 2c+1)}$$

and similarly

$$B_{2\nu} = -\frac{1}{4\pi\nu}\int_{|z|=1}\frac{(z^4-1)(z^{4\nu}+1)}{z^{2\nu+1}(z^4+2z^2\cosh 2c+1)}\,dz$$

Let $\beta = \cosh 2c$. Then

$$(z^4+2\beta z^2+1)^{-1} = (z^2+e^{2c})^{-1}(z^2+e^{-2c})^{-1}$$

$$= \sum_{\nu=0}^{\infty}(-1)^\nu(ze^{-c})^{2\nu}\sum_{\lambda=0}^{\infty}(-1)^\lambda(ze^{c})^{2\lambda}$$

The coefficient of $z^{2\nu}$ in the expansion of the above is

$$(-1)^\nu\left(e^{2\nu c}+e^{(2\nu-4)c}+\dots+e^{-2\nu c}\right) = (-1)^\nu e^{-2\nu c}\,\frac{1-e^{4c(\nu+1)}}{1-e^{4c}}$$

$$= (-1)^\nu\,\frac{\sinh(2\nu c+2c)}{\sinh 2c}$$

and hence the coefficient of $z^{2\nu}$ in $(z^4-1)(z^4+2\beta z^2+1)^{-1}$ is

$$(-1)^\nu\frac{\sinh(2\nu c+2c)}{\sinh 2c} + (-1)^\nu\frac{\sinh(2\nu c-2c)}{\sinh 2c}$$

$$= (-1)^\nu\,2\cosh 2\nu c$$

We shall now determine the residues inside the circle $|z| = 1$ of the function

$$h(z) = \frac{(z^{4v} - 1)(z^4 - 1)}{z^{2v+1}(z^4 + 2\beta z^2 + 1)} = \frac{z^{2v-1}(z^4 - 1)}{z^4 + 2\beta z^2 + 1} - \frac{z^4 - 1}{z^{2v+1}(z^4 + 2\beta z^2 + 1)}$$

$$= \frac{(z^{4v} - 1)(z^4 + 1)}{z^{2v+1}(z^2 + e^{2c})(z + ie^{-c})(z - ie^{-c})}$$

$$\mathrm{Res}\,(h(0)) = (-1)^v\, 2 \cosh 2vc$$

$$\mathrm{Res}\,(h(ie^{-c})) = \frac{(e^{-4vc} - 1)(e^{-4c} - 1)}{i(-1)^v e^{-(2v+1)c}(e^{2c} - e^{-2c})\, 2ie^{-c}}$$

$$= (-1)^{v+1} \sinh 2vc$$

$$\mathrm{Res}\,(h(-ie^{-c})) = \frac{(e^{-4vc} - 1)(e^{-4c} - 1)}{-i(-1)^v e^{-(2v+1)c}(e^{2c} - e^{-2c})(-2ie^{-c})}$$

$$= (-1)^{v+1} \sinh 2vc$$

Hence we get

$$\frac{1}{2\pi i}\int_{|z|=1} h(z)\,dz = 2(-1)^{v+1}(\sinh 2vc - \cosh 2vc)$$

$$= 2(-1)^v e^{-2vc}$$

$$A_{2v} = \frac{(-1)^v}{v} e^{-2vc}$$

The residues of the function

$$l(z) = \frac{(z^4 - 1)(z^{4v} + 1)}{z^{2v+1}(z^2 + e^{2c})(z + ie^{-c})(z - ie^{-c})}$$

$$= \frac{z^{2v-1}(z^4 - 1)}{z^4 + 2\beta z^2 + 1} + \frac{z^4 - 1}{z^{2v+1}(z^4 + 2\beta z^2 + 1)}$$

are given by

$$\mathrm{Res}\,(l(0)) = (-1)^{v+1}\, 2 \cosh 2vc$$

$$\mathrm{Res}\,(l(ie^{-c})) = \frac{(e^{-4vc} + 1)(e^{-4c} - 1)}{i(-1)^v e^{-(2v+1)c}(e^{2c} - e^{-2c})\, 2ie^{-c}}$$

$$= (-1)^v \cosh 2vc$$

$$\text{Res}\,(I(-ie^{-c})) = \frac{(e^{-4\nu c}+1)(e^{-4c}+1)}{-i(-1)^{\nu}e^{-(2\nu+1)c}(e^{2c}-e^{-2c})(-2ie^{-c})}$$

$$= (-1)^{\nu}\cosh 2\nu c$$

Hence we get

$$\frac{1}{2\pi i}\int_{|z|=1} I(z)\,dz = (-1)^{\nu+1}\,2\cosh 2\nu c + (-1)^{\nu}\,2\cosh 2\nu c$$

$$= 0$$

i.e. $\qquad B_{2\nu} = 0$

$$-4\pi(A_0 + \log a) + 2\pi\log 2 = \int_0^{2\pi}\log(\cos 2t + \cosh 2c)\,dt$$

$$= 2\int_0^{2\pi}\int_0^{c}\frac{\sinh 2c}{\cos 2t + \cosh 2c}\,dc\,dt + \int_0^{2\pi}\log(\cos 2t + 1)\,dt$$

$$= 2\int_0^{c}\int_0^{2\pi}\frac{\sinh 2c}{\cos 2t + \cosh 2c}\,dt\,dc + 2\int_{-\pi/2}^{\pi/2}\log(2\cos^2 t)\,dt$$

$$= \frac{2}{i}\int_0^{c}\sinh 2c\int_{|z|=1}\frac{z}{z^4 + 2\beta z^2 + 1}\,dz\,dc + 2\pi\log 2 + 4\int_0^{\pi}\log\sin x\,dx$$

But

$$\int_{|z|=1}\frac{z\,dz}{z^4 + 2\beta z^2 + 1} = \int_{|z|=1}\frac{z}{(z^2+e^{2c})(z+ie^{-c})(z-ie^{-c})}\,dz$$

$$= 2\pi i\left(\frac{ie^{-c}}{2ie^{-c}(e^{2c}-e^{-2c})} + \frac{-ie^{-c}}{-2ie^{-c}(e^{2c}-e^{-2c})}\right)$$

$$= \frac{\pi i}{\sinh 2c}$$

Hence

$$-4\pi(A_0 + \log a) = 2\int_0^{c} 2\pi\,dx + 4(-\pi\log 2)$$

$$= 4\pi c - 4\pi\log 2$$

and

$$A_0 + \log a = \log(2e^{-c})$$

From equation (1)

$$f(w) = \log w/a + \log(2e^{-c}) - \log a + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}e^{-2\nu c}}{\nu\,a\cosh 2\nu c}\cosh\left(2\nu\,\text{arccosh}\frac{w}{a}\right)$$

or

$$(2) \quad \log \frac{\zeta}{w} = \log \left(\frac{2e^{-c}}{a^2}\right) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}\, 2 \cosh 2\nu(\operatorname{arcosh} w/a)}{a\,\nu\,(e^{4\nu c} + 1)}$$

But $\cosh \nu z = 1/2\,(\cosh z + \sinh z)^{\nu} + 1/2\,(\cosh z - \sinh z)^{\nu}$

and we may rewrite (2) as

$$(3) \quad \log \frac{\zeta}{w} = \log \left(\frac{2e^{-c}}{a^2}\right) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}\left(\left(w + \sqrt{w^2 - a^2}\right)^{2\nu} + \left(w - \sqrt{w^2 - a^2}\right)^{2\nu}\right)}{2\,\nu\, a^{2\nu+1}\,(e^{4\nu c} + 1)}$$

If we put $a = 1$, $R = e^{c}$, then

$$\frac{1}{e^{4\nu c} + 1} = \frac{R^{-2\nu}}{R^{2\nu} + R^{-2\nu}}$$

and
$$\log \frac{2e^{-c}}{a^2} = \log \frac{2}{R}$$

Defining V by $\quad w = 1/2\,(V + V^{-1}) = \cosh \log V$,

$$1/2\,(V^{\nu} + V^{-\nu}) = \cosh \nu \log V = \cosh \nu(\operatorname{arcosh} w)$$
$$= T_{\nu}(w) \quad \text{(say)}$$

equation (2) becomes

$$(4) \quad \log \frac{\zeta}{w} = \log \frac{2}{R} + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}}{\nu} \frac{2R^{-2\nu}}{R^{2\nu} + R^{-2\nu}} T_{2\nu}(w)$$

This formula has been obtained, using a different method, by Szego, Am. Math. Mo. 57, pp 474-478, 1950.

(4) maps conformally the ellipse in the w-plane with major axis $R + 1/R$, minor axis $R - 1/R$, and foci at the points $w = \pm 1$ onto the unit circle $|\zeta| = 1$ in the ζ-plane.

Complete Sets on the Boundary of a Simple Domain.

Let G be a closed plane region bounded by a rectifiable curve γ. Since γ is a closed curve, any two different points P_1, P_2 on γ determine two distinct parts of γ. We shall denote (measuring in the positive sense) P_1P_2, P_2P_1 by γ_1, γ_2 respectively, and shall measure arc length s on γ from P_1 such that on γ_1 $0 \leqq s < a$ and on γ_2 $a \leqq s < b$.

We define

$$\lambda_1(s) = \begin{cases} 1 & \text{for } 0 \leqq s < a \\ 0 & \text{for } a \leqq s < b \end{cases}$$

$$\lambda_2(s) = 1 - \lambda_1(s) \quad \text{for} \quad 0 \leqq s < b.$$

Let $\{\psi_\nu ; \nu = 1, 2, \dots\}$ be an infinite set of functions harmonic in G; we may write ψ_ν as a function, $\psi_\nu(s)$, of the one variable s only on γ.

Let C be a certain class of functions defined on $0 \leqq s < b$ such that for any $f \in C$, $\lambda_\nu f$ may be uniformly approximated by the set of functions $\{\lambda_\nu \psi_n ; n = 1, 2, \dots\}$ for $\nu = 1, 2$. If we put certain restrictions on C we can use the functions ψ_n $(n = 1, 2, \dots)$ to uniformly approximate to elements of C on $0 \leqq s < b$.

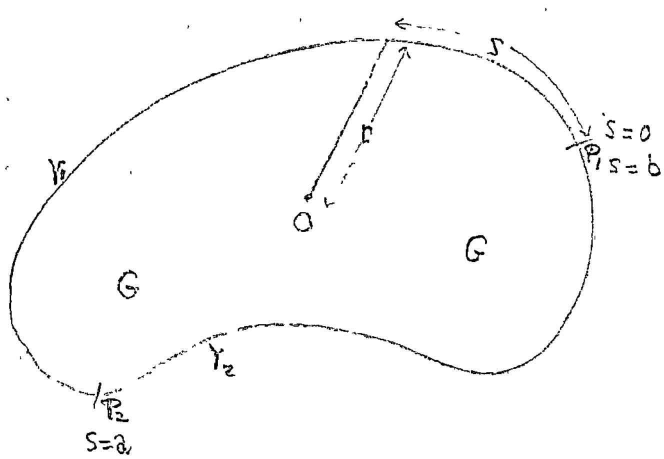
For example, if G is a circular disc of unit radius and r, θ are polar coordinates defined w.r.t. the centre O of G and the line OP_1,

$$\{\psi_\nu ; \nu = 1, 2, \dots\} = \{1, r^\nu \cos \nu\theta, r^\nu \sin \nu\theta ; \nu = 1, 2, \dots\}$$

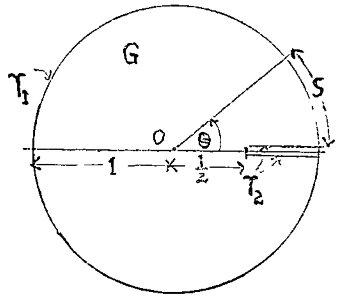
and $0 < a < 2\pi$, $b = 2\pi$, then C is the class of all functions

5)

which are absolutely continuous and periodic for $0 \leqq s < 2\pi$. Returning to the general case, let $r = r(s)$ be the distance measured from a fixed point O inside G of the point on γ at a distance s along γ from P_1. Now if log r is an element of the restricted class C we can, using the method previously established (pp5-7), map G conformally onto a circular disc. However, since the functions ψ_n $(n = 1,2,..)$ will in general not be orthogonal, the determination of the constant coefficients of the series appearing in the mapping formula will not be easy. Orthonormalis--ation of the functions ψ_n $(n = 1,2,..)$ will solve this problem, the new functions so formed replacing the ψ_n in the process. But since we can only find a finitenumber of the orthonormal functions, the final mapping formula so produced will only be approximate. In practical applications of any series formula, however, only approximations can be used, so that the above approxim--ate formula is sufficient.



The type of region for which sucg a situation occurs may be illustrated by the following. Let G be a circular disc of unit radius and centre at 0 with a slit along the line $\theta = 0$ from $r = 1/2$ to $r = 1$. Let γ_1 be the circumference of the circle and γ_2 be the slit, i.e. consist of the two borders which are coincidental, of the regions: $\pi/2 \geqq \theta > 0$, $1/2 \leqq r \leqq 1$, and $-\pi/2 \leqq \theta < 0$, $1/2 \leqq r \leqq 1$.



The functions defined by (1) are harmonic in G. On γ_1

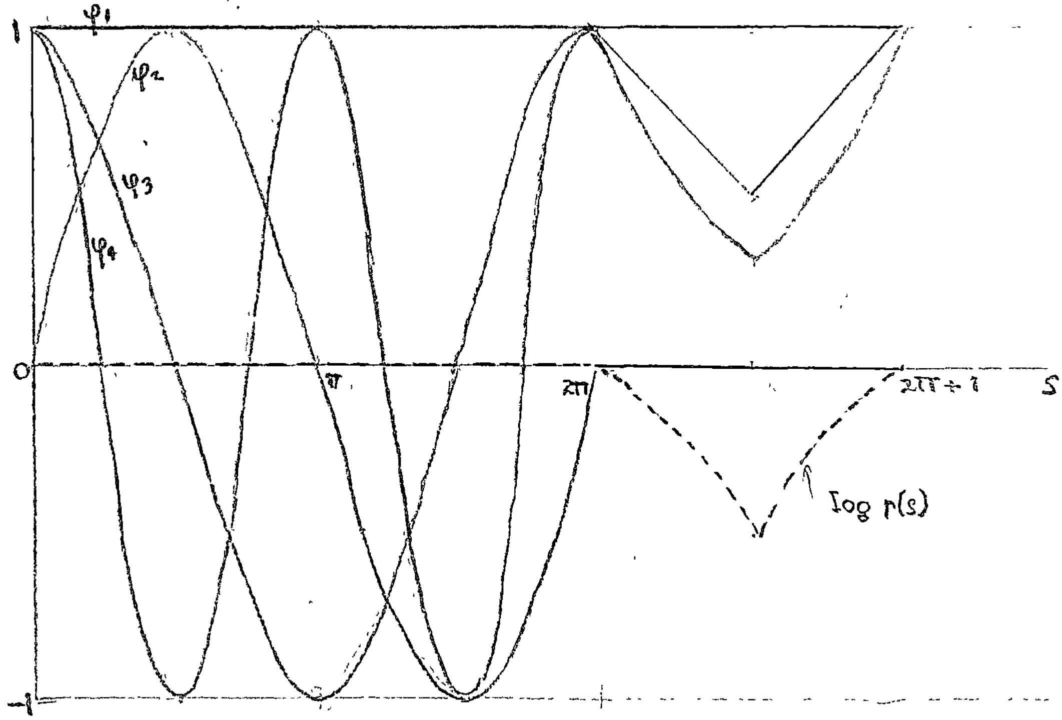
(2) $\{\varphi_n\} = \{1, \cos \nu s, \sin \nu s;\ \nu = 1,2,\ldots\} \quad 0 \leqq s < 2\pi$

and on γ_2

(3) $\{\varphi_n\} = \{1, r(s)^{\nu};\ \nu = 1,2,\ldots\} \qquad 2\pi \leqq s < 2\pi + 1$

where
$$r(s) = \begin{cases} 2\pi + 1 - s, & 2\pi \leqq s < 2\pi + 1/2 \\ s - 2\pi, & 2\pi + 1/2 \leqq s < 2\pi + 1. \end{cases}$$

On γ_1, $r = 1$, $\log r(s) = 0$, $0 \leqq s < 2\pi$, and the set of functions (2) may obviously be used to uniformly approximate this function, the coefficients are all zero. On γ_2 we are required to uniformly approximate the function $\{\log r(s);\ 2\pi \leqq s < 2\pi + 1\}$ by the set of functions (3). Weierstrass' theorem[6)] proves the feasibility of this operation since all the functions concerned are symmetric about the point $s = 2\pi + 1/2$. However it is not at all obvious that the functions (1) can be used to uniformly approximate to log $r(s)$ on γ. The sketch graph drawn below shows the type of functions with which we are concerned.



Transition to Hilbert Space.

Metric Spaces.

A set Ω is called a metric space provided there is defined on the set of all ordered pairs (x,y) of elements of Ω an everywhere finite, real valued function ρ satisfying the postulates[7]

I. For every $x \in \Omega$, $\rho(x,x) = 0$.

II. If $x, y, z \in \Omega$, then

$$\rho(x,y) \leq \rho(z,x) + \rho(z,y).$$

The function ρ is called the distance function. One of the simplest examples of a metric space is the real number system with ρ defined by putting

$$\rho(x,y) = |x - y|.$$

Let $\{x_\nu;\ \nu = 1,2,\ldots\}$ be a sequence of points in a metric space and let x_0 be a point in the space. We say that the sequences converges to x_0 provided that for every positive real number ε there exists a positive integer n_ε such that if $n > n_\varepsilon$,

$$\rho(x_n, x_0) < \varepsilon .$$

Hilbert Space.

A Hilbert space H is a set with elements satisfying the following postulates:

8)

I. The elements of H form a vector space.

II. To every ordered pair of elements of H corresponds a complex number (x,y), called the scalar product of x and y, with the properties:

(i) $(\alpha x, y) = \alpha(x,y)$, α complex.

(ii) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$

(iii) $(x,y) = \overline{(y,x)}$ where $\overline{(y,x)}$ is the complex conjugate of (x,y).

(iv) $(x,x) > 0$ for x different from the null element of H.

The non-negative number $(x,x)^{\frac{1}{2}}$ is called the norm of x and is denoted by $\|x\|$. If the distance from a vector x to a vector y is defined to be $\|x - y\|$, then, with respect to this distance function, H is a metric space. If $(x,y) = 0$, we say x is orthogonal to y, and write $x \perp y$.

III. (Axiom of completeness) If a sequence $\{x_\nu ; \nu = 1,2,..\}$ of elements of H satisfies the condition

$$\lim_{m,n\to\infty} \| x_m - x_n \| = 0$$

then there exists an element x of H such that

$$\lim_{n\to\infty} \| x - x_n \| = 0$$

IV. H is seperable, i.e., there exists an enumerable sequence $\{y_\nu ; \nu = 1,2,..\}$ in H such that, if $\varepsilon > 0$ and $y \in H$ are arbitrarily given, one at least of the elements y_n satisfies

$$\| y_n - y \| < \varepsilon .$$

Function Spaces.

Consider the set S of all real valued functions $f(x)$ defined and bounded for $0 \leqq x \leqq 1$. We can consider the elements of S as forming a vector space if we define $F = f + g$ as

$$\left.\begin{aligned} &F(x) = f(x) + g(x) \\ &\text{and } G = \lambda f \text{ as} \\ &G(x) = \lambda f(x) \end{aligned}\right\} \quad 0 \leqq x \leqq 1$$

The vector space so formed is called a function space. If we now introduce a metric in S the question arises: what does

$$\lim f_n = f$$

mean? Suppose we define the metric as

$$\rho(f,g) = \sup_{0 \leqq x \leqq 1} |f(x) - g(x)|$$

Then if the sequence $\{f_n\}$ converges w.r.t. this metric to f, the sequence of functions $\{f_n(x)\}$ converges uniformly to $f(x)$ for $0 \leqq x \leqq 1$. In Hilbert space the norm is related to the metric,

$$\|f\| = \rho(f,0).$$

Hence if we require a Hilbert space, we must define

$$\|f\| = \sup_{0 \leqq x \leqq 1} |f(x)|,$$

but this norm does not satisfy the parallelogram law[9] and hence cannot be derived from a scalar product. Hence the above metric space is not a Hilbert space.

The Hilbert Space L^2.

If $f(x)$ is a real valued Lebesgue measureable function on the interval (a,b), then we say that $f(x)$ belongs to the Lebesgue function space $L^2(a,b)$ if $\int_a^b f^2(x)\,dx$ exists as a Lebesgue integral. Since countable sets have zero Lebesgue measure we write (a,b) for the closed, half-open or open interval.

In $L^2(a,b)$ the scalar product is defined by

$$(f,g) = \int_a^b f(x)\, g(x)\, dx$$

and thus the norm is given by

$$\| f \| = \left(\int_a^b f^2(x)\, dx \right)^{\frac{1}{2}}$$

The metric is given by

$$\rho(f,g) = \left(\int_a^b (f(x) - g(x))^2 dx \right)^{\frac{1}{2}}$$

and $\qquad \lim f_n = f$

means that

$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x))^2 dx = 0.$$

$L^2(a,b)$ obviously satisfies the first two Hilbert space postulates, the Riesz-Fischer theorem[10] disposes of postulate III. $L^2(a,b)$ also satisfies postulate IV[11] and is thus a Hilbert space.

Complete Sets in Hilbert Space.

Since Hilbert space is a metric vector space, the following definition of completeness of a set in a metric vector space applies to Hilbert space.

If to every element x of a metric vector space E there corresponds a sequence of finite linear combinations of elements of a subset S of E which converges to x, we say that S is complete in E.

The Problem in Hilbert Space.

Consider the problem stated on pp.14-15. C can be considered as a function space. If we define the metric in this space by

$$(1) \qquad \rho(f,g) = \sup_{0\le x\le b} |f(x) - g(x)|$$

then $\qquad \lim f_n = f$

implies that $f_n(x) \to f(x)$ uniformly as $n \to \infty$. Thus we may restate the problem more generally as follows:

Let $\{\lambda_\nu \psi_n;\ n = 1,2,\ldots\}$ be complete in $\{\lambda_\nu f;\ f \in C\}$, $\nu = 1,2$. Does this imply that $\{\psi_n;\ n = 1,2,\ldots\}$ is complete in C?

However C defined with the metric (1) is not a Hilbert space. To obtain the analogous problem in Hilbert space we replace C and the metric (1) by the space $L^2(0,b)$ and its metric. The problem may then be written as follows:

Let $\psi_\nu \in L^2(0,b)$ $(\nu = 1,2,\ldots)$ such that $\{\lambda_\nu \psi_n;\ n = 1,2,\ldots\}$ is complete in $\{\lambda_\nu f;\ f \in L^2(0,b)\}$, $\nu = 1,2$. Does this imply that $\{\psi_n;\ n = 1,2,\ldots\}$ is complete in $L^2(0,b)$?

In order to express this problem in abstract Hilbert space it is necessary to find the abstract generalizations of the functions λ_1, λ_2.

Notation.

We shall use the following Hilbert space notation:

H denotes a Hilbert space over the field F of the real or complex nos

(x, y) denotes the scalar product of the elements x, y of H.

$\|x\|$ is the norm of the element x of H.

θ is the null element of H, O is the set of θ alone.

A transformation on H is a function T which is defined over H and which has one or more values Tf corresponding to each $f \in H$; if $M \subset H$ then $T(M) = \{Tx;\ x \in M\}$ where the symbol $\{\ldots;\ldots\}$ is used for "the set where..." so that $\{\alpha;\ \alpha > 0\}$ is the set of all positive real numbers.

A linear manifold is a non-empty subset M of H such that if $x, y \in M$ then $\alpha x + \beta y \in M$ for every pair of elements α, β of F.

The union of a linear manifold M and the set of its limit points is called a closed linear manifold or a subspace and is denoted by $\bar{M}$.

The dimension of a vector space equals the maximum number of linearly independent vectors in the space, this number is unique.

If S is a subspace of H, $[S]$ denotes the dimension of S, $S^{\perp}$ denotes the set of all elements of H which are orthogonal to every element of S.

If A is a transformation on H and I is the identity transformation, the transformation $I - A$ is denoted by $\underline{A}$ and is called the complement of A.

Definitions.

If M is a subspace of H, every z in H can be written in the form $x + y$, with $x \in M$ and $y \in M^{\perp}$; x is called the projection of z on M and we write

$$P z = x.$$

We define the subspace spanned by an arbitrary subset T of H (the span of T, in symbols $\vee T$) as the intersection of all subspaces containing T, or equivalently, the least subspace containing T.

Characterization of Projections.

The following theorem enables us to determine when a transformation is a projection.

Theorem 1.

A necessary and sufficient condition for a transformation P to be a projection on a subspace M of H is that P has the properties (1), (2), and (3):

(1) P is single valued and linear with domain H.

(2) For every $x, y \in H$

$$(Px, y) = (x, Py)$$

(3) For every $x \in H$

$$P(Px) = Px$$

M is uniquely determined by P.

Proof. Necessity: condition (1) is immediate.

(2) If $x, y \in H$, then

$$x = x_1 + x_2, \quad y = y_1 + y_2$$

where $x_1, y_1 \in M$ and $x_2, y_2 \in M^\perp$. Thus

$$(Px, y) = (x_1, y_1 + y_2) = (x_1, y_1)$$

$$(x, Py) = (x_1 + x_2, y_1) = (x_1, y_1)$$

(3) $$Px = x_1 \qquad \text{and} \qquad PPx = Px_1 = x_1$$

Uniqueness of M: if there exists a subspace M of H such that P is a projection on M, then, for any element x of M, $Px = x$; if $x \notin M$ then $Px \neq x$. Hence M is the set of all solutions of the equation $Px = x$, and $P(H) \supset M$. Now let y be any element of H and let $Py = z$. Then

$$Py = PPy = Pz.$$

Hence $\quad Pz = z \quad$ and $\quad z \in M.$

Hence $P(H) \subset M$ and we have proved that $P(H) = M$.

Sufficiency: Let

$$M = \{x;\ Px = x\}$$

Then $\quad M = P(H).$

Since P is linear, M is linear. By Schwarz' inequality

$$\|Px\| \cdot \|x\| \geq |(Px, x)| = \|Px\|^2$$

If $\|Px\| > 0$ we get

$$\|x\| \geq \|Px\|.$$

But this condition also holds if $\|Px\| = 0$, so that it holds for all $x \in H$. Hence for any $x, y \in H$

$$\|P(x - y)\| = \|Px - Py\| \leq \|x - y\|$$

and P is continuous over H. Hence M is closed, so that M is a subspace. Let $x \in H$. Then

$$x = Px + (x - Px)$$

Since $\quad PPx = Px, \quad$ if $y \in H$

$$0 = (Px - PPx, y) = (x - Px, Py)$$

If y runs through H, Py runs through M. Hence $x - Px \in M^{\perp}$

$Px \in M$ and P is the projection on M.

Hence if P is a projection on M, $\underline{P}$ is a projection on M. Thus P is a projection if and only if $\underline{P}$ is a projection.

Projections in L^2.

Let $a < c < b$ and P_1, P_2 be the characteristic functions of the intervals $a \leqq t < c$, $c \leqq t \leqq b$ respectively. Then P_1, P_2 are single valued with domain $L^2(a,b)$ and for $f, g \in L^2(a,b)$

$$
\begin{aligned}
(P_1 f, g) &= \int_a^b P_1(f(t))\, g(t)\, dt \\
&= \int_a^c f(t)\, g(t)\, dt \\
&= \int_a^b f(t)\, P_1(g(t))\, dt \\
&= (f, P_1 g)
\end{aligned}
$$

$$P_1(P_1 f) = P_1 f$$

Also $$P_1 = I - P_2 = \underline{P}_2$$

Thus by Th.1, P_1, P_2 are complementary projections on $L^2(a,b)$.

The Problem.

We can now say that the generalizations of the functions λ_1, λ_2 in Hilbert space are complementary projections in the space. Thus the problem may be restated:

Given $S \subset H$ and a projection P in H, and that $P(S)$, $\underline{P}(S)$ are complete in $P(H)$, $\underline{P}(H)$ respectively, when is S complete in H ?

Unitary Space.

A Hilbert space of finite dimension is called a unitary space. If H is an n-dimensional unitary space, there exist n linearly independent elements $e_1, \dots, e_n$, forming a complete set in H, and every $x \in H$ may be expressed in the form

$$x = \sum_1^n \alpha_\nu e_\nu$$

where $\alpha_1, \dots, \alpha_n \in F$.

Let $e_1, \dots, e_n$, be orthonormal, then

$$\alpha_\nu = (x, e_\nu)$$

and hence

$$x = \sum_1^n (x, e_\nu)\, e_\nu$$

Suppose $x_\nu \in H$ $(\nu = 1, \dots, n)$. Then we may write

$$x_\lambda = \sum_1^n (x_\lambda, e_\nu)\, e_\nu \qquad (\lambda = 1, \dots, n)$$

Let $c_\nu \in F$ $(\nu = 1, \dots, n)$ such that

$$\sum_1^n c_\nu x_\nu = \theta$$

or equivalently

$$\sum_1^n c_\lambda (x_\lambda, e_\nu) = 0 \qquad (\nu = 1, \dots, n)$$

This set of equations has a ~~only~~ a trivial solution if and only if

(1) $\qquad \det_1((x_r, e_s))_n \neq 0$

Hence $x_1, \dots, x_n$, are linearly independent if and only if (1) is true.

Suppose P is a projection on H such that $e_1, \dots, e_k \in P(H)$ and $e_{k+1}, \dots, e_n \in \underline{P}(H)$. Let $y_1, \dots, y_n \in H$ such that $Py_1, \dots, Py_n$ form a complete set in $P(H)$ and $\underline{P}y_1, \dots, \underline{P}y_n$ form a complete set in $\underline{P}(H)$. Then at least one of the $\binom{n}{k}$ determinants of order k formed from the matrix

$$\begin{bmatrix} (Py_1,e_1) & \cdot & \cdot & (Py_1,e_k) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ (Py_n,e_1) & \cdot & \cdot & (Py_n,e_k) \end{bmatrix}$$

is non-zero and at least one of the $\binom{n}{k}$ determinants of order $n - k$ formed from the matrix

$$\begin{bmatrix} (Py_1,e_{k+1}) & \cdot & \cdot & (Py_1,e_n) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ (Py_n,e_{k+1}) & \cdot & \cdot & (Py_n,e_n) \end{bmatrix}$$

is non-zero. From these two facts it is required to determine whether or not $\det_1((y_r,e_s))_n$ is zero.

We take as example the 3-dimensional Hilbert space with $k = 2$,

$$y_1 = e_1 + e_2 + e_3$$

$$y_2 = e_1 - e_2 - e_3$$

$$y_3 = \quad\quad e_2 + e_3$$

Then $$\begin{vmatrix} (Py_1,e_1) & (Py_1,e_2) \\ (Py_2,e_1) & (Py_2,e_2) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

and $$|(Py_3,e_3)| = 1 \neq 0$$

but $$\det_1((y_r,e_s))_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

and hence $\{Py_1,Py_2\}$, $\{\underline{P}y_3\}$ are complete in $P(H)$, $\underline{P}(H)$ respectively but $\{y_1,y_2,y_3\}$ is not complete in H. Thus, given $S \subset H$, completeness of $P(S)$, $\underline{P}(S)$ in $P(H)$, $\underline{P}(H)$ respectively does not generally imply completeness of S in H.

In future, the statement "... orthogonal to the set ..." will mean "... orthogonal to every element of the set ..."

Lemma 1.

Let $T \subset H$, $x \in H$. Then $x \perp T$ if and only if $x \perp VT$.

Proof. Let $x \perp VT$. $T \subset VT$, hence $x \perp T$.

Let $x \perp T$. Let N be the set of all finite linear combinations of the elements of T. Then $x \perp N$. Let y be a limiting point of N. Then there exists a sequence $\{y_\nu\}$, $y_\nu \in N$, such that

$$\lim \| y_n - y \| = 0.$$

Also, by Schwarz' inequality

$$\begin{aligned} |(x,y)| &= |(x,y_n) - (x,y_n - y)| \\ &= |(x,y_n - y)| \\ &\leq \|x\| \, \|y_n - y\| \end{aligned}$$

Taking limits on both sides we find

$$(x,y) = 0.$$

Hence $\qquad x \perp \bar{N}$

But $\bar{N}$ is the least subspace containing T, i.e.

$$\bar{N} = VT.$$

Hence $\qquad x \perp VT.$

Theorem 2.

Let $T \subset H$. Then the following postulates are equivalent:

I. T is complete in H.

II. $VT = H$.

III. θ is the only element of H which is orthogonal to every element of T.

Proof. Let I hold. Let $Q \subset \{1,2,\ldots\}$ and $T = \{x_\nu ; \nu \in Q\}$. Then for every $y \in H$, there exists $\alpha_\nu^y \in F$ $(\nu \in Q)$ such that

$$y = \sum_{\nu \in Q} \alpha_\nu^y x_\nu$$

Since H is a subspace containing T, $\left\{\sum_{\nu \in Q} \alpha_\nu^y x_\nu ; y \in H\right\}$ is a subspace containing T. But any other subspace containing T must contain this subspace, hence it is the least subspace containing T, i.e. VT. Hence $VT = H$ and II holds.

Obviously II implies I.

Let II hold. Then

$$(VT)^\perp = 0$$

$$T^\perp = 0 \quad \text{(Lemma 1)}$$

and III holds.

Let III hold, then by Lemma 1,

$$(VT)^\perp = 0$$

$$VT = H$$

and II holds.

<u>The Problem.</u>

Th.2 enables us to restate the problem:

Given $S \subset H$ and a projection P in H, and that

$$VP(S) = P(H)\,, \quad V\underline{P}(S) = \underline{P}(H)\,,$$

when is $$VS = H\,?$$

In some cases a solution to this problem may be easier to determine if projections are ignored. Thus we shall develop some theorems concerning conditions for the statement $VS = H$ to be true.

<u>Operators.</u>

A transformation A on H is bounded if there exists a positive real number α such that for every element x of H

$$\| Ax \| \leqq \alpha \, \| x \| \;;$$

the norm of A, in symbols $\| A \|$, is the infimum of all such values of α. <u>An operator</u> is a <u>bounded linear transformation</u> from H into H.

<u>Lemma 2.</u>

If M is a subspace of H and A is an operator on M such that for all $x \in M$ and some positive real number α,

$$\| Ax \| \geqq \alpha \| x \|$$

then $A(M)$ is a subspace of H.

<u>Proof.</u> Let $\{ x_\nu \}$ be a sequence of elements of M such that

$$y_n = Ax_n \to y.$$

$$\|y_n - y_m\| = \|Ax_n - Ax_m\|$$
$$\geqq \alpha\|x_n - x_m\|$$

Hence $\{x_n\}$ is a Cauchy sequence and there exists x such that $x_n \to x$. $x \in M$ since M is closed. But

$$\|y_n - Ax_n - y + Ax\| \leqq \|y_n - y\| + \|A\|.\|x_n - x\|$$

Thus we find

$$0 = \lim (y_n - Ax_n) = y - Ax,$$
$$y = Ax$$

and $A(M)$ is closed.

<u>Lemma 3.</u>

Let $T \subset H$. Let A be an operator on H, and α be a positive real number such that for $x \in VT$

$$\|Ax\| \geqq \alpha\|x\|$$

Then $$A(VT) = VA(T)$$

<u>Proof.</u> Let M, N be the sets of all finite linear combinations of the Ax, x, $(x \in T)$ respectively. Let $x \in VT$. Then

$$Ax \in A(VT)$$

and there exists a sequence $\{y_n\}$ of elements of N such that

$$\lim \|x - y_n\| = 0$$

But

$$0 \leqq \|Ax - Ay_n\| \leqq \|A\|.\|x - y_n\|$$

and hence because A is bounded

$$\lim \|Ax - Ay_n\| = 0$$

Also $Ay_\nu \in H$ and hence

$$Ax \in V\{Ay_\nu\},$$

(1) $\qquad A(VT) \subset VA(T)$

VT is a subspace, thus by lemma 2, $A(VT)$ is a subspace. But since

$$A(T) \subset A(VT)$$

the minimal property of $VA(T)$ implies that

(2) $\qquad VA(T) \subset A(VT)$

and from (1) and (2) the result follows.

<u>Theorem 3.</u>

Let A be an operator on H and α a positive real number such that for all $x \in H$

$$\|Ax\| \geqq \alpha \|x\|$$

Let T be complete in H. Then $A(T)$ is complete in $A(H)$.

<u>Proof.</u> $\qquad VA(T) = A(VT) = A(H)$

By lemma 2, $A(H)$ is a subspace of H, hence $A(H)$ is a Hilbert space. By Th.1, $A(T)$ is complete in $A(H)$.

An operator A is invertible if there exists an operator B, called the inverse of A and written A^{-1}, such that

$$AB = BA = I$$

where I is the identity transformation.

If A is invertible the conditions of Th.3 are satisfied by putting

$$\alpha = \|A^{-1}\|^{-1}$$

An automorphism U on H is a one-to-one linear transformation with H as range and domain such that for every $x, y \in H$

$$(Ux, Uy) = (x, y)$$

The inverse of U is its adjoint(13).

A basis of H is a linearly independent complete set in H.

By Th.3, if there exists an automorphism of $T \subset H$ onto an orthonormal basis of H, T is an orthonormal basis of H.

For example, let U be an operator on $L^2(0,\pi)$ such that for $f \in L^2(0,\pi)$

$$Uf = \lambda_1 f - \lambda_2 f$$

where λ_1, λ_2 are the characteristic functions of the sets $0 \leqq x < \frac{\pi}{2}$, $\frac{\pi}{2} \leqq x \leqq \pi$, respectively. Then

$$\begin{aligned}(Uf, Ug) &= \int_0^{\pi/2} f(x)\, g(x)\, dx + \int_{\pi/2}^{\pi} (-f(x))\,(-g(x))\, dx \\ &= \int_0^{\pi} f(x)\, g(x)\, dx \\ &= (f, g)\end{aligned}$$

and hence U is an automorphism. Thus since $\left\{\sqrt{\frac{2}{\pi}}\sin \nu x;\ \nu = 1, 2, \ldots\right\}$ is an orthonormal basis of $L^2(0,\pi)$, if

$$g_\nu(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\ \sin \nu x; & 0 \leqq x < \pi/2 \\ -\sqrt{\frac{2}{\pi}}\ \sin \nu x; & \frac{\pi}{2} \leqq x \leqq \pi, \end{cases}$$

$\left\{ g_\nu;\ \nu = 1, 2, \ldots \right\}$ is an orthonormal basis of $L^2(0,\pi)$.

If P is a projection on a subspace of H, different from H and 0, there exists $x \in H$ such that

$$\|x\| > 0 \, , \quad \|Px\| = 0.$$

Hence the relation

$$\|Px\| \geqq \alpha \|x\| \, , \; x \in H$$

implies that $\alpha = 0$. Hence Th.3 does not apply to projections.

<u>Theorem 4.</u>

Let $S \subset H$ and S be complete in H. Let P be a projection in H. Then $P(S)$ is complete in $P(H)$.

<u>Proof.</u> Let $y \in P(H)$ such that for all $x \in P(S)$, $(y,x) = 0$. Let $s \in S$ such that $s = x + x'$, where $x \in P(S)$, $x' \in \underline{P}(S)$. Then for every $s \in S$, $(y,s) = (y,x) = 0$. Hence $y = \theta$. See Th.2.

<u>Transitivity of Completeness.</u>

The next theorem shows that completeness is transitive.

<u>Theorem 5.</u>

Let $T \subset H$ and T be complete in H. Then if S is complete in T, S is complete in H.

<u>Proof.</u> Let M, N be the sets of all finite linear combinations of the elements of S, T respectively. Then for every $x \in H$ there exists $y \in M$, $z \in N$ such that for any $\varepsilon > 0$

$$\| x - z \| < \varepsilon/2$$

and $$\| z - y \| < \varepsilon/2$$

$$\begin{aligned}\| x - y \| &= \| (x - z) + (z - y) \| \\ &\leq \| x - z \| + \| z - y \| \\ &< \varepsilon\end{aligned}$$

The result follows from the first definition of completeness.

The Gram-Schmidt Orthogonalization Process.

Most tests for completeness are only applicable to orthonormal sets. As very few sets are not orthonormal we need some method of orthonormalising a set. The method outlined below is due to Gram and Schmidt. It is only applicable to enumerable sets. Suppose we want to orthonormalise the set $S = \{ x_\nu ; \nu = 1, 2, \ldots \}$. We consider the sequence S^* of elements $x_1^*, x_2^*, \ldots$ obtained from S by omitting all elements x_n depending linearly on $x_1, x_2, \ldots, x_{n-1}$. Then every finite subsequence of S^* is linearly independent. The orthonormal system Q with elements $y_1, y_2, \ldots$ is now defined as follows:

$$y_1 = x_1 / \| x_1 \|$$

$$y_2 = z_2 / \| z_2 \|$$

where $$z_2 = x_2^* - (y_1, x_2^*) y_1$$

Generally, if $y_1, y_2, \ldots, y_{n-1}$ are already defined

$$y_n = z_n / \| z_n \|$$

where $$z_n = x_n^* - \sum_{k=1}^{n-1} (y_k, x_n^*) \, y_k$$

To justify this definition of y_n it is necessary to show that $\| z_n \| \neq 0$ on the assumption that the existence of $y_1, \ldots, y_{n-1}$ has already been established. Every y_K is however, a linear combination of $x_1^*, \ldots, x_K^*$ so that z_n may be written in the form

$$z_n = x_n^* - \sum_{K=1}^{n-1} a_K x_K^* \ ;$$

the linear independence of $x_1^*, \ldots, x_K^*$ thus guarantees that $z_n \neq 0$.

We have

$$\begin{aligned}
(y_1, y_2) &= (x_1^*, x_2^* - (y_1, x_2^*)\, y_1)/\| x_1^* \| . \| z_2 \| \\
&= \left((x_1^*, x_2^*) - (x_1^*, x_2^*)\, (x_1^*, x_1^*)/\| x_1^* \|^2 \right)/\| x_1^* \| . \| z_2 \| \\
&= 0
\end{aligned}$$

Suppose $y_1, y_2, \ldots, y_n$ are orthonormal. Then for $m \leq n$

$$\begin{aligned}
(y_m, y_{n+1}) &= (y_m, x_{n+1}^* - \sum_{K=1}^{n} (y_K, x_{n+1}^*)\, y_K)/\| z_{n+1} \| \\
&= \left((y_m, x_{n+1}^*) - (y_m, x_{n+1}^*)\, (y_m, y_m) \right)/\ \| z_{n+1} \| \\
&= 0.
\end{aligned}$$

Hence $y_1, \ldots, y_{n+1}$ are orthonormal. By induction, Q is an orthonormal set.

<u>Orthonormal Sets.</u>

In the following we shall develop some tests for completeness of orthonormal sets in Hilbert space.

Theorem 6.

Let $Q \subset \{1, 2, \ldots\}$ Let $\{\varphi_\nu ; \nu \in Q\}$ be an orthonormal system of elements in H. Then conditions (1) to (4) are equivalent:

(1) $\{\varphi_\nu ; \nu \in Q\}$ is complete in H.

(2) If $x \in H$ is arbitrary, then

$$x = \sum_{\nu \in Q} (x, \varphi_\nu)\varphi_\nu$$

(3) (Parsevals relation). If $x, y \in H$ are arbitrary, then

$$(x, y) = \sum_{\nu \in Q} (x, \varphi_\nu)\,(\varphi_\nu, y)$$

(4) If $x \in H$ is arbitrary, then

$$\|x\|^2 = \sum_{\nu \in Q} |(x, \varphi_\nu)|^2$$

Proof. Let (1) hold. By Bessel's inequality

$$\sum_{\nu \in Q} |(x, \varphi_\nu)|^2 \leq \|x\|^2$$

Hence $\sum_{\nu \in Q} |(x, \varphi_\nu)|^2$ is convergent.

Let Q_1, Q_2 be finite subsets of Q such that $Q_1 \subset Q_2$. Then

$$\lim_{Q_1 \to Q} \Big\|\sum_{\nu \in Q_1}(x, \varphi_\nu)\varphi_\nu - \sum_{\nu \in Q_2}(x, \varphi_\nu)\varphi_\nu\Big\|^2 = \lim_{Q_1 \to Q} \sum_{\nu \in Q_2 - Q_1} |(x, \varphi_\nu)|^2$$
$$= 0$$

Hence the sequence of elements on the L.H.S. of the above is a Cauchy sequence and there exists $x_0 \in H$ such that

$$x_0 = \sum_{\nu \in Q} (x, \varphi_\nu)\varphi_\nu$$

Hence for $\nu \in Q$

$$(x_0, \varphi_\nu) = (x, \varphi_\nu),$$
$$(x_0 - x, \varphi_\nu) = 0$$

and $x_0 = x$. Hence (2) holds.

Let (2) hold. Then

$$\lim_{Q_1 \to Q} \sum_{\nu \in Q_1} (x, \varphi_\nu)\varphi_\nu = x$$
$$\lim_{Q_1 \to Q} \sum_{\nu \in Q_1} (y, \varphi_\nu)\varphi_\nu = y$$

and in consequence

$$\begin{aligned}(x,y) &= \lim_{Q_1 \to Q} \sum_{\nu \in Q_1} (x, \varphi_\nu)\, \overline{(y, \varphi_\nu)} \\ &= \sum_{\nu \in Q} (x, \varphi_\nu)\, (\varphi_\nu, y)\end{aligned}$$

and (3) holds.

Let (3) hold. Put $x = y$ and (4) follows immediately.

Let (4) hold. Assuming $\{\varphi_\nu ; \nu \in Q\}$ to be not complete, there exists an element z of H different from θ such that

$$(z, \varphi_\nu) = 0 \qquad (\nu \in Q)$$

and hence from (4),

$$\| z \|^2 = 0$$

which is a contradiction. Hence (1) holds and we have proved that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

<u>Theorem 7.</u>

Let $Q \subset \{1, 2, \ldots\}$. Let $y \in H$ such that for some complete set S in H, we have for all $x \in S$

$$(y,x) \neq 0$$

Let P_x be the projection on $V\, x ; (x \in S)$, and $\{\varphi_\nu ; \nu \in Q\}$ be an orthonormal set in H.

Then the necessary and sufficient condition for $\{\varphi_\nu ; \nu \in Q\}$ to be a basis of H is that for all $x \in S$

(1) $$\| P_x y \|^2 = \sum_{\nu \in Q} |(P_x y, \varphi_\nu)|^2$$

<u>**Proof.**</u> Necessity: $P_x y \in H$, hence (1) follows immediately from Parseval's relation.

Sufficiency: $P_x y \in Vx \qquad (x \in S)$

Hence there exists a real number α_x such that

$$P_x y = \alpha_x x \qquad \text{and}$$

$$(y, x) = (P_x y, x) = \alpha_x \|x\|^2$$

Hence $\alpha_x \neq 0$ and $\{P_x y;\ x \in S\}$ is complete in H.

From (1) we see that $\{\varphi_\nu ;\ \nu \in Q\}$ is complete in $\{P_x y;\ x \in S\}$.

By the transitivity of completeness, $\{\varphi_\nu ;\ \nu \in Q\}$ is complete in H.

<u>Lemma 4.</u>

Let the sequence $r_1, r_2, \ldots$ represent all the rational numbers in the interval (a,b). Then the functions defined by

$$s_{r_\nu}(x) = \begin{cases} 1 & \text{for } a \leqq x \leqq r_\nu \\ 0 & \text{for } r_\nu < x \leqq b \end{cases} \qquad (\nu = 1, 2, \ldots)$$

form a complete set in $L^2(a,b)$.

<u>**Proof.**</u> Let $g \in L^2(a,b)$ such that g is orthogonal to all the functions s_{r_ν} $(\nu = 1, 2, \ldots)$. Then

$$0 = \int_a^b g(x)\, s_{r_\nu}(x)\, dx = \int_a^{r_\nu} g(x)\, dx \qquad (\nu = 1, 2, \ldots)$$

i.e. the function

$$\Phi(x) = \int_a^x g(t)\, dt$$

is zero at all rational points of (a,b). But $\Phi(x)$ is continuous since it is a function of the upper limit of an integrable function. Hence $\phi(x) \equiv 0$ in (a,b) and $g(x) \equiv 0$ in (a,b). Thus the null function is the only element of $L^2(a,b)$ which is orthogonal to all of the s_{t_ν} $(\nu = 1,2,\dots)$. Q.E.D.

<u>Vitali and Dalzell Conditions for Completeness.</u>

In Th.7, put $H = L^2(a,b)$, $y = f$ where $f(x) > 0$ for $a \leq x \leq b$,
$S = \{ s_t ; a < t \leq b \}$ where $s_t(x)$ is the characteristic function of $a \leq x \leq t$.

Then by 14., S is complete in $L^2(a,b)$.

Also $$P_{S_t}(f(x)) = s_t(x)\, f(x) \qquad a \leq x \leq b$$

Hence the necessary and sufficient condition for the orthonormal set of functions $\varphi_\nu(x)$ $(\nu = 1,2,\dots)$ to be a basis of $L^2(a,b)$ is

(2) $$\int_a^t f^2(x)\, dx = \sum_1^\infty \left(\int_a^t f(x)\, \varphi_\nu(x)\, dx \right)^2 \qquad a \leq t \leq b.$$ (4)

If we put $f(x) \equiv 1$ we get Vitali's condition for completeness. We could replace the condition $f(x) > 0$ in the above by the condition: $f(x) > 0$ in any subset of (a,b) of non-zero measure, or: $\int_a^x f(x)\, dx = 0$ only if $x = a$.

The necessity of the condition stated in the following theorem follows almost immediately from (2).

Theorem 8.

Let $f \in L^2(a,b)$ and $f(x) > 0$ for $a \leq x \leq b$. Then the necessary and sufficient condition for an orthonormal set $\{\varphi_v ; v = 1,2,\dots\}$ of elements of $L^2(a,b)$ to be complete in $L^2(a,b)$ is

$$(3) \qquad \int_a^b \int_a^t f^2(x)dxdt = \sum_{v=1}^{\infty} \int_a^b \left(\int_a^t f(x)\, \varphi_v(x)dx \right)^2 dt$$

Dalzell's condition for completeness may be obtained from (3) by putting $f(x) \equiv 1$.[15]

Proof. Necessity: equation (2) holds, the terms on the R.H.S. are continuous, non-negative functions of t ; also the L.H.S. is a continuous function of t. Hence the series is uniformly convergent and integrating both sides we may interchange summation and integral signs on the R.H.S. to get equation (3).

Sufficiency: put

$$\Delta(t) \equiv \int_a^t f^2(x)dx - \sum_1^{\infty} \left(\int_a^t f(x)\, \varphi_v(x)dx \right)^2$$

Then (3) can be written

$$\int_a^b \Delta(t)dt = 0$$

Let $\lambda_t(x)$ be the characteristic function of $a \leq x \leq t$.

Then by Bessel's inequality

$$\int_a^b (\lambda_t(x)\, f(x))^2 dx \geq \sum_{v=1}^{\infty} \left(\int_a^b \lambda_t(x)\, f(x)\varphi_v(x)dx \right)^2$$

$$\text{i.e.} \quad \int_a^t f^2(x)dx \geq \sum_1^{\infty} \left(\int_a^t f(x)\, \varphi_v(x)dx \right)^2$$

$$\text{"} \qquad \Delta(t) \geq 0$$

Assuming that $\Delta(t)$ is continuous, since $\Delta(a) = 0$, we have

$$\Delta(t) \equiv 0$$

and sufficiency follows from the fact that (2) is sufficient for completeness.

Proving the continuity of $\Delta(t)$ reduces to proving the continuity of the series $\sum_{\nu=1}^{\infty}\left(\int_a^t f(x)\varphi_\nu(x)dx\right)^2$ for $a \leqq t \leqq b$.

Putting

$$a_{ij} = \int_a^{t_j} f(x)\varphi_i(x)\,dx \qquad (j = 1,2.)$$

we find, using Schwarz' inequality,

$$\left(\sum_1^\infty a_{i1}^2 - \sum_1^\infty a_{i2}^2\right)^2 = \left(\sum_1^\infty (a_{i1} + a_{i2})(a_{i1} - a_{i2})\right)^2$$
$$\leqq \sum_1^\infty (a_{i1} + a_{i2})^2 \sum_1^\infty (a_{i1} - a_{i2})^2$$
$$\leqq 2\sum_1^\infty (a_{i1}^2 + a_{i2}^2) \sum_1^\infty (a_{i1} - a_{i2})^2$$

By Bessel's inequality

$$\sum_{i=1}^\infty a_{ij}^2 \leqq \int_a^{t_j} f^2(x)\,dx \leqq \int_a^b f^2(x)\,dx$$

$\lambda_{t_j}(x)$ is the characteristic function of $a \leqq x \leqq t_j$, hence

$$\sum_{i=1}^\infty (a_{i1} - a_{i2})^2 = \sum_{i=1}^\infty \left\{\int_a^b (\lambda_{t_1}(x) - \lambda_{t_2}(x))\, f(x)\,\varphi_i(x)\,dx\right\}^2$$
$$\leqq \int_a^b (\lambda_{t_1}(x) - \lambda_{t_2}(x))^2 f^2(x)\,dx$$
$$= \left|\int_{t_1}^{t_2} f^2(x)\,dx\right|$$

Hence

$$\left|\sum_{i=1}^\infty a_{i1}^2 - \sum_{i=1}^\infty a_{i2}^2\right| \leqq 2\sqrt{\int_a^b f^2(x)\,dx \cdot \left|\int_{t_1}^{t_2} f^2(x)\,dx\right|}$$

and thus the L.H.S. $\to 0$ as $t_1 \to t_2$. Hence $\Delta(t)$ is continuous, and the theorem is proved.

Tests for Completeness Applicable to Non-orthonormal Sets.

In the following we shall develop some general criteria for completeness or non-completeness of sets in Hilbert space.

Theorem 9.

Let $Q, Q', Q'' \subset \{1, 2, \ldots\}$. Let $\{Q_n;\ n \in Q''\}$ be a monotonic increasing sequence of finite subsets of Q with limit Q. Let P be a projection in H such that $P \neq I$ or 0. Let $x_\nu \in H$ $(\nu \in Q)$ such that

$$(1) \qquad V\{Px_\nu;\ \nu \in Q\} = P(H).$$

Let $\quad y_k \in H \qquad\qquad (k \in Q')$

and $\quad z_{kn} \in V\{x_\nu;\ \nu \in Q_n\} \quad (n \in Q'')$

such that

$$(2) \qquad \lim_{Q_n \to Q} \| Py_k - Pz_{kn} \| = 0 \qquad\qquad (k \in Q')$$

Then if $\quad \lim_{Q_n \to Q} \underline{P}z_{kn} = z_k$ exists,

$$(3) \qquad V\{\underline{P}y_k - z_k;\ k \in Q'\} = \underline{P}(H)$$

implies

$$V\{x_\nu, y_k;\ \nu \in Q,\ k \in Q'\} = H.$$

Proof. Let $\varphi \in H$ such that

$$(\varphi, x_\nu) = 0 = (\varphi, y_k) \qquad\qquad (\nu \in Q,\ k \in Q')$$

We must prove that $\varphi = \theta$. We have

$$(P\varphi, Px_\nu) = -(\underline{P}\varphi, \underline{P}x_\nu),$$

$$(P\varphi, Py_K) = -(\underline{P}\varphi, \underline{P}y_K) \qquad (k \in Q')$$

Hence

$$(P\varphi, Pz_{kn}) = -(\underline{P}\varphi, \underline{P}z_{kn}) \qquad (n \in Q'', k \in Q')$$

$$|(\underline{P}\varphi, \underline{P}y_k - \underline{P}z_{kn})|^2 = |(P\varphi, Py_k - Pz_{kn})|^2 \qquad ''$$

$$\leq \|P\varphi\| . \|Py_k - Pz_{kn}\| \qquad ''$$

By (3)

$$\lim_{Q_n \to Q} (\underline{P}\varphi, \underline{P}y_k - \underline{P}z_{kn}) = 0 \quad \text{or} \qquad ''$$

$$(5). \qquad (\underline{P}\varphi, \underline{P}y_K) = \lim_{Q_n \to Q} (\underline{P}\varphi, \underline{P}z_{kn}) \qquad ''$$

Since

$$0 \leq \| \underline{P}z_{kn} - \underline{P}(\lim_{Q_n \to Q} z_{kn})\| \leq \|z_{kn} - \lim_{Q_n \to Q} z_{kn}\|, \qquad ''$$

$$\lim_{Q_n \to Q} \| \underline{P}z_{kn} - \underline{P}(\lim_{Q_n \to Q} z_{kn})\| = 0 \qquad ''$$

But

$$|(\underline{P}\varphi, \underline{P}z_{kn}) - (\underline{P}\varphi, \lim_{Q_n \to Q} \underline{P}z_{kn})|^2 = |(\underline{P}\varphi, \underline{P}z_{kn} - \lim_{Q_n \to Q} \underline{P}z_{kn})|^2 \qquad ''$$

$$\leq \|\varphi\| . \|\underline{P}z_{kn} - \lim_{Q_n \to Q} \underline{P}z_{kn}\| \qquad ''$$

Hence

$$\lim_{Q_n \to Q} (\underline{P}\varphi, \underline{P}z_{kn}) = (\underline{P}\varphi, \lim_{Q_n \to Q} \underline{P}z_{kn}) \qquad ''$$

and from (5)

$$(\underline{P}\varphi, \underline{P}y_K) = (\underline{P}\varphi, \lim_{Q_n \to Q} \underline{P}z_{kn}) \qquad ''$$

$$(\underline{P}\varphi, \underline{P}y_K - \lim_{Q_n \to Q} \underline{P}z_{kn}) = 0 \qquad ''$$

From (3) and Th.3, $\quad \underline{P}\varphi = \Theta$

From (4) $\quad (P\varphi, Pz_\nu) = 0 \qquad (\nu \in Q)$

By (1) $\quad P\varphi = \Theta$

Hence $\quad \varphi = \Theta$

Th.9 places no restrictions on the dimension of H, P(H) or $\underline{P}(H)$, hence it is true if $[H]$ is infinite and $[P(H)]$ is finite, or if $[H]$ is infinite and $[P(H)]$, $[\underline{P}(H)]$ are both infinite. It is also true if $[H]$ is finite.

We will now state a theorem which will be used to explain some examples later on.

Muntz-Szasz Theorem.[6]

In order that the set of functions $\{t^p ; 0 \leq t \leq 1\}$, be complete in $L^2(0,1)$, it is necessary and sufficient that the sequence of powers $\{p\}$ contains a sequence $\{p_\nu\}$ satisfying one of the conditions (1), (2) or (3):

(1) $\lim\limits_{\nu \to \infty} p_\nu = p_\infty$ is finite and $p_\infty > -1/2$

(2) $\lim\limits_{\nu \to \infty} p_\nu = -1/2$ and $\sum\limits_{\nu=1}^{\infty} | p_\nu + 1/2 | = \infty$

(3) $p_\nu \neq 0$, $\lim\limits_{\nu \to \infty} p_\nu = \infty$ and $\sum\limits_{\nu=1}^{\infty} 1/p_\nu = \infty$

Redundant Sets.

We define the following types of complete sets in infinite dimensional Hilbert space.

(i) A complete set S is finitely redundant if there exists a subset which is a basis and which has a finite complement w.r.t. S. A basis is finitely redundant.

e.g. $\{1, x, x^2, \sin x, \sin 2x, \ldots\}$ has three redundant elements in $L^2(0,\pi)$ since $\{\sin x, \sin 2x, \ldots\}$ is a basis of $L^2(0,\pi)$.

(ii) A complete S is infinitely redundant if there exists a subset which is a basis and which has an infinite complement w.r.t. S.

e.g. $\{\cos \nu x, \sin \nu x; \nu = 1,2,\ldots\}$ has an infinite number of redundant elements in $L^2(0,\pi)$.

(iii) A complete set is <u>essentially redundant</u> if there exists no subset which is a basis. This class of sets consists of two subclasses:

a) there exist infinite complete subsets and there exist infinite non-complete subsets.

e.g. Let $f_n = \{x^n; 0 \leq x \leq 1\}$

Then by condition (3) of M-S Th., $\{f_\nu; \nu = 2,4,6,8,\ldots\}$ is complete in $L^2(0,1)$ and $\{f_\nu; \nu = 2,4,8,16,\ldots\}$ is not complete in $L^2(0,1)$.

b) all infinite subsets are complete.

e.g. Let $g_n = \{x^{1/n}; 0 \leq x \leq 1\}$

Then by condition (3) of M-S Th., if Q is an infinite set of positive integers, $\{g_\nu; \nu \in Q\}$ is complete in $L^2(0,1)$.

(iv) A complete set is transfinitely redundant if it is non-denumerable and has a denumerable complete subset.

e.g. In the space $L^2(0,1)$, let $c_p(x)$ be the characteristic function of the set $0 \leq x \leq p$. Then $\{c_p; 0 \leq p \leq 1\}$ is complete and non-denumerable in $L^2(0,1)$ and $\{c_p; p \text{ rational}\}$ is complete and countable in $L^2(0,1)$.

<u>Theorem 10.</u>

Let H be of infinite dimension. Let $Q = \{1, 2, \dots\}$. Let P be a projection in H and x_ν $(\nu \in Q)$ be elements of H such that

$\{Px_\nu;\ \nu \in Q\}$ is finitely redundant in $P(H)$ and

$\{\underline{P}x_\nu;\ \nu \in Q\}$ is finitely redundant in $\underline{P}(H)$.

Then

$$\left[\left(V\{x_\nu;\ \nu \in Q\}\right)^{\perp}\right] = \infty$$

and hence $\{x_\nu;\ \nu \in Q\}$ is not complete in H.

<u>Proof.</u> There exist $Q_1, Q_2 \subset Q$ such that

$\{Px_\nu;\ \nu \in Q_1\}$ is a basis of $P(H)$ and

$\{\underline{P}x_\nu;\ \nu \in Q_2\}$ is a basis of $\underline{P}(H)$.

Let $Q' = Q_1 \cap Q_2$, then $Q - Q'$ is a finite set.

There exists $\varphi_{\mu 1} \in P(H)$, $\varphi_{\mu 2} \in \underline{P}(H)$ and a non zero complex number α_μ such that

$$(\varphi_{\mu 1}, Px_\nu) = \alpha_\mu \delta_{\mu\nu} \qquad (\mu, \nu \in Q')$$

$$(\varphi_{\mu 2}, \underline{P}x_\nu) = -\alpha_\mu \delta_{\mu\nu} \qquad \text{"}$$

and hence

$$(\varphi_{\mu 1}, Px_\nu) + (\varphi_{\mu 2}, \underline{P}x_\nu) = 0 \qquad \text{"}$$

Put $\qquad \varphi_\mu = \varphi_{\mu 1} + \varphi_{\mu 2} \qquad$ "

Then $\qquad P\varphi_\mu = \varphi_{\mu 1}, \ \underline{P}\varphi_\mu = \varphi_{\mu 1}$

and hence

$$(\varphi_\mu, x_\nu) = (\varphi_{\mu 1} + \varphi_{\mu 2}, Px_\nu + \underline{P}x_\nu) \qquad \text{"}$$

$$= (\varphi_{\mu 1}, Px_\nu) + (\varphi_{\mu 2}, \underline{P}x_\nu) \qquad \text{"}$$

$$= 0$$

Let Q'' be a finite subset of Q' and $a_\nu(\nu\in Q'')\in F$ such that

$$\sum_{\nu\in Q''} a_\nu\varphi_\nu = \theta$$

Then
$$P(\sum_{\nu\in Q''} a_\nu\varphi_\nu) = \sum_{\nu\in Q''} a_\nu\varphi_{\nu 1} = \theta$$

and hence

$$\begin{aligned}(\sum_{\mu\in Q''} a_\mu\varphi_{\mu 1}, Px_\nu) &= 0 && (\nu\in Q'')\\ &= \sum_{\mu\in Q''} a_\mu(\varphi_{\mu 1}, Px_\nu) && "\\ &= \sum_{\mu\in Q''} a_\mu\alpha_\mu\delta_{\mu\nu} && "\\ &= a_\nu\alpha_\nu && "\end{aligned}$$

Since $\alpha_\nu\neq 0$, $a_\nu = 0$ and $\{\varphi_\nu ; \nu\in Q''\}$ is linearly independent.
Hence $[V\{x_\nu ; \nu\in Q'\}] = \infty$
and $V\{\varphi_\nu ; \nu\in Q'\}$ is orthogonal to $V\{x_\nu ; \nu\in Q'\}$. Hence

$$[(V\{x_\nu ; \nu\in Q'\})^\perp] = \infty$$
$$[(V\{x_\nu ; \nu\in Q\})^\perp] = \infty$$
$$V\{x_\nu ; \nu\in Q\} \neq H.$$

It follows from the above theorem, that any set of functions which is orthogonal in $L^2(a,b)$ and in $L^2(b,c)$ cannot be complete in $L^2(a,c)$. For example, the sine fuctions form an orthonormal and complete set in $L^2(0,\pi)$, $L^2(\pi,2\pi)$, but are not complete in $L^2(0,2\pi)$.

Theorem 11.

Let H be of infinite dimension. Let $S\subset H$ such that $VS = H$, and P be a projection in H. Then it cannot occur that $P(S)$ is finitely redundant in $P(H)$ and $\underline{P}(S)$ is finitely redundant in $\underline{P}(H)$. These two redundacies are at least equal to the redundancy of S in H.

Hence if S is not finitely redundant in H, $P(S)$, $\underline{P}(S)$ are not finitely redundant in $P(H)$, $\underline{P}(H)$ respectively, and if one of the above projections of S forms a basis of its range, S is a basis of H.

Proof. Suppose S is more redundant in H than $P(S)$ is in $P(H)$. Discard the redundant elements of S to give S' which is a basis of H. Then by Th.4, $P(S')$ is complete in $P(H)$. But by assumption $P(S')$ is not complete in $P(H)$ and we have a contradiction. Hence the contrary statement is true. The proof of the remaining assertion follows from Ths.4 and 10.

Let L be the set of Legendre polynomials. Then L forms an orthogonal basis of $L^2(-1,1)$ and is (by Weierstrass' theorem and the transitivity of completeness) complete in $L^2(-1,2)$. By Th.11 L forms a basis of $L^2(-1,2)$ and is complete but not finitely redundant in $L^2(1,2)$.

Let T be the set of trigonometric functions. Then T forms an orthogonal basis of $L^2(0,2\pi)$ and is infinitely redundant in $L^2(0,\pi)$ and $L^2(\pi,2\pi)$.

The above two examples show that in general we cannot replace "finitely redundant" in the statement of Th.10 by "infinitely redundant" or the second "finitely redundant" by "complete but not finitely redundant".

<u>Theorem 12.</u>

Let P be a projection in H. Let $S \subset H$ such that $P(S)$, $\underline{P}(S)$ are complete in $P(H)$, $\underline{P}(H)$ respectively, and for some $y \in H - \theta$ we have for all $x \in S$

$$(Px, y) = (\underline{P}x, y)$$

Then S is not complete in H.

<u>Proof.</u> For all $x \in S$

$$(x, Py) = (x, \underline{P}y)$$

Put $\quad Py = \varphi_1, \quad -\underline{P}y = \varphi_2, \quad \varphi_1 + \varphi_2 = \varphi$

Then $\varphi_1 \in P(H)$, $\varphi_2 \in \underline{P}(H)$, $\varphi \in H$.

$\varphi_1 - \varphi_2 = y \neq \theta$, hence not both φ_1, φ_2 equal θ.

Hence $\varphi \neq \theta$ and for all $x \in S$, $(x, \varphi) = 0$.

The above theorem is true no matter what type of redundancy the sets $P(S)$, $\underline{P}(S)$ exhibit in $P(H)$, $\underline{P}(H)$ respectively. Hence we cannot deduce from the types of redundancy of two complete sets, say $P(T)$ and $\underline{P}(T)$, that T is complete in H.

Th.12 also proves that the following types of sets are not complete in Lebesgue function spaces.

(1) The elements are periodic with common period lessthan the length of their domain. Hence the functions 1, $\cos \nu x$, $\sin \nu x$ $(\nu = 1, 2, \ldots)$ do not form a complete set in $L^2(0, 2\pi + \varepsilon)$ $\varepsilon > 0$.

(ii) The elements are symmetric or skew symmetric about a point in some subset (of non-zero measure) of their domain. Hence the even or odd powers of x are not complete in $L^2(-1,1)$.

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1) I, p. 243.

2) VI, p. 246.

3) IX, p. 166.

4) IX, pp. 407, 410.

5) X, pp 48-49 and VIII, p. 304, 14.71.

6) IV, p. 286.

7) VII, p. 24.

8) II, p. 1.

9) Ann. Math., 2nd series, vol. 36, no. 3, July 1935, pp. 719-725.

10) VIII, p. 58.

11) XI, p. 75.

12) II, p. 10.

13) III, p. 43.

14) X, p. 28

15) X, p. 39.

16) V, pp. 86-91.