SOME TOPICS IN THE THEORY OF RING STRUCTURES ON ABELIAN GROUPS

by

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Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any university, and to the best of my knowledge and belief, contains no copy or paraphrase of material previously published or written by another person except where duly acknowledged.

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ABSTRACT

In recent years, Fuchs has described the absolute annihilator and the absolute (Jacobson) radical of a torsion group, and Gardner has characterised the absolute annihilator of a completely decomposable torsion-free group. In this thesis the problem of describing the absolute annihilator and the absolute radical of certain abelian groups is considered. This will involve a discussion of the rings on these groups, and the information so obtained allows us to answer several other questions from the theory of ring structures on abelian groups.

Complete descriptions of the absolute annihilator are given for vector groups, separable groups, certain mixed groups of torsion-free rank one, reduced algebraically compact groups, cohesive groups, and reduced groups whose quotients mod torsion subgroups are divisible. Partial characterisations are also provided for cotorsion groups, and torsion-free groups of rank two. For the absolute radical of a group, complete descriptions are provided for certain mixed groups of torsion-free rank one, reduced algebraically compact groups, certain strongly indecomposable torsion-free groups of finite rank, and partial descriptions are given for completely decomposable torsion-free groups, cotorsion groups, torsion-free groups of rank two, and cohesive groups.

The properties of rings on some of the forementioned torsion-free groups lead us to consider various aspects of nilpotence. Of particular interest are the T-nilpotent rings on completely decomposable torsion-free groups. A bound is also provided for the nil-degree, if it is finite, of certain torsion-free groups. The mixed groups of torsion-free rank one discussed in this thesis motivate an investigation of the additive group of a regular ring. A question of Fuchs concerning these groups is answered in the negative.

CONTENTS

INTRODUCTION

Throughout this thesis the word 'group' will always designate an additively written abelian group, and the word 'ring' will always designate a not necessarily associative ring. A *ring on a group* A is a ring whose additive group is (isomorphic to) A.

In 1948, Beaumont [1] investigated rings on direct sums of cyclic groups, and thereby initiated the more general problem of defining ring structures on an arbitrary group. Shortly afterwards Szele [1] studied trivial rings (rings in which the product of every pair of elements is equal to zero), and several years later Ree and Wisner [1], inspired by Szele's work, described the completely decomposable torsion-free groups that support only the trivial ring structure. In 1956, Fuchs [1] provided a more detailed account of constructing rings on a group. In part of his paper, Fuchs demonstrated a strong connection between a ring structure on a torsion group A and the partial multiplication on a basic subgroup of A. This knowledge enabled Fuchs to completely describe the absolute annihilator, and later in Fuchs [4], the absolute (Jacobson) radical of a torsion group. Recently, Gardner [5] has described the absolute annihilator of a completely decomposable torsion-free group.

In this thesis we investigate the structure of the absolute annihilator and the absolute radical of certain groups. In so doing we discuss the rings supported by these groups, and as a consequence are able to answer various other questions from the theory of ring structures on abelian groups.

After detailing a number of standard concepts, Chapter One introduces the absolute annihilator and the absolute radical of a group, and outlines the present knowledge of the structure of these subgroups. The remainder of the chapter deals with some results concerning these two concepts that will be useful in later work.

In Chapter Two the results of Ree and Wisner [1] and Gardner [5] are generalised to two other classes of torsion-free groups; the class of vector groups and the class of separable groups. We also provide some necessary and sufficient conditions for an arbitrary torsion-free group to have the property that it supports only the trivial ring.

In Chapter Three we describe the absolute radical of certain completely decomposable torsion-free groups. In order to do this a relation $\leqq'$ defined on a subset of the type set of a completely decomposable torsion-free group is introduced. It is shown that this relation has a significant connection with the T-nilpotent rings of Levitzki [1] and Bass [1]. More importantly however, is that if A is an arbitrary completely decomposable torsion-free group then the relation $\leqq'$ allows us to write A in a form that is different from the usual representation of A as a direct sum of rational groups. With this form of A we are able to characterise the absolute radical of A when the relation $\leqq'$ satisfies a certain chain condition.

In Chapter Four we initially concentrate our attention on a class $\mathcal{A}$ of mixed groups of torsion-free rank one. After discussing some of the rings on groups in this class, complete descriptions of the absolute annihilator and absolute radical of groups in $\mathcal{A}$ are given. The remainder of the chapter is concerned with other mixed groups that have properties similar to the groups in $\mathcal{A}$. Most notable amongst these is the reduced part of the additive group of a regular ring. A question of Fuchs [1] concerning these groups is answered in the negative. Partial descriptions of the absolute annihilator and the absolute radical of a cotorsion group, and complete descriptions of the absolute annihilator and the absolute radical of a reduced algebraically compact group are also provided.

In Chapter Five we turn our attention once again to torsion-free groups. Reid [1] has investigated associative rings on strongly indecomposable torsion-free groups of finite rank. Combining some of Reid's results with several of Beaumont and Pierce [1, 2], we obtain a partial description of the absolute radical of a strongly indecomposable torsion-free group of finite rank. We then discuss several classes of rings on strongly indecomposable torsion-free groups of finite rank. These include the almost nilpotent rings of van Leeuwen and Heyman [1], the unequivocal rings of Gardner [2], the rings with the finite norm property of Levitz and Mott [1], and the rings with the restricted minimum condition. The chapter concludes with a discussion of associative rings on a torsion-free group of rank two. A result of Freedman [1] is generalised, and a complete description of the absolute associative annihilator of a rank two torsion-free group is given.

In the first part of the final chapter we investigate rings on cohesive groups, and then describe the absolute annihilator of such a group. A partial description of the absolute radical of a cohesive group is also given. In the remainder of the chapter we generalise a recent result of Webb [1] by providing a bound for the nil-degree (if it is finite) of a torsion-free group A, not necessarily of finite rank, but with certain finiteness conditions on the rank of A/pA, for each prime p. This involves a discussion of the embedding of A in its p-adic completion $\hat{A}_{(p)} = \varprojlim_k A/p^kA$, for each prime p.

CHAPTER ONE

The purpose of this chapter is to introduce some basic definitions and results that will be used throughout the remainder of the thesis. We commence with some standard properties of groups and rings and then introduce the nil groups, the absolute annihilator of a group, and the absolute radical of a group. After surveying the current knowledge of these three concepts we conclude the chapter with some results that will prove to be particularly useful in later work.

1. BASIC DEFINITIONS AND NOTATION

Throughout this thesis the symbol A will be reserved for an Abelian group. T(A) will denote the torsion subgroup of A, and for each prime p, A_p will denote the p-component of A. The set of all primes is denoted by P, and a prime $p \in P$ is called a *relevant prime* of A if A_p is non-zero. We use the standard notation Q, Z, and for a prime p, $Z(p^\infty)$, $Z(p^k)$ and J_p for the group of rationals, the group of integers, the quasi-cyclic p-group, the cyclic p-group of order p^k and the group of p-adic integers, respectively.

A subgroup B of A is said to be *pure* in A if $nB = nA \cap B$ for all positive integers n. If S is a subset of A then $\langle S \rangle$ denotes the subgroup of A generated by S, and if A is torsion-free $\langle S \rangle_*^A$ (or simply $\langle S \rangle_*$ when the context is clear) denotes the unique minimal pure subgroup of A containing S; $\langle S \rangle_*$ is just the set of all elements of A that depend on S.

If A is a torsion-free group, $Q \otimes A$ is the *divisible hull* of A, and the *rank* of A, denoted by r(A), is the Q-rank of $Q \otimes A$. If A is a mixed group then the rank of the torsion-free group A/T(A) is called the *torsion-free rank* of A.

For each prime p and each ordinal σ the subgroup $p^\sigma A$ of A is defined inductively in the following manner: $p^0A = A$, $p^{\sigma+1}A = p(p^\sigma A)$ and $p^\sigma A = \bigcap_{\rho<\sigma} p^\rho A$ when σ is a limit ordinal. The *maximal p-divisible subgroup* of A is $p^\tau A$ where τ is the minimal ordinal such that $p^\tau A = p^{\tau+1}A$. These subgroups of A allow us to define the *p-height* $h_p^A(a)$ of an element a in A. If $a \in p^\tau A$ then $h_p^A(a) = \infty$, otherwise $h_p^A(a)$ is the unique ordinal σ such that $a \in p^\sigma A \backslash p^{\sigma+1}A$. If there is no danger of confusion we shall simply write $h_p(a)$ for the p-height of a in A. The subgroup of A consisting of all the elements of A with p-height an infinite ordinal for every prime p is called the *first Ulm subgroup* of A, and is denoted by A^1.

The *height matrix* $H_A(a)$ of an element a in the group A is defined as

$$H_A(a) = \begin{bmatrix} h_2(a), & h_2(2a), & \ldots, & h_2(2^ka), & \ldots \\ h_3(a), & h_3(3a), & \ldots, & h_3(3^ka), & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_p(a), & h_p(pa), & \ldots, & h_p(p^ka), & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = [\sigma_{pk}] .$$

The element σ_{pk} in the (p, k)-position of $H_A(a)$ records the p-height of p^ka, for all $p \in P$ and $k = 0, 1, 2, \ldots$. As usual we will write H(a) when the context is clear. If $p \in P$ then the p-*indicator* of a in A is the p-row of H(a), and it is denoted by $U_p(a)$. $U_p(a)$ is said to contain a *gap* if there is a non-negative integer k such that $h_p(p^ka) + 1 < h_p(p^{k+1}a)$. When dealing with p-indicators in p-groups the reference to the prime p will be deleted.

Two $\omega \times \omega$ matrices $[\sigma_{pk}]$ and $[\rho_{pk}]$ are called *equivalent* if for almost all primes p, the p-rows of the two matrices are identical, and for every remaining prime p there exist non-negative integers ℓ and m (depending on p) such that $\sigma_{p,k+\ell} = \rho_{p,k+m}$ for all $k = 0, 1, 2, \ldots$. If A has torsion-free rank one then any two elements a and b of infinite order in A are dependent, so the height matrices H(a) and H(b) are equivalent. Thus there is a uniquely determined equivalence class of height matrices associated with A; we denote this equivalence class by H(A). The importance of H(A) can be seen in the following structure theorem. We give the most general form of this theorem.

(1.1) (Rotman [1], Megibben [1], Myshkin [1], Wallace [1]). *Let* A *and* B *be mixed groups of torsion-free rank one with totally projective torsion subgroups. Then* A *and* B *are isomorphic if and only if*

(i) T(A) *and* T(B) *are isomorphic, and*

(ii) H(A) *and* H(B) *are equivalent.* //

It should be noted that Megibben [1] has proved (1.1) for mixed groups with torsion-complete torsion subgroups, and has also provided an example to show that (1.1) cannot be extended to arbitrary mixed groups of torsion-free rank one.

If a is an element of the torsion-free group A then $h_p(p^{k+1}a) = h_p(p^k a) + 1$ for every prime p and each $k = 0, 1, 2, \ldots$. Thus the first column of H(a) gives the same amount of information about A as the entire matrix H(a). The *characteristic* $\chi(a)$ of a in A is defined to be the first column of H(a); that is $\chi(a) = (h_2(a), h_3(a), \ldots, h_p(a), \ldots)$. Since $p^\omega A$ is the maximal p-divisible subgroup of A, $\chi(a)$ is an ordered sequence of non-negative integers and symbols ∞.

Two characteristics $(k_2, k_3, \ldots, k_p, \ldots)$ and $(\ell_2, \ell_3, \ldots, \ell_p, \ldots)$ are called *equivalent* if $k_p = \ell_p$ for almost all primes p, and if $k_p \neq \ell_p$ for some prime p then both k_p and ℓ_p are finite. An equivalence class of characteristics is called a *type*, and is represented by a characteristic in this class. Thus we write for the type t

$$t = (k_2, k_3, \ldots, k_p, \ldots) ,$$

where the characteristic can be replaced by an equivalent characteristic.

For an element a in a torsion-free group A, if $\chi(a)$ belongs to the type t, then we say that a *is of type* t and write $t(a) = t$. The set of types of all the elements of A is called the *type set* of A, and is denoted by $\mathcal{T}(A)$.

The *product* of the two characteristics $\chi = (k_2, k_3, \ldots, k_p, \ldots)$ and $\chi_1 = (\ell_2, \ell_3, \ldots, \ell_p, \ldots)$ is defined as

$$\chi\chi_1 = (k_2 + \ell_2, k_3 + \ell_3, \ldots, k_p + \ell_p, \ldots) ,$$

where the sum of ∞ and anything is ∞. A characteristic χ is called *idempotent* if $\chi^2 = \chi$. It is clear in this case that for every prime p either $k_p = 0$ or $k_p = \infty$. Since the multiplication of characteristics is compatible with the equivalence relation introduced above, we may speak of the product tt_1 of types t, t_1, and of an idempotent type $t = t^2$.

A torsion-free group A all of whose non-zero elements have the same type is called a *homogeneous group*. Evidently we are justified to speak of the *type* of A, which we shall denote by t(A). Whenever t(A) is an idempotent type we will represent t(A) by the characteristic containing 0's and ∞'s only. Also, in this case, we let P_1^A denote the primes p for which the p-component of t(A) is zero.

For the other elementary properties of type, particularly the quotient $t:t_1$ of the types t and t_1, and the relation $t \leqq t_1$ between the types t and t_1, we refer the reader to Fuchs [4], Section 85.

Jónsson [1] has introduced the following definitions. Suppose A and B are torsion-free groups of finite rank such that A is contained in the divisible hull of B. Then A is *quasi-contained* in B (denoted by $A \stackrel{\bullet}{\subseteqq} B$) if there is a positive integer n such that $nA \subseteqq B$. A is *quasi-equal* to B if $A \stackrel{\bullet}{\subseteqq} B$ and $B \stackrel{\bullet}{\subseteqq} A$, and A is *quasi-isomorphic* to B if A and

B are isomorphic to quasi-equal subgroups of some divisible group. A is called a *quasi-direct sum* of subgroups $B_1, B_2, \ldots, B_m$ of its divisible hull if A is quasi-equal to $B_1 \oplus B_2 \oplus \ldots \oplus B_m$, and this quasi-equality is called a *quasi-direct decomposition* of A. If A has only trivial quasi-direct decompositions then A is called *strongly indecomposable.*

The p-*adic topology* of a group A is the topology of A arising by declaring the subgroups p^kA ($k = 0, 1, 2, \ldots$) as a base of neighbourhoods about 0. For the group A, $\hat{A}_{(p)} \doteq \varprojlim_k A/p^kA$ is called the p-*adic completion* of A. It is well known that $\hat{A}_{(p)}$ is complete in its p-adic topology, and that $\hat{A}_{(p)}$ can be made into a p-adic module.

All other unexplained group theoretical terminology can be found in Fuchs [3, 4].

Since a ring on the group A is a ring whose additive group is isomorphic to A, there is no loss of generality in denoting a ring on A by $(A, \cdot)$. In this case we say that A *supports* $(A, \cdot)$. An ideal of $(A, \cdot)$ will always be written $(I, \cdot)$ and the factor ring on $(A, \cdot)/(I, \cdot)$ will be denoted by $(A/I, \cdot)$. A subgroup I of A is called an *absolute ideal* of A if $(I, \cdot)$ is an ideal of every ring $(A, \cdot)$ on A. Since the left and right multiplications by a fixed element of A are endomorphisms of A, every fully invariant subgroup of A is an absolute ideal of A.

When there is no danger of confusion we attribute to a ring $(A, \cdot)$ on A properties of the group A. Thus we are justified to use terms like torsion ring, divisible ring, pure ideal, etc. If R is a ring we will denote the additive group of R by R^+. The ring of p-adic integers will be denoted by Q_p^*, and the field of rationals by Q.

Every ring $(A, \cdot)$ on the torsion-free group A can be extended to a ring $(Q \otimes A, \cdot)$ on $Q \otimes A$ by defining, for q_1, q_2 in Q and a_1, a_2 in A,

$$(q_1 \otimes a_1) \cdot (q_2 \otimes a_2) = (q_1\, q_2) \otimes (a_1 \cdot a_2) \ .$$

Moreover, $(Q \otimes A, \cdot)$ can be made into a (not necessarily associative) algebra over the field Q by defining for q_1, q_2 in Q and $a \in A$,

$$q_1(q_2 \otimes a) = (q_1\, q_2) \otimes a \ .$$

Thus every torsion-free ring $(A, \cdot)$ can be viewed as a subring of an algebra over Q.

If $(A, \cdot)$ is a ring on a torsion-free group A, and a and b are any two elements of A, then for every prime p,

$$h_p(a) + h_p(b) \leqq h_p(a \cdot b) \ .$$

Thus $\chi(a)\ \chi(b) \leqq \chi(a \cdot b)$ and $t(a)\ t(b) \leqq t(a \cdot b)$ for all a and b in A.

An element a in the associative ring $(A, \cdot)$ on the group A is said to be *nilpotent* if there is a positive integer n such that $a^n = 0$. $(A, \cdot)$ is said it be *nil* if every element $a \in A$ is nilpotent, and $(A, \cdot)$ is called *nilpotent* if there is a positive integer n such that $(A, \cdot)^n = 0$. Following Levitzki [1] and Bass [1] we call the associative ring $(A, \cdot)$ *left T-nilpotent* if for every sequence $a_1, a_2, \ldots$ of its elements there is a positive integer n such that $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 0$. *Right T-nilpotence* is similarly defined. Clearly a nilpotent ring is left T-nilpotent, and a left T-nilpotent ring is nil.

If $(A, \cdot)$ is a ring on A then the *annihilator* of $(A, \cdot)$, denoted by $(0; (A, \cdot))$, is defined as $\{a \in A | a \cdot a' = a' \cdot a = 0 \text{ for all } a' \in A\}$. Clearly $((0; (A, \cdot)), \cdot)$ is an ideal of $(A, \cdot)$; furthermore if A is torsion-free then $((0; (A, \cdot)), \cdot)$ is a pure ideal of $(A, \cdot)$. The *annihilator* of the element a in $(A, \cdot)$, denoted by $(0; a)$, is similarly defined as $\{a' \in A | a \cdot a' = a' \cdot a = 0\}$.

If $(A, \cdot)$ is an associative ring on A then $a \in A$ is called *right quasi-regular* if there is an $a' \in A$ for which $a + a' + a\,a' = 0$. The right ideal $(I, \cdot)$ of $(A, \cdot)$ is called a *right quasi-regular right ideal* of $(A, \cdot)$ if every element of $(I, \cdot)$ is a right quasi-regular element in $(A, \cdot)$. A *left quasi-regular left ideal* of $(A, \cdot)$ is defined analogously.

The sum of all the right quasi-regular right ideals of $(A, \cdot)$ is also a right quasi-regular right ideal of $(A, \cdot)$; it is called the *Jacobson radical* of $(A, \cdot)$, and denoted by $J(A, \cdot)$. $(J(A, \cdot), \cdot)$ is a two sided ideal of $(A, \cdot)$ containing every other right quasi-regular right ideal of $(A, \cdot)$ (and in particular every nil right ideal of $(A, \cdot)$). It is well known that $(A/J(A, \cdot), \cdot)$ is an associative ring containing no proper right quasi-regular right ideals, and also that if $(I, \cdot)$ is an ideal of $(A, \cdot)$ then $J(I, \cdot) = (I, \cdot) \cap J(A, \cdot)$.

The Jacobson radical of an associative ring has many alternate descriptions. To give some of these we require some further definitions.

Suppose $(A, \cdot)$ is an associative ring on the group A. A right ideal $(I, \cdot)$ of $(A, \cdot)$ is said to be a *modular right ideal* of $(A, \cdot)$ if there is an $a \in A$ such that $a \cdot i - i \in I$ for all $i \in I$. *Modular left ideals* of $(A, \cdot)$ are defined analogously. The right $(A, \cdot)$-module M is *faithful* if the annihilator of M, $\{a \in (A, \cdot) | Ma = 0\}$, is trivial. M is called *irreducible* if it is not the trivial module and it contains no proper submodules. $(A, \cdot)$ is called *right primitive* if it admits a faithful irreducible right module. A two sided ideal $(I, \cdot)$ of $(A, \cdot)$ is called a *right primitive ideal* of $(A, \cdot)$ if $(A/I, \cdot)$ is a right primitive ring. A *left primitive ideal* of $(A, \cdot)$ is defined in a similar manner.

(1.2) (Jacobson [1]). *If* $(A, \cdot)$ *is an associative ring on the group* A *then the Jacobson radical of* $(A, \cdot)$ *is equal to*

(i) *the sum of all the right quasi-regular right ideals of* $(A, \cdot)$,

(ii) *the sum of all the left quasi-regular left ieals of* $(A, \cdot)$,

(iii) *the intersection of all the modular maximal right ideals of* $(A, \cdot)$,

(iv) *the intersection of all the modular maximal left ideals of* $(A, \cdot)$,

(v) *the intersection of all the right primitive ideals of* $(A, \cdot)$,

(vi) *the intersection of all the left primitive ideals of* $(A, \cdot)$,

(vii) $\{a \in A \mid a \cdot a'$ *is right quasi-regular, for every* $a' \in A\}$,

(viii) $\{a \in A \mid a' \cdot a$ *is left quasi-regular, for every* $a' \in A\}$. //

It is evident from (i) or (ii) above that if $(A, \cdot)$ and $(B, \cdot)$ are two associative rings and $\phi : (A, \cdot) \to (B, \cdot)$ is a surjective ring homomorphism then ϕ carries $J(A, \cdot)$ into $J(B, \cdot)$. If further ϕ is a ring isomorphism then $J(A, \cdot)$ is isomorphic to $J(B, \cdot)$. From (iii) or (iv) in (1.2) it is also clear that if $(A, \cdot)$ is an associative and commutative ring with identity on a group A then $J(A, \cdot)$ is the intersection of all the maximal ideals of $(A, \cdot)$.

A ring R of linear transformations of a vector space V is called *dense* if for every linearly independent subset $\{u_1, u_2, \ldots, u_n\}$ of V and any set $\{v_1, v_2, \ldots, v_n\}$ of V there is an $f \in R$ such that $f(u_i) = v_i$ for all $i = 1, 2, \ldots, n$.

(1.3) (Jacobson [1]). *If* $(A, \cdot)$ *is an associative ring on the group* A *then* $(A/J(A, \cdot), \cdot)$ *is isomorphic to*

(i) *a subdirect product of right primitive rings, and*

(ii) *a dense ring of linear transformations on a right vector space over a division ring.* //

We will also require the following well known result. First we need another definition. An associative ring $(A, \cdot)$ on a group A is called *Artinian* if it satisfies the minimum condition on left ideals.

(1.4) (Jacobson [1]). *If* $(A, \cdot)$ *is an Artinian ring on the group* A *then* $J(A, \cdot)$ *is nilpotent.* //

For the remainder of this thesis whenever we refer to the Jacobson radical of an associative ring $(A, \cdot)$ on the group A we shall call it the *radical* of $(A, \cdot)$. If $J(A, \cdot) = A$ then $(A, \cdot)$ is called a *radical ring*, while if $J(A, \cdot) = 0$ then $(A, \cdot)$ is called a *semisimple ring*. Accordingly Haimo [1] has called a group A a *radical group* if A supports a non-trivial radical ring, and an *anti-radical group* if A is not a radical group. Similarly, Beaumont and Lawver [1] have called a group A a *semisimple group* if A supports a semisimple ring, and a *strongly semisimple group* if A supports a non-trivial associative ring and every non-trivial associative ring on A is semisimple.

All other unexplained ring theoretical terminology can be found in Divinsky [1] or Jacobson [1].

2. THE ABSOLUTE ANNIHILATOR AND THE ABSOLUTE RADICAL

Following Fuchs [2] we introduce for the group A the set Mult A of all binary compositions on A which are both left and right distributive with respect to addition. An element of Mult A is called a *multiplication on* A. Clearly there is a one-to-one correspondence between Mult A and the set of all rings supported by A. For α and β in Mult A we define the *addition* of α and β by the rule

$$(\alpha + \beta)\ (a, b) = \alpha(a, b) + \beta(a, b)$$

for all a and b in A. With this definition of addition, Mult A is readily seen to be a group, called the *group of multiplications on* A. The zero of Mult A is the multiplication corresponding to the trivial ring on A.

Fuchs [2] has shown that Mult A is isomorphic to Hom(A, E(A)), where $E(A)$ is the endomorphism ring of A. Thus to every ring $(A, \cdot)$ on A there corresponds a map $\phi \in \mathrm{Hom}(A, E(A))$. This ϕ is defined *via* $\phi(a)b = a \cdot b$ for all a and b in A.

Szele [2] defined the *nil-degree* (*nilstufe*) of a group A to be ∞ or the largest integer n (if one exists) such that there is an associative ring $(A, \cdot)$ on A with $(A, \cdot)^n \neq 0$. Gardner [5] has defined the *strong nil-degree* of A similarly, where for the non-associative ring $(A, \cdot)$ on A, $(A, \cdot)^n$ is the subring of $(A, \cdot)$ generated by all products of the form $(\ldots((a_1 \cdot a_2) \cdot a_3) \cdot \ldots) \cdot a_n$. A group A is called *nil* (respectively *strongly nil*) if A has nil-degree one (respectively strong nil-degree one).

Szele [1] has shown that a torsion group is nil exactly if it is divisible, and that there exist no mixed nil groups. Included in his proof of the latter result is a method of defining a non-trivial associative ring on a mixed group with divisible torsion subgroup. Since this ring will prove to be particularly useful in Chapter Four, we outline Szele's construction.

(1.5) (Szele [1]). Suppose A is a mixed group with divisible torsion subgroup. Then A can be written as $A = T(A) \oplus A_1$, where A_1 is some proper subgroup of A.

Select a fixed but arbitrary element a of infinite order in A, and let the non-zero component of a in A_1 be a_1. If we embed A_1 in its divisible hull $Q \otimes A_1$ then we can write $Q \otimes A_1 = Q \oplus A_2$ where a_1 has non-zero component q_1 in Q, and A_2 is a suitable torsion-free divisible group. If now b is any element of A, b will have a unique representation

$$b = t_b + (n/m)q_1 + b_2 ,$$

where $t_b \in T(A)$, $b_2 \in A_2$ and n and $m \neq 0$ are integers.

For some prime $p \in P$ there is a quasi-cyclic subgroup $Z(p^\infty)$ of

$T(A)$. Suppose $Z(p^\infty)$ is generated by the elements $t_1, t_2, \ldots$ subject to the conditions

$$pt_1 = 0,\ pt_2 = t_1,\ \ldots,\ pt_k = t_{k-1},\ \ldots\ .$$

Suppose n and $m \neq 0$ are integers. If $(m, p) = 1$ define $(n/m)t_1$ to be the unique solution of the equation $mt = nt_1$ in $Z(p^\infty)$. On the other hand if $m = p^k m'$ where k is a positive integer and $(m', p) = 1$ then define $(n/m)t_1$ to be the unique solution of the equation $m't = n\ t_{k+1}$ in $Z(p^\infty)$. In either case $(n/m)t_1$ is a well defined element of $Z(p^\infty)$. Moreover, if n_1 and $m_1 \neq 0$ are also integers,

$$(*) \quad ((n/m) + (n_1/m_1))t_1 = (n/m)t_1 + (n_1/m_1)t_1\ .$$

If now b' is another element of A, then b' can be written uniquely

$$b' = t_{b'} + (n_1/m_1)q_1 + b_2'\ ,$$

where $t_{b'} \in T(A)$, $b_2' \in A_2$ and n_1 and $m_1 \neq 0$ are integers. Now define

$$b \cdot b' = ((nn_1)/(mm_1))t_1\ .$$

It is readily checked that $(*)$ ensures that $\cdot$ is both left and right distributive over $+$. Moreover, since every product of length three vanishes, $\cdot$ is associative. Thus $(A, \cdot)$ is an associative ring on A. Clearly $(A, \cdot)$ is not the trivial ring on A, since $a_1 \cdot a_1 = t_1 \neq 0$. //

The only other class of groups for which the nil groups have been characterised is the class of torsion-free completely decomposable groups. A torsion-free group A is called *completely decomposable* if it is a direct sum of rank one groups. We give the following paraphrase of the major result of Ree and Wisner [1].

(1.6) (Ree and Wisner [1]). *If* $A = \bigoplus_{i \in I} A_i$, *where the* A_i *are* *rational groups, then* A *is nil (equivalently strongly nil) if and only if* $t(A_i)\ t(A_j) \nleqq t(A_k)$ *for all* i, j *and* k *in* I. //

Contained in a subsequent proof of (1.6) by Gardner [5] is the following construction of associative rings on certain completely decomposable groups. These rings will prove useful in Chapter Two.

(1.7) (Gardner [5]). *Let* $A = \bigoplus_{i \in I} A_i$, *where the* A_i *are rational groups. If* $t(A_i)\ t(A_j) \leqq t(A_k)$ *for some* i, j *and* k *in* I *then there is an associative ring* $(A, \cdot)$ *on* A *with* $A_i \cdot A_\ell \neq 0$ *for some* $\ell \in I$, *and* $A_m \cdot A_\ell = 0$ *for all* $m \in I$, $m \neq i$. //

The *absolute annihilator* $A(*)$ of a group A is defined as the intersection of the annihilators of all rings on A. Analogously we define the *absolute associative annihilator* $A^{(a)}(*)$ of A as the intersection of the annihilators of all associative rings on A. Clearly $A(*) \subseteqq A^{(a)}(*)$ for all groups A. Also if A is a torsion-free group then both $A(*)$ and $A^{(a)}(*)$ are pure subgroups of A.

As with the nil groups, the present knowledge of the structure of the absolute annihilator (absolute associative annihilator) of a group does not extend beyond the torsion groups and the torsion-free completely decomposable groups.

(1.8) (Fuchs [1]). *If* A *is a torsion group then* $A(*) = A^{(a)}(*) = A^1$. *Moreover, there is an associative and commutative ring* $(A, \cdot)$ *on* A *such that* $(0; (A, \cdot)) = A^1$. //

(1.9) (Gardner [5]). *Let* $A = \bigoplus_{i \in I} A_i$ *where each* A_i *is a rational group, and let*

$I_1 = \{i \in I \mid$ *there do not exist* j *and* k *in* I *such that* $t(A_i)t(A_j) \leqq t(A_k)\}$.

Then $A(*) = A^{(a)}(*) = \bigoplus_{i \in I_1} A_i$. //

The *absolute radical* $J(A)$ of a group A is the intersection of the radicals $J(A, \cdot)$ of all associative rings $(A, \cdot)$ on A. As with the absolute annihilator, Fuchs [4] has been able to describe the absolute radical of a torsion group.

(1.10) (Fuchs [4]). *If* A *is a torsion group then* $J(A) = \bigcap_p pA$. *Furthermore, there is an associative and commutative ring* $(A, \cdot)$ *on* A *such that* $J(A, \cdot) = \bigcap_p pA$. //

3. PRELIMINARY RESULTS

The following result is of fundamental importance.

PROPOSITION 1.11. *Let* A *be a group and* a *an element of* A. *If, for some prime* p, $U_p(a)$ *commences with an integer and contains a gap, then* $a \notin A^{(a)}(*)$.

Proof: Assume p is the prime for which $U_p(a)$ commences with an integer and contains a gap. Then there is an integer i for which a gap occurs between $h_p(p^i a)$ and $h_p(p^{i+1} a)$, where we can assume $h_p(p^i a)$ is finite, say $h_p(p^i a) = k_i < \infty$. Thus there is an element $a' \in A$ such that $p^{i+1} a = pa'$ and $h_p(a') \geqq k_i + 1$. But then $(p^i a - a') \neq 0$ is an element of order p and of height $\min(h_p(p^i a), h_p(a')) = k_i$. Writing $p^i a - a' = p^{k_i} a''$ where $a'' \in A$, Corollary 27.2 of Fuchs [3] shows $\langle a'' \rangle$ is a direct summand of A, say $A = \langle a'' \rangle \oplus A_1$ for some subgroup A_1 of A. If we set $a'' \cdot a'' = a''$ we obtain an associative ring $(\langle a'' \rangle, \cdot)$ on $\langle a'' \rangle$. Letting $(A_1, \cdot)$ be the trivial ring on A_1, we now define $(A, \cdot)$ to be the ring direct sum of $(\langle a'' \rangle, \cdot)$ and $(A_1, \cdot)$. Clearly $(A, \cdot)$ is an associative ring on A. Now

$$(p^i a - a') \cdot a'' = (p^i a) \cdot a'' - a' \cdot a'' = p^i (a \cdot a'')$$

since $h_p(a') \geqq k_i + 1$ and a'' has order p^{k_i+1}. On the other hand

$$(p^i a - a') \cdot a'' = (p^{k_i} a'') \cdot a'' = p^{k_i} a'' \neq 0 .$$

Thus $a \cdot a'' \neq 0$, so $a \notin A^{(a)}(*)$. //

COROLLARY 1.12. *For a group* A, $T(A(*)) \subseteqq T(A^{(a)}(*)) \subseteqq (T(A))^1$. //

COROLLARY 1.13. *If* A *is a group such that* A/T(A) *is divisible then* $A(*) \subseteqq A^{(a)}(*) \subseteqq A^1$. *If further* A *is reduced* $A(*) = A^{(a)}(*) = A^1$.

Proof: Suppose a is an element of A such that $h_p(a) = k$ is finite for some prime p. If $U_p(a)$ does not contain a gap, then $h_p(p^n a) = k + n$ for every positive integer n. Let $\alpha : A \to A/T(A)$ denote the natural map. Since A/T(A) is divisible, $h_p(\alpha a) \not\geq k$, so there is an $a' \in A$ such that $\alpha a = p^{k+1}(\alpha a')$. Thus $a - p^{k+1} a' \in T(A)$. Suppose $O(a - p^{k+1} a') = p^\ell m$ where $(p, m) = 1$. Then $p^\ell m a = p^{k+\ell+1} m\, a'$, so

$$h_p(p^\ell a) = h_p(p^\ell m a) = h_p(p^{k+\ell+1} m\, a') \geqq k + \ell + 1 ,$$

contradicting the fact that $h_p(p^n a) = k + n$ for each positive integer n. Therefore $U_p(a)$ contains at least one gap. Proposition 1.11 now yields $a \notin A^{(a)}(*)$. Thus $A(*) \subseteqq A^{(a)}(*) \subseteqq A^1$.

To prove the final assertion it suffices to show that $A^1 \subseteqq A(*)$ for a reduced group A with A/T(A) divisible. First we show $(T(A))^1 \subseteqq A(*)$. Consider any ring $(A, \cdot)$ on A and let $a \in (T(A))^1$. It is clear that a annihilates T(A). If a' is any element of infinite order in A and $O(a) = n$ then the divisibility of A/T(A) shows $a' = na'' + t$, where a'' is an element of infinite order in A and $t \in T(A)$. But then

$$a \cdot a' = a \cdot na'' + a \cdot t = 0 ,$$

so $a \in A(*)$.

Next suppose a is an element of infinite order in A^1. Again let $(A, \cdot)$ be any ring on A and suppose $\phi \in \mathrm{Hom}(A, E(A))$ defines $(A, \cdot)$. Then $\phi(a) \in E(A)$, and since a has infinite p-height for all primes p, $\phi(a)|_{T(A)} = 0$. Hence $\phi(a)$ factors through $A/T(A)$,

$$\phi(A) : \begin{tikzcd} A \arrow[rr] \arrow[dr] & & A \\ & A/T(A) \arrow[ur] & \end{tikzcd} .$$

Since $A/T(A)$ is divisible and A is reduced, $\phi(a)$ is necessarily the zero map. That is $a \in A(*)$. //

The next lemma will be needed on several occasions.

LEMMA 1.14. *For the group* $A = \bigoplus_{i \in I} A_i$, $A(*) \subseteqq \bigoplus_{i \in I} A_i(*)$ *and* $A^{(a)}(*) \subseteqq \bigoplus_{i \in I} A_i^{(a)}(*)$. *If every* A_i *is an absolute ideal of* A *then* $A(*) = \bigoplus_{i \in I} A_i(*)$ *and* $A^{(a)}(*) = \bigoplus_{i \in I} A_i^{(a)}(*)$.

Proof: We shall only prove the assertions for the absolute annihilator of A, the proofs for the absolute associative annihilator being identical.

For a given $i \in I$, let $(A_i, \cdot)$ be a ring on A_i. By defining the trivial ring $(\bigoplus_{j \neq i} A_j, \cdot)$ on $\bigoplus_{j \neq i} A_j$ and taking the ring direct sum of $(A_i, \cdot)$ and $(\bigoplus_{j \neq i} A_j, \cdot)$ we obtain a ring $(A, \cdot)$ on A. Clearly

$$(0; (A, \cdot)) = (0; (A_i, \cdot)) \oplus \bigoplus_{j \neq i} A_j .$$

Since this is true for every ring $(A_i, \cdot)$ on A_i and for every $i \in I$, $A(*) \subseteqq \bigoplus_{i \in I} A_i(*)$.

If for each $i \in I$, A_i is an absolute ideal of A, then the equality $A(*) = \bigoplus_{i \in I} A_i(*)$ follows immediately from the fact that every ring $(A, \cdot)$ on A is the ring direct sum of the ideals $(A_i, \cdot)$, $i \in I$. //

Gardner [5] has introduced a chain

$$A(1) \subseteq A(2) \subseteq \ldots \subseteq A(\alpha) \subseteq \ldots$$

of subgroups of a group A, defined inductively as follows: $A(1) = A(*)$, $A(\alpha + 1)/A(\alpha) = [A/A(\alpha)](*)$, and $A(\beta) = \bigcup_{\alpha<\beta} A(\alpha)$ if β is a limit ordinal. It is clear that $A(\mu + 1) = A(\mu)$ for some ordinal μ.

PROPOSITION 1.15. *For any group* A, $A(\alpha)$ *is an absolute ideal of* A *for all ordinals* α.

Proof: First we show that $A(*)$ is a fully invariant subgroup of A. Let $f \in E(A)$ and $a \in A$. If $f(a) \notin A(*)$ then there is a homomorphism $\phi \in \mathrm{Hom}(A, E(A))$ such that $\phi(f(a)) \neq 0$. But $\phi f \in \mathrm{Hom}(A, E(A))$ and $(\phi f)(a) \neq 0$, so $a \notin A(*)$.

A trivial transfinite induction argument shows that $A(\alpha)$ is fully invariant in A for all ordinals α. The proposition now follows immediately. //

If we employ the same methods of proof as those used to prove Corollary 2.4 of Gardner [5] we can now establish

COROLLARY 1.16. *If* $A = A(\mu)$ *for some ordinal* μ *then every associative ring* $(A, \cdot)$ *on* A *is left and right T-nilpotent. If in addition* μ *is finite then* $(A, \cdot)^{\mu+1} = 0$. //

For the remainder of this chapter we shall be concerned with properties of the absolute radical of a group. We begin with some useful lemmas.

LEMMA 1.17. *If* A *and* B *are isomorphic groups then* J(A) *is isomorphic to* J(B).

Proof: Let $\phi : A \to B$ denote the isomorphism. If $(B, \cdot)$ is an associative ring on B then it is possible to define an associative ring $(A, \cdot)$ on A such that $\phi : (A, \cdot) \to (B, \cdot)$ is a ring isomorphism. Since ϕ is surjective $\phi(J(A, \cdot)) \subseteq J(B, \cdot)$, so $\phi(J(A)) \subseteq J(B, \cdot)$. This is true for every associative ring $(B, \cdot)$ on B, so $\phi(J(A)) \subseteq J(B)$. A similar argument verifies $\phi^{-1}(J(B)) \subseteq J(A)$. Thus $J(A) \cong J(B)$. //

LEMMA 1.18. *If* B *is an absolute ideal of the group* A *then* $J(B) \subseteq J(A)$.

Proof: If $(A, \cdot)$ is any associative ring on A then $(B, \cdot)$ is an ideal of $(A, \cdot)$. Since $J(B, \cdot) = (B, \cdot) \cap J(A, \cdot)$, $J(B) \subseteq J(B, \cdot) \subseteq J(A, \cdot)$. This is true for every associative ring $(A, \cdot)$ on A, so $J(B) \subseteq J(A)$. //

For any group A, T(A) is an absolute ideal of A. Thus Lemma 1.18 yields $J(T(A)) \subseteq J(A)$ for all groups A. This containment has previously been observed by Fuchs [4].

LEMMA 1.19. *If* $A = \bigoplus_{i \in I} A_i$ *then* $J(A) \subseteq \bigoplus_{i \in I} J(A_i)$. *If further* A_i *is an absolute ideal of* A *for every* $i \in I$, $J(A) = \bigoplus_{i \in I} J(A_i)$.

Proof: For $i \in I$, let $(A_i, \cdot)$ be an associative ring on A_i. If we extend this ring to an associative ring $(A, \cdot)$ on A in the usual way by defining the trivial ring on $\bigoplus_{j \neq i} A_j$, we have

$$J(A, \cdot) = J(A_i, \cdot) \oplus \bigoplus_{j \neq i} A_j .$$

Since this is true for every associative ring $(A_i, \cdot)$ on A_i and every $i \in I$, $J(A) \subseteq \bigoplus_{i \in I} J(A_i)$.

The last assertion of the Lemma follows immediately from Lemma 1.18. //

The next proposition will be extremely useful in describing the absolute radical of a group.

PROPOSITION 1.20. *If* $A/J(A, \cdot)$ *is a torsion group for every associative ring* $(A, \cdot)$ *on* A *then* $\bigcap_p pA \subseteq J(A)$.

Proof: Suppose $(A, \cdot)$ is an associative ring on A. Since $A/J(A, \cdot)$ is a torsion group, (1.10) yields $J(A/J(A, \cdot)) = \bigcap_p p(A/J(A, \cdot))$. Since also $J(A/J(A, \cdot), \cdot) = 0$ it is clear that $\bigcap_p p(A/J(A, \cdot)) = 0$. Now $[\bigcap_p pA + J(A, \cdot)]/J(A, \cdot) \subseteq \bigcap_p p(A/J(A, \cdot))$, so $\bigcap_p pA \subseteq J(A, \cdot)$. This is true for every associative ring $(A, \cdot)$ on A, so $\bigcap_p pA \subseteq J(A)$, as required. //

Beaumont and Lawver [1] have described the semisimple rational groups (and more generally the semisimple strongly indecomposable torsion-free groups of finite rank). In the next result we use Beaumont and Lawver's description to characterise the absolute radical of an arbitrary rational group. This characterisation plays an important role in subsequent chapters.

THEOREM 1.21. *Suppose* A *is a torsion-free group of rank one. Then exactly one of the following conditions holds:*

(i) $J(A) = A$;

(ii) $J(A) = 0$;

(iii) $J(A) = nA$ $(nA \neq 0,\ nA \neq A)$, *for a suitable integer* n.

(i) *holds exactly if* A *is a nil group,* (ii) *holds exactly if* $t(A)$ *is idempotent and* $|P_1^A| = 0$ *or* $|P_1^A| = \infty$, *and* (iii) *holds exactly if* $t(A)$ *is idempotent and* $0 \neq |P_1^A| < \infty$, *in which case* $n = \prod_{p \in P_1^A} p$. *Furthermore, there is an associative and commutative ring* $(A, \cdot)$ *on* A *for which*

$J(A, \cdot) = J(A)$.

Proof: It is clear that if A is a nil group then $J(A) = A$. Thus assume A is non-nil, in which case $t(A)$ is idempotent. From Theorem 3.2 of Beaumont and Lawver [1] if $|P_1^A| = 0$ or $|P_1^A| = \infty$ then $J(A) = 0$. Moreover, any non-trivial associative ring $(A, \cdot)$ on A (and such rings do exist) is commutative and has trivial radical.

Assume now that $t(A)$ is idempotent and $0 \neq |P_1^A| < \infty$. Consider an associative ring $(A, \cdot)$ on A. Theorem 3.2 of Beaumont and Lawver [1] shows $J(A, \cdot) \neq 0$, so since A has rank one, $A/J(A, \cdot)$ is necessarily a torsion group. This is true for every associative ring $(A, \cdot)$ on A, so Proposition 1.20 now yields $\bigcap_p pA \subseteqq J(A)$. Since $\bigcap_p pA = \bigcap_{p \in P_1^A} pA$, $\bigcap_{p \in P_1^A} pA \subseteqq J(A)$.

Next we give an example of an associative and commutative ring $(A, \cdot)$ on A for which $J(A, \cdot) = \bigcap_{p \in P_1^A} pA$. Since isomorphic rings have isomorphic radicals we lose no generality in assuming $A = \langle p^{-\infty} | p \in P \backslash P_1^A \rangle$. Consider the associative and commutative ring $(A, \cdot)$ defined by $a_1 \cdot a_2 = a_1 a_2$ for all a_1, a_2 in A, where the latter multiplication is multiplication in the field Q. For each $p \in P_1^A$ it is clear that $(A/pA, \cdot)$ is a non-trivial associative ring. Since A/pA has rank one, $(A/pA, \cdot)$ is now a field, so $J(A/pA, \cdot) = 0$. Therefore $J(A, \cdot) \subseteqq pA$, and so $J(A, \cdot) = \bigcap_{p \in P_1^A} pA$. (Beaumont and Lawver [1] have also observed the existence of such a ring on A).

Consequently if $t(A)$ is idempotent and $0 \neq |P_1^A| < \infty$, $J(A) = \bigcap_{p \in P_1^A} pA = nA$ where $n = \prod_{p \in P_1^A} p$. Evidently nA is a proper subgroup of A.

We complete the proof by observing that the three conditions A is a nil group, $t(A)$ is idempotent such that $|P_1^A| = 0$ or $|P_1^A| = \infty$, and $t(A)$ is idempotent such that $0 \neq |P_1^A| < \infty$ are mutually exclusive and exhaustive conditions for A. //

We close this chapter with the following result.

PROPOSITION 1.22. *For every group* A, $J(T(A/J(A))) = 0$. *Furthermore if* $A/T(A)$ *is divisible then* $J(A/J(A)) = 0$.

Proof: To prove the first assertion suppose $a + J(A) \in T(A/J(A))$ where $a \notin J(A)$. Assume n is the order of $a + J(A)$ in $T(A/J(A))$. For every associative ring $(A, \cdot)$ on A let $n_{(\cdot)}$ be the order of $a + J(A, \cdot)$ in $T(A/J(A, \cdot))$. Lemma 1.18 and the fact that $(A/J(A, \cdot), \cdot)$ is a semi-simple ring imply $J(T(A/J(A, \cdot))) \subseteq J(A/J(A, \cdot)) = 0$, so (1.10) shows $n_{(\cdot)}$ is a square-free integer. Since $na \in J(A, \cdot)$, $n_{(\cdot)}$ necessarily divides n. If $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the canonical representation of n as a product of powers of primes, let $n' = p_1 p_2 \cdots p_k$. Since $n_{(\cdot)}$ divides n', $n'a \in J(A, \cdot)$. This is true of every associative ring $(A, \cdot)$ on A, so $n'a \in J(A)$. Consequently $n = n'$, that is n is a square-free integer. Thus $T(A/J(A))$ is an elementary group, so (1.10) yields $J(T(A/J(A))) = 0$.

Next consider a group A such that $A/T(A)$ is divisible. Since $J(T(A/J(A))) = 0$, Lemma 1.18 shows $J((A/J(A))_p) = 0$, for every prime p. Consider a fixed prime p. (1.10) implies that $(A/J(A))_p$ is an elementary p-group, so

$$(*) \qquad A/J(A) = (A/J(A))_p \oplus (A^{(p)}/J(A)),$$

for some subgroup $A^{(p)}$ of A. Since $(T(A) + J(A))/J(A) \subseteq T(A/J(A))$,

$$[A/J(A)]/T(A/J(A)) \cong \frac{[A/J(A)]/[(T(A) + J(A))/J(A)]}{T(A/J(A))/[(T(A) + J(A))/J(A)]}.$$

But

$[A/J(A)]/[(T(A)+J(A))/J(A)] \cong A/(T(A)+J(A)) \cong [A/T(A)]/[(T(A)+J(A))/T(A)]$, so $[A/J(A)]/T(A/J(A))$ is a homomorphic image of $A/T(A)$. Thus $[A/J(A)]/T(A/J(A))$ is a p-divisible group. Now

$$A^{(p)}/J(A) \cong [A/J(A)]/(A/J(A))_p ,$$

and $[A/J(A)]/(A/J(A))_p$ is an extension of the p-divisible group $T(A/J(A))/(A/J(A))_p$ by the p-divisible group $[A/J(A)]/T(A/J(A))$. Thus $A^{(p)}/J(A)$ is a p-divisible subgroup of $A/J(A)$; moreover since $(A/J(A))_p$ is an elementary p-group, $A^{(p)}/J(A)$ is the maximal p-divisible subgroup of $A/J(A)$.

Suppose now $J(A/J(A)) \neq 0$. It is clear that $A/J(A) \neq T(A/J(A))$. Now for each prime p, $(A/J(A))_p$ and $A^{(p)}/J(A)$ are both absolute ideals of $A/J(A)$. Thus (1.10) and Lemma 1.19 applied to the decomposition (*) yield $J(A/J(A)) = J(A^{(p)}/J(A))$. Hence $J(A/J(A)) \subseteq A^{(p)}/J(A)$. Since this is true for every prime p, $J(A/J(A)) \subseteq \bigcap_p (A^{(p)}/J(A))$.

Denoting $\bigcap_p (A^{(p)}/J(A))$ by $A_1/J(A)$, for some subgroup A_1 of A, it is readily checked that $A_1/J(A)$ is torsion-free and divisible. Thus

$$A/J(A) = A_1/J(A) \oplus A_2/J(A)$$

for some subgroup A_2 of A. Lemma 1.19 and Theorem 1.21 now show $J(A/J(A)) \subseteq J(A_2/J(A))$. Consequently

$$0 \neq J(A/J(A)) \subseteq (A_1/J(A)) \cap (A_2/J(A)) .$$

We conclude therefore that $J(A/J(A)) = 0$. //

CHAPTER TWO

In Chapter One two major results from the literature concerning rings on completely decomposable torsion-free groups were given. (1.6) characterised the nil completely decomposable groups, while (1.9) gave a description of the absolute annihilator of a completely decomposable group. In this chapter we are concerned with generalising these results to other classes of torsion-free groups. It is pleasing to report that generalisations are possible to the class of vector groups and to the class of separable groups. We conclude the chapter by providing some necessary and sufficient conditions for an arbitrary torsion-free group to be strongly nil.

1. VECTOR GROUPS

Following Fuchs [4] we call a group V a *vector group* if V is a direct product of rank one torsion-free groups (that is, $V = \prod_{i \in I} R_i$ where the R_i are rational groups).

We begin this section by giving a description of the nil vector groups. To do this we need the following definitions, and the well known results (2.1) to (2.3).

A *slender group* A is a torsion-free group with the property that every homomorphism from a countable direct product of infinite cyclic groups $<e_n>$, $n = 1, 2, \ldots,$ into A sends almost all components $<e_n>$ into the zero of A. These groups were introduced and studied by J. Łoś (see Fuchs [4], Section 94).

A set I is *measurable* if I admits a countably additive measure μ such that μ assumes only the values 0 and 1, and

$$\mu(I) = 1, \ \mu(i) = 0 \text{ for all } i \in I .$$

(2.1) (Saşaida [1], Nunke [1]). *Every countable and reduced torsion-free group is slender.* //

(2.2) (Fuchs [2]). *A direct sum of slender groups is a slender group.* //

(2.3) (Łoś; see Fuchs [4], pp. 161,162). *If* A *is a slender group,* A_i $(i \in I)$ *are torsion-free groups, and the index set* I *is not measurable, then*

(i) *if* ϕ *is a homomorphism from* $\prod_{i \in I} A_i$ *into* A *such that* $\phi(\bigoplus_{i \in I} A_i) = 0$, *then* $\phi = 0$,

(ii) *there is a natural isomorphism*

$$\mathrm{Hom}(\prod_{i \in I} A_i, A) \cong \bigoplus_{i \in I} \mathrm{Hom}(A_i, A) \ . \ //$$

Whenever we represent a vector group as a direct product $V = \prod_{i \in I} R_i$ it is to be understood that the R_i are rational groups.

We are now in a position to prove

LEMMA 2.4. *If* $V = \prod_{i \in I} R_i$ *is a vector group such that the index set* I *is not measurable, and* $\mathrm{Hom}(R_i, \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k)) \neq 0$ *for some* i *and* k *in* I, *then for such an* i *and* k *there exists a* $j \in I$ *such that* $t(R_i)\ t(R_j) \leqq t(R_k)$.

Proof: $\mathrm{Hom}(R_i, \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k))$ is a subgroup of $\mathrm{Hom}(R_i, \prod_{j \in I} \mathrm{Hom}(R_j, R_k))$, so $\mathrm{Hom}(R_i, \mathrm{Hom}(R_j, R_k)) \neq 0$ for some $j \in I$. Now $\mathrm{Hom}(R_j, R_k)$ is a rank one torsion-free group whose type is $t(R_k) : t(R_j)$ (see Fuchs [4], p. 111). Thus $t(R_i)\ t(R_j) \leqq [t(R_k) : t(R_j)]\ t(R_j) \leqq t(R_k)$, as required. //

THEOREM 2.5. *Let* $V = \prod_{i \in I} R_i$ *be a vector group where the index set* I *is not measurable. Then the following conditions are equivalent:*

(i) V *is strongly nil;*

(ii) V *is nil;*

(iii) $t(R_i)\, t(R_j) \nleq t(R_k)$ *for all* i, j *and* k *in* I.

Proof: (i) ⇒ (ii) is immediate.

(ii) ⇒ (iii). Suppose $t(R_i)\, t(R_j) \leq t(R_k)$ for some i, j and k in I. It follows from (1.7) that we can define a non-trivial associative ring $(V', \cdot)$ on a completely decomposable direct summand V' of V. This ring can be extended to the whole of V by defining all other products to be zero, so V is non-nil.

(iii) ⇒ (i). If V is not strongly nil, then $\mathrm{Hom}(V, \mathrm{Hom}(V, V)) \neq 0$. Since $t(R_i)^2 \nleq t(R_i)$ for all $i \in I$ and I is not measurable, (2.1) and (2.3) (ii) yield $\mathrm{Hom}(V, V) \cong \prod_{k \in I} \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k)$. Now $\mathrm{Hom}(R_j, R_k)$ is either zero or a rank one torsion-free group whose type is less than or equal to $t(R_k)$. (2.1) and (2.2) now imply that $\bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k)$ is a slender group, for all $k \in I$. Applying (2.3) (ii) we obtain

$$\mathrm{Hom}(V, \mathrm{Hom}(V, V)) \cong \prod_{k \in I} \bigoplus_{i \in I} \mathrm{Hom}(R_i, \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k)) .$$

Hence $\mathrm{Hom}(R_i, \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k)) \neq 0$ for some i and k in I. From Lemma 2.4 we now conclude that $t(R_i)\, t(R_j) \leq t(R_k)$ for some $j \in I$, contradicting (iii). //

COROLLARY 2.6. *Let* $V = \prod_{i \in I} R_i$ *be a vector group, where* I *is not measurable. Then* V *is nil if and only if* $\bigoplus_{i \in I} R_i$ *is nil.* //

We now turn our attention to the absolute annihilator V(*) of a vector group V.

THEOREM 2.7. *Suppose* $V = \prod_{i\in I} R_i$ *is a vector group with the index set* I *not measurable, and*

$I_1 = \{i \in I \mid$ *there do not exist* j *and* k *in* I *such that* $t(R_i)\ t(R_j) \leqq t(R_k)\}$.

Then $V(*) = V^{(a)}(*) = \prod_{i\in I_1} R_i$.

Proof: Let v be a non-zero element of $V^{(a)}(*)$. Write $v = (\ldots, r_i, \ldots)$ where each $r_i \in R_i$ and some $r_i \neq 0$. Assume there exist j and k in I such that $t(R_i)\ t(R_j) \leqq t(R_k)$. Applying (1.7) we obtain an associative ring $(V_0, \cdot)$ on a finite rank completely decomposable direct summand $V_0 = \bigoplus_{i_0\in I_0} R_{i_0}$ of V such that $i \in I_0$, $R_i \cdot R_\ell \neq 0$ for some $\ell \in I_0$, and $R_m \cdot R_\ell = 0$ for all $m \in I_0$, $m \neq i$. Let $V = V_0 \oplus V'$ for some subgroup V' of V. By taking the ring direct sum of $(V_0, \cdot)$ and the trivial ring on V' we obtain an associative ring $(V, \cdot)$ on V. Now $v = \sum_{i_0\in I_0} r_{i_0} + v'$, where $v' \in V'$. Thus $0 = v \cdot r_\ell = r_i \cdot r_\ell$ for all $r_\ell \in R_\ell$. This cannot be the case since $R_i \cdot R_\ell \neq 0$, whence $v \in \prod_{i\in I_1} R_i$.

Conversely, suppose v is a non-zero element of $\prod_{i\in I_1} R_i$. It is clear that R_j is reduced for all $j \in I$. Assume $v \notin V(*)$. Then there is an element $\phi \in \mathrm{Hom}(V, \mathrm{Hom}(V, V))$ such that $\phi(v) \neq 0$. Thus $\mathrm{Hom}(\prod_{i\in I_1} R_i, \mathrm{Hom}(V, V)) \neq 0$. (2.1), (2.2) and (2.3) (ii) imply

$$\mathrm{Hom}(\prod_{i\in I_1} R_i, \mathrm{Hom}(\prod_{j\in I} R_j, \prod_{k\in I} R_k)) \cong \prod_{k\in I} \bigoplus_{i\in I_1} \mathrm{Hom}(R_i, \bigoplus_{j\in I} \mathrm{Hom}(R_j, R_k)),$$

so there exists an $i \in I_1$ and $k \in I$ such that $\mathrm{Hom}(R_i, \bigoplus_{j\in I} \mathrm{Hom}(R_j, R_k)) \neq 0$. From Lemma 2.4 we infer that $t(R_i)\ t(R_j) \leqq t(R_k)$ for some $j \in I$, contrary to our choice of v. Hence $v \in V(*)$. //

We conclude this section with some necessary conditions for a direct product of slender groups to be nil.

PROPOSITION 2.8. *Suppose* $A = \prod_{i\in I} A_i$, *where each* A_i *is a slender group, and the index set* I *is not measurable. Let* $(A, \cdot)$ *be a ring on* A. *If* $\bigoplus_{i\in I} A_i$ *is a subgroup of* $(0; (A, \cdot))$ *then* $(A, \cdot)$ *is the trivial ring on* A.

Proof: Let $\phi \in \mathrm{Hom}(\prod_{i\in I} A_i, \mathrm{Hom}(\prod_{j\in J} A_j, \prod_{k\in I} A_k))$ be the map defining $(A, \cdot)$, and consider an arbitrary element $a \in A$. Under the natural isomorphism

$$\mathrm{Hom}(\prod_{j\in I} A_j, \prod_{k\in I} A_k) \cong \prod_{k\in I} \mathrm{Hom}(\prod_{j\in I} A_j, A_k) ,$$

$\phi(a) \to (\ldots, \pi_k\, \phi(a), \ldots)$, where $\pi_k : \prod_{i\in I} A_i \to A_k$ is the natural projection for all $k \in I$. Now for each $a' \in \bigoplus_{i\in I} A_i$ we have $\pi_k\, \phi(a)a' = \pi_k\, (a \cdot a') = 0$ for all $k \in I$, so (2.3) (i) implies that $\pi_k\, \phi(a) = 0$ for all $k \in I$. Thus $\phi(a) = 0$. Since a was chosen to be an arbitrary element of A, $(A, \cdot)$ is the trivial ring on A. //

COROLLARY 2.9. *Let* $A = \prod_{i\in I} A_i$ *be a direct product of the slender groups* A_i, $i \in I$ *where* I *is not measurable. If* $\bigoplus_{i\in I} A_i$ *is a subgroup of* $A^{(a)}(*)$ *then* A *is nil.* //

We require the following result.

(2.10) (Gardner [1]). *Let* $\{A_n | n = 1, 2, \ldots\}$ *be a countable family of torsion-free groups, and suppose* B *is an arbitrary group. If* $\mathrm{Hom}(\bigoplus_{n=1}^{\infty} A_n, B) = 0$ *then* $\mathrm{Hom}(\prod_{n=1}^{\infty} A_n, B) = 0$. //

PROPOSITION 2.11. *Suppose* $A = \prod_{n=1}^{\infty} A_n$, *where each* A_n *is a slender* *group. If* Mult $(\bigoplus_{n=1}^{\infty} A_n) = 0$ *then* Mult $A = 0$, *in which case* A *is nil.*

Proof: Since $\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, \bigoplus_{n=1}^{\infty} A_n)) = 0$,

$\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, A_n)) = 0$ for each positive integer n. Consider such an integer n. Clearly $\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \prod_{m=1}^{\infty} \mathrm{Hom}(A_m, A_n)) = 0$. Thus, since $\bigoplus_{m=1}^{\infty} \mathrm{Hom}(A_m, A_n)$ is a subgroup of $\prod_{m=1}^{\infty} \mathrm{Hom}(A_m, A_n)$, it is evident that $\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \bigoplus_{m=1}^{\infty} \mathrm{Hom}(A_m, A_n)) = 0$. Since A_n is slender (2.3) (ii) now yields $\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \mathrm{Hom}(\prod_{m=1}^{\infty} A_m, A_n)) = 0$. Consequently (2.10) implies $\mathrm{Hom}(\prod_{k=1}^{\infty} A_k, \mathrm{Hom}(\prod_{m=1}^{\infty} A_m, A_n)) = 0$. This is true for every positive integer n, so clearly $\mathrm{Hom}(\prod_{k=1}^{\infty} A_k, \mathrm{Hom}(\prod_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n)) = 0$, as required. //

2. SEPARABLE GROUPS

The separable groups were introduced by Baer [1]. A torsion-free group A is called *separable* if every finite set of elements of A is contained in a completely decomposable direct summand of A. It is clear that this summand can be chosen to have finite rank.

We commence this section with a description of the nil separable groups. First however, we need to consider the following subgroups of a separable group.

Suppose $(A, \cdot)$ is a ring on the separable group A, and $A_1 \oplus A_2$ is a completely decomposable direct summand of A. We are permitted to write

$$A_1 = \langle a_1\rangle_* \oplus \langle a_2\rangle_* \oplus \ldots \oplus \langle a_{n_1}\rangle_*$$

and

$$A_2 = \langle a_{n_1+1}\rangle_* \oplus \langle a_{n_1+2}\rangle_* \oplus \cdots \oplus \langle a_{n_2}\rangle_*$$

for suitable elements $a_1, a_2, \ldots, a_{n_2}$ of A, and $A = A_1 \oplus A_2 \oplus A_2'$ for some subgroup A_2' of A. Since A_2' is a direct summand of A, Theorem 87.5 of Fuchs [4] shows it is separable, so there is a finite rank completely decomposable summand A_3 of A_2' with the property that $A_1 \oplus A_2 \oplus A_3$ contains all products of the form $a_i \cdot a_j$ where $i \in \{1, 2, \ldots, n_1\}$ and $j \in \{1, 2, \ldots, n_2\}$. Thus

$$A_3 = \langle a_{n_2+1}\rangle_* \oplus \langle a_{n_2+2}\rangle_* \oplus \cdots \oplus \langle a_{n_3}\rangle_*$$

for suitable elements $a_{n_2+1}, a_{n_2+2}, \ldots, a_{n_3}$ of A. Since $A_1 \oplus A_2 \oplus A_3$ is a pure subgroup of A it is clear that $a \cdot b \in A_1 \oplus A_2 \oplus A_3$ for all $a \in A_1$ and all $b \in A_1 \oplus A_2$.

LEMMA 2.12. *Let* $(A, \cdot)$ *be a ring on a separable group* A, *and let* A_1, A_2 *and* A_3 *be the subgroups of* A *defined as above. If* $\mathrm{Hom}(A_1, \mathrm{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \neq 0$ *then there exist* $i \in \{1, 2, \ldots, n_1\}$, $j \in \{1, 2, \ldots, n_2\}$ *and* $k \in \{1, 2, \ldots, n_3\}$ *such that* $t(a_i)\, t(a_j) \leqq t(a_k)$.

Proof: Clearly

$$\mathrm{Hom}(A_1, \mathrm{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \cong \bigoplus_{i=1}^{n_1} \bigoplus_{j=1}^{n_2} \bigoplus_{k=1}^{n_3} \mathrm{Hom}(\langle a_i\rangle_*, \mathrm{Hom}(\langle a_j\rangle_*, \langle a_k\rangle_*)).$$

Proceeding as in the proof of Lemma 2.4 we obtain the required result. //

THEOREM 2.13. *For a separable group* A *the following conditions are equivalent:*

(i) A *is strongly nil;*

(ii) A *is nil;*

(iii) *every rank* n $(n \leqq 3)$ *completely decomposable direct summand* *of* A *is nil.*

Proof: Clearly (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). It remains to show (iii) $\Rightarrow$ (i). Suppose there is a ring $(A, \cdot)$ on A, and elements a, b in A such that $a \cdot b \neq 0$. Let A_1 be a finite rank completely decomposable direct summand of A containing a and b, and let $A_2 = 0$. Define A_3 as we did prior to Lemma 2.12. For $e \in A_1$ define $\phi : A_1 \to \mathrm{Hom}(A_1, A_1 \oplus A_3)$ by $\phi(e)f = e \cdot f$ for all $f \in A_1$. Then $\phi \in \mathrm{Hom}(A_1, \mathrm{Hom}(A_1, A_1 \oplus A_3))$ and $\phi(a)b = a \cdot b \neq 0$. We now apply Lemma 2.12 and (1.7) to obtain a rank n ($n \leqq 3$) direct summand of A that is non-nil. //

Having characterised the nil separable groups we are now in a position to describe the structure of the absolute annihilator of a separable group. We need to make the following definitions.

A finite set of elements $\{a_1, a_2, \ldots, a_n\}$ of a separable group A is called *basic* if it is independent and $\langle a_1\rangle_* \oplus \langle a_2\rangle_* \oplus \ldots \oplus \langle a_n\rangle_*$ is a direct summand of A. An element $a \in A$ is called a *basic element* of A if the set $\{a\}$ is basic. For the separable group A we define

$$A' = \{a \in A \mid a \text{ is a basic element of } A \text{ with the property that there do not exist basic elements } b, c \text{ in } A \text{ with } \{a, b, c\} \text{ basic and } t(a)\, t(b) \leqq t(c)\}.$$

THEOREM 2.14. *Let* A *be a separable group and let* A' *be defined as above. Then* $A(*) = A^{(a)}(*)$ *is the pure subgroup of* A *generated by* A'.

Proof: If a is a non-zero element of $\langle A'\rangle_*$ then we can write

$$na = n_1a_1 + n_2a_2 + \ldots + n_ka_k$$

where $n \neq 0$, $n_1, n_2, \ldots, n_k$ are integers and $a_i \in A'$ for $i = 1, 2, \ldots, k$. If $a_i \notin A(*)$ for some $i \in \{1, 2, \ldots, k\}$ then there is a ring $(A, \cdot)$ on A

such that $a_i \cdot a' \neq 0$ for some $a' \in A$. Let $A_1 = \langle a_i \rangle_*$ and let

$$A_2 = \langle a_2 \rangle_* \oplus \langle a_3 \rangle_* \oplus \ldots \oplus \langle a_{n_2} \rangle_*$$

be such that $A_1 \oplus A_2$ is a completely decomposable direct summand of A containing a'. Define A_3 as we did prior to Lemma 2.12. As in the proof of Theorem 2.13, $a_1 \cdot a' \neq 0$ implies $\mathrm{Hom}(A_1, \mathrm{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \neq 0$, so Lemma 2.12 shows $t(a_i)\ t(a_j) \leqq t(a_k)$ for some $j \in \{i, 2, 3, \ldots, n_2\}$ and $k \in \{i, 2, 3, \ldots, n_3\}$. This contradicts the fact that $a_i \in A'$. Therefore for each $i \in \{1, 2, \ldots, k\}$, $a_i \in A(*)$, so $na \in A(*)$. Since $A(*)$ is pure in A it follows that $a \in A(*)$.

Next suppose a is a non-zero element of $A^{(a)}(*)$. Now a can be embedded in a finite rank completely decomposable direct summand A_1 of A,

$$A_1 = \langle a_1 \rangle_* \oplus \langle a_2 \rangle_* \oplus \ldots \oplus \langle a_{n_1} \rangle_* ,$$

and there exist integers $n \neq 0, n_1, n_2, \ldots, n_{n_1}$ such that

$$na = n_1\ a_1 + n_2\ a_2 + \ldots + n_{n_1}\ a_{n_1} .$$

If $a_i \notin A'$ for some $i \in \{1, 2, \ldots, n_1\}$ then there are basic elements b and c in A such that $\{a_i, b, c\}$ is basic and $t(a_i)\ t(b) \leqq t(c)$. By (1.7) there exists an associative ring $(A, \cdot)$ on A with $a_i \cdot a' \neq 0$ for some $a' \in A$. If we let

$$A_2 = \langle a_{n_1+1} \rangle_* \oplus \langle a_{n_1+2} \rangle_* \oplus \ldots \oplus \langle a_{n_2} \rangle_*$$

be such that $A_1 \oplus A_2$ is a completely decomposable summand of A containing a', and define A_3 as usual, then as in the proof of Theorem 2.13, $a_i \cdot a' \neq 0$ implies $\mathrm{Hom}(\langle a_i \rangle_*, \mathrm{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \neq 0$. A reference to Lemma 2.12 now yields $t(a_i)\ t(a_j) \leqq t(a_k)$ for some $j \in \{1, 2, \ldots, n_2\}$ and $k \in \{1, 2, \ldots, n_3\}$. (1.7) now shows that we can define an associative ring $(A_1 \oplus A_2 \oplus A_3, \times)$ on $A_1 \oplus A_2 \oplus A_3$ such that $<a_i>_* \times <a_\ell>_* \neq 0$ for some $\ell \in \{1, 2, \ldots, n_3\}$ and $<a_m>_* \times <a_\ell>_* = 0$ for all $m \in \{1, 2, \ldots, n_3\}$, $m \neq i$. $(A_1 \oplus A_2 \oplus A_3, x)$ can be extended to an associative ring (A, x) on A by setting all other products equal to zero. But then

$$0 = (na)x\, a_\ell = (n_i\, a_i)x\, a_\ell.$$

Thus if $n_i \neq 0$, $a_i \in A'$. Therefore $a \in <A'>_*$. //

Finally, we note how some of the methods of this chapter can be generalised to provide some necessary and sufficient conditions for an arbitrary torsion-free group A to be strongly nil. It is necessary to introduce certain rational subgroups of A. In the case that A has rank two these subgroups correspond to the 'groups of rank one' defined by Beaumont and Wisner [1].

Suppose A is a torsion-free group and $\{a_i | i \in I\}$ is a maximal independent set of elements of A. Each $a \in A$ can now be uniquely expressed as

$$(*) \qquad a = \alpha_{i_1}\, a_{i_1} + \alpha_{i_2}\, a_{i_2} + \ldots + \alpha_{i_n}\, a_{i_n}$$

for a suitable subset $\{a_{i_j} | j = 1, 2, \ldots, n\}$ of $\{a_i | i \in I\}$, and suitable rationals $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}$. For each $i \in I$ we now define two subgroups of the rationals, called the *rational groups belonging to* a_i, as follows:

$Q_{a_i} = \{\alpha \in Q |$ there is an expression of the form $(*)$ containing αa_i as a component$\}$

$Q'_{a_i} = \{\alpha \in Q | \alpha a_i \in A\}$.

Clearly $Q'_{a_i} \subseteq Q_{a_i}$, $1 \in Q'_{a_i}$ and $Q'_{a_i}\, a_i = <a_i>_*$, for each $i \in I$.

PROPOSITION 2.15. *Suppose* A *is a torsion-free group and* $\{a_i | i \in I\}$ *is a maximal independent set of elements of* A. *Let* Q_{a_i} *and*

Q'_{a_i} *be the rational groups belonging to* a_i, *for each* $i \in I$. *If* $t(Q'_{a_i})\ t(Q'_{a_j}) \nleq t(Q_{a_k})$ *for all* i, j *and* k *in* I *then* A *is strongly nil. Conversely if* A *is strongly nil then* $Q_{a_i}\ Q_{a_j} \nsubseteq Q'_{a_k}$ *for all* i, j *and* k *in* I.

Proof: Suppose $(A, \cdot)$ is a non-trivial ring on the group A. Since $\{a_i | i \in I\}$ is a maximal independent set of elements of A there are indices i and j in I such that $a_i \cdot a_j \neq 0$. Also, for suitable non-zero integers n, n_{i_1}, n_{i_2}, ..., n_{i_m} we can write

$$n(a_i \cdot a_j) = \sum_{\ell=1}^{m} n_{i_\ell}\ a_{i_\ell}$$

where $\{a_{i_\ell} | \ell = 1, 2, \ldots, m\}$ is some non-void subset of $\{a_i | i \in I\}$.

Now consider arbitrary non-zero β_i and β_j in Q'_{a_i} and Q'_{a_j} respectively. By definition, $\beta_i\ a_i$ and $\beta_j\ a_j$ are both elements of A, so $(\beta_i\ a_i) \cdot (\beta_j\ a_j) \in A$. It is readily checked that

$$\begin{aligned} n[(\beta_i\ a_i) \cdot (\beta_j\ a_j)] &= n[\beta_i\ \beta_j\ (a_i \cdot a_j)] \\ &= \beta_i\ \beta_j\ [n(a_i \cdot a_j)] \\ &= \beta_i\ \beta_j\ [\sum_{\ell=1}^{m} n_{i_\ell}\ a_{i_\ell}] \\ &= \sum_{\ell=1}^{m} n_{i_\ell}\ \beta_i\ \beta_j\ a_{i_\ell}\ . \end{aligned}$$

Thus if $k \in \{i_1, i_2, \ldots, i_m\}$, $n_k\ \beta_i\ \beta_j \in Q_{a_k}$. By consecutively choosing $\beta_j = 1$ and $\beta_i = 1$ it is immediate that $n_k\ \beta_i \in Q_{a_k}$ and $n_k\ \beta_j \in Q_{a_k}$. Hence $n_k\ Q'_{a_i} \subseteq Q_{a_k}$ and $n_k\ Q'_{a_j} \subseteq Q_{a_k}$, for each $k \in \{i_1, i_2, \ldots, i_m\}$.

Next consider fixed non-zero elements β_i and β_j in Q'_{a_i} and Q'_{a_j} respectively. Then for $k \in \{i_1, i_2, \ldots, i_m\}$

$$\begin{aligned} t(Q'_{a_i})\, t(Q'_{a_j}) &= t^{Q'_{a_i}}(\beta_i)\, t^{Q'_{a_j}}(\beta_j) \\ &= t^{n_k Q'_{a_i}}(n_k\, \beta_i)\, t^{n_k Q'_{a_j}}(n_k\, \beta_j) \\ &\leqq t^{Q_{a_k}}(n_k\, \beta_i)\, t^{Q_{a_k}}(n_k\, \beta_j) \\ &\leqq t^{Q_{a_k}}((n_k\, \beta_i)\,(n_k\, \beta_j)) \\ &= t(Q_{a_k}) , \end{aligned}$$

since $n_k^{\,2}\, \beta_i\, \beta_j$ is a non-zero element of Q_{a_k}. Evidently the first assertion of the Proposition has been verified.

To prove the converse statement suppose there exist i, j and k in I such that $Q_{a_i}\, Q_{a_j} \subseteq Q'_{a_k}$. If we embed A in its divisible hull $Q \otimes A$, then $\langle a_i \rangle_*^{Q \otimes A}$, $\langle a_j \rangle_*^{Q \otimes A}$ and $\langle a_k \rangle_*^{Q \otimes A}$ will all obviously be direct summands of $Q \otimes A$. Consequently we can define a ring $(Q \otimes A, \cdot)$ on $Q \otimes A$ by letting $a_i \cdot a_j = a_k$, and all other products not thus accounted for be zero. If now a and b are arbitrary elements of A then from the definition of $(Q \otimes A, \cdot)$ it is clear that $a \cdot b$ is either zero, or $a \cdot b = \alpha_i\, \alpha_j\, a_k$ where $\alpha_i \in Q_{a_i}$ and $\alpha_j \in Q_{a_j}$. Since $Q_{a_i}\, Q_{a_j} \subseteq Q'_{a_k}$, $a \cdot b \in A$. Thus $(A, \cdot)$ is a subring of $(Q \otimes A, \cdot)$, and since $a_i \cdot a_j = a_k \neq 0$, A is not strongly nil. //

Consider the completely decomposable group $A = \bigoplus_{i \in I} A_i$, where the A_i are rational groups. If for each $i \in I$ we select a non-zero element $a_i \in A_i$, then $\{a_i | i \in I\}$ is a maximal independent set of elements of A, and $Q'_{a_i} = Q_{a_i} \cong A_i$. A routine argument verifies that for all i, j and k in I, $t(A_i)\, t(A_j) \leq t(A_k)$ if and only if $A_i\, A_j \subseteq A_k$. Consequently the Ree and Wisner result (1.6) is a special case of Proposition 2.15.

Beaumont and Wisner [1] have shown that if $U_0 \subseteqq U$, $V_0 \subseteqq V$ are rational groups such that $U/U_0 \cong V/V_0$ then there exists a torsion-free group of rank two with independent elements {a, b} such that $Q_a' = U_0$, $Q_a = U$, $Q_b' = V_0$, and $Q_b = V$. Proposition 2.15 therefore also provides us with abundant examples of (strongly) nil torsion-free groups of rank two.

CHAPTER THREE

It is apparent from the previous chapter and the results (1.6) and (1.9) of Chapter One that for certain torsion-free groups A the condition that there exist elements a, b and c in A such that $t(a)\ t(b) \leqq t(c)$ is extremely useful. The first part of this chapter is devoted to a further analysis of this concept in completely decomposable torsion-free groups. We introduce a relation $\leqq'$ defined on a subset of the type set of a completely decomposable torsion-free group A. The relation $\leqq'$ has an interesting connection with the T-nilpotent rings of Levitzki [1] and Bass [1]. Also, it allows us to write A in a form that is different from the usual representation of A as a direct sum of rational groups. With A written in this form and the relation $\leqq'$ satisfying a certain chain condition we are then able to describe the absolute radical of A.

1. A RELATION ON A SET OF TYPES IN A COMPLETELY DECOMPOSABLE TORSION-FREE GROUP.

Throughout this chapter a completely decomposable torsion-free group will simply be called a completely decomposable group. For a completely decomposable group $A = \bigoplus_{i \in I} A_i$, where A_i is a rational group for each $i \in I$, let $\tilde{T}(A)$ denote the set $\{t(A_i) | i \in I\}$. We define a relation $\leqq'$ on $\tilde{T}(A)$ in the following manner: for i and j in I we say $t(A_i) \leqq' t(A_j)$ if there is an $i_1 \in I$ such that $t(A_i)\ t(A_{i_1}) \leqq t(A_j)$.

It is clear that $\leqq'$ is an antisymmetric and transitive, but not necessarily reflexive relation on $\tilde{T}(A)$.

By a *chain of length* n in $\tilde{T}(A)$ is meant a sequence of n (not necessarily distinct) types $t(A_{i_1}), t(A_{i_2}), \ldots, t(A_{i_n})$ in $\tilde{T}(A)$ with the property that

$$t(A_{i_1}) \leqq' t(A_{i_2}) \leqq' \cdots \leqq' t(A_{i_n}) .$$

The existence of chains of length n in $\tilde{T}(A)$ is not a new idea. Gardner [5] has defined a $2 \times n$ *π-matrix* to be a $2 \times n$ matrix of types

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \cdots\cdots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \cdots\cdots & \tau_{2n} \end{bmatrix}$$

such that $\tau_{1i}\,\tau_{2i} \leqq \tau_{1i+1}$ for $i = 1, 2, \ldots, n-1$. It is clear that for a completely decomposable group A the existence of a chain of length n in $\tilde{T}(A)$ is equivalent to the existence of a $2 \times n$ π-matrix over $\tilde{T}(A)$. Consequently

(3.1) (Gardner [5]). *Suppose* $A = \bigoplus_{i \in I} A_i$ *is a completely decomposable group where each* A_i *is a rational group, and let for each positive integer* n,
$I_n = \{i \in I \mid$ *there exists no chain of length* $n+1$ *in* $\tilde{T}(A)$ *commencing with* $t(A_i)\}$.
Then $A(n) = \bigoplus_{i \in I_n} A_i$, *and the following conditions are equivalent:*

(i) $A = A(n)$, $n < \infty$, *and* $A \neq A(n-1)$;

(ii) *there are chains of length* n *but no chains of length* $n+1$ *in* $\tilde{T}(A)$;

(iii) A *has strong nil-degree* n. //

If $A = \prod_{i \in I} A_i$ is a vector group, where A_i is a rational group for each $i \in I$, then we can define $\tilde{T}(A)$ and the relation $\leqq'$ on $\tilde{T}(A)$ in the same way as we defined the same concepts for a completely decomposable group. (3.1) now has an immediate extension to vector groups. The only difference in the statement of (3.1) is that $A = \prod_{i \in I} A_i$ where the index set I is not measurable, and for each n, $A(n) = \prod_{i \in I_n} A_i$. The proof of the corresponding statement of (3.1) for vector groups is identical to the original proof of (3.1). We now know, therefore, that if $A = \prod_{i \in I} A_i$ is a vector group where I is not measurable then A and $\bigoplus_{i \in I} A_i$ have the same strong nil-degree.

Vinsonhaler and Wickless [1] have studied another concept in a completely decomposable group A that is related to the existence of chains of length n in $\tilde{T}(A)$. Following Visonhaler and Wickless we define a *triangle of size* n to be a collection of $\frac{n(n+1)}{2}$ rational groups indexed by all sequences of the form $(i, i+1, \ldots, i+j)$ where $1 \leqq i \leqq n$, $0 \leqq j \leqq n - i$, such that

$$t(A_{(i,i+1,\ldots,i+j)})\, t(A_{(i+j+1,i+j+2,\ldots,i+j+k)}) \leqq t(A_{(i,i+1,\ldots,i+j+k)})$$

for all i, j as above and k with $i + j + k \leqq n$.

If $A = \bigoplus_{i \in I} A_i$ is a completely decomposable group, where each A_i is a rational group, then clearly the existence of a triangle of size n formed from the set of groups $\{A_i | i \in I\}$ implies the existence of a chain of length n in $\tilde{T}(A)$.

(3.2) (Vinsonhaler and Wickless [1]). *Let* $A = \bigoplus_{i \in I} A_i$ *be a completely decomposable group, where* A_i *is a rational group for each* $i \in I$. *Let* $(A, \cdot)$ *be an associative ring on* A *and suppose* $\tilde{T}(A)$ *is an ordered set. If* $(A, \cdot)^n \neq 0$ *for some positive integer* n, *then a triangle of size* n *can be formed by choosing groups from* $\{A_i | i \in I\}$. //

Vinsonhaler and Wickless [1] have stated that the requirement that $\tilde{T}(A)$ is an ordered set cannot be deleted in (3.2). However if we replace Vinsonhaler and Wickless' triangle of size n condition with our weaker condition then (3.2) can be substantially improved.

THEOREM 3.3. *If* $(A, \cdot)$ *is a ring on the completely decomposable group* A *such that* $(A, \cdot)^n \neq 0$ *for some positive integer* n *then there exists a chain of length* n *in* $\tilde{T}(A)$.

Proof: Suppose $A = \bigoplus_{i \in I} A_i$ where each A_i is a rational group, and $x_1, x_2, \ldots, x_n$ are elements of A such that

$$(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) \cdot x_n \neq 0 .$$

For each $i \in I$, let a_i be an arbitrary but fixed element of A_i. Then $t(a_i) = t(A_i)$, and each element of A_i can be written uniquely in the form αa_i for a suitable rational α. Thus for each $k \in \{1, 2, \ldots, n\}$ there exist elements $i(k, 1), i(k, 2), \ldots, i(k, m_k)$ in I such that

$$(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) \cdot x_k = \sum_{\ell=1}^{m_k} \alpha_{i(k,\ell)} a_{i(k,\ell)},$$

where $\alpha_{i(k,\ell)}$ is a non-zero rational for each $\ell \in \{1, 2, \ldots, m_k\}$. In particular

$$(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) \cdot x_{n-1} = \sum_{\ell=1}^{m_{n-1}} \alpha_{i(n-1,\ell)} a_{i(n-1,\ell)},$$

so

$$(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) \cdot x_n = \sum_{\ell=1}^{m_{n-1}} \alpha_{i(n-1,\ell)} (a_{i(n-1,\ell)} \cdot x_n).$$

On the other hand

$$(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) x_n = \sum_{\ell=1}^{m_n} \alpha_{i(n,\ell)} a_{i(n,\ell)} .$$

Therefore for every $\ell(n) \in \{1, 2, \ldots, m_n\}$ there exists an $\ell(n-1) \in \{1, 2, \ldots, m_{n-1}\}$ such that $a_{i(n-1,\ell(n-1))} \cdot x_n$ has a non-zero component in $A_{i(n,\ell(n))}$. In other words

$$t(a_{i(n-1,\ell(n-1))}) \leqq' t(a_{i(n,\ell(n))}) .$$

If we similarly analyse the product

$$(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) \cdot x_{n-1}$$

it is possible to choose an $\ell(n-2) \in \{1, 2, \ldots, m_{n-2}\}$ such that

$$t(a_{i(n-2,\ell(n-2))}) \leqq' t(a_{i(n-1,\ell(n-1))}) .$$

Repeating this procedure n times will produce a sequence of elements $a_{i(1,\ell(1))}, a_{i(2,\ell(2))}, \ldots, a_{i(n,\ell(n))}$ in A such that

$$t(a_{i(1,\ell(1))}) \leqq' t(a_{i(2,\ell(2))}) \leqq' \ldots \leqq' t(a_{i(n,\ell(n))}) ,$$

where $\ell(j) \in \{1, 2, \ldots, m_j\}$ for each $j \in \{1, 2, \ldots, n\}$. //

The definitions of a left T-nilpotent ring and a right T-nilpotent ring given by Levitzki [1] and Bass [1] (see Chapter One) can be generalised to include the non-associative rings. Indeed suppose $(A, \cdot)$ is a non-associative ring on the group A. $(A, \cdot)$ is called *left T-nilpotent* if for every sequence $a_1, a_2, \ldots$ of its elements there is a positive integer n such that $(\ldots((a_1 \cdot a_2) \cdot a_3) \cdot \ldots) \cdot a_n = 0$. *Right T-nilpotence* is defined similarly.

Gardner [4] has shown that if A is a group such that every associative ring on A is left T-nilpotent then every associative ring on A is right T-nilpotent. With a similar proof we can show that if every ring on A is left T-nilpotent then every ring on A is right T-nilpotent. In this case we are justified in saying that every ring on A is *T-nilpotent*.

Now for the major result of this section.

THEOREM 3.4. *Let* A *be a completely decomposable group. Then every ring on* A *is T-nilpotent if and only if* $\tilde{T}(A)$ *satisfies the ascending chain condition with respect to the relation* $\leqq'$.

Proof: Suppose $(A, \cdot)$ is a ring on A that is not left T-nilpotent.

Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a rational group, and select elements $x_1, x_2, \ldots$ of A such that for each positive integer n $(\ldots((x_1 \cdot x_2) \cdot x_3) \cdot \ldots) \cdot x_n \neq 0$. Theorem 3.3 yields, for each positive integer n, a chain of length n in $\tilde{T}(A)$. Moreover, an analysis of the proof of Theorem 3.3 shows that the k^{th} term (where k is a positive integer) of any one of these chains must be the type of one of at most m_k distinct rational groups A_i, $i \in I$.

With the notation of the proof of Theorem 3.3, define C to be the set of chains in $\tilde{T}(A)$ with respect to $\leqq'$ corresponding to the sequence $x_1, x_2, \ldots$ such that the first term of each of these chains is a type from $\{t(A_{i(1,\ell)}) | \ell \in \{1, 2, \ldots, m_1\}\}$. It is clear that C is an infinite set with the property that all the elements of C commence with a type from a finite subset of $\tilde{T}(A)$. Hence we can choose an $i_1 \in I$ such that there are infinitely many chains in C commencing with $t(A_{i_1})$. (Notice that $i_1 = i(1, \ell)$ for some $\ell \in \{1, 2, \ldots, m_1\}$). Each of these chains can have only a finite number of distinct second terms, so it is possible to choose an $i_2 \in I$ such that C contains infinitely many elements with first and second terms $t(A_{i_1})$ and $t(A_{i_2})$ respectively. Repeating this procedure it is possible to find an infinite chain

$$t(A_{i_1}) \leqq' t(A_{i_2}) \leqq' \ldots \leqq' t(A_{i_n}) \leqq' \ldots$$

in $\tilde{T}(A)$. Thus $\tilde{T}(A)$ will not satisfy the ascending chain condition. (It should be noted that the above argument is based upon the proof of a graph theoretical result, namely König's lemma (see Wilson [1], p. 40)).

Next suppose

$$t(A_{i_1}) \leqq' t(A_{i_2}) \leqq' \ldots \leqq' t(A_{i_n}) \leqq' \ldots$$

is an infinite ascending chain in $\tilde{T}(A)$. Then there exist indices

$j_1, j_2, \ldots$ in I such that $t(A_{i_k})\, t(A_{j_k}) \leqq t(A_{i_{k+1}})$ for all $k = 1, 2, \ldots$. Thus there are elements a_{i_1}, a_{j_1} and a_{i_2} in A_{i_1}, A_{j_1} and A_{i_2} respectively such that $\chi(a_{i_1})\, \chi(a_{j_1}) \leqq \chi(a_{i_2})$. Suppose now for each $k \in \{1, 2, \ldots, m\}$ we have selected elements a_{i_k}, a_{j_k} and $a_{i_{k+1}}$ in A_{i_k}, A_{j_k} and $A_{i_{k+1}}$ respectively, such that $\chi(a_{i_k})\, \chi(a_{j_k}) \leqq \chi(a_{i_{k+1}})$. Since $t(A_{i_{m+1}})\, t(A_{j_{m+1}}) \leqq t(A_{i_{m+2}})$, there are elements $a'_{i_{m+1}}$, $a_{j_{m+1}}$ and $a'_{i_{m+2}}$ in $A_{i_{m+1}}$, $A_{j_{m+1}}$ and $A_{i_{m+2}}$ respectively, such that $\chi(a'_{i_{m+1}})\, \chi(a_{j_{m+1}}) \leqq \chi(a'_{i_{m+2}})$. Since $a_{i_{m+1}}$ and $a'_{i_{m+1}}$ are dependent elements of $A_{i_{m+1}}$ it is now possible to choose an element $a_{i_{m+2}} \in A_{i_{m+2}}$ such that $\chi(a_{i_{m+1}})\, \chi(a_{j_{m+1}}) \leqq \chi(a_{i_{m+2}})$. Thus for each $k = 1, 2, \ldots$, there are elements a_{i_k}, a_{j_k} and $a_{i_{k+1}}$ in A_{i_k}, A_{j_k} and $A_{i_{k+1}}$ respectively, such that $\chi(a_{i_k})\, \chi(a_{j_k}) \leqq \chi(a_{i_{k+1}})$. Define $a_{i_k} \cdot a_{j_k} = a_{i_{k+1}}$ and let all other products in A not thus accounted for be zero. In this way we obtain a ring $(A, \cdot)$ on A for which

$$(\ldots(((a_{i_1} \cdot a_{j_1}) \cdot a_{j_2}) \cdot a_{j_3}) \cdot \ldots) \cdot a_{j_{n-1}} = a_{i_n} \neq 0$$

for each positive integer n. Thus $(A, \cdot)$ is not left T-nilpotent. //

2. THE ABSOLUTE RADICAL OF A COMPLETELY DECOMPOSABLE TORSION-FREE GROUP

If A is a completely decomposable group then the relation $\leqq'$ on $\tilde{T}(A)$ discussed in the previous section facilitates an alternate description of A, and it is this description of A that allows us to determine the structure of the absolute radical $J(A)$ when the non-idempotent types in $\tilde{T}(A)$ satisfy the ascending chain condition with respect to $\leqq'$.

For the completely decomposable group A, if we collect together all the rank one summands of A with the same type then it is possible to write A in the following form:

$$A = \bigoplus_{i\in I} A_i \oplus \bigoplus_{j\in J} B_j \ ,$$

for suitable index sets I and J, where each A_i, $i \in I$, and each B_j, $j \in J$ is a homogeneous completely decomposable group, all the components in this decomposition of A have distinct types, the type of each A_i, $i \in I$, is maximal in $\tilde{T}(A)$ with respect to the relation $\leqq'$, and the type of each B_j, $j \in J$, is not maximal in $\tilde{T}(A)$ with respect to $\leqq'$. It is clear that this decomposition of A is unique (up to rearrangements of the A_i terms or the B_j terms). This decomposition of A is called the *h-decomposition* of A.

From Lemma 1.19 it is immediate that

$$J(A) \subseteqq \bigoplus_{i\in I} J(A_i) \oplus \bigoplus_{j\in J} J(B_j) \ .$$

It is not difficult to establish that since A_i has maximal type in $\tilde{T}(A)$ with respect to $\leqq'$, A_i is an absolute ideal of A, for each $i \in I$. Thus Lemma 1.18 yields

$$\bigoplus_{i\in I} J(A_i) \subseteqq J(A) \ .$$

It is apparent from these two inequalities that to characterise the absolute radical of a completely decomposable group we initially need to concentrate our attention on describing the absolute radical of a homogeneous completely decomposable group. To do this we require the following two well known results.

(3.5) (Fuchs [4]). *Let* A *be a homogeneous completely decomposable group of finite rank. Then every pure subgroup of* A *is a direct summand of* A. //

(3.6) (Baer [1], Kulikov [1], Kaplansky [1]). *Direct summands of* *completely decomposable groups are completely decomposable.* //

For later purposes we will also need the following useful lemma.

LEMMA 3.7. *Suppose* $(A, \cdot)$ *is an associative ring on a homogeneous completely decomposable group* A *of finite rank,* $t(A)$ *is idempotent and* $0 \neq |P_1^A| < \infty$. *If* $J(A, \cdot) \neq 0$ *then either* $J(A, \cdot)$ *has rank equal to the rank of* A *or* A *contains a proper direct summand with trivial absolute radical.*

Proof: Let $(A, \cdot)$ be an associative ring on A for which $J(A, \cdot) \neq 0$. For notational convenience denote $\langle J(A, \cdot)\rangle_*$ by A_1. (3.5) shows that A_1 is a summand of A, so $A = A_1 \oplus A_2$ for some subgroup A_2 of A.

Since $J(A, \cdot)$ is an ideal of $(A, \cdot)$, $(A_1, \cdot)$ is also an ideal of $(A, \cdot)$. Therefore $J(A_1, \cdot) = (A_1, \cdot) \cap J(A, \cdot)$, so $J(A_1, \cdot) = J(A, \cdot)$. This means $\langle J(A_1, \cdot)\rangle_* = A_1$, so $A_1/\langle J(A_1, \cdot)\rangle$ is a torsion group. The proof of Proposition 1.20 now yields $\bigcap_p pA_1 \subseteq J(A_1, \cdot)$, so $0 \neq |P_1^A| < \infty$ indicates that $A_1/J(A_1, \cdot)$ is a bounded torsion group. Suppose n is a bound of $A_1/J(A_1, \cdot)$.

Now as groups $A/J(A, \cdot) \cong (A_1/J(A_1, \cdot)) \oplus A_2$. Since $nA_2 = n((A_1/J(A_1, \cdot)) \oplus A_2)$, nA_2 is an absolute ideal of $(A_1/J(A_1, \cdot)) \oplus A_2$. From Lemma 1.18, $J(nA_2) \subseteq J((A_1/J(A_1, \cdot)) \oplus A_2)$. $J(A/J(A, \cdot)) = 0$ and Lemma 1.17 now show $J(nA_2) = 0$. Since $A_2 \cong nA_2$, $J(A_2) = 0$. The assertion is now immediate. //

PROPOSITION 3.8. *Let* A *be a homogeneous completely decomposable group of finite rank such that* $t(A)$ *is idempotent and* $0 \neq |P_1^A| < \infty$. *Then* $J(A) = \bigcap_{p \in P_1^A} pA$.

Proof: We use induction on the rank of A. The case when A has rank one is settled by Theorem 1.21, so assume A is as stated in the Proposition and also that the Proposition is true for such groups whose ranks are strictly less than n, n a positive integer.

First, we claim that A cannot support a semisimple ring. Indeed suppose $(A, \cdot)$ is an associative ring on A such that $J(A, \cdot) = 0$. If $(I, \cdot)$ is a non-zero ideal of $(A, \cdot)$ then $(\langle I\rangle_*, \cdot)$ will be a non-zero pure ideal of $(A, \cdot)$. Owing to (3.5), $\langle I\rangle_*$ is a direct summand of A so (3.6) implies that $\langle I\rangle_*$ is a completely decomposable group. If A/I is not a torsion group then $r(\langle I\rangle_*) \lneq r(A)$. The induction hypothesis now shows $J(\langle I\rangle_*) \neq 0$, so $J(\langle I\rangle_*, \cdot) \neq 0$. Since $J(\langle I\rangle_*, \cdot) = (\langle I\rangle_*, \cdot) \cap J(A, \cdot)$, $J(A, \cdot) \neq 0$, contradicting the semisimplicity of $(A, \cdot)$. Therefore if $(I, \cdot)$ is a non-zero ideal of $(A, \cdot)$, A/I is a torsion group.

(1.3) shows that $(A, \cdot)$ is isomorphic to a subdirect product of right primitive rings $(A_i, \cdot)$, $i \in I$. A reference to Jacobson [1] shows that for each $i \in I$, $(A_i, \cdot)$ is isomorphic to $(A/P_i, \cdot)$, where $(P_i, \cdot)$ is a right primitive ideal of $(A, \cdot)$, and also that $(A_i, \cdot)$ is isomorphic to a dense ring of linear transformations on a right vector space over a division ring D_i. If, for $i \in I$, the characteristic of D_i is the prime p then clearly A_i is an elementary p-group, and furthermore, $p \in P_1^A$. On the other hand if the characteristic of D_i is zero then it is readily checked that A_i is torsion-free, which we know cannot be the case. Therefore $(A, \cdot)$ is isomorphic to a subdirect product of bounded rings $(A_i, \cdot)$, $i \in I$, where for each $i \in I$ the bound of A_i is a prime belonging to P_1^A. Since P_1^A is finite, A must be a torsion group. Evidently our claim is now established.

If now $(A, \cdot)$ is any associative ring on A then Lemma 3.7 shows that either $A/J(A, \cdot)$ is torsion, or A contains a proper direct summand with trivial absolute radical. The latter alternative can be deleted by virtue of (3.6) and the induction hypothesis, so Proposition 1.20 yields $\bigcap_{p \in P_1^A} pA \subseteqq J(A)$. The proof is completed by observing that Lemma 1.19 and Theorem 1.21 together imply $J(A) \subseteqq \bigcap_{p \in P_1^A} pA$. //

Wickless [1] has given an example of a countably infinite rank completely decomposable group supporting a semisimple ring. We can extend Wickless' argument to prove

LEMMA 3.9. *Suppose* $A = \bigoplus_{k=1}^{\infty} A_k$ *where each* A_k *is a rational group. If* $t(A_i)\, t(A_j) \leqq t(A_{i+j})$ *for all* i *and* j *in* $\{1, 2, \ldots\}$ *then* A *supports a semisimple ring.*

Proof: Assume $t(A_i)\, t(A_j) \leqq t(A_{i+j})$ for all i and j in $\{1, 2, \ldots\}$. As in the proof of Theorem 3.4 an induction argument shows that for each $i \in \{1, 2, \ldots\}$ we can select elements $a_i \in A_i$ such that $\chi(a_i)\, \chi(a_j) \leqq \chi(a_{i+j})$ for all i and j in $\{1, 2, \ldots\}$. We can now define a ring $(A, \cdot)$ on A by letting $a_i \cdot a_j = a_{i+j}$ for all i and j in $\{1, 2, \ldots\}$. It is clear that $(A, \cdot)$ is an associative and commutative ring on A.

Suppose now that a and b are non-zero elements of A. Let n be the largest positive integer such that a has a non-zero component in A_n, and m be the largest positive integer such that b has a non-zero component in A_m. Then $a \cdot b$ has a non-zero component in A_{m+n}, and since $n + m \gneqq \max\{n, m\}$, it is evident that $a + b + a \cdot b \neq 0$. Consequently $(A, \cdot)$ can never contain a non-zero right quasi-regular element. Thus $(A, \cdot)$ is semisimple. //

It should be noted that if A in Lemma 3.9 is also homogeneous with idempotent type, then Lemma 3.9 can be proved using a more general result proved by Amitsur [1]; if R is an associative ring with trivial nil radical then the polynomial ring over R is Jacobson semisimple.

Now for the description of the absolute radical of a homogeneous completely decomposable group.

THEOREM 3.10. *If* A *is a homogeneous completely decomposable group then exactly one of the following conditions holds:*

(i) $J(A) = A$;

(ii) $J(A) = 0$; *or*

(iii) $J(A) = \bigcap_{p \in P_1^A} pA$.

(i) *holds exactly when* $t(A)$ *is not idempotent*, (ii) *holds exactly when* $t(A)$ *is idempotent and either* $r(A)$ *is infinite*, $|P_1^A| = 0$ *or* $|P_1^A| = \infty$, *while* (iii) *holds exactly when* $t(A)$ *is idempotent*, $r(A)$ *is finite and* $0 \neq |P_1^A| < \infty$.

Proof: If $t(A)$ is not idempotent then A is a nil group, so $J(A) = A$. Suppose therefore that $t(A)$ is idempotent. If $r(A)$ is infinite and a is a non-zero element in $J(A)$, then it is possible to embed a in a countably infinite rank completely decomposable direct summand A_1 of A. Writing $A = A_1 \oplus A_2$ for some subgroup A_2 of A, Lemma 1.19 yields $J(A) \subseteqq J(A_1) \oplus J(A_2)$. From Lemma 3.9 it is evident that $J(A_1) = 0$, so $J(A) \subseteqq A_2$. Thus $0 \neq a \in A_1 \cap A_2$. We conclude therefore that $J(A) = 0$. If $|P_1^A| = 0$ or $|P_1^A| = \infty$ then Lemma 1.19 and Theorem 1.21 yield $J(A) = 0$. On the other hand if A has finite rank and $0 \neq |P_1^A| < \infty$ then Proposition 3.8 shows $J(A) = \bigcap_{p \in P_1^A} pA$. As in the proof of Theorem 1.21 the final assertion of the Theorem is evident from the fact that the three conditions mentioned there are mutually exclusive and exhaustive conditions for A. //

COROLLARY 3.11. *If* A *is a homogeneous completely decomposable group then* $J(A/J(A)) = 0$. //

Consider a completely decomposable group A with h-decomposition

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j \ .$$

Having completely characterised the absolute radical of A_i for each $i \in I$ we now turn our attention to defining subgroups $K(B_j)$ of B_j for each $j \in J$ that will render possible a description of the absolute radical of A when the non-idempotent types in $\tilde{T}(A)$ satisfy the ascending chain condition with respect to $\leqq'$. We need to develop some notation and give some preliminary results.

For each $j \in J$ define $\mathcal{B}^{(j)}$ to be the set of components in the h-decomposition of A whose types, with respect to $\leqq'$, are strictly greater than $t(B_j)$. Let $A^{(j)}$ denote the direct sum of the elements of $\mathcal{B}^{(j)}$. For the completely decomposable group A above we can prove

LEMMA 3.12. *For each* $j \in J$ *the subgroups* $A^{(j)}$ *and* $A^{(j)} \oplus B_j$ *of* A *are absolute ideals of* A.

Proof: Suppose $(A, \cdot)$ is a ring on A such that $(A^{(j)}, \cdot)$ is not an ideal of $(A, \cdot)$. Then we can assume without loss of generality that there exist elements $a_1 \in A^{(j)}$, $a_2 \in A$ and $a_3 \in A \backslash A^{(j)}$ such that $a_1 \cdot a_2 = a_3$. This amounts to the existence of two components of the h-decomposition of A, C and D say, such that $C \in \mathcal{B}^{(j)}$, $D \notin \mathcal{B}^{(j)}$ and $t(C) \leqq' t(D)$. But then $t(B_j) \lneqq' t(C)$ shows $t(B_j) \lneqq' t(D)$, that is $D \in \mathcal{B}^{(j)}$. This proves the first assertion of the Lemma. The proof that $A^{(j)} \oplus B_j$ is an absolute ideal of A is identical. //

For each $j \in J$ we now give four conditions on the completely decomposable group A that will make the definition of the subgroup $K(B_j)$

of B_j and the proof of our major result far easier. For each $j \in J$ we say that A satisfies

j(1) if $\mathcal{B}^{(j)} \cup \{B_j\}$ contains a group C such that $J(C) = 0$.

j(2) if $\mathcal{B}^{(j)} \cup \{B_j\}$ contains no group with idempotent type.

j(3) if A does not satisfy j(1) and B_j has idempotent type.

j(4) if B_j has non-idempotent type, $\mathcal{B}^{(j)}$ contains at least one group with idempotent type, and every group with idempotent type in $\mathcal{B}^{(j)}$ has non-trivial absolute radical.

For each $j \in J$ it is readily checked that the conditions j(1), j(2), j(3) and j(4) are mutually exclusive and exhaustive conditions for A.

We need one final piece of notation. If for some $j \in J$, $t(B_j)$ is not idempotent and $\mathcal{B}^{(j)}$ contains some elements with idempotent type then define

$$P_1^{(j)} = \cup \{P_1^C \mid C \in \mathcal{B}^{(j)} \text{ and } t(C) \text{ is idempotent}\}.$$

Now for the definitions of $K(B_j)$, $j \in J$. For each $j \in J$ define the subgroup $K(B_j)$ of B_j as follows

$$K(B_j) = \begin{cases} 0 & \text{if A satisfies condition j(1).} \\ B_j & \text{if A satisfies condition j(2).} \\ J(B_j) & \text{if A satisfies condition j(3).} \\ \bigcap_{p \in P_1^{(j)}} pB_j & \text{if A satisfies condition j(4).} \end{cases}$$

From previous comments it is clear that for each $j \in J$, $K(B_j)$ is a well defined subgroup of B_j. It should be noted that although the above definitions of the groups $K(B_j)$ for $j \in J$ are complicated these definitions do make the statement of the main Theorem of this chapter extremely simple: we will prove that if A is a completely decomposable group with h-decomposition

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j$$

such that the set of non-idempotent types in $\tilde{T}(A)$ satisfies the ascending chain condition with respect to $\leqq'$ then

$$J(A) = \bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} K(B_j) .$$

The containment

$$\bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} K(B_j) \subseteqq J(A)$$

is proved by a series of lemmas.

LEMMA 3.13. *Suppose* A *is a completely decomposable group with h-decomposition*

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j ,$$

and $j \in J$ *is such that* A *satisfies condition* j(3). *Then* $|\mathcal{B}^{(j)}| < \infty$ *and if* $C \in \mathcal{B}^{(j)} \cup \{B_j\}$ *then* C *has finite rank*, t(C) *is idempotent and* $0 \neq |P_1^C| < \infty$.

Proof: Since A satisfies j(3), $t(B_j)$ is idempotent and $J(B_j) \neq 0$. Thus Theorem 3.10 yields $0 \neq |P_1^{B_j}| < \infty$. If $C \in \mathcal{B}^{(j)} \cup \{B_j\}$ then clearly $t(B_j) \leqq t(C)$. Consequently C must also have idempotent type and, since A does not satisfy j(1), Theorem 3.10 shows that C has finite rank and that $0 \neq |P_1^C| < \infty$. If now $|\mathcal{B}^{(j)}| = \infty$ then $t(B_j) \leqq t(C)$ for all $C \in \mathcal{B}^{(j)}$ yields $|P_1^{B_j}| = \infty$. Thus $|\mathcal{B}^{(j)}| < \infty$, as required. //

LEMMA 3.14. *Suppose* A *is a completely decomposable group of finite rank with h-decomposition*

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j ,$$

where each A_i, $i \in \{1, 2, \ldots, n\}$, *has idempotent type. If, for each* $j \in \{1, 2, \ldots, m\}$, A *satisfies condition* $j(3)$ *then*

$$\bigoplus_{i=1}^{n} J(A_i) \oplus \bigoplus_{j=1}^{m} K(B_j) \subseteq J(A).$$

Proof: We induct on $n + m$. Clearly there is nothing to prove when $n + m = 1$. Therefore assume the Lemma is true for all groups of the stated kind with the number of components in their h-decompositions being less than $n + m$, and consider A as stated in the Lemma. For each $i \in \{1, 2, \ldots, n\}$, A_i is an absolute ideal of A, so Lemma 1.18 yields $J(A_i) \subseteq J(A)$. Thus it suffices to show that if $j \in \{1, 2, \ldots, m\}$ is such that $K(B_j) \neq 0$ then $K(B_j) \subseteq J(A)$. We consider two cases:

Case (i). B_j does not have minimal type in $\tilde{T}(A)$ with respect to $\leq'$. In this case it is possible to select $j_1 \in \{1, 2, \ldots, m\}$ such that $j_1 \neq j$, $t(B_{j_1}) \leq' t(B_j)$ and B_{j_1} does have minimal type in $\tilde{T}(A)$ with respect to $\leq'$. From Lemma 3.12, $A^{(j_1)}$ is an absolute ideal of A, so by Lemma 1.18, $J(A^{(j_1)}) \subseteq J(A)$. It is readily checked that $A^{(j_1)}$ satisfies all the hypotheses of the Lemma. Thus, since $A^{(j_1)}$ is a proper subgroup of A, the induction assumption yields $K(B_j) \subseteq J(A^{(j_1)})$. Therefore $K(B_j) \subseteq J(A)$.

Case (ii). B_j does have minimal type in $\tilde{T}(A)$ with respect to $\leq'$. If $A^{(j)} \oplus B_j$ is a proper subgroup of A then since $A^{(j)} \oplus B_j$ satisfies all the hypotheses of the Lemma we can again use the induction assumption on the absolute ideal $A^{(j)} \oplus B_j$ of A to obtain $K(B_j) \subseteq J(A)$. Hence assume $A = A^{(j)} \oplus B_j$.

From Lemma 3.13, every $C \in \mathcal{B}^{(j)}$ must have idempotent type and $0 \neq |P_1^C| < \infty$. Now $A^{(j)}$ satisfies all the hypotheses of the Lemma, so the induction assumption applied to $A^{(j)}$ and the definitions of $K(B_{j'})$

for $j' \in J$ yield $\langle J(A^{(j)})\rangle_* = A^{(j)}$.

Suppose now $(A, \cdot)$ is an associative ring on A. Lemma 3.12 shows $(A^{(j)}, \cdot)$ is an ideal of $(A, \cdot)$, so $J(A^{(j)}, \cdot) \subseteqq J(A, \cdot)$. Since $J(A^{(j)}) \subseteqq J(A^{(j)}, \cdot)$, $A^{(j)} \subseteqq \langle J(A, \cdot)\rangle_*$. But $\langle J(A, \cdot)\rangle_* \subseteqq A^{(j)} \oplus B_j$, so

$$\langle J(A, \cdot)\rangle_* = A^{(j)} \oplus (B_j \cap \langle J(A, \cdot)\rangle_*) .$$

Now $B_j \cap \langle J(A, \cdot)\rangle_*$ is pure in A, so it is pure in B_j. Since B_j has finite rank, (3.5) shows that $B_j \cap \langle J(A, \cdot)\rangle_*$ is a direct summand of B_j. Hence there is a summand B_j' of B_j such that $A = \langle J(A, \cdot)\rangle_* \oplus B_j'$. From Lemma 3.13 we know that every $C \in \mathcal{B}^{(j)} \cup \{B_j\}$ has idempotent type and $0 \neq |P_1^C| < \infty$. Thus A is reduced and, for almost all primes p, $pA = A$. Consequently for almost all primes p, $p\langle J(A, \cdot)\rangle_* = \langle J(A, \cdot)\rangle_*$. Hence the proof of Lemma 3.7 indicates that either $A = \langle J(A, \cdot)\rangle_*$ or $J(B_j') = 0$. (3.6) implies that B_j' is a homogeneous completely decomposable group. Since $r(B_j')$ is finite, $t(B_j')$ is idempotent and $0 \neq |P_1^{B_j'}| < \infty$, the conclusion $J(B_j') = 0$ above contradicts Theorem 3.10. Therefore $A = \langle J(A, \cdot)\rangle_*$, and since this is true for every associative ring $(A, \cdot)$ on A, Proposition 1.20 yields $\bigcap_p pA \subseteqq J(A)$. It is clear that

$$K(B_j) = \bigcap_{p \in P_1^{B_j}} pB_j = \bigcap_p pB_j \subseteqq \bigcap_p pA .$$

Thus $K(B_j) \subseteqq J(A)$. //

COROLLARY 3.15. *Suppose* A *is a completely decomposable group with h-decomposition*

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j .$$

If $j \in J$ *is such that* A *satisfies condition* $j(3)$ *then* $K(B_j) \subseteqq J(A)$.

Proof: Let $j \in J$ and suppose A satisfies j(3). From Lemma 3.13 it follows that $A^{(j)} \oplus B_j$ is a finite rank completely decomposable group satisfying the conditions of Lemma 3.14. Clearly B_j is one of the homogeneous components in the h-decomposition of $A^{(j)} \oplus B_j$. Moreover, since B_j has non-maximal type in $\tilde{T}(A)$ with respect to $\leqq'$, B_j is one of the homogeneous components with non-maximal type (with respect to $\leqq'$) in the h-decomposition of $A^{(j)} \oplus B_j$. Evidently $A^{(j)} \oplus B_j$ satisfies condition j(3), so Lemma 3.14 now yields $K(B_j) \subseteq J(A^{(j)} \oplus B_j)$. From Lemma 3.12 we see that $J(A^{(j)} \oplus B_j) \subseteq J(A)$, so $K(B_j) \subseteq J(A)$, as required. //

LEMMA 3.16. *Let* A *be a completely decomposable group with h-decomposition*

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j ,$$

and suppose the non-idempotent types in $\tilde{T}(A)$ *satisfy the ascending chain condition with respect to* $\leqq'$. *If* $j \in J$ *is such that* A *satisfies condition* j(4) *then* $K(B_j) \subseteq J(A)$.

Proof: Suppose $j \in J$ is such that A satisfies j(4). Let $A_1^{(j)}$ be the direct sum of those groups in $\mathcal{B}^{(j)} \cup \{B_j\}$ with idempotent types, and let $A_2^{(j)}$ be the direct sum of those groups in $\mathcal{B}^{(j)} \cup \{B_j\}$ with non-idempotent types. Clearly $A^{(j)} \oplus B_j = A_1^{(j)} \oplus A_2^{(j)}$.

Suppose $A_1^{(j)}$ has h-decomposition

$$A_1^{(j)} = \bigoplus_{i_1 \in I_1} A_{i_1} \oplus \bigoplus_{j_1 \in J_1} B_{j_1} .$$

We know that every homogeneous component of $A_1^{(j)}$ has idempotent type and non-zero absolute radical, so for each $j_1 \in J_1$, $A_1^{(j)}$ must satisfy $j_1(3)$. Since A_{i_1} is an absolute ideal of $A_1^{(j)}$ for each $i_1 \in I_1$, Lemma 1.18 and

Corollary 3.15 now yield

$$\bigoplus_{i_1\in I_1} J(A_{i_1}) \oplus \bigoplus_{j_1\in J_1} K(B_{j_1}) \subseteqq J(A_1^{(j)}) .$$

From the definition of $P_1^{(j)}$ it is immediate that

$$\bigcap_{p\in P_1^{(j)}} pA_1^{(j)} = \bigoplus_{i_1\in I_1} (\bigcap_{p\in P_1^{A_{i_1}}} pA_{i_1}) \oplus \bigoplus_{j_1\in J_1} (\bigcap_{p\in P_1^{B_{j_1}}} pB_{j_1}) .$$

Now for each $j_1 \in J_1$, $K(B_{j_1}) = J(B_{j_1})$, so Theorem 3.10 yields

$$\bigcap_{p\in P_1^{(j)}} pA_1^{(j)} = \bigoplus_{i_1\in I_1} J(A_{i_1}) \oplus \bigoplus_{j_1\in J_1} K(B_{j_1}) .$$

Thus

$$\bigcap_{p\in P_1^{(j)}} pA_1^{(j)} \subseteqq J(A_1^{(j)}) .$$

Now consider an associative ring $(A, \cdot)$ on A and suppose $a \in \bigcap_{p\in P_1^{(j)}} p(A^{(j)} \oplus B_j)$. We are permitted to write a in the form $a = a_1 + a_2$ where $a_1 \in \bigcap_{p\in P_1^{(j)}} pA_1^{(j)}$ and $a_2 \in \bigcap_{p\in P_1^{(j)}} pA_2^{(j)}$. Now every group $C \in \mathcal{B}^{(j)}$ with idempotent type satisfies $0 \neq |P_1^C| < \infty$, so an argument similar to the proof of Lemma 3.12 shows that $A_1^{(j)}$ is an absolute ideal of A. Consequently $(\bigcap_{p\in P_1^{(j)}} pA_1^{(j)}, \cdot)$ is an ideal of $(A, \cdot)$. Therefore for each positive integer n, $a^n = x + a_2^n$ where $x \in \bigcap_{p\in P_1^{(j)}} pA_1^{(j)}$.

For the positive integer n consider the element a_2^n. If $a_2^n \notin A_1^{(j)}$ then we can obtain, in the same way as in Theorem 3.3, a chain of length n in $\tilde{T}(A)$. Moreover, the last term of this chain can be chosen to be a non-idempotent type in $\tilde{T}(A)$. Since $a_2 \in A_2^{(j)}$ the first term of this chain is a non-idempotent type greater than or equal to $t(B_j)$ (with respect to $\leqq'$). As before, if $C \in \mathcal{B}^{(j)}$ and $t(C)$ is idempotent then $0 \neq |P_1^C| < \infty$. Hence every element of this chain of length n in $\tilde{T}(A)$ must be a non-idempotent type. Arguing as in the first half of the proof of Theorem 3.4 we see that if $a_2^n \notin A_1^{(j)}$ for all positive integers n then there exists an infinite ascending chain in the non-idempotent types in $\tilde{T}(A)$. This violates our assumed ascending chain condition, so there is a positive integer n such that $a_2^n \in A_1^{(j)}$.

Now $A_1^{(j)}$ is pure in A, so $a_2^n \in \bigcap_{p \in P_1^{(j)}} pA_1^{(j)}$. Thus $a_2^n \in J(A_1^{(j)}) \subseteqq J(A_1^{(j)}, \cdot)$. Also $x \in J(A_1^{(j)}, \cdot)$, so $a^n \in J(A_1^{(j)}, \cdot)$. Since $A_1^{(j)}$ is an absolute ideal of A, $A_1^{(j)}$ is an absolute ideal of $A^{(j)} \oplus B_j$. From Lemma 3.12, $(A^{(j)} \oplus B_j, \cdot)$ is a ring on $A^{(j)} \oplus B_j$, so $J(A_1^{(j)}, \cdot) \subseteqq J(A^{(j)} \oplus B_j, \cdot)$. Consequently for the integer n above

$$(a + J(A^{(j)} \oplus B_j, \cdot))^n = J(A^{(j)} \oplus B_j, \cdot) .$$

The element a was chosen to be an arbitrary element of $\bigcap_{p \in P_1^{(j)}} p(A^{(j)} \oplus B_j)$, so

$$\left(\frac{\bigcap_{p \in P_1^{(j)}} p(A^{(j)} \oplus B_j) + J(A^{(j)} \oplus B_j, \cdot)}{J(A^{(j)} \oplus B_j, \cdot)}, \cdot \right)$$

is a nil ideal of $\left(\dfrac{A^{(j)} \oplus B_j}{J(A^{(j)} \oplus B_j, \cdot)}, \cdot \right)$. Since the latter ring is semisimple

$$\bigcap_{p \in P_1^{(j)}} p(A^{(j)} \oplus B_j) \subseteqq J(A^{(j)} \oplus B_j, \cdot) .$$

From Lemma 3.12, $J(A^{(j)} \oplus B_j, \cdot) \subseteqq J(A, \cdot)$, so $\bigcap_{p \in P_1^{(j)}} p(A^{(j)} \oplus B_j) \subseteqq J(A, \cdot)$. This is true for every associative ring on A, so

$$K(B_j) = \bigcap_{p \in P_1^{(j)}} pB_j \subseteqq J(A) . \quad //$$

PROPOSITION 3.17. *Let A be a completely decomposable group with h-decomposition*

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j .$$

If the non-idempotent types in $\tilde{T}(A)$ *satisfy the ascending chain condition with respect to* $\leqq'$, *then*

$$\bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} K(B_j) \subseteqq J(A) .$$

Proof: Corollary 3.15, Lemma 3.16 and the comments prior to (3.5) show that it suffices to prove that if $j \in J$ is such that A satisfies condition j(2) then $K(B_j) \subseteqq J(A)$. However, this is immediate since in this case $K(B_j) = B_j$, and Theorem 3.4 and Lemma 3.12 imply that any associative ring $(A, \cdot)$ on A will contain $(A^{(j)} \oplus B_j, \cdot)$ as a T-nilpotent (and therefore radical) ideal. //

We now turn our attention to proving the reverse inclusion

$$J(A) \subseteqq \bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} K(B_j)$$

for the completely decomposable group A as described in Proposition 3.17. Again this will be achieved with a series of lemmas. First however, we outline a method of constructing non-trivial associative rings on certain completely decomposable groups. These rings are fundamental for our subsequent results.

EXAMPLE 3.18. As usual let A be a completely decomposable group with h-decomposition

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j .$$

Choose for a fixed $j \in J$, a non-zero element $b_j \in B_j$, and suppose there is a $C \in \mathcal{B}^{(j)} \cup \{B_j\}$ such that $t(C)$ is idempotent.

Since B_j is a completely decomposable group it is possible to choose a finite rank completely decomposable direct summand of B_j containing b_j. Also, since $t(C)$ is idempotent and $t(B_j) \leqq' t(C)$ we can select a basis $\{b_{j_1}, b_{j_2}, \ldots, b_{j_{k(j)}}\}$ of this summand, and a non-zero element $c \in C$ such that

(i) there are non-zero integers $n, n_1, n_2, \ldots, n_{k(j)}$ with $(n, n_1, n_2, \ldots, n_{k(j)}) = 1$ such that

$$n\, b_j = n_1\, b_{j_1} + n_2\, b_{j_2} + \ldots + n_{k(j)}\, b_{j_{k(j)}},$$

(ii) $\langle b_{j_1}\rangle_* \oplus \langle b_{j_2}\rangle_* \oplus \ldots \oplus \langle b_{j_{k(j)}}\rangle_*$ is a direct summand of B_j,

(iii) $\chi(b_{j_k}) \leqq \chi(c)$ for each $k \in \{1, 2, \ldots, k(j)\}$,

(iv) $\langle c\rangle_*$ is a direct summand of C, and

(v) $\chi(c)$ contains 0's and ∞'s only.

From conditions (iii) and (v) it is evident that for each $k \in \{1, 2, \ldots, k(j)\}$, $\chi(b_{j_k})\,\chi(b_{j_k}) \leqq \chi(c)$, $\chi(b_{j_k})\,\chi(c) \leqq \chi(c)$ and $\chi(c)\,\chi(c) \leqq \chi(c)$. Thus for each $k \in \{1, 2, \ldots, k(j)\}$ it is possible to define a ring $(A, \cdot)$ on A by letting

$$b_{j_k} \cdot b_{j_k} = b_{j_k} \cdot c = c \cdot b_{j_k} = c \cdot c = c\ ,$$

and letting all other products not thus accounted for be zero. It is readily checked that $(A, \cdot)$ is an associative ring on A, $(A, \cdot)^2 \subseteq (\langle c\rangle_*, \cdot)$ and $(\langle c\rangle_*, \cdot)$ is an ideal of $(A, \cdot)$. //

LEMMA 3.19. *Suppose* A *is a completely decomposable group with h-decomposition*

$$A = \bigoplus_{i\in I} A_i \oplus \bigoplus_{j\in J} B_j\ ,$$

and suppose the component of $a \in J(A)$ *in* B_j *is* b_j, *for a fixed* $j \in J$.

If there is a group $C \in \mathcal{B}^{(j)} \cup \{B_j\}$ *such that* $J(C) = 0$ *,then* $b_j = 0$.

Proof: Suppose $b_j \neq 0$. Since $J(C) = 0$ it follows from Lemma 1.19 that we can assume $C \neq B_j$. We consider two distinct cases:

Case (i). C has finite rank. With the same notation as Example 3.18, $n_1 \neq 0$. Thus for $k = 1$, Example 3.18 yields an associative ring $(A, \cdot)$ on A.

Now $J(C) = 0$ and $(\langle c\rangle_*, \cdot)^2 \neq 0$. Thus Theorem 3.10 and the proof of Theorem 1.21 show $J(\langle c\rangle_*, \cdot) = 0$. Hence $J(\langle c\rangle_*) = 0$, so Lemma 1.19 shows that the component of a in $\langle c\rangle_*$ must be zero. From the construction of $(A, \cdot)$ it is now evident that

$$(na) \cdot c = (n_1 \, b_{j_1}) \cdot c = n_1 \, c \; .$$

Therefore $(na) \cdot c \in J(A, \cdot) \cap (\langle c\rangle_*, \cdot)$, so since $(\langle c\rangle_*, \cdot)$ is an ideal of $(A, \cdot)$, $(na) \cdot c \in J(\langle c\rangle_*, \cdot) = 0$. Thus $n_1 \, c = 0$, contradicting the fact that $n_1 \neq 0$. We conclude therefore that $b_j = 0$.

Case (ii). C has infinite rank. As in Example 3.18 it is possible to select non-zero elements $c_2, c_3, \ldots$ in C such that

$$\langle c_2\rangle_* \oplus \langle c_3\rangle_* \oplus \cdots \oplus \langle c_n\rangle_* \oplus \cdots$$

is a direct summand of C, and, for each $i \in \{2, 3, \ldots\}$, $\chi(b_{j_1}) \leqq \chi(c_i)$ and $\chi(c_i)$ consists of 0's and ∞'s only. We can now define a ring $(A, \cdot)$ on A by letting

$$b_{j_1} \cdot b_{j_1} = c_2, \; b_{j_1} \cdot c_i = c_i \cdot b_{j_1} = c_{i+1}, \; c_i \cdot c_j = c_j \cdot c_i = c_{i+j},$$

for all i, j in $\{2, 3, \ldots\}$, and letting all other products not thus accounted for be zero. It is a routine matter to verify that $(A, \cdot)$ is an associative ring on A, and that

$$(\langle c_2\rangle_* \oplus \langle c_3\rangle_* \oplus \cdots \oplus \langle c_n\rangle_* \oplus \cdots, \cdot)$$

is an ideal of $(A, \cdot)$. Denoting this ideal by $(I, \cdot)$, an analysis of the proof of Lemma 3.9 shows that $(I, \cdot)$ is a semisimple ring. If we now proceed in the same way as in Case (i) of this Lemma we obtain

$$(na) \cdot c_2 = n_1 c_3 \in J(I, \cdot) = 0 .$$

Therefore $n_1 = 0$, so again we conclude $b_j = 0$.

COROLLARY 3.20. *If* A *is a non-reduced completely decomposable group then* $J(A) = 0$. //

LEMMA 3.21. *Let* A *be a completely decomposable group with* h-*decomposition*.

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j ,$$

and suppose the component of $a \in J(A)$ *in* B_j *is* b_j, *for a fixed* $j \in J$. *If* $t(B_j)$ *is not idempotent and there exists a* $C \in \mathcal{B}^{(j)}$ *with idempotent type such that* $0 \neq |P_1^C| < \infty$ *then* $b_j \in \bigcap_{p \in P_1^C} pB_j$.

Proof: Assume $b_j \neq 0$. From Lemma 3.19 it is clear that every group in $\mathcal{B}^{(j)} \cup \{B_j\}$ must have non-zero absolute radical. With the notation of Example 3.18, for $k \in \{1, 2, \ldots, k(j)\}$ define the associative ring $(A, \cdot)$ on A as in Example 3.18. Suppose the component of a in J(C) is c_1. If $C = A_i$ for some $i \in I$ then $J(C) \subseteq J(A)$. On the other hand if $C = B_{j_1}$ for some $j_1 \in J$ then the fact that every group in $\mathcal{B}^{(j)}$ has non-zero absolute radical implies that A satisfies condition $j_1(3)$. Consequently Corollary 3.15 shows $J(C) = K(C) \subseteq J(A)$. Thus in either case, $J(C) \subseteq J(A)$. Hence $a_1 = a - c_1 \in J(A)$. In particular $a_1 \in J(A, \cdot)$, so $(na_1) \cdot b_{j_k} \in J(A, \cdot)$. But

$$(na_1) \cdot b_{j_k} = (n_k b_{j_k}) \cdot b_{j_k} = n_k c ,$$

so

$$n_k c \in J(A, \cdot) \cap (\langle c\rangle_*, \cdot) = J(\langle c\rangle_*, \cdot).$$

The definition of the ring $(\langle c\rangle_*, \cdot)$ and the proof of Theorem 1.21 together imply that $J(\langle c\rangle_*, \cdot) = \bigcap_{p \in P_1^C} p \langle c\rangle_*$. Consequently $n_k c \in \bigcap_{p \in P_1^C} p \langle c\rangle_*$, so since $\chi(c)$ consists of 0's and ∞'s only, n_k is divisible by each $p \in P_1^C$. Since this is true for every $k \in \{1, 2, \ldots, k(j)\}$, $b_j \in \bigcap_{p \in P_1^C} p B_j$. //

Now for the promised characterisation of the absolute radical.

THEOREM 3.22. *Suppose* A *is a completely decomposable group with h-decomposition*

$$A = \bigoplus_{i \in I} A_i \oplus \bigoplus_{j \in J} B_j .$$

If the set of non-idempotent types in $\tilde{T}(A)$ *satisfies the ascending chain condition with respect to* $\leqq'$ *then*

$$J(A) = \bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} K(B_j) .$$

Proof: From Proposition 3.17 it suffices to prove

$$J(A) \subseteq \bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} K(B_j) .$$

Lemma 1.19 yields

$$J(A) \subseteq \bigoplus_{i \in I} J(A_i) \oplus \bigoplus_{j \in J} J(B_j) ,$$

so any element $a \in J(A)$ can be written uniquely as a finite sum

$$a = \sum_{i \in I} a_i + \sum_{j \in J} b_j ,$$

where each $a_i \in J(A_i)$ and each $b_j \in J(B_j)$. If $j \in J$ is such that A satisfies condition j(1) then Lemma 3.19 shows $b_j \in K(B_j)$. Alternatively if $j \in J$ is such that A satisfies either condition j(2) or j(3), then the definition of $K(B_j)$ yields $b_j \in K(B_j)$. Thus it suffices to show that if $j \in J$ is such that A satisfies condition j(4) then $b_j \in K(B_j)$. But this is immediate from Lemma 3.21 and the definitions of $K(B_j)$ and $P_1^{(j)}$. //

It should be noted that it is difficult to delete from Theorem 3.22 the condition that the non-idempotent types in $\tilde{T}(A)$ satisfy the ascending chain condition with respect to $\leqq'$. Indeed if we did discard this condition, two major problems would immediately arise. The first is that we would now not be able to use Theorem 3.4 as we have done at several stages in the proof of Theorem 3.22. The second problem is that it is extremely difficult to define a useful associative ring on a completely decomposable group A that has an infinite ascending chain in $\tilde{T}(A)$ with respect to $\leqq'$. This latter problem has also arisen in Vinsonhaler and Wickless [1].

We conclude this chapter by showing that a completely decomposable group satisfying the conditions of Theorem 3.22 has the property that $J(A/J(A)) = 0$. The following lemma is required.

LEMMA 3.23. *Let* $A = \langle a\rangle \oplus A_1$ *where* $\langle a\rangle$ *is a cyclic group of order* p, *and* A_1 *is a rational group such that* $pA_1 \neq A_1$. *Then* $J(A) \subseteqq pA_1$.

Proof: From Lemma 1.19 and (1.10) we see that

$$J(A) \subseteqq J(\langle a\rangle) \oplus J(A_1) = J(A_1) ,$$

so $J(A) \subseteqq A_1$. Consider an element $a_1 \in A_1 \setminus pA_1$. Since A_1 is a rational group every element of A_1 can be written uniquely in the form $(n/m)a_1$, where n and $m \neq 0$ are suitable integers, $(n, m) = 1$ and $(m, p) = 1$. Now $\langle a\rangle$ has no m-torsion, so clearly $(1/m)a$ is a well defined element of $\langle a\rangle$. Thus we can define a ring $(A, \cdot)$ on A by letting

$$a_1 \cdot a = a \cdot a_1 = a \cdot a = a .$$

It is readily checked that $(A, \cdot)$ is an associative ring on A, and that $(\langle a\rangle, \cdot)$ is an ideal of $(A, \cdot)$. If now $a_1 \in J(A, \cdot)$, then

$$a_1 \cdot a = a \in J(A, \cdot) \cap (\langle a\rangle, \cdot) = J(\langle a\rangle, \cdot) .$$

However $(\langle a\rangle, \cdot)$ is a field, so $J(\langle a\rangle, \cdot) = 0$. Thus $a_1 \notin J(A, \cdot)$, and so $a_1 \notin J(A)$. We conclude therefore that $J(A) \subseteq pA_1$. //

THEOREM 3.24. *If* A *is a completely decomposable group such that the set of non-idempotent types in* $\tilde{T}(A)$ *satisfies the ascending chain condition with respect to* $\leq'$, *then* $J(A/J(A)) = 0$.

Proof: Suppose A has h-decomposition

$$A = \bigoplus_{i\in I} A_i \oplus \bigoplus_{j\in J} B_j .$$

From Theorem 3.22 we know

$$J(A) = \bigoplus_{i\in I} J(A_i) \oplus \bigoplus_{j\in J} K(B_j) ,$$

so

$$A/J(A) \cong \bigoplus_{i\in I} (A_i/J(A_i)) \oplus \bigoplus_{j\in J} (B_j/K(B_j)) .$$

Consider the group G defined by

$$(*) \qquad G = \bigoplus_{i\in I} (A_i/J(A_i)) \oplus \bigoplus_{j\in J} (B_j/K(B_j)) .$$

Corollary 3.11 shows that for each $i \in I$, $J(A_i/J(A_i)) = 0$. Thus Lemma 1.19 applied to the decomposition (*) yields

$$J(G) \subseteq \bigoplus_{j\in J} J(B_j/K(B_j)) .$$

Suppose now there is a non-zero element $g \in J(G)$. If for each $j \in J$ the component of g in $J(B_j/K(B_j))$ is denoted by $\overline{b}_j$, where $b_j \in B_j$, then there is a fixed $j \in J$ such that $\overline{b}_j \neq \overline{0}$. From the definitions of $K(B_j)$ and Corollary 3.11 it is clear that A cannot satisfy conditions j(2) or j(3).

If A satisfies condition j(1) then there is a $C \in \mathcal{B}^{(j)} \cup \{B_j\}$ with idempotent type such that $J(C) = 0$. If C occurs as a B_{j_1} for some $j_1 \in J$ then $K(C) = 0$. Consequently, whether C occurs as an A_i for some $i \in I$ or as a B_{j_1} for some $j_1 \in J$, the component of the decomposition (*) corresponding to C is precisely C. Since $K(B_j) = 0$, the component of (*) corresponding to B_j is B_j. Now $J(C/J(C)) = 0$ and $J(B_j/J(B_j)) \neq 0$, so C and B_j are distinct groups. Since $g \in J(G)$ and $J(C) = 0$, Lemma 1.19 shows that the component of g in $C \oplus B_j$ in the decomposition (*) is necessarily b_j. Applying Lemma 1.19 again to the decomposition (*) yields $b_j \in J(C \oplus B_j)$. However, from Theorem 3.22 we know that $J(C \oplus B_j) = 0$. Thus A cannot satisfy j(1).

We now know that A must satisfy condition j(4). If $K(B_j) \neq 0$ then, since B_j is a homogeneous completely decomposable group,

$$B_j/K(B_j) = B_j/\bigcap_{p \in P_1^{(j)}} pB_j$$

is necessarily an elementary torsion group. Thus (1.10) shows $J(B_j/K(B_j)) = 0$. Hence we may assume that $K(B_j) = \bigcap_{p \in P_1^{(j)}} pB_j = 0$.

In this case $B_j/K(B_j) = B_j$, so the component of the decomposition (*) corresponding to B_j is precisely B_j. Since B_j is a homogeneous completely decomposable group we may write $B_j = \bigoplus_{k \in K} (B_j)_k$, where K is some index set and each $(B_j)_k$ is a rational group. For each $k \in K$ let $(b_j)_k$ denote the component of b_j in $(B_j)_k$. Consider now $k \in K$ such that $(b_j)_k \neq 0$.

Let $p \in P_1^{(j)}$. Then there is a $C \in \mathcal{B}^{(j)}$ such that t(C) is idempotent, $J(C) \neq 0$ and $p \in P_1^C$. If C occurs as a B_{j_1} for some $j_1 \in J$ then, since A satisfies j(4), A will satisfy $j_1(3)$. Consequently $K(C) = J(C)$. Thus whether C occurs as an A_i for some $i \in I$ or as a

B_{j_1} for some $j_1 \in J$, the component of (*) corresponding to C is $C/J(C)$. Since $p \in P_1^C$ and $J(C) \neq 0$, it is evident from Theorem 3.10 that $C/J(C)$ is an elementary torsion group with a non-zero p-component. Hence we can select a cyclic elementary p-group $(C/J(C))^{(p)}$ that is a direct summand of $C/J(C)$.

Since $g \in J(G)$, Lemma 1.19 shows that the component of g in $((B_j)_k \oplus (C/J(C))^{(p)})$ in the decomposition (*) is $(b_j)_k + \overline{c}$, where $\overline{c} \in J((C/J(C))^{(p)})$. Applying Lemma 1.19 again yields

$$(b_j)_k + \overline{c} \in J((B_j)_k \oplus (C/J(C))^{(p)}).$$

But $(C/J(C))^{(p)}$ is a cyclic elementary p-group and $p((B_j)_k) \neq (B_j)_k$, so Lemma 3.23 implies $J((B_j)_k \oplus (C/J(C))^{(p)}) \subseteqq p(B_j)_k$. Thus $\overline{c} = 0$ and $(b_j)_k \in p(B_j)_k$. Since this is true for every $k \in K$ with $(b_j)_k \neq 0$, it is clear that $b_j \in pB_j$. Now the prime p was chosen to be an arbitrary prime in $P_1^{(j)}$, so $b_j \in \bigcap_{p \in P_1^{(j)}} pB_j = 0$, contradicting our choice of b_j.

We conclude therefore that $J(G) = 0$. Since $G \cong A/J(A)$, a reference to Lemma 1.17 completes the proof. //

We close this chapter with the remark that the conclusions of Theorem 3.22 and Theorem 3.24 are valid for completely decomposable groups of finite rank (and more generally, completely decomposable groups with a finite number of components in their h-decompositions). We omit the proof since it is virtually identical to the proofs of the forementioned Theorems. The stages at which the ascending chain condition is required can be surmounted by straightforward induction arguments.

CHAPTER FOUR

In this chapter we investigate ring structures supported by certain mixed groups. A class $\mathcal{A}$ of mixed groups of torsion-free rank one is introduced, and several questions concerning rings on groups in this class are discussed. We then give complete descriptions of the absolute annihilator and the absolute radical of groups in $\mathcal{A}$. The remainder of the chapter deals with other mixed groups that have properties similar to groups in $\mathcal{A}$. Amongst these is the reduced part of the additive group of a regular ring. A question of Fuchs [1] concerning these groups is answered in the negative. We then conclude the chapter with partial descriptions of the absolute annihilator and the absolute radical of a cotorsion group, and complete descriptions of the absolute annihilator and the absolute radical of a reduced algebraically compact group.

1. A CLASS OF MIXED GROUPS

Let $\mathcal{A}$ denote the class of groups A such that A has torsion-free rank one and A can be embedded as a pure subgroup of the direct product of its p-components A_p.

That is a group A in $\mathcal{A}$ has the property that it can be viewed as a pure subgroup of $\prod_p A_p$. For the remainder of this chapter groups in the class $\mathcal{A}$ will be thought of in this context. Since $\prod_p A_p / \bigoplus_p A_p$ is torsion-free and divisible, it follows immediately that $A/\bigoplus_p A_p$ is torsion-free divisible and of rank one.

Suppose now A is an arbitrary mixed group with torsion subgroup $\bigoplus_p A_p$. For a in $\prod_p A_p$ let $\overline{a}$ denote the image of a under the natural map $\prod_p A_p \to \prod_p A_p / \bigoplus_p A_p$. A characterisation of the groups in $\mathcal{A}$ can now be given.

PROPOSITION 4.1. *Let* A *be a group in* $\mathcal{A}$ *and suppose* a *is an element of infinite order in* A. *Then* A *is the inverse image of* $\langle\bar{a}\rangle_*$ *under the natural map* $\prod_p A_p \to \prod_p A_p/\bigoplus_p A_p$. *Conversely for* p-*groups* A_p *and any element* a *in* $\prod_p A_p$ *of infinite order, the group* A *defined as the inverse image of* $\langle\bar{a}\rangle_*$ *under the natural map* $\prod_p A_p \to \prod_p A_p/\bigoplus_p A_p$ *is a member of* $\mathcal{A}$.

Proof: Let $A \in \mathcal{A}$ and suppose a is an element of infinite order in A. Since $A/\bigoplus_p A_p$ is pure in $\prod_p A_p/\bigoplus_p A_p$, $\langle\bar{a}\rangle_* \subseteq A/\bigoplus_p A_p$. But then $\langle\bar{a}\rangle_*$ is pure in $A/\bigoplus_p A_p$, and since they are both torsion-free and of rank one, $\langle\bar{a}\rangle_* = A/\bigoplus_p A_p$.

To prove the second statement it suffices to show that such a group A is pure in $\prod_p A_p$. This is immediate since $\bigoplus_p A_p$ is pure in $\prod_p A_p$ and $A/\bigoplus_p A_p$ is pure in $\prod_p A_p/\bigoplus_p A_p$. //

As mentioned earlier, a group $A \in \mathcal{A}$ has the property that it is a pure subgroup of $\prod_p A_p$. Consequently a strong connection exists between the height matrices of elements of A and the indicators of certain elements of the p-components A_p. This can be stated as follows.

LEMMA 4.2. *If* A *is a group in* $\mathcal{A}$ *and* $a = (a_2, a_3, \ldots, a_p, \ldots)$ *is an arbitrary element of* A *then the* p-*indicator of* a *in* A *is the indicator of* a_p *in* A_p, *for each prime* p.

Proof: The p-indicator of a in A is

$$(h_p(a), h_p(pa), \ldots, h_p(p^k a), \ldots)\ .$$

Now

$$h_p(p^k a) = h_p(p^k a' + p^k(0, 0, \ldots, 0, a_p, 0, \ldots))$$

for all positive integers k, where

$$a' = (a_2, a_3, \ldots, a_p, \ldots) - (0, 0, \ldots, 0, a_p, 0, \ldots)$$

obviously has infinite p-height in $\prod_p A_p$, and so has infinite p-height in A. Thus

$$\begin{aligned} h_p(p^k a) &= h_p((0, 0, \ldots, 0, p^k a_p, 0, \ldots)) \\ &= h_p^{A_p}(p^k a_p) , \end{aligned}$$

for each positive integer k. //

We give another characterisation of the groups in $\mathcal{A}$. The method of proof is known.

PROPOSITION 4.3. *If* A *is a group in* $\mathcal{A}$ *then* A_p *is a direct summand of* A *for each prime* p. *Conversely if* A *is a non-splitting mixed group such that* $A/\bigoplus_p A_p \cong Q$ *and* A_p *is a direct summand of* A *for each prime* p, *then the reduced part of* A *is a member of* $\mathcal{A}$.

Proof: The first assertion is obvious. To prove the second part of the Proposition let A be a non-splitting mixed group with the stated properties. We need only show that if A is reduced then A is in $\mathcal{A}$. For each prime p, A_p is a summand of A, so there is a subgroup $A^{(p)}$ of A such that $A = A_p \oplus A^{(p)}$. Now $A^{(p)} \cong A/A_p$, and the latter group is an extension of the p-divisible group $\bigoplus_q A_q/A_p$ by the p-divisible group $A/\bigoplus_q A_q$. Thus $A^{(p)}$ is a p-divisible subgroup of A, and consequently $\bigcap_p A^{(p)}$ is a divisible subgroup of A. Since A is reduced, $\bigcap_p A^{(p)} = 0$. Hence the projections $A \to A_p$ yield an embedding of A in $\prod_p A_p$ containing $\bigoplus_p A_p$. The purity of the embedding follows from the purity of $\bigoplus_p A_p$ in $\prod_p A_p$ and the purity of $A/\bigoplus_p A_p$ in $\prod_p A_p/\bigoplus_p A_p$. //

A consequence of Proposition 4.3 is that if A is a reduced mixed group of torsion-free rank one then various conditions on either the endomorphism ring E(A) of A or the rings supported by A force A to be in the class $\mathcal{A}$. We mention a few examples: E(A) is generalised left continuous (Rangaswamy [1]), E(A) is π-regular (Fuchs and Rangaswamy [1]), A is the additive group of a regular ring (Fuchs [1]), or A is the additive group of an ideal in a π-regular ring with identity (Fuchs and Rangaswamy [1]). Also a reduced mixed group A with A/T(A) torsion-free divisible and of rank one such that either A has no elements of infinite p-height for all primes p and E(A) is commutative (Szele and Szendrei [1]), A has no elements of infinite p-height for all primes p and A supports a ring (A, $\cdot$) such that (A, $\cdot$) $\cong$ E(A) (Schultz [1]), A is associative-closed (see Proposition 4.19), or A supports only commutative rings (see Proposition 4.21) is in $\mathcal{A}$. Other similar examples to the ones above are abundant.

If A is a group in the class $\mathcal{A}$ then Proposition 4.3 shows us that one method of defining a ring on A is to define a ring on the p-component A_p of A for some prime p, and then extend this ring in the obvious way to a ring on A. The ring so obtained can be useful, as is demonstrated by Proposition 1.11; however it is not too complicated, in the sense that the product of any two elements of A is in the p-component A_p of A. Notice that the only other rings that have been defined on mixed groups, namely Szele's ring in (1.5) and the ring defined in the proof of Lemma 3.23 also share this property. In the next result we outline a method of defining more complicated ring structures on certain groups A in the class $\mathcal{A}$. We require the following notation.

Let A be a group in $\mathcal{A}$ and suppose $a = (a_2, a_3, \ldots, a_p, \ldots)$ is an element of infinite order in A. Define

$$P_1^{(a)} = \{p \in P | a_p \neq 0\} .$$

PROPOSITION 4.4. *Let* A *be a group in* $\mathcal{A}$ *and suppose* $a = (a_2, a_3, \ldots, a_p, \ldots)$ *is an element of infinite order in* A. *If for almost all primes* $p \in P_1^{(a)}$, $\langle a_p \rangle$ *is a direct summand of* A, *then there is an associative and commutative ring* $(A, \cdot)$ *on* A *such that* $(A, \cdot)^2 \nsubseteq T(A)$.

Proof: For each prime p for which $\langle a_p \rangle$ is a summand of A_p we define an associative and commutative ring $(A_p, \cdot)$ on A_p in the following manner. Suppose $A_p = \langle a_p \rangle \oplus A^{(p)}$ for some subgroup $A^{(p)}$ of A. Define an associative and commutative ring $(\langle a_p \rangle, \cdot)$ on $\langle a_p \rangle$ by letting $a_p \cdot a_p = a_p$, and define the trivial ring on $A^{(p)}$. Take the ring direct sum of these two rings to obtain an associative and commutative ring $(A_p, \cdot)$ on A_p. If q is a prime for which $\langle a_q \rangle$ is not a summand of A_q then define $(A_q, \cdot)$ to be the trivial ring on A_q.

Now take the ring direct product of the rings $(A_p, \cdot)$ to obtain an associative and commutative ring $(\prod_p A_p, \cdot)$ on $\prod_p A_p$. We show $(A, \cdot)$ is a subring of $(\prod_p A_p, \cdot)$ with the property that $(A, \cdot)^2 \nsubseteq T(A)$. This amounts to showing that if b and c are elements of infinite order in A then $b \cdot c$ is likewise an element of infinite order in A.

Since A has torsion-free rank one there are non-zero integers n_1, n_2, m_1 and m_2 such that $n_1 a = m_1 b$, $n_2 a = m_2 c$. But then $m_1 m_2 (b \cdot c) = n_1 n_2 (a \cdot a)$. For almost all $p \in P_1^{(a)}$, the definition of $(A, \cdot)$ shows $a_p \cdot a_p = a_p$. Thus $a \cdot a = a + t$ where $t \in T(A)$. If $O(t) = n$ then $n m_1 m_2 (b \cdot c) = n n_1 n_2 a$, so Proposition 4.1 shows $b \cdot c$ is an element of infinite order in A. //

From the comments prior to (1.5) we know that there exist no mixed nil groups, and that a torsion group is nil exactly if it is divisible. It seems a reasonable question therefore, to ask which groups A in $\mathcal{A}$ have the property that if (A, •) is any ring on A then $(A, \cdot)^2 \subseteq T(A)$. Equivalently, which groups A in $\mathcal{A}$ have the property that the natural embedding of Hom(A, Hom(A, T(A)) in Mult A is an isomorphism?

To answer this question we need to make the following definitions. Before doing this, however, notice that Lemma 4.2 implies that for a group A in $\mathcal{A}$ and an element a of infinite order in A, if, for some prime p, $U_p(a)$ commences with a non-infinity (∞) ordinal then $U_p(a)$ contains at least one gap.

Let A be a group in the class $\mathcal{A}$, and a be an element of infinite order in A. For the prime $p \in P_1^{(a)}$ we say that $U_p(a)$ is *reasonable* if $U_p(a) = (\infty, \infty, \ldots)$ or $U_p(a)$ commences with zero and contains only one gap. The height matrix H(a) is called a *reasonable matrix* if for almost all primes $p \in P_1^{(a)}$, $U_p(a)$ is reasonable. H(a) is called a *very reasonable matrix* if for almost all primes $p \in P_1^{(a)}$, $U_p(a) = (\infty, \infty, \ldots)$ or for almost all primes $p \in P_1^{(a)}$, $U_p(a)$ commences with zero and contains only one gap. Notice that if b is another element of infinite order in A then H(a) is (very) reasonable exactly if H(b) is (very) reasonable.

THEOREM 4.5. *Suppose* A *is a group in* $\mathcal{A}$ *and* a *is an element of infinite order in* A. *If there is a ring* (A, •) *on* A *such that* $(A, \cdot)^2 \nsubseteq T(A)$, *then* H(a) *is a reasonable matrix. Conversely, if* H(a) *is a very reasonable matrix then there is an associative and commutative ring* (A, •) *on* A *for which* $(A, \cdot)^2 \nsubseteq T(A)$.

Proof: To prove the first assertion suppose $H(a)$ is not a reasonable matrix, and consider any ring $(A, \cdot)$ on A. Then for infinitely many primes $p \in P_1^{(a)}$ there exists an integer $k(p)$ and an ordinal $\alpha_{k(p)}$ such that

$$h_p(p^{k(p)}a) = \alpha_{k(p)} ,$$

where $k(p) < \alpha_{k(p)} < \infty$. Thus $p^{k(p)}a = p^{k(p)+1}a'$ for some $a' \in A$. But then $p^{k(p)}(a \cdot a) = p(a' \cdot p^{k(p)}a)$. Since $a' \cdot p^{k(p)}a$ is an element of $p^{\alpha_{k(p)}}A$ it is clear that

$$h_p(p^{k(p)}(a \cdot a)) \geqq \alpha_{k(p)} + 1 .$$

Hence $H(a \cdot a)$ is not equivalent to $H(a)$, so $a \cdot a$ cannot have infinite order in A. Thus $(A, \cdot)^2 \subseteqq T(A)$.

To prove the converse statement suppose $H(a)$ is a very reasonable matrix. We deal with the two cases separately.

Case (i). For almost all primes $p \in P_1^{(a)}$, $U_p(a) = (\infty, \infty, \ldots)$. In this case there is a positive integer n such that $U_p(na) = (\infty, \infty, \ldots)$ for every prime p. It now follows from Fuchs [4], p. 198, that na belongs to the divisible part of A, so A splits, $A = T(A) \oplus A_1$ where A_1 is some subgroup of A, $A_1 \cong Q$. Clearly by defining the field $(A_1, \cdot)$ on A_1 and extending this in the usual manner to a ring $(A, \cdot)$ on A, we obtain an associative and commutative ring $(A, \cdot)$ on A for which $(A, \cdot)^2 \nsubseteq T(A)$.

Case (ii). For almost all primes $p \in P_1^{(a)}$, $U_p(a)$ commences with zero and contains only one gap. If we write $a = (a_2, a_3, \ldots, a_p, \ldots)$ where $a_p \in A_p$ for each prime p, it is clear that for almost all $p \in P_1^{(a)}$

$$U_p(a) = (0, 1, 2, \ldots, n_p - 1, \infty, \infty, \ldots) ,$$

where $n_p \geqq 1$ and $O(a_p) = n_p$. Lemma 27.2 of Fuchs [3] now shows that for almost all primes $p \in P_1^{(a)}$, $\langle a_p \rangle$ is a direct summand of A_p. A reference to Proposition 4.4 completes the proof. //

From Proposition 4.1 and Lemma 4.2 it is evident that there are groups A in $\mathcal{A}$ for which the height matrix of any element of infinite order in A is reasonable but not very reasonable. If, however, we restrict our attention to the subclass $\mathcal{A}_s$ of $\mathcal{A}$ consisting of those groups A in $\mathcal{A}$ with the property that for all relevant primes p, A_p is a reduced separable p-group (that is, has no elements of infinite p-height), than the concepts of a reasonable matrix and a very reasonable matrix coincide. (In fact in this case, for the element a in A having infinite order, H(a) is a reasonable matrix if for almost all $p \in P_1^{(a)}$, $U_p(a)$ commences with zero and contains only one gap). Thus

COROLLARY 4.6. *Suppose* A *is a group in* $\mathcal{A}_s$ *and* a *is an element of infinite order in* A. *Then the following conditions are equivalent:*

(i) H(a) *is reasonable;*

(ii) *there is a ring* (A, •) *on* A *for which* $(A, \cdot)^2 \nsubseteq T(A)$;

(iii) *there is an associative and commutative ring* (A, •) *on* A *for which* $(A, \cdot)^2 \nsubseteq T(A)$. //

We require the following general result.

LEMMA 4.7. *Suppose* a *and* b *are elements in an arbitrary group* A. *If* (A, •) *is any ring on* A *then* $H(a) \leqq H(a \cdot b)$.

Proof: The assertion follows immediately from the observation that for each prime p and each integer n,

$$h_p(p^n a) \leqq h_p(p^n a \cdot b) \ . \ //$$

Fuchs [4] has shown that a ring with (left) identity element exists on a torsion group A exactly if A is bounded. This result has an immediate generalisation to groups in the class $\mathcal{A}$.

PROPOSITION 4.8. *Let* A *be a group in* $\mathcal{A}$ *and suppose* a *is an element of infinite order in* A . *A ring with (left) identity can be defined on* A *if and only if for all relevant primes* p *there is an integer* n_p *such that* A_p *is bounded by* p^{n_p}, *and for almost all relevant primes* p,

$$U_p(a) = (0, 1, 2, \ldots, n_p - 1, \infty, \infty, \ldots) .$$

Proof: Suppose (A, •) is a ring with left identity defined on A. For each relevant prime p, $A = A_p \oplus A^{(p)}$ for some subgroup $A^{(p)}$ of A. The p-divisibility of $A^{(p)}$ shows that A_p must be reduced; for otherwise (0; (A, •)) is non-trivial and (A, •) will consequently not have a left identity element. Thus for each relevant prime p, (A, •) splits as the ring direct sum of $(A_p, \cdot)$ and $(A^{(p)}, \cdot)$. Evidently $(A_p, \cdot)$ must be a ring with left identity element, so A_p is bounded. Suppose p^{n_p} is the minimal bound of A_p.

If 1 is the left identity of A, then 1 has infinite order in A, so H(1) and H(a) are equivalent. Lemma 4.7 yields $H(1) \leq H(a')$ for all $a' \in A$. For each relevant prime p, A_p is bounded (minimally) by p^{n_p}, so A_p contains a direct summand $\langle a_p \rangle$ of order p^{n_p}. Clearly

$$U_p(a_p) = (0, 1, 2, \ldots, n_p - 1, \infty, \infty, \ldots) .$$

Thus Lemma 4.2 and the fact that the component of 1 in A_p is bounded by p^{n_p} now yield

$$U_p(1) = (0, 1, 2, \ldots, n_p - 1, \infty, \infty, \ldots) .$$

Consequently the element a of infinite order in A will have the required form.

Next suppose for all relevant primes p there is an integer n_p such that $p^{n_p} A_p = 0$ and, for almost all relevant primes p,

$$U_p(a) = (0, 1, 2, \ldots, n_p - 1, \infty, \infty, \ldots) .$$

It suffices to show that A is isomorphic to a group A' upon which a ring with identity can be defined.

For each relevant prime p, A_p is a direct sum of cyclic groups with a cyclic summand of order p^{n_p}, say $A_p = \langle a_p \rangle \oplus A_p^*$, where $O(a_p) = n_p$ and A_p^* is some subgroup of A_p. Letting a_q be zero when q is not a relevant prime of A, consider the element $a' = (a_2, a_3, \ldots, a_p, \ldots)$ in $\prod_p A_p$. Define A' to be the inverse image of $\langle \bar{a}' \rangle_*$ under the natural map $\prod_p A_p \to \prod_p A_p / \bigoplus_p A_p$. It is immediate that H(a') is in the equivalence class H(A) and the equivalence class H(A'), so (1.1) shows $A \cong A'$. For each relevant prime p a ring with identity a_p can be defined on A_p by defining the trivial ring on A_p^* and letting a_p act as multiplication by 1. As in the proof of Proposition 4.4 these rings extend to a ring $(A', \cdot)$ on A' for which $a' = (a_2, a_3, \ldots, a_p, \ldots)$ will be the identity element. //

We now turn our attention to characterising the absolute annihilator of a group in the class $\mathcal{A}$. A complete description can be given.

THEOREM 4.9. *Let* A *be a group in the class* $\mathcal{A}$. *If* A *is reduced then* $A(*) = A^{(a)}(*) = A^1$. *Otherwise* $A(*) = A^{(a)}(*) = (T(A))^1$.

Proof: Since $A \in \mathcal{A}$, A/T(A) is divisible. Hence if A is reduced, Corollary 1.13 yields $A(*) = A^{(a)}(*) = A^1$. Assume therefore that A is a non-reduced group. As in the proof of Corollary 1.13, $(T(A))^1 \subseteq A(*)$, so the proof is completed by showing $A^{(a)}(*) \subseteq (T(A))^1$.

Consider the case when A contains a divisible torsion subgroup. We are permitted to write $A = D \oplus A'$, where D is the divisible torsion part of A and A' is some subgroup of A. Let a be an element of infinite order in A and suppose its (non-zero) component in A' is a_0. Embed A' in its divisible hull $D' \oplus Q$, where D' is a torsion divisible group. If the component of a_0 in Q is a_0', then define the associative ring $(D \oplus Q, \cdot)$ on $D \oplus Q$ as in Szele's example (1.5) such that $a_0' \cdot a_0' \neq 0$. This ring can be extended in the obvious manner to an associative ring $(D \oplus Q \oplus D', \cdot)$ on $D \oplus Q \oplus D'$ which will contain $(A, \cdot)$ as a subring, since the product of any two elements of A is in D. Clearly $a \cdot a_0 = a_0' \cdot a_0' \neq 0$, so $a \notin A^{(a)}(*)$. Consequently, Corollary 1.13 yields $A^{(a)}(*) \subseteq (T(A))^1$.

If A does not contain a divisible torsion subgroup then A must split, so $A = T(A) \oplus B$ for some subgroup B of A such that $B \cong Q$. Lemma 1.14, (1.8) and (1.9) now show

$$A^{(a)}(*) \subseteq (T(A))^{(a)}(*) \oplus B^{(a)}(*) = (T(A))^1,$$

as required. //

A complete description of the absolute radical of a group in the class $\mathcal{A}$ can also be given.

THEOREM 4.10. *Suppose* A *is a group in* $\mathcal{A}$ *and* a *is an element of infinite order in* A. *Then* $J(A) = \bigcap_p pA$ *exactly when* H(a) *is not a reasonable matrix and, for almost all primes* p, $U_p(a)$ *does not commence with zero. Otherwise* $J(A) = \bigcap_p p(T(A))$.

Proof: Since $A \in \mathcal{A}$, Proposition 4.3 shows that for each prime p, $A = A_p \oplus A^{(p)}$, where $A^{(p)}$ is some p-divisible subgroup of A. Thus

Lemma 1.19 and (1.10) imply

$$J(A) \subseteq J(A_p) \oplus J(A^{(p)}) \subseteq pA_p \oplus A^{(p)} ,$$

so $J(A) \subseteq pA$.

Suppose $H(a)$ is not a reasonable matrix and, for almost all primes p, $U_p(a)$ does not commence with zero. Consider an associative ring $(A, \cdot)$ on A. Since $H(a)$ is not a reasonable matrix, Proposition 4.5 implies $(A, \cdot)^2 \subseteq T(A)$. Since for almost all primes p, $U_p(a)$ does not commence with zero, there is an integer n for which $na \in \bigcap_p pA$. Thus for every $b \in A$,

$$na \cdot b \in T(A) \cap (\bigcap_p pA) = \bigcap_p p(T(A)) .$$

From Lemma 1.18 and (1.10), $\bigcap_p p(T(A)) \subseteq J(A) \subseteq J(A, \cdot)$, so $na \cdot b$ is a right quasi-regular element of $(A, \cdot)$. Consequently (1.2) shows $na \in J(A, \cdot)$, so $A/J(A, \cdot)$ is a torsion group. Since this is true of every associative ring $(A, \cdot)$ on A, Proposition 1.20 yields $\bigcap_p pA \subseteq J(A)$, as required.

The other case occurs when, for infinitely many primes p, $U_p(a)$ commences with zero, or for almost all primes p, $U_p(a) = (\infty, \infty, \ldots)$. In the first case $J(A) \subseteq \bigcap_p pA$ shows $J(A)$ must be torsion. Thus $J(T(A)) \subseteq J(A)$ implies $J(A) = J(T(A)) = \bigcap_p p(T(A))$. In the second case $D(A)$, the maximal divisible subgroup of A, contains an element of infinite order. Hence A splits, so $A = T(A) \oplus A_1$, where A_1 is some subgroup of A such that $A_1 \cong Q$. Lemma 1.19 and Theorem 1.21 show

$$J(A) \subseteq J(T(A)) \oplus J(A_1) = J(T(A)) ,$$

so again $J(A) = \bigcap_p p(T(A))$. //

It should be noted that Proposition 1.22 shows that if A is a group in the class $\mathcal{A}$ then $J(A/J(A)) = 0$.

Another question that can be partially settled for groups in the class $\mathcal{A}$ concerns the endomorphism ring. Fuchs [4] has asked whether two mixed groups of torsion-free rank one are necessarily isomorphic if their endomorphism rings are isomorphic and their quotients mod torsion subgroups are isomorphic to Q. For the elementary properties of endomorphism rings we refer the reader to Fuchs [4], Section 106.

PROPOSITION 4.11. *Let* A *and* B *be groups in* $\mathcal{A}$ *such that* T(A) *and* T(B) *are both totally projective (or torsion-complete). Then* A *and* B *are isomorphic if and only if*

(i) E(A) *and* E(B) *are isomorphic, and*

(ii) H(A) *and* H(B) *are equivalent.*

Proof: It is well known that if $A \cong B$ then $E(A) \cong E(B)$. Thus (1.1) shows it suffices to prove that if $E(A) \cong E(B)$ then $T(A) \cong T(B)$.

Since $A \in \mathcal{A}$, Proposition 4.3 shows that for each prime p, $A = A_p \oplus A^{(p)}$ where $A^{(p)}$ is a p-divisible subgroup of A. Consider this decomposition for a fixed prime p. From Fuchs [4] we know that B splits also, so we are permitted to write $B = B_1 \oplus B_2$ for subgroups B_1, B_2 of B such that $E(A_p) \cong E(B_1)$, $E(A^{(p)}) \cong E(B_2)$. If B_1 contains an element of order q, q a prime, $q \neq p$, then B_1 will contain a direct summand $(B_1)_q$ that is a cocyclic q-group. If $\varepsilon : B_1 \to (B_1)_q$ is the corresponding projection then ε is a primitive idempotent. Denoting the isomorphism $E(B_1) \cong E(A_p)$ by ϕ, $\phi(\varepsilon)$ is then a primitive idempotent of $E(A_p)$, so $\phi(\varepsilon)A_p$ is a cocyclic direct summand of A_p such that $E(\phi(\varepsilon)A_p) \cong E((B_1)_q)$. Clearly this cannot be the case, so $T(B_1)$ is necessarily a p-group. Thus $T(B_1)$ is a direct summand of B_1, so if B_1 is not torsion then B_1 will contain a torsion-free divisible summand, B_3 say. The argument above applied to B_3 again yields a contradiction, so B_1 is necessarily a torsion group. Theorem 108.1 of Fuchs [4] now shows $A_p \cong B_1$. An argument similar to the one above applied to the second summand B_2 establishes $B_1 = B_p$, so $A_p \cong B_p$. Since this is true for each prime p, the Proposition follows. //

2. THE ADDITIVE GROUP OF A REGULAR RING

An associative ring (A, ·) on a group A is *regular* (in the sense of von Neumann) if for each $a \in A$ there is some $b \in A$ such that $aba = a$.

Fuchs [1] has proved the following.

(4.12) (Fuchs [1]). *The additive group* A *of a regular ring is the direct sum of a torsion-free divisible group and a reduced group* C *such that*

$$\bigoplus_p A_p \subseteq C \subseteq \prod_p A_p ,$$

where the A_p *are elementary* p*-groups and* $C/\bigoplus_p A_p$ *is torsion-free and divisible.* //

Fuchs asks which groups with this additive structure support regular rings. He answers this in the affirmative when $C = \bigoplus_p A_p$ or $C = \prod_p A_p$ by defining fields on each A_p and the torsion-free divisible part.

It is possible to also answer Fuchs' question in the affirmative for a large class of groups. Before doing this we need an alternate description of groups that embed purely in the direct product of their p-components. This description is very similar to the description of the groups in $\mathcal{A}$ given in Proposition 4.1.

PROPOSITION 4.13. *Let* A *be a pure subgroup of* $\prod_p A_p$, *and let* $\{a_i | i \in I\}$ *be a maximal independent set of elements of infinite order in* A. *Then* A *is the inverse image of* $\langle \bar{a}_i | i \in I\rangle_*$ *under the natural map* $\prod_p A_p \to \prod_p A_p / \bigoplus_p A_p$. *Conversely for* p*-groups* A_p *and any independent set* $\{a_i | i \in I\}$ *of elements of infinite order in* $\prod_p A_p$, *the group* A *defined as the inverse image of* $\langle \bar{a}_i | i \in I\rangle_*$ *under the natural map* $\prod_p A_p \to \prod_p A_p / \bigoplus_p A_p$ *is a pure subgroup of* $\prod_p A_p$ *containing* $\bigoplus_p A_p$.

Proof: Suppose A is a pure subgroup of $\prod_p A_p$, and $\{a_i | i \in I\}$ is a maximal independent set of elements of infinite order in A. Since $A/\bigoplus_p A_p$ is pure in $\prod_p A_p / \bigoplus_p A_p$, $\langle \bar{a}_i | i \in I\rangle_* \subseteq A/\bigoplus_p A_p$. Suppose now $\bar{a} \neq \bar{0}$ and $\bar{a} \in A/\bigoplus_p A_p$. Since $\{a_i | i \in I\}$ is a maximal independent set of elements of infinite order in A there are integers $n \neq 0, n_1, n_2, \ldots, n_k$ such that

$$na = n_1 a_{i_1} + n_2 a_{i_2} + \ldots + n_k a_{i_k} ,$$

for some $i_1, i_2, \ldots, i_k \in I$. Consequently

$$n\bar{a} = n_1 \bar{a}_{i_1} + n_2 \bar{a}_{i_2} + \ldots + n_k \bar{a}_{i_k} ,$$

so, since $\langle \bar{a}_i | i \in I\rangle_*$ is precisely the set of all elements of $\prod_p A_p / \bigoplus_p A_p$ depending upon the set $\{\bar{a}_i | i \in I\}$, $\bar{a} \in \langle \bar{a}_i | i \in I\rangle_*$.

The second statement is proved in the same way as the analogous statement in Proposition 4.1. //

Suppose now A is a pure subgroup of $\prod_p A_p$, where each A_p is an elementary p-group. If $\{a_i | i \in I\}$ is a maximal independent set of elements of infinite order in A, then Proposition 4.13 shows

$A/\bigoplus_p A_p = \langle\bar{a}_i \mid i \in I\rangle_*$. Moreover, since $A/\bigoplus_p A_p$ is torsion-free and divisible, $A/\bigoplus_p A_p \cong \bigoplus_{i\in I} \langle\bar{a}_i\rangle_*$. For each $i \in I$, a_i may be written uniquely in the form

$$a_i = (a_i^{(2)}, a_i^{(3)}, \ldots, a_i^{(p)}, \ldots)$$

where $a_i^{(p)} \in A_p$ for each prime p. Consider a relevant prime p. Since A_p is an elementary p-group we can write

$$A_p = \bigoplus_{k\in K_p} (A_p)_k ,$$

where K_p is some index set and $(A_p)_k \cong Z(p)$ for each $k \in K_p$. Thus for each $i \in I$, $a_i^{(p)}$ can be uniquely expressed as

$$a_i^{(p)} = \sum_{k\in K_p} (a_i^{(p)})_k ,$$

where the sum is finite and $(a_i^{(p)})_k \in (A_p)_k$ for each $k \in K_p$. For i and j in I we say that $a_i^{(p)}$ and $a_j^{(p)}$ *overlap* if for some $k \in K_p$, $(a_i^{(p)})_k \neq 0$ and $(a_j^{(p)})_k \neq 0$. If j and j_1 are in I we now define

$$P_{j,j_1}(\{a_i \mid i \in I\}) = \{p \in P \mid a_j^{(p)} \text{ and } a_{j_1}^{(p)} \text{ overlap}\} .$$

Finally we say that the set $\{a_i \mid i \in I\}$ satisfies the *finiteness condition* if for distinct j and j_1 in I, $|P_{j,j_1}(\{a_i \mid i \in I\})| < \infty$.

THEOREM 4.14. *Suppose* $A = B \oplus C$ *where* B *is torsion-free and divisible, and* C *is reduced such that*

$$\bigoplus_p A_p \subseteqq C \subseteqq \prod_p A_p ,$$

where each A_p *is an elementary* p*-group and* $C/\bigoplus_p A_p$ *is torsion-free and divisible. If* $\{a_i \mid i \in I\}$ *is a countable maximal independent set of elements of infinite order in* C *satisfying the finiteness condition then* A *supports a commutative regular ring.*

Proof: We need only prove the case when C contains a countably infinite maximal independent set of elements of infinite order; the proof when C contains a finite maximal independent set of elements of infinite order is virtually identical.

By taking the direct sum of a field on B and a regular ring on C we obtain a regular ring on A. Thus we restrict our attention to defining a regular ring on a reduced group A that embeds purely in $\prod_p A_p$, where each A_p is an elementary p-group, $A/\bigoplus_p A_p$ is a torsion-free divisible group and $\{a_n \mid n \in Z^+\}$ is a countably infinite maximal independent set of elements of infinite order in A satisfying the finiteness condition.

For each relevant prime p, write

$$A_p = \bigoplus_{k \in K_p} (A_p)_k ,$$

as we have done above, and consider $k \in K_p$. If there does not exist an $n \in Z^+$ such that $(a_n^{(p)})_k \neq 0$ then let $((A_p)_k, \cdot)$ be a field on $(A_p)_k$, defined arbitrarily. If, however, there is an $n \in Z^+$ such that $(a_n^{(p)})_k \neq 0$ then select n minimal with respect to this property. Now define the field $((A_p)_k, \cdot)$ on $(A_p)_k$ by letting $(a_n^{(p)})_k \cdot (a_n^{(p)})_k = (a_n^{(p)})_k$. Taking the ring direct sum of these fields over $k \in K_p$ we obtain associative and commutative rings $(A_p, \cdot)$ on A_p, for every relevant prime p. Now define $(\prod_p A_p, \cdot)$ to be the ring direct product of the rings $(A_p, \cdot)$. Clearly $(\prod_p A_p, \cdot)$ is an associative and commutative ring on $\prod_p A_p$. We show $(A, \cdot)$ is a subring of $(\prod_p A_p, \cdot)$.

For any positive integer k, and for each positive integer $\ell < k$, $|P_{\ell,k}(\{a_n \mid n \in Z^+\})| < \infty$, so the definitions of $(A_p, \cdot)$ for each prime p involved in a_k shows $\bar{a}_k \cdot \bar{a}_k = \bar{a}_k$. Also for distinct positive integers k and ℓ, $|P_{\ell,k}(\{a_n | n \in Z^+\})| < \infty$ shows $\overline{a}_k \cdot \overline{a}_\ell = \overline{0}$. If now b and c are elements of infinite order in A then there are integers $n \neq 0$, $m \neq 0$ and sets of integers $\{n_k | k \in Z^+\}$, $\{m_k | k \in Z^+\}$ such that

$$n\overline{b} = n_1\, \overline{a}_1 + n_2\, \overline{a}_2 + \ldots + n_k\, \overline{a}_k + \ldots$$

and

$$m\overline{c} = m_1\, \overline{a}_1 + m_2\, \overline{a}_2 + \ldots + m_k\, \overline{a}_k + \ldots$$

where almost all of the n_k's and m_k's are zero. Thus

$$nm(\overline{b} \cdot \overline{c}) = n_1\, m_1\, \overline{a}_1 + n_2\, m_2\, \overline{a}_2 + \ldots + m_k\, n_k\, \overline{a}_k + \ldots$$

where again almost all of the $n_k\, m_k$ are zero. Proposition 4.13 now yields $b \cdot c \in A$. It follows therefore that $(A, \cdot)$ is a subring of $(\prod_p A_p, \cdot)$.

As mentioned earlier, $A/ \bigoplus_p A_p = \bigoplus_{n=1}^{\infty} \langle \overline{a}_n \rangle_*$. From the definition of $(A, \cdot)$ it is clear that $(A/ \bigoplus_p A_p, \cdot)$ is a ring direct sum of the fields $(\langle a_n \rangle_*, \cdot)$, so $(A/ \bigoplus_p A_p, \cdot)$ is a regular ring. Thus $(A, \cdot)$ is an extension of the regular ring $(\bigoplus_p A_p, \cdot)$ by the regular ring $(A/ \bigoplus_p A_p, \cdot)$. A reference to Kaplansky [2], p. 112 now shows $(A, \cdot)$ is regular. //

COROLLARY 4.15. *Let* $A = B \oplus C$ *where* B *is torsion-free and divisible and* C *is reduced such that*

$$\bigoplus_p A_p \subseteq C \subseteq \prod_p A_p \ ,$$

where each A_p *is an elementary* p*-group and* $C/ \bigoplus_p A_p$ *is torsion-free divisible and of rank one. Then* A *supports a commutative regular ring.* //

It is not true in general that a group with the structure described in (4.12) is the additive group of a regular ring. To provide counter examples we need the following general result.

PROPOSITION 4.16. *If* $A = \prod_p A_p$ *where each* A_p *is a reduced* p-*group, then every ring* $(A, \cdot)$ *on* A *is the direct product of the rings* $(A_p, \cdot)$.

Proof: Let A and $(A, \cdot)$ be as stated in the Proposition. For each prime p, A_p is p-reduced, so $\prod_{q \neq p} A_q$ is the maximal p-divisible subgroup of A. As such $\prod_{q \neq p} A_q$ is an absolute ideal of A. Hence

$$(A, \cdot) = (A_p, \cdot) \oplus (\prod_{q \neq p} A_q, \cdot) ,$$

where the direct sum is the ring direct sum. The proof is now completed by observing that if a and b are elements of A with p-components a_p and b_p respectively, then the p-component of $a \cdot b$ is $a_p \cdot b_p$. //

We also require

LEMMA 4.17. *Suppose* A *is a reduced group such that* $A/T(A)$ *is divisible and* A_p *is bounded, for each prime* p. *Then* A *is a pure subgroup of* $\prod_p A_p$ *with the property that every ring on* A *extends uniquely to a ring on* $\prod_p A_p$.

Proof: The first claim follows from the proof of Proposition 4.3. To prove the second assertion of the Lemma we use the proof of Theorem 119.3 of Fuchs [4]. All that we need verify is that $\prod_p A_p$ is a reduced algebraically compact group for which $\prod_p A_p/A$ is divisible. This is immediate from the fact that each A_p is bounded and $\prod_p A_p/\bigoplus_p A_p$ is divisible.//

Consider now $\prod_p A_p$, where for each prime p, A_p is a p-group.

If $a \in \prod_p A_p$ then as usual write

$$a = (a^{(2)}, a^{(3)}, \ldots, a^{(p)}, \ldots) ,$$

where $a^{(p)} \in A_p$ for each prime p. For elements $a_1, a_2, \ldots, a_n$ of A, define, for each $i \in \{1, 2, \ldots, n\}$

$$P_i = \{p \in P | a_i^{(p)} \neq 0\}$$

and

$$P_i' = \{p \in P | a_i^{(p)} \neq 0 \text{ and } a_j^{(p)} = 0 \text{ for } j \neq i\}$$

EXAMPLE 4.18. For each integer n greater than 1 we give examples of reduced groups with structures as described in (4.12) that cannot support a regular ring.

Consider the group $\prod_{p \in P} Z(p)$ and elements $a_1, a_2, \ldots, a_n$ in $\prod_{p \in P} Z(p)$ such that $|P_1 \cap P_2| = \infty$ and, for each $i \in \{1, 2, \ldots, n\}$, $|P_i'| = \infty$. It is readily checked that for each integer $n > 1$ such a choice of $a_1, a_2, \ldots, a_n$ is possible. Next define A to be the inverse image of $\langle \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \rangle_*$ under the natural map $\prod_{p \in P} Z(p) \to \prod_{p \in P} Z(p) / \bigoplus_{p \in P} Z(p)$. It is clear that A is reduced containing $\bigoplus_{p \in P} Z(p)$ as torsion subgroup, A has torsion-free rank n, and A/T(A) is divisible, so A indeed has the structure described in (4.12).

Suppose now $(A, \cdot)$ is a regular ring on A. Lemma 4.17 shows that $(A, \cdot)$ extends to a ring $(\prod_{p \in P} Z(p), \cdot)$ on $\prod_{p \in P} Z(p)$. From Proposition 4.16 we see that $(\prod_{p \in P} Z(p), \cdot)$ is the ring direct product of the rings $(Z(p), \cdot)$, $p \in P$. Now for each prime $p \in P$, $(Z(p), \cdot)$ is an ideal of $(A, \cdot)$, so $(Z(p), \cdot)$ must also be a regular ring. Thus $(Z(p), \cdot)$ is a field, so $|P_1 \cap P_2| = \infty$ implies $a_1 \cdot a_2$ is an element of infinite order in A. Hence there are integers $\ell \neq 0, \ell_1, \ell_2, \ldots, \ell_n$ such that

$$\ell(a_1 \cdot a_2) = \ell_1 a_1 + \ell_2 a_2 + \dots + \ell_n a_n ,$$

or equivalently, for each prime p,

$$(*) \qquad \ell(a_1^{(p)} \cdot a_2^{(p)}) = \ell_1 a_1^{(p)} + \ell_2 a_2^{(p)} + \dots + \ell_n a_n^{(p)} .$$

If now $p \in P_k'$ for some $k \in \{1, 2, \dots, n\}$, then (*) reduces to

$$0 = \ell_k a_k^{(p)} .$$

Since for each $k \in \{1, 2, \dots, n\}$, $|P_k'| = \infty$, it is clear that for each $k \in \{1, 2, \dots, n\}$, $\ell_k = 0$. Therefore $a_1 \cdot a_2 \in T(A)$, contradicting the fact that it has infinite order in A. We conclude therefore that A can never support a regular ring. //

Another class of groups some of whose members resemble those of the class $\mathcal{A}$ is the class of associative-closed groups. A group A is *associative-closed* if the set of associative multiplications in Mult A is a subgroup of Mult A. These groups have been extensively studied by Hardy [1], and as usual the torsion case has been well described. Hardy has shown that a torsion group A is associative-closed if and only if A_p has cyclic reduced component for each prime p. If now A is a mixed associative-closed group then for each prime p, the reduced part of A_p must be cyclic. Indeed if p is a prime for which the reduced part of A_p is not cyclic, then A decomposes as

$$A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus A_1$$

where $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are cyclic p-groups and A_1 is some subgroup of A. Evidently $\langle a_1 \rangle \oplus \langle a_2 \rangle$ is not associative-closed, so A itself cannot be associative-closed. A partial converse is contained in the next result.

PROPOSITION 4.19. *Suppose* A *is a reduced group such that* A/T(A) *is divisible. Then* A *is associative-closed if and only if* A_p *is cyclic for each prime* p.

Proof: From the above comments we need only show that if A_p is cyclic for each prime p then A is associative-closed. Consider an associative ring (A, •) on A. It follows from Lemma 4.17 that A is a pure subgroup of $\prod_p A_p$ with the property that (A, •) extends to a ring $(\prod_p A_p, \cdot)$ on $\prod_p A_p$. From Proposition 4.16 we know that $(\prod_p A_p, \cdot)$ is the direct product of the associative rings $(A_p, \cdot)$, $p \in P$. Consequently $(\prod_p A_p, \cdot)$ is an associative ring. The proof is now completed by observing that A_p associative-closed for each prime p implies $\prod_p A_p$ is associative-closed. //

Hardy [1] has also investigated the Lie-closed groups; those groups whose set of Lie-multiplications in Mult A form a subgroup of Mult A. Similar arguments to the ones above verify

PROPOSITION 4.20. *Let* A *be a reduced group such that* A/T(A) *is divisible. Then* A *is Lie-closed if and only if* A_p *has rank less than or equal to two, for all primes* p. //

Finally we consider groups that support only commutative rings. If A is such a group then clearly, for each prime p, the reduced part of A_p must be cyclic. Indeed we can prove

PROPOSITION 4.21. *Suppose* A *is a reduced group such that* A/T(A) *is divisible. Then* A *supports only commutative rings if and only if* A_p *is cyclic for each prime* p.

Proof: The proof is very similar to the proof of Proposition 4.19. Again we need only verify that if A_p is cyclic for each prime p then A supports only commutative rings. Let (A, •) be a ring on A.

From Proposition 4.16 and Lemma 4.17, $(A, \cdot)$ can be considered as a subring of the ring $\prod_p (A_p, \cdot)$ on $\prod_p A_p$. Since for each prime p, A_p is cyclic, $\prod_p (A_p, \cdot)$ is a commutative ring. Thus $(A, \cdot)$ will also be a commutative ring. //

3. COTORSION GROUPS, ALGEBRAICALLY COMPACT GROUPS

We commence this section with a discussion of the structures of the absolute annihilator of a cotorsion group and the absolute annihilator of a reduced algebraically compact group. For the elementary properties of cotorsion groups and algebraically compact groups we refer the reader to the appropriate sections of Fuchs [3].

A similarity exists between the above mentioned groups and groups in the class $\mathcal{A}$; namely, if A is a reduced cotorsion group then A may be written uniquely in the form $A = \prod_p A_{(p)}$ where, for each prime p, $A_{(p)}$ is a reduced cotorsion group which is a p-adic module. In the special case that A is a reduced algebraically compact group, each $A_{(p)}$ is a reduced algebraically compact group that is also complete in its p-adic topology.

THEOREM 4.22. *If* A *is a cotorsion group then* $A(*) \subseteqq A^{(a)}(*) \subseteqq A^1$. *If, further,* A *is an adjusted cotorsion group then* $A(*) = A^{(a)}(*) = A^1$.

Proof: If A is a cotorsion group, write $A = D \oplus R$ where D is divisible and R is reduced. From Lemma 1.14, $A^{(a)}(*) \subseteqq D^{(a)}(*) \oplus R^{(a)}(*)$. Now since $D^{(a)}(*) \subseteqq D = D^1$, it is clear that to prove the first assertion of the Theorem it suffices to show $R^{(a)}(*) \subseteqq R^1$. Thus assume A is a reduced group. Now A may be written uniquely in the form $A = \prod_p A_{(p)}$, where, for each prime p, $A_{(p)}$ is a reduced cotorsion group which is a p-adic module. Since for every prime p, $\prod_{q \neq p} A_{(q)}$ is a p-divisible subgroup of A, an argument similar to the one above shows that we may further restrict our attention to cotorsion groups A that are also p-adic modules, for some prime p.

Let a be an element of A that has finite p-height, say $h_p(a) = n$. We show $a \notin A^{(a)}(*)$. If B is a p-basic submodule of A then $A = B + p^{n+1}A$, so $a = b + p^{n+1}a'$ where $b \in B$, $b \neq 0$ and $a' \in A$. Now B is a direct sum of cyclic p-adic modules, so it is possible to choose a cyclic submodule (and summand) B_1 of B for which b has a non-zero component b_1 in B_1. Since B_1 is a cyclic p-adic module, B_1 is either a cyclic p-group or the p-adic integers. In either case B_1 is a pure subgroup of A that is algebraically compact, so B_1 is a direct summand of A. Suppose $A = B_1 \oplus A_1$ for some subgroup A_1 of A.

If B_1 is a cyclic p-group then it is possible to define an associative and commutative ring $(B_1, \cdot)$ on B_1 for which $b_1 \cdot b_1 \neq 0$. On the other hand if B_1 is a copy of the p-adic integers then define $(B_1, \cdot)$ to be the ring of p-adic integers. In either case $(B_1, \cdot)$ is an associative and commutative ring on B_1 such that $b_1 \cdot b_1 \neq 0$. Letting $(A_1, \cdot)$ be the trivial ring on A define $(A, \cdot)$ to be the ring direct sum of $(B_1, \cdot)$ and $(A_1, \cdot)$. Clearly, $(A, \cdot)$ is an associative and commutative ring on A such that

$$a \cdot b_1 = b_1 \cdot b_1 \neq 0 ,$$

so $a \notin (0; (A, \cdot))$. Thus $a \notin A^{(a)}(*)$, whence $A(*) \subseteq A^{(a)}(*) \subseteq A^1$.

Suppose now A is an adjusted cotorsion group. Then A is reduced and A/T(A) is divisible, so the second assertion of the Theorem follows immediately from Corollary 1.13. //

COROLLARY 4.23. *If* A *is a reduced algebraically compact group then* $A(*) = A^{(a)}(*) = 0$.

Proof: For a reduced algebraically compact group A, $A^1 = 0$. //

With methods of proof similar to those used above we can give the corresponding description of the absolute radical of a cotorsion group.

THEOREM 4.24. *If* A *is a cotorsion group then* $J(A) \subseteq \bigcap_p pA$.

Proof: As in the proof of Theorem 4.22, Lemma 1.19 allows us to restrict our attention to cotorsion groups A that are also p-adic modules, for some prime p. In this case we need to prove $J(A) \subseteq pA$. We proceed as in the proof of Theorem 4.22.

Suppose $a \notin pA$. Again let B be a p-basic submodule of A. Then $A = B + pA$, so a must have a non-zero component in some direct summand B_1 of B. Furthermore we can assume that B_1 is either a cyclic p-group or a copy of the p-adic integers, and that the component of a in B_1 is not in pB_1. As before B_1 is a direct summand of A. From (1.10) and the fact that $J(Q_p^*) = pQ_p^*$, (Q_p^* being the ring of p-adic integers), no matter what form B_1 takes, it is possible to define an associative and commutative ring $(B_1, \cdot)$ on B_1 such that $J(B_1, \cdot) = pB_1$. If we extend this ring in the usual manner to an associative and commutative ring $(A, \cdot)$ on A, it is clear that $a \notin J(A, \cdot)$. Consequently $J(A) \subseteq pA$. //

COROLLARY 4.25. *If* A *is a reduced algebraically compact group then* $J(A) = \bigcap_p pA$.

Proof: If A is a reduced algebraically compact group then A can be written as $A = \prod_p A_{(p)}$ where each $A_{(p)}$ is a p-adic module complete in its p-adic topology. For each prime p, $A_{(p)}$ is p-reduced and $\prod_{q \neq p} A_{(q)}$

is p-divisible, so the proof of Proposition 4.16 shows that every associative ring $(A, \cdot)$ on A is the direct product of the associative rings $(A_{(p)}, \cdot)$. Thus it suffices to prove $pA \subseteq J(A)$ when A is a p-adic module complete in its p-adic topology, for some prime p.

Since A is complete, $A \cong \varprojlim_k A/p^kA$. If now $(A, \cdot)$ is any associative ring on A, (1.10) yields $p(A/p^kA) \subseteq J(A/p^kA, \cdot)$, for all positive integers k. It is now readily checked that since the natural maps $A/p^kA \to A/p^\ell A$, k and ℓ positive integers, $\ell < k$, are epimorphisms, $A_1 = \{p(A/p^kA) | k \in Z^+\}$ and $A_2 = \{J(A/p^kA, \cdot) | k \in Z^+\}$ together with the maps of the inverse system $\{A/p^kA | k \in Z^+\}$ form two inverse systems for which there is a monomorphism $\phi : A_1 \to A_2$. Theorem 12.2 of Fuchs [3] now shows

$$\varprojlim_k p(A/p^kA) \subseteq \varprojlim_k J(A/p^kA, \cdot) ,$$

and Theorem 1 of Ion [1] yields

$$\varprojlim_k J(A/p^kA, \cdot) = J(\varprojlim_k (A/p^kA, \cdot)) .$$

A trivial calculation proves

$$p(\varprojlim_k A/p^kA) \subseteq \varprojlim_k p(A/p^kA) ,$$

so

$$p(\varprojlim_k A/p^kA) \subseteq J(\varprojlim_k(A/p^kA, \cdot)) .$$

Thus $pA \subseteq J(A, \cdot)$, and since this is true for every associative ring $(A, \cdot)$ on A, $pA \subseteq J(A)$, as required. //

COROLLARY 4.26. *If* A *is a reduced algebraically compact group then* $J(A/J(A)) = 0$. //

It should also be noted that if A is an adjusted cotorsion group then $J(A/J(A)) = 0$. This is an immediate consequence of Proposition 1.22.

Our knowledge of the absolute radical of a reduced algebraically compact group allows us to answer in the negative a question raised by Rotman [2]. First however, notice that Theorem 119.5 of Fuchs [4] shows that every ring on a reduced group A can be extended uniquely to a ring on $\mathrm{Ext}(Q/Z, A)$. Rotman now asks the following question. If $(A, \cdot)$ is a semisimple ring on a reduced group A then is the induced ring $(\mathrm{Ext}(Q/Z, A), \cdot)$ on $\mathrm{Ext}(Q/Z, A)$ also semisimple? We close this chapter with the following proposition.

PROPOSITION 4.27. *Suppose* $(A, \cdot)$ *is a semisimple ring on the reduced group* A. *If* A *is torsion-free then* $J(\mathrm{Ext}(Q/Z, A), \cdot) \neq 0$. *However, if* A *is a torsion group or* A *is a mixed group such that* $A/T(A)$ *is divisible then* $J(\mathrm{Ext}(Q/Z, A), \cdot) = 0$.

Proof: First suppose A is torsion-free, in which case $\mathrm{Ext}(Q/Z, A)$ is a reduced algebraically compact group. Write

$$\mathrm{Ext}(Q/Z, A) = \prod_p (\mathrm{Ext}(Q/Z, A))_{(p)} ,$$

where each $(\mathrm{Ext}(Q/Z, A))_{(p)}$ is a reduced algebraically compact group complete in its p-adic topology. Since each $(\mathrm{Ext}(Q/Z, A))_{(p)}$ is also a p-adic module, Corollary 4.25 yields

$$J(\mathrm{Ext}(Q/Z, A)) = \prod_p p(\mathrm{Ext}(Q/Z, A))_{(p)} .$$

Thus

$$\prod_p p(\mathrm{Ext}(Q/Z, A))_{(p)} \subseteq J(\mathrm{Ext}(Q/Z, A), \cdot) .$$

Now $p(\mathrm{Ext}(Q/Z, A))_{(p)} \neq 0$ for at least one prime p, so

$J(\mathrm{Ext}(Q/Z, A), \cdot) \neq 0$.

Next suppose A is a torsion group, or A is a mixed group such that $A/T(A)$ is divisible. In either case Fuchs [3], p. 234 shows that $\mathrm{Ext}(Q/Z, A)$ can be written uniquely in the form

$$\mathrm{Ext}(Q/Z, A) = \prod_p \mathrm{Ext}(Z(p^\infty), A) .$$

Consider a fixed prime p. It is readily checked that

$$\begin{aligned}\mathrm{Ext}(Z(p^\infty), A) &\cong \mathrm{Ext}(Z(p^\infty), T(A)) \\ &\cong \mathrm{Ext}(Z(p^\infty), A_p) .\end{aligned}$$

Since $J(A, \cdot) = 0$ it follows that $J(A_p, \cdot) = 0$. Hence (1.10) shows that A_p is an elementary p-group. Consequently $\mathrm{Ext}(Z(p^\infty), A)$ is a subgroup of the p-component $(\mathrm{Ext}(Q/Z, A))_p$ of $\mathrm{Ext}(Q/Z, A)$. Since $\prod_{q \neq p} \mathrm{Ext}(Z(q^\infty), A)$ is p-divisible, $\mathrm{Ext}(Z(p^\infty), A) = (\mathrm{Ext}(Q/Z, A))_p$. Now

$$((\mathrm{Ext}(Q/Z, A))_p, \cdot) \cong (A_p, \cdot) ,$$

so Proposition 4.16 shows that $(\mathrm{Ext}(Q/Z, A), \cdot)$ is the ring direct product of the semisimple rings $((\mathrm{Ext}(Q/Z, A))_p, \cdot)$. Consequently $J(\mathrm{Ext}(Q/Z, A), \cdot) = 0$, as required. //

CHAPTER FIVE

We return once again to our study of rings on torsion-free groups. For the majority of this chapter we will be concerned with strongly indecomposable torsion-free groups of finite rank. Reid [1] has given a partial survey of rings on such groups, and we combine his results with several of Beaumont and Pierce [1, 2] to obtain a partial description of the absolute radical of a strongly indecomposable torsion-free group of finite rank. This description extends Theorem 1.21 of Chapter One. Our discussion of these groups then continues by relating several so far unrelated concepts: the almost nilpotent rings of van Leeuwan and Heyman [1] with the unequivocal rings of Gardner [2], and the rings with the finite norm property of Levitz and Mott [1] with the rings with the restricted minimum condition.

We then explore associative rings on a torsion-free group of rank two. A result of Freedman [1] is generalised, and a complete description of the absolute associative annihilator of a torsion-free group of rank two is given. Our knowledge of the structure of the absolute radical of a completely decomposable group together with Theorem 5.8 also enable us to give a partial characterisation of the absolute radical of a torsion-free group of rank two.

1. STRONGLY INDECOMPOSABLE TORSION-FREE GROUPS OF FINITE RANK

We commence this section with a survey of the known results concerning the associative rings supported by a strongly indecomposable torsion-free group of finite rank. For the sake of convenience let S denote the class of all strongly indecomposable torsion-free groups of finite rank. We begin with the following definitions, first introduced by Reid [1].

A group A is *faithful* if there is some associative ring on A with trivial left annihilator. A group A is *fully faithful* if A is non-nil and every associative ring on A has trivial left annihilator. An associative ring (A, ·) on A is called a *faithful ring* if the left annihilator of (A, ·) is trivial.

Reid has shown that fully faithful groups cannot be mixed, a torsion group is fully faithful exactly if it is p-elementary and cyclic for some prime p, and torsion-free fully faithful groups are strongly indecomposable. He has also shown

(5.1) (Reid [1]). *Let* A *be a group in the class* $\mathcal{S}$. *Then exactly one of the following conditions holds:*

(i) A *is fully faithful;*

(ii) A *is nil;*

(iii) A *is non-nil and every associative ring on* A *is nilpotent.* //

For a group A in $\mathcal{S}$ satisfying (ii) or (iii) above, it is clear that J(A) = A. Thus in our attempt to describe J(A) for a group A in $\mathcal{S}$ we may confine our attention to the case when A is fully faithful. Later, in Chapter Six, we will give bounds for the nil-degree of A in Case (iii) above.

Reid has also given a complete description of the fully faithful groups in $\mathcal{S}$. His description involves full subrings of algebraic number fields. All the information we require in this regard is contained in Beaumont and Pierce [2].

Suppose J is the ring of integers in an algebraic number field K. Beaumont and Pierce have shown that there is a one-to-one correspondence between the quasi-equality classes of full subrings of K and the sets of prime ideals of J. This correspondence may be stated as follows. For a prime ideal P_J of J define

$J_{P_J} = \{x/y | x, y \in J \text{ and } y \notin P_J\}$, and for a set Π of prime ideals of J

define $J_\Pi = \bigcap_{P_J \in \Pi} J_{P_J}$. Then each quasi-equality class of full subrings of K contains a unique integrally closed ring J_Π which is the largest ring in the quasi-equality class.

Beaumont and Pierce say that a set Π of prime ideals of J *satisfies condition* U if

(i) for every rational prime p, $\Pi_p = \{P_J \in \Pi | p \in P_J\}$ contains at most one prime ideal, and

(ii) if $P_J \in \Pi$ then P_J is unramified and of degree one.

We will need the following paraphrase of several of Beaumont and Pierce's result.

(5.2) (Beaumont and Pierce [2]). *Let* J *be the ring of integers in an algebraic number field* K, *and suppose* Π *is a set of prime ideals of* J. *Then*

(i) *if* Π *satisfies condition* U *then every subgroup of finite index in* J_Π *is of the form* nJ_Π, *where* n *is a rational integer,*

(ii) *if* J_Π *is strongly indecomposable and* Π *does not satisfy condition* U *then the quasi-equality class of* J_Π *in* K *contains infinitely many rings with identity no two of which are group isomorphic.* //

Notice that for any set Π of prime ideals of J, J_Π denotes an associative ring on the group J_Π^+. Consequently $J(J_\Pi)$ is the radical of the associative ring J_Π, and not the absolute radical of J_Π^+. As usual, the latter radical is denoted by $J(J_\Pi^+)$.

We are now in a position to state Reid's results on fully faithful groups in the class $\mathcal{S}$. Part of this contains the major result of Beaumont and Lawver [1].

(5.3) (Reid [1]). *Suppose* A *is a faithful group in the class* $\mathcal{S}$. *Then* A *is fully faithful and*

(i) $K = Q \otimes E(A)$ *is an algebraic number field whose degree is the rank of* A,

(ii) *if* J_Π *is the integrally closed subring of* K *corresponding to* E(A) *then* A *is quasi-isomorphic to* J_Π^+,

(iii) *the* quasi-equality *class of non-trivial associative rings on* A *coincides with the quasi-equality class of* J_Π *in* K,

(iv) A *is (strongly) semisimple exactly when* Π *is either an empty or infinite set.* //

For the remainder of this chapter whenever A is a fully faithful group in $\mathcal{S}$, K, J and Π will respectively denote the algebraic number field $Q \otimes E(A)$, the ring of integers in K, and the set of prime ideals of J corresponding to the quasi-equality class of E(A) in K. In this case we say that Π is the set of prime ideals of J corresponding to the group A. See page 95A.

Important for this section and for others following (and interesting in itself) is the structure of the ideals of a faithful ring on a group in $\mathcal{S}^*$. To specify this we require the following well known definitions and results.

Beaumont and Pierce [1] have called an associative ring (A, $\cdot$) on a torsion-free group A *of simple* or *of semisimple algebra type* if the associative ring ($Q \otimes A$, $\cdot$) on the divisible hull $Q \otimes A$ of A is a simple or semisimple Q-algebra.

(5.4) (Beaumont and Pierce [1]). *Let* (A, $\cdot$) *be an associative ring on a torsion-free group* A *of finite rank. Then as a group* A *is quasi-equal to the group direct sum of its maximal nilpotent ideal*

Define $\mathcal{S}^*$ to be the subclass of $\mathcal{S}$ consisting of those groups A in $\mathcal{S}$ which satisfy the following property

(*) either A is not a faithful group or, if A is a faithful group (and Π is the set of prime ideals of J corresponding to A) then the additive group of any ring in the quasi-equality class of J_Π in K is isomorphic to A.

A reference to Corollary 4 of Murely [1] verifies that $\mathcal{S}_1$ (for the definition of $\mathcal{S}_1$ see page 107) is a subclass of $\mathcal{S}^*$, so $\mathcal{S}^*$ is a non-trivial class of groups. Also it is evident from (5.2)(ii) that if A is a faithful group in $\mathcal{S}^*$ then Π satisfies condition U.

and a subring of semisimple algebra type. //

(5.5) (Beaumont and Pierce [1]). *Suppose* A *is a torsion-free group of finite rank. If* (A, •) *is an associative ring on* A *of semisimple algebra type then* (A, •) *contains a subring* (C, •) *of finite index such that*

$$(C, \cdot) = (S_1, \cdot) \oplus (S_2, \cdot) \oplus \ldots \oplus (S_m, \cdot) \quad \textit{(ring direct sum)}$$

where each $(S_i, \cdot)$ *is a ring of simple algebra type.* //

We can now prove

PROPOSITION 5.6. *If* A *is a fully faithful group in* $\mathcal{S}$ *then* A *is homogeneous with idempotent type. If* A *is in* $\mathcal{S}^*$*, the non-zero ideals of any non-trivial associative ring* (A, •) *are of the form* (nA, •), n *an integer.*

Proof: Let (A, •) be a faithful ring on A. Since A is strongly indecomposable, (5.4) and (5.5) imply that (A, •) contains a subring of finite index and of simple algebra type. Clearly (A, •) will also have simple algebra type, so a reference to Lemma 121.6 of Fuchs [4] now shows that A is homogeneous with idempotent type and that the non-zero ideals of (A, •) are of finite index in A. (Reid [1] has also noticed that the non-zero ideals of (A, •) are of finite index in A).

Suppose (A, •) is defined by the map $\phi \in \mathrm{Hom}(A, \mathrm{Hom}(A, A))$. If now (I, •) is a non-zero ideal of (A, •), $\phi(I)$ is a non-zero ideal of $\phi(A)$. Furthermore, since (I, •) has finite index in (A, •), $\phi(I)$ has finite index in $\phi(A)$. Let Π be the set of prime ideals of J corresponding to A, and let $A \in \mathcal{S}^*$. From previous comments it is evident that Π satisties condition U. Thus, since $\phi(A)$ has finite index in J_Π, (5.2) (i) yields $\phi(A) = mJ_\Pi$ for some integer m. Now $J_\Pi^+ \cong mJ_\Pi^+$, so (5.2) (i) also shows that every subgroup of finite index in mJ_Π is of the form $n(mJ_\Pi)$, for some integer n. Consequently, $\phi(A) = mJ_\Pi$ and the fact that $\phi(I)$ has finite index in $\phi(A)$ imply $\phi(I) = n\phi(A)$, for some integer n. Finally, since $\ker\phi = 0$, $I = nA$, as required. //

It follows from Proposition 5.6 that since a fully faithful group A in $\mathcal{S}$ is necessarily homogeneous with idempotent type, we are justified to use the notation P_1^A of Chapter One. There is now a strong connection between the rational primes in P_1^A and the prime ideals in the set Π corresponding to A. This may be stated as follows.

LEMMA 5.7. *Let* A *be a fully faithful group in* $\mathcal{S}$. *Then the rational prime* p *is in* P_1^A *if and only if there is a prime ideal* P_J *in* Π *containing* p.

Proof: Suppose $(A, \cdot)$ is a faithful ring defined on the group A in $\mathcal{S}$. From (5.3) (ii) we know that A is quasi-isomorphic to J_Π^+. Thus J_Π^+ is homogeneous with the same idempotent type as t(A). A direct proof (or reference to the proof of Theorem 5.5 of Beaumont and Lawver [1]) now shows that $p \in P_1^A$ if and only if there is a prime ideal $P_J \in \Pi$ containing p. //

We are now in a position to prove the main result of this section.

THEOREM 5.8. *Suppose* A *is a group in the class* $\mathcal{S}^*$. *Then exactly one of the following conditions holds*:

(i) $J(A) = A$;

(ii) $J(A) = 0$;

(iii) $J(A) = nA$ $(nA \neq 0,\ nA \neq A)$, *for a suitable integer* n.

(i) *holds exactly if every associative ring on* A *is nilpotent*,

(ii) *holds exactly if* A *is fully faithful and* $|P_1^A| = 0$ *or* $|P_1^A| = \infty$, *and* (iii) *holds exactly if* A *is fully faithful and* $0 \neq |P_1^A| < \infty$, *in which case* $n = \prod_{p \in P_1^A} p$. *Moreover there is an associative and commutative ring* $(A, \cdot)$ *on* A *for which* $J(A, \cdot) = J(A)$.

Proof: Let A be a group in $\mathcal{S}^*$. From (5.1) we need only consider the case when A is fully faithful. Let Π be the set of prime ideals of J corresponding to A. We consider two distinct cases.

Case (i). $|P_1^A| = 0$ or $|P_1^A| = \infty$. From Lemma 5.7 and the fact that there is at most one rational prime in each prime ideal in Π it is clear that Π is either an empty or infinite set. Since J_Π is commutative, (5.3) (iii) and (5.3) (iv) now show that every non-trivial associative ring on A (and such rings do exist) is commutative and has zero radical.

Case (ii). $0 \neq |P_1^A| < \infty$.

Since A is in $\mathcal{S}^*$, Π necessarily satisfies condition U. Lemma 5.7 now yields $|\Pi| = |P_1^A|$, so $0 \neq |\Pi| < \infty$.

First we prove the inclusion $\bigcap_{p \in P_1^A} pA \subseteq J(A)$. From Proposition 1.20 and Proposition 5.6 this amounts to showing that any associative ring $(A, \cdot)$ on A has non-zero radical. But this is immediate from (5.3) (iv) and the fact that $0 \neq |\Pi| < \infty$.

To prove the reverse inclusion $J(A) \subseteq \bigcap_{p \in P_1^A} pA$, property $(*)$ and the fact that isomorphic rings have isomorphic radicals show that it suffices to prove the inclusion $J(J_\Pi) \subseteq \bigcap_{p \in P_1^A} pJ_\Pi$. (Again notice that $J(J_\Pi)$ is the radical of the associative ring J_Π). Since the non-zero prime ideals of J_Π are precisely the ideals $P_J\, J_\Pi$, where $P_J \in \Pi$, and the non-zero prime ideals of J_Π are maximal, it follows that $J(J_\Pi) = \bigcap_{P_J \in \Pi} P_J\, J_\Pi$. Now $0 \neq |\Pi| < \infty$ yields $J(J_\Pi) \neq 0$, so (5.2) (i) shows $J(J_\Pi) = nJ_\Pi$ for some integer n.

Suppose now $p \in P_1^A$ is such that $(n, p) = 1$. From Lemma 5.7 it is possible to choose a prime ideal $P_J \in \Pi$ such that $p \in P_J$. But then $pJ_\Pi \subseteqq P_J J_\Pi$, so $0 \neq J_\Pi/P_J J_\Pi$ is an elementary p-group. Alternatively, since $nJ_\Pi = J(J_\Pi) \subseteqq P_J J_\Pi$, $J_\Pi/P_J J_\Pi$ is a bounded group with trivial p-component. We conclude therefore that every $p \in P_1^A$ divides n. Since $J((J_\Pi/J(J_\Pi))^+) = 0$, it is clear from (1.10) that n is the product of the distinct primes in P_1^A. Consequently $J(J_\Pi) = nJ_\Pi = \bigcap_{p \in P_1^A} pJ_\Pi$.

Evidently we have now shown that $J(A) = \bigcap_{p \in P_1^A} pA = nA$, and that there is an associative and commutative ring $(A, \cdot)$ on A for which $J(A, \cdot) = nA$.

The proof of the Theorem is completed by observing that the three conditions mentioned in the Theorem are mutually exclusive and exhaustive conditions for A. //

COROLLARY 5.9. *If* A *is a group in* $\mathcal{S}^*$ *then* $J(A/J(A)) = 0$. //

COROLLARY 5.10. *If* A *is a fully faithful group in* $\mathcal{S}^*$ *such that* $0 \neq |P_1^A| < \infty$, *then the proper subgroups of* A *that can occur as the radical of some non-trivial associative ring on* A *are precisely the subgroups* mA, *where* m *is a square-free product of primes from* P_1^A.

Proof: Suppose n is the product of all the distinct primes from P_1^A. If $(A, \cdot)$ is a non-trivial associative ring on A then Theorem 5.8 yields $nA \subseteqq J(A, \cdot)$. Thus Proposition 5.6 shows $J(A, \cdot) = mA$, where m is some square-free product of primes from P_1^A.

Conversely, suppose m is a square-free product of primes from P_1^A. Since n/m is a well defined integer, $(n/m)J_\Pi$ is the quasi-equality class of J_Π, and

$$J((n/m)J_\Pi) = (n/m)J_\Pi \cap J(J_\Pi) \ .$$

Now $J(J_\Pi) = nJ_\Pi$, so

$$J((n/m)J_\Pi) = nJ_\Pi = m((n/m)J_\Pi) \ .$$

From property (*) we conclude that there is an associative ring $(A, \cdot)$ on A for which $J(A, \cdot) = mA$. //

Haimo [1] has described the radical and antiradical rational groups. In our terminology he has shown that a rational group A is radical if and only if t(A) is idempotent and $0 \neq |P_1^A| < \infty$, and A is antiradical and non-nil if and only if t(A) is idempotent and $|P_1^A| = 0$ or $|P_1^A| = \infty$. Furthermore, Haimo has shown that if A is a rational radical group then A supports a non-radical ring $(A, \cdot)$ such that $J(A, \cdot)^+ \cong A$. If we now define an arbitrary group A to be radical if A supports a non-nilpotent radical ring, and A to be antiradical if it is not radical, then we can extend Haimo's results to groups A in $\mathcal{S}^*$.

COROLLARY 5.11. *Let* A *be a group in* $\mathcal{S}^*$. *Then* A *is a radical group if and only if* A *is fully faithful and* $0 \neq |P_1^A| < \infty$. A *is an antiradical group that supports a non-nilpotent ring if and only if* A *is fully faithful and* $|P_1^A| = 0$ *or* $|P_1^A| = \infty$. *If* A *is a radical group then* A *supports at least one non-radical ring* $(A, \cdot)$ *such that* $J(A, \cdot)^+ \cong A$.

Proof: If A is a radical group then the proof of Theorem 5.8 shows A is fully faithful and $0 \neq |P_1^A| < \infty$. Conversely, if A is fully faithful such that $0 \neq |P_1^A| < \infty$ then Corollary 5.10 shows that A supports an associative ring $(A, \cdot)$ such that $J(A, \cdot) = nA$, where $n = \prod_{p \in P_1^A} p$. Now

$$J(A, \cdot) = (nA, \cdot) \cap J(A, \cdot) = nA \ ,$$

so $(nA, \cdot)$ is a radical ring. Since $nA \cong A$, it is now possible to define an associative ring $(A, \times)$ on A such that $(A, \times) \cong (nA, \cdot)$. Clearly $(A, \times)$ is not nilpotent and $J(A, \times) = A$, so A is a radical group.

The second assertion of the Corollary now follows from the proof of Theorem 5.8, while Corollary 5.10 implies the final statement. //

2. SOME CLASSES OF RINGS ON STRONGLY INDECOMPOSABLE TORSION-FREE GROUPS OF FINITE RANK

We begin with some definitions.

A class R of rings is called a *radical class* of rings if the following three conditions hold:

(i) a homomorphic image of an R-ring is an R-ring;

(ii) every ring $(A, \cdot)$ contains an R-ideal $R(A, \cdot)$ which contains every other R-ideal of $(A, \cdot)$; and

(iii) the factor ring $(A/R(A, \cdot), \cdot)$ is R-semisimple

Evidently the class of right quasi-regular rings is a radical class of rings.

A ring $(A, \cdot)$ is a *prime ring* if $I \cdot J = 0$ implies $I = 0$ or $J = 0$, where $(I, \cdot)$ and $(J, \cdot)$ are ideals of $(A, \cdot)$.

The first class of rings we wish to investigate were first introduced by van Leeuwen and Heyman [1]. An associative ring $(A, \cdot)$ is said to be *almost nilpotent* if every proper homomorphic image of $(A, \cdot)$ is nilpotent. The prime simple rings are assumed not to be almost nilpotent.

A major result in van Leeuwen and Heyman [1] is that an almost nilpotent ring is either β-radical or β-semisimple, where β denotes the Baer radical class. It is this result we wish to extend for associative rings on a group A in S^*.

We require the following lemma.

LEMMA 5.12. *Let* A *be a group in* $\mathcal{S}^*$. *If* (A, ·) *is a faithful ring on* A *such that* $(A, \cdot)^n \subseteq \bigcap_{p \in P_1^A} pA$ *for some positive integer* n, *then* $(A, \cdot)^2 \subseteq \bigcap_{p \in P_1^A} pA$.

Proof: Suppose $\phi \in \mathrm{Hom}(A, E(A))$ defines the faithful ring (A, ·). Since (A, ·) is associative and $\ker \phi = 0$, it is clear that $\phi : (A, \cdot) \to \phi(A)$ is a ring isomorphism. Thus $(\phi(A))^n \subseteq \bigcap_{p \in P_1^A} p(\phi(A))$.

Let Π be the set of prime ideals of the ring of integers J of the algebraic number field $K = Q \otimes E(A)$ corresponding to A. $\phi(A)$ is then a full subring of J_Π, so as in the proof of Proposition 5.6, $\phi(A) = kJ_\Pi$ for some rational integer k. If 1 is the identity element of K then $k.1 \in \phi(A)$. But then

$$(k.1)^n = k^n.1 \in \bigcap_{p \in P_1^A} p(\phi(A)) .$$

For each $p \in P_1^A$, $h_p^{J_\Pi}(1) = 0$, so every $p \in P_1^A$ divides k. Also, since $1 \in J_\Pi$, $(kJ_\Pi)^2 = k(kJ_\Pi)$, so $(\phi(A))^2 = k(\phi(A))$. Hence $(\phi(A))^2 \subseteq \bigcap_{p \in P_1^A} p(\phi(A))$, so $(A, \cdot)^2 \subseteq \bigcap_{p \in P_1^A} pA$, as required. //

We can now describe the almost nilpotent rings on a strongly indecomposable torsion-free group in $\mathcal{S}^*$.

THEOREM 5.13. *Suppose* (A, ·) *is an associative ring on the group* A *in the class* $\mathcal{S}^*$. *Then* (A, ·) *is almost nilpotent if and only if*

(i) (A, ·) *is nilpotent, or*

(ii) (A, ·) *is faithful,* $0 \neq |P_1^A| < \infty$ *and* $(A, \cdot)^2 \subseteq \bigcap_{p \in P_1^A} pA$.

Proof: Suppose (A, ·) is a faithful almost nilpotent ring on the group A in $\mathcal{S}^*$. For each $p \in P_1^A$, (A/pA, ·) is a nilpotent ring on

A/pA. Since A is torsion-free of finite rank n say, A/pA will have rank at most n. Hence $(A, \cdot)^{n+1} \subseteqq \bigcap_{p \in P_1^A} pA$, so Lemma 5.12 shows $(A, \cdot)^2 \subseteqq \bigcap_{p \in P_1^A} pA$. If $|P_1^A| = 0$ then $A \cong Q$, so $(A, \cdot)$, as a field, cannot be almost nilpotent. If $|P_1^A| = \infty$ then $(A, \cdot)$ will be the trivial ring on A. Indeed, if $(A, \cdot)^2 \neq 0$ then Proposition 5.6 yields $(A, \cdot)^2 = mA$ for some integer m. But then $mA \subseteqq \bigcap_{p \in P_1^A} pA$, which clearly cannot be the case. We conclude therefore that $0 \neq |P_1^A| < \infty$.

For the converse we need only consider the case when $(A, \cdot)$ is as described in (ii). Proposition 5.6 shows that if $(I, \cdot)$ is a non-zero ideal of $(A, \cdot)$ then there is an integer m such that $I = mA$. Since m will be a product of powers of primes from P_1^A, the condition $(A, \cdot)^2 \subseteqq \bigcap_{p \in P_1^A} pA$ yields $(A, \cdot)^{m_1} \subseteqq mA$, for some integer m_1. Consequently $(A, \cdot)$ is an almost nilpotent ring. //

Consider an almost nilpotent ring on a group A in $\mathcal{S}^*$. If $(A, \cdot)$ is nilpotent then $J(A, \cdot) = A$. On the other hand if $(A, \cdot)$ is faithful then Theorem 5.13 and Theorem 5.8 show $0 \neq \bigcap_{p \in P_1^A} pA \subseteqq J(A, \cdot)$. Thus $(A/J(A, \cdot), \cdot)$ is a nilpotent ring. But then $(A, \cdot)$ is an extension of the (right) quasi-regular ring $(J(A, \cdot), \cdot)$ by the quasi-regular ring $(A/J(A, \cdot), \cdot)$. Consequently $(A, \cdot)$ is itself a quasi-regular ring. It follows therefore that an almost nilpotent ring on a group A in $\mathcal{S}^*$ is radical.

We are naturally led to the class of rings we wish to relate to the almost nilpotent rings; the so called unequivocal rings. They were introduced in Gardner [2], and later studied in Divinsky [2]. An associative ring $(A, \cdot)$ is *unequivocal* if for every radical class $\mathcal{R}$, $(A, \cdot)$ is either $\mathcal{R}$-radical or $\mathcal{R}$-semisimple.

The following definition is required. A subring $(B, \cdot)$ of a ring $(A, \cdot)$ is a *subideal (accessible subring)* of $(A, \cdot)$ if there is a finite chain

$$(B, \cdot) = (A_1, \cdot) \subseteqq (A_2, \cdot) \subseteqq \ldots \subseteqq (A_n, \cdot) = (A, \cdot)$$

of subrings of $(A, \cdot)$ such that $(A_i, \cdot)$ is an ideal of $(A_{i+1}, \cdot)$ for all $i = 1, 2, \ldots, n - 1$.

Divinsky has shown that unequivocality can be characterised by the relationship between a ring and its subideals, rather than by the family of all radical classes. We give this relationship in the next proposition, and we provide a proof since the exact statement of the proposition has not appeared in the literature.

PROPOSITION 5.14. *Suppose* $(A, \cdot)$ *is an associative ring on the group* A. *Then* $(A, \cdot)$ *is unequivocal if and only if for every proper ideal* $(I, \cdot)$ *of* $(A, \cdot)$ *there is a non-zero ring homomorphism from* $(I, \cdot)$ *onto a subideal of* $(A/I, \cdot)$.

Proof: Theorem 1 of Divinsky [2] shows that a proper ideal $(I, \cdot)$ of an unequivocal ring $(A, \cdot)$ can be homomorphically mapped onto a non-zero subideal of $(A/I, \cdot)$. To verify the converse assertion suppose that for every proper ideal $(I, \cdot)$ of $(A, \cdot)$ there is a non-zero homomorphism from $(I, \cdot)$ onto a subideal of $(A/I, \cdot)$. If now the proper ideal $(I, \cdot)$ can be represented as $\mathcal{R}(A, \cdot)$ for some radical class $\mathcal{R}$, then Lemma 2 of Divinsky [1] shows that $(I, \cdot)$ cannot be homomorphically mapped onto a non-zero $\mathcal{R}$-semisimple ring. This cannot be the case however, since any non-zero subideal of the $\mathcal{R}$-semisimple ring $(A/I, \cdot)$ will itself be $\mathcal{R}$-semisimple. //

Divinsky [2] easily establishes that there are four kinds of unequivocal rings; divisible torsion-free rings, reduced torsion-free rings, divisible p-rings, and reduced p-rings. Since a divisible p-ring is necessarily the trivial ring, Proposition 5.14 shows that a divisible p-ring is both unequivocal and almost nilpotent. Lemma 1 of Divinsky [2] provides examples of arbitrary rank divisible torsion-free rings and arbitrary rank elementary p-rings that are unequivocal but not almost nilpotent. Turning our attention to the unequivocal rings on groups in $\mathcal{S}^*$ we can prove

THEOREM 5.15. *Suppose* $(A, \cdot)$ *is an associative ring on the group* A *in the class* $\mathcal{S}^*$. *If* $(A, \cdot)$ *is unequivocal then either*

(i) $(A, \cdot)$ *is nilpotent, or*

(ii) $(A, \cdot)$ *is faithful,* $|P_1^A| < \infty$ *and* $(A, \cdot)^2 \subseteqq \bigcap_{p \in P_1^A} pA$.

Conversely, if $(A, \cdot)$ *satisfies condition* (ii) *then* $(A, \cdot)$ *is unequivocal.*

Proof: Let $(A, \cdot)$ be an unequivocal ring on the group A in $\mathcal{S}^*$. If $(A, \cdot)$ is not nilpotent then obviously $(A, \cdot)$ is faithful. Suppose the rank of A is n, and consider a prime $p \in P_1^A$. Since A/pA is a torsion group of rank at most n, $(A/pA, \cdot)$ is an Artinian ring. Thus (1.4) implies $J(A/pA, \cdot)$ is nilpotent. Suppose now $J(A/pA, \cdot) \neq A/pA$. Then since $(A/pA)/J(A/pA, \cdot)$ is finite, it follows from (1.3) (ii) that $(A/pA)/J(A/pA, \cdot)$ is a finite direct sum of full matrix rings over division rings, each of characteristic p. Hence A can be mapped homomorphically onto a simple ring of characteristic p. If now M is the upper radical class defined by the class of all simple rings with identity and characteristic p, it is clear that $(A, \cdot)$ is not M-radical.

Hence $(A, \cdot)$ is M-semisimple, and as such it is a subdirect product of simple rings of characteristic p, (see Theorem 46 of Divinsky [1]). Consequently A will have characteristic p, which cannot be the case since A is torsion-free. Therefore $J(A/pA, \cdot) = A/pA$, so $(A/pA, \cdot)$ will be nilpotent. As in the proof of Theorem 5.13, $(A, \cdot)^2 \subseteq \bigcap_{p \in P_1^A} pA$ and $|P_1^A| < \infty$.

Conversely, suppose $(A, \cdot)$ satisfies (ii). Let $(I, \cdot)$ be a proper ideal of $(A, \cdot)$. From Proposition 5.14 we need only show that there is a non-zero ring homomorphism from $(I, \cdot)$ onto a subideal of $(A/I, \cdot)$.

Proposition 5.6 yields $I = mA$ for some non-zero integer m, $m \neq 1$. Suppose $m = p^k q$ where $p \in P_1^A, (p, q) = 1$ and $k \geq 1$. Consider $m_1 = p^{k-1}q$. Clearly $(m_1A/mA, \cdot)$ is an ideal of $(A/mA, \cdot)$. Also the condition $(A, \cdot)^2 \subseteq \bigcap_{p \in P_1^A} pA$ shows $(m_1A/mA, \cdot)$ is the trivial ring on m_1A/mA, and $(mA/pmA, \cdot)$ is the trivial ring on mA/pmA. Now, as groups, $mA/pmA \cong m_1A/mA$, since they are both elementary p-groups such that

$$r(mA/pmA) = r(A/pA) = r(m_1A/mA) .$$

By taking the composition of the natural map $mA \to mA/pmA$ with the isomorphism $mA/pmA \cong m_1A/mA$, we obtain a non-zero ring homomorphism from mA onto m_1A/mA, as required. //

COROLLARY 5.16. *For a faithful ring* $(A, \cdot)$ *on a reduced group* A *in* S^* *the following conditions are equivalent:*

(i) $(A, \cdot)$ *is almost nilpotent;*

(ii) $(A, \cdot)$ *is unequivocal;*

(iii) A *is homogeneous with idempotent type,* $0 \neq |P_1^A| < \infty$ *and*

$(A, \cdot)^2 \subseteq \bigcap_{p \in P_1^A} pA$. //

It is not known in general whether a nilpotent ring on a group in $\mathcal{S}^*$ is unequivocal. However we can answer this in the affirmative for a suitable subclass of $\mathcal{S}^*$.

Define $\mathcal{S}_1$ to be the set of homogeneous groups A in $\mathcal{S}$ with the property that $r(A/pA) \leqq 1$ for all $p \in P$. These groups belong to a larger class of groups that have been studied extensively by Murley [1]. We will not, however, need to draw on any results from Murley's work.

It follows from Theorem 2.5 of Gardner [4] that a nilpotent ring on a group A is unequivocal if and only if the trivial ring on A is unequivocal. Also, Proposition 2.9 of Gardner [4] shows that if $\mathcal{R}$ is any radical class and $(A, \cdot)$ is a nilpotent ring on A then $\mathcal{R}(A, \cdot)^+$ is a pure subgroup of A. Consequently, to prove a nilpotent ring on the group A in $\mathcal{S}_1$ is unequivocal we need to show the trivial ring on A is unequivocal, and to do this, when we apply Proposition 5.14, we need only consider pure subgroups (that is, ideals) of A. These observations form the basis for the proof of our next result.

PROPOSITION 5.17. *A nilpotent ring on a group* A *in* $\mathcal{S}_1$ *is unequivocal.*

Proof: From the remarks above we need only verify that every proper pure subgroup B of A has a non-zero homomorphism into A/B. But this is immediate since $r(A/pA) \leqq 1$ for all $p \in P$ implies that A is a cohesive group (see Chapter Six), and so A/B is torsion-free and divisible.//

Divinsky [2] has shown that the set of all $2m/(2n + 1)$, where m and n are integers, and the usual multiplication operation form an unequivocal ring. The proof of this follows from Theorem 5.15.

More generally we can ask the following question. Given a faithful ring $(A, \cdot)$ on the group A in $\mathcal{S}^*$, which ideals of $(A, \cdot)$ are unequivocal?

To facilitate a complete answer to this question we make the following definition. For a faithful ring $(A, \cdot)$ on the group A in $\mathcal{S}^*$ let

$$P_2^A = \{p \in P \mid (A, \cdot)^2 \subseteqq pA\} .$$

We can now prove

PROPOSITION 5.18. *Let* $(A, \cdot)$ *be a faithful ring on a group* A *in* $\mathcal{S}^*$. *Then* $(nA, \cdot)$ *is unequivocal if and only if* $p \in P_1^A$ *and* $(n, p) = 1$ *imply* $p \in P_2^A$.

Proof: Suppose $(nA, \cdot)$ is unequivocal and $p \in P_1^A$ is such that $(n, p) = 1$. If $p \notin P_2^A$ then $(A, \cdot)^2 \nsubseteq pA$, so there are elements a and b in A for which $h_p(a \cdot b) = 0$. Since $(n, p) = 1$, $h_p(na \cdot nb) = 0$, so $(nA, \cdot)^2 \nsubseteq p(nA)$. But this contradicts Theorem 5.15 since $(nA, \cdot)$ is a faithful ring on nA, and nA is strongly indecomposable with the same rank and type as A.

Conversely, suppose n is an integer such that if $p \in P_1^A$ with $(n, p) = 1$ then $p \in P_2^A$. It is clear that $(nA, \cdot)^2 \subseteqq n(nA)$. Also if $p \in P_1^A$ is such that $(n, p) = 1$ then $(nA, \cdot)^2 \subseteqq p(nA)$. Thus $(nA, \cdot)^2 \subseteqq \bigcap_{p \in P_1^A} p(nA)$. As before, $(nA, \cdot)$ is a faithful ring on nA, nA is in $\mathcal{S}^*$ and $t(nA) = t(A)$, so it now follows from the proof of Theorem 5.13 that $|P_1^{nA}| < \infty$. Thus Theorem 5.15 shows $(nA, \cdot)$ is unequivocal. //

There is another question that naturally arises from our discussion of unequivocal rings. Given a faithful ring $(A, \cdot)$ on a group A in $\mathcal{S}^*$, which proper subroups of A can occur as $\mathcal{R}(A, \cdot)$ for some radical class $\mathcal{R}$? A partial answer can be provided when A is in $\mathcal{S}^*$, and a complete answer when A is in $\mathcal{S}_1$.

PROPOSITION 5.19. *Let* $(A, \cdot)$ *be a faithful ring on a group* A *in* $\mathcal{S}^*$. *If* $nA = \mathcal{R}(A, \cdot)$ *for some radical class* $\mathcal{R}$ *and* $n \neq 1$, *then* n *is a square-free product of primes from* $P_1^A \backslash P_2^A$. *Conversely, if* A *is in* $\mathcal{S}_1$ *and* n *is a square-free product of primes from* $P_1^A \backslash P_2^A$ *then* $nA = \mathcal{R}(A, \cdot)$ *for some radical class* $\mathcal{R}$.

Proof: Suppose $nA = \mathcal{R}(A, \cdot)$ for some radical class $\mathcal{R}$, and $n = p^k q$ where $p \in P_1^A$, $(p, q) = 1$ and $k > 1$. It is clear that $(p^{k-1}qA/nA, \cdot)$ and $(nA/pnA, \cdot)$ are trivial rings. Thus, since

$$r(p^{k-1}qA/nA) = r(A/pA) = r(nA/pnA) ,$$

$(p^{k-1}qA/nA, \cdot) \cong (nA/pnA, \cdot)$. Also, $(p^{k-1}qA/nA, \cdot)$ is an ideal of $(A/nA, \cdot)$, so as in the proof of Theorem 5.15 there is a non-zero ring homomorphism from $(nA, \cdot)$ onto an ideal of $(A/nA, \cdot)$. This is impossible since $(nA, \cdot)$ is $\mathcal{R}$-radical and $(A/nA, \cdot)$ is $\mathcal{R}$-semisimple. Consequently n is square-free.

Suppose now $p \in P_2^A$ is such that $(n, p) \neq 1$. Write $n = pq$ where $(p, q) = 1$. Since $(A, \cdot)^2 \subseteq pA$, the above proof with $p^{k-1}qA$ replaced by qA leads to the same contradiction. We conclude therefore that n is a square-free product of primes from $P_1^A \backslash P_2^A$.

To prove the converse assertion let n be a square-free product of primes from $P_1^A \backslash P_2^A$. Assume there is a non-zero ring homomorphism f from $(nA, \cdot)$ onto a subideal of $(A/nA, \cdot)$. Since A is in $\mathcal{S}_1$, A/nA is a finite direct sum of cyclic p-elementary groups. Each of these cyclic p-elementary groups is a field, since if p divides n then $p \notin P_2^A$, and so $(A/pA, \cdot)$ is a non-trivial ring. If we denote the non-zero image of $(nA, \cdot)$ under f by $(A_1/nA, \cdot)$, for a suitable subgroup A_1 of A, it is clear that $(A_1/nA, \cdot)$ will contain an identity element $1_{A_1/nA}$. But then there is an element $a \in A$ for which $f(na) = 1_{A_1/nA}$. We now have

$$1_{A_1/nA} = f(na \cdot na) = nf(a \cdot na) = 0,$$

a contradiction. Consequently there does not exist a non-zero homomorphism from $(nA, \cdot)$ and a subideal of $(A/nA, \cdot)$. If we now let M denote the upper radical class defined by $(A/nA, \cdot)$, then it follows from Theorem 2.5 of Gardner [3] and the fact that $(A/nA, \cdot)$ is M-semisimple that $M(A, \cdot) = nA$. //

We now turn our attention to two other classes of rings on groups in $\mathcal{S}$. The first of these that we consider are the rings with the restricted minimum condition.

An associative ring $(A, \cdot)$ on a group A satisfies the *restricted minimum condition* on left ideals if the minimum condition holds modulo every non-zero ideal of $(A, \cdot)$.

Clearly our discussion of rings with the restricted minimum condition will involve a knowledge of the structure of the additive group of an Artinian ring. Szele and Fuchs [1] have provided a complete description of these groups.

(5.20) (Szele and Fuchs [1]). *A group* A *is the additive group of an Artinian ring if and only if* A *has the form*

$$(*) \qquad A = \bigoplus_{m} Q \oplus \bigoplus_{finite} Z(p_i^\infty) \oplus \bigoplus_{n} Z(p_j^{k_j}) \quad \textit{with } p_j^{k_j} \textit{ dividing } m,$$

where m and n are arbitrary cardinals, p_i *and* p_j *are distinct primes and* m *is a fixed integer.* //

Fuchs [2] has also given a partial structure theorem for the additive group of a ring with the restricted minimum condition.

(5.21) (Fuchs [2]). *A not torsion-free group* A *is the additive group of a ring with the restricted minimum condition if and only if it is of the form* (*) *of* (5.20). *The additive group of a torsion-free ring with the restricted minimum condition is homogeneous with idempotent type.* //

Fuchs asks which homogeneous torsion-free groups with idempotent types admit rings with the restricted minimum condition. It is well known that a rational group is the additive group of a ring with the restricted minimum condition exactly if its type is idempotent. More generally we can prove

THEOREM 5.22. *Suppose* $(A, \cdot)$ *is an associative ring on a group* A *in the class* $\mathcal{S}$. *Then* $(A, \cdot)$ *satisfies the restricted minimum condition if and only if either*

(i) $(A, \cdot)$ *is a faithful ring on* A, *or*

(ii) $(A, \cdot)$ *is the trivial ring on* A, *and* A *is a rational group with idempotent type such that* $|P \backslash P_1^A| < \infty$.

Proof: Suppose $(A, \cdot)$ is a non-trivial associative ring on A with the restricted minimum condition. It is clear that $(0; (A, \cdot)) \neq A$. If now $(A, \cdot)$ is nilpotent then $(0; (A, \cdot)) \neq 0$, so $(A/(0; (A, \cdot)), \cdot)$ will be a nilpotent Artinian ring. Since $(0; (A, \cdot))$ is a pure subgroup of A, $A/(0; (A, \cdot))$ is torsion-free, so (5.20) shows it will be divisible. On the other hand Proposition 122.1 of Fuchs [4] shows $A/(0; (A, \cdot))$ must satisfy the minimum condition on subgroups. We conclude therefore that $(A, \cdot)$ is a faithful ring.

If $(A, \cdot)$ is the trivial ring on A and $(A, \cdot)$ satisfies the restricted minimum condition, then it is clear that $(A, \cdot)$ must have rank one. If t(A) is not idempotent or t(A) is idempotent and $|P \backslash P_1^A| = \infty$ then it is possible to find a factor group of A that is an infinite direct sum of cocyclic groups. It is well known that such a group does not satisfy the minimum condition on subgroups. Since (A, •) is the trivial ring on A it is now evident that A is as stated in (ii).

Conversely if (A, •) is a faithful ring on A then Proposition 5.6 implies (A, •) must satisfy the restricted minimum condition. Alternatively, if (A, •) has the form stated in (ii) then any non-zero homomorphic image of (A, •) will be the trivial ring on a finite direct sum of cocyclic groups. Since such a group satisfies the minimum condition on subgroups, (A, •) is necessarily a ring with the restricted minimum condition. //

Levitz and Mott [1] have studied the properties of rings in a subclass of the class of rings with the restricted minimum condition. Following Levitz and Mott we say an associative ring (A, •) on the group A has the *finite norm property* if each of its proper homomorphic images is a finite ring.

It is clear that a ring (A, •) that has the finite norm property also satisfies the restricted minimum condition. From Proposition 5.6 it is apparent that a faithful ring on a group A in $\mathcal{S}$ also has the finite norm property.

As with Divinsky and the unequivocal rings, Levitz and Mott [1] have shown that the additive group of a ring with the finite norm property can have one of three possible structures; divisible torsion-free group, reduced torsion-free group, or bounded torsion group. They have also shown that if (A, •) is an infinite ring (so in particular a torsion-free ring) with the finite norm property then (A, •) is either a prime ring or the trivial ring on A. More generally, Hill [1] has shown that if (A, $\cdot$) is a torsion-free ring with the property that A/I is bounded for every non-zero ideal (I, $\cdot$) of (A, $\cdot$), then (A, $\cdot$) is either a prime ring or the trivial ring on the infinite cyclic group.

For torsion-free rings with the restricted minimum condition we can prove the following.

PROPOSITION 5.23. *Let* (A, $\cdot$) *be an associative and commutative ring on the torsion-free group* A. *If* (A, $\cdot$) *satisfies the restricted minimum condition then* (A, $\cdot$) *is either a prime ring or the trivial ring on a rational group* A *with idempotent type for which* $|P\backslash P_1^A| < \infty$.

Proof: Suppose (A, $\cdot$) satisfies the restricted minimum condition and (A, $\cdot$) is not a prime ring. Since a commutative ring is a prime ring exactly if it has no divisors of zero, there are elements a and b of A for which $a \cdot b = 0$. Thus $(0; b) \neq 0$. Suppose $(0; b) \neq A$. Then since ((0; b), $\cdot$) is a pure two sided ideal of (A, $\cdot$), (A/(0; b), $\cdot$) is a torsion-free Artinian ring. From (5.20) it is now clear that A/(0 ; b) is a torsion-free divisible group. If $\phi \in \mathrm{Hom}(A, E(A))$ defines the ring (A, $\cdot$) then $\phi(b) \in E(A)$ and $\phi(b)$ factors through A/(0 ; b),

$$\begin{tikzcd} \phi(b) : A \arrow[rr] \arrow[dr] & & A \\ & A/(0\ ;\ b) \arrow[ur] & \end{tikzcd} .$$

Since A is reduced, $\phi(b)$ must be the zero map. Thus $b \in (0; (A, \cdot))$, and so $(0; b) = A$, a contradiction. Therefore $(0; b) = A$, in which case $(0; (A, \cdot)) \neq 0$.

If we now assume that $(0; (A, \cdot)) \neq A$ then an argument similar to the one above easily verifies that A/(0; (A, $\cdot$)) is torsion-free and divisible, and consequently that if a' is an arbitrary element of A then $\phi(a')$ is the zero map. We therefore conclude that (A, $\cdot$) is the trivial ring on A.

Finally, if (A, $\cdot$) is the trivial ring on A and (A, $\cdot$) satisfies the restricted minimum condition then, as in the proof of Theorem 5.22, A is necessarily a rational group with idempotent type and $|P \backslash P_1^A| < \infty$. //

For commutative finite rank torsion-free rings with the finite norm property we have

THEOREM 5.24. *Suppose* (A, $\cdot$) *is an associative and commutative ring on a finite rank torsion-free group* A. *Then* (A, $\cdot$) *has the finite norm property if and only if* (A, $\cdot$) *is a prime ring or the trivial ring on the infinite cyclic group.*

Proof: We need only verify that a prime ring (A, $\cdot$) on A has the finite norm property. Since (A, $\cdot$) is a commutative prime ring, (A, $\cdot$) will have no divisors of zero. Consequently $Q \otimes A$ can be made into a finite dimensional algebra $(Q \otimes A, \cdot)$ over Q that will also contain no divisors of zero. It is well known that $J(Q \otimes A, \cdot)$ is now a nil ideal of $(Q \otimes A, \cdot)$, so $J(Q \otimes A, \cdot) = 0$. Consequently, $(Q \otimes A, \cdot)$ is a direct sum of simple subalgebras. Since $(Q \otimes A, \cdot)$ has no divisors of zero, $(Q \otimes A, \cdot)$ will itself be simple. Hence (A, $\cdot$) will have simple algebra type. It now follows from Lemma 121.6 of Fuchs [4] that the non-zero ideals of (A, $\cdot$) are of finite index in A. The proof is completed by observing that A has finite rank. //

COROLLARY 5.25. *Suppose* (A, $\cdot$) *is a non-trivial associative and commutative ring on a reduced torsion-free group* A *of finite rank. Then the following conditions are equivalent:*

(i) (A, $\cdot$) *satisfies the restricted minimum condition;*

(ii) (A, $\cdot$) *has the finite norm property;*

(iii) $(A, \cdot)$ *is a prime ring, in which case* $(A, \cdot)$ *is a full subring of a finite dimensional algebraic number field.*

Proof: If $(A, \cdot)$ is a prime ring on A then the proof of Theorem 5.24 shows that $J(Q \otimes A, \cdot) = 0$ and that $(Q \otimes A, \cdot)$ is a simple algebra over Q. Thus $(Q \otimes A, \cdot)$ is isomorphic to a ring of $n \times n$ matrices over a division algebra over Q. The commutativity of $(A, \cdot)$ and the fact that $(Q \otimes A, \cdot)$ has no divisors of zero now show that $(Q \otimes A, \cdot)$ is a field. Moreover, since $Q \otimes A$ has finite dimension over Q, $(Q \otimes A, \cdot)$ is an algebraic number field. Thus Theorem 5.24 shows (ii) and (iii) are equivalent.

Clearly (ii) implies (i), so from the comments above it suffices to show that (i) implies $(A, \cdot)$ is a prime ring. This is immediate from Proposition 5.23. //

It is obvious that any associative ring on a finite torsion group satisfies the restricted minimum condition and has the finite norm property. The comments prior to Proposition 5.23, and Proposition 1.3 of Levitz and Mott [1] show that the only other groups that can possibly support associative rings satisfying both the restricted minimum condition and the finite norm property are the infinite rank reduced torsion-free groups, the divisible torsion-free groups, or the infinite rank elementary p-groups, for some prime p. We conclude this section with examples to show that the equivalence of (i) and (ii) in Corollary 5.25 cannot be proved for divisible torsion-free groups of rank strictly greater than one, or for infinite rank elementary p-groups, for any prime p.

EXAMPLE 5.26. Suppose A is a torsion-free divisible group of rank greater than one. Then we can write $A = A_1 \oplus A_2$ where A_1 and A_2

are both non-zero torsion-free divisible groups. Lemma 122.2 of Fuchs [4] shows that we can define fields $(A_1, \cdot)$ and $(A_2, \cdot)$ on A_1 and A_2 respectively. By taking the ring direct sum of $(A_1, \cdot)$ and $(A_2, \cdot)$ we obtain an associative and commutative ring $(A, \cdot)$ on A. Since the only proper ideals of $(A, \cdot)$ are $(A_1, \cdot)$ and $(A_2, \cdot)$, it is clear that $(A, \cdot)$ satisfies the restricted minimum condition but does not have the finite norm property. (Notice that if the rank of A above is one then any non-trivial ring $(A, \cdot)$ on A satisfies the restricted minimum condition and has the finite norm property, while the trivial ring on A does not satisfy the restricted minimum condition and does not have the finite norm property).

If A is an infinite rank elementary p-group for some prime p, then we can write $A = A_1 \oplus A_2$ where $A_1 \cong \bigoplus_{\aleph_0} Z(p)$ and $A_2 \cong \bigoplus_{m} Z(p)$ for some infinite cardinal m. From Lemma 122.3 of Fuchs [4] we can define associative and commutative rings $(A_1, \cdot)$ and $(A_2, \cdot)$ with identities on A_1 and A_2 respectively, such that $(A_1, \cdot)$ and $(A_2, \cdot)$ both have no proper ideals. (Notice that $(A_1, \cdot)$ and $(A_2, \cdot)$ are now both fields). Again let $(A, \cdot)$ be the ring direct sum of $(A_1, \cdot)$ and $(A_2, \cdot)$. As in the torsion-free divisible case $(A, \cdot)$ satisfies the restricted minimum condition but does not have the finite norm property. //

3. TORSION-FREE GROUPS OF RANK TWO

Freedman [1] has shown that a torsion-free group A of rank two supporting an associative ring with identity has the property that its type set, $\mathcal{T}(A)$, contains at most three elements. The major part of Freedman's proof consists in showing that, for such a group A, $\mathcal{T}(A)$ contains at most two maximal elements. More generally we can prove

PROPOSITION 5.27. *Let* A *be a torsion-free group of rank* n *with the property that every pure subgroup of* A *of rank greater than one is non-nil. Then* $\mathcal{T}(A)$ *contains at most* n *maximal elements.*

Proof: We use an induction argument. Clearly the Proposition is true for a rational group, so assume that every non-nil group of rank k ($k < n$) satisfying the conditions of the Proposition has the property that its type set contains at most k maximal elements. Suppose A is as stated in the Proposition, and let $a_1, a_2, \ldots, a_{n+1}$ be $n + 1$ distinct elements of A such that $t(a_i) \neq t(a_j)$ for $i \neq j$, and $t(a_i)$ is maximal in $\mathcal{T}(A)$ for each $i = 1, 2, \ldots, n + 1$.

First we show that any subset of n distinct elements from $\{a_1, a_2, \ldots, a_{n+1}\}$ is a maximal independent set of elements of A. Clearly this amounts to showing that $\{a_1, a_2, \ldots, a_n\}$ is an independent set of elements of A. If $\{a_1, a_2, \ldots, a_n\}$ is not independent then there exists a $k \leqq n$ for which $\{a_1, a_2, \ldots, a_{k-1}\}$ is independent but $\{a_1, a_2, \ldots, a_k\}$ is not. If $A_1 = <\bigoplus_{i=1}^{k-1} <a_i>>_*$ then $a_k \in A_1$ and, since A_1 is pure in A, $\mathcal{T}(A_1) \subseteqq \mathcal{T}(A)$. But then A_1 is a rank $(k - 1)$ torsion-free group satisfying the conditions of the Proposition for which $\mathcal{T}(A_1)$ contains the k maximal elements $t(a_1), t(a_2), \ldots, t(a_k)$. Consequently $\{a_1, a_2, \ldots, a_n\}$ is a maximal independent set of elements of A.

We can now choose a non-zero integer m, and integers $m_1, m_2, \ldots, m_n$ such that

$$ma_{n+1} = m_1 a_1 + m_2 a_2 + \ldots + m_n a_n .$$

If $i \in \{1, 2, \ldots, n\}$ then the set $\{a_1, a_2, \ldots, a_{n+1}\}\backslash\{a_i\}$ is independent, so $m_i \neq 0$.

Consider now any ring $(A, \cdot)$ on A. For distinct i and j in $\{1, 2, \ldots, n + 1\}$, the maximality of $t(a_i)$ and $t(a_j)$ in $\mathcal{T}(A)$ shows $a_i \cdot a_j = 0$. In particular for any $i \in \{1, 2, \ldots, n\}$

$$0 = m(a_{n+1} \cdot a_i) = m_i\, a_i^2 \,.$$

Thus $m_i \neq 0$ yields $a_i^2 = 0$. Hence $(A, \cdot)$ must be the trivial ring on A. Since A is non-nil it now follows that $\mathcal{T}(A)$ contains at most n maximal elements. //

Beaumont and Wisner [1] have studied associative rings on torsion-free groups of rank two. Following Beaumont and Wisner we make the following definitions for the torsion-free group A of rank two. If $a \neq 0$ is an element of A then let

$$Q_a' = \{\alpha \in Q \mid \alpha a \in A\} \,.$$

Now define the *nucleus* D of A to be the subgroup $D = \bigcap_{a \in A} Q_a'$ of Q.

The following results were obtained by examining the possible multiplication tables for a distinguished pair of independent elements in a torsion-free group of rank two.

(5.28) (Beaumont and Wisner [1]). *Suppose* A *is a torsion-free group of rank two. Then* $(A, \cdot)$ *is a non-commutative associative ring on* A *if and only if* $(A, \cdot)$ *is defined by* $a_1 \cdot a_2 = \phi(a_1)a_2$ *or* $a_1 \cdot a_2 = \phi(a_2)a_1$ *for all* a_1, a_2 *in* A, *where* ϕ *is a non-trivial homomorphism of* A *into the nucleus* D *of* A. //

(5.29) (Beaumont and Wisner [1]). *Let* A *be a torsion-free group of rank two. If* $(A, \cdot)$ *is an associative and commutative ring on* A *and* A *contains an element* a *such that* $a^2 \neq 0$, *then there exists an element* b *in* A *such that* b *and* b^2 *are independent elements of* A. //

With the aid of (5.28) and (5.29) the major result of Freedman [1] can now be generalised.

THEOREM 5.30. *Suppose* A *is a torsion-free group of rank two that supports a non-trivial associative ring* (A, $\cdot$). *Then* $\mathcal{T}$(A) *contains at most three elements.*

Proof: We consider two cases separately.

Case (i). (A, $\cdot$) is non-commutative. (5.28) now gives the structure of (A, $\cdot$); suppose $a_1 \cdot a_2 = \phi(a_1)a_2$ for all a_1, a_2 in A, where $0 \neq \phi \in \mathrm{Hom}(A, D)$. It is clear that $D = \langle p^{-\infty} | pA = A \rangle$ and also that Im ϕ is a rank one torsion-free group with the same type as t(D). Thus Im $\phi \cong D$. Hence there is a non-zero $\theta \in \mathrm{Hom}(A, D)$ such that θ maps A onto D. We can now define a non-commutative associative ring (A, $\times$) on A by letting $a_1 \times a_2 = \theta(a_1)a_2$ for all a_1, a_2 in A. Since $1 \in D$ there is an element $a \in A$ for which $\theta(a) = 1$. But then a will be a left identity of (A, $\times$) so for every $a' \in A$, $t(a) \leqq t(a')$. (Notice that if (A, $\cdot$) has the alternate description in (5.28) then we can argue as above to again obtain $t(a) \leqq t(a')$).

Case (ii). (A, $\cdot$) is commutative. It is readily checked that (A, $\cdot$) non-trivial and commutative implies the existence of an element $a \in A$ such that $a^2 \neq 0$. Thus (5.29) shows that we can choose an element $a_1 \in A$ such that a_1 and a_1^2 are independent. If a_2 is a non-zero element of A then there are integers $m \neq 0$, m_1 and m_2 such that $ma_2 = m_1 a_1 + m_2 a_1^2$. Consequently

$$t(a_1) = t(a_1) \cap t(a_1^2) \leqq t(a_2) .$$

In either case $\mathcal{T}$ (A) contains a smallest element. We now argue as in Freedman [1]. Since A has rank two, each chain in $\mathcal{T}$(A) is of length at most two. Proposition 5.27 shows $\mathcal{T}$ (A) contains at most two maximal elements. Consequently $|\mathcal{T}(A)| \leqq 3$. //

A consequence of the proof of Case (i) above is the following observation.

PROPOSITION 5.31. *Suppose* $(A, \cdot)$ *is a non-commutative associative ring on a torsion-free group* A *of rank two. Then* A *is completely decomposable.*

Proof: It is clear that D can be made into a rank one module over itself, that is D is a projective D-module. As in the proof of Theorem 5.30 there is a non-zero $\theta \in \mathrm{Hom}(A, D)$ such that θ maps A onto D. It is readily checked that A is a D-module and $\theta \in \mathrm{Hom}_D(A, D)$. Consequently, A will contain a D-direct summand isomorphic to D. Thus A is completely decomposable. //

Turning our attention to the absolute (associative) annihilator of a torsion-free group A of rank two, (1.9) provides us with a complete description when A is completely decomposable. The next result shows us that this knowledge is also sufficient to describe the absolute (associative) annihilator when A has a quasi-direct decomposition.

PROPOSITION 5.32. *Suppose* A *and* B *are two torsion-free groups of finite rank such that* A *is quasi-equal to* B. *If* n *is a non-zero integer for which* $nA \subseteqq B$ *and* $nB \subseteqq A$ *then*

$$A(*) = \langle nB(*)\rangle_*^A ,$$

and

$$A^{(a)}(*) = \langle n(B^{(a)}(*))\rangle_*^A .$$

Proof: We will prove

$$A^{(a)}(*) = \langle n(B^{(a)}(*))\rangle_*^A ,$$

the proof for the absolute annihilator being identical.

First suppose $a \in A^{(a)}(*)$. If $(B, \cdot)$ is an associative ring on B, then we can define an associative ring $(A, \times)$ on A by letting $a_1 \times a_2 = n(na_1 \cdot na_2)$ for all a_1, a_2 in A. For any element $b \in B$,

$$n^3(na \cdot b) = n(na \cdot n^2 b) = a \times nb = 0.$$

Thus $na \cdot b = 0$ for all $b \in B$. Since this is true for every associative ring $(B, \cdot)$ on B, $na \in B^{(a)}(*)$. Therefore $n^2 a \in nB^{(a)}(*)$, and so $a \in \langle nB^{(a)}(*)\rangle_*^A$.

Conversely, suppose $a \in \langle nB^{(a)}(*)\rangle_*^A$. Then there is an integer m such that $ma = nb$, where $b \in B^{(a)}(*)$. If $(A, \cdot)$ is an associative ring on A then we can define an associative ring $(B, \times)$ on B by letting $b_1 \times b_2 = n(nb_1 \cdot nb_2)$ for all b_1, b_2 in B. Clearly for all $a' \in A$,

$$n^4 m(a \cdot a') = ma \times na' = nb \times na' = 0$$

As before we conclude that $a \in A^{(a)}(*)$. //

Evidently we need now only consider strongly indecomposable torsion-free groups of rank two. If A is such a group then it is clear from the lack of a non-associative analogue of (5.1) that we can really only hope to describe the absolute associative annihilator of A. The following result from Wickless [1] will be required.

(5.33) (Wickless [1]). *Suppose* A *is a torsion-free group of rank two. Then* A *is non-nil and every associative ring on* A *is nilpotent if and only if there exist independent elements* a *and* b *in* A *such that if* Q_a, Q'_a, Q_b *and* Q'_b *are the rational groups belonging to* a *and* b *respectively, then*

(i) Q_a, Q_b *and* Q_b' *all have non-idempotent types,*

(ii) $Q_a' \subseteq Q_a \subseteq Q_b' \subseteq Q_b$, *and*

(iii) $t(Q_a)^2 \leq t(Q_b')$.

Furthermore, if $(A, \cdot)$ *is a non-trivial associative ring on* A *then the multiplication table for* $(A, \cdot)$ *is*

$$(*) \qquad \begin{array}{c|cc} \cdot & a & b \\ \hline a & \alpha b & \beta b \\ b & \gamma b & 0 \end{array}$$

where α, β *and* γ *are suitable rationals.* //

THEOREM 5.34. *Let* A *be a strongly indecomposable torsion-free group of rank two. Then exactly one of the following holds:*

(i) $A^{(a)}(*) = 0$;

(ii) $A^{(a)}(*) = A$;

(iii) $A^{(a)}(*)$ *is the pure subgroup of* A *generated by any element with maximal type.*

(i) *holds exactly when* A *is fully faithful,* (ii) *holds exactly when* A *is nil, and* (iii) *holds exactly when* A *is non-nil and every associative ring on* A *is nilpotent.*

Proof: Clearly we need only consider the case when A is non-nil and every associative ring on A is nilpotent. Choose independent elements a and b in A satisfying (i), (ii) and (iii) of (5.33). If $(A, \cdot)$ is a non-trivial associative ring on A then $(A, \cdot)$ will have the multiplication table (*) of (5.33), for suitable rationals α, β and γ. Since $(a \cdot a) \cdot b = 0$, the associative law yields $a \cdot (a \cdot b) = \beta^2 b = 0$. Thus $\beta = 0$. Similarly $\gamma = 0$, and so $\alpha \neq 0$. It is now clear that $(0; (A, \cdot)) = \langle b \rangle_*$. This is true for every non-trivial associative ring $(A, \cdot)$ on A, so $A^{(a)}(*) = \langle b\rangle_*$.

From (i), (ii) and (iii) of (5.33) it is clear that $t(a) \lneq t(b)$, so if a' is any element of A with maximal type, $\langle a'\rangle_* = \langle b\rangle_*$. Consequently $A^{(a)}(*)$ is the pure subgroup of A generated by any element with maximal type. //

Consider now the absolute radical of a torsion-free group A of rank two. In the cases that A is completely decomposable or A is in S^*, the final remarks of Chapter Three and Theorem 5.8 respectively provide complete descriptions of J(A). Thus apart from some strongly indecomposable groups, we need only consider the case when A has a quasi-direct decomposition $A \doteq A_1 \oplus A_2$, where A_1 and A_2 are rational groups. Although $J(A_1 \oplus A_2)$ has been completely described, the shortage of a result similar to Proposition 5.32 for the absolute radical limits the information about the structure of J(A) that can be deduced from our knowledge of $J(A_1 \oplus A_2)$.

CHAPTER SIX

In this final chapter we provide a bound for the nil-degree (if it is finite) of a torsion-free group A, not necessarily of finite rank, but with certain finiteness conditions on the rank of A/pA for each prime p. We also prove that an associative ring on such a group is nilpotent exactly if it is nil. These results extend some similar results obtained quite recently by Webb [1] for torsion-free groups of finite rank.

The methods of proof employed in this chapter involve a discussion of the embedding of a torsion-free group A in its p-adic completion $\hat{A}_{(p)}=\lim\limits_{\substack{\leftarrow \\ k}} A/p^kA$, for each prime p. In the case that A/pA has finite rank we exhibit a strong relationship between $\hat{A}_{(p)}$ and any p-basic subgroup of A. It is this connection, and an analogue of a result of Fuchs [4] concerning the Z-adic completions of certain rings, that enable us to prove our main result.

Evidently, it is natural to begin this chapter by exploring the rings supported by p-pure subgroups of the p-adic integers J_p for each prime p, and consequently the rings supported by cohesive groups.

1. COHESIVE GROUPS

Dubois [1] has introduced and investigated the class of cohesive groups. Following Dubois we call a torsion-free group A *cohesive* if, for every non-zero pure subgroup S of A, A/S is divisible.

It is clear that a cohesive group is either divisible or reduced. Consequently, in our attempt to describe the absolute annihilator and the absolute radical of a cohesive group we are permitted to restrict our attention to reduced cohesive groups. The following result is vital for our discussion of rings on such groups.

(6.1) (Dubois [1]). *A torsion-free group* A *is cohesive if and only if for every prime* p *either* A/pA *is zero or* A *is isomorphic to a* p*-pure subgroup of the* p*-adic integers* J_p. //

It is clear from (6.1) that we need to consider the rings supported by p-pure subgroups of the p-adic integers J_p, for each prime p. A complete description of these rings can be given.

PROPOSITION 6.2. *Suppose* A *is a* p*-pure subgroup of the* p*-adic integers* J_p*, for some prime* p. *If* $(A, \times)$ *is any ring on* A *then either* $(A, \times)$ *is the trivial ring, or* $(A, \times)$ *is a subring of a suitable ring on* J_p*, in which case*

$$a_1 \times a_2 = j \cdot a_1 \cdot a_2$$

for every a_1, a_2 *in* A*, where* j *is some element of* J_p *and* $\cdot$ *is the multiplication operation in the ring* Q_p^* *of* p*-adic integers.*

Proof: Let $(A, \times)$ be a non-trivial ring on A and suppose $\phi \in \mathrm{Hom}(A, E(A))$ is the map defining $(A, \times)$. Then for any $a \in A$, $\phi(a) \in \mathrm{Hom}(A, A) \subseteq \mathrm{Hom}(A, J_p)$. Armstrong [1] has shown that every homomorphism from A into J_p is the restriction of an endomorphism of J_p. Since every endomorphism of J_p is multiplication in Q_p^* by a suitable element of J_p, there is a unique $j_a \in J_p$ such that $\phi(a)a' = j_a \cdot a'$ for all $a' \in A$. Consider now the map $\phi_1 \in \mathrm{Hom}(A, J_p)$ defined by $\phi_1(a) = j_a$. Again ϕ_1 is the restriction of some endomorphism of J_p, and as such $\phi_1(a) = j \cdot a$ for some element $j \in J_p$, for each $a \in A$. But then for a_1, a_2 in A we have

$$a_1 \times a_2 = \phi(a_1)a_2 = j_{a_1} \cdot a_2 = \phi_1(a_1) \cdot a_2 = j \cdot a_1 \cdot a_2 .$$

It is evident that $(A, \times)$ is now a subring of a suitable ring on J_p. //

The structure of the absolute annihilator of a cohesive group is now immediate.

THEOREM 6.3. *For a reduced cohesive group* A, $A(*) = A^{(a)}(*) = A$ *exactly when* A *is nil, otherwise* $A(*) = A^{(a)}(*) = 0$. //

Dubois [1] has shown that in a reduced cohesive group all the non-zero elements have types with the same set of infinity places. Thus we are justified to use the notation P_1^A to denote the set of primes p for which the p-component of the type of every non-zero element of A is not infinity (∞). Proposition 6.2 now enables us to exhibit a relationship between $J(A)$ and $\bigcap_{p \in P_1^A} pA$.

PROPOSITION 6.4. *Let* A *be a reduced cohesive group. If* A *is non-nil then* $J(A) \subseteqq \bigcap_{p \in P_1^A} pA$.

Proof: For each $p \in P_1^A$, (6.1) shows that $A \cong A^{(p)}$ where $A^{(p)}$ is a p-pure subgroup of the p-adic integers J_p. From Lemma 1.17 it suffices to prove the inclusion $J(A^{(p)}) \subseteqq pA^{(p)}$ for each $p \in P_1^A$.

Since A is non-nil, each $A^{(p)}$ is a non-nil subgroup of J_p. Consider now a fixed but arbitrary prime $p \in P_1^A$. Proposition 6.2 shows the existence of a ring $(A^{(p)}, \times)$ on $A^{(p)}$ defined in the following manner: for all a_1, a_2 in $A^{(p)}$,

$$a_1 \times a_2 = j \cdot a_1 \cdot a_2 ,$$

where $j \in J_p$ and $\cdot$ is the multiplication operation of Q_p^*. Writing

$$j = s_0 p^k + s_1 p^{k+1} + \ldots$$

with $s_0 \neq 0$,

$$a_1 \times a_2 = p^k((s_0 + s_1p + \ldots) \cdot a_1 \cdot a_2) \in A^{(p)}$$

for all a_1, a_2 in $A^{(p)}$. Since $A^{(p)}$ is p-pure in J_p

$$(s_0 + s_1p + \ldots) \cdot a_1 \cdot a_2 \in A^{(p)}$$

for all a_1, a_2 in $A^{(p)}$. Thus we can define an associative ring $(A^{(p)}, \circ)$ on $A^{(p)}$ by letting

$$a_1 \circ a_2 = (s_0 + s_1p + \ldots) \cdot a_1 \cdot a_2$$

for all a_1, a_2 in $A^{(p)}$.

Now $A^{(p)}$ is p-pure in J_p, so it is possible to select $a_0 \in A^{(p)}$ such that a_0 is a p-adic unit. Then since $A^{(p)}/pA^{(p)} \cong Z(p)$, $a_0 \circ a_0 \notin pA^{(p)}$ shows $(A^{(p)}/pA^{(p)}, \circ)$ is a field. Therefore $J(A^{(p)}/pA^{(p)}, \circ) = 0$, and so $J(A^{(p)}, \circ) \subseteqq pA^{(p)}$, as required. //

Dubois [1] has shown that a reduced cohesive group is strongly indecomposable. Consequently, Theorem 5.8 is an improvement on Proposition 6.4 in the case that A is a reduced cohesive group of finite rank. It is an open question whether the corresponding statement of Theorem 5.8 holds for reduced cohesive groups of infinite rank.

Every group A considered thus far for which some information about the structure of J(A) has been given also has the property that J(A/J(A)) = 0. The next proposition shows that cohesive groups are no exception to the conjecture that J(A/J(A)) = 0 for all groups A.

PROPOSITION 6.5. *If* A *is a cohesive group then* J(A/J(A)) = 0.

Proof: Clearly we may assume that A is a non-nil reduced cohesive group for which J(A) ≠ 0. From Proposition 1.22 we may also assume A/J(A) ≠ T(A/J(A)). Now, since [A/J(A)]/T(A/J(A)) is torsion-free and A is cohesive, [A/J(A)]/T(A/J(A)) is divisible. If we now proceed as in the proof of Proposition 1.22 we obtain the required result. //

2. THE NIL-DEGREE OF TORSION-FREE GROUPS

Feigelstock [1] has introduced a concept very similar to the strong nil-degree of a torsion-free group. Following Feigelstock we define the *extra strong nil-degree (strong nilstufe)* of a torsion-free group A to be the largest positive integer n such that there is a ring on A with a non-zero product of length n (all possible bracketings considered), but no ring on A with non-zero products of length greater than n. If no such n exists then the extra strong nil-degree is defined to be ∞. For a torsion-free group A we let $N(A)$, $N_S(A)$ and $N_E(A)$ respectively denote the nil-degree, the strong nil-degree and the extra strong nil-degree of A.

Feigelstock [2] has claimed that if A is a torsion-free group of rank two then $N_E(A)$ is 1, 2 or ∞, but appears to have only shown that $N(A)$ is 1, 2 or ∞; his proof relies on Lemma 1 of Beaumont and Wisner [1] that requires consideration of associative rings. Feigelstock has also conjectured that if A is a torsion-free group of finite rank n then $N_E(A)$ is 1, 2, ..., n or ∞.

Recently Webb [1] has shown that if A is a torsion-free group of rank n then $N(A)$ is 1, 2, ..., n or ∞ and $N_E(A)$ is 1, 2, ..., 2^{n-1} or ∞. Also, he has provided an example of a torsion-free group A of rank three for which $N_E(A) = 4$. Thus Feigelstock's conjecture is not true. However, if we replace $N_E(A)$ with $N_S(A)$, the conjecture can be proved.

Recall that for a ring $(A, \cdot)$ on the group A and a positive integer k, $(A, \cdot)^k$ is the subring of $(A, \cdot)$ generated by all products of the form $(\ldots((a_1 \cdot a_2) \cdot a_3)\ldots) \cdot a_k$.

THEOREM 6.6. *Let* $(A, \cdot)$ *be a ring on a torsion-free group* A *of finite rank* n. *If* $(A, \cdot)^m = 0$ *for some positive integer* m *then* $(A, \cdot)^{n+1} = 0$.

Proof: Suppose $(A, \cdot)^m = 0$ for some positive integer m, and k is a positive integer for which $(A, \cdot)^{k+1} \neq 0$. We show $(A, \cdot)^k/(A, \cdot)^{k+1}$ is not a torsion group.

Indeed, suppose $(A, \cdot)^k/(A, \cdot)^{k+1}$ is torsion. If we choose a non-zero element $a \in (A, \cdot)^k$ then there is an integer $n_1 \neq 0$ such that $n_1 a \in (A, \cdot)^{k+1}$. Thus

$$0 \neq n_1 a = a_1' \cdot a_1 + a_{1_2}' \cdot a_{1_2} + a_{1_3}' \cdot a_{1_3} + \ldots + a_{1_{n(1)}}' \cdot a_{1_{n(1)}} ,$$

where a_1 and a_{1_i} are in A, and a_1' and a_{1_i}' are in $(A, \cdot)^k$, for each $i \in \{2, 3, \ldots, n(1)\}$. Without loss of generality we can assume $a_1' \cdot a_1 \neq 0$.

Since $a_1' \in (A, \cdot)^k$ it is possible to choose a non-zero integer n_2 such that $n_2 a_1' \in (A, \cdot)^{k+1}$. Hence

$$0 \neq n_2(a_1' \cdot a_1) = (a_2' \cdot a_2 + a_{2_2}' \cdot a_{2_2} + a_{2_3}' \cdot a_{2_3} + \ldots + a_{2_{n(2)}}' \cdot a_{2_{n(2)}}) \cdot a_1,$$

where a_2 and a_{2_i} are in A, and a_2' and a_{2_i}' are in $(A, \cdot)^k$, for each $i \in \{2, 3, \ldots, n(2)\}$. Again we can assume $(a_2' \cdot a_2) \cdot a_1 \neq 0$.

If we repeat this procedure we can obtain elements $a_1, a_2, \ldots, a_{m-k}$ in A, and an element a_{m-k}' in $(A, \cdot)^k$ such that

$$(\ldots((a_{m-k}' \cdot a_{m-k}) \cdot a_{m-k-1}) \cdot \ldots) \cdot a_1 \neq 0 .$$

Clearly

$$(\ldots((a_{m-k}' \cdot a_{m-k}) \cdot a_{m-k-1}) \cdot \ldots) \cdot a_1 \in (A, \cdot)^m ,$$

contradicting the fact that $(A, \cdot)^m = 0$. We conclude that $(A, \cdot)^k/(A, \cdot)^{k+1}$ cannot be a torsion group.

Consequently, for each positive integer k for which $(A, \cdot)^{k+1} \neq 0$, $(A, \cdot)^k/(A, \cdot)^{k+1}$ has torsion-free rank greater than zero. That is, $r((A, \cdot)^k)$ is strictly greater than $r((A, \cdot)^{k+1})$. Since A has finite rank n, $(A, \cdot)^{n+1} = 0$. //

COROLLARY 6.7. *If* A *is a torsion-free group of rank* n *then* $N_S(A)$ *is* 1, 2, ..., n *or* ∞. //

The following example shows that for each positive integer n the bound of n for $N_S(A)$ in Corollary 6.7 is actually attained.

EXAMPLE 6.8. For each $i \in \{1, 2, \ldots, n\}$ let A_i be a rational group with type $(2i, 2i, \ldots, 2i, \ldots)$. Consider the rank n completely decomposable group $A = \bigoplus_{i=1}^{n} A_i$. Clearly there exists a chain of length n but no chain of length n + 1 in $\tilde{T}(A)$ with respect to the relation $\leqq'$. Thus (3.1) yields $N_S(A) = n$. //

The remainder of this chapter is concerned with extending the associative case of Corollary 6.7 to other classes of torsion-free groups. Our aim is two-fold: we wish to find some infinite rank torsion-free groups whose nil-degrees, if finite, are bounded, and we would also like under certain circumstances to lower the bound on the finite nil-degrees mentioned in the Corollary.

A method of achieving these ends has already been suggested in our previous discussion of p-pure subgroups of the p-adic integers. It is evident that if $A^{(p)}$ is such a group then $N(A^{(p)})$, $N_S(A^{(p)})$ and $N_E(A^{(p)})$ are all either 1 or ∞. We therefore concentrate our attention on torsion-free groups A with the property that for each prime p, $r(A/pA)$ is bounded by some positive integer n (not depending on p).

This amounts to considering torsion-free groups whose p-basic subgroups all have rank $\leq$ n.

Recall that for a group A and a prime p, $\hat{A}_{(p)} = \varprojlim_k (A/p^kA)$ denotes the p-adic completion of A. If A is torsion-free and p-reduced then clearly $\hat{A}_{(p)}$ is torsion-free. Also, $\hat{A}_{(p)}$ can be made into a module over the ring of p-adic integers Q_p^* by defining, for $j = s_0 + s_1 p + \ldots + s_k p^k + \ldots$ in Q_p^* and $(a_1 + pA, a_2 + p^2A, \ldots, a_k + p^kA, \ldots)$ in $\hat{A}_{(p)}$,

$$j(a_1 + pA, a_2 + p^2A, \ldots, a_k + p^kA, \ldots)$$
$$= (j^{(1)}(a_1 + pA), j^{(2)}(a_2 + p^2A), \ldots, j^{(k)}(a_k + p^kA), \ldots)$$

where $j^{(k)} = s_0 + s_1 p + \ldots + s_{k-1} p^{k-1}$ for each positive integer k.

The next result enables us to extend rings on certain groups to rings on their p-adic completions.

PROPOSITION 6.9. *Suppose* A *is a group with no elements of infinite* p*-height for some prime* p, *and* (A, ·) *is a ring on* A. *Then there is exactly one ring structure* $(\hat{A}_{(p)}, \cdot)$ *on* $\hat{A}_{(p)}$ *which extends that of* (A, ·), *and this preserves associativity and commutativity in* (A, ·). *Furthermore* $(\hat{A}_{(p)}, \cdot)$ *becomes a* Q_p^**-algebra.*

Proof: The proof of the Proposition is analogous to the proof of Corollary 119.4 of Fuchs [4]. The only statements that require verification are that the extension $(\hat{A}_{(p)}, \cdot)$ of (A, ·) is unique, and that $(\hat{A}_{(p)}, \cdot)$ becomes a Q_p^*-algebra. Since A can be regarded as a p-pure and p-dense subgroup of the p-reduced group $\hat{A}_{(p)}$ the proof of Lemma 119.2 of Fuchs [4] applies to show that $(\hat{A}_{(p)}, \cdot)$ is unique. That $(\hat{A}_{(p)}, \cdot)$ is a Q_p^*-algebra follows at once from the definition of the Q_p^*-module $\hat{A}_{(p)}$ given prior to the Proposition. //

The following well known result is required.

(6.10) (Fuchs [3], p. 166). *Let* $0 \to B \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$ *be a* p-*pure exact sequence. Then the sequence*

$$0 \to \hat{B}_{(p)} \xrightarrow{\hat{\alpha}} \hat{A}_{(p)} \xrightarrow{\hat{\beta}} \hat{C}_{(p)} \to 0$$

is splitting exact. //

LEMMA 6.11. *Suppose* A *is a torsion-free group and* B *is a* p-*basic subgroup of* A. *Then* $\hat{A}_{(p)}$ *and* $\hat{B}_{(p)}$ *are isomorphic* p-*adic modules. Furthermore,* $\hat{A}_{(p)}$ *has finite rank over* Q_p^* *if and only if* B *has finite rank over* Z, *and in this case the* Q_p^*-*rank of* $\hat{A}_{(p)}$ *and the* Z-*rank of* B *coincide.*

Proof: Consider the p-pure exact sequence

$$0 \to B \xrightarrow{\alpha} A \to A/B \to 0$$

where α is the inclusion map. (6.10) shows that the sequence

$$0 \to \hat{B}_{(p)} \xrightarrow{\hat{\alpha}} \hat{A}_{(p)} \to (\widehat{A/B})_{(p)} \to 0$$

is splitting exact, so $\hat{A}_{(p)} \cong \operatorname{Im} \hat{\alpha} \oplus (\widehat{A/B})_{(p)}$. Since A/B is p-divisible, $(\widehat{A/B})_{(p)} = 0$, whence $\hat{A}_{(p)} \cong \hat{B}_{(p)}$ (as groups).

Next let $(b_1 + pB, b_2 + p^2 B, \ldots, b_k + p^k B, \ldots)$ be an arbitrary element of $\hat{B}_{(p)}$, and let j be a p-adic integer. Then

$$\begin{aligned}
&\hat{\alpha}(j(b_1 + pB, b_2 + p^2B, \ldots, b_k + p^kB, \ldots)) \\
&\qquad = \hat{\alpha}(j^{(1)}b_1 + pB, j^{(2)}b_2 + p^2B, \ldots, j^{(k)}b_k + p^kB, \ldots) \\
&\qquad = (j^{(1)}b_1 + pA, j^{(2)}b_2 + p^2A, \ldots, j^{(k)}b_k + p^kA, \ldots) \\
&\qquad = j(b_1 + pA, b_2 + p^2A, \ldots, b_k + p^kA, \ldots) \\
&\qquad = j\, \hat{\alpha}(b_1 + pB, b_2 + p^2B, \ldots, b_k + p^kB, \ldots)\ ,
\end{aligned}$$

so $\hat{A}_{(p)}$ and $\hat{B}_{(p)}$ are isomorphic Q_p^*-modules.

Suppose now the rank of B is finite. A trivial induction argument together with (6.10) show that the rank of $\hat{B}_{(p)}$ over Q_p^* is precisely the rank of B. Thus the Q_p^*-rank of $\hat{A}_{(p)}$ is the rank of B. To prove the converse suppose $\hat{A}_{(p)}$ has finite rank n over Q_p^*, and B has rank strictly greater than n. Then B contains a p-pure free summand of rank (n + 1), so (6.10) shows that $\hat{A}_{(p)} \cong \hat{B}_{(p)}$ contains a summand isomorphic to the direct sum of (n + 1) copies of J_p. This is clearly impossible. //

Suppose A is a torsion-free group and $\alpha : A \to \hat{A}_{(p)}$ is the canonical map from A into its p-adic completion. If a is an arbitrary element of A then let $\hat{a}$ denote the image of a under the map α. Similarly if B is a p-basic subgroup of A and $\beta : B \to \hat{B}_{(p)}$ is the canonical map from B into its p-adic completion, then let $\overline{b}$ denote the image of $b \in B$ under the map β. We can now improve the final assertion in Lemma 6.11.

LEMMA 6.12. *Let* A *be a torsion-free group with finite rank* p-*basic subgroup* $B = \langle b_1\rangle \oplus \langle b_2\rangle \oplus \ldots \oplus \langle b_n\rangle$. *Then the elements* $\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n$ *of* $\hat{A}_{(p)}$ *form a basis of* $\hat{A}_{(p)}$ *over* Q_p^*.

Proof: From Lemma 6.11 it suffices to show that the set $S = \{\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_n\}$ of elements of $\hat{B}_{(p)}$ form a basis of $\hat{B}_{(p)}$ over Q_p^*.

First we show that S is independent over Q_p^*. Indeed suppose

$$(*) \qquad j_1 \overline{b}_1 + j_2 \overline{b}_2 + \ldots + j_n \overline{b}_n = \overline{0}$$

for some p-adic integers $j_1, j_2, \ldots, j_n$. With $j_i^{(k)}$ defined as usual for $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots\}$, (*) becomes

$$(j_1^{(1)}b_1 + pB,\ j_1^{(2)}b_1 + p^2B,\ \ldots,\ j_1^{(k)}b_1 + p^kB,\ \ldots) +$$
$$(j_2^{(1)}b_2 + pB,\ j_2^{(2)}b_2 + p^2B,\ \ldots,\ j_2^{(k)}b_2 + p^kB,\ \ldots) + \ldots +$$
$$(j_n^{(1)}b_n + pB,\ j_n^{(2)}b_n + p^2B,\ \ldots,\ j_n^{(k)}b_n + p^kB,\ \ldots)$$
$$= (pB,\ p^2B,\ \ldots,\ p^kB,\ \ldots)\ .$$

Thus

$$j_1^{(k)}b_1 + j_2^{(k)}b_2 + \ldots + j_n^{(k)}b_n \in p^kB$$

for each $k \in \{1, 2, \ldots\}$. Hence for every $k \in \{1, 2, \ldots\}$ there are integers $\ell_1^{(k)}, \ell_2^{(k)}, \ldots, \ell_n^{(k)}$ such that

$$j_1^{(k)}b_1 + j_2^{(k)}b_2 + \ldots + j_n^{(k)}b_n = \ell_1^{(k)}p^k\, b_1 + \ell_2^{(k)}p^k\, b_2 + \ldots + \ell_n^{(k)}p^k\, b_n.$$

Consequently $j_i^{(k)} = \ell_i^{(k)}p^k$ for each $i \in \{1, 2, \ldots, n\}$. But then

$$\begin{aligned} j_i\ \overline{b}_i &= (j_i^{(1)}b_i + pB,\ j_i^{(2)}b_i + p^2B,\ \ldots,\ j_i^{(k)}b_i + p^kB,\ \ldots) \\ &= (\ell_i^{(1)}p\ b_i + pB,\ \ell_i^{(2)}p^2\ b_i + p^2B,\ \ldots,\ \ell_i^{(k)}p^k\ b_i + p^kB,\ \ldots) \\ &= \overline{0}\ , \end{aligned}$$

for each $i \in \{1, 2, \ldots, n\}$. Since $\hat{B}_{(p)}$ is torsion-free as a Q_p^*-module, we conclude that S is independent over Q_p^*.

Next we show that S generates $\hat{B}_{(p)}$. Let

$$(b^{(1)} + pB,\ b^{(2)} + p^2B,\ \ldots,\ b^{(k)} + p^kB,\ \ldots)$$

be an arbitrary element of $\hat{B}_{(p)}$. Then for each $k \in \{1, 2, \ldots\}$ there are suitable integers $m_i^{(k)}$, $i \in \{1, 2, \ldots, n\}$, such that

$$b^{(k)} + p^kB = (m_1^{(k)}b_1 + m_2^{(k)}b_2 + \ldots + m_n^{(k)}b_n) + p^kB\ ,$$

and $0 \leqq m_i^{(k)} < p^k$. Now for each $k \in \{1, 2, \ldots\}$

$$b^{(k+1)} + p^kB = b^{(k)} + p^kB\ ,$$

so $b^{(k+1)} - b^{(k)} \in p^kB$. It follows that for each $i \in \{1, 2, \ldots, n\}$ and each $k \in \{1, 2, \ldots\}$, $(m_i^{(k+1)} - m_i^{(k)})b_i \in p^k \langle b_i\rangle$. Thus for each $i \in \{1, 2, \ldots, n\}$, the sequence $m_i^{(1)}, m_i^{(2)}, \ldots, m_i^{(k)}, \ldots$ has the property that $m_i^{(k+1)} \equiv m_i^{(k)} \pmod{p^k}$, for each $k \in \{1, 2, \ldots\}$. Hence $m_i^{(1)}, m_i^{(2)}, \ldots, m_i^{(k)}, \ldots$ determines a p-adic integer j_i for which $j_i^{(k)} = m_i^{(k)}$ for each $k \in \{1, 2, \ldots\}$. But then

$$\begin{aligned}
&j_1 \overline{b}_1 + j_2 \overline{b}_2 + \ldots + j_n \overline{b}_n \\
&\quad = (m_1^{(1)} b_1 + pB,\ m_1^{(2)} b_1 + p^2 B,\ \ldots,\ m_1^{(k)} b_1 + p^k B,\ \ldots) + \\
&\qquad\quad (m_2^{(1)} b_2 + pB,\ m_2^{(2)} b_2 + p^2 B,\ \ldots,\ m_2^{(k)} b_2 + p^k B,\ \ldots) + \ldots + \\
&\qquad\qquad (m_n^{(1)} b_n + pB,\ m_n^{(2)} b_n + p^2 B,\ \ldots,\ m_n^{(k)} b_n + p^k B,\ \ldots) \\
&\quad = ((m_1^{(1)} b_1 + m_2^{(1)} b_2 + \ldots + m_n^{(1)} b_n) + pB, \\
&\qquad\quad (m_1^{(2)} b_1 + m_2^{(2)} b_2 + \ldots + m_n^{(2)} b_n) + p^2 B,\ \ldots \\
&\qquad\qquad \ldots,\ (m_1^{(k)} b_1 + m_2^{(k)} b_2 + \ldots + m_n^{(k)} b_n) + p^k B,\ \ldots) \\
&\quad = (b^{(1)} + pB,\ b^{(2)} + p^2 B,\ \ldots,\ b^{(k)} + p^k B,\ \ldots)\ ,
\end{aligned}$$

so S indeed generates $\hat{B}_{(p)}$. //

A consequence of Lemma 6.12 and Proposition 6.9 is the following.

PROPOSITION 6.13. *Suppose* A *is a torsion-free group with no elements of infinite* p*-height for some prime* p, *and* r(A/pA) *is finite. Then any ring* (A, $\cdot$) *on* A *is completely determined by its effect upon any* p*-basic subgroup of* A. *If* A *has finite rank and* r(A) = r(A/pA), *then it is possible to choose a* p*-basic subgroup of* A *that is also a subring of* (A, $\cdot$).

Proof: Let $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \ldots \oplus \langle b_n \rangle$ be a p-basic subgroup of A. If $\hat{A}_{(p)}$ is the p-adic completion of A then Proposition 6.9 show that (A, $\cdot$) may be viewed as a subring of ($\hat{A}_{(p)}$, $\cdot$). Lemma 6.12 now shows that the ring ($\hat{A}_{(p)}$, $\cdot$), and hence the ring (A, $\cdot$), is determined by the effect of (A, $\cdot$) on the set $\{b_1, b_2, \ldots, b_n\}$.

To prove the final assertion of the Proposition we use an argument similar to the proof of Lemma 4.3 of Beaumont and Pierce [1]. Suppose $r(A) = r(A/pA) = n$. Then $\{b_1, b_2, \ldots, b_n\}$ is a maximal independent set of elements of A, so for all i and $j \in \{1, 2, \ldots, n\}$ there exists an integer m with $(m, p) = 1$, and integers $m_1, m_2, \ldots, m_n$ such that

$$m\,(b_i \cdot b_j) = m_1\, b_1 + m_2\, b_2 + \ldots + m_n\, b_n .$$

Consequently $(mB, \cdot) = (\langle mb_1, mb_2, \ldots, mb_n\rangle, \cdot)$ is a subring of $(A, \cdot)$. Finally since B is p-pure in A and $(m, p) = 1$ it follows that mB is a p-basic subgroup of A. //

Fuchs [1] has demonstrated a strong connection between the ring structures supported by a torsion group A and the partial multiplications on a basic subgroup B of A. Fuchs has shown that a ring $(A, \cdot)$ on A is completely determined by its effect on B, and conversely that any partial multiplication structure on B extends uniquely to a ring on A.

It is not difficult to find examples to show that the partial similarity of Proposition 6.13 with Fuch's result cannot be strengthened. Indeed, simply let A be a rational group with non-idempotent type. It is clear that there is a prime p for which A satisfies the conditions of Proposition 6.13. However, since A is a nil group and any p-basic subgroup of A is cyclic, not every partial multiplication on a p-basic subgroup of A will extend to a ring on A.

Returning once again to our main discussion, suppose A is a torsion-free group with no elements of infinite p-height, for some prime p, and $(A, \cdot)$ is an associative ring on A. Proposition 6.9 shows that $(A, \cdot)$ can be viewed as a subring of an associative ring $(\hat{A}_{(p)}, \cdot)$ on $\hat{A}_{(p)}$. If we let K denote the quotient field of Q_p^*, then $K \otimes_{Q_p^*} \hat{A}_{(p)}$ can be made into an associative algebra $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$ over K by defining, for k_1, k_2 in K and $\hat{a}_1$, $\hat{a}_2$ in $\hat{A}_{(p)}$,

$$(k_1 \otimes \hat{a}_1) \cdot (k_2 \otimes \hat{a}_2) = (k_1\ k_2) \otimes (\hat{a}_1 \cdot \hat{a}_2)$$

and

$$k_1(k_2 \otimes \hat{a}_1) = (k_1\ k_2) \otimes \hat{a}_1 .$$

It is clear that if $\hat{A}_{(p)}$ has finite rank over Q_p^* then $K \otimes_{Q_p^*} \hat{A}_{(p)}$ will have finite dimension over K. Also the map $\hat{a} \to 1 \otimes \hat{a}$ for each $\hat{a} \in \hat{A}_{(p)}$ is an embedding of $(\hat{A}_{(p)}, \cdot)$ in $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$, so $(A, \cdot)$ can be viewed as a subring of the algebra $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$.

These comments form the basis for the proof of our next result.

PROPOSITION 6.14. *Let* A *be a torsion-free group with no elements of infinite* p*-height for some prime* p*, and suppose* $r(A/pA) = n$. *If* $(A, \cdot)$ *is a nil ring then* $(A, \cdot)^{n+1} = 0$.

Proof: $(A, \cdot)$ can be embedded in the associative algebra $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$ over the field K. If B is a p-basic subgroup of A then there exist elements $b_1, b_2, \ldots, b_n$ of A such that $B = \langle b_1\rangle \oplus \langle b_2\rangle \oplus \ldots \oplus \langle b_n\rangle$. Lemma 6.12 shows that $\{\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n\}$ is now a basis of $\hat{A}_{(p)}$ over Q_p^*, so $\{1 \otimes \hat{b}_1, 1 \otimes \hat{b}_2, \ldots, 1 \otimes \hat{b}_n\}$ is a basis of $K \otimes_{Q_p^*} \hat{A}_{(p)}$ over K. For each $i \in \{1, 2, \ldots, n\}$, b_i is a nilpotent element of $(A, \cdot)$, so $1 \otimes \hat{b}_i$ is a nilpotent element of $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$. Since $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$ has finite dimension n over K, a reference to Abian [1], p. 155, now shows $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)^{n+1} = 0$.

Thus $(A, \cdot)^{n+1} = 0$, as desired. //

Now for the main results of this section.

THEOREM 6.15. *Suppose* $A = D \oplus R$ *is a torsion-free group, where* D *is a divisible group and* R *is a reduced group. Suppose further that* D *has finite rank* d *and the rank of* A/pA *is bounded by the integer* n, *for every prime* p. *If* $(A, \cdot)$ *is a nil ring on* A *then* $(A, \cdot)^{(d+1)(n+1)} = 0$.

Proof: Let $(A, \cdot)$ be a nil ring on A. If there is a prime p for which A has no elements of infinite p-height, then Proposition 6.14 shows $(A, \cdot)^{n+1} = 0$. Hence we can assume that A has elements of infinite p-height for every prime p.

Consider a fixed prime p. It is readily checked that $A/p^{\omega}A$ is a torsion-free group with no elements of infinite p-height such that $r([A/p^{\omega}A]/p(A/p^{\omega}A)) \leqq n$. Also, since $p^{\omega}A$ is a fully invariant subgroup of A, the nil ring $(A, \cdot)$ on A yields a nil ring $(A/p^{\omega}A, \cdot)$ on $A/p^{\omega}A$. Thus Proposition 6.14 implies $(A/p^{\omega}A, \cdot)^{n+1} = 0$. Since this is true for every prime p, $(A, \cdot)^{n+1} \subseteqq \bigcap_p p^{\omega}A = D$.

Now $(D, \cdot)$ is an ideal of $(A, \cdot)$, so $(D, \cdot)$ is also a nil ring. Since $(D, \cdot)$ can be made into a finite dimensional algebra over the field Q, $(D, \cdot)$ is a nilpotent ring. Theorem 6.6 now shows $(D, \cdot)^{d+1} = 0$, so $(A, \cdot)^{(n+1)(d+1)} = 0$, as required. //

COROLLARY 6.16. *Let* A *be a reduced torsion-free group with the property that* $r(A/pA)$ *is bounded by the positive integer* n, *for every prime* p. *Then* $N(A)$ *is* $1, 2, \ldots, n$ *or* ∞. //

We conclude this final chapter by noting that certain results in Webb [1] enable us to give the non-associative analogues of the previous

Theorem and its Corollary. The proofs are omitted since they are direct consequences of the non-associative results in Webb's work and the arguments used to prove Theorem 6.15.

THEOREM 6.17. *Let* $A = D \oplus R$ *be a torsion-free group where* D *is a divisible group and* R *is a reduced group. Suppose* D *has finite rank* d *and the rank of* A/pA *is bounded by the integer* n, *for every prime* p. *If* $(A, \cdot)$ *is a ring on* A *for which there is a positive integer* m *such that every product of length* m *is zero, then every product of length* $(2^{n-1} + 1)(2^{d-1} + 1)$ *is zero.* //

COROLLARY 6.18. *Suppose* A *is a reduced torsion-free group with the property that* $r(A/pA)$ *is bounded by the positive integer* n, *for every prime* p. *Then* $N_E(A)$ *is* $1, 2, \ldots, 2^{n-1}$ *or* ∞. //

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TABLE OF NOTATION

We generally follow the notation of Fuchs [3, 4] or Divinsky [1]. For the readers convenience we list the unusual notation used in this thesis.

$(A, \cdot)$, $(A/I, \cdot)$	5
$(A, \cdot)^n$	10
$A(*)$, $A^{(a)}(*)$	12
$A(\alpha)$	16
$A^{(j)}$	47
$\mathcal{A}$	64
$(0; (A, \cdot))$	6
$\mathcal{B}^{(j)}$	47
$(I, \cdot)$	5
J, J_{Π}	94
$J(A)$	13
$J(A, \cdot)$	7
$K(B_j)$	48
$N(A)$, $N_E(A)$, $N_S(A)$	128
P	1
P_1^A	4
$P_1^{(a)}$	68
$P_1^{(j)}$	48
Π	94
$\mathcal{R}(A, \cdot)$	101
$\mathcal{S}$	92
$\mathcal{T}(A)$	4
$\tilde{T}(A)$	35
$\leqq'$	35

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We list the important new or uncommon terms only.