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CERTAIN AUTOMORPHISMS OF FREE RANK PLANES

BY

Graham S. Kelly B.Sc.(Hons.)

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Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any university, and, to the best of my knowledge or belief, it contains no copy or paraphrase of material previously published or written by an other person, except when due reference is made in the text.

Graham Kelly

CONTENTS

SUMMARY

Projective planes of finite free rank are planes freely generated by openly finite configurations (see Hughes and Piper, 1972, Chapter XI). We use the concept of a hyperfree extension process to obtain some properties of the finite collineation groups and polarities of such planes.

We first obtain some basic properties of projective planes, free completions, hyperfree extension processes and free rank planes, together with some properties useful for our investigation.

The main work of the thesis is concerned with finite collineation groups which fix elementwise the confined core of a plane of finite free rank. Most of the known properties of such groups are obtained, as well as some which, as far as is known to the author, have not previously been obtained. If G is such a group, we determine $|G|$ when G is cyclic, we obtain upper bounds for both $|G|$ and the number of conjugacy classes to which G can belong, and we investigate the subplane of elements fixed by G. As our basic tool, we use the existence of a hyperfree completion process Q for the plane from its confined core, such that each configuration of Q is invariant under G.

We then use similar methods to prove most known results about polarities of planes of finite free rank. Finally, we consider planes not having free rank, such as open, non-free planes. We give a generalization of a theorem of Kopejkina and use it to prove a theorem about some collineation groups of such planes.

NOTES ON THE EXPOSITION

The first three chapters are divided into sections, which are numbered serially and prefixed with the chapter number. For example, the third section of chapter 2 is denoted by 2.3. Within a section, the results are numbered serially and prefixed by the section and the chapter number. For example, the fourth result of section 1.7 is denoted by 1.7.4. Chapter 4 contains only one section and two results, which are denoted by 4.1 and 4.2.

We use multiplication on the right to denote the action of a permutation or automorphism α on an element x ; i.e. the image of x under α is denoted by $x\alpha$. For the most part, we use multiplication on the right to denote the action of other mappings too. However, for convenience, we have in a few instances used $\sigma(x)$ to denote the image of x under a mapping σ.

Most of the notation we use is explained in the text. Elements of configurations or sets are denoted by lower case Latin letters, and sets, groups and extension processes by upper case Latin letters. Lower case Greek letters are used for mappings and configurations. The following notation is used in the text without explanation:

$\{x \in X \; ; \; x \text{ satisfies } P\}$	the subset of elements of a set X which satisfy condition P.
$\sigma : X \to Y, \quad \sigma : \pi \to \rho$	σ maps set X into set Y, or configuration π into configuration ρ.

$\lvert X \rvert$, $\lvert \pi \rvert$	the cardinality of a set X or configuration π.
$X \cup Y$, $X \cap Y$	the union and intersection, respectively, of sets X and Y.
N	the set of non-negative integers.
m.n or mn	the product of numbers m and n.
$\sum_{i=1}^{n} x_i$, $\prod_{i=1}^{n} x_i$	the sum and product, respectively, of numbers x_i, $1 \leq i \leq n$.
S_r	the symmetric group of degree r.
$G_1 \times G_2$	the direct product of groups G_1 and G_2.
ϕ	the empty set or configuration.
$[n]$	the integral part of a (real) number n.

INTRODUCTION

In this thesis, we define a projective plane to be a <u>free rank plane</u> if it is the union of a hyperfree extension process from its confined core. Its <u>rank</u> is defined as the number of hyperfree elements plus twice the number of isolated elements in such a process. Our aim is to investigate certain finite collineation groups and polarities of free rank planes of finite rank. As our main tool, we use the existence of a hyperfree extension process for the plane canonically associated with the collineation group or polarity. Although both degenerate planes and projective planes equal to their core are free rank planes, we are not interested in such planes. For the remainder of the introduction, we use "free rank plane" to mean "non-degenerate free rank plane not equal to its core".

We first give an outline of the literature of free rank planes and their automorphisms. The first free rank planes to be defined and studied were those having empty core. They are called <u>free planes</u> and were defined by Hall (10) in 1943. Much of the literature of free rank planes is written for free planes only. For an integer $r \geq 8$, Hall defined a free plane of rank r to be the free completion of a line, two points off the line, and r-6 points on the line. He proved that the free completion of any finite configuration having empty core is a free plane, provided it is non-degenerate. He also showed that free planes have empty core, that finitely generated subplanes of free planes are free, and that free planes are isomorphic if and only if they have the same rank.

Hall's work was continued by Kopejkina (17) and Dembowski (5). Kopejkina extended Hall's definition by defining free planes of infinite rank. He then showed that all subplanes are free. Dembowski proved that a free plane of rank r has free subplanes of all possible ranks up to, and including, r. $\aleph_0$. These theorems are analagous to theorems about free groups. Kopejkina also gave a construction for a plane having empty core but which is not free.

Collineations and correlations of free planes were first investigated by Dembowski (5). He proved that the orbits of the full collineation group of a free plane are all infinite, and that there are infinitely many distinct such orbits. The full collineation group of the free plane of rank 8 has been determined by Sandler (24, 25), but the full collineation groups of free planes of higher rank are not known.

The non-trivial finite collineation groups of free planes have been investigated by Lippi (18,19), Alltop (2), Iden (13,14,15,16) and Sandler (27). The first three of the authors, independently, proved that for any finite collineation group G of a free plane π of finite rank, π has a finite subconfiguration invariant under G and freely generating π. This result has been the basic tool in the study of such groups. In (19), Lippi considered the subplane π' of elements fixed by a collineation of prime power order p^k of a free plane π. He proved the following : If $p = 2$, then π' has infinite rank. If $p > 2$ and π has finite rank r, then either π' has finite rank $r' \equiv r(\mathrm{mod}\,p)$, or π' is degenerate and finite. Alltop (2) proved that m_r is an upper bound for the orders of finite collineation groups of a free plane of finite rank r, where $m_8 = m_9 = 4!$, $m_{10} = 5!$, and $m_r = 2\left[(r-6)!\right] \; \forall \; r \geq 11$. He also showed that m_r is the least upper bound when $r \neq 9$, and conjected that 12 is the lead upper bound when $r = 9$. This conjecture was proved by Sandler (27). Alltop's result implies that free planes of finite rank have maximal finite collineation groups. No characterization of these has been obtained, but Iden (16) has shown that the number of isomorphism classes of them increases rapidly with rank. In another paper, Iden (15) obtained strong results about the normalizers of certain finite collineations groups of free planes.

Polarities of free planes have been studied by Abbiw-Jackson (1) and Glock (8,9). Let π be a free plane of finite rank r. Abbiw-Jackson proved the following : If α is a polarity of π with j absolute points, the $j \equiv r(\text{mod}2)$ and $0 \leq j \leq r - 6$. When $r > 8$, π has a polarity with j absolute points for each such j. When $r = 8$, all polarities of π have two points and are of the same type (i.e. conjugate by a collineation of π). In (8), Glock extended these results by classifying all types of polarity of π with j absolute points, for all possible r and j. To do this, he developed a theory of symmetric incidence structures. With each such structure is associated a unique polarity of a free plane and, for each polarity of a free plane, there is at least one symmetric incidence structure associated with it. For $r \geq 9$, he concluded that there is only one type of polarity when either $j = 0$, or $r = 9$ and $j = 1$, and infinitely many types otherwise. In (9), he obtained similar results for polarities of free planes of infinite rank.

Most of the above results about free planes were obtained using Hall's original definition. An alternative has been given by Siebenmann (29), who proved that a plane is free exactly when it is the union of a hyperfree completion process from the empty configuration. He also gave a simple proof, using this characterization, that subplanes of free planes are free. Although Siebenmann was the first author to define a hyperfree extension process, a type of hyperfree extension process had been considered earlier by Ditor (6). Further results about such processes have been obtained by Ellers and Row (7). They showed that any hyperfree extension process for a configuration π can be replaced by another indexed by the natural numbers. They then proved that if π is the union of a hyperfree completion process from ρ, then π is the free completion of a configuration obtained from ρ in a natural way. Hence "hyperfree completions" are not essentially distinct from "free completions".

The first authors to investigate free rank planes having non-empty core were Hughes and Piper (12, chapter XI). Their "openly finitely generated planes" are our "free rank planes of finite rank". They showed that two such planes are isomorphic if and only if they have isomorphic cores and the same rank, thereby generalizing Hall's result for free planes. They also proved that the full collineation group G of such a plane π is the semi-direct product of the full collineation group of the core of π and the normal subgroup of G consisting of all collineations of π which fix the core elementwise.

Many results for free planes have analogues for free rank planes with non-empty core. For example, if π is a free rank plane with non-empty core, then any subplane of π containing the core of π is a free rank plane. We note, however, that there exist free rank planes having subplanes which are not free rank planes. As another example, it was shown by O'Gorman (22) that if a free rank plane π has non-empty core K and finite rank r, and α is a polarity of π with j absolute points outside K, then $j \equiv r (\mathrm{mod} 2)$ and $0 \leq j \leq r$. This is similar to Abbiw-Jackson's result (1) for polarities of free planes (stated above).

Just as any free plane has non-trivial finite collineation groups, so does any free rank plane π with rank $r \geq 2$ and non-empty core K. For r finite, the finite collineation groups of π which fix K elementwise have been studied by Hughes and Piper (12, chapter XI) and O'Gorman (21). Hughes and Piper proved that, for any line ℓ of K, such a group acts faithfully on a set X of r points, each incident with ℓ, such that $K \cup X$ freely generates π. This result has the corollary that all maximal finite collineation groups of π fixing K elementwise have order $r!$ and are conjugate (within the full collineation group of π). These maximal finite collineation groups were investigated further by O'Gorman (21). She determined the stabilizers of all elements of the plane with respect to such a group, and used this to obtain results about their orbit lengths and subplanes generated by their orbits. Because the proofs of all results mentioned in this paragraph rely on the existence of elements in the core fixed by G, similar results do not hold for finite collineation groups of free planes.

The main work in this thesis is concerned with the finite collineation groups which fix elementwise the core of a free rank plane of finite rank. Many of the known properties of such groups are obtained, together with some which, so far as is known to the author, are new. We also generalize results of Abbiw-Jackson (1) and Glock (8) to results about polarities of all free rank planes of finite rank. Perhaps our most important achievement lies not in our new results, but in that most of our results and arguments hold for all free rank planes, rather than just for free planes or just those having non-empty core. We are therefore able to present a unified account of much of the literature on finite collineation groups and polarities of free rank planes.

We now examine the contents of the thesis in more detail. Sections 1.1, 1.2 and 1.3 are preliminary and consist mainly of definitions. In 1.4, we define a free completion of a configuration, and show that it always exists and is unique up to isomorphism. We also prove the well known result that any automorphism group G of a configuration extends uniquely to an automorphism group G' of its free completion such that $G \cong G'$. This is useful in giving examples of automorphisms and automorphism groups of free rank planes.

In 1.5, we define hyperfree extension processes (henceforth abbreviated to "HF processes") and obtain some of their elementary properties. We prove a result due to Ellers and Row (7, theorem 2) which implies that we may, without loss of generality, work only with those HF processes indexed by the non-negative integers. We give methods of obtaining one HF process from another (or others). One of these (1.5.8) involves obtaining a HF process from the intersection of a given set of such processes. Finally, we define the rank of a HF process as the number of HF elements plus twice the number of isolated elements in the process, and we show that it depends only on the union and intersection of the configurations of the process. This implies that the rank of a free rank plane (defined above) is well defined.

In 1.6, we define a free rank plane and its rank and show that our definitions are equivalent to the usual ones (see, for example, (12, chapter XI)). The main theorem of the section states that two free rank planes are isomorphic if, and only if, they have isomorphic cores and the same rank. Because this result includes free rank planes of infinite rank, it is more general than the corresponding result in (12, chapter XI). We conclude the section by proving an existence theorem for free rank planes.

Section 1.7 is devoted to proving properties of free rank planes needed in later chapters. We first show that if a subplane of a free rank plane π either contains, or has empty intersection with, the core of π, then it is a free rank plane. We than prove a result due to Dembowski (5, theorem 1.1) that if a subplane of a free plane is generated by a four-point or four-line η, then it is freely generated by η. After this, we consider the Baer subplanes of free rank planes. Two of the results we prove are well known. The third, which is new, states that if a is a point incident with two lines x and y of a free rank plane π, then π has a Baer subplane containing a and x, but not y. The proof uses a variation of a method due to Row (23). Finally, we define almost-confined configurations and use them to prove that the full collineation group of a free rank plane π has infinitely many orbits outside the core of π. This is a generalization of a result of Dembowski (5).

In chapter 2, we investigate the collineation groups G which have finite orbits and fix elementwise the core K of a free rank plane π of finite rank r. In 2.1 we prove, using the intersection theorem 1.5.8 mentioned above, that to each such G there is a HF process Q for π from K such that each configuration of Q is invariant under G. This result is basic and is used throughout the chapter. We show that G acts faithfully as a permutation group of the isolated and hyperfree elements in Q; i.e. as a permutation group of at most r elements. Hence such a G is finite and has order at most r!.

The representation of G as a permutation group is used both in 2.2 and 2.3. In 2.2 we use it, together with a lemma characterizing orders of permutations of a finite set, to characterize $|G|$ when G is cyclic. The proof contains an examination of certain special cases. Most of 2.3 is devoted to obtaining least upper bounds for $|G|$. These have been obtained by other authors (Alltop (2), Sandler (27), and Hughes and Piper (12, chapter XI)), but our proof for them is new. Again, an examination of special cases is necessary, but the number in our proof is much smaller than the number considered by Alltop and Sandler in their proofs.

In 2.4, we consider the subplane of elements fixed by such a G (denoted by $\pi(1,G)$). We show that it is a free rank plane, determine its possible ranks, and show that it has infinite rank exactly when G has infinitely many orbits of length 2. We also generalize the results of Lippi (19) mentioned above. Necessary and sufficient conditions are obtained for a finitely generated subplane of π to be $\pi(1,G)$ for some such G, and we give an example of a Baer subplane which is not $\pi(1,G)$ for any such G. Finally, we show that when $\pi(1,G)$ is degenerate, there is no relationship between the numbers of points and lines fixed by G, provided r is sufficiently large. This result is motivated by an example of Lippi (19) of a collineation of a free plane with two fixed lines and one fixed point.

In 2.5 we investigate the conjugacy, within the full collineation group of π, of finite collineation groups of π fixing K elementwise. We first give an example which shows that conditions both necessary and sufficient for conjugacy of such groups may be difficult to obtain. Most of the section is devoted to obtaining a finite upper bound for the number of conjugacy classes of such groups. For this, it is necessary to consider separately the cases K empty and K non-empty. A number of other results are proved, including the result of Hughes and Piper (12, chapter XI) that when $K \neq \phi$, any two maximal finite collineation groups of π fixing K elementwise are conjugate.

In chapter 3, we investigate the polarities of free rank planes of finite rank. In 3.1, we prove first that to each polarity α of such a plane π, there is a HF process for π canonically associated with α. We then prove the results of Abbiw-Jackson (1) and

O'Gorman (22) about the number of absolute points a polarity of π may have outside the core of π. In 3.2, we consider the possible types of polarities of π (i.e. the conjugacy classes of such polarities, within the full automorphism group of π). We obtain the results of Abbiw-Jackson (1) and Glock (8) for free planes of finite rank (stated above), and analogous results for polarities of free rank planes having non-empty core.

In chapter 4, we first prove that if m is an integer and π the union of a strictly increasing sequence of free rank planes, each having rank $<$ m and the same core, then π is not a free rank plane. This is a generalization of Kopejkina's construction (17) for a non-free plane with empty core. We then prove that any free rank plane π' of finite rank can be embedded in a non-free rank plane π such that any collineation group of π' extends to a collineation group of π.

We now consider the originality of our work. Sections 1.1 to 1.6 contain no results obtained by the author, but some are unpublished and the treatment of much of the material is new. Proposition 1.7.7 and its corollary 1.7.8 are new. Corollary 1.7.11 generalizes to free rank planes a result proved by Dembowski (5) for free planes. In chapters 2 and 3, we use HF processes as a tool to investigate the finite collineation groups and polarities of free rank planes. No previous author has used HF processes for this. Consequently, many of our proofs for known results are new. The method of obtaining the (known) least upper bounds of 2.3 is new. In 2.4, new proofs are given for some theorems of Lippi (19). The proofs of all the results of chapter 3, with the exception of 3.2.1 and partial exception of 3.1.1, are new.

Chapters 2, 3 and 4 also contain some new results and generalizations of results of other authors. Theorem 2.1.1 is new. Theorem 2.1.3 generalizes to all free rank planes a result due to Hughes and Piper (12, chapter XI) for those having non-empty core. Corollary 2.1.4 and theorem 2.2.3 are new. Theorems 2.4.1, 2.4.6 and 2.4.11 are generalizations of results of Lippi (19), and 2.4.8, 2.4.9 and 2.4.10 are new. The example given after 2.5.1 is new, as is all of 2.5 after the proof of 2.5.6. Results 3.2.5, 3.2.6, and 3.2.10 are new, and 3.2.11 generalizes to all free rank planes of finite rank a result of Glock (8) for free planes of finite rank. Theorem 4.1 is a generalization of a result due to Kopejkina (17). The proof given here is due to the author and not based on Kopejkina's proof. Theorem 4.2 is new.

Except for some elementary results about projective planes and groups, for which we refer the reader to (12) and (11) respectively, the thesis is self-contained. The references at the back contain only those works referred to in the thesis.

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CHAPTER 1

HYPERFREE EXTENSION PROCESSES AND FREE RANK PLANES.

In this chapter, we define hyperfree extension processes and free rank planes. We prove some of their elementary properties, together with some which are used in later chapters.

Sections 1.1, 1.2 and 1.3 are preliminary and consist mainly of definitions. In 1.4, we define a free completion and prove some properties of the free completion process. We prove that any automorphism group of a configuration extends uniquely to an automorphism group of its free completion. In 1.5, we define a hyperfree extension process, prove some of its properties, and show how new hyperfree extension processes can be obtained from given ones. We also define the rank of such a process, and show that it depends only on the union and intersection of the configurations of the process. In 1.6, we define a free rank plane and show that two such planes are isomorphic if and only if they have the same rank and isomorphic cores. Finally, in 1.7, we prove properties of free rank planes useful in later chapters. Many of these are generalizations of well known properties of non-degenerate free planes.

1.1 Configurations and Planes

A configuration ρ is a set of points and lines together with a symmetric incidence relation between the points and lines such that

(a) the sets of points and lines are disjoint;

(b) for any two points of ρ , there is at most one line of ρ incident with both.

It follows that for any two lines of ρ, there is at most one point incident with both. The points and lines of ρ are called the <u>elements</u> of ρ and we write $x \in \rho$ for x is an element of ρ. Other conventional set notation and terminology are also used for configurations.

Let ρ be a configuration. If $x, y \in \rho$, we write $x \, I \, y$ if x is incident with y, and $x \not I y$ otherwise. If x_1 and x_2 are points of ρ and there is a line y of ρ incident with both, then y is uniquely determined by x_1 and x_2 (by (b)), so we denote it by $x_1 \circ x_2$. We say y <u>joins</u> x_1 and x_2. Analagously, if x_1 and x_2 are lines of ρ and there is a point y of ρ incident with both x_1 and x_2, then we denote it by $x_1 \circ x_2$ and we say that x_1 and x_2 <u>intersect in</u> y.

A <u>subconfiguration</u> ρ' of a configuration ρ is a subset of ρ, together with the restriction of the incidence relation of ρ. We say that ρ <u>contains</u> ρ' and write $\rho' \subseteq \rho$. Clearly, ρ' is itself a configuration. Any set of elements of ρ is the set of elements of a unique subconfiguration of ρ. If ρ_1 and ρ_2 are subconfigurations of ρ, let $\rho_1 - \rho_2$ be the subconfiguration of ρ with elements $\{x \in \rho_1 \, ; \, x \notin \rho_2\}$. The <u>intersection</u> $\bigcap_{\rho' \in C} \rho'$ and <u>union</u> $\bigcup_{\rho' \in C} \rho'$ of a family C of subconfigurations of ρ are the subconfigurations of ρ with elements $\bigcup_{\rho' \in C} \{x \, ; \, x \in \rho'\}$ and $\bigcap_{\rho' \in C} \{x \, ; \, x \in \rho'\}$ respectively. We also need to define the union of certain families C of configurations for which there is no configuration containing all of them. If no two configurations of C have elements in common and there are no incidences between elements of distinct configurations of C, then define $\rho = \bigcup_{\rho' \in C} \rho'$ to have elements $\bigcup_{\rho' \in C} \{x \; ; \; x \in \rho'\}$, and define two elements x and y of ρ to be incident if there is a $\rho' \in C$ for which $x, y \in \rho'$ and x is incident with y in ρ'. If C is linearly ordered by the subconfiguration relation, then we define the union of C in the same way. If the union of C is defined then so is the intersection of C, because C is a family of subconfigurations of its union.

A set of points (resp. lines) of a configuration ρ is <u>collinear</u> (<u>concurrent</u>) if every point (line) of the set is incident with the same line (point). A subconfiguration of ρ consisting of four points (resp. lines), no three of which are collinear (concurrent), is a <u>four-point</u> (<u>four-line</u>).

A <u>plane</u> is a configuration π satisfying

(1) Any two distinct points of π are both incident with exactly one line of π.

(2) Any two distinct lines of π are both incident with exactly one point of π.

A plane is <u>non-degenerate</u> if it contains a four-point. Otherwise it is <u>degenerate</u>.

We assume that the reader is familiar with both the principle of duality for planes and

<u>Theorem 1.1.1</u> : If π is a non-degenerate plane then there is a cardinal $n \geq 2$ for which π has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, and every element of π is incident with $n + 1$ elements of π.

The cardinal n is the <u>order</u> of the plane. If a non-degenerate plane π is infinite, then it has order $|\pi|$.

For both the principle of duality and the proof of 1.1.1, we refer the reader to (12, chapter 3). We note that the "projective planes" of (12) are our "non-degenerate planes".

Degenerate planes are completely classified by

<u>Theorem 1.1.2</u> : A plane π is degenerate if, and only if, one of

(a) π is empty;

(b) the points of π are collinear and the linesconcurrent;

(c) π has a point p and a line ℓ for which $p \not I \ell$, and all other points and lines of π are incident with ℓ and p respectively.

This classification is based upon that of (30).

1.2 Mappings of Configurations

An <u>isomorphism</u> (resp. <u>duality</u>) α from a configuration ρ onto a configuration ρ' is a bijection of the points of ρ onto the points (lines) of ρ' and the lines of ρ onto the lines (points) of ρ' such that, for any $x, y \in \rho$, we have $x \mathrm{I} y \Leftrightarrow x\alpha \mathrm{I} y\alpha$. We write $\rho' = \rho\alpha$. Two configurations are <u>isomorphic</u> if there is an isomorphism between them. Isomorphism is an equivalence relation on any set of configurations. We often consider isomorphic configurations to be equal.

A <u>collineation</u> (resp. <u>correlation</u>) of a configuration ρ is an isomorphism (duality) from ρ onto itself. Under composition, the collineations of ρ form a group, the <u>full collineation group</u> of ρ. Any subgroup of this group is a <u>collineation group</u> of ρ. The composition of a collineation and a correlation of ρ is a correlation of ρ, and the composition of two correlations is a collineation of ρ. Hence the set of all collineations and correlations of ρ form a group under composition, the <u>full automorphism group</u> of ρ. Elements of this group are <u>automorphisms</u> of ρ and any subgroup is an <u>automorphism group</u> of ρ. The full collineation group is a normal subgroup of index two of the full automorphism group of ρ. Group notation and terminology are used when referring to automorphisms of ρ. For example, the identity automorphism of ρ is denoted by 1, and $\langle\alpha\rangle$ denotes the automorphism group generated by an automorphism α. If G is a collineation group of ρ and $x \in \rho$, then the set $\{x\alpha \,;\, \alpha \in G\}$ is the <u>G-orbit</u> of x, or the <u>orbit</u> of x <u>under</u> G, and is denoted by xG. When $G = \langle\alpha\rangle$ for some α, the G-orbits are referred to as <u>α-orbits</u>.

Suppose that ρ is a configuration and ρ' is a subconfiguration of ρ. If $\beta : \rho \to \eta$ is an isomorphism (duality), then $\rho'\beta$ is a subconfiguration of η, and β induces an isomorphism (duality) of ρ'

onto $\rho'\beta$, denoted by $\beta|_{\rho'}$. If G is an automorphism group of ρ, then we denote $\bigcup_{\alpha \in G} \rho'\alpha$ by $\rho'G$. If $\rho'G = \rho'$, then ρ' is <u>invariant</u> under G. If $\rho'G = \rho'$, then $\rho'\alpha = \rho'$ for each $\alpha \in G$, so $\alpha|_{\rho'}$ is an automorphism of ρ'. Thus G induces an automorphism group of ρ', denoted by $G|_{\rho'}$.

Suppose that ρ_i is a subconfiguration of π_i, i = 1,2, and that $\beta: \rho_1 \to \rho_2$ is an isomorphism. If $\alpha: \pi_1 \to \pi_2$ is an isomorphism for which $\rho_1\alpha = \rho_2$ and $\alpha|_{\rho_1} = \beta$, then α is an <u>extension</u> of β. We also say that β <u>extends to</u> α. If G and G' are automorphism groups of ρ_1 and π_1 respectively for which $\rho_1 G' = \rho_1$ and $G'|_{\rho_1} = G$, then G' is an <u>extension</u> of G, and G <u>extends to</u> G'.

1.3 Extension Processes

An <u>extension process</u> E is a set $E = \{E_w ; w \in W\}$ of configurations, where W is well-ordered by some partial order $<$, and $u < v$ implies $E_u \subseteq E_v$. We say that W <u>indexes</u> E. We write $\overline{E} = \bigcup_{w \in W} E_w$ and $\underline{E} = \bigcap_{w \in W} E_w$, and say that E is an extension process <u>for</u> $\overline{E}$ <u>from</u> $\underline{E}$. We denote the least element of W by 0, so $\underline{E} = E_0$. If $x \in \overline{E}$, then the <u>E-stage</u> of x is the least element of $\{w \in W ; x \in E_w\}$. We denote it by $st_E(x)$. If $x, y \in \overline{E}$, x I y and $st_E(x) < st_E(y)$, then x is an <u>E-bearer</u> of y. We write $x \prec y$ (E). The relation $\prec(E)$ is the <u>bearer relation</u> of E.

If $\{E^{(i)} ; i \in I\}$ is a family of extension processes, each indexed by the same W, then the extension process $E' = \left\{ \bigcap_{i \in I} E_w^{(i)} ; w \in W \right\}$ is the intersection of the family. If E and F are extension processes for which $\overline{E} = \underline{F}$, then E + F denotes the extension process with configurations $E \cup F$. It is indexed by a well-ordered set having ordinal equal to the sum of the ordinals of the indexing sets of E and F.

Let E be an extension process indexed by W. If $\rho \subseteq \overline{E}$, then the extension processes with configurations $\{\rho \cap E_w ; w \in W\}$ and $\{\rho \cup E_w ; w \in W\}$ are denoted by $\rho \cap E$ and $\rho \cup E$ respectively. They are also indexed by W.

1.4 Generation of Planes and Free Completions

A subplane of a plane π is a subconfiguration of π which is also a plane. The intersection of any family of subplanes of π is also a subplane of π. Thus, if ρ is any subconfiguration of π, we may define the subplane of π generated by ρ to be the intersection of all subplanes of π containing ρ. It is denoted by $[\rho]_\pi$. We note that there is always at least one subplane of π containing ρ, namely π itself.

One can construct $[\rho]_\pi$ from ρ in the following way : Let $\rho_0 = \rho$. For $n \geq 0$, define $\rho_{n+1} = \rho_n \cup \{x \in \pi ; x$ is incident with at least two elements of $\rho_n\}$. Clearly $\bigcup_{n=0}^{\infty} \rho_n$ is a subplane of π containing ρ , so $[\rho]_\pi \subseteq \bigcup_{n=0}^{\infty} \rho_n$. For any subplane π' of π containing ρ, one shows by induction on n that $\rho_n \subseteq \pi'$ for each n. Hence $\bigcup_{n=0}^{\infty} \rho_n = [\rho]_\pi$. The extension process $E = \{ \rho_i ; i=0,1,\ldots \}$ is called the <u>generation process</u> for $[\rho]_\pi$ from ρ. For $x \in [\rho]_\pi$, the E-stage and E-bearers of x are called the <u>ρ-stage</u> and <u>ρ-bearers</u> of x respectively.

Let π be a plane and ρ a subconfiguration of π. If $n \geq 0$, then each element of $\rho_{n+1} - \rho_n$ is incident with at least two elements of ρ_{n+1} (by definition). We say that ρ <u>freely generates</u> $[\rho]_\pi$ if each element of $\rho_{n+1} - \rho_n$ is incident with exactly two elements of ρ_{n+1}, for each $n \geq 0$. Note that if ρ freely generates $[\rho]_\pi$, then each $x \in \rho_{n+1} - \rho_n$ is incident with two elements of ρ_n and no elements of $\rho_{n+1} - \rho_n$, for each $n \geq 0$. We therefore have

<u>Lemma 1.4.1</u> : If π is a plane and ρ a subconfiguration of π , then $[\rho]_\pi$ is freely generated by ρ if, and only if, both

(a) every element of $[\rho]_\pi - \rho$ has at most two ρ-bearers;

(b) no two elements of equal non-zero ρ-stage are incident .

If π is a plane freely generated by ρ , then π is a <u>free completion</u> of ρ, and the generation process for π from ρ is a <u>free completion process</u>. These concepts were first defined by M. Hall (10) in 1943.

Our next result is well known.

Proposition 1.4.2 : Suppose that π and π' are free completions of ρ and ρ' respectively. If $\alpha : \rho \longrightarrow \rho'$ is an isomorphism (duality), then α extends uniquely to an isomorphism (duality) of π onto π' for which $\rho_i\alpha = \rho_i'$, $i = 0,1,\ldots$.

Proof : ρ_1 is obtained from ρ by adding elements $x.y$ to ρ, where x and y are distinct points or distinct lines of ρ and $x.y \notin \rho$. Since α is an isomorphism (duality), we have $x.y \notin \rho$ if, and only if, $(x\alpha).(y\alpha) \notin \rho'$. We extend α to an isomorphism of ρ_1 onto ρ_1' by defining $(x.y)\alpha = (x\alpha).(y\alpha)$ for each such pair of points or lines. This extension of α is well defined, because $x.y$ (resp. $(x\alpha).(y\alpha)$) is incident in ρ_1 (resp. ρ_1') only with x and y (resp. $x\alpha$ and $y\alpha$). Similarly, we extend α to an isomorphism of ρ_2 onto ρ_2', etc. Thus α extends to an isomorphism of π onto π' for which $\rho_i\alpha = \rho_i'$, $i=0,1,\ldots$.

It remains to show that this extension is unique. Suppose α_1 and α_2 are two such extensions of α and that $\alpha_1 \neq \alpha_2$. Choose an $x \in \pi$ of minimal ρ-stage for which $x\alpha_1 \neq x\alpha_2$. Then $x \notin \rho$, because $\alpha_1|_\rho = \alpha_2|_\rho = \alpha$. Therefore $x = y.z$ where y and z are the two ρ-bearers of x. By the minimality of $st_\rho(x)$, we have $y\alpha_1 = y\alpha_2$ and $z\alpha_1 = z\alpha_2$. This implies $x\alpha_1 = x\alpha_2$, a contradiction. Thus the extension of α is unique.

Suppose that ρ is any configuration. We demonstrate that a free completion of ρ exists. Define an extension process $F = \{F_n(\rho) ; n \in N\}$ as follows : Let $F_0(\rho) = \rho$. Assume $F_n(\rho)$ has been defined. Obtain $F_{n+1}(\rho)$ from $F_n(\rho)$ by adding new elements x.y for each pair of points x,y not joined by a line of $F_n(\rho)$ and each pair of lines x,y not intersecting in a point of $F_n(\rho)$. The new elements x.y are defined to be incident with x and y and no other element of $F_{n+1}(\rho)$. Define $F(\rho) = \overline{F}$. Clearly F is a free completion process and $F(\rho)$ is a free completion of ρ.

By 1.4.2, there is, up to isomorphism, at most one free completion of ρ and free completion process from ρ. Because these always exist, we refer to <u>the</u> free completion of ρ, denoted by $F(\rho)$, and of <u>the</u> free completion process for $F(\rho)$ from ρ, the configurations of which we denote by $F_n(\rho)$, n = 0,1,..., as defined above. Some elementary properties of the free completion process are combined in

<u>Proposition 1.4.3</u> : Let ρ be any configuration.

(a) $F_{m+n}(\rho) = F_m(F_n(\rho)) \quad \forall\, m, n \geq 0$.

(b) $F(F_n(\rho)) = F(\rho) \;\forall\, n \geq 0$.

(c) $|F_1(\rho)| \leq |\rho|^2$.

(d) If ρ is not a plane and $F(\rho)$ is non-degenerate, then $F(\rho) - \rho$ is infinite.

<u>Proof</u> : (a) It suffices to show $F_1(F_n(\rho)) = F_{n+1}(\rho)$. This is an immediate consequence of the definition of $F_i(\rho)$ for each i.

(b) This follows from (a).

(c) Let $|\rho| = n$. Elements of $F_1(\rho) - \rho$ are of the form $x.y$ where $x, y \in \rho$ and $x \neq y$. Thus $|F_1(\rho) - \rho| \leq \binom{n}{2}$, and $|F_1(\rho)| = |F_1(\rho) - \rho| + |\rho| \leq \binom{n}{2} + n = \frac{n(n+1)}{2} \leq n^2 = |\rho|^2$.

(d) Because ρ is not a plane, there is a pair of points or lines $x, y \in \rho$ for which $x.y \notin \rho$. Hence $|F(\rho) - \rho| \geq 1$. If $F(\rho) - \rho$ is finite, there is an $x \in F(\rho)$ of maximal ρ-stage $m > 0$. This implies $F(\rho) = F_m(\rho)$, and x is incident with only two elements of $F(\rho)$. But $F(\rho)$ is non-degenerate, and every element of a non-degenerate plane is incident with at least three other elements of the plane (by 1.1.1). Hence $F(\rho) - \rho$ is infinite.

Our next result, which is well known (see, for example, (13, lemma 3)), provides a tool for obtaining examples of automorphism groups of planes which are free completions.

<u>Theorem 1.4.4</u> : If α is any automorphism of a configuration ρ, then α extends uniquely to an automorphism of $F(\rho)$ for which $F_n(\rho)\alpha = F_n(\rho)$ $\forall\ n \geq 0$. If G is any automorphism group of ρ, then G extends to a unique automorphism group G' of $F(\rho)$ which is isomorphic to G and satisfies $F_n(\rho)G' = F_n(\rho)$ $\forall\ n \geq 0$.

<u>Proof</u> : The first statement is an immediate consequence of 1.4.2. Suppose now that G is any automorphism group of ρ. For each $\alpha \in G$, there is a unique extension of α to an automorphism α' of $F(\rho)$ for

which $F_n(\rho)\alpha' = F_n(\rho)\ \forall\ n \geq 0$. Let $G' = \{\alpha' ; \alpha \in G\}$. By the uniqueness of α' for each $\alpha \in G$, we have $\beta'\gamma' = (\beta\gamma)'$ and $(\beta^{-1})' = (\beta')^{-1}$ for any $\beta, \gamma \in G$. Thus G' is a group and the map of G onto G' defined by $\alpha \rightarrow \alpha'$ is a group isomorphism.

1.5 Hyperfree Extension Processes

An extension process P is <u>hyperfree</u> if

(a) no two elements of $\overline{P}$ of equal non-zero P-stage are incident, and

(b) no element of $\overline{P}$ has more than two P-bearers.

We abbreviate "hyperfree" to "HF" and "hyperfree extension process" to "HF process". By 1.4.1, free completion processes are HF processes.

Let P be a HF process. An element of $\overline{P}$ is <u>P-free</u> if it has two P-bearers and <u>P-HF</u> if it has only one. If it has none, and is not incident with any element of the same P-stage, then it is <u>P-isolated</u>. We also say that P has free, HF and isolated elements.

If P is a HF process and $x \in \overline{P}$, then a <u>P-chain</u> of x is a set $\{x_0, \ldots, x_n\}$ of elements of $\overline{P}$ for which $x_n = x$ and, if $n > 0$, x_i is a P-bearer of x_{i+1}, $0 \leq i \leq n-1$. The number n is the <u>length</u> of the P-chain. For each $x \in \overline{P}$, we define the <u>P-socle</u> of x to be the subconfiguration of $\overline{P}$ having as its elements the union of all P-chains of x. We denote it by $P(x)$.

The prefix "P-" is sometimes dropped from the above definitions if it is clear to which HF process we are referring.

We combine some elementary properties of HF processes together in the following proposition. Part (b) was first proved by Ellers and Row (7 , theorem 1).

Proposition 1.5.1 : Let P be a HF process.

(a) If $x, y \in \overline{P}$ and $x \text{ I } y$, then either $x < y \ (P)$ or $y < x(P)$ or $st_P(x) = st_P(y) = 0$.

(b) $P(x)$ is finite $\forall \ x \in \overline{P}$.

(c) If $x \in \overline{P}$, then $P(y) \subseteq P(x) \ \forall \ y \in P(x)$.

Let X be the set of P-isolated and P-HF elements. If $\overline{P}$ is a plane, then

(d) $P_0 \cup X$ generates $\overline{P}$;

(e) if α_1 and α_2 are collineations of $\overline{P}$ for which

$$\alpha_1\Big|_{P_0 \cup X} = \alpha_2\Big|_{P_0 \cup X}, \text{ then } \alpha_1 = \alpha_2.$$

Proof : (a) We cannot have $st_P(x) = st_P(y) > 0$, because no two elements of equal non-zero P-stage are incident. Thus either $st_P(x) = st_P(y) = 0$ or $x < y(P)$ or $y < x(P)$.

(b) We proceed by transfinite induction on $st_P(x)$. If $st_P(x) = 0$, then $P(x) = \{x\}$, which is finite. Suppose now that $st_P(x) > 0$ and that $P(y)$ is finite for all y having P-stage $< st_P(x)$. For any P-chain C of x of length > 0, $C - \{x\}$ is a P-chain of a P-bearer of x. Thus $P(x) = \{x\} \cup \Big(\bigcup_{y \in B} P(y) \Big)$, where B is the set of P-bearers of x. Because $|B| \leq 2$ and $P(y)$ is finite $\forall\, y \in B$ (by the induction assumption), $P(x)$ is finite. By induction, $P(x)$ is finite $\forall\, x \in \overline{\overline{P}}$.

(c) If $y \in P(x)$, then there is a P-chain $C = \{x_0, \dots, x_n\}$ with $y = x_0$, $x = x_n$, and, if $n > 0$, x_i a P-bearer of x_{i+1}, $0 \leq i \leq n$. If C' is any P-chain of y, then $C \cup C'$ is a P-chain of x. This implies $C' \subseteq P(x)$. Hence $P(y) \subseteq P(x)$.

(d) Suppose $[P_0 \cup X]_{\overline{\overline{P}}} \neq \overline{\overline{P}}$. Choose an $x \in \overline{\overline{P}} - [P_0 \cup X]_{\overline{\overline{P}}}$ of minimal P-stage. Because $x \notin X$ and $x \notin P_0$, x is P-free and has two P-bearers y and z. By the minimality of $st_P(x)$, both $y, z \in [P_0 \cup X]_{\overline{\overline{P}}}$. This implies $x = y \circ z \in [P_0 \cup X]_{\overline{\overline{P}}}$, a contradiction. Hence $[P_0 \cup X]_{\overline{\overline{P}}} = \overline{\overline{P}}$.

(e) Let ρ be the maximal subconfiguration of $\overline{\overline{P}}$ for which $\alpha_1|_\rho = \alpha_2|_\rho$ (ρ is the union of all subconfigurations with this property). As in the proof of (d), one shows $\rho = \overline{\overline{P}}$. Hence $\alpha_1 = \alpha_2$.

Let P be a HF process and $x \in \overline{\overline{P}}$. Because $P(x)$ is finite (by 1.5.1 (b)), there is a P-chain of x having maximal length n. Define n to be the <u>P-length</u> of x. We denote it by $\ell_P(x)$. A HF process P is <u>standard</u> if it is indexed by the non-negative integers and $\ell_P(x) = st_P(x)$ for all $x \in \overline{\overline{P}}$.

Two HF processes P and Q are <u>similar</u> if $\overline{P} = \overline{Q}$ and they have the same bearer relation. If P and Q are similar, then $P(x) = Q(x)$ and $\ell_P(x) = \ell_Q(x)\ \forall\ x \in \overline{P}$. Similarity is an equivalence relation on any set of HF processes.

Our next theorem is due to Ellers and Row (7 , theorem 2)

<u>Theorem 1.5.2</u> : For any HF process P, there exists a unique standard HF process similar to P.

<u>Proof</u> : Define the extension process $Q = \{Q_n\ ;\ n \in N\}$ by $Q_n = \{x \in \overline{P}\ ;\ \ell_P(x) \leq n\}$, $n \in N$. Then $\overline{P} = \overline{Q}$. We first show that Q is a HF process similar to P. We have $y < x(P) \Rightarrow \ell_P(y) < \ell_P(x) \Rightarrow st_Q(y) < st_Q(x) \Rightarrow y < x(Q)$, and $y < x(Q) \Rightarrow \ell_P(y) < \ell_P(x)$ and $y\ I\ x \Rightarrow y < x(P)$. Thus P and Q have the same bearer relation. Thus each $x \in \overline{Q}$ has at most two Q-bearers. To show that Q is HF, it remains to show that no two incident elements of $\overline{Q}$ have equal non-zero Q-stage. Suppose $x, y \in \overline{Q}$ and $x\ I\ y$. By 1.5.1 (a), either $x < y(P)$ or $y < x(P)$ or $st_P(x) = st_P(y) = 0$. These imply, respectively, $x < y(Q)$, $y < x(Q)$, and $\ell_P(x) = \ell_P(y) = 0$; i.e. $st_Q(x) < st_Q(y)$, $st_Q(x) > st_Q(y)$ and $st_Q(x) = st_Q(y) = 0$. Thus x and y do not have equal non-zero Q-stage. Q is therefore a HF process. It is similar to P, because they have the same bearer relation.

Because P and Q are similar, $\ell_P(x) = \ell_Q(x)\ \forall\ x \in \overline{Q}$. By definition, $st_Q(x) = \ell_P(x)\ \forall\ x \in \overline{Q}$. Hence $st_Q(x) = \ell_Q(x)\ \forall\ x \in \overline{Q}$, so Q is standard. Q is unique, because if R is any standard HF process similar to P, we have $st_R(x) = \ell_R(x) = \ell_P(x) \quad \forall x \in \overline{R}$, and hence

$R_n = \{ x \in \overline{R} ;\ st_R(x) \leq n \} = \{ x \in \overline{R} ;\ \ell_P(x) \leq n \} = Q_n$ for each n.

<u>Corollary 1.5.3</u> : For any HF process P, there exists a HF process Q similar to P for which $\underline{P} = \underline{Q}$, $\overline{P} = \overline{Q}$, and Q is indexed by the non-negative integers.

<u>Proof</u> : The extension process P' with configurations $P - \{ P_0 \}$ is HF. Let P" be the unique standard HF process similar to P'. Then the extension process Q with configurations $\{ P_0 \} \cup P''$ can be indexed by the non-negative integers, and has the required properties.

By 1.5.3, we may, without loss of generality, adopt the following

<u>Convention</u> : Henceforth, unless stated otherwise, all HF processes are indexed by the non-negative integers.

For later use, we now state two trivially proved properties of P-length (for a HF process P).

<u>Lemma 1.5.4</u> : If P is a HF process and $x \in \overline{P}$, then $\ell_P(x) \leq st_P(x)$. If B is the set of P-bearers of x, then $\ell_P(x) = \max \{ \ell_P(y) + 1 ;\ y \in B \}$.

Our next result characterizes standard HF processes.

Proposition 1.5.5 : A HF process P is standard if, and only if, $P_n = F_1(P_{n-1}) \cup \{$P-HF elements of P-length n$\}$, $n = 1,2,\ldots$.

Proof : Suppose first that P is standard. A P-isolated element x has no P-bearers, so $\ell_P(x) = 0 = st_P(x)$. Therefore $P_n - P_{n-1}$ has only P-HF and P-free elements $\forall\ n \geq 1$. The HF elements of $P_n - P_{n-1}$ are exactly those which have P-stage n, and hence P-length n. The free elements are of the form $x.y$, where x and y are points (lines) of P_{n-1} not both incident with a line (point) of P_{n-1}, and $x.y$ is incident in P_n only with x and y. Hence $P_{n-1} \cup \{$P-HF elements of length n$\} \subseteq P_n \subseteq F_1(P_{n-1}) \cup \{$P-HF elements of length n$\}$, $n = 1,2,\ldots$. It remains to show that $z \in F_1(P_{n-1})$ and $z \notin P_{n-1}$ imply $z \in P_n$. Such a z has P-bearers x and y in P_{n-1}, which implies $\ell_P(z) = \max\{\ell_P(x) + 1,\ \ell_P(y) + 1\} \leq n$ (using 1.5.4). Hence $st_P(z) = \ell_P(z) \leq n$, and $z \in P_n$.

Conversely, assume that $P_n = F_1(P_{n-1}) \cup \{$P-HF elements of length n$\}$ $\forall\ n \geq 1$. We show $st_P(x) = \ell_P(x)$ by induction on $st_P(x)$. If $st_P(x) = 0$, then $\ell_P(x) = 0 = st_P(x)$. Assume $st_P(x) = \ell_P(x)$ $\forall$ x for which $st_P(x) \leq n - 1$; i.e. $\forall x \in P_{n-1}$. Let $z \in P_n - P_{n-1}$. If z is P-HF, then $\ell_P(z) = n = st_P(z)$ (by assumption). Suppose z is P-free with P-bearers $x, y \in P_{n-1}$. By 1.5.4, $\ell_P(z) \leq st_P(z) = n$.

Suppose $\ell_P(z) < n$. Then, by 1.5.4, $n > \max\{1 + \ell_P(x), 1 + \ell_P(y)\}$, which implies both $\ell_P(y) < n - 1$ and $\ell_P(z) < n - 1$. Thus both $x, y \in P_{n-2}$ (by the induction assumption). But this implies $z = x.y \in F_1(P_{n-2}) \subseteq P_{n-1}$ (by assumption), a contradiction. Hence $\ell_P(z) = n = st_P(z)$. By induction, $st_P(x) = \ell_P(x) \ \forall \ x \in \overline{P}$. Thus P is standard.

It follows from 1.5.5 that a free completion process is a standard HF process (because it has no HF elements and $F_n(\rho) = F_1(F_{n-1}(\rho))$, by 1.4.3 (a)).

We now prove a series of results which give methods of obtaining new HF processes from given ones.

<u>Lemma 1.5.6</u> : If P and Q are HF processes for which $\overline{P} = \underset{\sim}{Q}$, then P + Q is a HF process.

<u>Proof</u> : Suppose $x, y \in \overline{P + Q}$ and $st_{P+Q}(x) = st_{P+Q}(y) > 0$. Then either both $x, y \in \overline{P}$ and $st_P(x) = st_P(y) > 0$, or both $x, y \in \overline{Q} - \overline{P} = \overline{Q} - \underset{\sim}{Q}$, and $st_Q(x) = st_Q(y) > 0$. Thus $x \not\sim y$. Hence no two elements of equal non-zero (P+Q)-stage are incident. If $x \in \overline{P + Q}$, then the (P + Q)-bearers of x are the P- or Q-bearers of x according as $x \in \overline{P}$ or $x \in \overline{Q} - \underset{\sim}{Q}$. Hence no elements of $\overline{P + Q}$ have more than two (P+Q)-bearers. Thus P + Q is a HF process.

The following observation was first made by Siebenmann (29, proof of Theorem I).

<u>Lemma 1.5.7</u> : If P is a HF process and $\rho \subseteq \bar{P}$, then $\rho \cap P$ is a HF process for ρ.

<u>Proof</u> : Let $R = \rho \cap P$. Clearly $\bar{R} = \rho$. We have $st_R(x) = st_P(x) \; \forall \; x \in \rho$. Hence no two elements of ρ of equal non-zero R-stage are incident. We also have $x < y(R) \Rightarrow x < y(P)$. Thus no element of ρ has more than two R-bearers. Hence R is a HF process.

The intersection of a family of HF processes is not necessarily a HF process, as the following example shows : Define

$P_0 = \{ a_1, \ldots, a_4 \}$, where $a_1, \ldots, a_4$ are P-isolated points,

$P_1 = P_0 \cup \{ a_i \circ a_{1+(i \bmod 4)}, \; i = 1, \ldots, 4 \}$, where $a_i \circ a_{1+(i \bmod 4)}$ are lines incident in P_1 only with a_i and $a_{1+(i \bmod 4)}, i = 1, \ldots, 4$.

$P_n = F_{n-1}(P_1), \; n > 1$.

Define $Q_0 = P_1 - P_0$, and $Q_n = P_n \; \forall \; n > 0$. Clearly P and Q are HF processes. However $(P \cap Q)_0 = \emptyset$ and $(P \cap Q)_1 = P_1$. Thus elements of $(P \cap Q)$-stage one are incident. Therefore $P \cap Q$ is not a HF process.

Although the intersection of HF processes is not necessarily HF, we do have

<u>Proposition 1.5.8</u> : Let $\{P^{(i)};\ i \in I\}$ be a family of HF processes for which $\overline{P^{(i)}} = \overline{P^{(j)}}$ and $P_0^{(i)} = P_0^{(j)}$ for any $i,j \in I$. Let the extension processes R and Q be defined by

$$R_n = \bigcap_{i \in I} P_n^{(i)},\ \forall\, n \in N,$$

$$Q_{2n} = R_n,\ \forall\, n \in N,$$

$$Q_{2n+1} = R_n \cup \{\text{points of } R_{n+1}\},\ \forall\, n \in N.$$

Then Q is a HF process and $Q_0 = R_0 = P_0^{(i)}\ \forall\ i \in I$.

<u>Proof</u> : Let $n \in N$ and $x \in R_{n+1} - R_n$. There is a $j \in I$ for which $x \in P_{n+1}^{(j)} - P_n^{(j)}$. Since x is not incident with any elements of the same $P^{(j)}$-stage and has at most two $P^{(j)}$-bearers, x is incident with at most two elements of $P_{n+1}^{(j)}$. Hence x is incident with at most two elements of R_{n+1}, for each $x \in R_{n+1}-R_n$ and $n \in N$. It follows that elements of $Q_{2n+1}-Q_{2n}$ (resp. $Q_{2n+2}-Q_{2n+1}$) are incident with at most two elements of Q_{2n} (resp. Q_{2n+1}), for each $n \in N$. Hence elements of $\overline{Q}$ have at most two Q-bearers. From the definition of Q, elements of equal non-zero Q-stage are either all points or all lines. Thus no two of them are incident. Hence Q is a HF process. Because $P_0^{(i)} = P_0^{(j)}$ for each $i,j \in I$, we have $Q_0 = R_0 = \bigcap_{i \in I} P_0^{(i)} = P_0^{(j)}\ \forall\, j \in I$.

We now give a method of obtaining a new HF process R from a given HF process P by changing the sets of isolated and HF elements. Let P be a HF process, k a positive integer, and V a set of P-HF and P-isolated elements, each of P-stage $\geq k$. Suppose that $\lambda : V \to P_{k-1}$ maps points and lines of V into the points and lines respectively of P_{k-1} such that $v\lambda$ is not incident with the P-bearer of v (if it exists), for each $v \in V$. Define $W = \{ v \circ v\lambda \; ; \; v \in V \}$ and define the extension process R by

$$R_n = \begin{cases} P_n, & 0 \leq n \leq k-1, \\ P_{n-1} \cup W, & n \geq k. \end{cases}$$

For P, k, V, λ and W defined as above, we denote R by $\Gamma(k, V, \lambda, W)(P)$.

<u>Proposition 1.5.9</u> : If $R = \Gamma(k, V, \lambda, W)(P)$, then

(a) $v < v \circ v\lambda(P)$ and $v \circ v\lambda < v(R) \; \forall \; v \in V$;

(b) except for the relations of (a), P and R have the same bearer relation;

(c) R is a HF process and $R_0 = P_0$, $\overline{R} = \overline{P}$;

(d) the sets of R-isolated and R-HF elements are I-V and $(H-V) \cup (V \cap I) \cup W$ respectively, where I and H are the sets of P-isolated and P-HF elements respectively;

(e) if G is a collineation group of $\overline{P}$ for which $P_n G = P_n$ $\forall \; n \in N$ and $WG = W$, then $R_n G = R_n \; \forall \; n \in N$.

<u>Proof</u> : During the proof, we make observations (i), (ii), etc., to which we refer later in the proof.

(a) Let $v \in V$. Either $v < v \circ v\lambda(P)$, $v \circ v\lambda < v(P)$, or $st_P(v) = st_P(v.v\lambda) = 0$ (by 1.5.1(a)). The last two are not possible, because $v\lambda$ is not incident with the P-bearer of v, and $st_P(v) \geq k > 0$. Hence $v < v \circ v\lambda$ (P), as required. It now follows that

(i) $st_P(v\lambda) < k \leq st_P(v) < st_P(v \circ v\lambda) \; \forall \; v \in V$.

From (i), we have

(ii) $v \circ v\lambda$ is P-free with P-bearers v and $v\lambda$ for all $v \in V$.

Hence $v \circ v\lambda = v' \circ v'\lambda$ implies $v = v'$, so the mapping of V onto W given by $v \to v \circ v\lambda$ is a bijection. Furthermore, because elements of W are P-free (by (ii)) and elements of V are either P-HF or P-isolated, we have

(iii) $V \cap W = \phi$.

For each $v \in V$, we have $v \notin P_{k-1}$ (as $st_P(v) \geq k$), and $v \notin W$ (by (iii)). Thus $v \notin P_{k-1} \cup W = R_k$ and $st_R(v) > k$. Because $v.v\lambda \in W$, $st_R(v \circ v\lambda) = k$. Thus $st_R(v.v\lambda) = k < st_R(v)$. Hence $v \circ v\lambda < v(R)$ as required.

(b) From the definition of R, we have

(iv) $x < y \; (R) \Leftrightarrow x < y(P) \; \forall \; x \notin W, \; y \notin W$.

We now show

(v) there are no incidences between elements of W.

Suppose, on the contrary, that there exist $v, v' \in V$ for which $v.v\lambda \; I \; v'.v'\lambda$. Then one of $v \circ v\lambda$ or $v'.v'\lambda$ is a P-bearer of the other, because $st_P(v \circ v\lambda) > 0$ (by (i)). We may assume $v \circ v\lambda < v'.v'\lambda$ (P). By (ii), either $v.v\lambda = v'$, or $v \circ v\lambda = v'\lambda$. Neither is possible, because $V \cap W = \phi$ and $st_P(v'\lambda) \leq k < st_P(v.v\lambda)$ (by (i)). Thus (v) is proved.

From (i), (ii) and (v), it follows that

(vi) $v.v\lambda$ is incident in R_k only with $v\lambda$, and $st_R(v\lambda) < k = st_R(v.v\lambda)$

Suppose that $x \in \overline{R}$ and $v \in V$. We have the equivalences

(vii) $x < v.v\lambda(R) \Leftrightarrow x = v\lambda \Leftrightarrow x \neq v$ and $x < v.v\lambda(P)$ (by (ii));

(viii) $v.v\lambda < x(P) \Leftrightarrow v.v\lambda\ I\ x$ and $x \neq v$, $x \neq v\lambda$ (by (ii) and 1.5.1(a)),

$\Leftrightarrow v.v\lambda < x(R)$ and $x \neq v$ (by (vi)).

From (iv), (vii) and (viii), it follows that the bearer relation of R and P is the same, except for the relations of (a).

(c) Clearly $\overline{R} = \overline{P}$ and $R_0 = P_0$. Elements of $\overline{R}$ of equal non-zero R-stage either have equal non-zero P-stage, or are both in W. In neither case are they incident (by (v)). Thus it remains to show that any $x \in \overline{R}$ has at most two R-bearers. If $x \notin V \cup W$ then the P- and R-bearers of x coincide (by (b)), so x has at most two P-bearers. Suppose now that $x \in V$. Then x has at most one P-bearer. From (a) and (b), $x.x\lambda$ is the only R-bearer of x which is not a P-bearer. Thus x has at most two R-bearers. Finally, suppose $x \in W$. Then $x = v.v\lambda$ for some $v \in V$, so x has at most one R-bearer (by (vi)). Hence every element of $\overline{R}$ has at most two R-bearers. R is therefore a HF process.

(d) Let I' be the set of R-isolated elements. Each $v \in V$ has $v.v\lambda$ as an R-bearer (by (a)), and elements $v.v\lambda$ of W have $v\lambda$ as an R-bearer (by (vi)). Thus $I' \cap (V \cup W) = \phi$. Elements not in $V \cup W$ are R-isolated exactly when they are P-isolated (by (b)). Hence $I' = I - V \cup W = I - V$ (as elements of W are P-free, by (ii)).

Let H' be the set of R-HF elements. By (vi), each $v.v\lambda \in W$ is R-HF with R-bearer $v\lambda$. Hence $W \subseteq H'$. By (a) and (b), the R-bearers of each $v \in V$ are $v.v\lambda$ and the P-bearer of v (if it exists). Thus $v \in V$ is R-HF exactly when v is P-isolated. Therefore $V \cap H' = V \cap I$. By (b), elements not in $V \cup W$ are R-HF exactly when they are P-HF (i.e. in H-V). Hence $H' = (H-V) \cup (V \cap I) \cup W$.

(e) is an immediate consequence of the definition of R.

<u>Example</u> : Let P be a HF process and ℓ a line of P_0. Let V be the set of P-HF lines having P-bearer not incident with ℓ. Define $\lambda : V \to P_0$ by $v\lambda = \ell$ for each $v \in V$. Let $W = \{v.\ell \; ; \; v \in V\}$. Then $R = \Gamma(1,V,\lambda,W)(P)$ is defined. From 1.5.9(d), the R-HF elements are $W \cup (V \cap I) \cup (H - V)$. Because W has only points and $V \cap I = \emptyset$, all R-HF lines are contained in H-V; i.e. they are P-HF. By 1.5.9(b), the R- and P-bearers of these R-HF lines are the same. All P-HF lines in H-V have P-bearer incident with ℓ. Thus we have obtained a new HF process R for $\overline{P}$ from P_0 such that all R-HF lines have R-bearer incident with ℓ.

<u>Proposition 1.5.10</u> : Suppose $R = \Gamma(k,V,\lambda,W)(P)$ and that there are only finitely many P-HF elements of P-length $> k$. If V consists entirely of such elements, then R has strictly fewer HF elements of length $> k$ than P.

<u>Proof</u> : Let I and H be the sets of P-isolated and P-HF elements

respectively. Because V has no P-isolated elements, $V \cap I = \phi$.
Thus, by 1.5.9(d), the set of R-HF elements is $(H-V) \cup W$. It therefore suffices to show

(i) $\ell_R(x) \leq \ell_P(x) \ \forall \ x \in H - V$

(ii) $\ell_R(v.v\lambda) \leq k < \ell_P(v) \ \forall \ v \in V.$

We first show (ii). By assumption, $k < \ell_P(v) \ \forall \ v \in V$. By 1.5.4, we have $\ell_R(v.v\lambda) \leq st_R(v.v\lambda) \ \forall \ v \in V$. But $st_R(v.v\lambda) = k$ $\forall \ v \in V$ (by the definition of R). Hence $\ell_R(v.v\lambda) \leq st_R(v.v\lambda) = k < \ell_P(v) \ \forall \ v \in V$, and (ii) is proved.

We now show (i). We show $\ell_R(x) \leq \ell_P(x) \ \forall \ x \in \overline{R}$. Suppose, on the contrary, that there is a $y \in \overline{R}$ for which

(iii) $\ell_P(y) < \ell_R(y)$.

Choose such a y of minimal R-stage. If y has no R-bearer, then it has no P-bearer (by 1.5.9(a) and (b)). This implies $\ell_R(y) = \ell_P(y) = 0$, contradicting (iii). Thus y has an R-bearer. Let u be an R-bearer of y of maximal R-length. Then $\ell_R(y) = \ell_R(u) + 1$ (by 1.5.4). By the minimality of $st_R(y)$, $\ell_R(u) \leq \ell_P(u)$. If u is also a P-bearer of y, then $\ell_R(y) = \ell_R(u) + 1 \leq \ell_P(u) + 1 \leq \ell_P(y)$ (by 1.5.4), contradicting (iii). Thus u is an R-bearer of y but not a P-bearer. By 1.5.9(a) and (b), $y = v$ and $u = v.v\lambda$ for some $v \in V$. Hence $\ell_R(u) \leq k$ (by (ii)). This implies $\ell_R(y) = \ell_R(u) + 1 \leq k + 1$. Because $y \in V$, $\ell_P(y) \geq k + 1$ (as all elements of V have P-length $> k$).

Hence $\ell_R(y) \leq k + 1 \leq \ell_P(y)$, again contradicting (iii). Hence $\ell_R(x) \leq \ell_P(x) \ \forall\, x \in \overline{R}$ and (i) is proved.

Our next proposition contains, in a general form, some technical results of previous authors about HF processes and free completions. Some of these are proved as corollaries.

If ρ' is a subconfiguration of a configuration ρ, then ρ' is <u>closed in</u> ρ if, for any x, y $\in \rho'$ for which $x \circ y \in \rho$, we have $x.y \in \rho'$.

<u>Proposition 1.5.11</u> : Suppose that P is a HF process for a plane and ρ is a subconfiguration of $\overline{P}$ satisfying

(i) $P(x) \subseteq \rho \ \forall\, x \in \rho$;

(ii) $\rho \cap P_0$ is closed in P_0.

Then $[\rho]_{\overline{P}}$ is freely generated by ρ and

(a) the ρ- and P-bearers of each $x \in [\rho]_{\overline{P}} - \rho$ coincide ;

(b) $P(x) \subseteq [\rho]_{\overline{P}} \ \forall\, x \in [\rho]_{\overline{P}}$;

(c) if $Q = [\rho]_{\overline{P}} \cap P$, then Q is a HF process and $Q(x) = P(x) \ \forall\, x \in [\rho]_{\overline{P}}$;

(d) if $R = [\rho]_{\overline{P}} \cup P$, then R is a HF process and the R- and P-bearers of each $x \in \overline{R} - R_0$ coincide ;

(e) if $P_0 \subseteq \rho$, then $Q + R$ is defined, satisfies $\underline{Q + R} = P_0$, $\overline{Q + R} = \overline{P}$, and is similar to P.

Proof : Each element of $[\rho]_{\overline{P}} - \rho$ has at least two ρ- bearers and at most two P-bearers. Thus to show that ρ freely generates $[\rho]_{\overline{P}}$, it suffices (by 1.4.1) to show (a) and that no two elements of equal ρ-stage are incident.

We first show (a). Suppose that $x \in [\rho]_{\overline{P}} - \rho$. We consider two cases.

(1) $st_P(x) > 0$: Suppose that the P- and ρ-bearers of x are not the same. We may assume $st_\rho(x)$ is minimal with respect to this property. Because x has at least two ρ-bearers and at most two P-bearers, it has a ρ-bearer y which is not a P-bearer. Because neither $y < x(P)$ nor $st_P(x) = 0$, we have $x < y(P)$ (by 1.5.1(a)). If $y \in \rho$, then $x \in P(y) \subseteq \rho$ (by (i)), contradicting $x \in [\rho]_{\overline{P}} - \rho$. Hence $y \in [\rho]_{\overline{P}} - \rho$. By the minimality of $st_\rho(x)$, the P- and ρ-bearers of y coincide. Because $x < y(P)$, this implies that x is a ρ-bearer of y, a contradiction. Hence the ρ- and P-bearers of each such x coincide.

(2) $st_P(x) = 0$: We have $x \in [\rho]_{\overline{P}} \cap P_0 - \rho \cap P_0$. Suppose x is of minimal ρ-stage with respect to this property. Let x have ρ-bearers y and z. If both $y, z \in \rho \cap P_0$, then $x = y \circ z \in \rho \cap P_0$ (since $\rho \cap P_0$ is closed in P_0), a contradiction. Hence one of $y, z \notin \rho \cap P_0$.

Suppose $y \notin \rho \cap P_0$. By the minimality of $st_\rho(x)$, we have $y \notin [\rho]_{\overline{P}} \cap P_0 - \rho \cap P_0$, and hence $y \notin P_0$. Thus $x < y(P)$. By case (1), the P- and ρ-bearers of y coincide. This implies x is a ρ-bearer of y, a contradiction. Hence no such x can exist. This completes both case (2) and the proof of (a).

We next show that elements of equal non-zero ρ-stage are not incident. Suppose that $x,y \in [\rho]_{\overline{P}}$, $st_\rho(x) = st_\rho(y) > 0$, and $x \mathrel{I} y$. Then x is not a ρ-bearer of y, and vice-versa, and $x,y \notin \rho$. By (a), x is not a P-bearer of y, and vice-versa. Hence $st_P(x) = st_P(y) = 0$ (by 1.5.1(a)). This implies both $x,y \in [\rho]_{\overline{P}} \cap P_0 - \rho \cap P_0$. But this set is empty (by case (2) of the previous paragraph). Hence no two elements of equal non-zero ρ-stage are incident. This completes the proof that $[\rho]_{\overline{P}} = F(\rho)$.

We prove (b) by induction on $st_\rho(x)$. If $st_\rho(x) = 0$, then $x \in \rho$ and $P(x) \subseteq \rho \subseteq [\rho]_{\overline{P}}$ (by (ii)). Assume now that $st_\rho(x) = n > 0$, and that $P(y) \subseteq [\rho]_{\overline{P}}$ for all $y \in [\rho]_{\overline{P}}$ of ρ-stage $< n$. If x has ρ-bearers u and v, then $P(u) \subseteq [\rho]_{\overline{P}}$ and $P(v) \subseteq [\rho]_{\overline{P}}$. By (a), u and v are also the P-bearers of x. Thus $P(x) = \{x\} \cup P(u) \cup P(v) \subseteq [\rho]_{\overline{P}}$. By induction, (b) is proved.

(c) is an immediate consequence of 1.5.7 and (b).

We now show (d). For $n > 0$, $R_n - R_{n-1} \subseteq P_n - P_{n-1}$. Hence no two elements of equal non-zero R-stage are incident. To show that R is a HF process, it now suffices to prove that the R- and P-bearers of each $x \in \overline{R} - R_0$ coincide. Suppose that there exist $x, y \in \overline{R} - R_0$ such that $x < y(P)$ and $y < x(R)$. There is an m for which $x \in P_m$ and $y \notin P_m$. Thus $x \in R_m = P_m \cup [\rho]_{\overline{P}}$, which implies $y \in R_m$ (since $y < x(R)$). Because $y \notin P_m$, we have $y \in [\rho]_{\overline{P}}$. Thus $P(y) \subseteq [\rho]_{\overline{P}}$, by (b). Hence $x \in P(y) \subseteq [\rho]_{\overline{P}} \subseteq R_0$, contradicting $x \in \overline{R} - R_0$. Thus no such x,y exist, and (d) is proved.

Finally, suppose $P_0 \subseteq \rho$. Then $\overline{Q} = [\rho]_{\overline{P}}$ and $R_0 = [\rho]_{\overline{P}} \cup P_0 = [\rho]_{\overline{P}}$. Hence $\overline{Q} = R_0$, and Q + R is defined. We have $\underline{Q + R} = [\rho]_{\overline{P}} \cap P_0 = P_0$, and $\overline{Q + R} = \overline{R} = \overline{P}$. By (c), $P(x) = Q(x)$ $\forall x \in [\rho]_{\overline{P}}$, so the P- and Q-bearers of each $x \in \overline{Q}$ coincide. By (d), the R- and P-bearers of each $x \in \overline{R} - R_0$ coincide. Thus Q + R is similar to P, and (e) is proved.

<u>Corollary 1.5.12</u> : Let P be a HF process for a plane. If X is a set of elements of $\overline{P}$ and $\rho = P_0 \cup \left(\bigcup_{x \in X} P(x) \right)$, then (i) and (ii) of 1.5.11 are satisfied. Hence $[\rho]_{\overline{P}} = F(\rho)$ and (a) to (e) of 1.5.11 are satisfied. If X contains all the P-isolated and P-HF elements, then $\overline{P} = F(\rho)$.

<u>Proof</u> : We have $\rho \cap P_0 = P_0$, which is closed in P_0. If $y \in \rho$, then either $y \in P_0$, which implies $P(y) = \{y\} \subseteq \rho$, or $y \in P(x)$

for some $x \in X$, which implies $P(y) \subsetneq P(x) \subseteq \rho$ (by 1.5.1(c)). Thus (i) and (ii) of 1.5.11 are satisfied. Hence $[\rho]_{\overline{P}} = F(\rho)$ and (a) to (e) of 1.5.11 are satisfied. If X contains the P-isolated and P-HF elements, then $P_0 \cup X$ generates $\overline{P}$ (by 1.5.1(d)). Hence ρ generates $\overline{P}$, so $\overline{P} = [\rho]_{\overline{P}} = F(\rho)$.

<u>Corollary 1.5.13</u> (Siebenmann (29, lemma 2)) : If P is any HF process for a plane, then $[P_n]_{\overline{P}} = F(P_n)\ \forall\ n \in N$. If P_n contains all the P-isolated and P-HF elements, then $\overline{P} = F(P_n)$.

<u>Proof</u> : Let $\rho = P_n$ and apply 1.5.11 to obtain $[P_n]_{\overline{P}} = F(P_n)$. If P_n contains all the P-isolated and P-HF elements, then P_n generates $\overline{P}$ (by 1.5.1(d)), so $\overline{P} = [P_n]_{\overline{P}} = F(P_n)$.

<u>Corollary 1.5.14</u> (Hall (10, theorem 4.3)) : If ρ is a subconfiguration of a configuration ρ' and ρ is closed in ρ', then ρ freely generates $[\rho]_{F(\rho')}$. If ρ is a proper subconfiguration of ρ', then $F(\rho)$ is a proper subplane of $F(\rho')$.

<u>Proof</u> : Let $P = \{ F_n(\rho') ; n \in N \}$. By 1.5.11, $[\rho]_{F(\rho')}$ equals $F(\rho)$ and the P- and ρ-bearers of each $x \in F(\rho)$ coincide. Thus, if $x \in F(\rho) - \rho$, then x has P-bearers, so $x \notin \rho'$. Hence $\rho' \cap F(\rho) = \rho$ and $\rho' - \rho \subseteq F(\rho') - F(\rho)$. Thus $F(\rho)$ is proper when ρ is proper.

Corollary 1.5.15 : Suppose that P is a standard HF process for a plane and that P_0 has only P-isolated elements. Then $[P(v)]_{\overline{P}} = F(P(v))$ for each $v \in \overline{P}$. If $R = [P(v)]_{\overline{P}} \cap P$, then R is also a standard HF process and $\ell_R(x) = \ell_P(x) \; \forall \; x \in \overline{R}$.

Proof : Let $\rho = P(v)$. By 1.5.11, $[\rho]_{\overline{P}} = F(\rho)$.
By 1.5.11(c), R is a HF process for which $R(x) = P(x) \; \forall x \in [\rho]_{\overline{P}}$.
Hence $\ell_R(x) = \ell_P(x) \; \forall x \in \overline{R}$. We also have $st_R(x) = st_P(x) \; \forall \; x \in \overline{R}$ (by the definition of R), and $st_P(x) = \ell_P(x) \; \forall x \in \overline{R}$ (as P is standard).
Therefore $st_R(x) = \ell_R(x) \; \forall \; x \in \overline{R}$, and R is standard.

Let P be any HF process (not necessarily indexed by the non-negative integers). If there are i P-isolated elements not in $\underline{P}$ and h P-HF elements, define the rank of P to be 2i + h. We denote it by r(P). Our next two results are elementary.

Lemma 1.5.16 : Let P and Q be similar HF processes, not necessarily indexed by the non-negative integers. If $\underline{P} = \underline{Q}$ and $\overline{P} = \overline{Q}$, then $r(P) = r(Q)$.

Lemma 1.5.17 : If P and Q are HF processes for which $\overline{P} = \underline{Q}$, then $r(P+Q) = r(P) + r(Q)$.

<u>Lemma 1.5.18</u> : If P is a HF process for which $\overline{P} - P_0$ is finite, then

$$r(P) = 2\left|\overline{P} - P_0\right| - f(P_0,\overline{P}), \qquad \ldots. \text{ (i)}$$

where $f(P_0,\overline{P})$ is the number of point-line pairs (p, ℓ) for which $p \mathrel{I} \ell$ and at least one of $p, \ell \in \overline{P} - P_0$.

<u>Proof</u> : The proof is by induction on $\left|\overline{P} - P_0\right|$. If $\left|\overline{P} - P_0\right| = 0$, then $f(P_0,\overline{P}) = 0$ and $r(P) = 0$, satisfying (i).
Suppose now that $\left|\overline{P} - P_0\right| = j > 0$, and that (i) is true for HF processes Q satisfying $\left|\overline{Q} - Q_0\right| < j$. Because $\overline{P} - P_0$ is finite and non-empty, there exists an $x \in \overline{P}$ of maximal P-stage > 0. By 1.5.7, P' is a HF process, where $P' = (\overline{P} - \{x\}) \cap P$. We have $P_0' = P_0$ and $\overline{P'} = \overline{P} - \{x\}$. Thus $\left|\overline{P'} - P_0'\right| = j - 1$, so

$$r(P') = 2(j-1) - f(P_0',\overline{P'}) \qquad \ldots. \text{ (ii).}$$

Let x be incident with k elements of $\overline{P}$. We have

$$f(P_0,\overline{P}) = f(P_0',\overline{P}) + k \qquad \ldots. \text{ (iii).}$$

Because x has maximal P-stage, it is incident in $\overline{P}$ only with its P-bearers. Thus k = 0, 1 or 2, according as x is P-isolated, P-HF or P-free. From the definition of P', and because x has maximal P-stage, the P-isolated (resp. P-HF) elements in $\overline{P} - \{x\}$ are exactly the P'-isolated (P'-HF) elements. Thus

$$r(P) = r(P') + \begin{cases} 0 \text{ if } x \text{ is P-free,} \\ 1 \text{ if } x \text{ is P-HF,} \\ 2 \text{ if } x \text{ is P-isolated,} \end{cases}$$

$$= r(P') + (2-k)$$

$$= 2(j-1) - f(P_0', \overline{P'}) + 2 - k \quad \text{(by (ii))}$$

$$= 2j + f(P_0, \overline{P}) \quad \text{(by (iii))}$$

$$= 2\left|\overline{P} - P_0\right| + f(P_0, \overline{P}).$$

Hence (i) has been proved, by induction.

It follows from the above lemma that the rank of a HF process P for which $\overline{P} - P_0$ is finite depends only on P_0 and $\overline{P}$. We use this observation in the proof of

Proposition 1.5.19 : If P and Q are HF processes for which $P_0 = Q_0$ and $\overline{P} = \overline{Q}$, then $r(P) = r(Q)$.

Proof : We assume first that $\overline{P}$ is a plane, and prove the proposition for that case. We may assume $r(P) \leq r(Q)$. Let X be the set of P-isolated and P-HF elements not in P_0, and Y be the set of Q-isolated and Q-HF elements not in Q_0.

Suppose first that both r(P) and r(Q) are finite. Then X and Y are finite. Define $\pi = \left(\bigcup_{x \in X \cup Y} P(x)\right) \cup P_0$ and $P' = \pi \cap P$, $Q' = \pi \cap Q$.

Then P' and Q' are HF processes (by 1.5.7) and $P_0 = P_0' = Q_0'$, $\overline{P'} = \overline{Q'} = \pi$. Because $X \cup Y$ is finite and all P-socles are finite (by 1.5.1(b)), $\pi - P_0$ is finite. Therefore $\overline{P'} - P_0'$ and $\overline{Q'} - Q_0'$ are finite, and $r(P') = r(Q')$ (by 1.5.18). We have $P(x) \subseteq \pi \; \forall x \in \pi$, so $P(x) = P'(x) \; \forall x \in \pi$. Thus elements of π are P-isolated or P-HF exactly when they are P'-isolated or P'-HF respectively. Because $X \subseteq \pi'$, this implies $r(P) = r(P')$. For $x, y \in \pi$, we have $x < y(Q') \Rightarrow x < y(Q)$. Thus Q-isolated elements in π are Q'-isolated, and Q-HF elements in π are either Q'-HF or Q'-isolated. Since $Y \subseteq \pi$, this implies $r(Q') \geq r(Q)$. Hence $r(Q') \geq r(Q) \geq r(P) = r(P')$. Because $r(P') = r(Q')$, we have $r(P) = r(Q)$.

Suppose now that $r(Q)$ is infinite. Then Y is infinite, $|Y| = r(Q)$, and $|X| \leq |Y|$. Define $\rho = \bigcup_{x \in X} P(x)$ and $\rho' = \left(\bigcup_{y \in \rho} Q(y)\right) \cup Q_0$.

Because all P- and Q-socles are finite, we have X is finite $\Leftrightarrow$ ρ is finite $\Leftrightarrow$ $\bigcup_{y \in \rho} Q(y)$ is finite, and if X is infinite, then $|X| = |\rho| = \left|\bigcup_{y \in \rho} Q(y)\right|$. By 1.5.12, $[\rho']_{\overline{Q}} = F(\rho')$, and the ρ'- and Q-bearers of each $x \in F(\rho') - \rho'$ coincide. Thus elements of $F(\rho') - \rho'$ are Q-free. But $P_0 \cup X \subseteq \rho'$, so ρ' generates $\overline{P}$ (by 1.5.1(d)). Thus $F(\rho') = \overline{P} = \overline{Q}$, and all elements of $\overline{Q} - \rho'$ are Q-free. Therefore $Y \subseteq \rho'$. Furthermore, $Y \cap Q_0 = \phi$, so $Y \subseteq \bigcup_{y \in \rho} Q(y)$. Since Y is infinite, $\bigcup_{y \in Y} Q(y)$ is infinite. Thus X is infinite and $|X| = \left|\bigcup_{y \in \rho} Q(y)\right| \geq |Y|$. Therefore $|X| = |Y|$, which implies $r(P) = r(Q)$.

We have now proved the proposition for the case $\overline{P}$ is a plane. Suppose $\overline{P}$ is not a plane. Let F be the free completion process for $F(\overline{P})$ from $\overline{P}$. Then $\overline{P+F}$ is a plane, $\overline{P+F} = \overline{Q+F}$, and $\underline{P+F} = \underline{Q+F}$. Hence $r(P+F) = r(Q+F)$. By 1.5.17, this implies $r(P) = r(Q)$.

The above theorem is proved under the assumption that P and Q are indexed by the non-negative integers. However, by 1.5.16 and 1.5.3, the theorem is true without this assumption.

1.6 Free Rank Planes

A configuration ρ' is confined if it is finite and each element of ρ' is incident with at least three other elements of ρ'. The union of finitely many confined configurations is also confined. For any configuration ρ, the core of ρ, denoted by $\kappa(\rho)$, is the union of all confined subconfigurations of ρ. Our first result is a generalization of Theorem 4.8 of (10).

Lemma 1.6.1 : If P is any HF process and ρ' any confined subconfiguration of $\overline{P}$, then $\rho' \subseteq P_0$. Hence $\kappa(\overline{P}) \subseteq P_0$. For any configuration ρ, $\kappa(F(\rho)) = \kappa(\rho)$.

Proof : Suppose $\rho' \nsubseteq P_0$. As ρ' is finite, there is an $x \in \rho'$ of maximal P-stage > 0. There are at least three elements of ρ' incident with x, all of P-stage $\leq st_P(x)$. But x is not incident with any element of equal P-stage and has at most two P-bearers. This contradiction implies $\rho' \subseteq P_0$. Thus the first assertion is proved. The last two assertions follow immediately from the first.

<u>Lemma 1.6.2</u> : If $\alpha : \rho \longrightarrow \rho'$ is an isomorphism or duality of configurations, then $\kappa(\rho)\alpha = \kappa(\rho')$. In particular, $\kappa(\rho)\alpha = \kappa(\rho)$ for any automorphism α of ρ.

<u>Proof</u> : The result follows from the observation that γ is a confined subconfiguration of ρ if and only if $\gamma\alpha$ is a confined subconfiguration of ρ'.

By 1.6.1, if we wish to construct a configuration using a HF process, we can at best construct it from its core. A configuration ρ <u>has free rank</u> if there is a HF process for ρ from $\kappa(\rho)$. The <u>free rank</u> (or just <u>rank</u>) of ρ is the rank of any HF process for ρ from $\kappa(\rho)$. By 1.5.19, the free rank of ρ is well defined. A plane which has free rank is called a <u>free rank plane</u>. Trivially, any plane equal to its core has free rank. In the next proposition, we prove some elementary and well known results about free rank planes. Part (a) was first proved by Schleiermacher and Strambach (28 , theorem 1(ii)). They proved that (a) holds in any plane.

<u>Proposition 1.6.3</u> : If ρ is a configuration having free rank, then $F(\rho)$ is a free rank plane having the same core and rank as ρ. If π is a free rank plane with core κ and rank r, then

(a) $[\kappa]_\pi = F(\kappa)$;

(b) if P is a HF process for π from κ, then $R = F(\kappa) \cup P$ is a HF process for π from $F(\kappa)$, and $r(R) = r$;

(c) if $\pi \neq \kappa$ and π is non-degenerate, then $\pi - \kappa$ is infinite.

<u>Proof</u> : By 1.6.1, $\kappa(F(\rho)) = \kappa(\rho)$. Let Q be a HF process for ρ from $\kappa(\rho)$ and F be the free completion process for $F(\rho)$ from ρ. Then Q + F is a HF process for $F(\rho)$ from $\kappa(F(\rho))$, so $F(\rho)$ has free rank. Because there are no F-isolated or F-HF elements outside ρ, we have $r(F) = 0$. Hence $F(\rho)$ has rank $r(Q+F) = r(Q) + r(F) = r(Q)$, which is the rank of ρ.

(a) Let P be a HF process for π from κ. Then $[\kappa]_\pi = [P_0]_\pi = F(P_0)$ (by 1.5.13).

(b) We have $\kappa \cap P_0 = \kappa$ and $P(x) = \{x\} \subseteq \kappa \ \forall x \in \kappa$. Hence (i) and (ii) of 1.5.11 are satisfied (with $\rho = \kappa$). Let $S = F(\kappa) \cap P$ and $R = F(\kappa) \cup P$. By 1.5.11, R and S are HF processes and the κ-, P- and S-bearers of each element of $F(\kappa) - \kappa$ coincide. Thus each element of $F(\kappa) - \kappa$ has two S-bearers, which implies $r(S) = 0$. By 1.5.11(e), P is similar to S + R, and $\underline{P} = \underline{S + R}$, $\overline{P} = \overline{S + R}$. Thus $r = r(P) = r(S+R) = r(S) + r(R) = r(R)$.

(c) Let P be a HF process for π from κ. If $\pi - \kappa$ is finite, then there is an $x \in \pi$ of maximal P-stage > 0, and x is incident only with its P-bearers; i.e. with at most two elements of π. This contradicts 1.1.1. Hence $\pi - \kappa$ is infinite.

Our next theorem shows that our definition of rank coincides with the usual definition (see for example (12, page 220)).

Theorem 1.6.4 : If ρ is a configuration for which $\rho - \kappa(\rho)$ is finite, then ρ has free rank. Its rank is $2\,|\rho - \kappa(\rho)| - f(\kappa(\rho), \rho)$, where $f(\kappa(\rho), \rho)$ is the number of incident point-line pairs (p, ℓ) for which $p\ I\ \ell$ and at least one of $p, \ell \in \rho - \kappa(\rho)$.

Proof : If $\rho = \kappa(\rho)$, then ρ has free rank 0 and the theorem holds. Assume $\rho - \kappa(\rho) \neq \emptyset$. We first show there exists on $x \in \rho - \kappa(\rho)$ incident with ≤ 2 elements of ρ. Suppose, on the contrary, that each element of $\rho - \kappa(\rho)$ is incident with ≥ 3 elements of ρ. For each $y \in \rho - \kappa(\rho)$, define a configuration ρ_y as follows :

choose three elements $z_1^{(y)}$, $z_2^{(y)}$, $z_3^{(y)}$ of ρ incident with y. If $i \in \{1, 2, 3\}$ and $z_i^{(y)} \in \kappa(\rho)$, choose a confined configuration $\rho_i^{(y)}$ containing $z_i^{(y)}$. If $z_i^{(y)} \in \rho - \kappa(\rho)$, let $\rho_i^{(y)} = \{z_i^{(y)}\}$. Define $\rho_y = \bigcup_{i=1}^{3} \rho_i^{(y)}$. Then $(\rho - \kappa(\rho)) \cup \left(\bigcup_{y \in \rho - \kappa(\rho)} \rho_y\right)$ is a confined configuartion containing $\rho - \kappa(\rho)$, a contradiction. Hence there is an $x \in \rho - \kappa(\rho)$ incident with ≤ 2 elements of ρ.

Suppose $|\rho - \kappa(\rho)| = n$. Let $\rho = \rho_n$ and choose an $x_n \in \rho - \kappa(\rho)$ incident with ≤ 2 elements of ρ. Define $\rho_{n-1} = \rho_n - \{x_n\}$. Then $\kappa(\rho_{n-1}) = \kappa(\rho_n)$. If $n > 1$, then by the argument of the previous paragraph, there is an $x_{n-1} \in \rho_{n-1} - \kappa(\rho_{n-1})$ incident with only two elements of ρ_{n-1}. Define $\rho_{n-2} = \rho_{n-1} - \{x_{n-1}\}$. Continuing in this way, we define $x_{n-2},\ldots,x_0$ and $\rho_{n-2},\ldots,\rho_0$. We have $\rho_0 = \kappa(\rho)$. The extension process $P = \{\rho_i ; 0 \leq i \leq n\}$ is a HF process for ρ from $\kappa(\rho)$. Thus ρ has free rank. By 1.5.18, its rank is $r(P) = 2|\overline{P} - P_0| - f(P_0,\overline{P}) = 2|\rho - \kappa(\rho)| - f(\kappa(\rho),\rho)$.

A question which naturally arises is : do all planes have free rank? This question was answered in the negative by Kopejkina (17), who proved that the union of a strictly increasing chain of non-degenerate free rank planes, each having rank 8 and empty core, is not a free rank plane. Such a chain exists. We give a generalisation of Kopejkina's result in Chapter 4.

A <u>free plane</u> is a free rank plane with empty core. Free planes were first defined by Hall and they were the first free rank planes to be studied. The definition of a free plane as the union of a HF process from ϕ is due to Siebenmann (29). We prove some elementary and well known results about free planes in the following proposition. Parts (a) and (e) were first proved by Kopejkina (17) and Hall (10) respectively. Our proof for (a) is due to Siebenmann (29).

<u>Proposition 1.6.5</u> : (a) Subplanes of free planes are free.

(b) Degenerate planes are free.

(c) A degenerate plane is finite if and only if it has finite rank.

(d) A subplane of rank r_0 of a finite degenerate plane of rank r_1 is proper if, and only if, $r_0 < r_1$.

(e) A non-degenerate free plane has rank ≥ 8.

Proof : (a) Let π_0 be a subplane of a free plane π. If P is a HF process for π from $\emptyset$, then $\pi_0 \cap P$ is a HF process for π_0 from $\emptyset$ (by 1.5.7). Hence π_0 is free.

(b), (c) and (d) are trivially proved using 1.1.2.

(e) This is shown by an inspection of possible types of HF processes P with $r(P) \leq 7$ and $P_0 = \emptyset$. One shows that $F(\overline{P})$ is degenerate in each case. This suffices, because $F(\overline{P}) = \overline{P}$ for a plane. We omit the inspection of cases because the result is well known (see for example (10, Theorem 4.11)).

We note that any plane having non-empty core is not free and hence is non-degenerate.

We now work towards proving our main theorem of this section, which gives necessary and sufficient conditions for two non-degenerate free rank planes to be isomorphic.

Proposition 1.6.6 : Let π be a free rank plane and π_0 be a proper non-degenerate subplane of π such that there exists a HF process P for π from π_0. Then, for any line ℓ of π_0, there is a HF process Q for π from π_0 for which there are no Q-isolated elements, and all Q-HF elements are points with Q-bearer ℓ .

<u>Proof</u> : We define HF processes R,S,T,Q such that $R_0 = S_0 = T_0 = Q_0 = \pi_0$, $\bar{R} = \bar{S} = \bar{T} = \bar{Q} = \pi$, and

(i) R has no isolated elements,

(ii) S has no isolated elements and no HF lines,

(iii) T has no isolated elements, no HF points having T-bearer other than ℓ , and no HF lines having T-bearer incident with ℓ ,

(iv) Q has the required property.

R is obtained from P, S from R, etc. using 1.5.9. For this proof, we refer to 1.5.9(a), (b) etc. simply as (a), (b) etc.

Let V_P be the set of P-isolated elements. P_0 has no isolated elements as it is a non-degenerate plane. Thus $st_P(x) \geq 1 \ \forall x \in V_P$. Choose a point s and a line t of $\mathcal{P}_0$. Define $\lambda_1 : V_P \to P_0$ by $x\lambda_1 = s$ if x is a point and $x\lambda_1 = t$ if x is a line. As elements of V_P have no P-bearers, $x\lambda_1$ is not incident with the P-bearer of x for any $x \in V_P$. Define $W_R = \{x.x\lambda_1 \ ; \ x \in V_P\}$. Then $R = \Gamma(1, V_P, \lambda_1, W_R)(P)$ is defined and $\bar{R} = \bar{P} = \pi$, $R_0 = P_0 = \pi_0$ (by (c)). By (d), the set of of R-isolated elements is $V_P - V_P = \emptyset$. Thus (i) is satisfied.

We now define S. Choose distinct lines m and n of π_0 for which $m \neq \ell$, $n \neq \ell$ and $m.n \not I \ell$. Such lines exist as π_0 is a non-degenerate plane. Let V_R be the set of R-HF lines. Because each $x \in V_R$ has an R-bearer, $st_R(x) \geq 1$ for all $x \in V_R$. Define $\lambda_2 : V_R \to R_0$ by

$$x\lambda_2 = \begin{cases} \ell, & \text{if the R-bearer of } x \text{ is not incident with } \ell, \\ m, & \text{if the R-bearer of } x \text{ is incident with } \ell \text{ and} \\ & \text{is not } \ell.m, \\ n, & \text{if the R-bearer of } x \text{ is } \ell.m. \end{cases}$$

Then $x\lambda_2$ is not incident with the R-bearer of x $\forall x \in V_R$. Define $W_S = \{x.x\lambda_2 \,;\; x \in V_R\}$ and $S = \Gamma(1, V_R, \lambda_2, W_S)(R)$. By (c), $\overline{S} = \overline{R} = \pi$ and $S_0 = R_0 = \pi_0$. By (d), S has no isolated elements (as R has none) and S has HF elements $(H_R - V_R) \cup W_S$, where H_R is the set of R-HF elements. W_S has only points and, by the definition of V_R, $H_R - V_R$ has only points. Thus S has no HF lines, and (ii) is satisfied.

We now define T. Choose distinct non-collinear points p,q,r of π_0, none incident with ℓ. Such points exist as π_0 is a non-degenerate plane. Let V_S be the set of S-HF points not having ℓ as S-bearer. Define $\lambda_3 : V_S \to S_0$ by

$$x\lambda_3 = \begin{cases} p, \text{ if the S-bearer of } x \text{ is not incident with } p, \\ q, \text{ if the S-bearer of } x \text{ is incident with } p \text{ and is} \\ \quad \text{not } p \circ q, \\ r, \text{ if the S-bearer of } x \text{ is } p \cdot q. \end{cases}$$

Define $W_T = \{x \circ x\lambda_3 \ ; \ x \in V_S\}$ and $T = \Gamma(1, V_S, \lambda_3, W_T)(S)$. By (c), $T_0 = S_0 = \pi_0$ and $\overline{T} = \overline{S} = \pi$. By (d), T has no isolated elements and has HF elements $(H_S - V_S) \cup W_T$, where H_S is the set of S-HF points. To show that T satisfies (iii), we need to show that the points of $H_S - V_S$ have T-bearer ℓ, and the lines of W_T have T-bearer not incident with ℓ. By the definition of V_S, points of $H_S - V_S$ have S-bearer ℓ. Hence they have T-bearer ℓ (by(b)). Because $\{p, q, r\} \subseteq T_0$ and each line of W_T is incident with one of these points, each line of W_T has p, q or r as T-bearer. None of p, q or r is incident with ℓ. Hence (iii) is satisfied.

Finally, we define Q. Let V_T be the set of T-HF lines. Define $\lambda_4 : V_T \to T_0$ by $x\lambda_4 = \ell \ \forall \ x \in V_T$. Because T satisfies (iii), $x\lambda_4$ is not incident with the T-bearer of x. Define $W_Q = \{x \circ x\lambda_4 ; \ x \in V_T\}$ and $Q = \Gamma(1, V_T, \lambda_4, W_Q)(T)$. By (c), $\overline{Q} = \overline{T} = \pi$ and $Q_0 = T_0 = \pi_0$. By (d), Q has no isolated elements and has HF elements $(H_T - V_T) \cup W_Q$, where H_T is the set of T-HF elements. All elements of $H_T - V_T$ and W_Q are points. It remains to show they have Q-bearer ℓ. All T-HF points have T-bearer ℓ (by (iii)), so the points of $H_T - V_T$ have T-bearer ℓ.

By (b), they also have Q-bearer ℓ. All points in W_T are incident with $\ell \in Q_0$, so they too have Q-bearer ℓ. Hence all Q-HF elements are points with Q-bearer ℓ.

The following corollary is a generalization of lemma 11.10 of (12).

Corollary 1.6.7 : Let π be a free rank plane with rank r and non-empty core K. For any line $\ell \in F(K)$, π is the free completion of a configuration ρ consisting of $F(K)$ and r other points, each incident with ℓ and no other line of ρ.

Proof : Because $F(K)$ has non-empty core K, it is non-degenerate. By 1.6.3(b), there is a HF process Q for π from $F(K)$ with $r(Q) = r$. By 1.6.6, we may assume that Q has no isolated elements, and all Q-HF elements are points with Q-bearer ℓ. Let X be the set of Q-HF points. Because $r(Q) = r$, we have $|X| = r$. Let $\rho = F(K) \cup X$. Then $[\rho]_\pi = \pi$ (by 1.5.1(d)). Furthermore, $Q(z) \subseteq \rho \ \forall z \in \rho$ and $\rho \cap Q_0 = Q_0$. Hence, by 1.5.11, $\pi = [\rho]_\pi = F(\rho)$.

The above corollary is the main tool in proving our isomorphism theorem for non-degenerate free rank planes for the case when the core is non-empty. We now prove a series of lemmas leading to an analagous result for free planes.

Lemma 1.6.8 : Suppose that $\{\rho_1, \ldots, \rho_n\}$ is a set of configurations for which $\rho_{i+1} = \rho_i \cup \{x_i\}$, where x_i is incident with exactly two elements of ρ_i, $i = 1, \ldots, n-1$. Then $F(\rho_1) = F(\rho_2) = \ldots = F(\rho_n)$.

Proof : We need only show $F(\rho_1) = F(\rho_2)$. There exists a HF process P for $F(\rho_2)$ from ρ_1 defined by $P_0 = \rho_1$, $P_1 = \rho_1 \cup \{x_1\} = \rho_2$, $P_k = F_{k-1}(\rho_2)$, $k > 1$. Then $\overline{P} = F(\rho_2)$. Every element of $\overline{P} - \rho_1$ is P-free. Hence the standard HF process P' similar to P is the free completion process from ρ_1 (by 1.5.5). Thus $F(\rho_1) = \overline{P'} = \overline{P} = F(\rho_2)$.

Lemma 1.6.9 : Let ρ be a degenerate plane and $x \notin \rho$ be incident with at most one element of ρ. If $F(\rho \cup \{x\})$ is non-degenerate, then there exists a HF process P for $F(\rho \cup \{x\})$ from ϕ having at least four isolated points or four isolated lines.

Proof : Because $F(\rho \cup \{x\})$ is non-degenerate, ρ is non-empty and contains a point and a line. Hence, by 1.1.2, ρ has a point p and a line ℓ for which all lines (resp. points), except possibly ℓ (resp p), are incident with p (resp. ℓ).

We assume first that x is a point and x is incident with exactly one line y of ρ. Because $F(\rho \cup \{x\})$ is non-degenerate, we have $y \neq \ell$. In addition, ρ has at least three points q,r,s (possibly including p) incident with ℓ and three lines m,n,o (possibly including y or ℓ) incident with p. We consider two cases. These correspond to Figs. (a) and (b) below.

<u>(a) $p \, I \, \ell$</u> : We may assume $n = y$, $o = \ell$ and $s = p$. Let $\rho' = \rho \cup \{x, x \circ q, (x \circ q) \circ m\}$. By 1.6.8, $F(\rho') = F(\rho \cup \{x\})$. Define $P_0 = \phi$, $P_1 = \{x, r, p, m \circ (x \circ q)\}$, $P_2 = P_1 \cup \{\ell, x \circ q\}$, $P_n = F_{n-3}(\rho')$, $n \geq 3$.

<u>(b) $p \, \not{I} \, \ell$</u> : We may assume $n = y$, $q = n \circ \ell$, $r = m \circ \ell$, $s = o \circ \ell$. Define $P_0 = \phi$, $P_1 = \{x, p, r, s\}$, $P_2 = P_1 \cup \{\text{lines of } \rho\}$, $P_n = F_{n-3}(\rho \cup \{x\})$, $n \geq 3$.

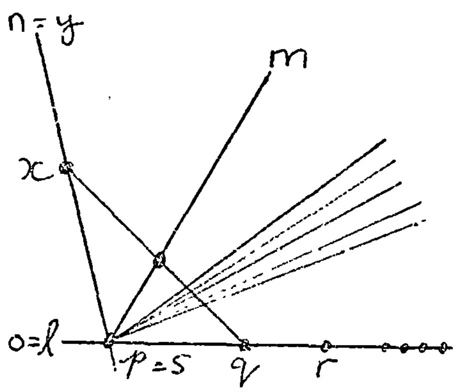


Fig. (a)

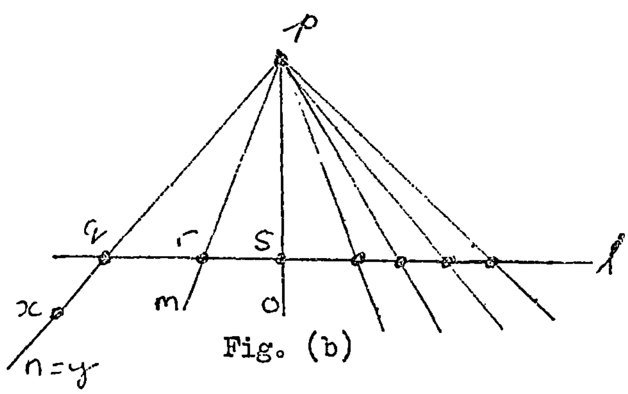


Fig. (b)

Hence the required HF process P exists in both cases.

Suppose now that x is a point incident with no line of ρ . Define a configuration ρ_0 by $\rho_0 = \rho \cup \{x \circ p\}$ if $p \, I \, \ell$, and $\rho_0 = \rho \cup \{x \cdot p, (x \circ p) \cdot \ell\}$ if $p \, \not{I} \, \ell$. Then ρ_0 is a degenerate plane, x is incident with one line of ρ_0, and $F(\rho_0 \cup \{x\}) = F(\rho \cup \{x\})$

(by 1.6.8). From the previous paragraph, there is a HF process P for $F(\rho_0 \cup \{x\})$ from $\emptyset$ with at least four isolated points.

Finally, if x is a line, one uses the dual of the above arguments to obtain a HF process P for $F(\rho \cup \{x\})$ from $\emptyset$ having at least four isolated lines.

<u>Lemma 1.6.10</u> : For any non-degenerate free plane π', there exists a HF process for π' from $\emptyset$ having at least four isolated points or four isolated lines.

<u>Proof</u> : Let π' have rank r'. To prove the lemma, it suffices to show that for any non-degenerate free plane π of rank $r \leq r'$, there exists a HF process for π from $\emptyset$ having at least four isolated points or four isolated lines. We prove this by induction on r. Assume first that π has rank $r \leq r'$ and that, for any non-degenerate free plane π_0 of rank $< r$, there exists a HF process for π_0 from $\emptyset$ with the required property. Let P be a HF process for π from $\emptyset$. We consider two cases.

(1) <u>r is infinite</u> : Let η be a four-point of π and define $\rho_0 = \bigcup_{y \in \eta} P(y)$. By 1.5.12, $[\rho_0]_\pi = F(\rho_0)$. $F(\rho_0)$ is non-degenerate because it contains η. Because all P-socles are finite, so is ρ_0. Hence ρ_0 has finite free rank r_0 (by 1.6.4). Thus $F(\rho_0)$ is a non-degenerate free plane of rank $r_0 < r$ (by 1.6.3).

Hence there is a HF process R for $F(\rho_0)$ from ϕ having at least four isolated points or four isolated lines. Let $Q = F(\rho_0) \cup P$. Then Q is a HF process for π from $F(\rho_0)$ (by 1.5.11(d)). Hence $R + Q$ is a HF process for π from ϕ having the required property.

(2) r is finite : In this case, there are only finitely many P-HF and P-isolated elements. Choose one, x, of maximal P-stage. Let Y be the set of P-isolated and P-HF elements other than x, together with the P-bearer of x(if x is P-HF). Define $\rho_1 = \bigcup_{y \in Y} P(y)$. By 1.5.12, we have

(i) $[\rho_1]_\pi = F(\rho_1)$, the ρ_1- and P-bearers of each element of $F(\rho_1) - \rho_1$ coincide, and $P(z) \subseteq F(\rho_1)$ $\forall\, z \in F(\rho_1)$.

Because the ρ_1-and P-bearers of each $x \in F(\rho_1) - \rho_1$ coincide, all elements of $F(\rho_1) - \rho_1$ are P-free. Hence $x \notin F(\rho_1) - \rho_1$. Because $x \notin Y$, and by the maximality of $st_P(x)$, we have $x \notin \rho_1$. Therefore

(ii) $x \notin F(\rho_1)$.

Now $F(\rho_1) \cup \{x\}$ contains $Y \cup \{x\}$; ie. all P-isolated and P-HF elements. Hence $[F(\rho_1) \cup \{x\}]_\pi = \pi$ (by 1.5.1(d)). Because Y contains the P-bearer of x (if it exists), $P(x) \subseteq \rho_1 \cup \{x\} \subseteq F(\rho_1) \cup \{x\}$. Therefore $P(z) \subseteq F(\rho_1) \cup \{x\}$ $\forall\, z \in F(\rho_1) \cup \{x\}$ (using the third statement of (i)). By 1.5.11, this implies

$$\text{(iii)}\quad \pi = \left[F(\rho_1) \cup \{x\}\right]_{\pi} = F(F(\rho_1) \cup \{x\}).$$

We now consider two subcases.

(a) <u>$F(\rho_1)$ is non-degenerate</u> : Let $Q = F(\rho_1) \cap P$ and $R = F(\rho_1) \cup P$. By 1.5.11, Q and R are HF processes, Q + R is similar to P, $\underline{Q + R} = P_0 = \phi$, and $\overline{Q + R} = \overline{P} = \pi$. Therefore $r = r(P) = r(Q+R) = r(Q) + r(R)$. Because x is P-isolated or P-HF, x is (Q+R)-isolated or (Q+R) -HF. Since $x \notin F(\rho_1) = \overline{Q}$ (by (ii)), x is R-isolated or R-HF. Therefore $r(R) > 0$. Hence $r(Q) = r - r(R) < r$. Thus $F(\rho_1)$ is a free plane of rank $r(Q) < r$. Because $F(\rho_1)$ is non-degenerate, there is a HF process S for $F(\rho_1)$ from ϕ having at least four isolated points or four isolated lines (by the induction assumption). Hence S + R is a HF process for π having the required property.

(b) <u>$F(\rho_1)$ is degenerate</u> : Because $x \notin F(\rho_1)$ (by (ii)), x is incident with at most one element of $F(\rho_1)$. By (iii), $\pi = F(F(\rho_1) \cup \{x\})$. Hence, by 1.6.9, a HF process exists for π having the required properties.

It now only remains to prove that when π is a non-degenerate free plane of rank 8 (the minimum possible rank, by 1.6.5(e)), there exists a HF process for π from ϕ having the required property. Let P be a HF process for π from ϕ. Define ρ_1 as in case (2) above. Then $F(\rho_1)$ is a free plane of rank < 8 (this is shown in subcase (a) above). Hence $F(\rho_1)$ is degenerate. By the argument of subcase (b) above, a HF process for π from ϕ exists which has the required property.

<u>Lemma 1.6.11</u> : Any two non-degenerate free planes of rank 8 are isomorphic.

<u>Proof</u> : Since any two four-points are isomorphic, so are their free completions (by 1.4.2). Hence it suffices to show that any free plane π of rank 8 is the free completion of a four-point. Let P be a HF process for π from ϕ. Let η be the set of P-isolated ~~or~~ ~~P-isolated~~ elements. By 1.6.10, we may assume η contains a four-point or a four-line. But because π has rank 8, η <u>is</u> a four-point or four-line, and there are no P-HF elements. Thus $[\eta]_\pi = \pi$ (by 1.5.1(d)). Because $P(x) = \phi \subseteq \eta \;\; \forall \;\; x \in \eta$, we have $\pi = [\eta]_\pi = F(\eta)$ (by 1.5.11). If η is a four-point, there is nothing further to prove. Suppose η is a four-line with lines a, b, c and d. Then Q is a HF process for π, where $Q_0 = \{a.b,\ b.c,\ c.d,\ d.a\}$, $Q_1 = Q_0 \cup \eta$, $Q_2 = F_1(\eta)$, $Q_n = F_{n-1}(\eta)$, $n \geq 3$. By 1.5.13, $\pi = F(Q_0)$; ie. π is the free completion of a four point.

We now prove a result for free planes analagous to 1.6.7.

<u>Lemma 1.6.12</u> : Any non-degenerate free plane π of rank r has a non-degenerate subplane π_0 of rank 8 such that, for any line ℓ of π_0, π is the free completion of a configuration consisting of π_0 and a set X_ℓ of r - 8 points, each incident with ℓ and no other line of π_0.

<u>Proof</u> : Let P be a HF process for π from ϕ. By 1.6.10, we may assume that the set of P-isolated elements contains a four-point or four-line γ. Because $P(x) = \phi \subseteq \gamma \ \forall \ x \in \gamma$, we have $[\gamma]_\pi = F(\gamma)$ (by 1.5.11). Let $\pi_0 = F(\gamma)$. Then π_0 is a non-degenerate subplane of π of rank 8. Let $Q = \pi_0 \cap P$ and $R = \pi_0 \cup P$. By 1.5.11, Q and R are HF processes, Q + R is similar to P, $\underline{Q + R} = \phi = P_0$, and $\overline{Q + R} = \overline{P} = \pi$. We have $r = r(P) = r(Q+R) = r(Q) + r(R)$. Since $Q_0 = \phi$ and $\overline{Q} = \pi_0$, we have $r(Q) = 8$. Therefore $r(R) = r - 8$.

Choose any line ℓ of π_0. By 1.6.6, we may assume that R has no isolated elements and that all R-HF elements are points with R-bearer ℓ. Let X_ℓ be the set of R-HF points. Because $r(R) = r - 8$, we have $|X_\ell| = r - 8$. Let $\rho = \pi_0 \cup X_\ell$. Then $[\rho]_\pi = \pi$ (by 1.5.1(d)). Furthermore, $R(z) \subseteq \rho \ \forall \ z \in \rho$, and $\rho \cap R_0 = R_0$. Hence, by 1.5.11, $\pi = [\rho]_\pi = F(\rho)$.

We now state and prove our isomorphism theorem for non-degenerate free rank planes. The theorem is well known (see, for example, (12, chapter XI)). It was first proved by Hall (10) for free planes of finite rank.

<u>Theorem 1.6.13</u> : Two non-degenerate free rank planes are isomorphic if, and only if, their cores are isomorphic and they have the same rank.

<u>Proof</u> : Suppose first that two free rank planes π and π' are isomorphic. Let $\alpha : \pi \to \pi'$ be an isomorphism. By 1.6.2, $K(\pi') = K(\pi)\alpha \cong K(\pi)$. Let P be a HF process for π from $K(\pi)$. Define the HF process P' by $P_i' = P_i\alpha$, $i = 0,1,\ldots$. Then x is P-isolated (resp. P-HF) if and only if $x\alpha$ is P'-isolated (resp. P'-HF). Therefore $r(P) = r(P')$, and π and π' have the same rank.

Conversely, assume π and π' are free rank planes for which $K(\pi) \cong K(\pi')$ and both π and π' have rank r. We consider two cases.

(1) <u>$K(\pi) \neq \emptyset$</u> : Since $K(\pi) \cong K(\pi')$, we have $F(K(\pi)) \cong F(K(\pi'))$ (by 1.4.2). Let $\alpha : F(K(\pi)) \to F(K(\pi'))$ be an isomorphism. Choose a line ℓ of $F(K(\pi))$. By 1.6.7, $\pi = F(\rho)$, where ρ consists of $F(K(\pi))$ together with r points $\left\{x_i \ ; \ i \leq i \leq r\right\}$ incident only with ℓ. Also by 1.6.7, $\pi' = F(\rho')$, where ρ' consists of $F(K(\pi'))$ and r points $\left\{x_i' \ ; \ 1 \leq i \leq r\right\}$ incident only with $\ell\alpha$. Extend α to an isomorphism of ρ onto ρ' by defining $x_i\alpha = x_i'$, $1 \leq i \leq r$. By 1.4.2, α extends to an isomorphism of π onto π'.

(2) $\underline{K(\pi) = \phi}$: One shows that $\pi \cong \pi'$ in the same way as case (1), using 1.6.11 and 1.6.12 instead of 1.6.7.

Because of the above theorem, we refer to <u>the</u> non-degenerate free rank plane having a given core K and rank r. When r is <u>finite</u>, we denote it by π_r^K. Thus π_r^K can be regarded as a representative from the non-degenerate free rank planes having core isomorphic to K and finite rank r. We denote π_r^ϕ by π_r. Because non-degenerate free planes have rank ≥ 8, we use the notation π_r only when $r \geq 8$. We denote the non-degenerate free rank plane having core K and countably infinite rank by $\pi_{\aleph_0}^K$ (or $\pi_{\aleph_0}$ if $K = \phi$).

<u>Theorem 1.6.14</u> : For any non-empty configuration K equal to its core, π_r^K exists for all non-negative integers r. π_r exists for all non-negative integers $r \geq 8$.

<u>Proof</u> : Suppose first that K is a non-empty configuration equal to its core. Choose a line ℓ of K. The free completion of $K \cup X$, where X is a set of r points incident only with ℓ, is a free rank plane having core K and rank r (by 1.6.4 and 1.6.3). Thus π_r^K exists for all non-negative integers r. For $r \geq 8$, let ρ be a configuration having a line ℓ, two points not incident with ℓ, and r - 6 points incident with ℓ. Then $F(\rho)$ is a non-degenerate free plane of rank r. Hence π_r exists for all integers $r \geq 8$.

1.7 Some Properties of Free Rank Planes

In this section we prove a number of properties of free rank planes which are used in later chapters. Many of these are generalizations of well-known properties of free planes.

We first consider the subplanes of free rank planes. We have shown, in 1.6.5(a), that subplanes of free planes are free. We now generalize this to

Theorem 1.7.1 : Subplanes of a free rank plane π which contain $K(\pi)$ are free rank planes with core $K(\pi)$. Subplanes of π having empty intersection with $K(\pi)$ are free planes.

Proof : Let P be a HF process for π from $K(\pi)$. For any subplane π' of π, we have $K(\pi') \subseteq \pi' \cap K(\pi)$. Suppose first that π' contains $K(\pi)$. Then $K(\pi') = K(\pi)$, and $\pi' \cap P$ is a HF process for π' from $K(\pi')$ (by 1.5.7). Hence π' is a free rank plane with core $K(\pi)$. Suppose now that $\pi' \cap K(\pi) = \phi$. Then $K(\pi') = \phi$ and $\pi' \cap P$ is a HF process for π' from ϕ. Hence π' is free.

We note that, in general, subplanes of free rank planes are not necessarily free rank planes. It is possible, for example, for a free rank plane π to have a subplane π' for which $\pi' \subseteq K(\pi)$ and π' does not have free rank.

We now consider subplanes of free planes generated by four-points or four-lines. Our next result was proved by Dembowski (5 , theorem 1.1) for any non-degenerate plane having empty core. Our proof is that of Dembowski.

Theorem 1.7.2 : If η is a four-point or four-line of a free plane π, then $[\eta]_\pi$ is freely generated by η.

Proof : Consider the generation process $(\eta_i)_{i=0}^{\infty}$ for $[\eta]_\pi$ from η. Each η_i is a finite configuration with empty core. By 1.6.4, it has free rank $r_i = 2|\eta_i| - f_i$, where f_i is the number of incidences is η_i. Because each element of $\eta_{i+1} - \eta_i$ is incident with at least two elements of η_{i+1}, we have $f_{i+1} \geq f_i + 2|\eta_{i+1} - \eta_i|$. Equality holds if, and only if, each element of $\eta_{i+1} - \eta_i$ is incident with exactly two elements of η_{i+1} ; ie. when $\eta_{i+1} = F_1(\eta_i)$. Hence

$$
\begin{aligned}
r_{i+1} &= 2|\eta_{i+1}| - f_{i+1} \\
&= 2|\eta_i| - (f_{i+1} - 2|\eta_{i+1} - \eta_i|) \\
&\leq 2|\eta_i| - f_i \text{ (equality holding if, and only if, } \eta_{i+1} = F_1(\eta_i)) \\
&= r_i .
\end{aligned}
$$

Hence $(r_i)_{i=0}^{\infty}$ is a decreasing sequence of integers, and $r_{i+1} = r_i$ if and only if $\eta_{i+1} = F_1(\eta_i)$ $\forall\, i \in N$.

By 1.6.3, $F(\eta_i)$ is a free plane of rank r_i. Because η_i contains η, $F(\eta_i)$ is non-degenerate. Therefore $r_i \geq 8 \ \forall i \in N$ (by 1.6.5(e)). Thus $(r_i)_{i=0}^{\infty}$ is bounded below by 8, which implies $8 \leq r_i \leq r_0 \ \forall \ i \in N$. But $\eta = \eta_0$ has rank 8. Hence $r_0 = 8$ and $r_i = 8 \ \forall \ i \in N$. Thus $r_{i+1} = r_i \ \forall \ i \in N$, implying $\eta_{i+1} = F_1(\eta_i)$ $\forall \ i \in N$. Therefore $\eta_i = F_i(\eta) \ \forall \ i \in N$. Hence $[\eta]_\pi$ is freely generated by η.

We next prove a technical result useful in chapter 2.

<u>Proposition 1.7.3</u> : If P is a HF process for a free plane π, and η is a four-point or four-line of π, then $\ell_P(x) \leq st_\eta(x) + m$ $\forall x \in [\eta]_\pi$, where $m = \max\{\ell_P(y); y \in \eta\}$.

<u>Proof</u> : We proceed by induction on $st_\eta(x)$. $st_\eta(x) = 0$, then $x \in \eta$ and $\ell_P(x) \leq m$ (by the definition of m). Suppose now that $st_\eta(x) = n > 0$ and that $\ell_P(u) \leq st_P(u) + m$ for all $u \in [\eta]_\pi$ of η-stage $< n$. By 1.7.2, $[\eta]_\pi = F(\eta)$. Thus x has exactly two η-bearers y and z. Both y and z have lower η-stage than x, and thus

(i) $\quad st_\eta(y) \leq st_\eta(x) - 1, \ st_\eta(z) \leq st_\eta(x) - 1.$

By the induction assumption, we have

(ii) $\quad \ell_P(y) \leq m + st_\eta(y), \quad \ell_P(z) \leq m + st_\eta(z)$.

We consider two cases :

(a) y and z are the P-bearers of x : By 1.5.4, we have

$$\begin{aligned}\ell_P(x) &= \max\left\{\ell_P(y),\ \ell_P(z)\right\} + 1 \\ &\leq \max\left\{m + st_\eta(y),\ m + st_\eta(z)\right\} + 1 \quad \text{(by (ii))} \\ &\leq (m + st_\eta(x) - 1) + 1 \quad \text{(by (i))} \\ &= m + st_\eta(x).\end{aligned}$$

(b) At least one of y, z, say y, is not a P-bearer of x :

By 1.5.1(a), either x is a P-bearer of y or $st_P(x) = st_P(y) = 0$.

In either case,

$$\begin{aligned}\ell_P(x) \leq \ell_P(y) &\leq st_\eta(y) + m \quad \text{(by (ii))} \\ &\leq st_\eta(x) + m \quad \text{(by (i))}.\end{aligned}$$

In both cases (a) and (b), we have $\ell_P(x) \leq st_\eta(x) + m$. By induction, the proposition has been proved.

If α is a collineation of a plane π, then the subconfiguration of π with elements $\{x \in \pi \ ; \ x\alpha = x\}$ is a subplane of π. It is called the <u>subplane of fixed elements</u> of α. We denote it by $\pi(1, \alpha)$.

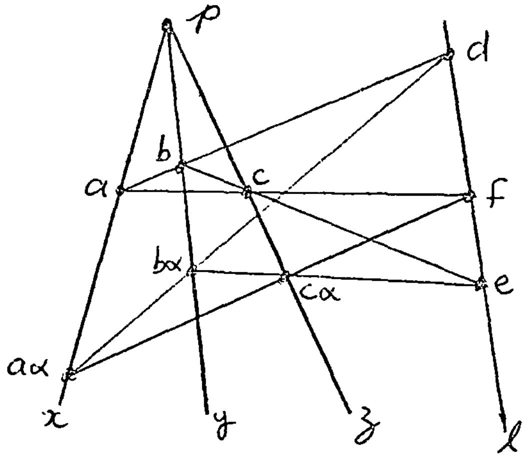
A <u>Baer subplane</u> π_0 of a plane π is a proper subplane of π for which every element of $\pi - \pi_0$ is incident with an element of π_0 (note that, for any subplane π' of π, every element of $\pi - \pi'$ is incident with at most one element of π'). Subplanes of fixed elements of collineations of order 2 are Baer subplanes.
We first prove the well known

<u>Lemma 1.7.4</u> : Suppose π is a non-degenerate plane for which $\pi \neq K(\pi)$, and α is a collineation of π fixing $K(\pi)$ elementwise. If $\pi(1,\alpha)$ is a Baer subplane of π, then $\pi(1,\alpha)$ is non-degenerate.

<u>Proof</u> : Because $K(\pi)$ is fixed elementwise by α, we have $K(\pi) \subseteq \pi(1,\alpha)$. Thus $\pi(1,\alpha)$ has core $K(\pi)$. If $K(\pi)$ is non-empty, then $\pi(1,\alpha)$ is not free and hence is non-degenerate (by 1.6.5(b)). Thus we assume $K(\pi) = \emptyset$; ie. π is free.

Suppose $\pi(1,\alpha)$ is degenerate. By 1.1.2, and because $\pi(1,\alpha)$ is a Baer subplane, $\pi(1,\alpha)$ contains a point p, a line ℓ, all points of π incident with ℓ, and all lines of π incident with p. Choose lines x,y and z incident with p, none equal to ℓ. Choose points a,b,c such that a I x, b I y, c I z, none of a, b, c are incident with ℓ, and a, b, c are not collinear. Because x, y, z are incident with p, they are in $\pi(1,\alpha)$. Hence $a\alpha$ I x, $b\alpha$ I y, $c\alpha$ I z. Because a, b, c are not incident with ℓ, they are not in $\pi(1,\alpha)$. Hence $a\alpha \neq a$, $b\alpha \neq b$ and $c\alpha \neq c$. Let $d = (a \circ b) \circ \ell$, $e = (b \circ c) \circ \ell$, $f = (a \circ c) \circ \ell$. Then $d, e, f \in \pi(1,\alpha)$, as they are incident with ℓ. Since d I a∘b, we have d I $(a \circ b)\alpha$. Similarly e I $(b \circ c)\alpha$ and f I $(a \circ c)\alpha$.

Define a subconfiguration ρ of π by



$\rho = \{p, \ell, x, y, z, z, b, c, a\alpha, b\alpha, c\alpha, d, e, f, a.b, b.c, c.a, (a.b)\alpha, (b.c)\alpha, (c.a)\alpha\}$.

ρ is illustrated opposite. Clearly ρ is a confined configuration. This contradicts our assumption that π is free.

In our next result we generalize to free rank planes a result first proved by Lippi(19) for free planes. The proof given here is due to Row (23, proof of theorem 2).

Theorem 1.7.5 : Let π be a non-degenerate free rank plane for which $\pi \neq K(\pi)$. Any non-degenerate Baer subplane of π containing $K(\pi)$ has core $K(\pi)$ and rank $|\pi|$.

Proof : Let π_0 be a Baer subplane of π containing $K(\pi)$. Then π_0 has core $K(\pi)$ and has free rank (by 1.7.1). It remains to show that it has rank $|\pi|$. Let P be a HF process for π from $K(\pi)$. Then $Q = \pi_0 \cap P$ is a HF process for π_0 from $K(\pi)$ (by 1.5.7). We need to show there are $|\pi|$ elements of π_0 which are Q-isolated or Q-HF.

Choose a point p and a line ℓ for which $p \not I \ell$ and both $p, \ell \notin \pi_0 \cup K(\pi)$

Both p and ℓ are incident with only one element of π_0 and they both have at most two P-bearers. Thus there are at most six lines x I p for which any of $x < p(P)$, $x \circ \ell < \ell(P)$, $x \in \pi_0$ or $x \circ \ell \in \pi_0$. Hence there are $|\pi|$ lines x I p for which $p < x$ (P), $\ell < x \circ \ell$ (P), $x \notin \pi_0$ and $x \circ \ell \notin \pi_0$. For each such x, either $x \circ \ell < x(P)$ or $x < x \circ \ell(P)$. Thus either x is P-free with bearers $x \circ \ell$ and p, or $x \circ \ell$ is P-free with bearers x and ℓ. Thus each pair $(x, x \circ \ell)$ contains a P-free element not in π_0 and having P-bearers not in π_0. There are $|\pi|$ such pairs. Thus there is a set X of $|\pi|$ elements which contains only lines incident with p and points incident with ℓ, and for which each $x \in X$ is not in π_0 and is P-free with P-bearers not in π_0.

Each $x \in X$ is incident with some $\lambda(x) \in \pi_0$, because π_0 is a Baer subplane. Because each $x \in X$ is P-free with P-bearers not in π_0, $\lambda(x)$ is not a P-bearer of x. Therefore $x < \lambda(x)(P)$ $\forall x \in X$. Because each $\lambda(x)$ has a P-bearer not in π_0, it is either Q-HF or Q-isolated. Because the lines of X are concurrent and the points collinear, the mapping $\lambda: X \to \pi_0$ is one-to-one. Hence $\{\lambda(x) \; ; \; x \in X\}$ has $|\pi|$ elements. Thus there are $|\pi|$ Q-HF or Q-isolated elements. Hence π_0 has rank $|\pi|$.

Let π be a non-degenerate free rank plane for which $\pi \neq K(\pi)$. Then all non-degenerate Baer subplanes which contain $K(\pi)$ have core

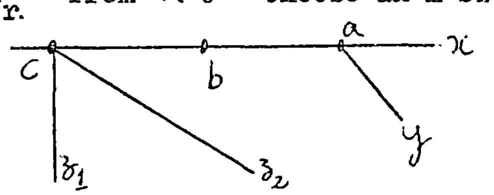
$\kappa(\pi)$ and the same rank $|\pi|$. We therefore have

<u>Corollary 1.7.6</u> : Any two non-degenerate Baer subplanes of a free rank plane π which contain $\kappa(\pi)$ are isomorphic.

We next prove a result which ensures that π_r^{κ} has infinitely many distinct Baer subplanes when $\pi_r^{\kappa} \neq \kappa$. The proof uses a variation of a method due to Row (23, theorem 1).

<u>Proposition 1.7.7</u> : Suppose that $\pi_r^{\kappa} \neq \kappa$. If a is any point of π_r^{κ} and x and y are lines of π_r^{κ} incident with a, then π_r^{κ} has a non-degenerate Baer subplane containing a and x, but not y.

<u>Proof</u> : Choose points b and c distinct from a which are incident with x, and lines z_1 and z_2 distinct from x and incident with c. Let $L = \{a, b, c, x, z_1, z_2\}$. Let P be a HF process for π_r^{κ} from κ. Choose an m such that P_m contains $L \cup \{y\}$ and



all P-isolated and P-HF elements (this is possible, as r is finite). Let $\rho = P_m$. By 1.5.13, $\pi_r^{\kappa} = F(\rho)$.

We obtain the required Baer subplane as the union of an extension process $B = \{B_i \,;\, i \in N\}$, where B_i is a subconfiguration of $F_i(\rho)$ and, for each $i \geq 1$,

(a) B_i is closed in $F_i(\rho)$,

(b) $B_i - B_{i-1} \subseteq F_i(\rho) - F_{i-1}(\rho)$,

(c) each element of $F_{i-1}(\rho)$ is incident with an element of B_i.

Define B_1 to consist of elements of L together with all lines of $F_1(\rho)$ incident with c and all points of $F_1(\rho)$ incident with x. Note that $y \notin B_1$, because $y \not{I}\, c$. Define $B_0 = B_1 \cap \rho$. Then $y \notin B_0$. Clearly (a) and (b) are satisfied when i = 1. We show (c). Let $u \in F_0(\rho) = \rho$. If u is a point, then either $u.c \in \rho$, or $u.c \in F_1(\rho) - \rho$. In either case $u.c \in B_1$ and u I u.c. If u is a line, then $u \text{ I } u.x \in B_1$ (similarly). Thus (c) is true when i = 1.

Assume that B_i has been defined and satisfies (a), (b) and (c) for $0 < i \leq n$. In particular, if $z \in F_n(\rho)$ and z is not incident with any element of B_n, then $z \in F_n(\rho) - F_{n-1}(\rho)$ (by (c)). Let B_{n+1} consist of elements of B_n, together with

(i) elements of $F_{n+1}(\rho) - F_n(\rho)$ incident with two elements of B_n;

(ii) elements $z.\lambda(z)$, where $z \in F_n(\rho) - F_{n-1}(\rho)$ is not incident with any element of B_n, and $\lambda(z) \in L$

is chosen such that z. $\lambda(z) \in F_{n+1}(\rho) - F_n(\rho)$.
The choice of $\lambda(z)$ is possible, because z is incident in $F_n(\rho)$ only with its two ρ-bearers, and L contains three collinear points and three concurrent lines.

Using the induction assumption and the definition of B_{n+1}, it is easily verified that (a), (b) and (c) are satisfied with $i = n + 1$. By induction, B_i is defined $\forall\ i \geq 0$ such that (a), (b) and (c) are satisfied when $i \geq 1$.

Define $\pi = \bigcup_{i=0}^{\infty} B_i$. Then π is a subplane of π_r^{K}, because B_i is closed in $F_i(\rho)$ $\forall\ i \in N$ (by (a)). By (b), we have $\pi \cap F_i(\rho) = B_i$ $\forall\ i \in N$. Hence $\pi \cap \rho = B_0$, which implies both x, a $\in \pi$ and y $\notin \pi$. By (c), π is a Baer subplane. We ensure that π is non-degenerate by choosing $\lambda(z) \in \{a, b\}$ for at least one z (see (ii) above).

<u>Corollary 1.7.8</u> : If $K \neq \pi_r^K$, then π_r^K has a non-degenerate free Baer subplane π for which $\pi \cap K = \emptyset$.

<u>Proof</u> : We use the notation developed in the above proposition and its proof. Choose a, b, c, x, z_1 and z_2 to be not incident with any element of K. Then $B_0 \cap K = \emptyset$. Because $K \subseteq P_0 \subseteq \rho$, we have

$\kappa \cap \pi = \kappa \cap (\pi \cap \rho) = \kappa \cap B_0 = \phi$. Hence π_r^κ has a Baer subplane π for which $\pi \cap \kappa = \phi$. By 1.7.1, π is free.

We note that the Baer subplane constructed in the proof of 1.7.7 does not contain κ when $\kappa \neq \phi$. It is possible to show that for certain a, x and y, and for $r \geq 1$, π_r^κ has a Baer subplane containing κ, a and x, but not y. The construction for this subplane is similar to the construction used in the proof of 1.7.7.

A finite non-empty configuration ρ is <u>almost-confined</u> if it has an element x incident with exactly two elements of ρ, and every other element is incident with at least three elements of ρ. The element x is the <u>vertex</u> of ρ.

<u>Lemma 1.7.9</u> (Dembowski (5 , lemma 3.3)) : If P is a HF process and ρ an almost-confined configuration of $\overline{P}$ with vertex x, then $\rho \subseteq P_0 \cup P(x)$.

<u>Proof</u> : Suppose $\rho \not\subseteq P_0 \cup P(x)$. As ρ is finite, there is a $y \in \rho$ of maximal P-stage with respect to the property $y \notin P_0 \cup P(x)$. Because $x \in P(x)$, $y \neq x$. Thus y is incident with at least three elements of ρ. As y has at most two P-bearers, there is a $z \in \rho$ for which z I y and z is not a P-bearer of y. Because $y \notin P_0$, y is a P-bearer of z (by 1.5.1(a)). Thus $y \in P(z)$. By the maximality of $st_P(y)$, $z \in P(x)$. But $y \in P(z)$ and $z \in P(x)$ imply $y \in P(x)$ (by 1.5.1(c)), a contradiction. Hence $\rho \subseteq P_0 \cup P(x)$.

The following proposition and its corollary demonstrate that when $\pi_r^K \neq K$, π_r^K also possesses properties proved by Dembowski (5 , section 3.3) for non-degenerate planes having empty core (including non-degenerate free planes).

Proposition 1.7.10 : Suppose $\pi_r^K \neq K$. Then, for any integer m, π_r^K has an almost-confined configuration ρ for which $|\rho| > m$ and $\rho \cap K = \phi$. Furthermore, ρ can be chosen to have either a point or line as vertex.

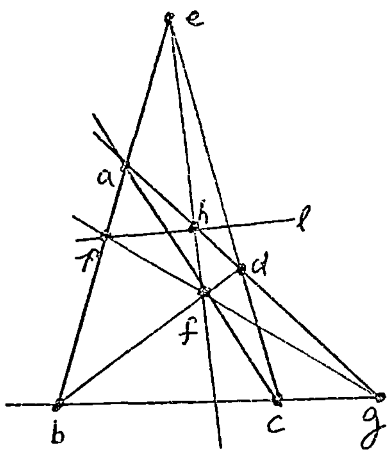
Proof : By 1.7.8, π_r^K has a non-degenerate free subplane π for which $\pi \cap K = \phi$. π has a non-degenerate (free) subplane of rank 8 (by 1.6.12). Thus π_r^K contains a non-degenerate free subplane π' of rank 8 for which $\pi' \cap K = \phi$. It therefore suffices to prove the theorem for $K = \phi$ and r = 8. We prove

(a) π_8 has an almost-confined configuration ;

(b) if ρ is any almost-confined configuration of π_8 and ρ has as vertex the point (resp. line) x, then π_8 has an almost-confined configuration ρ' having a line (resp. point) as vertex and satisfying $|\rho'| = |\rho| + 1$.

Clearly (a) and (b) suffice for the proof of the theorem.

We have $\pi_8 = F(\eta)$, where $\eta = \{a,b,c,d\}$ is a four point.



Define $e = (a.b).(c.d)$, $f = (a.c).(b.d)$,

$g = (a.d).(b.c)$, $h = (e.f).(a.d)$,

$p = (a.b).(f.g)$, $\ell = p.h.$

Then $\rho = \{a, b, c, d, a.b, b.c, c.d, d.a, a.c, b.d, e, f, g, e.f, f.g, p, h, \ell\}$ is an almost-confined configuration of π_8 with vertex ℓ. Thus (a) is proved.

We next show (b). Let ρ be any almost-confined configuration of π_8, and x be its vertex. We may assume that x is a point (if x is a line, use the dual of the following argument). These are two lines u and v of ρ incident with x. ρ has a line ℓ not incident with x. Since ℓ is incident with ≥ 3 points of ρ, there is a $y \in \rho$ for which $y \mathrm{I} \ell$, $y \not\mathrm{I}\, u$, $y \not\mathrm{I}\, v$. Thus $x.y \notin \rho$. Define $\rho' = \rho \cup \{x.y\}$. Then ρ' is an almost-confined configuration with vertex x.y, a line, and $|\rho'| = |\rho| + 1$. Thus (b) is proved.

<u>Corollary 1.7.11</u> (Dembowski (5)): If $\pi_r^K \neq K$, then the full automorphism group of π_r^K has infinitely many orbits outside K.

<u>Proof</u>: It suffices to define a sequence $(x_i)_{i=0}^{\infty}$ of elements of $\pi_r^K - K$ for which, for any $i \neq j$, there is no automorphism α of π_r^K satisfying $x_i \alpha = x_j$. Let P be a HF process for π_r^K from K.

Define x_0 to be the vertex of an almost-confined configuration ρ_0 for which $\rho_0 \cap K = \emptyset$. Assume x_i has been defined for $0 \leq i < n$. Let x_n be the vertex of an almost-confined configuration ρ_n for which $\rho_n \cap K = \emptyset$ and $|\rho_n| \geq |P(x_i)| \ \forall\, i < n$. Such a ρ_n exists, by 1.7.10. By induction, the sequence $(x_i)_{i=0}^{\infty}$ is defined.

Assume $i \neq j$ and α is an automorphism of π_r^K for which $x_i\alpha = x_j$. We may assume $i < j$. Then $\rho_j\alpha^{-1}$ is an almost-confined configuration with vertex x_i. Therefore $\rho_j\alpha^{-1} \subseteq P(x_i) \cup K$ (by 1.7.9). Because $\rho_j \cap K = \emptyset$ and $K\alpha = K$, we have $\rho_j\alpha^{-1} \cap K = \emptyset$. Therefore $\rho_j\alpha^{-1} \subseteq P(x_i)$. Hence $|\rho_j| = |\rho_j\alpha^{-1}| \leq |P(x_i)|$, contradicting the definition of ρ_j. Thus no such α exists.

Finally, we consider the cardinality of non-degenerate free rank planes.

<u>Theorem 1.7.13</u> : Let π be a non-degenerate free rank plane having core K and rank k. Provided $\pi \neq K$, we have $|\pi| = \max(k, |K|, \aleph_0)$.

<u>Proof</u> : We have $\pi = F(\rho)$, where ρ is defined as follows :

If $\kappa = \phi$, then ρ has a line ℓ, two points not incident with ℓ, and k - 6 points incident with ℓ. If $\kappa \neq \phi$, then ρ contains κ and k other points, each incident with exactly one line of κ. By 1.4.3(c), $|F_n(\rho)| \leq |\rho|^{2^n}$ for each $n \in N$. We consider two cases :

(1) <u>$|\kappa|$, k are finite</u> : This implies ρ is finite and hence $F_n(\rho)$ is finite $\forall$ $n \in N$. Therefore $\left| \bigcup_{n=0}^{\infty} F_n(\rho) \right| \leq \aleph_0$.
But π is infinite (by 1.6.3(c)), so $|\pi| \geq \aleph_0$. Hence
$|\pi| = \aleph_0 = \max(k, |\kappa|, \aleph_0)$.

(2) <u>Either $|\kappa|$ or k is infinite</u> : We have $|\rho| = \max(k, |\kappa|)$ and $|F_n(\rho)| = |\rho|$ $\forall$ $n \in N$. Hence $\left| \bigcup_{n=0}^{\infty} F_n(\rho) \right| \leq \max(k, |\kappa|)$.
Hence $|\pi| = \max(k, |\kappa|) = \max(k, |\kappa|, \aleph_0)$.

CHAPTER 2

FINITE COLLINEATION GROUPS

In this chapter, we investigate collination groups G of π_r^K which fix K elementwise and for which all G-orbits are finite. All such groups are finite (this is one of the first results we obtain). As our basic tool, we use the existence, for each G, of a HF process Q for π_r^K from K such that each configuaration of Q is invariant under G.

In 2.1, we prove the existence, for each G, of such a HF process Q, and we obtain some properties of Q. We also show that there is a faithful representation of G as a permutation group of the Q-isolated and Q-HF elements. This representation of G is used in 2.2 to characterize the n for which there is a collineation of π_r^K having order n and fixing K elementwise. It is also used in 2.3 to obtain least upper bounds for $|G|$. For $K = \phi$, these upper bounds were obtained by Alltop (2) for $r \neq 9$ and Sandler (27) for $r = 9$.

In 2.4, we obtain some results concerning the elements of π_r^K fixed by G, including some theorems of Lippi (19). Finally, in 2.5, we obtain upper bounds for the number of conjugacy classes, within the full collineation group of π_r^K, of certain finite collineation groups of π_r^K.

2.1 G-invariant HF Processes

Suppose ρ is a configuration and G is a collineation group of ρ. If Q is a HF process for ρ such that $Q_n G = Q_n$ for each $n \in N$, then Q is <u>G-invariant</u>. If α is a collineation of ρ and $Q_n \alpha = Q_n$ for each

$n \in N$, then Q is <u>α-invariant</u>.

<u>Theorem 2.1.1</u> : If G is a collineation group of π_r^K fixing K elementwise, and all G-orbits are finite, then there exists a G-invariant HF process Q for π_r^K from K.

<u>Proof</u> : Let P be a HF process for π_r^K from K. For each $\alpha \in G$, $P^{(\alpha)} = \{P_n\alpha ; \alpha \in G\}$ is a HF process for π_r^K from K. For $n \in N$, define $R_n = \bigcap_{\alpha \in G} P_n^{(\alpha)}$. Define an extension process Q by $Q_{2n} = R_n$ and $Q_{2n+1} = R_n \cup \{\text{points of } R_{n+1}\}$, for each $n \in N$. By 1.5.8, Q is a HF process for $\overline{Q}$ from K. For $\beta \in G$,
$Q_{2n}\beta = R_n\beta = \left(\bigcap_{\alpha \in G} P_n\alpha\right)\beta = \bigcap_{\alpha \in G} (P_n\alpha\beta) = \bigcap_{\alpha \in G} P_n\alpha = R_n = Q_{2n}$, since $G\beta = G$. Thus $R_n = Q_{2n}$ is invariant under G for all $n \in N$. Since the points of R_{n+1} are permuted by G, Q_{2n+1} is invariant under G for each $n \in N$. Therefore Q is a G-invariant HF process for $\overline{Q}$ from K. It remains to show $\overline{Q} = \pi_r^K$. Clearly $\overline{Q} \subseteq \pi_r^K$. Let $x \in \pi_r^K$. Because xG is finite, $xG \subseteq P_m$ for some m. Thus $xG \subseteq P_m\alpha$ for each $\alpha \in G$. Consequently $x \in xG \subseteq \bigcap_{\alpha \in G} P_m\alpha = R_m = Q_{2m} \subseteq \overline{Q}$. Thus $\overline{Q} = \pi_r^K$, and Q is the required G-invariant HF process for π_r^K from K.

<u>Example</u> : For $r \geq 8$, define a HF process Q for π_r as follows:

$Q_0 = \{a, b, \ell\}$, where a and b are Q-isolated points and ℓ is a Q-isolated line.

$Q_1 = Q_0 \cup \{x_1, \ldots, x_{r-6}\}$, where x_i is a Q-HF point with Q-bearer ℓ, $1 \leq i \leq r-6$.

$Q_n = F_{n-1}(Q_1)$, $n > 1$.

Consider the full collineation group G of Q_1. Clearly $G \cong S_2 \times S_{r-6}$ and $Q_0\ G = Q_0$. By 1.4.4, G extends to a collineation group of $F(Q_1) = \pi_r$ for which $F_n(Q_1)G = F_n(Q_1)\ \forall\ n \geq 0$; i.e. $Q_n\ G = Q_n$ $\forall\ n \geq 1$. Hence Q is a G-invariant HF process for π_r.

For later use, we combine some elementary properties of G-invariant HF processes together in

Proposition 2.1.2 : If G is a collineation group of π_r^{κ} fixing κ elementwise and Q is a G-invariant HF process for π_r^{κ}, then

(a) $st_Q(x) = st_Q(x\alpha)$ for all $x \in \pi_r^{\kappa}$, $\alpha \in G$.

(b) $Q(x\alpha) = Q(x)\alpha$ ....

(c) $\ell_Q(x) = \ell_Q(x\alpha)$ ....

(d) if $Q_0 = \kappa$, then $R = F(\kappa) \cup Q$ is a G-invariant HF process for π_r^{κ} from $F(\kappa)$.

Suppose that I and H are the sets of Q-isolated and Q-HF elements respectively. Then

(e) $IG = I$ and $HG = H$,

(f) if $Q_0 = \kappa$ or $Q_0 = F(\kappa)$, then $G \cong G|_{H \cup I}$.

Proof : (a) Let $x \in \pi_r^{\kappa}$. If $st_Q(x) = 0$ then $x \in Q_0$ and thus $x\alpha \in Q_0$ $\forall \alpha \in G$. Thus $st_Q(x\alpha) = 0 = st_Q(x)\ \forall\ \alpha \in G$. Suppose now that $st_Q(x) = n > 0$. Because $Q_{n-1}\ G = Q_{n-1}$ and $Q_nG = Q_n$, we have $(Q_n - Q_{n-1})G = Q_n - Q_{n-1}$. Therefore $xG \subseteq Q_n - Q_{n-1}$ and $st_Q(x\alpha) = n = st_Q(x)\ \ \forall\ \alpha \in G$.

(b) Suppose $x \in \pi_r^{\kappa}$ and $\alpha \in G$. If $C = \{x_0, x_1, \dots, x\}$ is a Q-chain of x, then $C\alpha$ is a Q-chain of $x\alpha$, since α preserves Q-stage and incidence. Hence $Q(x\alpha) = Q(x)\alpha$, by the definition of a Q-socle.

(c) This also follows immediately from the result that if C is a Q-chain of x, then $C\alpha$ is a Q-chain of $x\alpha$, for any $x \in \pi_r^K$ and $\alpha \in G$.

(d) By 1.6.3 (b) R is a HF process for π_r^K from $F(K)$. Because G fixes K elementwise, G fixes $F(K)$ elementwise (by 1.4.4). Therefore $F(K)G = F(K)$, and $R_n G = F(K)G \cup Q_n G = F(K) \cup Q_n = R_n$ for each n. Thus R is G-invariant.

(e) Suppose $x \in I$ (resp. $x \in H$), $st_Q(x) = n$ and $\alpha \in G$. Then x is incident with no(one) element of Q_n. Since $Q_n\alpha = Q_n$, $x\alpha$ is also incident with no(one) element of Q_n, and we have $st_Q(x\alpha) = n$. Thus $x\alpha$ is also Q-isolated (Q-HF). Hence $IG = I$ and $HG = H$.

(f) By (e), G permutes $H \cup I$. Define $\sigma: G \to G|_{H \cup I}$ by $\alpha\sigma = \alpha|_{H \cup I}$. Clearly σ is a surjective group homomorphism. Suppose $\alpha_1\sigma = \alpha_2\sigma$. Then $\alpha_1|_{H \cup I} = \alpha_2|_{H \cup I}$. Since G fixes both K and $F(K)$ elementwise, G fixes Q_0 elementwise. Hence $\alpha_1|_{Q_0} = \alpha_2|_{Q_0}$. Because $\alpha_1|_{Q_0 \cup H \cup I} = \alpha_2|_{Q_0 \cup H \cup I}$, we have $\alpha_1 = \alpha_2$ (by 1.5.1(e)). σ is therefore an injection. Hence σ is a group isomorphism, and $G \cong G|_{H \cup I}$.

Suppose G is a collineation group of π_r^K fixing K elementwise. If all G-orbits are finite, then a G-invariant HF process Q for π_r^K from K exists, by 2.1.1. By 2.1.2 (f), G is isomorphic to a permutation group of $H \cup I$, where H and I are the sets of Q-HF and

Q-isolated elements respectively. By the definition of rank, $r = 2|I| + H$. Hence $|H \cup I| \leq r$. We have therefore proved

<u>Theorem 2.1.3.</u> : If G is a collineation group of π_r^{κ} fixing κ elementwise and all G-orbits are finite, then G is isomorphic to a permutation group of a set of at most r elements.

We note that, for $\kappa \neq \phi$, this theorem is proved in (12, chapter XI), and it has been used by O'Gorman (21) for the study of finite collineation groups of π_r^{κ}, where $\kappa \neq \phi$. In this thesis, we use it only to prove properties possessed by collineation groups of π_r^{κ} for all κ. The first of these is

<u>Corollary 2.1.4.</u> : Suppose G is a collineation group of π_r^{κ} fixing κ elementwise. Then G is finite if, and only if, every G-orbit is finite.

We note that 2.1.4 does not hold for collineation groups of planes having infinite free rank. For example, $\pi_{\aleph_0}$ is freely generated by a configuration ρ having denumerably many points $\{1,2,\ldots\}$ and no lines. Define a collineation α of ρ by $(2^i+j)\alpha = 2^i + ((j+1) \bmod 2^i)$, $j = 0,1,\ldots,2^i-1$, $i = 0,1,\ldots$. By 1.4.4, α extends uniquely to a collineation of $F(\rho) = \pi_{\aleph_0}$. It has infinite order, because $\alpha|_{\rho}$ has infinite order. However, each element of ρ has a finite α-orbit. By induction, one shows that each element of $F_n(\rho)$ has a finite α-orbit, for all $n > 0$. The group $G = \langle \alpha \rangle$ is therefore infinite, but all G-orbits are finite.

Although our next theorem is not used later in the chapter, it is of interest because it has been the basic tool for the study, by all previous authors, of the finite collineation groups of $\pi_r^{\mathcal{K}}$. It was first proved by Lippi (19) for $\mathcal{K} = \phi$ and G cyclic, and has been generalized to the form stated here by later authors (Alltop (2), Iden (14), and Hughes and Piper (12, chapter XI)). Our proof for it uses the existence of a G-invariant HF process for $\pi_r^{\mathcal{K}}$ from $\mathcal{K}$. One can also simply prove the existence of such a G-invariant HF process using this theorem.

<u>Theorem 2.1.5</u> : If G is a finite collineation group of $\pi_r^{\mathcal{K}}$ fixing $\mathcal{K}$ elementwise, then $\pi_r^{\mathcal{K}}$ has a subconfiguration ρ which freely generates $\pi_r^{\mathcal{K}}$ and is invariant under G. Furthermore, $\rho - \mathcal{K}$ may be assumed finite and ρ minimal.

<u>Proof</u> : By 2.1.1, a G-invariant HF process Q for $\pi_r^{\mathcal{K}}$ from $\mathcal{K}$ exists. Let H and I be the set of Q-HF and Q-isolated elements respectively. Define $\rho_0 = \mathcal{K} \cup \left(\bigcup_{x \in H \cup I} Q(x) \right)$. By 1.5.12, $\pi_r^{\mathcal{K}} = F(\rho_0)$. For $\alpha \in G$, we have $\rho_0\alpha = \rho_0$, because $\mathcal{K}\alpha = \mathcal{K}$, $(H \cup I)\alpha = H \cup I$ and $Q(x)\alpha = Q(x\alpha)$ for $x \in H \cup I$ (using 2.1.2(e) and (b)). Therefore ρ_0 freely generates $\pi_r^{\mathcal{K}}$ and is G-invariant.

Because $H \cup I$ is finite and all Q-socles are finite (1.5.1(b)), $\rho_0 - \mathcal{K}$ is finite. Therefore ρ_0 satisfies all the requirements of the theorem except (possibly) minimality. Since $\rho_0 - \mathcal{K}$ is finite, there is a minimal configuration ρ such that $\mathcal{K} \subseteq \rho \subseteq \rho_0$ and ρ satisfies the requirements of the theorem.

We now prove some technical results concerning the orbits of finite collineation groups of π_r^K. The following result and its proof are due to Dembowski (5, lemma 2.2).

<u>Lemma 2.1.6</u> : Suppose that G is a finite collineation group of π_r^K, Q is a G-invariant HF process for π_r^K, and u is a Q-bearer of v. If $|uG|$ odd or v is Q-HF, then $|uG|$ divides $|vG|$. If $|uG|$ is even, then $\frac{|uG|}{2}$ divides $|vG|$.

<u>Proof</u> : Suppose u is incident with j elements of vG and v is incident with k elements of uG. Because G is transitive on both uG and vG, every element of uG is incident with j elements of vG and every element of vG is incident with k elements of uG. We may count the incidences of the configuration $uG \cup vG$ in two ways, obtaining the equation $j|uG| = k|vG|$. Because uIv, both j, $k \geq 1$. All elements of uG have the same Q-stage as u (by 2.1.1(a)) and therefore have lower Q-stage than v. Thus $k \leq 2$, as v has at most two Q-bearers. If v is Q-HF, then k = 1. The conclusions of the lemma now follow.

We combine the remaining results concerning orbits together in our next proposition. Part (a) of this is an elementary result in the theory of finite permutation groups (see Wielandt (31, theorem 3.2)).

If G is a collineation group of a configuration ρ, and $x \in \rho$, then we denote the subgroup $\langle \alpha \in G; \ x\alpha = x \rangle$ of G by G_x.

<u>Proposition 2.1.7</u> : Suppose G is a finite collineation group of π_r^K and Q is a G-invariant HF process. Then

(a) $|xG| \cdot |G_x| = |G|$ for all $x \in \pi_r^K$,

(b) if $x \in Q(y)$ and $|xG|$ is divisible by an odd m, then so is $|yG|$,

(c) if G has odd order and $x \in Q(y)$, then $|xG|$ divides $|yG|$,

(d) if u is a Q-bearer of v, then either $G_v \subseteq G_u$ or v is Q-free and its Q-bearers form a G_v-orbit.

Proof : (a) The G-orbit of x is $\{x\alpha_1, \ldots, x\alpha_n\}$, where $\{\alpha_1, \ldots, \alpha_n\}$ is a set of left coset representatives for G_x in G. We therefore have $|G| = n|G_x| = |xG| \cdot |G_x|$.

(b) There exists a Q-chain $\{x_0, \ldots, x_n\}$, where $x = x_0$, $y = x_n$, $x_i \mathrel{I} x_{i+1}$ and $st_Q(x_i) < st_Q(x_{i+1})$, $i = 0, \ldots, n-1$. By 2.1.6, either $|x_i G|$ divides $|x_{i+1} G|$, or $\frac{|x_i G|}{2}$ divides $|x_{i+1} G|$, $i=0,1,\ldots,n-1$. Thus, if m divides $|x_i G|$, then it divides $|x_{i+1} G|$, $0 \leq i \leq n-1$. Since m divides $|x_0 G|$, it divides $|x_i G|$, $i = 0,1,\ldots,n-1$. Thus m divides $|yG|$.

(c) By (a), $|xG|$ divides $|G|$ and is thus odd. It now follows from (b) that $|xG|$ divides $|yG|$.

(d) Suppose either that the Q-bearers of v do not form a G_v-orbit, or that v is Q-HF (with Q-bearer u). We show $G_v \subseteq G_u$. Let $\alpha \in G_v$. Then $v\alpha = v$. Because $u \mathrel{I} v$ and $st_Q(u) < st_Q(v)$, we have $u\alpha \mathrel{I} v$ and $st_Q(u\alpha) < st_Q(v)$. Thus $u\alpha$ is a Q-bearer of v. Since we are assuming that the Q-bearers of v do not form a G_v-orbit, or that v has only one Q-bearer u, we have $u\alpha = u$, i.e. $\alpha \in G_u$. Hence $G_v \subseteq G_u$.

2.2 Finite order Collineations

Throughout this section, we let $n = p_1^{s_1} \dots p_k^{s_k}$, where $p_1, \dots, p_k$ are distinct primes and $s_i \geq 1$, $i = 1, \dots, k$. We characterize the n for which there is a collineation of π_r^{κ} having order n and fixing κ elementwise. We first prove an analagous result for permutations of finite sets.

Lemma 2.2.1 : There is a permutation of order n of a finite set X if, and only if, $|X| \geq \sum_{i=1} p_i^{s_i}$.

Proof : We assume first that X has a permutation α of order n. Let $Y = \{x \in X \; ; \; x\alpha = x\}$ and $X' = X - Y$. It suffices to show that $|X'| \geq \sum_{i=1}^{k} p_i^{s_i}$. Let $O_1, \dots, O_m$ be the α-orbits contained in X'. Then $|O_j| \geq 2$ for each j. Clearly $|O_j|$ divides n, so we may factorize $|O_j|$ as follows :

$$|O_j| = \prod_{h=1}^{\ell(j)} p_{j_h}^{r_{j_h}}, \text{ where } 1 \leq r_{j_h} \leq s_{j_h}, \; h = 1, \dots, \ell(j), \quad j = 1, \dots, m.$$

Because $p_{j_h}^{r_{j_h}} \geq 2$ for each h, we may use the inequality $a.b \geq a + b$ (when $a \geq 2$, $b \geq 2$) to obtain $|O_j| \geq \sum_{h=1}^{\ell(j)} p_{j_h}^{r_{j_h}}$. Therefore

$$|X'| = \sum_{j=1}^{m} |O_j| \geq \sum_{j=1}^{m} \sum_{h=1}^{\ell(j)} p_{j_h}^{r_{j_h}} \quad \dots \qquad (i)$$

Because α has order n, n is the least common multiple of $\{|O_j|, \; j = 1, \dots, m\}$. Thus for each $i \in \{1, \dots, k\}$, $p_i^{s_i}$ divides $|O_{j(i)}|$ for some $j(i) \in \{1, \dots, m\}$. Therefore $p_i^{s_i}$ appears in the right side of (i) at least once for each $i \in \{1,\ldots,k\}$. Thus $|X'| \geq \sum_{i=1}^{k} p_i^{s_i}$.

Conversely, assume $|X| \geq \sum_{i=1}^{k} p_i^{s_i}$. Then there is a set $\{O_1,\ldots,O_k\}$ of pairwise disjoint subsets of X such that $|O_i| = p_i^{s_i}$. Suppose $O_i = \{x_1^{(i)},\ldots,x_{p_i^{s_i}}^{(i)}\}$, $i = 1,\ldots,k$. Define a permutation α of X by

$$x\alpha = \begin{cases} x, & \text{if } x \in X - \bigcup_{i=1}^{n} O_i, \\ x_{1+(j \bmod p_i^{s_i})} & \text{if } x = x_j^{(i)} \in \bigcup_{i=1}^{n} O_i . \end{cases}$$

Clearly, α has order n.

Suppose α is a collineation of π_r^{K} having order n and fixing K elementwise. By 2.1.2(f), α is determined by $\alpha|_X$, where X is the set of Q-isolated and Q-HF elements of any α-invariant HF process Q for π_r^{K} from K. We have $|X| \leq r$. We therefore expect our characterization of n to be similar to that of 2.2.1, possibly with $|X| = r$. In fact, our characterization differs from this only because of the geometric nature of α. For example, when $K = \phi$, there must exist Q-isolated elements and this implies $|X| < r$ in this case. Furthermore, there may be incidences between the Q-isolated and Q-HF elements which must be preserved by α.

Before we prove our characterization theorem, we need

Lemma 2.2.2 : Suppose α is a permutation of order n of a finite set X with α-orbits $O_1, \ldots, O_m$. If $|O_i| \geq 2$ for each i, then one of the following is true :

(1) $|X| \geq 2 + \sum_{j=1}^{k} p_j^{s_j}$;

(2) $|O_i|$ does not divide $|O_j|$ for any $i \neq j$.

Proof : Suppose (2) is not true. We show (1) holds. We may assume $|O_1|$ divides $|O_2|$. Let $X' = X - O_1$. The least common multiple (LCM) of $\{|O_i|, i = 1, \ldots, m\}$ is n. Because $|O_1|$ divides $|O_2|$, the LCM of $\{|O_i|, i = 2, \ldots, m\}$ is also n. Thus $\alpha|_{X'}$ is a permutation of order n of X'. By 2.2.1, $|X'| \geq \sum_{j=1}^{k} p_j^{s_j}$.

Thus $|X| = |O_1| + |X'| \geq 2 + |X'| \geq 2 + \sum_{j=1}^{k} p_j^{s_j}$. Hence (1) holds.

Theorem 2.2.3 : There is a collineation of π_r^K having order n and fixing K elementwise if, and only if, $r \geq e(n) + \sum_{i=1}^{k} p_i^{s_i}$, where

$e(n) = 0$ if $K \neq \phi$ and, for $K = \phi$,

$$e(n) = \begin{cases} 6 & \text{if } k = 1,\ p_1 > 5 \text{ or } k = 1,\ p_1 = 5,\ s_1 > 1, \\ 5 & \text{if } k = 1,\ p_1 = 3 \text{ or } k = 1,\ p_1 = 5,\ s_1 = 1, \\ 4 & \text{otherwise.} \end{cases}$$

Proof : We assume first that α is such a collineation of π_r^{K}, and we show $r \geq e(n) + \sum_{i=1}^{k} p_i^{s_i}$.

Let Q be an α-invariant HF process for π_r^{K} from K. Let I and H be the sets of Q-isolated and Q-HF elements respectively. We have $r = 2|I| + |H| = |I| + |H \cup I|$. By 2.1.2(f), $(H \cup I)\alpha = H \cup I$ and $\alpha|_{H \cup I}$ is a permutation of order n of $H \cup I$. There is an $x_i \in H \cup I$ such that $p_i^{s_i}$ divides $|x_i\langle\alpha\rangle|$, $i = 1,\ldots,k$. We note that it is possible that $x_i = x_j$ for $i \neq j$. Let $X = \bigcup_{i=1}^{k} x_i\langle\alpha\rangle$. Then $X\alpha = X$ and $\alpha|_X$ has order n. By 2.2.1, we have

$$|X| \geq \sum_{i=1}^{k} p_i^{s_i}. \qquad \ldots \qquad \text{(i)}$$

We also have

$$r = |I| + |H \cup I| = |I| + |H \cup I - X| + |X| \quad \ldots \quad \text{(ii)}$$

It follows immediately from (i) and (ii) that $r \geq \sum_{i=1}^{k} p_i^{s_i}$. Thus we have proved $r \geq e(n) + \sum_{i=1}^{k} p_i^{s_i}$ when $K \neq \phi$. Henceforth, we assume $K = \phi$.

Because $|x_i\langle\alpha\rangle| \geq p_i^{s_i} \geq 2$ for each i, we may apply 2.2.2. Either (1) or (2) of 2.2.2 is satisfied. Suppose first that (1) is satisfied, i.e. $|X| \geq 2 + \sum_{i=1}^{k} p_i^{s_i}$. If $k = 1$, then

$|X| = |x_1\langle\alpha\rangle| = p_1^{s_1} < 2 + p_1^{s_1}$, contradicting (1). Thus $k \geq 2$ and $e(n) = 4$. From (ii), we have

$$r \geq |I| + |H \cup I - X| + 2 + \sum_{i=1}^{k} p_i^{s_i} .$$

Thus, to show $r \geq e(n) + \sum_{i=1}^{k} p_i^{s_i}$, we need to show that $|I| + |H \cup I - X| \geq 2$. If $|I| \geq 2$, this is so. If $|I| = 1$, then the one Q-isolated element is fixed and is therefore not in X. Hence $I \subseteq H \cup I - X$ and $|H \cup I - X| \geq 1$. Thus $|I| + |H \cup I - X| \geq 2$, as required.

We now suppose that (1) is not satisfied. By 2.2.2, condition (2) is satisfied. We consider two cases (A) and (B).

<u>(A) $k \geq 2$</u> : In this case $e(n) = 4$. From (i) and (ii), we need to show $|I| + |H \cup I - X| \geq 4$. This is so if $|I| \geq 4$. We consider cases $|I| = 1$, 2 and 3.

<u>(a) $|I| = 3$</u> : Let $I = \{u,v,w\}$. If $I \cap (H \cup I - X)$ is non-empty, then $|H \cup I - X| \geq 1$ and thus $|I| + |H \cup I - X| \geq 4$. If $I \cap (H \cup I - X)$ is empty, then $I \subseteq X$ and thus u,v,w forms an α-orbit. Every element of π_r has one of u,v or w in its Q-socle and thus $|x\langle\alpha\rangle|$ is divisible by 3 for each $x \in \pi_r$ (by 2.1.7(b)). Because X satisfies (2), $I = X$. Since π_r is non-degenerate, there are Q-HF elements and these are not in X. Thus $|H \cup I - X| \geq 1$ and $|I| + |H \cup I - X| \geq 4$.

<u>(b) $|I| = 2$</u> : Let $I = \{u,v\}$. Either both $u\alpha = u$ and $v\alpha = v$, or $\{u,v\}$ is an α-orbit. In the former case $I \subseteq H \cup I - X$, which implies $|H \cup I - X| \geq 2$ and thus $|H \cup I - X| + |I| \geq 4$. If $\{u,v\}$ is an α-orbit, then $(u.v)\alpha = u.v$. We may redefine Q, making u.v Q-isolated and u and v Q-HF with bearer u.v. We then have $|I| = 1$.

<u>(c) $|I| = 1$</u> : Let $I = \{u\}$. We may assume u is a point. Since $u\alpha = u$, $u \notin X$. Thus $|H \cup I - X| \geq 1$. Suppose $|H \cup I - X| \leq 2$. Then there exists at most one element of $H - X$ which, if it exists, is fixed by α.

Let U be the set of Q-HF lines with Q-bearer u. Because $\overline{Q} = \pi_r$ is non-degenerate, there is a Q-HF point y with Q-bearer $x \in U$. By 2.1.6, $|x\langle\alpha\rangle|$ divides $|y\langle\alpha\rangle|$. If $x\alpha \neq x$, then both $x, y \in X$ (because any element of $H - X$ is fixed by α). But $|x\langle\alpha\rangle|$ dividing $|y\langle\alpha\rangle|$ would then contradict (2) of 2.2.2. Hence $x\alpha = x$ and $x \in H - X$.

Let V be the set of Q-HF points with Q-bearer x and let $V' = U \cup V - \{x\}$. Then $V' \subseteq X$, since both $x, u \in H \cup I - X$ and $|H \cup I - X| \leq 2$. Therefore V' contains no elements fixed by α. However, by an argument similar to that of the previous paragraph, with V' replacing U, one shows that V' contains elements fixed by α, a contradiction. Hence $|H \cup I - X| \geq 3$ and $|I| + |H \cup I - X| \geq 4$.

<u>(B) $k = 1$</u> : In this case $e(n) = 4, 5$ or 6. X consists of one α-orbit of $p_1^{s_1}$ elements. We must show that $|I| + |H \cup I - X| \geq e(n)$. If $|I| \geq 6$, this is so. We consider the cases $1 \leq |I| \leq 5$.

(a) $\underline{3 \leq |I| \leq 5}$: Since X consists of one α-orbit, and $I\alpha = I$, either $X \subseteq I$ or $X \cap I$ is empty. If $X \cap I$ is empty, then $I \subseteq H \cup I - X$ and thus $|I| + |H \cup I - X| \geq 2|I| \geq 6 \geq e(n)$. Thus we assume $X \subseteq I$. This implies $3 \leq p_1^{s_1} \leq 5$. We consider the possible values of p_1 and s_1. In each case, we show that either $|I| \geq e(n)$ or $|I - X| + |I| \geq e(n)$, both of which imply $|H \cup I - X| + |I| \geq e(n)$.

(i) $p_1 = 2$, $s_1 = 1$: $|I - X| \geq 1$, $|I| \geq 3 \Rightarrow |I - X| + |I| \geq 4 = e(n)$;

(ii) $p_1 = 2$, $s_1 = 2$: $|I| \geq 4 = e(n)$;

(iii) $p_1 = 5$, $s_1 = 1$: $|I| \geq 5 = e(n)$;

(iv) $p_1 = 3$, $s_1 = 1$: If $|I| \geq 4$, then $|I - X| \geq 1$ and $|I| + |I - X| \geq 5 = e(n)$. If $|I| = 3$, then H is non-empty because π_r is non-degenerate. Let $y \in H$. Then $y\langle\alpha\rangle \subseteq H$, and $x \in Q(y)$ for some $x \in I$, so $3 = |x\langle\alpha\rangle|$ divides $|y\langle\alpha\rangle|$ (by 2.1.7(b)). Hence $|H| \geq |y\langle\alpha\rangle| \geq 3$. Since $X \subseteq I$, we have $H \subseteq H \cup I - X$. Thus $|H \cup I - X| \geq 3$ and $|I| + |H \cup I - X| \geq 6 = e(n)$.

(b) $\underline{|I| = 2}$: Let $I = \{u,v\}$. Suppose first that u and v are fixed by α. Then $I \subseteq H \cup I - X$. It is clear that because π_r is non-degenerate, there are at least two other elements of H not in X. Therefore $|H \cup I - X| \geq 4$ and $|H \cup I - X| + |I| \geq 6 \geq e(n)$. If $\{u,v\}$ forms an α-orbit, then we may redefine Q, making u.v Q-isolated and u,v Q-HF with bearer u.v. We then have $|I| = 1$.

(c) $|I| = 1$: Let $I = \{u\}$. Since u is fixed by α, $u \in H \cup I - X$. Thus $|H \cup I - X| \geq 1$. We must show $|H - X| \geq 2$, 3 or 4, depending on p_1 and s_1. An inspection of the few possible cases shows that this is so, because π_r is non-degenerate.

This completes cases (A) and (B), and hence we have shown $r \geq e(n) + \sum_{i=1}^{k} p_i^{s_i}$.

Conversely, assume $r \geq e(n) + \sum_{i=1}^{k} p_i^{s_i}$. We show there is a collineation of π_r^{κ} fixing κ elementwise and having order n. We consider cases (A) to (D). In each, we define a configuration ρ freely generating π_r^{κ} (this can be verified using 1.6.4 and 1.6.3). We then define a collineation α of ρ having order n and fixing κ elementwise. By 1.4.4, α extends to the required collineation of π_r^{κ}. In each case, we let $t = r - e(n) - \sum_{i=1}^{k} p_i^{s_i}$.

(A) $\kappa \neq \emptyset$: In this case $e(n) = 0$. Define ρ to consist of κ and a set X of r points, each incident with one line ℓ of κ. Since $r \geq \sum_{i=1}^{k} p_i^{s_i}$, there is a permutation α' of X having order n (by 2.2.1). Define the collineation α of ρ by $x\alpha = x$ for $x \in \kappa$, and $x\alpha = x\alpha'$ for $x \in X$.

(B) $\kappa = \emptyset$, $k \geq 2$: In this case $e(n) = 4$. Define ρ to consist of

(a) two points x and y,

(b) k sets of lines $L^{(1)}, \ldots, L^{(k)}$ which are pairwise disjoint and

$L^{(i)} = \{ \ell_1^{(i)}, \ldots, \ell^{(i)}_{p_i^{s_i}} \}$, and

(c) if $t > 0$, a set of lines $L = \{ \ell_1, \ldots, \ell_t \}$.

We define the lines of $L^{(1)}$ to be incident only with x, and all other lines of ρ to be incident only with y. Define α by $x\alpha = x$, $y\alpha = y$, $\ell_j^{(i)}\alpha = \ell^{(i)}_{1+(j \bmod p_i^{s_i})}$, $j = 1,\ldots,p_i^{s_i}$, $i = 1,\ldots,k$, and if $t > 0$, $\ell_j\alpha = \ell_j$, $j = 1,\ldots,t$.

<u>(C) $K = \phi$, $k = 1$, $p_1 \leq 3$</u> : We have $e(n) = 4$ or 5 as $p_1 = 2$ or 3. Suppose first that $p_1 = 2$ and $s_1 = 1$ (i.e. $n=2$). Define ρ to have four points x_1, x_2, x_3 and x_4, and if $r > 8$, $r - 8$ lines $\ell_1, \ldots, \ell_{r-8}$, where ℓ_i is incident with x_1 only. Define $x_2\alpha = x_3$, $x_3\alpha = x_2$, and all other elements of ρ to be fixed by α. Suppose now that either $p_1 = 3$ or $s_1 > 1$. Define ρ to have

(a) a point x,

(b) p_1 lines $y_1, \ldots, y_{p_1}$ incident with x,

(c) $p_1^{s_1}$ points $z_1, \ldots, z_{p_1^{s_1}}$, where z_i is incident with $y_{1+[(i-1) \bmod p_1]}$, and

(d) if $t > 0$, t lines $\ell_1', \ldots, \ell_t'$ incident with x.

Define α by $x\alpha = x$, $y_i\alpha = y_{1+(i \bmod p_1)}$, $z_i\alpha = z_{1+(i \bmod p_1^{s_1})}$, and if $t > 0$, $\ell_i'\alpha = \ell_i'$, $i = 1,\ldots,t$.

(D) $\kappa = \phi$, $p_1 \geq 5$, $k = 1$: We have $e(n) = 5$ if $p_1 = 5$ and $s_1 = 1$, and $e(n) = 6$ otherwise. Furthermore $r \geq e(n) + p_1^{s_1} \geq 5 + 5 = 10$. If $r = 10$, $p_1 = 5$ and $s_1 = 1$, then let ρ have five points $x_1, x_2, \ldots, x_5$ and no lines, and define α by $x_i\alpha = x_{1+(i \bmod 5)}$, $i = 1, \ldots, 5$. Suppose now that either $r > 10$ or $p_1 > 5$ or $s_1 > 1$. Define ρ to have

(a) two points x and y,

(b) two lines u and v incident with x,

(c) $p_1^{s_1}$ lines $\ell_1, \ldots, \ell_{p_1^{s_1}}$ incident with y,

(d) if $t > 0$, t lines $\ell_1', \ldots, \ell_t'$ incident with y.

Define α by $x\alpha = x$, $y\alpha = y$, $u\alpha = u$, $v\alpha = v$, $\ell_i\alpha = \ell_{1+(i \bmod p_1^{s_1})}$, $i = 1, \ldots, p_1^{s_1}$ and, if $t > 0$, $\ell_i'\alpha = \ell_i'$, $i = 1, \ldots, t$.

This completes the proof of theorem 2.2.3.

2.3 Maximal Finite Collineation Groups

Our first and main aim in this section is to give a least upper bound for $|G|$, where G is a collineation group of π_r^{κ} fixing κ elementwise and having finite orbits. It follows from 2.1.3 that $|G| \leq r!$. However, $r!$ is not always the least possible upper bound. For $r \geq 8$, define a sequence of numbers m_r by $m_8 = 4!$, $m_{10} = 5!$, and $m_r = 2\left[(r-6)!\right]$ otherwise. We prove

Theorem 2.3.1 : If G is a collineation group of π_r^{κ} fixing κ elementwise and having finite orbits, then $|G| \leq \begin{cases} r!, \text{if } \kappa \neq \emptyset . \\ m_r, \text{if } \kappa = \emptyset . \end{cases}$

These numbers are the least upper bounds for $|G|$.

The proof of 2.3.1 is given later, after some preliminary lemmas. The numbers m_r, $r \geq 8$, were first obtained by Alltop (2) for $r \neq 9$ and by Sandler (27) for r = 9. For their proof that $|G| \leq m_r$ when $\kappa = \emptyset$, both these authors used 2.1.5, together with an extensive case analysis of possible minimal finite configurations ρ for which $\rho G = \rho$ and $\pi_r = F(\rho)$. The number of cases we have to consider is much smaller.

Our first lemma shows that the upper bounds of 2.3.1 are best possible.

Lemma 2.3.2 : If $\kappa \neq \emptyset$, then there is a collineation group of π_r^{κ} fixing κ elementwise and having order r! . For each $r \geq 8$, there is a collineation group of π_r of order m_r.

Proof : Suppose first that $\kappa \neq \emptyset$. Choose a line ℓ of κ. Then $\pi_r^{\kappa} = F(\kappa \cup X)$, where X is a set of r points incident with ℓ and no other line of κ. There is a collineation group G of $\kappa \cup X$ which fixes κ elementwise and such that $G/_X$ is the full permutation group of X. Because $|X| = r$, we have $|G| = r!$. By 1.4.4 , G extends to a collineation group of π_r^{κ} of order r!

Suppose now that $K = \phi$. For $r \geq 8$ and $r \neq 8, 10$, the example given after the proof of 2.1.1 is a collineation group of π_r isomorphic to $S_2 \times S_{r-6}$, which has order $2\left[(r-6)!\right] = m_r$. Suppose $r = 8$ or $r = 10$. Then $\pi_r = F(\rho)$, where ρ has $\frac{r}{2}$ points and no lines. The full permutation group of these points has order $(\frac{r}{2})! = m_r$ and is a collineation group of ρ. By 1.4.4, this extends to a collineation group of π_r of order m_r.

Lemma 2.3.3 : If a and b are positive integers, then $a!\, b! \leq (a+b-1)!$. If, in addition, $a \geq 2$ and $b \geq 2$, then $a!\, b! \leq 2\left[(a+b-2)!\right]$.

Proof : By induction on b for a fixed arbitrary a.

Lemma 2.3.4 : If $n_1, \ldots, n_k$ are positive integers, then

$$\text{(i)} \quad \prod_{i=1}^{k} n_i! \leq \left(\sum_{i=1}^{k} n_i - k + 1\right)! \;.$$

If, in addition, $n_i \geq 2$ for each i, and $k \geq 2$, then

$$\text{(ii)} \quad \prod_{i=1}^{k} n_i! \leq 2\left[\left(\sum_{i=1}^{k} n_i - k\right)!\right] .$$

Proof : By induction on k, using the inequalities in 2.3.3.

Lemma 2.3.5 : Suppose G is a permutation group of a finite set X. If

$(X_i)_{i=1}^{n}$ is a set of pairwise disjoint subsets of X such that

$X_i\, G = X_i$, $1 \leq i \leq n$, and $\bigcup_{i=1}^{n} X_i = X$, then $|G| \leq \prod_{i=1}^{n} |G|_{X_i}|$.

Proof : The map $\sigma : G \to G|_{X_1} \times G|_{X_2} \times \cdots \times G|_{X_n}$ defined by

$\sigma(\alpha) = (\alpha|_{X_1}, \ldots, \alpha|_{X_n})$, $\alpha \in G$, is clearly a group monomorphism.

If G is a permutation group of a set X, then a G-orbit xG is trivial or non-trivial according as $|xG| = 1$ or $|xG| > 1$ respectively.

Lemma 2.3.6 : Suppose G is a permutation group of a finite set X and there are j trivial and k non-trivial G-orbits, where $k \geq 1$. Then

$$|G| \leq \begin{cases} (|X| - j)!, & \text{if } k = 1, \\ 2\left[(|X| - j - k)!\right], & k \geq 2. \end{cases}$$

Proof : Let the G-orbits in X be $O_1, \ldots, O_{j+k}$, where $|O_i| > 1$ for $1 \leq i \leq k$ and, if $j > 0$, $|O_i| = 1$ for $1 + k \leq i \leq j + k$. We have $O_i\, G = O_i$ for each i and $\bigcup_{i=1}^{k+j} O_i = X$. By 2.3.5, $|G| \leq \prod_{i=1}^{j+k} |G|_{O_i}|$.

Because $G|_{O_i}$ is a subgroup of the full permutation group of O_i, we have $|G|_{O_i}| \leq |O_i|!$

Hence $|G| \leq \prod_{i=1}^{j+k} (|O_i|!)$

$$= \prod_{i=1}^{k} (|O_i|!), \text{ since } |O_i| = 1 \text{ for any } 1+k \leq i \leq j+k,$$

$$\leq \begin{cases} |O_1|!, & \text{if } k = 1, \\ 2\left[\left(\sum_{i=1}^{k} |O_i| - k\right)!\right] & \text{if } k \geq 2, \text{ by 2.3.4(ii).} \end{cases}$$

This is the required inequality, because $\sum_{i=1}^{k} |O_i| = |X| - j$.

Lemma 2.3.7 : Suppose that G is a finite collineation group of $\pi_r^{\mathcal{K}}$ fixing $\mathcal{K}$ elementwise, Q is a G-invariant HF process for $\pi_r^{\mathcal{K}}$ from $\mathcal{K}$, and I and H are the sets of Q-isolated and Q-HF elements respectively. If H_1 is a G-orbit of Q-HF elements, then the set B of Q-bearers of elements of H_1 also forms a G-orbit, and each element of B is incident with the same number $b \geq 1$ of elements of H_1. If, in addition, $B \subseteq H \cup I$, then

$$|G| \leq |B|! \ (b!)^{|B|} \ |I - B|! \ |H - H_1 \cup B|! \ .$$

Proof : We first show that B is a G-orbit. Let x and y be elements of B. We show that $x\alpha = y$ for some $\alpha \in G$. By the definition of B, x and y are Q-bearers of some u and v respectively in H_1. As H_1 is a G-orbit, there is an $\alpha \in G$ such that $u\alpha = v$. Since Q is G-invariant, x being a Q-bearer of u implies $x\alpha$ is a Q-bearer of $u\alpha = v$. Thus both $x\alpha$ and y are Q-bearers of v. As v is Q-HF, it has only one Q-bearer. Thus $x\alpha = y$, as required.

Suppose $x \in B$. Let x be incident with b elements of H_1. Then $x\alpha$ is also incident with b elements of H_1 for each $\alpha \in G$, since H_1 is a G-orbit. As B is a G-orbit, $B = xG$. Hence each element of B is incident with the same number b of elements of H_1.

Finally, we prove the inequality. By 2.1.2(f), $G \cong G/_{H \cup I}$. We have $IG = I$, $HG = H$, $H_1G = H_1$ and $BG = B$. Consequently $(H_1 \cup B)G = H_1 \cup B$, $(I - B)G = I - B$ and $(H - H_1 \cup B)G = H - H_1 \cup B$.

By 2.3.5, we have

$$|G| \leq |G|_{H_1 \cup B}| \cdot |G|_{I - B}| \cdot |G|_{H - H_1 \cup B}|$$

$$\leq |G|_{H_1 \cup B}| \cdot |I - B|! \; |H - H_1 \cup B|! \qquad \ldots (i)$$

Now $G|_{H_1 \cup B}$ is a subgroup of the full collineation group of $H_1 \cup B$, a configuration in which elements of B and H_1 are all points and all lines respectively, or the dual, such that each element of H_1 is incident with one element of B and each element of B is incident with b elements of H_1. The full collineation group of this configuration is isomorphic to $S_B \times T$, where S_B is the full permutation group of B and $T = S_b \times S_b \ldots \times S_b$ ($|B|$ times), where S_b is the symmetric group on B letters. Hence $|G|_{H_1 \cup B}| \leq |S_B| \cdot (|S_b|)^{|B|} = |B|!\,(b!)^{|B|}$. Substituting in (i), we obtain the required inequality.

<u>Proof of 2.3.1</u> : It follows from 2.3.2 that the upper bounds given are best possible. It remains to show that they are upper

bounds. It follows from 2.1.3 that $|G| \leq r!$ Thus we only need show that $|G| \leq m_r$ when $K = \phi$. Since $1 \leq m_r$ for each $r \geq 8$, we assume G is non-trivial.

Let Q be a G-invariant HF process for π_r from ϕ with sets of isolated and HF elements I and H respectively. Since $G \cong G|_{H \cup I}$ (by 2.1.2(f)), we show that $|G|_{H \cup I}| \leq m_r$. Let there be j trivial and k non-trivial G-orbits in $H \cup I$. Because G is non-trivial, $k \geq 1$. From 2.3.6,

$$|G|_{H \cup I}| \leq \begin{cases} (|H \cup I| - j)! & \text{if } k = 1, \\ 2\left[(|H \cup I| - j - k)!\right] & \text{if } k \geq 2. \end{cases}$$

By the definition of rank, $|H \cup I| = r - |I|$. Hence, using these inequalities, we obtain $|G| \leq 2\left[(r-6)!\right] \leq m_r$ if either

(a) $|I| + j + k \geq 6$ and $k \geq 2$

or (b) $|I| + j \geq 6$ and $k = 1$.

If $|I| \geq 6$, then either (a) or (b) holds. We consider the cases $1 \leq |I| \leq 5$.

(1) $\underline{|I| = 5}$: If $k \geq 2$, then (a) holds. If $k = 1$ and $j \geq 1$ then (b) holds. Thus we assume $k = 1$ and $j = 0$. This implies that there are no Q-HF elements, and one G-orbit of 5 Q-isolated elements. Therefore $r = 10$ and $|G|_{H \cup I}| = |G|_I| \leq |I|! = 5! = m_{10}$.

(2) $\underline{|I| = 4}$: If $k \geq 2$, then (a) is satisfied. Thus we assume $k = 1$. If $j \geq 2$, then (b) holds. Hence we assume $j \leq 1$. Thus the four Q-isolated elements form a G-orbit and there is at most one Q-HF element which, if it exists, is fixed by G. Hence $r = 8$ or $r = 9$. If $r = 8$, then there is no Q-HF element and $|G|_{H \cup I}| = |G|_I| \leq |I|! = 4! = m_8$.

Suppose now that $r = 9$. There is one Q-HF element x, and $|xG| = 1$. Let x have Q-bearer u. By 2.1.6, $|uG|$ divides $|xG|$ and thus $|uG| = 1$. We have $|G|_{H \cup I}| = |G|_I|$. Suppose first that $G|_I = S_I$, the full permutation group of I. Then G has an element α of order 3 and there is an α-orbit $0 \subseteq I$ such that $|0| = 3$. Because $|uG| = 1$, $u \notin I$. Hence u has at least two elements of I in its Q-socle, at least one of which is in 0. By 2.1.7(b), $|0| = 3$ divides $|u\langle\alpha\rangle|$. But $|u\langle\alpha\rangle| \leq |uG| = 1$, a contradiction. Thus it is not possible for $G|_I = S_I$. Hence $G|_I$ is a proper subgroup of S_I, and $|G|_I|$ properly divides $|S_I| = 24$. Therefore $|G| = |G|_I| \leq 12 = m_9$.

(3) $\underline{|I| = 3}$: Because $r = 2|I| + |H| \geq 8$, we have $|H| \geq 2$. Suppose first that $k = 1$. Then either H or I is fixed elementwise by G, since $HG = H$, $IG = I$ and $H \cup I$ contains only one non-trivial G-orbit. If I is fixed elementwise, or if H is fixed elementwise and $|H| \geq 3$, then $j \geq 3$ and (b) holds. The only other possibility is that H is fixed elementwise and $|H| = 2$. In this case $G|_{H \cup I} = G|_I$, $r = 2|I| + |H| = 8$, and we have $|G| = |G|_I| \leq |I|! = 3! < m_8$.

Suppose now that $k \geq 2$. If $j + k \geq 3$, then (a) is satisfied. Hence we assume $k = 2$ and $j = 0$. It follows that I and H are the two non-trivial G-orbits in $H \cup I$. Let $I = \{x_1, x_2, x_3\}$. As I is a G-orbit, I consists entirely of points or entirely of lines, and $Y = \{x_1 \cdot x_2, x_2 \cdot x_3, x_3 \cdot x_1\}$ is also a G-orbit. Because H is a G-orbit, the set B of Q-bearers of H forms a G-orbit, by 2.3.7. There is an element of H which has a Q-bearer in I or Y. Therefore $B = I$, or $B = Y$. We may assume $B = I$ (if B were Y, then we could redefine Q, making elements of Y Q-isolated and elements of I Q-free). Each element of H is incident with exactly one element of I and, by 2.3.7, each element of I is incident with the same number $b \geq 1$ of elements of H. Therefore $|H| = 3b = r - 2|I| = r - 6$. By

2.3.7 (with $H_1 = H$, $B = I$ and $b = \frac{r-6}{3}$), we have

$$|G| \leq 3! \left[\left(\frac{r-6}{3}\right)! \right] \leq (r-6)! \quad \text{(by 2.3.4(i))}$$

$$\leq m_r \,.$$

<u>(4) $|I| = 2$</u> : Let $I = \{u, v\}$. Suppose first that G fixes both u and v. Then $j \geq 2$. If $k \geq 2$, then (a) is satisfied. Assume $k = 1$, i.e. there is only one non-trivial G-orbit of Q-HF elements. In order for $\bar{Q}$ to be non-degenerate, it is clear that there exist at least two Q-HF elements fixed by G. Thus $j \geq 4$ and (b) is satisfied. Suppose now that G does not fix u and v. Then I forms a G-orbit and u and v are either both points or both lines. We redefine Q, making $u \cdot v$ isolated and u and v hyperfree with Q-bearer $u \cdot v$. We then have only one Q-isolated element.

<u>(5) $|I| = 1$</u>: Let $I = \{u\}$. Then u is fixed by G. Thus $j \geq 1$. Suppose first that $k = 1$, i.e. there is only one non-trivial G-orbit of Q-HF elements. An inspection of the few possible cases shows that, in order for $\bar{Q}$ to be non-degenerate, there are at least four Q-HF elements fixed by G. Thus $j \geq 5$ and (b) is satisfied.

Suppose now that $k \geq 2$. If $j + k \geq 5$ then (a) is satisfied, so we assume $j + k \leq 4$. Since $j \geq 1$, we have $k = 2$ or $k = 3$. In either case $j \leq 2$, and thus there is at most one Q-HF element fixed by G. Consequently, we have either

(i) there exist non-trivial G-orbits H_1 and H_2 of Q-HF elements such that elements of H_1 have u as Q-bearer, and there is an element of H_2 having an element of H_1 as Q-bearer, or

(ii) Q may be redefined such that (i) is satisfied.

We therefore assume that (i) is satisfied. By 2.3.7, the Q-bearers of H_2 form a G-orbit B. Since there is an element in H_2 with Q-bearer in H_1, we have $B = H_1$. Each element of H_1 is incident with the same number $b \geq 1$ of elements of H_2, and

$$\text{(iii)} \quad |G| \leq |H_1|\,!\,(b!)^{|H_1|}\,|H - H_1 \cup H_2|\,!\,|I|\,! \;.$$

Let $h = |H_1|$. As each element of H_2 is incident with one element of H_1, we have $|H_2| = b|H_1| = bh$. We also have $|I| = 1$, $|H| = r - 2$ and $|H_1| = h \geq 2$. Substituting these in (iii) we obtain

(iv) $|G| \leq h!\ (b!)^h\ (r-2-h-bh)!,\ \ h \geq 2.$

Suppose first that $r - 2 - h - bh \geq 2$. Then from (iv)

$|G| \leq 2\left[(r-4-h)!\right]$ (using 2.3.4(ii) if $b \geq 2$, and the second inequality of 2.3.3. if $b = 1$).

Since $h \geq 2$ we have $|G| \leq 2\left[(r-6)!\right] \leq m_r$ in this case. Assume now that $r - 2 - h - bh \leq 1$. Then from (iv), we have

$$|G| \leq h!\ (b!)^h$$

$$\leq \begin{cases} h!, \text{ if } b = 1, \\ 2\left[(hb-1)!\right] \text{ if } b \geq 2 \text{ (by 2.3.4(ii))} \end{cases}$$

$$\leq \begin{cases} \left(\frac{r-2-x}{2}\right)!, \quad b = 1, \\ 2\left[(r-3-x-h)!\right], \quad b \geq 2, \end{cases}$$

where $x = r - 2 - h - bh$. It is easily shown that $\left(\frac{r-2-x}{2}\right)! \leq m_r$ for $x \leq 1$. Thus $|G| \leq m_r$ if $b = 1$. If $b \geq 2$ and $x + h \geq 3$, then $|G| \leq 2\left[(r-6)!\right] \leq m_r$. The only other possibility is that $b \geq 2$ and $x + h \leq 2$. This implies $x = 0$ and $h = 2$, since $h = |H_1| \geq 2$. Hence $0 = x = r - 2 - 2b - 2$ and $b = \frac{r-4}{2}$. Substituting in (iv), we have $|G| \leq 2\left[\left(\frac{r-4}{2}\right)!\right]^2$. One shows that $2\left[\left(\frac{r-4}{2}\right)!\right]^2 \leq m_r$ by inspection for $r = 8$, 9 and 10, and by induction for $r > 10$.

This completes the proof of 2.3.1.

In section 2.5, we show that when $\kappa \neq \phi$, all maximal finite collineation groups of π_r^κ fixing κ elementwise are conjugate. This is not true when $\kappa = \phi$. In fact, maximal finite collineation groups of π_r do not even have the same order. For example, $\pi_{11} = F(\rho)$, where ρ has two points x and y, three lines incident with x but not y, and four lines incident with y and not x. The full collineation group G_1 of ρ has order $(3!)(4!) = 144$. By 1.4.4, G_1 extends to a collineation group of π_{11} of order 144. Assume G_1 is not maximal. Then G_1 is a proper subgroup of a finite collineation group G of π_{11}. Thus $|G_1|$ divides $|G|$ properly. Hence $|G| \geq 2 \times 144 = 288$. But from 2.3.1, $|G| \leq 2.5! = 240$, a contradiction. Thus G_1 is maximal but does not have order $m_{11} = 240$. By 2.3.2, π_{11} has a collineation group of order 240. Thus maximal finite collineation groups of π_r do not have the same order.

By the above example, maximal finite collineation groups of π_r are not necessarily isomorphic. Iden (16) has shown that, for $r \geq 20$, there are at least p(r-19) isomorphism classes of such groups, where p(r-19) is the number of unrestricted partitions of the integer r - 19. Because p(k) tends asymptotically to $\frac{1}{4k\sqrt{3}} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$, the number of isomorphism classes of maximal finite collineation groups of π_r increases rapidly with r.

We note that, by 2.1.3, any finite collineation group of π_r is a subgroup of S_r, the symmetric group of degree r. Thus the number of isomorphism classes of such subgroups (including maximal ones) is at most the number of isomorphism classes of subgroups of S_r.

2.4 Subplanes of Fixed Elements

If α (resp. G) is a collineation (collineation group) of a plane π, then the set of elements of π fixed by α (resp. G) forms a subplane of π, denoted by $\pi(1,\alpha)$ (resp. $\pi(1,G)$). In this section, we consider two questions. Given a finite collineation group G of π_r^K fixing K elementwise, what is the nature of $\pi_r^K(1,G)$? Secondly, which subplanes of π_r^K are $\pi_r^K(1,G)$ for some such G?

Lippi (18,19) has shown that if α is a collineation of π_r of prime power order p^s, where $s > 0$, then

(a) if $p \neq 2$, then $\pi_r(1,\alpha)$ is a possibly degenerate free plane of finite rank r', where $r' \equiv r(\bmod\ p)$;

(b) if $p = 2$, then $\pi_r(1,\alpha) \cong \pi_{\aleph_0}$.

These results, in a more general form, as well as others, are proved in this section.

Theorem 2.4.1 : If G is a non-trivial finite collineation group of π_r^K fixing K elementwise, then $\pi_r^K(1,G)$ is a free rank plane having core K and rank r_1, where either

(1) $r_1 = |\pi_r^{\kappa}|$ and $\pi_r^{\kappa}(1,G)$ is non-degenerate, or

(2) $0 \leq r_1 \leq r - 3$ and $\pi_r^{\kappa}(1,G)$ is possibly degenerate.

Before proving 2.4.1, we prove a series of lemmas, some of which are needed in the proofs of later theorems.

Lemma 2.4.2 : If α is a collineation of a non-degenerate free rank plane π fixing $\kappa(\pi)$ elementwise and having order 2, then $\pi(1,\alpha)$ is a non-degenerate free rank plane having core $\kappa(\pi)$ and rank $|\pi|$.

Proof : $\pi(1,\alpha)$ contains $\kappa(\pi)$, since α fixes $\kappa(\pi)$ elementwise. It is therefore a free rank plane (by 1.7.1) and has core $\kappa(\pi)$. If $x \in \pi - \pi(1,\alpha)$ then $x.x\alpha$ is fixed by α, since α has order two, and $x \; I \; x.x\alpha$. Thus $\pi(1,\alpha)$ is a Baer subplane of π. The result now follows from 1.7.4 and 1.7.5.

Lemma 2.4.3 : If G is a finite collineation group of π_r^{κ} fixing κ elementwise and having order 2^j, where $j > 0$, then $\pi_r^{\kappa}(1,G)$ is a non-degenerate free rank plane having core κ and rank $|\pi_r^{\kappa}|$.

Proof : We proceed by induction on j. If $j = 1$, then $G = \{1, \alpha\}$, where α has order two. Thus $\pi_r^{\kappa}(1,G) = \pi_r^{\kappa}(1,\alpha)$, and the result follows from 2.4.2. Suppose that the lemma has been proved for $1 \leq n < j$ and that $|G| = 2^j$. Then G has a normal subgroup G_0 of order 2^{j-1} (see, for example, (11)). Define $\pi = \pi_r^{\kappa}(1,G_0)$. By the induction assumption, π is a free rank plane having core κ

and rank $|\pi_r^{\kappa}|$. Clearly $\pi_r^{\kappa}(1,G) \subseteq \pi$. If $\pi_r^{\kappa}(1,G) = \pi$, then we are finished. Suppose $\pi_r^{\kappa}(1,G) \subsetneq \pi$. From the normality of G_o in G, it follows that $\pi G = \pi$ (because $x \in \pi$, $\alpha \in G \Rightarrow (x\alpha)G_o = (xG_o)\alpha = x\alpha \Rightarrow x\alpha \in \pi$). Hence $\pi_r^{\kappa}(1,G) = \pi(1,G|_{\pi})$. Let β be a coset representative for G_o in G. Then $G = G_o \cup \beta G_o$ and $\beta^2 \in G_o$. This implies that $G|_{\pi} = \{1, \beta|_{\pi}\}$ and that $\beta|_{\pi}$ has order 2. Hence $\pi_r^{\kappa}(1,G) = \pi(1, \beta|_{\pi})$, and it follows from 2.4.2 that $\pi_r^{\kappa}(1,G)$ has core κ and rank $|\pi|$. By 1.7.13, $|\pi| = |\pi_r^{<}|$. By induction, the lemma is true for all j.

We note that 2.4.3 was first proved by Lippi (19) for the case of G cyclic and $\kappa = \phi$. Our proof for 2.4.3 is based upon that of Lippi, except that we prove directly in 1.7.5 that a non-degenerate Baer subplane of π_r^{κ} containing κ has rank $|\pi_r^{\kappa}|$ (Lippi proved that a maximal proper non-degenerate subplane of π_r is not finitely generated and used a result of Baer (3) that Baer subplanes are maximal).

We now consider finite collineation groups G for which $|G|$ is divisible by an odd number.

Lemma 2.4.4 : Suppose that G is a finite collineation group of π_r^{κ} fixing κ elementwise such that $|G|$ is divisible by an odd number > 1.

Let C be the set of odd order subgroups of G and $\pi' = \bigcap_{P \in C} \pi_r^{K}(1,P)$.

Then

(a) $\pi' G = \pi'$;

(b) if $x \in \pi_r^{K} - \pi'$, then $|xG| \geq 3$;

(c) if Q is a G-invariant HF process for π_r^{K}, then

$Q(x) \subseteq \pi'$ for each $x \in \pi'$;

(d) π' is a free rank subplane of π having core K and rank r', where $0 \leq r' \leq r - 3$;

(e) if $G' = G|_{\pi'}$, then $\pi_r^{K}(1,G) = \pi'(1,G')$ and $|G'| = 2^i$, for some $i \geq 0$.

Proof : (a) It suffices to show that $x \in \pi'$, $\alpha \in G$ and $\beta \in \bigcup_{P \in C} P$ imply $x\alpha\beta = x\alpha$. Because β has odd order, so has $\alpha\beta\alpha^{-1}$. Thus $\alpha\beta\alpha^{-1} \in \bigcup_{P \in C} P$. This implies $x\alpha\beta\alpha^{-1} = x$, i.e. $x\alpha\beta = x\alpha$, as required.

(b) Suppose $x \in \pi_r^{K} - \pi'$. Then there is a $P \in C$ of odd order ≥ 3 for which $|xP| > 1$. By 2.1.7(a), $|xP|$ divides $|P|$. Hence $|xP| \geq 3$, and $|xG| \geq |xP| \geq 3$.

(c) It suffices to show that $Q(x) \subseteq \pi_r^{K}(1,P)$ for each $P \in C$ and each $x \in \pi_r^{K}(1,P)$. Let $u \in Q(x)$. By 2.1.7(c), $|uP|$ divides $|xP|$, because P has odd order. Because $|xP| = 1$, we have $|uP| = 1$. Hence $u \in \pi_r^{K}(1,P)$. Thus $Q(x) \subseteq \pi_r^{K}(1,P)$, as required.

(d) π' is a subplane of π_r^{κ}, because the intersection of any set of subplanes is a subplane. Since κ is fixed elementwise by G, $\kappa \subseteq \pi_r^{\kappa}(1,P)$ for each $P \in C$. Thus $\kappa \subseteq \pi'$ and π' has core κ. By 1.7.1, π' has free rank. It remains to show that its rank r' satisfies $0 \leq r' \leq r - 3$.

Let Q be a G-invariant HF process for π_r^{κ} from κ. The extension process $S = \pi' \cap Q$ is a HF process for π' from κ (by 1.5.7). By (c), $S(x) = Q(x)$ for all $x \in \pi'$. Thus every S isolated (resp. S-HF) element is also Q-isolated (Q-HF). Thus $r' \leq r$. Because $|G|$ is divisible by an odd prime, there is a non-trivial $P \in C$ (for example, a Sylow subgroup). Since Q is P-invariant, there is at least one non-trivial P-orbit xP of Q-isolated and Q-HF elements (by 2.1.2(f)). Because $|xP|$ divides $|P|$, we have $|xP| \geq 3$. Because $\pi' G = \pi'$, we have $xP \cap \pi' = \phi$. Hence there are at least three Q-isolated or Q-HF elements which are not S-isolated or S-HF. Therefore $r' \leq r - 3$.

(e) Because $\pi_r^{\kappa}(1,G) \leq \pi_r^{\kappa}(1,P)$ for each $P \in C$, we have $\pi_r^{\kappa}(1,G) \leq \pi'$. Hence $\pi_r^{\kappa}(1,G) = \pi'(1,G|_{\pi'}) = \pi'(1,G')$. It remains to show that $|G'| = 2^i$ for some $i \geq 0$. If $\alpha \in G$, then α has order $2^j m$ for some odd m and integer $j \geq 0$. Thus α^{2^j} has odd order m and $\alpha^{2^j} \in P$ for some $P \in C$. Hence $\pi' \subseteq \pi_r^{\kappa}(1,P) \subseteq \pi_r^{\kappa}(1, \alpha^{2^j})$, which implies $x\alpha^{2^j} = x$ for all $x \in \pi'$. Therefore $\alpha|_{\pi'}$ has order dividing 2^j, i.e. $\alpha|_{\pi'}$ has order a power of two. This is true for each $\alpha \in G$. Hence $G' = G|_{\pi'}$ is a 2-group and $|G'| = 2^i$ for some $i \geq 0$.

<u>Proof of 2.4.1</u> : Because G fixes κ elementwise, we have $\kappa \subseteq \pi_r^\kappa(1,G)$. Hence $\pi_r^\kappa(1,G)$ has core κ and free rank (by 1.7.1). It remains to show that either (1) or (2) is satisfied. If $|G| = 2^j$ for any $j > 0$, then (1) is satisfied. Thus we assume $|G|$ is divisible by an odd number > 1. We use the notation and results of 2.4.4. In particular, $\pi_r^\kappa(1,G) = \pi'(1,G')$, where $|G'| = 2^i$ for some $i \geq 0$. We consider three cases

(i) <u>$|G'| = 1$</u> : In this case $\pi_r^\kappa(1,G) = \pi'(1,G') = \pi'$. By 2.4.4(d), (2) is satisfied.

(ii) <u>$|G'| > 1$, π' is degenerate</u> : By 1.6.5, π' is a free plane and is finite, as it has rank $\leq r - 3$ (by 2.4.4(d)). Because $\pi'(1,G')$ is a proper subplane of π', it has rank r_1, where $0 \leq r_1 < r - 3$ (by 1.6.5(d)). Thus (2) is satisfied.

(iii) <u>$|G'| > 1$, π' is non-degenerate</u> : By 2.4.4(d), $\pi' \cong \pi_{r'}^\kappa$. Since $|G'| = 2^i$, $i > 0$, $\pi'(1,G')$ has rank $|\pi_{r'}^\kappa|$ and is non-degenerate, by 2.4.3. Because both r and r' are finite, $|\pi_r^\kappa| = \max(\kappa, \aleph_0) = |\pi_{r'}^\kappa|$ (using 1.7.13). Hence $\pi'(1,G') = \pi_r^\kappa(1,G)$ has rank $|\pi_r^\kappa|$, and (1) is satisfied.

This completes the proof of 2.4.1.

We note that 2.4.1 is not a best possible result. For example, there are no free planes of rank one, so $\pi_r(1,G)$ does not have rank one for any G. By an inspection of cases one can also show, for example, that $\pi_{13}(1,G)$ does not have rank 0 for any finite collineation group G of π_{13}. We show later that 2.4.1 is a best possible result for $K \neq \phi$ and $r \geq 2$, and that it is also very close to best possible when $K = \phi$ and r is sufficiently large.

We next prove a lemma which is needed later, when we characterize non-degenerate subplanes of π_r^K which have finite rank and are $\pi_r^K(1,G)$ for some finite collineation group G fixing K elementwise.

Lemma 2.4.5 : Suppose G is a non-trivial finite collineation group of π_r^K fixing K elementwise and that $\pi_r^K(1,G)$ has finite rank and is non-degenerate. If Q is a G-invariant HF process for π_r^K, then $Q(x) \subseteq \pi_r^K(1,G)$ for each $x \in \pi_r^K(1,G)$.

Proof : By 2.4.3, $|G| \neq 2^j$ for any $j > 0$. Hence $|G|$ is divisible by an odd number > 1. We use the notation of 2.4.4. If $|G'| > 1$, then it follows from cases (ii) and (iii) of the proof of 2.4.1 that $\pi_r^K(1,G)$ is either degenerate or has infinite rank, which is not so. Hence $|G'| = 1$ and $\pi_r^K(1,G) = \pi'$. The result now follows from 2.4.4(c).

Our next theorem gives more information about $\pi_r^K(1,G)$ for particular $|G|$.

Theorem 2.4.6 : Suppose that G is a finite collineation group of π_r^K fixing K elementwise, and that $\pi_r^K(1,G)$ has rank r_1.

(a) If $|G| = 2^j$, $j > 0$, then $r_1 = |\pi_r^K|$.

(b) If $|G| = p_1^{s_1} \dots p_k^{s_k}$, where p_i is an odd prime > 1 and $s_i \geq 1$, $i = 1,\dots,k$, then there exist integers $t_1,\dots,t_k$ such that $r-r_1 = \sum_{i=1}^{k} t_i p_i$.

Proof : By 2.4.3 we need only prove (b). We use the notation and results of 2.4.4. Because G is an odd order subgroup of itself, we have $\pi_r^K(1,G) = \pi'$. Let Q be a G-invariant HF process for π_r^K from K, and I and H be the sets of Q-isolated and Q-HF elements respectively. By 1.5.7, the extension process $S = \pi_r^K(1,G) \cap Q$ is a HF process for $\pi_r^K(1,G)$ from K. Let S have isolated elements I_S and HF elements H_S. Because $Q(x) \subseteq \pi_r^K(1,G)$ for each $x \in \pi_r^K(1,G)$ (by 2.4.4(c)), $S(x) = Q(x)$ for each $x \in \pi_r^K(1,G)$. Thus $H_S \subseteq H$ and $I_S \subseteq I$, and

$$r - r_1 = (2|I| + |H|) - (2|I_S| + |H_S|)$$

$$= 2|I - I_S| + |H - H_S| \qquad \dots\dots\dots (i)$$

By 2.1.2(e), $IG = I$ and $HG = H$. Also, $I_S G = I_S$ and $H_S G = H_S$. Thus $(H-H_S)G = H - H_S$ and $(I-I_S)G = I - I_S$. The sets $I - I_S$ and $H - H_S$ may therefore be partitioned into G-orbits, each of cardinality > 1 and dividing $|G|$ (by 2.1.7(a)). Thus there exist non-negative integers $t_1,\dots,t_k$ for which

$$2\,|I - I_S| + |H - H_S| = \sum_{i=1}^{k} t_i\, p_i \qquad \text{........(ii).}$$

The result now follows from (i) and (ii).

<u>Corollary 2.4.7 (Lippi (19))</u> : If α is a collineation of π_r having order p^s, where p is a prime and $s > 0$, then

(a) if $p = 2$, then $\pi_r(1, \alpha) \cong \pi_{\aleph_0}$;

(b) if $p \neq 2$, then $\pi_r(1, \alpha)$ is a (possibly degenerate) free plane of rank r_1, where $r_1 \equiv r \pmod{p}$.

We next give a necessary and sufficient condition for $\pi_r^{\kappa}(1, G)$ to have infinite rank (i.e. for (1) of 2.4.1 to be satisfied).

<u>Theorem 2.4.8</u> : Let G be a finite collineation group of π_r^{κ} fixing κ elementwise. Then $\pi_r^{\kappa}(1, G)$ has infinite rank if, and only if, there are infinitely many G-orbits of cardinality two.

<u>Proof</u> : Let $X = \left\{ x \in \pi_r^{\kappa} \,;\, |xG| = 2 \right\}$. Assume first that X is infinite. If $|G| = 2^j$, some $j > 0$, then $\pi_r^{\kappa}(1, G)$ has infinite rank (by 2.4.3). Thus we assume $|G|$ is divisible by an odd number > 1. We use the notation and results of 2.4.4. By 2.4.4(b), $X \subseteq \pi'$. Thus π' is infinite and $G' = G/_{\pi'}$ is non-trivial. By 2.4.4(e), $|G'| = 2^i$, some $i > 0$. Since π' is infinite but has

finite rank r' (by 2.4.4(d)), π' is non-degenerate (by 1.6.5(c)). Hence $\pi'(1,G')$ has infinite rank $|\pi'|$, by 2.4.3. Since $\pi_r^{\kappa}(1,G) = \pi'(1,G')$, $\pi_r^{\kappa}(1,G)$ has infinite rank.

Conversely, assume $\pi_r^{\kappa}(1,G)$ has infinite rank. Let Q be a G-invariant HF process for π_r^{κ} from κ. The extension process $S = \pi_r^{\kappa}(1,G) \cap Q$ is a HF process for $\pi_r^{\kappa}(1,G)$ from κ (by 1.5.7). Because $\pi_r^{\kappa}(1,G)$ has infinite rank, there are infinitely many S-isolated or S-HF elements. By 2.1.7(d), either the Q-bearers of an $x \in \pi_r^{\kappa}(1,G)$ are also in $\pi_r^{\kappa}(1,G)$, or they form a G-orbit of two elements (because $G_x = G$). Thus if x is S-isolated or S-HF, then either x is also Q-isolated or Q-HF, or x is Q-free and the Q-bearers of x form a G-orbit of two elements. There are infinitely many S-HF or S-isolated elements, but only finitely many Q-HF or Q-isolated elements. Thus there are infinitely many $x \in \pi_r^{\kappa}(1,G)$ for which the Q-bearers of x form a G-orbit of two elements. Thus there are infinitely many G-orbits of cardinality two.

We note that 2.4.8 is not true for planes having infinite free rank. For example, define as the configuration ρ a line ℓ, three points x_1, x_2 and x_3 not incident with ℓ, and a denumerable set of points $\{y_1, y_2, \dots\}$, each incident with ℓ. Clearly ρ has rank $\aleph_0$. Therefore $F(\rho) = \pi_{\aleph_0}$. Let $G = \langle \alpha \rangle$, where α is the collineation of ρ defined by $\ell\alpha = \ell$, $x_i\alpha = x_{(i+1)\mathrm{mod}3}$, $i = 1,2,3$, $y_j\alpha = y_{j+1}$, $y_{j+1}\alpha = y_j$, for each $j \equiv 0(\mathrm{mod}2)$. By 1.4.4, G extends to a collineation group of order 6 of $\pi_{\mathcal{N}_0}$ for which $F_n(\rho)G = F_n(\rho)$ for each $n \in N$. Therefore $F = \{F_n(\rho)\ ;\ n \in N\}$ is a G-invariant HF process for $\pi_{\mathcal{N}_0}$. Every element of $\pi_{\mathcal{N}_0} - \rho$ has one of x_1, x_2 or x_3 in its F-socle. Since $3 = |x_1 G| = \{x_1,x_2,x_3\}$, $|xG|$ is divisible by 3 for each $x \in \pi_{\mathcal{N}_0} - \rho$ (by 2.1.7(b)). Thus $\pi_{\mathcal{N}_0}(1,G) \subseteq \rho$. Hence $\pi_{\mathcal{N}_0}(1,G) = \{\ell\}$. However, there are infinitely many points x incident with ℓ for which $|xG| = 2$. Thus 2.4.8 is not true for finite collineation groups of $\pi_{\mathcal{N}_0}$.

We now consider the second of the questions mentioned in the introduction to this section. Firstly, we characterize non-degenerate subplanes of π_r^K which have finite rank, core K, and are $\pi_r^K(1,G)$ for some G.

<u>Theorem 2.4.9</u> : Suppose π is a non-degenerate subplane of π_r^K containing K and having finite rank r_1. Then $\pi = \pi_r^K(1,G)$ for some non-trivial finite collineation group G of π_r^K if, and only if, both $0 \leq r_1 \leq r - 3$ and there is a HF process P for π_r^K from π.

<u>Proof</u> : Firstly, suppose $\pi = \pi_r^K(1,G)$ for some non-trivial finite collineation group G. Then G fixes K elementwise, since

$\kappa \subseteq \pi$. Hence $0 \leq r_1 \leq r - 3$, by 2.4.1. Let Q be a G-invariant HF process for π_r^κ from κ. By 2.4.5, $Q(x) \subseteq \pi$ for each $x \in \pi$. We have $Q_0 = \kappa \subseteq \pi$. Hence (i) and (ii) of 1.5.11 are satisfied (with $\rho = \pi$). By 1.5.11(d), the extension process $P = \pi \cup Q$ is a HF process for π_r^κ from π. Note that since $\pi G = \pi$ and Q is G-invariant, so is P.

Conversely, assume that $0 \leq r_1 \leq r - 3$ and that there is a HF process P for π_r^κ from π. Choose a line ℓ of π. By 1.6.6, there is a HF process Q for π_r^κ from π such that all Q-HF elements are points with Q-bearer ℓ, and there are no Q-isolated elements. Because $\kappa \subseteq \pi$, there is a HF process R for π from κ with $r_1 = r(R)$. Then R + Q is a HF process for π_r^κ from κ, and $r = r(R+Q) = r(R) + r(Q) = r_1 + r(Q)$. Thus $r(Q) = r - r_1$ and there are $k = r - r_1$ Q-HF points $\{x_1, \ldots, x_k\}$ with Q-bearer ℓ. We may assume that $Q_1 = \pi \cup \{x_1, \ldots, x_k\}$. By 1.5.13, $\pi_r^\kappa = F(Q_1)$. Define a collineation group G of Q_1 by $xG = \{x\}$ for each $x \in \pi$, and $G|\{x_1, \ldots, x_k\} \cong S_k$, the symmetric group on k letters. By 1.4.4, G extends to a finite collineation group of π_r^κ for which $F_n(Q_1)G = F_n(Q_1)$, $\forall\, n \geq 0$.

We now show that $\pi_r^\kappa(1,G) = \pi$. By definition, $\pi \subseteq \pi_r^\kappa(1,G)$. Suppose there is an $x \in \pi_r^\kappa(1,G) - \pi$. Because $k = r - r_1 \geq 3$, no element of $\{x_1, \ldots, x_k\}$ is fixed by G. Thus $x \in \pi_r^\kappa - Q_1$. The HF process $F = \{ F_n(Q_1) \; ; \; n \in N \}$ is G-invariant and x has x_i in its F-socle, for some $i \in \{1,\dots,k\}$. Since $k \geq 3$, there are distinct $h,j \in \{1,\dots,k\}$, each distinct from i. Because $G|_{\{x_1,\dots,x_k\}} \cong S_k$, there is an $\alpha \in G$ such that $O = \{x_i, x_j, x_k\}$ is an α-orbit. By 2.1.7(b), $|O| = 3$ divides $|x\langle\alpha\rangle|$. Hence $|xG| \geq |x\langle\alpha\rangle| \geq 3 > 1$, contradicting $x \in \pi_r^{\kappa}(1,G)$. Thus no such x exists, and $\pi_r^{\kappa}(1,G) = \pi$.

<u>Note</u> : It follows from the proof of 2.4.9 that the HF process P for π_r^{κ} from $\pi_r^{\kappa}(1,G)$ may be assumed G-invariant.

We now use 2.4.9 to show that 2.4.1 is the best possible result when $\kappa \neq \phi$ and $r \geq 2$; i.e. for each r_1 satisfying (1) or (2) of 2.4.1, there is finite collineation group G of π_r^{κ} such that $\pi_r^{\kappa}(1,G)$ has rank r_1 and core κ. Suppose first that r_1 satisfies (1), i.e. $r_1 = |\pi_r^{\kappa}|$. By 2.3.2, there is a collineation group of π_r^{κ} having order r! and fixing κ elementwise. Since $r \geq 2$, this has a subgroup G of order 2^j, some $j > 0$ (for example, the 2-Sylow subgroup). By 2.4.3, $\pi_r^{\kappa}(1,G)$ has core κ and rank r_1. Suppose now that r_1 satisfies (2), i.e. $0 \leq r_1 \leq r - 3$. Because $\kappa \neq \phi$, $\pi_{r_1}^{\kappa}$ exists. Let $\pi = \pi_{r_1}^{\kappa}$. By 2.4.9, to show the existence of a finite G satisfying $\pi = \pi_r^{\kappa}(1,G)$, it suffices to find a HF process P for π_r^{κ} from π. Define $P_0 = \pi$, $P_1 = \pi \cup X$, where X is a set of $r - r_1$ P-HF lines with bearers in $\mathcal{K}$, and $P_n = F_{n-1}(P_1)$, $n > 1$. Then $r(P) = r - r_1$. Let R be a HF process for π from $\mathcal{K}$. Then $r(R) = r_1$, $\overline{R+P} = \overline{P}$, $\underline{R+P} = \mathcal{K}$ and $r(R+P) = r(R) + r(P) = r$. Hence $\overline{P} = \overline{R+P} = \pi_r^{\mathcal{K}}$, and P is the required HF process. Thus 2.4.1 is the best possible result for $\mathcal{K} \neq \phi$ and $r \geq 2$.

We now consider subplanes of $\pi_r^{\mathcal{K}}$ which have rank $|\pi_r^{\mathcal{K}}|$ and core $\mathcal{K}$ and which are $\pi_r^{\mathcal{K}}(1,G)$ for some finite G. No satisfactory characterization of these subplanes has been obtained. However, we do have

<u>Theorem 2.4.10</u> : For each $r \geq 8$, π_r has a non-degenerate Baer subplane which is not $\pi_r(1,G)$ for any finite collineation group of π_r (in particular, for any collineation of order 2).

Before proving this theorem, we note that such a subplane has rank $|\pi_r|$, by 1.7.5.

<u>Proof</u> : Choose an almost-confined configuration ρ of π_r with vertex point a and bearer lines x and y. By 1.7.7, there is a Baer subplane π of π_r containing a and x, but not y. Suppose $\pi = \pi_r(1,G)$ for some finite collineation group G of π_r. Let Q be a G-invariant HF process for π_r from ϕ. Since $a \in \pi_r(1,G)$, either the Q-bearers of a form a G-orbit, or they are both in $\pi_r(1,G)$ (by 2.1.7(d),

since $G_a = G$). By 1.7.9, the Q-bearers of a are x and y. Thus either $xG = yG = \{x,y\}$ or both $x, y \in \pi_r(1,G)$. Neither is possible, because $x \in \pi$, $y \notin \pi$. This contradiction implies $\pi \neq \pi_r(1,G)$ for any finite collineation group G of π_r.

The above theorem is proved only for free planes, because the Baer subplane constructed in the proof of 1.7.7 does not necessarily contain K when $K \neq \phi$. However, it is possible to show that for $K \neq \phi$, $r \geq 1$ and certain a, x and y, π_r^K has a Baer subplane containing K, a and x, but not y (see note after 1.7.8). Hence, it is possible to extend the above theorem to : provided $r \geq 1$, π_r^K has a Baer subplane π having core K such that $\pi \neq \pi_r^K(1,G)$ for any finite G. We do not prove this, because the above theorem suffices to provide an example of such a subplane.

Finally, we consider degenerate subplanes of π_r^K which are $\pi_r^K(1,G)$ for some finite collineation group G fixing K elementwise. Note that since K is fixed elementwise, we have $K \subseteq \pi_r^K(1,G)$. If $K \neq \phi$, then K contains a four point and thus cannot be contained in a degenerate plane. We therefore assume $K = \phi$.

If α is a collineation of a finite non-degenerate plane π, then $\pi(1,\alpha)$ contains equally many points and lines, even though $\pi(1,\alpha)$ may be degenerate (see (4 , theorem 4.1.2)). This is not true if is a collineation of π_r and $\pi_r(1,\alpha)$ is degenerate. This was first observed by Lippi (19), who gave an example of a collineation of π_9 with two fixed lines and one fixed point. In fact, there is not much relationship between the numbers of fixed lines and fixed points of α, provided $\pi_r(1,\alpha)$ is degenerate and r sufficiently large. This is shown by

<u>Theorem 2.4.11</u> : If m and n are integers for which $m \geq 3$, $n \geq 3$ and $r \geq m+n+7$, then there is a collineation α of π_r with m fixed points and n fixed lines.

<u>Proof</u> : Let $t = r - m - n - 1$. Define a configuration ρ to have points $u_1,\ldots,u_m$ and $w_1,\ldots,w_t$, lines $v_1,\ldots,v_n$ and incidences $u_1 \text{ I } v_i$, $1 \leq i \leq n$, $v_1 \text{ I } u_i$, $1 \leq i \leq m$, and $v_2 \text{ I } w_i$, $1 \leq i \leq t$. Define a collineation α of ρ as follows : the points $u_1,\ldots,u_m$ and lines $v_1,\ldots,v_n$ are fixed by α. If t is odd, define $w_i\alpha = w_{1+(i \bmod t)}$, $1 \leq i \leq t$. If t is even, let $t = t_1 + t_2$, where t_1 and t_2 are odd and $t_1, t_2 \geq 3$. Such a t_1 and t_2 exist, because $t = r - m - n - 1 \geq 6$. Define $w_i\alpha = w_{1+(i \bmod t_1)}$, $1 \leq i \leq t_1$, and $w_{t_1+i}\alpha = w_{t_1+1+(i \bmod t_2)}$, $1 \leq i \leq t_2$. Thus if $W = \{w_1,\ldots,w_t\}$, then $|w\langle\alpha\rangle|$ is odd and >2 for each $w \in W$.

By 1.4.4, α extends uniquely to a collineation of $F(\rho) = \pi_r$ for which $F = \{F_n(\rho);\ n \in N\}$ is an α-invariant HF process. If $U = \{u_1,\ldots,u_m\}$ and $V = \{v_1,\ldots,v_n\}$, then every element x of

$\pi_r - U \cup V$ has a $w \in W$ in its F-socle. Since $|w\langle\alpha\rangle|$ is odd, $|w\langle\alpha\rangle|$ divides $|x\langle\alpha\rangle|$ (by 2.1.7(b)), and hence $|x\langle\alpha\rangle| \geq |w\langle\alpha\rangle| > 1$. Thus $x \notin \pi_r(1,\alpha)$ for all $x \in \pi_r - U \cup V$. Hence $\pi_r(1,\alpha) = U \cup V$, i.e. it contains m points and n lines.

From the above theorem and its proof, it follows that for each r_1 satisfying $7 \leq r_1 \leq r - 6$, π_r has a degenerate subplane of rank r_1 which is $\pi_r(1,G)$ for some finite G. By a similar method, one can show that for any $r \geq 11$ and r_1 satisfying $2 \leq r_1 \leq r - 3$, π_r has such a subplane. Thus 2.4.1 is almost a best possible result for free planes.

3.5 Conjugacy classes

In this section we study the conjugacy, within the full collineation group of π_r^K, of finite collineation groups of π_r^K fixing K elementwise. Unless stated otherwise, "conjugacy" will mean "conjugacy within the full collineation group".

We first give some necessary conditions for the conjugacy of two such groups. Define two HF processes P and Q to be <u>isomorphic</u> if $\overline{P} = \overline{Q}$ and there is a collineation ψ of $\overline{P}$ for which $P_n\psi = Q_n$, $n \in N$. An alternative definition is that $\overline{P} = \overline{Q}$ and, for each $n \in N$, there is an isomorphism $\psi_n : P_n \longrightarrow Q_n$ for which $\psi_n|_{P_m} = \psi_m$, $\forall\ m \leq n$. If ρ is a configuration, G a collineation group of ρ, and X a set of integers, denote the subconfiguration $\{x \in \rho ; |xG| \in X\}$ of ρ by $\rho(X,G)$. For $n \in N$, write $\rho(\{n\},G)$ as $\rho(n,G)$.

Lemma 2.5.1 : If G and G' are conjugate finite collineation groups of π_r^K fixing K elementwise, then

(1) $G \cong G'$,

(2) $\pi_r^K(X,G) \cong \pi_r^K(X,G') \quad \forall X \subseteq N$,

(3) there exist isomorphic HF processes Q and Q' for π_r^K from K which are G- and G'-invariant respectively.

Proof : Let $G' = \psi^{-1} G \psi$. The map $\sigma : G \to G'$ defined by $\sigma(\alpha) = \psi^{-1} \alpha \psi$, $\alpha \in G$, is a group isomorphism. Hence (1).
If $x \in \pi_r^K$, then $|xG| = |(xG)\psi| = |x\psi(\psi^{-1}G\psi)| = |(x\psi)G'|$.
Therefore $\pi_r^K(X,G)\psi = \pi_r^K(X,G')$ for each $X \subseteq N$. Hence (2).
Finally, if Q is a G-invariant HF process, then $Q' = \{Q_n\psi ; n \in N\}$ is a G'-invariant HF process isomorphic to Q. Hence (3).

We now give an example of two finite collineation groups G and G' of π_{34} for which (1) and (3) of 2.5.1 are satisfied, and $\pi_{34}(1,G) = \pi_{34}(1,G')$, but G and G' are not conjugate. From this example, it would seem that conditions which are both necessary and sufficient for conjugacy may be difficult to obtain.

Define a HF process Q for π_{34} by

$Q_0 = \{a,b\}$, where a,b are Q-isolated lines,

$Q_1 = Q_0 \cup \{1,\ldots,30\}$, where $1,\ldots,18$ are Q-HF points with Q-bearer a, and $19,\ldots,30$ are Q-HF points with Q-bearer b,

$Q_n = F_{n-1}(Q_1)$, $n \geq 2$.

Define collineations α and α' of Q_1 as follows :

$a = a\alpha = a\alpha'$, $b = b\alpha = b\alpha'$,

$i\alpha = i\alpha' = 1+(i \bmod 9)$, $i = 1,\ldots,9$,

$(j+i)\alpha = j+1+(i \bmod 3)$, $\quad i = 1,2,3$, $j = 9,12,15,18$,

$(21+i)\alpha = 21+1+(i \bmod 9)$, $\quad i = 1,\ldots,9$,

$(9+i)\alpha' = 9+1+(i \bmod 9)$, $\quad i = 1,\ldots,9$,

$(j+i)\alpha' = j+1+(i \bmod 3)$, $\quad i = 1,2,3$, $j = 18,21,24,27$.

By 1.4.4 , α and α' extend uniquely to collineations of π_{34} for which Q is both α- and α'-invariant. Define $G = \langle\alpha\rangle$ and $G' = \langle\alpha'\rangle$. Because both α and α' have order 9, 2.5.1(1) is satisfied. Because Q is both G- and G'-invariant, 2.5.1(3) is satisfied. For each $x \in \pi_{34} - \{a,b,a\circ b\}$, we have $i \in Q(x)$ for at least one $i \in \{1,\ldots,30\}$. By 2.1.7(c), $|iG|$ divides $|xG|$ and $|iG'|$ divides $|xG'|$. Since both $|iG|$, $|iG'| > 1$, both $|xG|, |xG'| > 1$. Hence $\pi_{34}(1,G) = \pi_{34}(1,G') = \{a,b,a.b\}$. Furthermore, $G|_X$ and $G'|_X$ are conjugate as permutation groups of X, the set of Q-HF and Q-isolated elements. However, all these are not sufficient for the conjugacy of G and G'. Let $Y = \{1,3\}$. We show $\pi_{34}(Y,G) \not\cong \pi_{34}(Y,G')$. If $x \in \pi_{34} - \{a,b,a.b\}$, it follows from 2.1.7(c) that $|xG| = 3$ if and only if $|iG| = 3$ for every $i \in Q(x) \cap \{1,\ldots,30\}$, and that $|xG'| = 3$ if and only if $|iG'| = 3$ for every $i \in Q(x) \cap \{1,\ldots,30\}$. Hence $\pi_{34}(Y,G)$ is the subplane of π_{34} freely generated by $\{a,b,i;\ 10 \leq i \leq 21\}$, and $\pi_{34}(Y,G') = \{a, b, a.b, 19, \ldots, 30\}$. Thus $\pi_{34}(Y,G) \not\cong \pi_{34}(Y,G')$. By 2.5.1(2), G and G' are not conjugate.

It is not knownwhether (1), (2) and (3) of 2.5.1 are together sufficient for the conjugacy of G and G'.

We now work towards obtaining upper bounds for the number of conjugacy classes of finite collineation groups of π_r^{κ} which fix κ elementwise. Our best results are for certain groups G for which $\pi_r^{\kappa}(1,G)$ is non-degenerate. Our investigation of these groups is based on

<u>Proposition 2.5.2</u> : Let G be a finite collineation group of π_r^{κ} fixing κ elementwise. Suppose that $\pi_r^{\kappa}(1,G)$ has a non-degenerate subplane π_0 for which there is a G-invariant HF process P for π_r^{κ}.

from π_0. Then, for any line ℓ of π_0, there is a G-invariant HF process Q for π_r^K from π_0 for which

(a) there are no Q-isolated elements, and

(b) all Q-HF elements are points with Q-bearer ℓ.

<u>Proof</u> : By 1.6.6, there is a Q satisfying (a) and (b). We show that the Q constructed in the proof of 1.6.6 is G-invariant, if P is. For this, we use 1.5.9(f). We use the notation of the proof of 1.6.6.

We first show if P is G-invariant, then so is R. By 1.5.9(f) it suffices to show $W_R G = W_R$. Because V_P is the set of P-isolated elements, we have $V_P G = V_P$ (by 2.1.2(e)). It remains to prove that for each $x \in V_P$ and $\alpha \in G$, we have $(x.x\lambda_1)\alpha = x\alpha\circ(x\alpha)\lambda_1 \in W_R$. Because $x\lambda_1 \in \pi_0 \subseteq \pi_r^K(1,G)$, we have $(x\lambda_1)\alpha = x\lambda_1$. If x is a point (resp. line), then $x\alpha$ is also a point (line) of V_P and $s = x\lambda_1 = (x\alpha)\lambda_1$ (resp. $t = x\lambda_1 = (x\alpha)\lambda_1$). Hence $(x\alpha)\lambda_1 = x\lambda_1 = (x\lambda_1)\alpha$. This implies $(x.x\lambda_1)\alpha = (x\alpha)\circ(x\lambda_1)\alpha = (x\alpha)\circ(x\alpha)\lambda_1 \in W_R$, as required.

Similarly, one uses 1.5.9(f) and the definitions of S,T and Q to show that S,T and Q are G-invariant.

Lemma 2.5.3 : Suppose that G is a finite collineation group of π_r^K fixing K elementwise and Q is a G-invariant HF process for π_r^K . Then the (unique) standard HF process similar to Q is also G-invariant.

Proof : By 2.1.2(f), $\ell_Q(x\alpha) = \ell_Q(x) \ \forall \alpha \in G, \ x \in \pi_r^K$. Thus the HF process Q' defined by $Q'_n = \{x \in \pi_r^K \ ; \ \ell_Q(x) \leq n\}$ is G-invariant.

We are now able to prove three nice theorems about the finite collineation groups of π_r^K when $K \neq \phi$. The first result is proved in (12, chapter XI).

Theorem 2.5.4 : Suppose $K \neq \phi$ and G is a finite collineation group of π_r^K fixing K elementwise. Then, for any line ℓ of $F(K)$, there is a set $\{x_1, \ldots, x_r\}$ of points incident with ℓ and a G-invariant HF process Q for π_r^K given by

$Q_0 = F(K)$,

$Q_1 = F(K) \cup \{x_1, \ldots, x_r\}$, where x_i is a Q-HF point with Q-bearer ℓ ,

$Q_n = F_{n-1}(Q_1)$, $n > 1$.

Proof : Because G fixes K elementwise, G fixes $F(K)$ elementwise (by 1.4.4). Thus $F(K)$ is a non-degenerate subplane of $\pi_r^K (1,G)$. By 2.1.2(d), there is a G-invariant HF process Q for π_r^K from $F(K)$. By 2.5.2, we may assume there are no Q-isolated elements and that all Q-HF elements are points with Q-bearer ℓ. We may also assume Q is standard (by 2.5.3). Hence Q satisfies the requirements of the theorem (by 1.5.5).

Theorem 2.5.5 : If $K \neq \phi$, then the number of conjugacy classes of finite collineation groups of π_r^K which fix K elementwise and have order m is at most the number of conjugacy classes of subgroups of order m of S_r, the symmetric group of degree r.

Proof : Choose a line ℓ of $F(K)$, and define a HF process P by $P_0 = F(K)$, $P_1 = F(K) \cup X$, where X is a set of r P-HF points with P-bearer ℓ, and $P_i = F_{i-1}(P_1)$ $\forall i > 1$. Let G be any finite collineation group of π_r^K fixing K elementwise and having order m. Then there is a G-invariant HF process for π_r^K from $F(K)$ as given in 2.5.4. There is an isomorphism ψ of Q_1 onto P_1 fixing $F(K)$ elementwise. By 1.4.2, this extends to an isomorphism of $F(Q_1)$ onto $F(P_1)$ (i.e. to a collineation of π_r^K) for which $Q_i\psi = P_i$ $\forall i \geq 1$. Define $G' = \psi^{-1} G \psi$. Then P is G'-invariant. Thus the number of conjugacy classes of such G equals the number of conjugacy classes of finite collineation groups G' fixing K elementwise, having order m, and for which P is G'-invariant. Thus, to prove the theorem, it suffices to show that for any two such G_1' and G_2', the conjugacy of $G_1'|_X$ and $G_2'|_X$ with respect to a permutation ψ of X implies the conjugacy of G_1' and G_2'. Suppose that $G_2'|_X = \psi^{-1}(G_1'|_X)\psi$. Then ψ extends to a collineation of P_1 fixing $F(K)$ elementwise and satisfying $G_2'|_{P_1} = \psi^{-1}\left(G_1'|_{P_1}\right)\psi$. Hence, by 1.4.4, ψ extends to a collineation of $F(P_1) = \pi_r^K$ for which $G_2' = \psi^{-1} G_1' \psi$. Thus G_1' and G_2' are conjugate.

The following theorem is also proved in (12, chapter XI).

<u>Theorem 2.5.6</u> : If $K \neq \phi$, then all maximal finite collineation groups of π_r^K fixing K elementwise have order r! , and they are all conjugate.

<u>Proof</u> : By 2.5.5, we only need to prove that any such group G has order r! . There is a G-invariant HF process Q for π_r^K from $F(K)$ as given in 2.5.4. Define a finite collineation group G_0 of Q_1 to fix $F(K)$ elementwise and satisfy $G_0|_{\{x_1,\ldots,x_r\}} \cong S_r$. Then $G|_{Q_1} \subseteq G_0$. By 1.4.4 , G_0 extends to a collineation of π_r^K fixing K elementwise, having order r! , and for which $G \subseteq G_0$. By the maximality of G, we have $G = G_0$. Thus $|G| = |G_0| = r!$.

By 2.5.1(2), it is meaningful to investigate the conjugacy classes of finite collineation groups G of π_r^K for which $\pi_r^K(1,G)$ has core K and finite rank r_1, and is non-degenerate (i.e. $\pi_r^K(1,G) \cong \pi_{r_1}^K$). From 2.4.9 and the note after its proof, we may also use 2.5.2 to investigate the conjugacy of these groups. We prove theorems analogous to 2.5.4, 2.5.5. and 2.5.6.

<u>Theorem 2.5.7</u> : Suppose G is a finite collineation group of π_r^K for which $\pi_r^K(1,G) \cong \pi_{r_1}^K$. Then, for any line ℓ of $\pi_r^K(1,G)$, there is a set $\{x_1,\ldots,x_{r-r_1}\}$ of points incident with ℓ and a G-invariant HF process Q for π_r^K given by

$$Q_0 = \pi_r^K(1,G),$$

$$Q_1 = Q_0 \cup \{x_1,\ldots,x_{r-r_1}\}, \text{ where } x_i \text{ is Q-HF with Q-bearer } \ell,$$

$$Q_n = F_{n-1}(Q_1),\ n > 1.$$

<u>Proof</u> : By 2.4.9 and the note after 2.4.9, there is a G-invariant HF process Q for π_r^K from $\pi_r^K(1,G)$. The rank of Q is $r-r_1$, since if R is a HF process for $\pi_r^K(1,G)$ from K, we have $r = r(R+Q) = r(R) + r(Q) = r_1 + r(Q)$. By 2.5.2, we may assume that all Q-HF elements are points with Q-bearer ℓ and that there are no Q-isolated elements. By 2.5.3, we may assume Q is standard. Thus Q is the required HF process (by 1.5.5).

Theorem 2.5.8 : The number of conjugacy classes of finite collineation groups G of π_r^κ having order m and for which $\pi_r^\kappa(1,G) \cong \pi_{r_1}^\kappa$ is at most the number of conjugacy classes of subgroups of order m of S_{r-r_1}, the symmetric group of degree $r - r_1$.

Proof : Choose a line ℓ' of $\pi_{r_1}^\kappa$ and define a HF process P by $P_0 = \pi_{r_1}^\kappa$, $P_1 = \pi_{r_1}^\kappa \cup X$, where X is a set of $r - r_1$ P-HF points with P-bearer ℓ', and $P_{i-1} = F_{i-1}(P_1)$ $\forall\, i > 1$. Let G be a collineation group of order m of π_r^κ for which $\pi_r^\kappa(1,G) \cong \pi_{r_1}^\kappa$. Let $\psi : \pi_r^\kappa(1,G) \to \pi_{r_1}^\kappa$ be an isomorphism. There is a G-invariant HF process Q as given in 2.5.7, with $\ell = \ell'\psi^{-1}$ (note that the line ℓ of 2.5.7 may be any line of $\pi_r^\kappa(1,G)$). Thus ψ extends to an isomorphism of Q_1 onto P_1. By 1.4.2 , it extends to isomorphism of $F(Q_1)$ onto $F(P_1)$ (i.e. to a collineation of π_r^κ) for which $Q_i\psi = P_i$, $\forall\, i \geq 0$. Define $G' = \psi^{-1}G\psi$. Then G' fixes $\pi_{r_1}^\kappa$ elementwise and P is G'-invariant. Thus the number of conjugacy classes of such groups G is at most the number of conjugacy classes of collineation groups G' having order m, fixing $\pi_{r_1}^\kappa$ elementwise and for which P is G'-invariant. The proof of the theorem is now completed in an identical manner to that of 2.5.5, with "$\pi_{r_1}^\kappa$" replacing "$F(\kappa)$".

Theorem 2.5.9 : All finite collineation groups G of π_r^{κ}, maximal with respect to the property $\pi_r^{\kappa}(1,G) \cong \pi_{r_1}^{\kappa}$, have order $(r-r_1)!$, and they are all conjugate.

Proof : By 2.5.8, we only need to show that any such group G has order $(r-r_1)!$. There is a G-invariant HF process Q as given in 2.5.7. Define a finite collineation group G_0 of Q_1 to fix $\pi_r^{\kappa}(1,G)$ elementwise and satisfy $G_0|\{x_1,\ldots,x_{r-r_1}\} \cong S_{r-r_1}$. Then $G|_{Q_1} \subseteq G_0$. By 1.4.4, G_0 extends to a collineation group of π_r^{κ} fixing $\pi_r^{\kappa}(1,G)$ elementwise, having order $(r-r_1)!$, and for which $G \subseteq G_0$. We have $\pi_r^{\kappa}(1,G) \subseteq \pi_r^{\kappa}(1,G_0)$. But $G \subseteq G_0$ implies $\pi_r^{\kappa}(1,G_0) \subseteq \pi_r^{\kappa}(1,G)$. Hence $\pi_r^{\kappa}(1,G) = \pi_r^{\kappa}(1,G_0)$. By the maximality of G, $G = G_0$. Thus $|G| = |G_0| = (r-r_1)!$.

We now consider conjugacy classes of finite collineation groups of π_r for which $\pi_r(1,G)$ is either degenerate or has infinite rank; i.e. those groups for which we cannot apply 2.5.2. We obtain an upper bound for the number of conjugacy classes of these groups. It is evident from our methods that this is far from being a least upper bound.

Suppose k is a non-negative integer. A HF process P is

<u>k-standard</u> if P is standard, all P-HF elements have P-length $\leq k$, and P_0 has only P-isolated elements. The last condition ensures that P is similar to a HF process from ϕ, i.e. that $\overline{P}$ is a free plane. If Q is any HF process for π_r from ϕ, then the standard HF process similar to Q is k-standard, for some k. If P is a k-standard HF process and Q is any HF process isomorphic to P, then Q is k-standard. Furthermore, if P has t isolated elements, then so has Q. Thus isomorphism is an equivalence relation on k-standard HF processes for π_r with t isolated elements. Denote the number of isomorphism classes of such HF processes by $f_r(k,t)$.

Let C_k be the set of finite collineation groups of π_r for which there exists a k-standard G-invariant HF process. If $G \in C_k$ and G' is conjugate to G, then $G' \in C_k$ (this follows from 2.5.1(3)).

<u>Proposition 2.5.10</u> : The number of conjugacy classes of collineation groups of π_r contained in C_k is at most

$$\sum_{t=1}^{\left[\frac{r}{2}\right]} d_{r-t}\, f_r(k,t),$$ where d_{r-t} is the number of subgroups of S_{r-t}, the symmetric group of degree $r - t$.

<u>Proof</u> : For $1 \leq t \leq \left[\frac{r}{2}\right]$, choose a set $\mathcal{E}_r(k,t)$ of representatives from the isomorphism classes of k-standard HF processes for π_r with t isolated elements. Then $\left|\mathcal{E}_r(k,t)\right| = f_r(k,t)$.

Suppose $Q' \in \mathcal{E}_r(k,t)$. Let Q' have isolated elements I' and HF elements H'. Let G' be a finite collineation group of π_r for which Q' is G'-invariant. By 2.1.2(f), $G' \cong G'|_{H' \cup I'}$. Hence the number of such groups G' is at most $d_{|H' \cup I'|} = d_{r-t}$.

Let $G \in C_k$. Then there is a k-standard G-invariant HF process Q for π_r. Suppose Q has t isolated elements. Then there is a $Q' \in \mathcal{E}_r(k,t)$ which is isomorphic to Q and a collineation ψ of π_r for which $Q'_n = Q_n\psi, \forall n \in N$. Define $G' = \psi^{-1} G \psi$. Then Q' is G'-invariant. Thus, for each $G \in C_k$, there is a $G' \in C_k$ which is conjugate to G and for which Q' is G'-invariant, for some

$Q' \in \bigcup_{t=1}^{[\frac{r}{2}]} \mathcal{E}_r(k,t)$. Hence the number of conjugacy classes of collineation groups contained in C_k is at most $\sum_{i=1}^{[\frac{r}{2}]} d_{r-t} \, |\mathcal{E}_r(k,t)|$,

which equals $\sum_{i=1}^{[\frac{r}{2}]} d_{r-t} \, f_r(k,t)$.

From the above proposition, we must find an integer k_r such that, for any finite collineation group G of π_r, there is a k_r-standard G-invariant HF process. We first obtain an upper bound for $f_r(k,t)$ when $k > 1$.

<u>Proposition 2.5.11</u> : $f_r(k,t) \leq (t+1)2^{r-2t+k}(r-t)^{(r-2t)2^{k-1}}$, $k > 1$.

<u>Proof</u> : Suppose $(h_i)_{i=1}^{k}$ is a sequence of integers for which $h_i \geq 0$, $i = 1,\ldots,k$, and $\sum_{i=1}^{k} h_i = r - 2t$. We first determine an upper bound B for the number of isomorphism classes of k-standard HF processes with t isolated elements and h_i HF elements of length i, $i = 1,\ldots,k$.

Let P be such a HF process and $|P_i| = n_i$, $0 \leq i \leq k-1$. We obtain an upper bound for n_i. By 1.5.5, we have

$$P_i = F_1(P_{i-1}) \cup \{\text{P-HF elements of length } i\},\ i = 1,2,\ldots \qquad \ldots (i)$$

From this and 1.4.3(c), $n_i \leq n_{i-1}^2 + h_i$, $i = 1,\ldots,k-1$. Since P_0 has only P-isolated elements, $n_0 = t$. We show by induction that

$$n_i \leq \left(t + \sum_{j=1}^{i} h_j\right)^{2^i},\ i = 1,\ldots,k-1. \qquad \ldots(ii)$$

For $i = 1$, $n_1 \leq n_0^2 + h_1 \leq (n_0+h_1)^2 = (t+h_1)^2$. Suppose now that

$$n_{i-1} \leq \left(t + \sum_{j=1}^{i-1} h_j\right)^{2^{i-1}}, \text{ for some } i \geq 2. \quad \text{Then}$$

$$n_i \leq n_{i-1}^2 + h_i \leq \left(t + \sum_{j=1}^{i-1} h_j\right)^{2^i} + h_i \leq \left(t + \sum_{j=1}^{i} h_j\right)^{2^i}.$$

Thus, by induction, (ii) is true. From (ii), we obtain

$$|P_i| \leq \left(t + \sum_{j=1}^{i} h_j\right)^{2^i} \leq \left(t + \sum_{j=1}^{k} h_j\right)^{2^{k-1}} = (r-t)^{2^{k-1}}, i=1,\ldots,k-1. \qquad \ldots(iii)$$

We note that (iii) is also true for $i = 0$.

We now determine B. Let P_0 have m isolated points and $t - m$ isolated lines. There are $t + 1$ possibilities for m, so there are $t + 1$ isomorphism classes for the configuration P_0. Define $w_0 = t + 1$. For each $i \geq 1$, we now determine an upper bound w_i for the number of possible isomorphism classes of P_i, for a given isomorphism class of P_{i-1}. Because w_i is independent of the isomorphism class of P_{i-1}, we may let $B = \prod_{i=0}^{\infty} w_i$. Suppose $i \geq 1$ and the isomorphism class of P_{i-1} is given. If $i > k$, then $P_i = F_1(P_{i-1})$ (by (i)), and hence the isomorphism class of P_i is uniquely determined. Let $w_i = 1$ for $i > k$. Suppose now that $1 \leq i \leq k$. It follows from (i) that the isomorphism class of P_i depends upon the choice of bearers in P_{i-1} for the h_i HF elements of length i. There are at most $|P_{i-1}|^{h_i}$ ways of choosing the bearers. Because $|P_{i-1}| \leq (r-t)^{2^{k-1}}$ (from (iii)), we let $w_i = (r-t)^{2^{k-1}h_i}$, $1 \leq i \leq k$. We therefore have

$$B = \prod_{i=0}^{\infty} w_i = (t+1) \prod_{i=1}^{k} (r-t)^{2^{k-1}h_i} = (t+1)(r-t)^{2^{k-1}(r-2t)}, \text{ because}$$

$\sum_{i=1}^{k} h_i = r - 2t.$

The upper bound B is independent of the particular sequence $(h_i)_{i=1}^{k}$. Hence $f_r(k,t) \leq B.W$, where W is the number of ways of choosing such a sequence. We need to show $W \leq 2^{r-2t+k}$. If $t = \frac{r}{2}$, then there are no HF elements. This implies $h_i = 0$, $i = 1,\ldots,k$ and $W = 1 \leq 2^k = 2^{r-2t+k}$.

Suppose now that $t < r/2$. Let T_1 and T_2 be the power sets of $\{1,\ldots,k\}$ and $\{1,\ldots,r-2t\}$ respectively. Let H be the set of sequences $(h_i)_{i=1}^{k}$ under consideration. Define $\sigma : H \to T_1 \times T_2$ by $\sigma\left[(h_i)_{i=1}^{k}\right] = \left(\{i \; ; \; h_i \neq 0\}, \left\{\sum_{j=1}^{i} h_j \; ; \; 1 \leq i \leq k\right\}\right)$. Clearly σ is one-to-one. Hence $W = |H| \leq |T_1| \cdot |T_2| = 2^k . 2^{r-2t}$, as required.

We note that the inequalities used in the proof of 2.5.11 are quite crude. One can obtain a better upper bound for $f_r(k,t)$ than that of 2.5.11 by using more precise inequalities. However, the proof and the eventual upper bound obtained become so complicated that it is not worthwhile.

We now work towards obtaining an integer k_r such that, for any finite collineation group G of π_r, a k_r-standard G-invariant HF process exists.

<u>Proposition 2.5.12</u> : Suppose that G is a finite collineation group of π_r and that k is an integer with the following property : for each standard G-invariant HF process Q for π_r such that Q_0 has only isolated elements, and for each Q-HF point (resp. line) v of Q-length $> k$, there is a point (line) $f_Q(v)$ of π_r such that

(a) $\ell_Q(f_Q(v)) < k$;

(b) $f_Q(v)$ is not incident with the Q-bearer of v;

(c) $f_Q(v) \in \pi_r(1, G_v)$, where $G_v = \{\alpha \in G;\ v\alpha = v\}$.

Then there exists a k-standard G-invariant HF process for π_r.

Proof : Let Q be a standard G-invariant HF process for π_r such that Q_0 has only isolated elements. Such a Q exists, as we may take it to be the standard HF process similar to a G-invariant HF process for π_r from ϕ. If there are no Q-HF elements of Q-length $> k$, then Q is k-standard there is nothing to prove. Suppose now that there is a Q-HF element v of Q-length $> k$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of coset representatives for G_v in G. Let $V = vG = \{v\alpha_i; 1 \leq i \leq n\}$ and $U = \{f_Q(v)\alpha_i;\ 1 \leq i \leq n\}$. Define $\lambda : V \to U$ by $(v\alpha_i)\lambda = f_Q(v)\alpha_i$. For each i, we have $st_Q(f_Q(v)\alpha_i) = st_Q(f_Q(v)) = \rho_Q(f_Q(v)) < k$ (by (a)), and $f_Q(v)\alpha_i$ is not a Q-bearer of $v\alpha_i$ (by (b)). Thus we may define $Q' = \Gamma(k, V, \lambda, W)$, where $W = \{(v\alpha_i)\circ(f_Q(v)\alpha_i)\ ;\ 1 \leq i \leq n\}$. By 1.5.9(c), Q' is a HF process for π_r satisfying $Q_0' = Q_0$.

We now show that Q' is G-invariant. By 1.5.9(e), it suffices to show $WG = W$. We show that $x \in W$ and $\alpha \in G$ imply $x\alpha \in W$. Suppose $x = v\alpha_i \circ f_Q(v)\alpha_i$. Then $v\alpha_i\alpha = v\alpha_j$, for some j. Thus $\alpha_i\alpha\alpha_j^{-1} \in G_v$. By (c), this implies $f_Q(v)\alpha_i\alpha\alpha_j^{-1} = f_Q(v)$; i.e. $f_Q(v)\alpha_i\alpha = f_Q(v)\alpha_j$. Hence $x\alpha = (v\alpha_i\alpha)\circ(f_Q(v)\alpha_i\alpha) = v\alpha_j \circ f_Q(v)\alpha_j \in W$.

Thus $WG = W$, and Q' is G-invariant. Let Q'' be the standard HF process similar to Q'. By 2.5.3, Q'' is also G-invariant.

Because $\ell_Q(v) > k$ and $V = vG$, all elements of V have Q-length $> k$ (by 2.1.2(c)). Thus, by 1.5.10, Q' has fewer HF elements of length $> k$ than Q. Hence Q'' has fewer HF elements of length $> k$ than Q. Because $Q_0' = Q_0$, both Q_0' and Q_0'' have only isolated elements. If Q'' has no HF elements of length $> k$, then it is k-standard, and there is nothing further to prove. If there are Q''-HF elements of length $> k$, then we repeat the above argument to obtain a standard G-invariant HF process Q''' such that Q_0''' has only isolated elements and Q''' has fewer HF elements of length $> k$ than Q''. Because there are only finitely many Q-HF elements of length $> k$, the required HF process is obtained after finitely many steps.

Lemma 2.5.13 : If P is any HF process, $x \in \overline{P}$ and $\ell_p(x) \geq 9$, then $P(x)$ contains a four-point and four-line.

Proof : It suffices to prove the lemma for the case $\ell_p(x) = 9$, because if $\ell_p(x) > 9$, then $P(x)$ has an element y of P-length 9, and $P(y) \subseteq P(x)$. We may assume that x is a line. There is a P-chain $C = \{x_0, x_1, \ldots, x_9\}$ for which $x_i I x_{i+1}$, and $\ell_P(x_i) = i$, $i = 0, 1, \ldots, 8$, and $x_9 = x$. We show that C contains a four-point and a four-line. Because x_9 is a line, $x_0, x_2, \ldots, x_8$ are points and $x_1, x_3, \ldots, x_9$ are lines. Because $x_0 \circ x_2 = x_1 \neq x_3 = x_2 \circ x_4$, x_0, x_2 and x_4 are not collinear. Similarly x_2, x_4 and x_6 are not collinear and x_0, x_2, x_6, x_8 are not collinear. Hence there is a four-point contained in $\{x_0, x_2, \ldots, x_8\}$. Similarly, $\{x_1, x_3, \ldots, x_9\}$ contains a four-line.

Lemma 2.5.14 : Suppose G is a collineation group of order 2^j of a non-degenerate free plane π. For any G-invariant HF process Q for π, there is a four-point $\eta \subseteq \pi(1,G)$ for which each element of η has Q-length $\leq 9 + 6j$.

Proof : We proceed by induction on j. Suppose $j = 0$. Then $G = \{1\}$. Choose an element x of Q-length 9. Then Q(x) contains a four-point η (by 2.5.13). Each element of η has Q-length ≤ 9 and $\eta \subseteq \pi(1,G)$ (trivially).

Assume that the lemma is true for j satisfying $0 \leq j < n$ and that G has order 2^n. Then G has a normal subgroup G' of order 2^{n-1}. Let β be a coset representative for G' in G. Then $G = G' \cup G'\beta$. Let $\pi' = \pi(1,G')$. If $x \in \pi'$, then either $x \in \pi(1,G)$ or $xG = \{x, x\beta\} \subseteq \pi'$. Thus $\pi'G = \pi'$ and $G\,|_{\pi'} = \{1, \beta|_{\pi'}\}$, where $\left(\beta\,|_{\pi'}\right)^2 = 1$. Let Q be any G-invariant HF process for π.

Then Q is also G'-invariant. By the induction assumption, there is a four-point $\eta' \subseteq \pi'$ such that $\ell_Q(x) \leq 9 + 6(n-1)$ for each $x \in \eta'$. We now consider two cases. In each, we obtain a four-point $\eta \subseteq \pi(1,G)$ for which $\ell_Q(z) \leq 9 + 6n \quad \forall z \in \eta$.

(1) There is an $\{x,y\} \subset \eta'$ for which $\{x\beta, y\beta, x, y\}$ is a four-point : Let $\rho = \{x\beta, y\beta, x, y\}$. Then $\rho \subset \pi'$. By 1.7.2, $[\rho]_{\pi} = [\rho]_{\pi'} = F(\rho)$. Because $\ell_Q(x\beta) = \ell_Q(x) \leq 3 + 6n$ and $\ell_Q(y\beta) = \ell_Q(y) \leq 3 + 6n$, it follows from 1.7.3 that

$$\ell_Q(z) \leq st_{\rho}(z) + 3 + 6n \;\forall\; z \in F(\rho) \qquad \dots \text{(i)}$$

Because $\rho G = \rho$ and $\beta|_{\rho}$ has order two, $F(\rho)G = F(\rho)$ and $\beta|_{F(\rho)}$ has order two (by 1.4.4). Let $\eta = \{a,b,c,e\}$, where $a = (x \circ y)\circ(x\beta \circ y\beta)$, $b = (x\circ y\beta)\circ(y\circ x\beta)$, $c = (x\circ x\beta)\circ(y\circ y\beta)$, $d = (b\circ c)\circ(x\circ y)$ and $e = (x\circ d\beta)\circ(x\beta\circ d)$. Then $\eta \subseteq F(\rho) \subseteq \pi'$ and each point of η is fixed by β and has ρ- stage ≤ 6. From (i) $\ell_Q(z) \leq 9 + 6n \;\forall\; z \in \eta$. Because $G|_{\pi'} = \{1, \beta|_{\pi'}\}$ and η is fixed elementwise by β, we have $\eta \subseteq \pi(1,G)$.

(2) No such $\{x,y\} \subset \eta'$ exists : Either η' is fixed elementwise by β, in which case we let $\eta = \eta'$, or there is an $\{x,y,z\} \subset \eta'$ for which $x,y \in \pi(1,G)$ and $\{x,y,z,z\beta\}$ is a four point. In this case $\{x,y,z,z\beta\}$ freely generates a subplane of π' (by 1.7.2), and we let $\eta = \{d,x,y,f\}$, where $a = (x\circ y)\circ(z\cdot z\beta)$, $b = (z\circ x)\circ(y\circ z\beta)$, $c = (z\circ y)\cdot(x\circ z\beta)$, $d = (z\circ z\beta)\circ(b\circ c)$, $e = (a\circ b)\circ(x\circ z\beta)$, $f = (z\cdot z\beta)\circ(e\circ e\beta)$. By the same argument as case (1), η satisfies our requirements.

By induction, the lemma is true for all j.

We note that 2.5.14 is true with "four-point" replaced by "four-line" (by the dual argument and the duals of 1.7.2 and 1.7.3).

For each $r \geq 8$, let $j_r = \max\{j ; j \text{ divides } r!\}$ and $k_r = 10 + 6j_r$.

Proposition 2.5.15 : For any finite collineation group G of π_r, there exists a k_r-standard G-invariant HF process for π_r.

Proof : By 2.5.12, it suffices to show that, for any standard G-invariant HF process Q for π_r such that Q_0 has only isolated elements, and any Q-HF point (line) v of Q-length $> k_r$, there is a point (line) $f_Q(v)$ satisfying 2.5.12 (a),(b) and (c) (with $k = k_r$). We assume v is a point. For the case v is a line, the dual of the following argument, and the dual of 2.5.14, are used.

We first show that $Q(v)G_v = Q(v)$ and $|G_v|Q(v)| = 2^j$, for some $j \geq 0$. Choose any $\alpha \in G_v$. By 2.1.2(b), $Q(v)\alpha = Q(v\alpha) = Q(v)$. Thus $Q(v)G_v = Q(v)$. Let α have order $2^s m$, for some odd m and integer $s \geq 0$. Then α^{2^s} has order m. By 2.1.7(c), $|x\langle\alpha^{2^s}\rangle|$ divides $|v\langle\alpha^{2^s}\rangle|$ for each $x \in Q(v)$. Because $|v\langle\alpha^{2^s}\rangle| = 1$, we have $x\alpha^{2^s} = x \; \forall x \in Q(v)$. Thus $\alpha|_{Q(v)}$ has order a power of two, for each $\alpha \in G_v$. Thus $G_v|_{Q(v)}$ is a 2-group and has order 2^j, for some $j \geq 0$. We note that $j \leq j_r$, since 2^j divides $|G_v|$ and $|G_v|$ divides r! (by 2.1.3).

Let $\pi = [Q(v)]_{\pi_r}$. By 1.5.15, $\pi = F(Q(v))$. Because $Q(v)G_v = Q(v)$ and $|G_v|_{Q(v)}| = 2^j$, some j, we have $\pi G_v = \pi$ and $|G_v/_\pi| = 2^j$ (by 1.4.4). By 1.5.15, the extension process

$R = \pi \cap Q$ is a standard HF process for π for which $\ell_R(x) = \ell_Q(x)$ $\forall x \in \pi$. Because $\pi G_v = \pi$ and Q is G_v-invariant, R is $(G_v/_\pi)$-invariant. π is non-degenerate, because $\ell_Q(v) > k_r \geq 10$, and thus Q(v) contains a four-point (by 2.5.13). By 2.5.14, there is a four-point $\eta \subseteq \pi(1, G_v/_\pi)$, each element of which has R-length $\leq 9 + 6j \leq 9 + 6j_r < k_r$. Because $\ell_Q(x) = \ell_R(x)$ $\forall x \in \pi$, each element of η has Q-length $< k_r$. Because $\pi(1, G_v/_\pi) \subseteq \pi_r(1, G_v)$, we have $\eta \subseteq \pi_r(1, G_v)$. Since η contains 3 non-collinear points, at least one point of η is not incident with the Q-bearer of v. Let this point be $f_Q(v)$. Then $f_Q(v)$ satisfies 2.5.12(a), (b) and (c), with $k = k_r$.

Combining 2.5.10, 2.5.11 and 2.5.15, we obtain

<u>Theorem 2.5.16</u> : The number of conjugacy classes of finite collineation groups of π_r is at most

$$\sum_{t=1}^{[\frac{r}{2}]} (t+1)\, d_{r-t}\, 2^{r-2t+k_r}\, (r-t)^{(r-2t)2^{k_r-1}}$$, where d_{r-t} is the number of subgroups of S_{r-t}, and $k_r = 10 + 6j_r$, where $j_r = \max\{j;\ 2^j \text{ divides } r!\}$.

From 2.5.5 and 2.5.16, it follows that for any r and κ for which π_r^κ exists, there are only finitely many conjugacy classes of finite collineation groups of π_r^κ fixing κ elementwise. In conclusion, we note that, although they occur in only finitely many conjugacy classes, there are infinitely many such groups, unless $\kappa \neq \phi$ and r = 1 (in which case the only such group is the identity, by 2.1.3). To show this, it suffices to find an $x \in \pi_r^\kappa$ which has infinitely many distinct images under finite order collineations of π_r^κ fixing κ elementwise. The existence of such an x follows from theorem 11 of (26) when either $\kappa \neq \phi$ and $r \geq 2$, or $\kappa = \phi$ and $r \geq 9$ (because in these cases π_r^κ is the free extension of rank one of π_{r-1}^κ - using the terminology of (26)). The only other possibility is $\kappa = \phi$ and r = 8. π_8 is freely generated by a four-point $\{x,y,z,u\}$. In (24 , section 3), a method is given for obtaining all four-points which freely generate π_8. It is possible to show that, for fixed x,y and z, there are infinitely many possibilities for u. Since each such u is the image of x under a finite order collineation of π_8, π_8 has infinitely many distinct finite collineation groups.

CHAPTER 3

POLARITIES

A polarity is a correlation of order two. Abbiw-Jackson (1) first showed that π_r has polarities for each $r \geq 8$, and O'Gorman (22) showed that, for $\mathcal{K} \neq \phi$, any polarity of $\mathcal{K}$ extends to a polarity of $\pi_r^{\mathcal{K}}$ for each $r \geq 0$. In this chapter, we obtain some properties of polarities of free rank planes which have either previously been obtained by other authors, or which follow immediately from their work.

In 3.1, we prove first that to each polarity α of $\pi_r^{\mathcal{K}}$ there is a HF process for $\pi_r^{\mathcal{K}}$ canonically associated with α. We then investigate the possible numbers of absolute points outside $\mathcal{K}$ that a polarity of $\pi_r^{\mathcal{K}}$ may have. In 3.2, we investigate the conjugacy classes of polarities of $\pi_r^{\mathcal{K}}$, within the full automorphism group of $\pi_r^{\mathcal{K}}$.

Throughout this chapter, we assume that the empty configuration has a trivial polarity.

3.1 Absolute Points of Polarities

Suppose that α is a polarity of a configuration ρ and that $x \in \rho$. If $x \mathrel{I} x\alpha$, then x is α-absolute (or just absolute, if it is clear to which polarity we are referring). If $x \not\mathrel{I} x\alpha$, then x is non-α-absolute (or just non-absolute). We also say that α has absolute or non-α-absolute elements. Clearly, x is α-absolute if, and only if, $x\alpha$ is α-absolute, and each α-absolute element x is incident with exactly one α-absolute element, namely $x\alpha$.

If α is a polarity of π_r^{κ}, then there does not necessarily exist a HF process Q for π_r^{κ} from κ such that each configuration of Q is invariant under α. If such a HF process existed, and x were an α-absolute point outside κ, then x and $x\alpha$ would have equal non-zero Q-stage and be incident, contradicting the definition of a HF process. Although such an "α-invariant" HF process Q does not always exist, we do have

<u>Proposition 3.1.1</u> : For each polarity α of π_r^{κ} there exists an integer $m \geq 0$ and a HF process Q for π_r^{κ} satisfying

(a) $Q_0 = \kappa$;

(b) $\pi_r^{\kappa} = F(Q_{2m})$ and $Q_n\alpha = Q_n \;\; \forall\, n \geq 2m$;

(c) If $m > 0$, then there is a sequence $a_1, \ldots, a_m$ of points for which

$$Q_{2n-1} = Q_{2n-2} \cup \{a_n\}, \quad Q_{2n} = Q_{2n-1} \cup \{a_n\alpha\}, \quad 1 \leq n \leq m \;;$$

(d) All Q-isolated and Q-HF points are contained in Q_{2m}.

<u>Proof</u> : We first define a subconfiguration ρ of π_r^{κ} for which $\kappa \subseteq \rho$, $\rho - \kappa$ is finite, $\pi_r^{\kappa} = F(\rho)$ and $\rho\alpha = \rho$. Let P be a HF process for π_r^{κ} from κ, and let X be the set of P-isolated and P-HF elements. Define $\rho = \kappa \cup \left(\bigcup_{x \in X \cup X\alpha} P(x) \right)$. Because X is finite and all P-socles are finite (by 1.5.1(b)), $\rho - \kappa$ is finite. Because $X \subseteq \rho$, we have from 1.5.12 that $\pi_r^{\kappa} = \overline{P} = F(\rho)$ and

(*) $\qquad P(x) \subseteq \rho \quad \forall\, x \in \rho$.

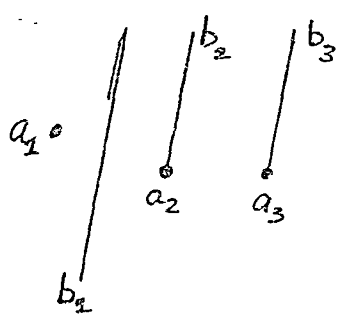
It remains to show that $\rho\alpha = \rho$. Because $\kappa\alpha = \kappa$ (by 1.6.2), it suffices to show that $(\rho - \kappa)\alpha \subseteq \rho$. Suppose, on the contrary, that there is an $x \in \rho - \kappa$ for which $x\alpha \notin \rho$. Let $st_P(x\alpha)$ be maximal with respect to these properties. Because $X \cup X\alpha \subseteq \rho$, $x \notin X \cup X\alpha$.

Hence x is P-free with two P-bearers u and v, which are both in ρ (by ($*$)). Also, $x \in P(y)$ for some $y \in X \cup X\alpha$, and hence x is in a P-chain of y. Therefore x is incident with some $w \in P(y)$ of higher P-stage than x. Thus x is incident with at least three elements u, v and w of ρ. Thus $x\alpha$ is with $u\alpha$, $v\alpha$ and $w\alpha$, at least one of which, say $u\alpha$, is of higher P-stage than $x\alpha$. Either $u\alpha \notin \rho$, contradicting the maximality of $st_P(x\alpha)$, or $u\alpha \in \rho$, implying $x\alpha \in P(u\alpha) \subseteq \rho$ (by ($*$)) and thus contradicting $x\alpha \notin \rho$. In either case, we obtain a contradiction. Hence $\rho\alpha = \rho$.

We now define m and Q. Let $\rho - \kappa$ have m points. Since $\rho\alpha = \rho$ and $\kappa\alpha = \kappa$, we have $(\rho - \kappa)\alpha = \rho - \kappa$. Hence $\rho - \kappa$ has m lines. For $n \geq 2m$, define $Q_n = F_{n-2m}(\rho)$. Suppose $m > 0$. Then there is an element x of $\rho - \kappa$ of maximal P-stage and x is incident with at most two elements of $\rho - \kappa$. Hence $x\alpha$ is also incident with at most two elements of $\rho - \kappa$. If x is a point, define $a_m = x$. If x is a line, define $a_m = x\alpha$. Define $Q_{2m-1} = Q_{2m} - \{a_m\alpha\}$ and $Q_{2m-2} = Q_{2m-1} - \{a_m\}$. Continuing in this way, we define a_{m-1}, Q_{2m-3}, Q_{2m-4}, ..., a_1, Q_1, Q_0. We have $Q_0 = \kappa$, since $\rho - \kappa$ has m points. Hence Q satisfies (a). Q clearly satisfies (c). Because $\pi_r^{\kappa} = F(\rho) = F(Q_{2m})$ and $Q_n\alpha = F_{n-2m}(\rho)\alpha = F_{n-2m}(\rho) = Q_n$ (using 1.4.4), Q satisfies (b). Q satisfies (d), since all elements of Q-stage $> 2m$ are Q-free.

We note that the configuration ρ obtained in the proof of 3.1.1 is, in the terminology of (22), an "openly finite self-polar (under α) free generating configuration". The existence of such configurations when $\kappa = \phi$ was first proved by Abbiw-Jackson (1). For $\kappa \neq \phi$, their existence was shown by O'Gorman (22). The investigation by all previous authors of the polarities of free rank planes has been based upon the existence of such configurations.

If α is a polarity of π_r^{κ} and Q is a HF process satisfying (a), (b), (c) and (d) of 3.1.1, then Q is an <u>α-canonical</u> HF process for π_r^{κ}. We show by example that an α-canonical HF process for π_r^{κ} is not uniquely determined by α. Define the configuration ρ to equal



$\{a_1, a_2, a_3, b_1, b_2, b_3\}$, where a_i is a point and b_i is a line, $i = 1,2,3$, and $a_2 \mathrm{I} b_2$, $a_3 \mathrm{I} b_3$. Clearly $F(\rho) = \pi_{10}$. Define a polarity of ρ by $a_i \alpha = b_i$, $i = 1,2,3$.

By 1.4.4, α extends uniquely to a polarity of $F(\rho) = \pi_{10}$. Define two HF processes P and Q as follows :

$P_0 = \phi$, $P_{2i-1} = P_{2i-2} \cup \{a_i\}$, $P_{2i} = P_{2i-1} \cup \{b_i\}$, $i = 1,2,3$, and $P_i = F_{i-6}(P_i)$, $i > 6$;

$Q_0 = \phi$, $Q_1 = \{b_2, b_3\}$, $Q_2 = Q_1 \cup \{a_2, a_3\}$, $Q_{2i-1} = Q_{2i-2} \cup \{a_i\}$, $Q_{2i} = Q_{2i-1} \cup \{b_i\}$, $i = 2,3,4$, and $Q_i = F_{i-8}(Q_8)$, $i > 8$.

Then P and Q are distinct α-canonical HF processes for π_{10}.

<u>Lemma 3.1.2</u> : If α is a polarity of the free plane π_r, then there exists an α-canonical HF process Q for which there is at least one non-α-absolute Q-isolated point.

Proof : By 3.1.1, an α-canonical HF process Q for π_r exists. Suppose there are no non-α-absolute Q-isolated points. Since $Q_0 = \phi$, there is at least one α-absolute Q-isolated point. If there are two such points a and b, then we redefine Q, making $(a.b)\alpha$ and a.b Q-isolated, and $a, b, a\alpha, b\alpha$ Q-HF with Q-bearers a.b and $(a.b)\alpha$. If there is only one α-absolute Q-isolated point a, then $Q_2 = \{a, a\alpha\}$ and $Q_4 = Q_2 \cup \{b, b\alpha\}$, where b and $b\alpha$ are Q-HF with bearers $a\alpha$ and a respectively. We redefine Q, letting $Q_2 = \{b, b\alpha\}$ and $Q_4 = \{a, a\alpha\} \cup Q_2$. Thus redefined, Q has b as a non-α-absolute isolated point.

Lemma 3.1.3 : If α is a polarity of π_r^{κ} and Q is an α-canonical HF process for π_r^{κ}, then all α-absolute points outside κ are either Q-HF or Q-isolated.

Proof : Suppose a is an α-absolute point outside κ. Since no two elements of equal non-zero Q-stage are incident, we have $st_Q(a) \neq st_Q(a\alpha)$. By (b) and (c) of 3.1.1, there is an $n \geq 0$ for which $st_Q(a) = 2n + 1$ and $st_Q(a\alpha) = 2n + 2$. Suppose a is Q-free. Then its Q-bearers x and y have Q-stage $\leq 2n$. Thus $st_Q(x\alpha) \leq 2n$ and $st_Q(y\alpha) \leq 2n$, because $Q_{2n}\alpha = Q_{2n}$. This implies $a\alpha$ is incident with three elements of lower Q-stage, namely $x\alpha$, $y\alpha$ and a. This contradicts the definition of a HF process. Hence a is either Q-HF or Q-isolated.

Since any α-canonical HF process for π_r^{κ} has at most r isolated or HF elements, it follows from 3.1.3 that any polarity of π_r^{κ} has at most r absolute points outside κ. The main result of this section, which we now prove, specifies more closely the possible number of absolute points outside κ that a polarity of π^{κ} may have. It was first proved by Abbiw-Jackson (1) for free planes and was extended to the case $\mathcal{K} \neq \phi$ by O'Gorman (22).

Theorem 3.1.4 : If α is a polarity of $\pi_r^{\mathcal{K}}$ with j absolute points outside $\mathcal{K}$, then

(a) $j \equiv r (\mathrm{mod}\ 2)$;

(b) $0 \leq j \leq r$;

(c) if $\mathcal{K} = \phi$, then $j \leq r - 6$.

Unless $j = 0$, $r = 8$ and $\mathcal{K} = \phi$, there exists, for each polarity α of $\mathcal{K}$ and for each j satisfying (a), (b) and (c), an extension of α to $\pi_r^{\mathcal{K}}$ with j absolute points outside $\mathcal{K}$.

Proof : Let Q be an α-canonical HF process for $\pi_r^{\mathcal{K}}$ and let

n_1 = number of α-absolute, Q-isolated points,

n_2 = number of α-absolute, Q-HF points,

i = number of non-α-absolute, Q-isolated points,

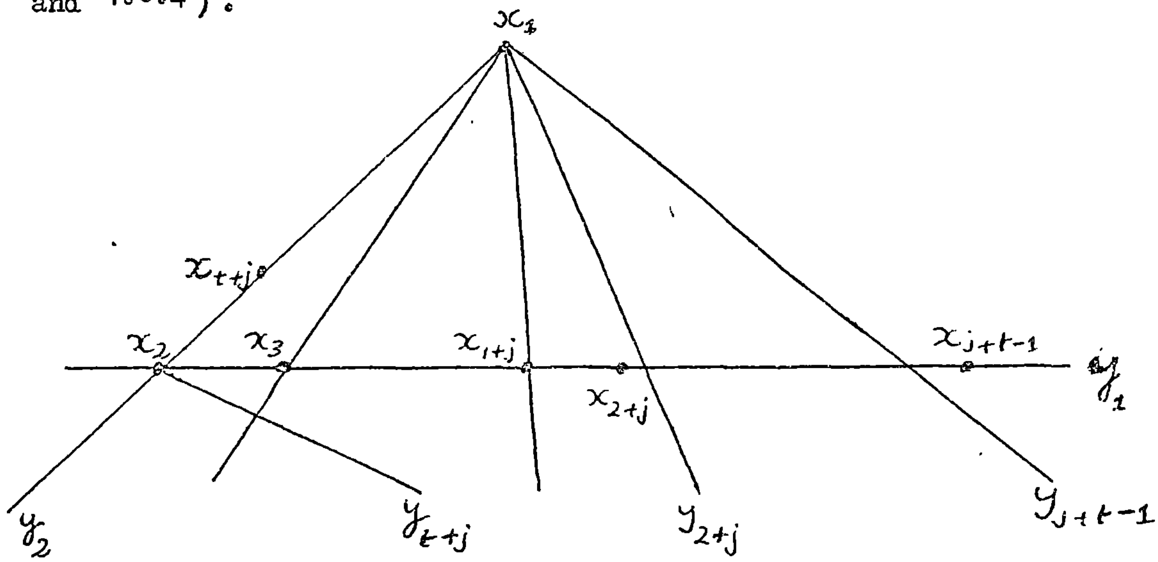
h = number of non-α-absolute, Q-HF points.

By 3.1.1, there is an integer m such that all Q-HF and Q-isolated points are contained in Q_{2m}, and, if $m > 0$, then $Q_{2n-1} = Q_{2n-2} \cup \{a_n\}$, $Q_{2n} = Q_{2n-1} \cup \{a_n\alpha\}$, $1 \leq n \leq m$, where $a_1, \ldots, a_m$ is a sequence of points. Hence $Q_{2n}^{\alpha} = Q_{2n}$ for $1 \leq n \leq m$. It follows that a line $x\alpha$ is Q-isolated if, and only if, x is a Q-isolated non-α-absolute point, and that a line $x\alpha$ is Q-HF if, and only if, x is either Q-isolated and α-absolute or Q-HF and non-α-absolute. Thus there are i Q-isolated lines and $n_1 + h$ Q-HF lines. Altogether, there are $n_1 + 2i$ Q-isolated elements and $n_1 + n_2 + 2h$ Q-HF elements. Hence $r = 3n_1 + n_2 + 4i + 2h$. By 3.1.3, $j = n_1 + n_2$. We therefore have $r = j + 2(n_1+2i+h)$. Thus $r \equiv j$ (mod 2) and $0 \leq j \leq r$. It remains to show (c). Suppose

$K = \phi$. To show $j \leq r - 6$, we need to show $n_1 + 2i + h \geq 3$. By 3.1.2, we may assume $i \geq 1$. It is easily verified that if $i = 1$, $n_1 = 0$ and $h = 0$, then $\overline{Q}$ is degenerate, contradicting $\overline{Q} = \pi_r$. Therefore $n_1 + 2i + h \geq 3$, and (c) is proved.

The second part of the theorem is proved by giving examples. Let α be any polarity of K and assume j satisfies (a), (b) and (c) and we do not have $r = 8$, $j = 0$ and $K = \phi$. We first define a configuration ρ containing K and freely generating π_r^K. We consider two cases.

(1) $\underline{K = \phi}$: Let $t = \frac{r-j-6}{2} + 2$. Let ρ have points x_i and lines y_i, $i = 1,\ldots,j+t$, with the following incidences : $x_1 \; I \; y_1$ and $y_1 \; I \; x_i$, $2 \leq i \leq j + t - 1$, $x_i \; I \; y_i$, $2 \leq i \leq 1 + j$, $x_{t+j} \; I \; y_2$, $y_{t+j} \; I \; x_2$. It is quickly verified that $F(\rho) = \pi_r$ (using 1.6.3 and 1.6.4).



(2) $\underline{K \neq \phi}$: Let $t = \frac{r-j}{2}$. Choose a non-α-absolute line ℓ of K. Define ρ to consist of K together with points x_i and lines y_i, $i = 1,2,\ldots,j+t$, where $x_i \; I \; \ell$, $y_i \; I \; \ell\alpha$, $i = 1,2,\ldots,j+t$, and $x_i \; I \; y_i$, $i = 1,2,\ldots,j$. Then $F(\rho) = \pi_r^K$.

In both cases (1) and (2), we extend α from κ to ρ by defining $x_i\alpha = y_i$, $i = 1,\ldots,j+t$. By 1.4.4 α extends to a polarity of $F(\rho) = \pi_r^\kappa$. Define an α-canonical HF process Q for π_r^κ as follows :

$Q_0 = \kappa$, $Q_{2i-1} = Q_{2i-2} \cup \{x_i\}$, $Q_{2i} = Q_{2i-1} \cup \{y_i\}$, $i = 1,\ldots,j+t$, $Q_n = F_{n-2(j+t)}(Q_{2(j+t)})$, $n > 2(j+t)$. The α-absolute points not in κ are either Q-HF or Q-isolated (by 3.1.3) and are therefore contained in $Q_{2(j+t)}$, which equals ρ. There are j absolute points in $\rho - \kappa$ (namely $x_1,\ldots,x_j$ if $\kappa \neq \phi$, or $x_2,\ldots,x_{1+j}$ if $\kappa = \phi$). Hence α has j absolute points outside κ, as required.

We note that if, in the second part of 3.1.4, we allowed the possibility of $j = 0$, $r = 8$ and $\kappa = \phi$, then the configuration ρ defined in the above proof would be such that $F(\rho)$ is degenerate and not equal to π_8. This exceptional case is considered more fully in the next section.

3.2 Conjugacy Classes of Polarities

Two polarities α and α' of a configuration ρ are <u>conjugate</u> if there is an automorphism δ of ρ for which $\alpha' = \delta^{-1}\alpha\delta$. We say that α and α' are conjugate <u>with respect to</u> δ. If $\alpha' = \delta^{-1}\alpha\delta$, then we also have $\alpha' = (\alpha\delta)^{-1}\alpha(\alpha\delta)$. Therefore, if δ is a correlation, then α and α' are also conjugate with respect to the collineation $\alpha\delta$. Thus we need only consider conjugacy with respect to a collineation.

Suppose α and α' are polarities of π_r^κ and ψ is a collineation of

π_r^K for which $\alpha' = \psi^{-1}\alpha\psi$. If $\{x_1,\dots,x_j\}$ is the set of α-absolute points outside K, then the set of α'-absolute points outside K is $\{x_1\psi,\dots,x_j\psi\}$, since $K\psi = K$. Thus conjugate polarities of π_r^K have the same number of absolute points outside K. Hence, for a given j, conjugacy is an equivalence relation on the polarities of π_r^K with j absolute points outside K. Its equivalence classes are called <u>conjugacy classes</u>.

The problem of determining conjugacy classes of polarities of free planes was first considered by Abbiw-Jackson (1). He solved the problem completely for the case of rank 8. Glock (8) extended this work by determining the number of conjugacy classes of polarities of π_r with j absolute points, for each $r \geq 8$ and j satisfying $0 \leq j \leq r - 6$ and $j \equiv r \pmod 2$. These results, and similar ones for $K \neq \phi$, are obtained in this section. We also give a necessary and sufficient condition for the conjugacy of two polarities of π_r^K having no absolute points outside K.

The following useful sufficient condition for the conjugacy of two polarities of π_r^K is based upon theorem 4.1 of (1).

<u>Lemma 3.2.1</u> : Suppose α and α' are polarities of π_r^K and ρ and ρ' are subconfigurationsof π_r^K which freely generate π_r^K and for which $\rho\alpha = \rho$ and $\rho'\alpha' = \rho'$. If there is an isomorphism $\psi : \rho \longrightarrow \rho'$ for which $\psi^{-1}(\alpha|_\rho)\psi = \alpha'|_{\rho'}$, then α and α' are conjugate.

<u>Proof</u> : By 1.4.2, the isomorphism ψ of ρ onto ρ' extends uniquely to an isomorphism of $F(\rho)$ onto $F(\rho')$, i.e. to a collineation of π_r^K.

By assumption, $(\psi^{-1}\alpha\psi)|_{\rho'} = \alpha'|_{\rho'}$. Since any polarity of ρ' extends uniquely to a polarity of $F(\rho') = \pi_r^{\mathcal{K}}$ (by 1.4.4), we have $\psi^{-1}\alpha\psi = \alpha'$.

We now consider conjugacy classes of polarities of $\pi_r^{\mathcal{K}}$ which have no absolute points outside $\mathcal{K}$. To determine these conjugacy classes we show that, for each such polarity α, there is an α-canonical HF process satisfying certain conditions. This is done in 3.2.3 for the case $\mathcal{K} \neq \phi$ and in 3.2.4 for $\mathcal{K} = \phi$. For the proof of these, we need the methods of obtaining a new α-canonical HF process from a given one provided in

<u>Lemma 3.2.2</u> : Suppose α is a polarity of $\pi_r^{\mathcal{K}}$ and P is an α-canonical HF process for $\pi_r^{\mathcal{K}}$.

(a) If x is a non-α-absolute P-HF point with P-bearer u, and a is a point of lower P-stage than x for which $a \not I u$, then there is an α-canonical HF process Q for which x is Q-free with Q-bearers u and x.a, $(a.x)\alpha$ is Q-HF with Q-bearer $x\alpha$, and the P- and Q-bearers of all other points of $\pi_r^{\mathcal{K}}$ coincide.

(b) If x is a non-α-absolute P-isolated point and a is a point of lower P-stage than x, then there is an α-canonical HF process Q for which x is Q-HF with Q-bearer a.x, $(a.x)\alpha$ is Q-HF with Q-bearer $a\alpha$, and the P- and Q-bearers of all other points of $\pi_r^{\mathcal{K}}$ coincide.

<u>Proof</u> : Suppose $P_0 = \mathcal{K}$ and m is an integer for which $\pi_r^{\mathcal{K}} = F(P_{2m})$, all P-isolated and P-HF points are contained in P_{2m}, and $P_{2n-1} = P_{2n-2} \cup \{a_n\}$, $P_{2n} = P_{2n-1} \cup \{a_n\alpha\}$, for $1 \leq n \leq m$.

(a) Because x is P-HF with P-bearer u and is non-α-absolute, $x\alpha$ is P-HF with P-bearer $u\alpha$, and both $x, x\alpha \in P_{2m}$. Suppose x and $x\alpha$ have P-stages $2k+1$ and $2k+2$ respectively. Define Q by

$$Q_i = \begin{cases} P_i, & 0 \leq i \leq 2k, \\ P_{2k} \cup \{(a.x)\alpha\}, & i = 2k+1, \\ P_{i-2} \cup \{(a.x)\alpha, a.x\}, & i \geq 2k+2. \end{cases}$$

It is easily verified that Q is an α-canonical HF process satisfying the given conditions.

(b) Because x is P-isolated and non-α-absolute, $x\alpha$ is also P-isolated, and both $x, x\alpha \in P_{2m}$. Suppose x and $x\alpha$ have P-stages $2k+1$ and $2k+2$ respectively. Define Q as in part (a). Again it is easily verified that Q satisfies the requirements of the lemma.

We denote the HF process defined in the proof of parts (a) and (b) of 3.2.2 by $\Delta_1(P,x,a)$ and $\Delta_2(P,x,a)$ respectively. Thus $\Delta_1(P,x,a)$ and $\Delta_2(P,x,a)$ satisfy (a) and (b) respectively of 3.2.2.

Lemma 3.2.3 : Suppose $K \neq \phi$ and α is a polarity of π_r^K with no absolute points outside K. Then, for any line ℓ of K, there is an α-canonical HF process Q for which

(a) there are no Q-isolated points;

(b) all Q-HF points have Q-bearer ℓ;

(c) π_r^K is the free completion of $K \cup \{$Q-HF elements$\}$.

Proof : Let P be an α-canonical HF process for π_r^K. We first obtain an α-canonical HF process R having no isolated points. If there are no P-isolated points, let R = P. Suppose now that x is a

P-isolated point. Then $st_P(\ell\alpha) < st_P(x)$. Let $P^{(1)} = \Delta_2(P, x, \ell\alpha)$. It follows from 3.2.2 (b) that $P^{(1)}$ has one less isolated point than P. Continuing in this way, we obtain the required HF process R after finitely many steps (as there are only finitely many P-isolated points).

We next obtain an α-canonical HF process S for which there are no S-isolated points, and no S-HF points having an S-bearer incident with $\ell\alpha$. If R satisfies these properties, let S = R. Suppose now that y is an R-HF point with R-bearer u, where u I $\ell\alpha$. There exists a point $a \in K$ for which $a \not I u$ and $a \not I \ell$. Define $R^{(1)} = \Delta_1(R, y, a)$. From 3.2.2 (a), $R^{(1)}$ has no isolated points and has one less HF point, the bearer of which is incident with $\ell\alpha$, than R. Continuing in this way, we obtain the required HF process S after finitely many steps.

We now obtain Q. If all S-HF points have ℓ as S-bearer, let Q = S. Suppose now that z is an S-HF point with S-bearer v, where $v \neq \ell$. Since there are no S-HF points having S-bearer incident with $\ell\alpha$, we have $v \not I \ell\alpha$. Define $S^{(1)} = \Delta_1(S, z, \ell\alpha)$. From 3.2.2 (a), there are no $S^{(1)}$-isolated points and $S^{(1)}$ has one less HF point, not having $\ell\alpha$ as bearer, than S. After finitely many steps, we obtain an α-canonical HF process Q satisfying (a) and (b). The configuration $\rho = K \cup \{\text{Q-HF elements}\}$ generates π_r^K (by 1.5.1(d)). From (a) and (b), it satisfies $Q(x) \subseteq \rho$ for all $x \in \rho$. Hence $[\rho]_{\pi_r^K} = F(\rho)$ (by 1.5.11) Thus $\pi_r^K = F(\rho)$ and (c) is satisfied.

We now prove a similar type of result for the case $K = \phi$. For later use, we also allow the possibility of α having one absolute point.

Lemma 3.2.4 : If α is a polarity of π_r with at most one absolute point, then there is an α-canonical HF process Q for π_r satisfying :

(a) There are exactly two Q-isolated, non-α-absolute points a_1 and a_2 .

(b) All non-α-absolute Q-HF points have Q-bearer $a_1\alpha$.

(c) If there are no α-absolute points, then $\rho = \{$Q-isolated and Q-HF elements$\}$ freely generates π_r.

Proof : Let P be an α-canonical HF process for π_r. Then $P_0 = \phi$ and there is an integer m for which $\pi_r = F(P_{2m})$, all P-isolated and P-HF elements are contained in P_{2m}, and $P_{2n-1} = P_{2n-2} \cup \{a_n\}$, $P_{2n} = P_{2n-1} \cup \{a_n\alpha\}$, $1 \leq n \leq m$.

We first show that we may assume the existence of at least two non-α-absolute P-isolated elements. By 3.1.2, we may assume a_1 is non-α-absolute. Suppose that a_1 is the only non-α-absolute P-isolated point. Let there be k non-α-absolute P-HF points with P-bearer $a_1\alpha$. If $k \geq 1$, then we may assume that $a_2, \ldots, a_{k+1}$ have this property. If $k \geq 2$, then we may redefine P, making a_2, $a_2\alpha$, a_3 and $a_3\alpha$ P-isolated, and a_1 and $a_1\alpha$ P-free with P-bearers $a_2\alpha, a_3\alpha$ and a_2, a_3 respectively. If $k = 1$, and there is a non-α-absolute P-HF point with P-bearer $a_2\alpha$, then we may assume a_3 has this property. We redefine P, making a_3 and $a_3\alpha$ P-isolated, and a_2 and $a_2\alpha$ P-free with P-bearers $a_1\alpha$, $a_3\alpha$ and a_1, a_3 respectively. Suppose now that either $k = 1$ and there is no non-α-absolute P-HF point with P-bearer a_2 , or $k = 0$. Then an inspection of the few possible cases, using the existence of at most one α-absolute point, shows that P can always be redefined to have at least two non-α-absolute isolated points.

From the previous paragraph, we may assume that a_1 and a_2 are non-α-absolute P-isolated points. If x is another non-α-absolute P-isolated point, then define $P^{(1)} = \Delta_2(P,x,a_1)$. From 3.2.2 (b), $P^{(1)}$ has one fewer non-α-absolute isolated point than P. We continue in this way. After finitely many steps, we obtain an α-canonical HF process R for which there are exactly two non-α-absolute R-isolated points a_1 and a_2.

We next obtain an α-canonical HF process S for which there are exactly two non-α-absolute S-isolated points a_1 and a_2, and all S-HF points have either $a_1\alpha$ or $a_1 \cdot a_2$ as S-bearer. If all non-α-absolute R-HF points have either $a_1\alpha$ or $a_1 \cdot a_2$ as S-bearer, let S = R. Suppose now that there is a non-α-absolute R-HF point x with R-bearer u, where $u \neq a_1\alpha$ and $u \neq a_1 \cdot a_2$. For $u \not I a_1$, define $R^{(1)} = \Delta_1(R,x,a_1)$. If $u \, I \, a_1$, then $u \not I a_2$ (as $u \neq a_1 \cdot a_2$). Thus, for $u \, I \, a_1$, we may define $R^{(1)'} = \Delta_1(R,x,a_2)$ and $R^{(1)} = \Delta_1(R^{(1)'},(a_2 \cdot x)\alpha,a_1)$. In either case, it follows from 3.2.2 (a) that $R^{(1)}$ has one fewer HF point, not having either $a_1\alpha$ or $a_1 \cdot a_2$ as bearer, than R. Continuing in this way, we obtain the required HF process S after finitely many steps (as there are only finitely many R-HF elements).

Finally, we obtain the required HF process Q. If there are no non-α-absolute S-HF points with S-bearer $a_1 \cdot a_2$, let Q = S. Suppose now there is a non-α-absolute S-HF point y with S-bearer $a_1 \cdot a_2$. We consider three cases. In each, we define an α-canonical HF process $S^{(1)}$ for which there are no $S^{(1)}$-isolated points, all non-α-absolute $S^{(1)}$-HF points have either $a_1 \cdot a_2$ or $a_1\alpha$ as $S^{(1)}$-bearer, and $S^{(1)}$ has one fewer non-α-absolute HF point with bearer $a_1 \cdot a_2$ than has S.

<u>Case (1)</u> : There exists an S-HF point a with S-bearer $a_1\alpha$:
In this case $a \not\subset a_1 \cdot a_2$ and we may assume $st_S(a) < st_S(y)$. We may therefore define $T = \Delta_1(S, y, a)$, $T^{(1)} = \Delta_1(T, (a \cdot y)\alpha, a_2)$, and $S^{(1)} = \Delta_1(T^{(1)}, (a \cdot y) \cdot (a_2 \alpha), a_1)$.

<u>Case (2)</u> : There are no S-HF points with S-bearer $a_1\alpha$, but there is an α-absolute point b for which $b \not\subset a_1 \cdot a_2$, $b \not\subset y\alpha$, $b \not\subset a_1\alpha$:
It is easily verified that we may assume $st_S(b) < st_S(y)$. Since $b \not\subset a_1 \cdot a_2$, we may define $T = \Delta_1(S, y, b)$ and $S^{(1)} = \Delta_1(T, (y \cdot b)\alpha, a_1)$.

In both cases (1) and (2), one uses 3.2.2 (a) to verify that $S^{(1)}$ has the required properties. If neither case (1) nor (2) is satisfied, then the following case (3) holds, because otherwise $\overline{S}$ would be degenerate, contradicting $\overline{S} = \pi_r$.

<u>Case (3)</u> : There is an α-absolute point c with $y\alpha$ as S-bearer :
In this case $st_S(y) < st_S(c)$. Let $st_S(y) = 2n + 1$. Define $S^{(1)}$ by

$$S_i^{(1)} = \begin{cases} S_i\,, & 0 \leq i \leq 2n, \\ S_{2n} \cup \{c\}\,, & i = 2n+1, \\ S_{i-2} \cup \{c, c\alpha\}, & i > 2n+1. \end{cases}$$

With this definition, c is $S^{(1)}$-isolated, y is $S^{(1)}$-free with bearers $a_1 \cdot a_2$ and $c\alpha$, and the $S^{(1)}$- and S-bearers of all other elements of π_r coincide. Thus $S^{(1)}$ satisfies the required conditions in this case.

If there are no non-α-absolute $S^{(1)}$-HF elements with $a_1 \cdot a_2$ as

$S^{(1)}$-bearer, define $Q = S^{(1)}$. Otherwise, we repeat the above process, and after finitely many steps we obtain an α-canonical HF process Q satisfying (a) and (b). We show Q satisfies (c). Suppose there is no α-absolute point. Then all Q-HF and Q-isolated elements are non-α-absolute. Let ρ be the configuration consisting of these elements. Since $Q_o = \phi$, we have $[\rho]_{\pi_r} = \pi_r$ (by 1.5.1(d)). It follows from (a) and (b) that $Q(x) \subseteq \rho$ for all $x \in \rho$. Thus $[\rho]_{\pi_r} = F(\rho)$ (by 1.5.11). Hence $\pi_r = F(\rho)$, and (c) is satisfied.

The major consequence of 3.2.3 and 3.2.4 is

<u>Theorem 3.2.5</u> : Two polarities α and α' of π_r^K, both having no absolute points outside K, are conjugate with respect to a collineation of π_r^K if, and only if, $\alpha|_K$ and $\alpha'|_K$ are conjugate with respect to a collineation of K.

<u>Proof</u> : First assume α and α' are conjugate with respect to a collineation ψ of π_r^K. Since $K\alpha = K\alpha' = K\psi = K$ (by 1.6.2), α and $\alpha'|_K$ are conjugate with respect to the collineation $\psi|_K$ of K.

Conversely, assume $\alpha|_K$ and $\alpha'|_K$ are conjugate with respect to the collineation ψ of K. Suppose $\alpha'|_K = \psi^{-1}(\alpha|_K)\psi$. We consider two cases.

(1) <u>$K \neq \phi$</u> : Because there are no α-absolute points, $r \equiv 0 \pmod 2$ (by 3.1.4 (a)). Choose a line ℓ of K. By 3.2.3, there is an

α-canonical HF process Q for which $\pi_r^K = F(Q_r)$ and, if $r > 0$, then $Q_{2n-1} = Q_{2n-2} \cup \{a_n\}$, $Q_{2n} = Q_{2n-1} \cup \{a_n\alpha\}$, where a_n (resp. $a_n\alpha$) is a Q-HF point (line) with Q-bearer ℓ ($\ell\alpha$), $1 \leq n \leq \frac{r}{2}$. Also by 3.2.3, there is an α'-canonical HF process Q' for which $Q_0' = K$, $\pi_r^K = F(Q_r')$ and, if $r > 0$, then $Q_{2n-1}' = Q_{2n-2}' \cup \{a_n'\}$, $Q_{2n}' = Q_{2n-1}' \cup \{a_n'\alpha'\}$, where a_n' (resp. $a_n'\alpha'$) is a Q'-HF point (line) with Q'-bearer $\ell\psi$ ($\ell\psi\alpha'$), $1 \leq n \leq \frac{r}{2}$. We have $\pi_r^K = F(Q_r) = F(Q_r')$ and $Q_r\alpha = Q_r$, $Q_r'\alpha' = Q_r'$. The collineation ψ of K extends to an isomorphism of Q_r onto Q_r' by defining $a_n\psi = a_n'$ and $(a_n\alpha)\psi = a_n'\alpha'$. From this definition, we obtain $a_n'(\psi^{-1}\alpha\psi) = a_n'\alpha'$ and $(a_n'\alpha')(\psi^{-1}\alpha\psi) = a_n' = (a_n'\alpha')\alpha'$. We therefore have $\psi^{-1}(\alpha|_{Q_r})\psi = \alpha'|_{Q_r}$, (using the assumption $\psi^{-1}(\alpha|_K)\psi = \alpha'|_K$). By 3.2.1, α and α' are conjugate with respect to a collineation of π_r^K.

(2) <u>$K = \phi$</u> : In this case $\alpha|_K$ and $\alpha'|_K$ are both the trivial polarity. One proves that α and α' are conjugate in the same way as case (1), except that one uses 3.2.4 instead of 3.2.3.

<u>Corollary 3.2.6</u> : If $r \equiv 0 \pmod 2$ and $K \neq \phi$, then the number of conjugacy classes of polarities of π_r^K having no absolute points outside K equals the number of conjugacy classes of polarities of K.

<u>Proof</u> : By the previous theorem, it suffices to show that any polarity of K extends to a polarity of π_r^K having no absolute points outside K. This is shown in 3.1.4.

Corollary 3.2.7 (Glock (8)) : If $r > 8$ and $r \equiv 0 (\mathrm{mod}\ 2)$, then the polarities of π_r with no absolute points form one conjugacy class.

Proof : By the second part of 3.1.4, there exists a polarity of π_r with no absolute point. Thus there is at least one conjugacy class of such polarities. By 3.2.5, there is at most one conjugacy class, since any two polarities of the empty configuaration are conjugate (trivially).

We now use 3.2.4 to help prove two theorems concerning polarities of π_8 and π_9. In the first of these, we consider a possibility excluded from the second part of 3.1.4 (namely $k = \phi$, $r = 8$ and $j = 0$).

Theorem 3.2.8 (Abbiw-Jackson (1)) : All polarities of π_8 have two absolute points and they form one conjugacy class.

Proof : By 3.1.4 (a), (b) and (c), a polarity of π_8 has either two absolute points or none. Suppose α is a polarity of π_8 with no absolute points. By 3.2.4, there is an α-canonical HF process Q for which there are two non-α-absolute Q-isolated points a_1 and a_2, and for which π_8 is the free completion of the Q-isolated and Q-HF elements. Since there are four Q-isolated elements ($a_1, a_2, a_1\alpha$ and $a_2\alpha$), there are no Q-HF elements, as $r = 8$. Hence $\pi_8 = F(\rho)$, where $\rho = \{a_1, a_2, a_1\alpha, a_2\alpha\}$. But $F(\rho)$ is degenerate and π_8 is non-degenerate, a contradiction. Thus no such polarity α exists. All polarities of π_8 therefore have two absolute points.

By the second part of 3.1.4, there is a polarity of π_8 with two absolute points. Thus there is at least one conjugacy class of polarities of π_8. It remains to show that any two polarities α and α' of π_8 are conjugate. Suppose Q is an α-canonical HF process for π_8 for which $Q_0 = \phi$, $Q_{2n-1} = Q_{2n-2} \cup \{a_n\}$, $Q_{2n} = Q_{2n-1} \cup \{a_n\alpha\}$, $0 < n \leq m$, and $\pi_r = F(Q_{2m})$. We may assume m is minimal with respect to the property $\pi_r = F(Q_{2m})$. By 3.1.2, we may assume that the Q-isolated point a_1 is non-α-absolute. An inspection of possible cases, using 1.5.13 , shows that $m \leq 5$ and that Q_{2m} is one of the configurations illustrated (see figs. (1) to (4); a relabelling of the a_i's, $1 \leq i \leq 5$, may be necessary for Q_{2m} to equal one of these four configurations). In each of figs. (1) to (4), there is a relabelling of the a_i's (and a consequent redefinition of Q) such that a_1 and a_2 are α-absolute Q-isolated points, a_3 is a non-α-absolute Q-HF point with Q-bearer a_1, and $\pi_8 = F(Q_6)$. Similarly, one proves the existence of an α'-canonical HF process Q' satisfying $\pi_8 = F(Q_6')$ and $Q_6' = \{a_i', a_i'\alpha';\ 1 \leq i \leq 3\}$, where a_1' and a_2' are α'-absolute Q'-isolated points, and a_3' is a non-α'-absolute Q'-HF point with Q'-bearer $a_1'\alpha'$. The isomorphism $\psi : Q_6 \to Q_6'$ defined by $a_n\psi = a_n'$ and $a_n\alpha\psi = a_n'\alpha'$ satisfies $\psi^{-1}(\alpha|_{Q_6}) = \alpha'|_{Q_6'}$. By 3.2.1, α and α' are conjugate.

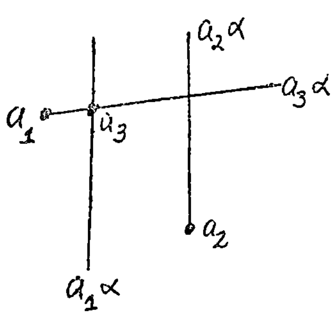


Fig.(1)

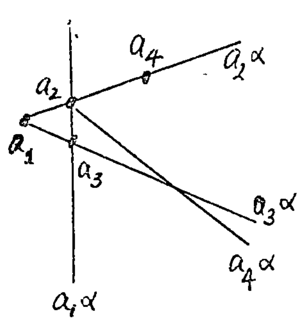


Fig.(2)

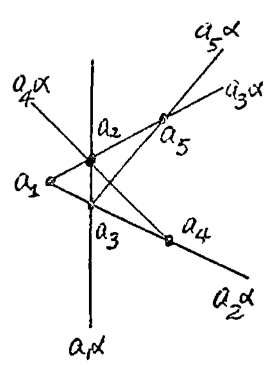


Fig.(3)

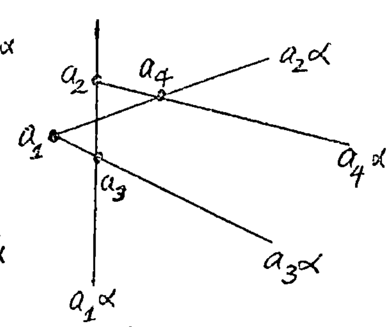


Fig.(4)

Theorem 3.2.9 (Glock (8)) : The polarities of π_9 having one absolute point form one conjugacy class.

Proof : By the second part of 3.1.4, there exists a polarity α of π_9 with one α - absolute point. Thus there is at least one conjugacy class of such polarities. It remains to show that any two polarities α and α' of π_9, both having one absolute point, are conjugate. Let Q and Q' be α- and α'-canonical HF processes respectively. Suppose $Q_{2n-1} = Q_{2n-2} \cup \{a_n\}$, $Q_{2n} = Q_{2n-1} \cup \{a_n\alpha\}$, $1 \leq n \leq m$, and $\pi_9 = F(Q_{2m})$. By 3.2.4, we may assume a_1 and a_2 are Q-isolated and non-α-absolute. Since $a_1, a_2, a_1\alpha$ and $a_2\alpha$ are all Q-isolated, there is only one Q-HF point b, which is α-absolute (by 3.1.3). Because $\pi_9 = \overline{Q}$ is non-degenerate, b does not have Q-bearer $a_1 . a_2$. Thus b has either $a_1\alpha$ or $a_2\alpha$ as Q-bearer. We may assume that $a_1\alpha$ is the Q-bearer and that $b = a_3$. Thus $\pi_9 = F(Q_6)$, by 1.5.13. Similarly, we may assume that $\pi_9 = F(Q_6')$ and $Q_6' = \{a_i', a_i\alpha'; 1 \leq i \leq 3\}$, where a_1', a_2' are non-α'-absolute Q'-isolated points, and a_3' is an α'-absolute Q'-HF point with Q'-bearer $a_1'\alpha'$. The isomorphism $\psi : Q_6 \to Q_6'$ defined by $a_i\psi = a_i'$ and $a_i\alpha\psi = a_i'\alpha'$, $i = 1,2,3$, satisfies $\psi^{-1}(\alpha|Q_6)\psi = \alpha'|Q_6'$. Thus α and α' are conjugate, by 3.2.1.

We note that although there is only one conjugacy class of polarities of π_9 having one absolute point, there are infinitely many such polarities. To show this, we note that if such a polarity α has x as its absolute point, and ψ is any collineation of π_9, then $x\psi$ is an absolute point of $\psi^{-1}\alpha\psi$. If $x\psi \neq x$, then $\psi^{-1}\alpha\psi \neq \alpha$, since they have different absolute points. Thus, if G_9 is the full collineation group of π_9, then to each $y \in xG_9$ there is at least one polarity of π_9 having y as its absolute point. By theorem 2.3 of (5), xG_9 is infinite. Hence there are infinitely many polarities of π_9 having one absolute point. Similarly, it follows that C is infinite for any conjugacy class C of polarities of π_r having at least one absolute point.

<u>Theorem 3.2.10</u> : If κ is a non-empty subplane of π_1^κ, then there are at most $\frac{n\,|\kappa|}{2}$ conjugacy classes of polarities of π_1^κ, where n is the number of conjugacy classes of polarities of κ.

<u>Proof</u>: If n = 0, then κ has no polarities and therefore π_1^κ has no polarities (as if α is a polarity of π_1^κ, then $\alpha|_\kappa$ is a polarity of κ). Thus the theorem is true for n = 0. Henceforth we assume $n > 0$.

We now define a set X of $\frac{n\,|\kappa|}{2}$ polarities of π_1^κ. Choose representatives $\alpha_1,\dots,\alpha_n$ from the conjugacy classes of polarities of . For each line ℓ of κ and $i \in \{1,\dots,n\}$, define a configuration $\rho_i^{(\ell)}$ as follows :

$\rho_i^{(\ell)} = \kappa \cup \{a_i^{(\ell)}, b_i^{(\ell)}\}$, where $a_i^{(\ell)}$ is a point incident only with ℓ, and $b_i^{(\ell)}$ is a line incident with $a_i^{(\ell)}$ and $\ell\alpha_i$ only.

Clearly $F(\rho_i^{(\ell)}) = \pi_1^\kappa$. Define a polarity $\alpha_i^{(\ell)}$ of $\rho_i^{(\ell)}$ by $x\alpha_i^{(\ell)} = x\alpha_i$ for $x \in \kappa$, and $a_i^{(\ell)}\alpha_i = b_i^{(\ell)}$. By 1.4.4, $\alpha_i^{(\ell)}$ extends uniquely to a polarity of $F(\rho_i^{(\ell)}) = \pi_1^\kappa$.

Let $X = \left\{\alpha_i^{(\ell)} \; ; \quad 1 \leq i \leq n, \; \ell \text{ a line of } \kappa\right\}$. Since κ is a non-degenerate plane, κ has $\frac{|\kappa|}{2}$ lines (by 1.1.1). Thus $|X| = \frac{n.|\kappa|}{2}$.

We next show that any polarity β of π_1^κ is conjugate to a polarity in X. Let Q be a β-canonical HF process for π_1^κ. We have $F(Q_0) = F(\kappa) \subsetneqq \pi_1^\kappa$. Hence there is an $m > 0$ for which

$Q_{2k-1} = Q_{2k-2} \cup \{a_k\}$, $Q_{2k} = Q_{2k-1} \cup \{a_k\beta\}$, $0 < k \leq m$, and

$F(Q_{2m}) = \pi_1^\kappa$. If a_1 is Q-free, then $a_1 = x.y$ for some lines $x, y \in \kappa$. Since κ is a plane, $x.y \in \kappa$, contradicting $a_1 \notin \kappa$. Thus a_1 is either Q-HF or Q-isolated. As $r = 1$, there are no Q-isolated elements and one Q-HF element. Thus a_1 is the only Q-HF element. Suppose it has line $\ell \in \kappa$ as Q-bearer. Then $a_1\beta$ is Q-free with Q-bearers $\ell\beta$ and a_1, and $\pi_1^\kappa = F(Q_2)$ (by 1.5.13).

By the definition of $\alpha_1, \ldots, \alpha_n$, α_i and $\beta|_\kappa$ are conjugate with respect to some collineation ψ of κ, for some i. Let $\beta|_\kappa = \psi^{-1} \alpha_i \psi$. Then ψ extends to an isomorphism of $\rho_i^{(\ell\psi^{-1})}$ onto Q_2 by defining $a_i^{(\ell\psi^{-1})}\psi = a_1$ and $b_i^{(\ell\psi^{-1})}\psi = a_1\beta$. This isomorphism satisfies $\psi^{-1}(\alpha_i^{(\ell\psi^{-1})}\Big|_{\rho_i^{(\ell\psi^{-1})}})\psi = \beta|_{Q_2}$.

Hence $\alpha_i^{(\ell\psi^{-1})}$ and β are conjugate (by 3.2.1).

Since every polarity of π_1^κ is conjugate to a polarity in X, there are at most $|X| = \frac{n.|\kappa|}{2}$ conjugacy classes of polarities of π_1^κ.

We note that $\frac{n.|K|}{2}$ is only an upper bound because it is possible for two polarities in X to be conjugate. In fact, $\alpha_i^{(\ell)}$ and $\alpha_i^{(\ell')}$ are conjugate if and only if there is a collineation ψ of K such that $\ell\psi = \ell'$ and $\ell\alpha_i\psi = \ell'\alpha_i$. Such a ψ may, but does not necessarily, exist.

In our final theorem, we consider the conjugacy of polarities of π_r^K having j absolute points outside K, for all possible r, j and K satisfying (a), (b) and (c) of 3.1.4 but not dealt with in theorems 3.2.5 to 3.2.10.

Theorem 3.2.11 : Suppose that j and r are positive integers for which $1 \leq j \leq r$ and $j \equiv r(\text{mod}2)$, and that π_r^K satisfies :

(a) If $K \neq \phi$ and $r = 1$, then K is not a plane;

(b) If $K = \phi$, then $j \leq r - 6$ and $r + j \geq 12$.

Then, to each conjugacy class C of polarities of K, there are infinitely many conjugacy classes of polarities α of π_r^K satisfying $\alpha|_K \in C$ and having j absolute points outside K.

Proof: Let C' be a conjugacy class of polarities α of π_r^K satisfying $\alpha|_K \in C$ and having j absolute points outside K. If α and α' are two polarities in C', then $\alpha' = \psi^{-1}\alpha\psi$ for some $\psi \in G_r^K$, the full collineation group of π_r^K. If α has absolute points $a_1, \ldots, a_j$ outside K, then α' has absolute points $a_1\psi, \ldots, a_j\psi$ outside K. Thus the set $X = \{a;\ a$ is an α-absolute point outside K for some $\alpha \in C'\}$ is contained in at most j G_r^K-orbits. Suppose there were only finitely many conjugacy classes of polarities α

satisfying $\alpha|_K \in C$ and having j absolute points outside K. Then $Y = \{a;$ a is an α-absolute point outside K of such an $\alpha\}$ would be contained in only finitely many G_r^K-orbits. Thus, to prove the theorem, it suffices to find a sequence $(\alpha_i)_{i=0}^{\infty}$ of polarities of π_r^K such that

(1) each α_i has j absolute points outside K and $\alpha_i|_K \in C$;

(2) there is a sequence $(a_i)_{i=0}^{\infty}$ of points of π_r^K for which a_i is an α_i-absolute point $\forall\ i \in N$, and if $i \neq k$, there is no collineation ψ of π_r^K satisfying $a_i\psi = a_k$.

We first show that π_{r-1}^K exists and $\pi_{r-1}^K \neq K$. If $K \neq \phi$, then π_{r-1}^K exists because $r \geq 1$ and by 1.6.14. If $K = \phi$, then from assumption (b) we have $2r - 6 = r + (r-6) \geq r + j \geq 12$. Hence $r \geq 9$ and π_{r-1} exists. The possibility of $\pi_{r-1}^K = K$ occurs only when $r = 1$ and $\pi_0^K = F(K) = K \neq \phi$. But K is not a plane when $r = 1$ (by assumption (a)), so $F(K) \neq K$ (by 1.4.3(d)). Thus $\pi_{r-1}^K \neq K$.

Let P be a HF process for π_{r-1}^K from K. Suppose $z \in \pi_{r-1}^K$ and ρ is an almost-confined configuration with vertex z such that $\rho \cap K = \phi$. Then $\rho \subseteq P(z)$, by 1.7.9. Because $\pi_{r-1}^K \neq K$, it follows from 1.7.10 that π_{r-1}^K has a sequence of lines $(z_i)_{i=0}^{\infty}$ and almost confined configurations $(\rho_i)_{i=0}^{\infty}$ such that ρ_i has vertex z_i, $\rho_i \cap K = \phi$, and $|\rho_i| > |P(z_k)|$, $\forall\ k < i$, $i = 1,2,\ldots$.

We next show that π_{r-1}^K has a polarity α such that $\alpha|_K \in C$ and α has $j - 1$ absolute points outside K. Choose a polarity α' of K

such that $\alpha' \in C$. Because $0 < j \leq r$ and $j \equiv r(\text{mod}2)$, we have $0 \leq j - 1 \leq r - 1$ and $(j-1) \equiv (r-1)$ mod2. Thus, by 3.1.4, α' extends to a polarity α of π^{κ}_{r-1} with $j - 1$ absolute points outside κ, unless $r - 1 = 8$, $\kappa = \phi$ and $j - 1 = 0$. But $r - 1 = 8$, $\kappa = \phi$ and $j - 1 = 0$ imply $r + j = 10 < 12$, contradicting (b). Thus, in all cases, α' extends to the required polarity of π^{κ}_{r-1}.

We next show that π^{κ}_{r-1} is a subplane of π^{κ}_{r}, and we define a sequence of polarities $(\alpha_i)_{i=0}^{\infty}$ of π^{κ}_{r} such that $\alpha_i \big| \pi^{\kappa}_{r-1} = \alpha$ for each i. Suppose $i \in N$. Then $\rho_i\alpha$ is an almost-confined configuration with vertex $z_i\alpha$. If Q is an α-canonical HF process for π^{κ}_{r-1} from κ, then $\rho_i\alpha \subseteq Q(z_i\alpha)$ (by 1.7.9) and thus $z_i\alpha$ is incident with two lines of its Q-socle. Hence $z_i\alpha$ is Q-free. By 3.1.3, $z_i\alpha$ is non-α-absolute. Hence $z_i \not I z_i\alpha$. Define a HF process $Q^{(i)}$ by

$Q_0^{(i)} = \pi^{\kappa}_{r-1}$,

$Q_1^{(i)} = \pi^{\kappa}_{r-1} \cup \{a_i\}$, where a_i is $Q^{(i)}$-HF with bearer z_i,

$Q_2^{(i)} = Q_1^{(i)} \cup \{b_i\}$, where b_i is a $Q^{(i)}$-free line with $Q^{(i)}$-bearers $z_i\alpha$ and a_i (this is well defined because $z_i \not I z_i\alpha$),

$Q_n^{(i)} = F_{n-2}(Q_2^{(i)})$, $n > 2$.

Clearly $r(Q^{(i)}) = 1$. Define $R^{(i)} = P + Q^{(i)}$. Then $r(R^{(i)}) = r(P) + r(Q^{(i)}) = (r-1) + 1 = r$. Hence $R^{(i)}$ is a HF process for π_r^κ from κ. Thus $\overline{Q^{(i)}} = F(Q_2^{(i)}) = \pi_r^\kappa$. We extend α to a polarity α_i of $Q_2^{(i)}$ by defining $a_i \alpha_i = b_i$. By 1.4.4, α_i extends uniquely to a polarity of π_r^κ such that $Q_n^{(i)}\alpha_i = Q_n^{(i)}$ $\forall\, n \geq 2$.

We now show that $(\alpha_i)_{i=0}^{\infty}$ satisfy (1) and (2). We have $\alpha_i|_\kappa = \alpha|_\kappa = \alpha' \in C$. Because $Q_n^{(i)}\alpha_i = Q_n^{(i)}$ $\forall\, n \geq 2$, we have $st_{Q^{(i)}}(u) = st_{Q^{(i)}}(u\alpha_i)$ $\forall\, u \notin Q_2^{(i)}$. Hence $u \not\mathrel{I} u\alpha_i$ $\forall\, u \notin Q_2^{(i)}$, and all α_i-absolute points are contained in $Q_2^{(i)}$. Thus α_i has j absolute points outside κ, since a_i is an α_i-absolute point and there are $j-1$ α-absolute points in π_{r-1}^κ outside κ. Hence (1) is satisfied. We show (2). By definition, a_i is an α_i-absolute point outside κ for each $i \in N$. Suppose that $i \neq k$ and there is a collineation ψ of π_r^κ for which $a_i\psi = a_k$. We may assume $i > k$. We first show that $z_i\psi = z_k$. Because $a_i \mathrel{I} z_i$, we have $a_k \mathrel{I} z_i\psi$. Thus either a_k is an $R^{(k)}$-bearer of $z_i\psi$, or vice-versa. Assume the former. Then $a_k \in \rho_i\psi$, since $\rho_i\psi$ is an almost-confined configuration with vertex $z_i\psi$, and $\rho_i\psi \subseteq R^{(k)}(z_i\psi) \cup \kappa$ (by 1.7.9). But $a_k \in \rho_i\psi$ implies $a_k\psi^{-1} = a_i \in \rho_i$, a contradiction (because $\rho_i \subseteq \pi_{r-1}^\kappa$ and $a_i \notin \pi_{r-1}^\kappa$). Hence $z_i\psi$ is an $R^{(k)}$-bearer of a_k. But a_k has only one $R^{(k)}$-bearer, namely z_k. Therefore $z_i\psi = z_k$. Hence $\rho_i\psi$ has vertex z_k. By 1.7.9, $\rho_i\psi \subseteq R^{(k)}(z_k) \cup \kappa$. Because $\rho_i \cap \kappa = \phi$ and $\kappa\psi = \kappa$, we have $\rho_i\psi \cap \kappa = \phi$. In addition, $R^{(k)}(z_k) = P(z_k)$, because $z_k \in \overline{P} = \pi_{r-1}^\kappa$. Hence $\rho_i\psi \subseteq P(z_k)$. Therefore $|\rho_i| = |\rho_i\psi| \leq |P(z_k)|$. But $i > k$ implies

$|\rho_i| > |P(z_k)|$ (by definition of ρ_n for each $n \in N$). This is a contradiction, so there is no collineation ψ satisfying $a_i\psi = a_k$. Thus (2) is satisfied.

This completes the proof of the theorem.

Corollary 3.2.12 (Glock (8)) : If $r \equiv j \pmod 2$, $1 \leq j \leq r - 6$, and $r + j \geq 12$, then there are infinitely many conjugacy classes of polarities of π_r having j absolute points.

Hence, in 3.2.11, we have extended Glock's result to all planes having finite free rank.

CHAPTER 4

NON-FREE RANK PLANES

In this chapter, we consider planes π for which $\pi \neq \kappa(\pi)$ and which do not have free rank. The existence of such planes was first proved by Kopejkina (17), who gave a construction for a plane having empty core and not having free rank (such planes, along with free planes, are called <u>open</u>). We show that any π_r^κ can be embedded in a plane π not having free rank in such a way that any collineation group G of π_r^κ extends to a collineation group of π.

We first give a generalization of Kopejkina's construction.

<u>Theorem 4.1</u> : Suppose that $(\pi^{(i)})_{i=0}^{\infty}$ is a strictly increasing sequence of non-degenerate free rank planes for which $\kappa = \kappa(\pi^{(1)}) = \kappa(\pi^{(2)}) = \dots$. If $\pi^{(i)}$ has rank r_i and there is a finite m for which $r_i \leq m$ for all i, then $\pi = \bigcup_{i=0}^{\infty} \pi^{(i)}$ is a plane having core κ and not having free rank.

<u>Proof</u>: We first show that π is a plane having core κ. Let x and y be distinct points of π. Then both $x, y \in \pi^{(i)}$ for some i. As $\pi^{(i)}$ is a plane, there is a unique line $\ell \in \pi^{(i)}$ incident with both x and y. Since $\ell \in \pi^{(j)}$ $\forall$ $j \geq i$ and each $\pi^{(j)}$ is a plane, ℓ is the only line of π incident with both x and y. Similarly, any two lines of π intersect in exactly one point. Thus π is a plane. Clearly $\kappa \subseteq \kappa(\pi)$. If ρ is any confined configuration of π, then ρ is finite and is therefore contained in $\pi^{(i)}$ for some i. Hence $\rho \subseteq \kappa$. This implies $\kappa(\pi) \subseteq \kappa$. Hence $\kappa(\pi) = \kappa$.

We now show π does not have free rank. Assume, on the contrary, that π has free rank. Then there is a HF process P for π from κ. Let X be the set of P-isolated and P-HF elements. By 1.5.1(d), $[\kappa \cup X]_\pi = \pi$. If π has finite rank, then X is finite, and hence $\kappa \cup X \subseteq \pi^{(i)}$ for some i. This implies $\pi = [\kappa \cup X]_\pi = \pi^{(i)}$, a contradiction. Thus π has infinite rank k. Hence we may assume P is given by $P_0 = \kappa$, $P_1 = \kappa \cup X$, where X is a set of k P-isolated points, and $P_n = F_{n-1}(P_1)$, $n > 1$. By 1.5.7, $Q^{(i)} = \pi \cap P$ is a HF process for $\pi^{(i)}$ from κ. For each i, $X \cap \pi^{(i)}$ is a set of $Q^{(i)}$-isolated points. Since $r_i \leq m$, we have $|X \cap \pi^{(i)}| \leq \frac{m}{2}$ for each i. Hence the increasing sequence $(X \cap \pi^{(i)})_{i=0}^{\infty}$ of sets satisfies

$\left| \bigcup_{i=0}^{\infty} (X \cap \pi^{(i)}) \right| \leq \frac{m}{2}$. But we also have $\bigcup_{i=0} (X \cap \pi^{(i)}) =$

$X \cap (\bigcup_{i=0}^{\infty} \pi^{(i)}) = X \cap \pi = X$, and $|X| = k$. Since k is infinite and

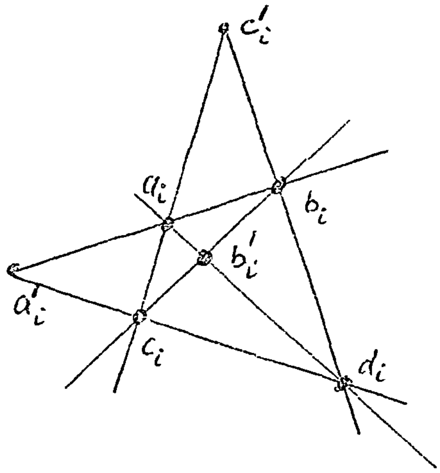
$\frac{m}{2}$ is finite, this is a contradiction. Thus π does not have free rank.

We note that Kopejkina proved the above theorem for $\kappa = \phi$ and $r_i = 8$ for each i.

If, in the above theorem, all r_i's are equal to some r, then the plane π is <u>r-uniform</u>.

<u>Theorem 4.2</u> : For each r and κ for which π_r^κ exists, there is an (r+8)-uniform plane π which contains π_r^κ, has core κ, and has the property that any collineation group of π_r^κ extends to a collineation group of π.

Proof : We first define an infinite sequence $(\pi^{(i)})_{i=0}^{\infty}$ of free rank planes, each having core κ and rank r + 8 and containing π_r^{κ}. Let P be a HF process for π_r^{κ} from κ. For each $i \geq 0$, define a HF process $Q^{(i)}$ by



$Q_0^{(i)} = \pi_r^{\kappa}$,

$Q_1^{(i)} = \pi_r^{\kappa} \cup \{a_i, b_i, c_i, d_i\}$, where $a_i, \ldots, d_i$ are isolated points,

$Q_2^{(i)} = Q_1^{(i)} \cup \{a_i \cdot b_i, a_i \circ c_i, a_i \circ d_i, b_i \circ c_i, b_i \circ d_i, c_i \circ d_i\}$,

$Q_3^{(i)} = Q_2^{(i)} \cup \{a_i', b_i', c_i'\}$, where $a_i' = (a_i \cdot b_i) \circ (c_i \cdot d_i)$, $b_i' = (a_i \cdot d_i) \circ (b_i \cdot c_i)$, $c_i' = (a_i \circ c_i) \circ (b_i \circ d_i)$,

$Q_n^{(i)} = F_{n-3}(Q_3^{(i)})$ $n > 3$.

We observe that $\overline{Q^{(i)}} = F(Q_1^{(i)})$. There are four $Q^{(i)}$-isolated elements (a_i, b_i, c_i and d_i) and no $Q^{(i)}$-HF elements. Therefore $r(Q^{(i)}) = 8$. We have $\underline{P + Q^{(i)}} = \kappa$, $\overline{P + Q^{(i)}} = \overline{Q^{(i)}}$ and $r(P + Q^{(i)}) = r(P) + r(Q^{(i)}) = r + 8$. Hence $\overline{Q^{(i)}}$ is a free rank plane having core κ and rank r + 8. It contains π_r^{κ} and so is non-degenerate. Let $\pi^{(i)} = \overline{Q^{(i)}}$, $i = 0, 1, \ldots$.

We next show that we may assume $\pi^{(i-1)}$ is a proper subplane of $\pi^{(i)}$, $i = 1, 2, \ldots$. For each such i, define

$\rho_i = \pi_r^{\kappa} \cup \{a_i', b_i', c_i', d_i', a_i' \circ d_i', b_i' \circ d_i', c_i' \cdot d_i'\}$, where $d_i' = b_i$.

Then ρ_i is a proper closed subconfiguration of $Q_3^{(i)}$. Hence

$[\rho_i]_{\pi^{(i)}} = [\rho_i]_{F(Q_3^{(i)})} = F(\rho_i) \subsetneqq \pi^{(i)}$ (by 1.5.14). Let $\pi^{(i)'} = F(\rho_i)$. Then $\pi^{(i)'}$ is the free completion of $\pi_r^{\kappa} \cup \{a_i', b_i', c_i', d_i'\}$, which is isomorphic to $Q_1^{(i-1)}$. Hence, by 1.4.2, $\pi^{(i)'} \cong F(Q_1^{(i-1)}) = \pi^{(i-1)}$, the isomorphism being uniquely determined by the mapping $a_i' \to a_{i-1}$, $b_i' \to b_{i-1}$, $c_i' \to c_{i-1}$, $d_i' \to d_{i-1}$. Thus we may identify $\pi^{(i)'}$ and $\pi^{(i-1)}$, and assume $a_i' = a_{i-1}$, $b_i' = b_{i-1}$, etc. Hence we may assume $\pi^{(i-1)}$ is a proper subplane of $\pi^{(i)}$. Define $\pi = \bigcup_{i=0}^{\infty} \pi^{(i)}$. By 4.1, π has core κ and is an (r+8)-uniform plane not having free rank and containing π_r^{κ}.

Finally, we show that any collineation group G of π_r^{κ} extends to a collineation group of π. For each i, G extends to a collineation group $G^{(i)}$ of $Q_1^{(i)}$, where $\{a_i, b_i, c_i, d_i\}$ is fixed elementwise by $G^{(i)}$. By 1.4.4, $G^{(i)}$ extends to a collineation group of $F(Q_1^{(i)}) = \pi^{(i)}$. Suppose $i \geq 1$. Because $\{a_i, \ldots, d_i\}$ is fixed elementwise by $G^{(i)}$, so is $F(\{a_i, \ldots, d_i\})$ (by 1.4.4). Hence $a_{i-1}, \ldots, d_{i-1}$ are fixed by $G^{(i)}$. It follows that $\pi^{(i-1)} G^{(i)} = \pi^{(i-1)}$ and $G^{(i)}|_{\pi^{(j)}} = G^{(j)}$, $0 \leq j < i$ and $i \geq 1$. Hence G extends to a collineation group G' of π defined by $G'|_{\pi^{(i)}} = G^{(i)}$, $i \geq 0$.

We have used a generalization of Kopejkina's construction to prove the above theorem. However, if one lets $\pi = F(\pi_r^K \cup \pi')$, where π' is an open non-free plane obtained from Kopejkina's construction, then π also satisfies the requirements of the above theorem (one shows that π does not have free rank using a lemma due to O'Gorman (20)). Thus our generalization is not strictly necessary for the above proof.

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