SOME ASPECTS OF THE MATHEMATICAL THEORY OF STORAGE

BY

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Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of the author's knowledge and belief, contains no copy or paraphrase of material previously published or written by another person, except when due reference is made in the text of the thesis.

Signed: A. M. Hasofer

(A. M. Hasofer)

PREFACE

The research work embodied in this thesis was done during the years 1961 - 1964 at the Department of Mathematics, the University of Tasmania. The following parts of the thesis are original: Chapter 1, section 3; Chapter 2, sections 2, 3 and 11; Chapter 3, section 1, Theorems 3.3, 3.5 and 3.6 of section 4, sections 5 to 11; Chapter 4, most of sections 2 to 6; Chapter 5, most of sections 4 and 5; the whole of Chapters 6 and 7. Most of the new results have been published and may be found in papers [32], [33], [34] and [35].

I would like to take this opportunity to thank my supervisor, Professor L. S. Goddard, who has untiringly given me much of his time as well as valuable advice throughout the preparation of this thesis.

I would also like to thank Professor E. J. G. Pitman, who introduced me to Mathematical Statistics and who taught me the importance of rigour in Probability Theory. This thesis is imbued throughout with the spirit of his teaching.

Finally I would like to thank my wife Atara, who has cheerfully put up for the last three years with all the sleepless nights and fits of temper incident to the writing of the thesis and has unceasingly helped me with typing, checking and duplicating.

SUMMARY

In this thesis, a storage model with infinite capacity, additive stochastic input and unit release per unit time is investigated. The content of the store in the deterministic case is defined as the unique solution of an integral equation. Properties of non-negative additive stochastic processes are obtained. These properties are used to study the distribution of the time of first emptiness when the input is stationary, and the distribution of the content. Applications to dams and queues with specific input laws are given. In particular, the waiting time for the queues M/M/1 and M/G/1, and the content of the dam with Gamma input are studied in detail. The dam with Inverse Gaussian input is introduced and its transient solution obtained explicitly.

Finally, in the case of a Compound Poisson input, the continuity and differentiability of the distribution of the content are investigated.

A non-stationary Compound Poisson input is considered, and it is shown that the probability of the store being empty and the Laplace transform of the content can be expanded in a power series. When the parameter of the input is periodic, it is shown that all terms of the series expansion are asymptotically periodic, and explicit expressions for the leading terms are obtained.

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INTRODUCTION

AND SURVEY OF THE LITERATURE

1. General outline of the thesis

The theory of storage has attracted much attention in recent years. Although the impetus was first given by economic problems of inventory and provisioning and engineering problems in dam design, it soon appeared that storage models had an intrinsic mathematical interest. Storage models with stochastic input are analogous to models in queueing and renewal theory, and provide interesting examples of Markov processes having unusual properties.

In this thesis, the following model is investigated: an input $\xi(t)$, which is a stochastic process with independent increments, is fed into a store, over an interval of time t. The output from the store is of one unit per unit time, except when the store is empty. The two main processes investigated are the time of first emptiness and the content of the store at any time t.

This abstract model contains, as special cases, the following models which have been extensively studied :

(a) the single-server queue with Poisson arrivals and exponential service times (M/M/1).

(b) the single-server queue with Poisson arrivals and general service times (M/G/1).

(c) the single-server queue with bulk arrivals at points of time which follow a Poisson distribution and either exponential or general service times.

(d) the infinite dam with Poisson input and constant-rate release.

(e) the infinite dam with constant inputs at equidistant points of time and releases following the negative exponential distribution.

(f) the infinite dam with a Gamma-distributed input, and constant-rate release.

The advantage of using a single abstract model for these various situations is, of course, that of being able to use a unified technique to obtain results which have been established previously by widely varying methods, and then only for special cases. By using the general method, we are also able to obtain new results not previously published.

Another feature of this thesis is the emphasis on continuous parameter methods. Many of the important results in the field under investigation have been obtained by limiting methods e.g. Moran [52], Gani and Prabhu [29]. However, it seems simpler to study the continuous-time model directly, and it turns out in fact that the required results can be obtained just as easily in this way as with limiting methods.

It should be emphasised that only results relating to the waiting time can be obtained by our technique, when it is applied to queueing models. The queue length cannot be studied by this method.

The main contents of this thesis are as follows:

Chapter 1 deals with the deterministic version of the model under investigation. It is interesting to note that the problem of the definition of the content of a store with completely general inputs and outputs, though a natural one, has not been given attention until quite recently. In Chapter 1, we show that the content of a store can be defined in two equivalent ways:

(a) as the unique non-negative measurable solution of an integral equation,

(b) by a formula involving the maximum and the supremum functionals.

The relevance of the deterministic model to the stochastic one is due to the fact that almost all sample functions of the stochastic model obey the formulae derived in Chapter 1. From these formulae, we can obtain immediately all the basic probabilistic formulae required in the succeeding chapters. The results of this chapter were given by Kingman in a paper [41] submitted to the Journal of the Australian Mathematical Society in July 1963. Identical results were obtained independently by the author in June 1963, and a paper embodying these results was submitted in September 1963 to the Proceedings of the Cambridge Philosophical Society. The Editor of the Proceedings, however, pointed out that the author's main result was identical with Kingman's result.

There is, nevertheless, a substantial part of this chapter, giving a detailed motivation for the form of the result, which is original. It is the author's view that the form of the integral equation defining the content of the store can be best motivated when input functions having continuous derivatives are considered, even though in the application to stochastic models, only inputs whose sample functions are almost surely step functions are encountered. The case of an input having a continuous derivative is considered in detail in Chapter 1.

Chapter 2 is an exposition of some results in the theory of non-negative processes with independent increments. The general theory of such processes has been investigated at length by Lévy [44], [45]. A restatement of his results is to be found in Doob [17]. However, the special case of non-negative processes does not seem to have been studied on its own merits.

The spur to try special techniques for studying non-negative random variables was given to the author at the beginning of 1962, in a private communication from D. G. Kendall. Professor Kendall pointed out, in commenting on some work of the author which used characteristic functions, that it was "always a mistake" to use characteristic functions when dealing with non-negative random variables, and that Laplace-Stieltjes transforms should be used instead. Laplace transforms are used throughout this thesis and their use has made many results much more easy to obtain than with other techniques. This is true in particular for the results of Chapter 2. New methods include the use of the notion of complete monotonicity of Laplace-Stieltjes transforms and the expansion of the distribution function of the Compound Poisson process as a power series in the time parameter t.

A detailed investigation of the Compound Poisson process is carried out. Special attention, however, is given to a type of process which is specific to dam theory, namely, stochastic processes which are almost surely increasing in every time interval. The prototype of such processes is Moran's [52] Gamma input, which is investigated in detail. It is shown by an elementary method that almost all sample functions of such processes have zero derivative almost everywhere, although their discontinuities form an everywhere dense set.

Finally a new process with independent increments, the Inverse Gaussian process, is introduced. This process was constructed by the author [35] so that a store with such an input would have a time-dependent content distribution which could be expressed in closed form. It was pointed out by P.A. P. Moran, in a private communication, that the process distribution was identical with a distribution introduced by Tweedie [73], who had not, however, pointed out its infinitely divisible character.

In Chapter 3, the first passage time for the content of the store is investigated. The basic method is that devised by Kendall [39] p. 209. However, Kendall made it clear in his paper that he was only sketching his results. In particular, as pointed out by Lloyd, [46] p. 133, Kendall's main formula giving the density function of the first passage time in terms of the density function of the input was only conjectured, and has since been used repeatedly in the literature on dams without proof.

Chapter 3 gives in detail a rigorous treatment of the first passage time distribution in the case of a stationary input which was first given by the author [34]. It is first proved, using the results of Chapter 1, that the first passage time is in fact a random variable (possibly defective). Then Kendall's main formula is proved under very weak conditions. A general formula is obtained for the first passage time distribution in the case of a Compound Poisson input, and this is shown to specialise to the formula obtained by Lloyd [46] and Mott [54] by inductive and combinatorial methods respectively.

Finally, the asymptotic behaviour of the distribution of the time of first emptiness for large values of the initial content is investigated and some explicit expressions for special types of inputs are obtained.

In Chapter 4, the distribution of the content of the store is investigated. First a formula for the Laplace-Stieltjes transform of the time-dependent distribution of the store content in terms of the distribution of the input and the probability of emptiness is obtained. The formula is inverted and special forms investigated in the case of discrete and absolutely continuous inputs.

Specializing to a stationary input, the Laplace transform of the probability of emptiness is obtained and the formula is inverted, expressing the probability of emptiness as an integral of the probability of first emptiness.

Next, it is proved by probability methods that, with a stationary input, the store content distribution tends, as t tends to infinity, to a stationary distribution, which is independent of initial conditions, provided the mean value of the input per unit time is less than one. The behaviour of the content distribution in other cases is also investigated. The celebrated Pollaczek-Khintchine formula for the Laplace transform of the limiting distribution is obtained for a general form of input, and is inverted in two different ways. Finally, various asymptotic formulae for the behaviour of the limiting distribution are obtained.

The work in this chapter is based on results by various authors scattered in the literature. The proof of existence of a limiting distribution is, however, original in its use for a general input, although based on an idea of Takács [71] p. 52. Various proofs of known results are also original.

Chapter 5 is a collection of results obtained by applying the formulae of Chapters 3 and 4 to special types of inputs. The following models are considered:

(a) the queue M/M/1,

(b) the queue with Poisson input and fixed service time,

(c) the dam with Gamma input,

(d) the dam with inverse Gaussian input.

The results for the first two models are well-known and have been obtained many times by a wide variety of methods in recent literature. Those for the fourth model have been obtained by the author [35], while most of those for the third have not so far been published. The interest of this chapter lies mainly in the fact that all the various results, with applications to very different situations, can all be obtained by one and the same method.

Chapter 6 investigates the continuity and differentiability properties of the distribution function $W(t,x) = P\{\zeta(t) \leq x\}$, where $\zeta(t)$ is the store content, as a function of the two variables t and x. The main result is that if the input distribution is of Compound Poisson type, and if the jump distribution has a bounded derivative, then $W(t,x)$ is a differentiable function of both t and x, and satisfies an integro-differential equation. This chapter is an amplified exposition of a paper by the author [32].

Finally Chapter 7 investigates the case of a storage model with a Compound Poisson input, when the arrival density varies periodically with time. It is shown that the probability of emptiness can be represented as a power series in a suitable parameter whose coefficients can be calculated by recurrence as the solutions of convolution-type integral equations.

The asymptotic behaviour of the probability of emptiness is then investigated, using Abelian theorems for the Laplace transform inversion formula. It is shown that the probability of emptiness is asymptotically periodic and can be represented by a Fourier series.

Finally it is shown that the Laplace transform of the waiting time is also asymptotically periodic and can be represented by a Fourier series.

Various mathematical results which are required for establishing the results of this chapter are also proved.

2. A survey of recent literature on storage

The following survey of the literature on storage does not claim to be exhaustive, but only to outline the main stages of the development of the theory of storage with additive stochastic input when the store has infinite capacity.

Moreover, it should be noted that many of the important formulae in the theory of storage were actually developed within the framework of queueing theory, so that a large part of this survey will deal with papers on queueing theory which do not contain any mention of a more general interpretation.

The first important formula in the theory of storage was given by Pollaczek [55] in 1930 and Khintchine [40] in 1932, who gave a formula for the Laplace transform of the limiting distribution of the waiting time in the queue M/G/1. In 1933, the first investigation of the transient behaviour of the queue M/M/1 was made by Kolmogorov [42].

Little work of importance in the field was done until 1951, when Kendall [38], recognizing the fact that the length of the queue at time t in the queue M/G/1 was not a Markovian process, introduced the method of the imbedded Markov chain, and was the first to prove that when the mean rate of arrival is less than the mean rate of service, the waiting-time distribution tends to a limit distribution as t tends to infinity.

The next step in the development of the transient theory of storage was the work of Lederman and Reuter [43] in 1954. Lederman and Reuter studied the queue M/M/1 by the method of the birth and death equations, and, using the spectral theory of differential equations, obtained explicit expressions for the time-dependent queue size in terms of modified Bessel functiohs. Soon after, their results were obtained by a number of different methods:

In 1954, Bailey [1] obtained Lederman and Reuter's results by using generating functions. He also obtained further results in 1957 [2]. Further refinements of the technique are to be found in Cox and Smith [12].

In 1956, Champernowne [9] used random-walk methods to solve the queue M/M/1.

Also in 1956, Clarke [10] studied the queue with non-stationary Poisson arrivals and exponential service times, using generating functions. A similar problem was investigated by Luchak [48] in 1956, using spectral theory.

Conolly [11], in 1958, solved the birth and death equations by Laplace transforms and difference-equation methods.

Finally Karlin and McGregor [37], in the same year, applied an orthogonal polynomial method, reducing the solution of the queue M/M/1 to the finding of a suitable measure that would make a given sequence of polynomials orthogonal.

While the investigation of the queue M/M/1 was being carried out, the queue M/G/1 was also given much attention.

Pollaczek [56] studied in 1952 the transient behaviour of the queue M/G/1, using complex-variable methods. His results are available in expanded form in [57].

The turning point, however, in the study of the queue M/G/1, was Takács' paper [70] in 1955, where the notion of virtual waiting time was first introduced. Takács remarked that, although the length of the queue at time t was not a Markov process, the time required to complete the service of all customers in the queue at time t had the Markov property. However, the considered process was not of a type which had been previously studied in detail. The introduction of the virtual waiting time concept also heralded the combination of queueing and storage theories in one abstract model.

The virtual waiting time process was further investigated by Beneš [3] in 1957 and Reich [61], [62], while Descamps [13] in France seems to have rediscovered the idea independently (see Saaty [65] p. 198).

Further results, concerning the analytical properties of the waiting-time distribution, and the queue with non-stationary Poisson arrivals and general service times were obtained by the author in 1963 [32], [33].

Other developments related to the queue M/G/1 were the use of Spitzer's identity (see Spitzer [67], [68]) in 1957 to prove the Pollaczek-Khintchine formula by combinatorial methods, and the use of Dantzig's method of "marks" by Runnenberg in 1960 to obtain by probabilistic methods Takács' time-dependent formula for the Laplace-Stieltjes transform of the waiting-time distribution [64].

Combinatorial methods were also used by Beneš [4], [5], [6], and Reich [63] to obtain various formulae related to the virtual time in the queue M/G/1.

All the methods just quoted are of course of much wider applicability but are mentioned here only in so far as they can be applied to our storage model.

The great development of queueing theory in the 1950's can be gauged by the fact that Miss Doig's bibliography [16] (1957) lists about seven hundred papers on queueing theory, while the author's own bibliography lists several hundred subsequent papers.

While the queueing aspects of storage were being developed at such a rate, the theory of dams was making its appearance on the stage.

Smith [66] had described in 1953 an analogy between the single-server queue and an infinite dam model.

Moran [50], [51] in 1954 and 1955 gave a simple and practical formulation of the finite dam problem, as well as several solutions, in which extensive use was made of Markov chain methods. Gani [22] obtained an exact solution of the finite dam when the input is of Poisson type and the release is at a constant rate.

The most interesting problem in the theory of storage, however, with no direct analogue in queueing theory was formulated by Moran [52] in 1956. This was the infinite dam with an input following the infinitely divisible Gamma distribution, and a constant-rate release, whose solution Moran obtained by using a discrete approximation, and then passing to the limit.

The great turning point in dam theory was the meeting of the Research Section of the Royal Statistical Society on 6th March, 1957, where a review paper on storage systems by Gani [23] and a paper by Kendall [39] entitled "Some problems in the theory of dams" were read, and a lively discussion ensued, with many noteworthy contributions to dam theory, including in particular those of Foster, Lindley, Downton, Smith, Thatcher and Daniels. Not only was Moran's Gamma input dam investigated, but dams with general additive inputs were considered there for the first time. At the end of the discussion, Kendall gave a summary of the main results, and concluded: "This is a most satisfactory state of affairs. There is still a great deal that we do not know about the Moran process, but there are very few processes about which we know so much".

Another important development in 1957 was the realisation of the close relationship between the content of a dam and Takács' virtual waiting time mentioned earlier (see e.g. Downton [18]). The following is a quotation from Gani and Prabhu [27] p. 114:

"The application of limiting methods to Markov chain models has in some ..... cases led to solutions of the storage problem in continuous time. However, the procedure has proved cumbersome and has partly obscured the simplicity of the underlying Markov processes. Surprisingly enough, it escaped both Moran and Gani that the problems they first considered, formulated as they were in widely different terms and with apparently distinct solutions, were identical, and closely connected with Takács' [70] elegant work on analogous queueing processes". The reference to Gani's and Moran's models which are the duals of each other is to Gani [22] and Moran [51]. A systematic exposition of storage theory up to 1959 was given by Moran [53].

Using Takács' integro-differential equation, Gani and Prabhu, who had already published a joint paper on dam theory [25], obtained a large number of new results for the dam with a simple Poisson input and constant-rate release in 1958 and 1959 [26], [27], [28]. Prabhu, who had done some previous work on discrete dams [58], now used the new formulae in queueing theory [59], [60].

However, in applying the Takács technique to dam models, especially those with a Gamma-type input, it soon became apparent that the intuitive definition of the content of the dam led to difficulties. This led Gani and Pyke [30] to redefine the content of a dam, using the supremum and maximum functionals. A review of the known results, in the light of the new definition, using limits of discrete approximations to obtain the Takács integro-differential equation, was presented by Gani and Prabhu [29], and the results extended by Gani [24] to dams with non-stationary Compound Poisson inputs.

The return to the method of discrete approximations was not, however, fully satisfactory, and Kingman [41] in 1963, showed that a direct treatment was possible in the general case also. The following is a quotation from his paper:

"The problem of storage in an infinite dam with a continuous release has been studied by a number of authors ..., who have formulated it in probabilistic terms by supposing the input to be a continuous-time stochastic process. These authors have encountered difficulties which they have overcome by regarding the continuous-time problem as a limit of discrete-time analogues. The purpose of this paper is to suggest that these difficulties are the result of an unfortunate specification of the problem, and to show that the adoption of a slightly different (and more realistic) formulation avoids the difficulties and allows a treatment which does not have recourse to discrete time analogues."

As mentioned earlier, the author obtained the same results as Kingman independently in 1963.

The problem of the distribution of the time to first emptiness, for which Kendall [39] had conjectured a solution, still remained to be solved. A complete solution for the discrete case was given by Lloyd [46] and Mott [54] in 1963, while a solution for the absolutely continuous case and the general case of a Compound Poisson input was given by the author [34].

CHAPTER 1

A DETERMINISTIC INVESTIGATION OF THE STORAGE MODEL WITH INFINITE CAPACITY

1. Definition and elementary properties of the input and output functions.

In this chapter, we shall be concerned with the following deterministic model: we have a "store" with an infinite capacity.

The "input" in the interval $(0, t]$ is given by the function $\xi(t)$. We shall make the following assumptions:

(a) $\xi(0) = 0$,

(b) $\xi(t)$ is non-decreasing,

(c) $\xi(t)$ is continuous to the right,

(d) $\xi(t) \to \infty$ as $t \to \infty$.

From these assumptions, we can deduce that $\xi(t)$ has the following properties:

(a) The discontinuities of $\xi(t)$ are all of the first kind, i.e. $\xi(t+0) = \lim_{h \downarrow 0} \xi(t+h) = \xi(t)$, $\xi(t-0) = \lim_{h \downarrow 0} \xi(t-h)$ both exist, and $\xi(t-0) \leq \xi(t)$.

(b) The discontinuities of $\xi(t)$ are denumerable.

(c) $\xi(t)$ is the sum of three non-decreasing functions $\xi_1(t)$, $\xi_2(t)$; $\xi_3(t)$, where $\xi_1(t)$ is a step-function, $\xi_2(t)$ is absolutely continuous, i.e.

$$\xi_2(t) = \int_0^t f(u)\, du,$$

and $\xi_3(t)$ is a singular function, i.e. a continuous function whose derivative vanishes almost everywhere.

(d) $\xi(t)$ is differentiable almost everywhere.

(e) $\xi(t)$ determines a measure over the σ-field of Borel sets of the non-negative axis $(0, \infty)$.

We shall say that the point t is a <u>point of stationarity</u> of $\xi(t)$ if there exists a one-sided neighbourhood $[t, t+\varepsilon)$ of t where $\xi(t)$ is constant. Otherwise we shall say that t is a <u>point of increase</u>.

The measure of any Borel set B induced by $\xi(t)$ will be called the <u>input over the set B</u>.

We now introduce the notion of <u>output</u>. This is not as straight forward as the notion of input, because, while the input is independent of what is happening in the store, the possibility of realizing an output depends on the periods of emptiness of the store in $[0, t]$. Moreover, we must lay down a rule for determining to what extent a desired output rate will be realized when the store is empty and the input rate is lower than the desired output rate.

As a first step, we shall introduce a "planned output function", $\eta(t)$. This will have the same properties as $\xi(t)$, i.e., it will also be a non-decreasing, right-continuous function with $\eta(0) = 0$, $\eta(\infty) = \infty$. We shall call the content of the store $\zeta(t)$. It is clear that $\zeta(t)$ is not equal to $\xi(t) - \eta(t)$, because of the fact that the planned output $\eta(t)$ cannot be fully realized if the store becomes empty, i.e., if $\zeta(u) = 0$ for a sufficiently large set of values of $u \in (0, t]$. However, the function $\xi(t) - \eta(t)$ will play an important part in the determination of $\zeta(t)$. We shall denote it by $\nu(t)$, and call it the "net planned input". The function $\nu(t)$ has the following properties:

(a) $\nu(t)$ is a function of bounded variation, and its total variation $|\nu|(t)$ in $(0, t]$ is certainly smaller than $\xi(t) + \eta(t)$.

(b) $\nu(t)$ can be decomposed into its positive variation $\nu_+(t)$ and its negative variation $\nu_-(t)$. These are defined as follows: We put

$$(x)_+ = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(x)_- = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$x = (x)_+ - (x)_- ,$$

$$|x| = (x)_+ + (x)_- .$$

Let $0 = t_0 < t_1 < t_2 \dots < t_n = t$ be any dissection of the interval $(0, t]$.

We define

$$\nu_+(t) = \sup \sum_{k=0}^{n-1} \left[\nu(t_{k+1}) - \nu(t_k) \right]_+ ,$$

$$\nu_-(t) = \sup \sum_{k=0}^{n-1} \left[\nu(t_{k+1}) - \nu(t_k) \right]_- ,$$

where the supremum is taken over all possible dissections of $(0,t]$. We then have

$$\nu(t) = \nu_+(t) - \nu_-(t)$$
$$|\nu|(t) = \nu_+(t) + \nu_-(t).$$

It will be shown in the sequel that the key to a correct definition of the content function, $\zeta(t)$, lies in finding the difference between the two functions $\xi(t)$ and $\eta(t)$, thus obtaining the function of bounded variation $\nu(t)$, and then representing the latter as the difference of two non-decreasing functions. The new decomposition, $\nu(t) = \nu_+(t) - \nu_-(t)$, however, has a very important property. The function of bounded variation $\nu(t)$ defines a signed measure μ on the Borel sets of the positive t-axis. It can be shown (see Loève [47] p.86, theorem A) that there exists a Borel subset D of the real axis, having the following property:

$$\nu_+(t) = \mu(A \cap D),$$
$$-\nu_-(t) = \mu(A \cap D^c),$$

where A is the interval $(0, t]$ and D^c is the complement of D.

We shall now introduce a general restriction on the form of $\eta(t)$. We shall assume that $\eta(t)$ has no discontinuities, and that $t = 0$ is not a point of stationarity of $\eta(t)$. Let us remark that the last restriction does not affect the generality of the argument. For suppose that the first point of increase of $\eta(t)$ is t_0. Then in the interval $(0, t_0]$, there is no planned output, and therefore the content of the store is exactly $\zeta(0) + \xi(t_0)$ at t_0. Thus, by changing the origin to t_0 and taking the initial content of the store to be $\zeta(0) + \xi(t_0)$ instead of $\zeta(0)$, we have reduced the problem to one where the point $t = 0$ is not a point of stationarity of $\eta(t)$.

We shall show that, if we make the two preceding assumptions, then by changing the way of measuring time, we can reduce the function $\eta(t)$ to the form $\eta(t) = t$. In fact, the assumption of no discontinuities ensures that there is at least one value of t for which $\eta(t) = x$, where x is any non-negative number. Whenever there is more than one value of t corresponding to x, we take the supremum of these values as $\eta^{-1}(x)$.

We next take as our new measure of time $u = \eta(t)$. Then $\xi(t) = \xi\left[\eta^{-1}(u)\right]$ is the new imput function, which has obviously all three properties required at the beginning of the section. For the above reason, we shall consider in the remainder of this chapter only planned output functions of the form $\eta(t) = t$.

2. The case of an input function which is a step function.

The simplest case that we shall consider is that of an input $\xi(t)$ which is a step function having only a finite number of jumps in every finite interval of time. Let $t_1, t_2, \dots, t_n, \dots$ be the points of discontinuity of $\xi(t)$, and $X_1, X_2, \dots, X_n, \dots$ be the magnitude of the jumps. Then $\xi(t) = \sum_{t_n \le t} X_n$.

DIAGRAM

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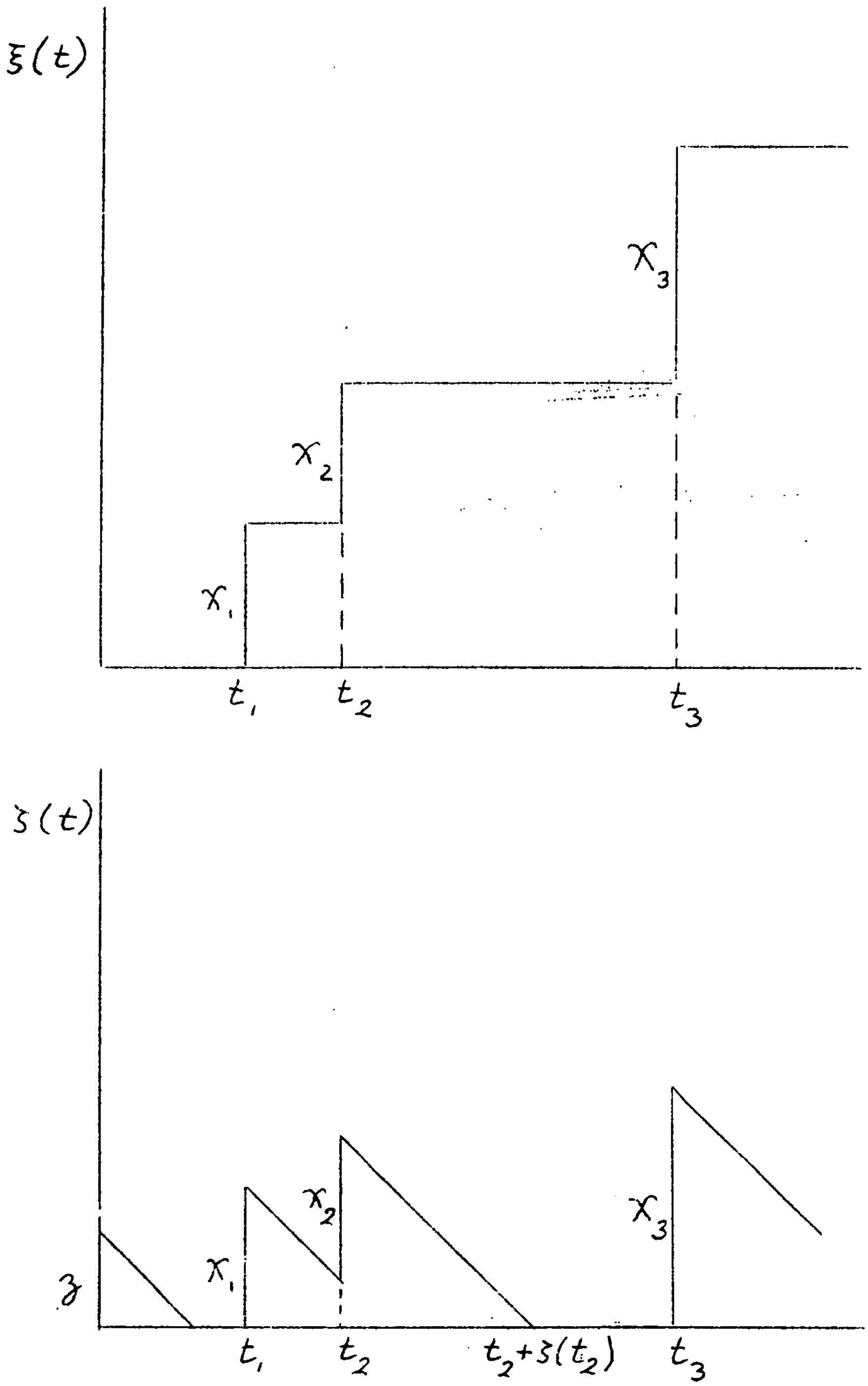

Figure 1.1

To face p. 6

In this case, it is not difficult to define $\zeta(t)$ by recurrence in a natural way. We shall assume that the initial content of the store is z. Then

(a) If $t_1 \leq z$, then, for $0 \leq t < t_1$, $\zeta(t) = z - t$, and for $t = t_1$, $\zeta(t) = z - t_1 + \chi_1$.

(b) If, on the other hand, $t_1 > z$, we have

$$\zeta(t) = \begin{cases} z - t & \text{for} \quad 0 \leq t < z, \\ 0 & \text{for} \quad z \leq t < t_1, \\ \chi_1 & \text{for} \quad t = t_1. \end{cases}$$

In the same way, we can define $\zeta(t)$ recursively as follows: given $\zeta(t) = \zeta(t_n) > 0$ at $t = t_n$, we have:

(a) If $t_{n+1} - t_n \leq \zeta(t_n)$, then, for $t_n \leq t < t_{n+1}$, $\zeta(t) = \zeta(t_n) - (t - t_n)$ and for $t = t_{n+1}$, $\zeta(t) = \zeta(t_n) - (t_{n+1} - t_n) + \chi_{n+1}$.

(b) If $t_{n+1} - t_n > \zeta(t_n)$, we have

$$\zeta(t) = \begin{cases} \zeta(t_n) - (t - t_n) & \text{for} \quad t_n \leq t < t_n + \zeta(t_n), \\ 0 & \text{for} \quad t_n + \zeta(t_n) \leq t < t_{n+1}, \\ \chi_{n+1} & \text{for} \quad t = t_{n+1}. \end{cases}$$

The situation is depicted in Figure 1.1.

We shall now show that $\zeta(t)$ satisfies the following integral equation:

$$\zeta(t) = z + \xi(t) - t + \int_0^t U[-\zeta(u)]\, du, \qquad (1.1)$$

where $U(x)$ is the Heaviside unit function, defined by

$$U(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We first note that

$$U[-\zeta(t)] = \begin{cases} 0 & \text{if } \zeta(t) > 0 \text{, i.e. if the store is not empty,} \\ 1 & \text{if } \zeta(t) = 0 \text{, i.e. if the store is empty.} \end{cases}$$

Consider first the interval $[0, t_1]$. We have two cases:

(a) $t_1 \leq z$. Then for $0 \leq t < t_1$, $\zeta(t) > 0$, $U[-\zeta(t)] = 0$, $\xi(t) = 0$, and $\zeta(t) = z - t$,

as required by the formula.

At $t = t_1$, $\xi(t) = X_1$, so that

$\zeta(t) = z - t_1 + X_1$ as required.

(b) $t_1 > z$. Then for $0 \leq t < z$, $\zeta(t) = z - t > 0$, and $U[-\zeta(t)] = 0$, so that $\int_0^t U[-\zeta(u)]\,du = 0$. On the other hand, if $z \leq t < t_1$, $\zeta(t) = 0$, so that $\int_0^t U[-\zeta(u)]\,du = t - z$. Therefore

$$\zeta(t) = \begin{cases} z - t & \text{for } 0 \leq t < z, \\ (z-t) + (t-z) = 0 & \text{for } z \leq t < t_1, \\ (z-t) + X_1 + (t-z) = X_1 & \text{for } t = t_1. \end{cases}$$

Turning now to the interval $[t_n, t_{n+1}]$, suppose that $\zeta(t)$ satisfies $\zeta(t_n) = z + \xi(t_n) - t_n + \int_0^{t_n} U[-\zeta(u)]\,du$.

Then it suffices to show that, for $t_n < t \leq t_{n+1}$,

$$\zeta(t) = \zeta(t_n) + [\xi(t) - \xi(t_n)] - (t - t_n) + \int_{t_n}^{t} U[-\zeta(u)]\,du.$$

We must consider two cases

(a) $t_{n+1} - t_n \leq \zeta(t_n)$. Then, for $t_n \leq t < t_{n+1}$,
$\zeta(t) > 0$, $U[-\zeta(t)] = 0$, $\xi(t) - \xi(t_n) = 0$,
and $\zeta(t) = \zeta(t_n) - (t - t_n)$.
Moreover, $\xi(t_{n+1}) - \xi(t_n) = \chi_{n+1}$, so that
$\zeta(t_{n+1}) = \zeta(t_n) + \chi_{n+1} - (t_{n+1} - t_n)$, as required.

(b) $t_{n+1} - t_n > \zeta(t_n)$. Then, for
$t_n \leq t < t_n + \zeta(t_n)$, $\zeta(t) = \zeta(t_n) - (t - t_n)$, and
$U[-\zeta(t)] = 0$, so that $\int_{t_n}^{t} U[-\zeta(u)]\,du = 0$. On the other hand, if $t_n + \zeta(t_n) \leq t < t_{n+1}$,
$\zeta(t) = 0$, so that $\int_{t_n}^{t} U[-\zeta(u)]\,du = t - [t_n + \zeta(t_n)]$.
It follows that

$$\zeta(t) = \begin{cases} \zeta(t_n) - (t - t_n) & \text{for } t_n \leq t < t_n + \zeta(t_n), \\ \zeta(t_n) - (t - t_n) + \{t - [t_n + \zeta(t_n)]\} = 0 & \text{for } t_n + \zeta(t_n) \leq t < t_{n+1} \\ \zeta(t_n) - (t_{n+1} - t_n) + \{t_{n+1} - [t_n + \zeta(t_n)]\} + \chi_{n+1} = \chi_{n+1} & \text{for } t = t_{n+1} \end{cases}$$

Thus the values of $\zeta(t)$ as defined at the beginning of the section satisfy equation (1.1) for all values of $t \geq 0$.

Let us now consider what the expressions for $\nu_+(t)$ and $\nu_-(t)$ are in the case under consideration. By inspection, it is easy to see that in this case we simply have $\nu_+(t) = \xi(t)$ and $\nu_-(t) = t$. The set $\mathcal{D}$ described at the end of section 1 is simply the denumerable set $\{t_n\}$, where the t_n are the points of discontinuity of $\xi(t)$. We thus see that equation (1.1) can be rewritten as

$$\zeta(t) = z + \nu(t) + \int_0^t U[-\zeta(u)]\, d\nu_-(u) \qquad (1.2)$$

We shall show later that this integral equation has a unique non-negative measurable solution. From the above considerations, we conclude that this solution coincides with the natural definition given at the beginning of this section.

Equation (1.2) has a simple intuitive meaning. The content of the store is equal to the initial content, plus the net planned input, plus a correction term. The correction term represents that part of the planned output which could not be realised, due to the store being empty. If μ_- is the measure induced by $\nu_-(t)$ on the positive axis, and B the set of points in $(0,t]$ for which $\zeta(t)=0$, the correction term is $\mu_-(B)$. In the case of a step input function, the set B is the union of a finite number of disjoint intervals.

3. <u>The case of an input function which has a continuous derivative</u>.

The case of an input function which has a continuous derivative already exhibits all the difficulties of the general case. However, these difficulties can be easily overcome in that case.

It is natural to assume that the rate of change of $\zeta(t)$ will be given by the following equations:

(a) If $\zeta(t)>0$, $\zeta'(t) = \xi'(t)-1 = \nu'(t)$.

(b) If $\zeta(t)=0$, there are two possibilities :

(i) if $\xi'(t) \geqslant 1$, i.e. if the rate of input is larger than or equal to the planned rate of output, then the store will start filling at the rate of $\xi'(t) - 1$, i.e.

$$\zeta'(t) = \xi'(t) - 1 = \nu'(t).$$

(ii) if $\xi'(t) < 1$, i.e. if the rate of input is smaller than the planned rate of output, then the store will remain empty, i.e.

$$\zeta'(t) = 0.$$

The formulae for $\zeta'(t)$ in both cases (i) and (ii) can be summarised as

$$\zeta'(t) = [\xi'(t) - 1]_+ = [\nu'(t)]_+ .$$

We shall need the following lemma.

Lemma: For every function $\nu(t)$ having a continuous derivative,

$$\nu_+(t) = \int_0^t [\nu'(u)]_+ \, du .$$

Proof:

$$\nu_+(t) = \sup \sum_{k=0}^{m-1} [\nu(t_{k+1}) - \nu(t_k)]_+ .$$

But

$$[\nu(t_{k+1}) - \nu(t_k)]_+ = \left[\int_{t_k}^{t_{k+1}} \nu'(u)\,du \right]_+ \leq \int_{t_k}^{t_{k+1}} [\nu'(u)]_+ \, du ,$$

so that

$$\sum_{k=0}^{m-1} [\nu(t_{k+1}) - \nu(t_k)]_+ \leq \int_0^t [\nu'(u)]_+ \, du . \qquad (1.3)$$

On the other hand, because of the continuity of $\nu'(t)$, the inverse image of the set $(0, \infty)$ under the mapping $\nu'(u)$,

DIAGRAM

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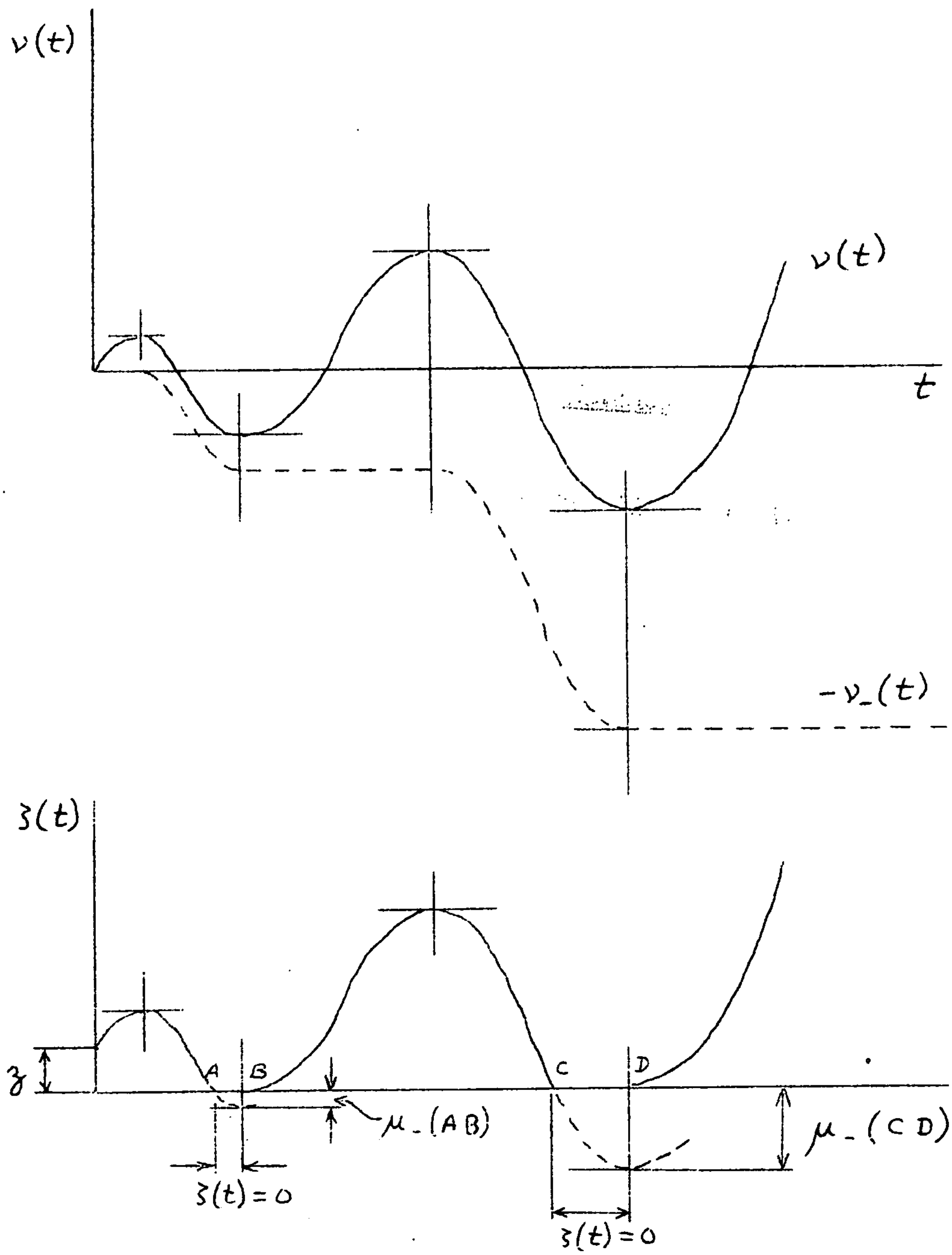

Figure 1.2

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$u \in (o,t)$, is an open set, which is the union of a denumerable number of disjoint open intervals I_n. Let t_n, t_n' be the end points of these intervals. Then

$$\int_0^t [\nu'(u)]_+ du = \sum_n \int_{t_n}^{t_n'} \nu'(u)du = \sum_n \left[\nu(t_n') - \nu(t_n)\right],$$

$$= \sum_n \left[\nu(t_n') - \nu(t_n)\right]_+ .$$

The last equality follows from the fact that on the intervals I_n, $\nu'(t)$ is positive. Finally, because $|\nu(t)|$ is bounded on $[o,t]$, we can choose a finite subset of the I_n's such that, if the summation is extended to that subset only,

$$\sum_n \left[\nu(t_n') - \nu(t_n)\right] > \int_0^t [\nu'(u)]_+ du - \varepsilon,$$

where ε is arbitrarily small. For the dissection of (o,t) determined by the t_n, t_n', we obviously have

$$\sum_k \left[\nu(t_{k+1}) - \nu(t_k)\right]_+ > \int_0^t [\nu'(u)]_+ du - \varepsilon. \qquad (1.4)$$

Using equations (1.3) and (1.4), we conclude that

$$\sup \sum_k \left[\nu(t_{k+1}) - \nu(t_k)\right]_+ = \int_0^t [\nu'(u)]_+ du. \qquad (1.5)$$

This completes the proof of the lemma.

<u>Corollary</u>: <u>Under the same conditions</u>, $\nu_-(t) = \int_0^t [\nu'(u)]_- du.$

Using the above lemma, we can summarize all the equations giving $\zeta'(t)$ as follows:

$$\zeta'(t) = \nu'(t)\left\{1 - U[-\zeta(t)]\right\} + [\nu'(t)]_+ U[-\zeta(t)] \qquad (1.6)$$

Remembering that $\nu'(t) = [\nu'(t)]_+ - [\nu'(t)]_-$, we can re-write(1.6) as

$$\zeta'(t) = \nu'(t) + U[-\zeta(t)]\,[\nu'(t)]_- \qquad (1.7)$$

Equation (1.7) can be easily integrated, using the corollary to the lemma. We find

$$\zeta(t) = z + \nu(t) + \int_0^t U[-\zeta(u)]\, d\nu_-(u), \qquad (1.8)$$

which is the same equation as equation (1.2).

The formation of $\zeta(t)$ in the case of an input function having a continuous derivative is depicted in Fig. 1.2.

4. The function $\nu^*(t)$ and its use in the definition of $\zeta(t)$.

We shall now return to the case of a general input function, and we shall introduce a function $\nu^*(t)$ defined by the formula

$$\nu^*(t) = -\inf_{0 \le u \le t} \nu(u).$$

This function has the following properties:

(a) It is non-decreasing.

(b) $\nu^*(0) = 0$.

(c) It is continuous. This property follows from the fact that $\nu(t)$ has no downward jumps.

To gain some insight into the relationship between $\zeta(t)$ and $\nu^*(t)$, we shall consider a special case depicted in Figure 1.3. In this figure, $\nu(t)$ has two minima at t_1 and t_3, and we assume that $0 > \nu(t_1) > \nu(t_3)$. The

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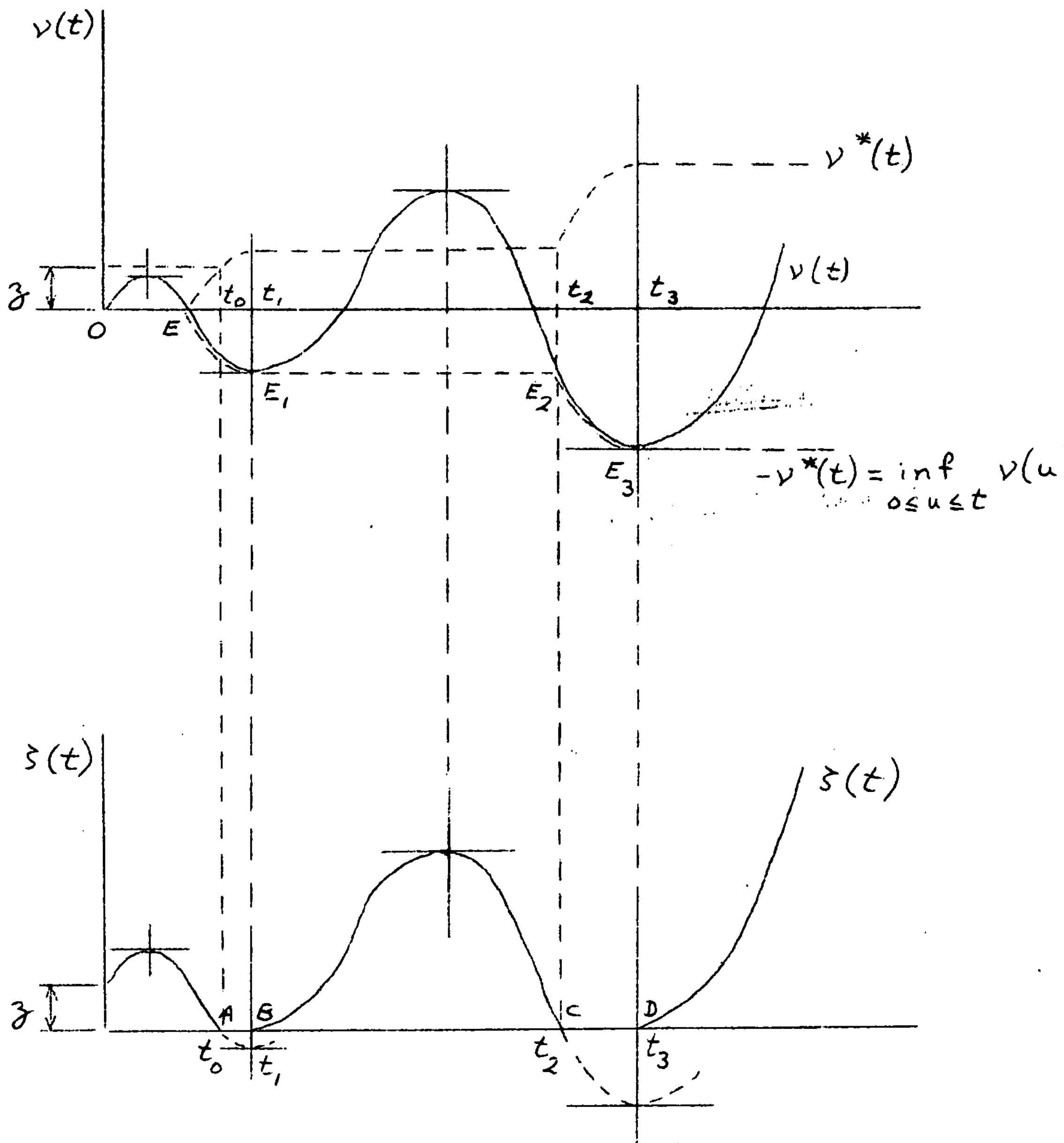

Figure 1.3

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initial content of the store, ζ , is assumed to be smaller than $|\nu(t_1)|$.

Let us draw the function $-\nu^*(t)$. It remains zero as long as $\nu(t)$ remains above the t-axis, i.e. until the point E . At E it starts decreasing, remaining equal to $\nu(t)$ as long as $\nu(t)$ is decreasing. Then at t_1 , when $\nu(t)$ starts increasing, it remains constant, until $\nu(t)$ has reached again the value of the first minima, $\nu(t_1)$, at t_2 , and then decreases, remaining equal to $\nu(t)$, until t_3 , the second minimum.

Let us now consider the content of the store. As long as the store has never become empty, we have $\zeta(t) = \zeta + \nu(t)$. However, it is clear that the store will become empty for the first time when $\nu(t)$ has reached for the first time the value $-\zeta$. This will happen at the smallest value of t for which $\inf_{0 \le u \le t} \nu(u) = -\zeta$, in other words when $\nu^*(t) = \zeta$ for the first time. This happens at t_0 . The store will then remain empty as long as $\nu'(t) \le 0$, i.e. as long as $\nu(t) = -\nu^*(t)$. We thus see that for $t_0 \le t < t_1$, $\zeta(t) = 0 = \nu(t) + \nu^*(t)$. After t_1 , $\zeta(t)$ will increase from zero at the same rate as $\nu(t)$, and then decrease at the same rate as $\nu(t)$, until t_2 , when $\nu(t)$ reaches a negative value equal to that at t_1 . During the time interval (t_1, t_2), $\nu^*(t)$ has remained constant, so that again we can write $\zeta(t) = \nu(t) + \nu^*(t)$. In general, $\nu^*(t)$ will remain constant after t_0 in all those periods where the store is not empty, and will be equal to $-\nu(t)$ for all periods where the store is empty, so that, in general, for $t \geq t_0$, we have $\zeta(t) = \nu(t) + \nu^*(t)$. We thus have

$$\zeta(t) = \begin{cases} \nu(t) + z & \text{for} \quad t < t_0\,, \\ \nu(t) + \nu^*(t) & \text{for} \quad t \geq t_0\,. \end{cases}$$

However, for $t < t_0$, we have $z > \nu^*(t)$, and for $t \geq t_0$, we have $z \leq \nu^*(t)$, so that we can combine the two formulae for $\zeta(t)$ into the one formula

$$\zeta(t) = \nu(t) + \max\left[z, \nu^*(t)\right] \tag{1.9}$$

Let us also note here that t_0, the "time of first emptiness" of the store, is the smallest value of t for which $\nu^*(t) = z$.

We have thus obtained intuitively for the special case under consideration, an explicit formula for $\zeta(t)$ in terms of $\nu^*(t)$ and $\nu(t)$. It remains to establish formally the equivalence of the two formulae (1.8) and (1.9) in the general case. This will be done in the next section.

5. The formal definition of the content of the store.

We shall now show that equation (1.8) can be used to define the content of a store in a unique way. We shall need the following theorem:

Theorem 1.1: Let z be an arbitrary positive number, $\nu(t)$ a function of bounded variation which has only upward jumps, that is, $\nu(t) - \nu(t-0) \geq 0$, and is continuous to the right. Let $\nu_-(t)$ be the negative variation of $\nu(t)$. Then the integral equation

$$\zeta(t) = z + \nu(t) + \int_0^t U[-\zeta(u)]\, d\nu_-(u) \qquad (1.8)$$

has a unique non-negative measurable solution, given by the formula

$$\zeta(t) = \nu(t) + \max[z, \nu^*(t)] \qquad (1.9)$$

where $\nu^*(t) = -\inf_{0 \le u \le t} \nu(u)$.

Proof: We recall that $\nu^*(0) = 0$, and that $\nu^*(t)$ is a non-decreasing, right-continuous function of t. It is therefore measurable and non-negative, and it follows that $\zeta(t)$, as defined by equation (1.9), is measurable.

Further, as $\nu(t)$ has only upward jumps, $\nu^*(t)$ is continuous.

We shall first show that $\zeta(t)$, as defined by equation (1.9) satisfies the integral equation (1.8).

Let us consider the various possible cases:

(a) $\nu^*(t) < z$.

Then (1.9) gives us

$\zeta(u) = z + \nu(u)$ for all u such that $0 \le u \le t$,

and this value satisfies equation (1.8) as the integrand vanishes at every point of $(0, t]$, for, as $\nu^*(t) < z$, is follows that

$z + \inf_{0 \le u \le t} \nu(u) > 0$, and in particular,

$\zeta(u) = z + \nu(u) > 0$, from which we deduce that $\nu[-\zeta(u)] = 0$.

(b) $\nu^*(t) \geq \zeta$. Let t_0 be the first value of t for which $\nu^*(t) = \zeta$. The existence of t_0 is ensured by the hypotheses on $\nu(t)$, in particular by the restriction to upward jumps only, which make $\nu^*(t)$ continuous as well as non-decreasing. It is this step in the proof which makes it imperative to assume that the planned output function has no jumps.

Returning to the proof of the theorem, we now see that, for $t \geq t_0$,

$$\zeta(t) = \nu(t) + \nu^*(t) = \nu(t) - \inf_{0 \leq u \leq t} \nu(u) \geq 0 \qquad (1.10)$$

Now, at all points t such that $\zeta(t) = 0$, (and this implies that $t \geq t_0$, by the definition of t_0), we have

$$\nu^*(t) = -\nu(t).$$

If, on the other hand, $\zeta(t)$ is positive, we have

$$\nu(t) > -\nu^*(t) = \inf_{0 \leq u \leq t} \nu(u).$$

As $\nu(t)$ is continuous to the right, and $\nu^*(t)$ is continuous, there exists $\delta > 0$ such that $\nu(t+h) > \inf_{0 \leq u \leq t+h} \nu(u)$ for all positive $h < \delta$. It follows that t is not a point of increase of $\nu^*(t)$. We have thus established three important facts:

(i) The first point at which $\zeta(t) = 0$ is the smallest solution of the equation $\nu^*(t) = \zeta$.

(ii) Every point for which $\zeta(t) > 0$ is a point of stationarity of $\nu^*(t)$.

(iii) All points at which $\zeta(t) = 0$ are points for which $\nu(t) = -\nu^*(t)$.

Let now $E = \{ u ; \quad t_0 \leq u < t , \ \nu(t) = -\nu^*(t) \}$, and let D be the (denumerable) set of discontinuities of $\nu(t)$. If $t \in D$,

$$\nu(t) > \nu(t-0) \geq -\nu^*(t-0) = -\nu^*(t),$$

and so $t \notin E$, so that D and E are disjoint.

Let now $E_1 = \{ u ; t_0 \leq u < t , \nu(t-0) = -\nu^*(t) \}$.

Then $E_1 = E \cup D_1$ where D_1 is a subset of D.

If $t \notin E_1$, then $\nu(t) \geq \nu(t-0) > -\nu^*(t)$, and hence there exists an open interval I containing t in which $\nu(t) > -\nu^*(t)$. It follows that, in I, $\nu^*(t)$ is constant. Hence, since $t_0 \in E_1$, the complement E_1^c of E_1 in $[t_0, t)$ is open, and every point of E_1^c has a neighbourhood in which $\nu^*(t)$ is constant. Thus $\nu^*(t)$ is constant on every connected component of E_1^c. But the connected components of E_1^c form an at most countable family $\{I_n\}$ of open intervals. It follows that

$$\int_{E_1^c} d\nu^*(u) = \sum_n \int_{I_n} d\nu^*(u) = 0 .$$

But

$$\int_{E_1} d\nu^*(u) = \int_E d\nu^*(u) + \int_{D_1} d\nu^*(u)$$

and as $\nu^*(t)$ is continuous and D_1 is denumerable,

$\int_{D_1} d\nu^*(u) = 0$. We conclude that

$$\int_E d\nu^*(u) = \int_{E_1} d\nu^*(u)$$

Now as, on E, $\nu(t) = -\nu^*(t)$, we have

$$\int_E d\nu^*(u) = -\int_E d\nu(u) .$$

Finally,

$$\nu^*(t) - \nu^*(t_0) = \int_E d\,\nu(u),$$

which can be written

$$\nu^*(t) - \nu^*(t_0) = \int_{t_0}^{t} U[-\zeta(u)]\, d\nu(u) \qquad (1.10)$$

We now notice that as $\nu^*(t)$ is non-decreasing, the restriction to E of the Stieltjes measure μ determined by $\nu(t)$ is negative. Hence we must have $\mu_+(S) = 0$, for every measurable subset S of E . Therefore

$$\nu^*(t) - \nu^*(t_0) = \int_{t_0}^{t} U[-\zeta(u)]\, d\nu_-(u) .$$

Using now the relation $\nu^*(t_0) = z$, we conclude that in case (b) also

$$\begin{aligned} \zeta(t) &= \nu(t) + \nu^*(t) , \\ &= z + \nu(t) + \int_{t_0}^{t} U[-\zeta(u)]\, du , \\ &= z + \nu(t) + \int_0^t U[-\zeta(u)]\, du , \end{aligned}$$

since $\zeta(t) = 0$ for $0 \leq t < t_0$.

This completes the proof that $\zeta(t)$, as defined by equation (1.9) satisfies the integral equation (1.8).

We shall now show that if $\zeta(t)$ is non-negative and satisfies equation (1.8), it must be of the form given by equation (1.9), and equation (1.8) can therefore have at most one non-negative solution.

We obviously have, for every u such that $0 \le u \le t$,

$$\zeta(u) \geq z + \nu(u),$$

and also

$$\zeta(t) = \zeta(u) + \nu(t) - \nu(u) + \int_u^t U[-\zeta(v)]\, d\nu_-(v) \geq \nu(t) - \nu(u)$$

so that

$$\zeta(t) \geq \nu(t) - \inf_{0 \le u \le t} \nu(u) = \nu(t) + \nu^*(t).$$

Suppose now that $\zeta(u) > 0$ for all $u \in [0, t]$. Then $U[-\zeta(u)] = 0$ for all $u \in [0, t]$, and $\zeta(t) = z + \nu(t)$. On the other hand, if $\zeta(u) = 0$ for some $u \in [0, t]$, and if we denote $\sup\{u ; \zeta(u) = 0\}$ by y, then there exists a non-decreasing sequence $\{u_n\}$ tending to y, such that $\zeta(u_n) = 0$. For every u_n, we have

$$\begin{aligned} \zeta(t) &= \zeta(u_n) + \nu(t) - \nu(u_n) + \int_{u_n}^t U[-\zeta(u)]\, d\nu_-(u) \\ &= \nu(t) - \nu(u_n) + \int_{u_n}^y U[-\zeta(u)]\, d\nu_-(u) \\ &\geq \nu(t) - \nu(u_n). \end{aligned} \tag{1.11}$$

Finally, letting n tend to infinity, we find that

$$0 \le \zeta(t) - [\nu(t) - \nu(u_n)] = \int_{u_n}^y U[-\zeta(u)]\, d\nu_-(u) \le \nu_-(y) - \nu_-(u_n),$$

and as $\nu_-(u)$ has no discontinuities, it follows that

$$\zeta(t) = \nu(t) - \lim_{n\to\infty} \nu(u_n).$$

From this result and (1.1.1), we conclude that

$$\zeta(t) = \nu(t) - \inf_{0\le u\le t} \nu(u) = \nu(t) + \nu^*(t) > z + \nu(t).$$

This formula holds if $t \geq t_0$, the first zero of $\zeta(t)$.

For $t < t_0$, we have seen that

$$\zeta(t) = z + \nu(t) \geqslant \nu(t) + \nu^*(t).$$

Combining the two formulae, we find that

$$\zeta(t) = \nu(t) + \max\left[z,\ \nu^*(t)\right].$$

This completes the proof of uniqueness.

Using the theorem, we can now give a formal definition of the content of a store.

Definition: Let $\xi(t)$ be the input function to a store, satisfying the following conditions:

(a) $\xi(0) = 0$,

(b) $\xi(t)$ is non-decreasing,

(c) $\xi(t)$ is continuous to the right,

(d) $\xi(t) \to \infty$ as $t \to \infty$.

Let $\eta(t)$ be the planned output function, satisfying conditions (a) to (d), and satisfying in addition the following condition

(e) $\eta(t)$ is continuous.

Put $\nu(t) = \xi(t) - \eta(t)$, and let $\nu_-(t)$ be the negative variation of $\nu(t)$.

Finally put

$$\nu^*(t) = -\inf_{0 \le u \le t} \nu(t).$$

Then the content of the store, $\zeta(t)$, is the unique measurable, non-negative solution of the equation

$$\zeta(t) = z + \nu(t) + \int_0^t U[-\zeta(u)]\, d\nu_-(u) \quad (1.8)$$

where z is the initial content of the store.
This solution can be expressed in the form

$$\zeta(t) = \nu(t) + \max[z, \nu^*(t)]. \quad (1.9)$$

To complete this section, we state and prove a formula initially obtained by Beneš [5] for step-function inputs, and proved in its general form by Kingman [41] .

Theorem 1.2: For any value of s , we have

$$e^{-s\zeta(t)} = e^{-sz - s\nu(t)} - s\int_0^t e^{-s[\nu(t)-\nu(u)]}\, U[-\zeta(u)]\, d\nu_-(u). \quad (1.10)$$

Proof: This is based on the obvious identity

$$e^{-sf(t)} = 1 - s\int_0^t e^{-sf(u)}\, df(u),$$

where $f(t)$ is a non-decreasing function.
If we put

$$f(t) = \int_0^t U[-\zeta(u)]\, d\nu_-(u)$$

we can write

$$e^{-s\zeta(t)} = e^{-sz - s\nu(t)} \cdot e^{-sf(t)}$$

$$= e^{-sz - s\nu(t)}\left[1 - s\int_0^t e^{-sf(u)} U[-\zeta(u)]\, d\nu_-(u)\right] \quad (1.12)$$

We now note that the integrand vanishes unless $\zeta(u) = 0$.
But when $\zeta(u) = 0$, it follows from equation (1.8) that

$$f(u) = \int_0^u U[-\zeta(v)]\, d\nu_-(v) = -z - \nu(u).$$

Thus (1.12) can be written

$$e^{-s\zeta(t)} = e^{-sz - s\nu(t)} - s\int_0^t e^{-s[\nu(t) - \nu(u)]} U[-\zeta(u)]\, d\nu_-(u).$$

This completes the proof of the theorem.

6. Various interpretations of the model.

The model outlined in section 1 of this chapter can have various interpretations. We shall give here one interpretation in queueing theory and three interpretations in the theory of dams.

(a) An interpretation in queueing theory.

Let customers arrive at times $\{t_1, t_2, \ldots, t_n, \ldots\}$ at a service point with a single server. Let $X_1, X_2, \ldots, X_n, \ldots$ be the service times of these customers. Let us write

$$\xi(t) = \sum_{t_n \le t} X_n.$$

We define the "virtual waiting time" at any time t as the time that a customer arriving at time t would have to wait until he began to be served. This is equal to the total time required to complete the service of all customers in the queue at time t, including the customer being served. It is clear that if we put $\eta(t) = t$, the virtual waiting time is equal to $\zeta(t)$. Let us note that, as the input function is a step function, $\nu(t) = t$, so that $\zeta(t)$ satisfies the integral equation

$$\zeta(t) = z + \xi(t) - t + \int_0^t U[-\zeta(u)]\,du \,. \tag{1.1}$$

(b) A dam of infinite capacity with steady release.

We consider a dam having infinite capacity, i.e. a dam such that no overflow ever occurs. Let $\xi(t)$ be the amount of water flowing into the dam in the interval $(0, t]$. Let the release rule be as follows:

If the dam is not empty, the release is at a rate of one unit of water per unit time.

Then $\zeta(t)$ represents the amount of water in the dam at time t.

(c) A dam with infinite depth with steady input.

Let us consider a dam with infinite depth, i.e. a dam such that it is never empty. Overflow, however, can occur. In other words, the dam under consideration operates in a nearly full condition, while the dam in (b) operated in a nearly empty condition. Let the input to the dam be steady, while the output in $(0, t]$ is given by $\xi(t)$. Then $\zeta(t)$ represents the amount of water required to fill the dam completely. Thus model (c) is the exact dual of model (b).

(d) Let the inputs and releases to the dam occur at fixed points of time, which we shall denote by $\{0, 1, 2, 3, \dots\}$ Let the input at time n be X_n and the amount of water in the dam at time n after the input has flowed in be Z_n. (This assumption corresponds to the right-continuity of the input function in the continuous time model).

Let the releases occur at the times n before the inputs, and let Y_n be the planned release, i.e. the release at time n is equal to $\min(Z_{n-1}, Y_n)$.

To reduce this problem to the preceding one, we change our method of measuring time. In our new scale of measurement, the time elapsed up to instant n is the total amount of planned release up to and including instant n, i.e. $t_n = \sum_{k=0}^{n} Y_k$. Put $\xi(t) = \sum_{t_n \leq t} X_n$. Then the content of the dam at time n, Z_n, is given by $\xi(t_n)$.

CHAPTER 2

THE STRUCTURE OF NON-NEGATIVE ADDITIVE PROCESSES

1. Definitions and elementary properties.

In the following chapters, we shall be concerned with non-negative, additive, stationary stochastic processes. Let $\xi(t)$ be such a process. It will have the following properties:

(1) $\xi(t) \geqslant 0$ a.s. (Non-negativity).

(2) If $t_1 < t_2 < \ldots < t_n$, $(n \geq 3)$, the differences $\xi(t_2) - \xi(t_1), \ldots, \xi(t_n) - \xi(t_{n-1})$ are mutually independent (Additivity).

(3) For every set of points $\{t_1, t_2, \ldots, t_n\}$, and every value of u, the joint distribution of the random variables

$$\{\xi(t_1), \xi(t_2), \ldots, \xi(t_n)\}$$

is the same as that of

$$\{\xi(t_1+u), \xi(t_2+u), \ldots, \xi(t_n+u)\} \quad \text{(Stationarity)}.$$

It follows from non-negativity and stationarity that almost all sample functions of the process are non-decreasing.

We shall write $K(t, x) = P\{\xi(t) \leq x\}$.

$K(t,x)$ is the distribution function of $\xi(t)$, and we have $K(t, +\infty) = 1$, $K(t, 0-) = 0$.

We shall also write

$$\Theta(t, s) = E\left[e^{-s\xi(t)}\right] = \int_{0-}^{+\infty} e^{-sx} d_x K(t, x), \qquad (2.1)$$

where $s = \sigma + i\omega$ is a complex number.

$\Theta(t,s)$ is the Laplace-Stieltjes transform of $K(t,x)$ with respect to x. It has the following properties:

(1) The integral (2.1) converges for every s such that $\mathrm{Re}(s) \geq 0$.

This follows immediately from the relation

$$\left| \int_{0-}^{+\infty} e^{-sx} d_x K(t,x) \right| \leq \int_{0-}^{+\infty} e^{-\sigma x} d_x K(t,x) \leq 1 \quad , \sigma \geq 0 .$$

(2) For every fixed value of t, $\Theta(t,s)$ is an analytic function of s in the region $\mathrm{Re}(s) > 0$.

This follows immediately from well-known properties of the Laplace-Stieltjes transform (see Widder [74] p. 57, Theorem 5a).

(3) $$\Theta(t+u,s) = \Theta(t,s)\,\Theta(u,s) . \qquad (2.2)$$

This follows from the additivity and stationarity of the process $\xi(t)$.

<u>Theorem 2.1:</u> $\Theta(t,s)$ is of the form

$$\Theta(t,s) = \exp\{-\alpha(s)\,t\} .$$

<u>Proof:</u> Let s take a fixed real value such that $s \geq 0$. For every value of t, we have $0 \leq \Theta(t,s) \leq 1$. Put $f(t) = \log \Theta(t,s)$. Then $f(t) \leq 0$ and we have

$$f(t+u) = f(t) + f(u) .$$

We now prove the following lemma:

<u>Lemma:</u> Let $f(t)$ satisfy the equation

$$f(t+u) = f(t) + f(u)$$

for every t and u. Moreover let $f(t)$ be bounded, either from above or from below in $0 \leq t \leq c$ for some c. Then $f(t) = \alpha t$ for some fixed α.

<u>Proof of the lemma</u>: We first prove that if r is a rational number, then $f(r) = r f(1)$. In fact let $r = m/n$.
Then $f(m/n) = f(\sum_1^m 1/n) = m f(1/n)$. But
$f(1) = f(\sum_1^n 1/n) = n f(1/n)$, i.e. $f(1/n) = f(1)/n$.
We conclude that $f(m/n) = (m/n) f(1)$.
Consider now $\varphi(t) = f(t) - [f(c)/c]\, t$.
Then $\varphi(c) = f(c) - [f(c)/c]\, c = 0$,
and $\varphi(t+c) = \varphi(t) + \varphi(c) = \varphi(t)$, i.e. $\varphi(t)$ is periodic of period c. It follows that $\varphi(t)$ is bounded either from above or from below over the entire t-axis.

Suppose now that there exists t_0 such that $\varphi(t_0) \neq 0$. We have $\varphi(n t_0) = n \varphi(t_0)$, and this is bounded neither from above nor from below. It follows that $\varphi(t)$ is identically zero and $f(t) = [f(c)/c]\, t$. Put now $t = 1$. Then $f(c)/c = f(1)$, so that finally $f(t) = t f(1)$.

This completes the proof of the lemma.

Returning to the main theorem, we conclude from (2.2) that $\log \Theta(t,s) = \{-\alpha(s)\}\, t$, where $\alpha(s) = -\log\{\Theta(1,s)\}$. As this formula is true for all real values of s, it must be true for all complex values as well, because of the uniqueness of the Laplace transform.

This completes the proof of the theorem.

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2. Complete monotonicity of $\Theta(t, s)$ in s.

We shall start by recalling certain definitions and theorems relating to absolute and complete monotonicity. For proofs of the theorems stated, see Widder [74].

Definition 2.1: A function $f(s)$ is absolutely monotonic in the interval $a \leq x < b$ if it has non-negative derivatives of all orders for $a < x < b$:

$$f^{(k)}(s) \geqslant 0 \qquad (a < s < b \; ; k = 0, 1, 2, \ldots),$$

and is continuous at a .

Definition 2.2: A function $f(s)$ is completely monotonic in $[a, b)$ if and only if $f(-s)$ is absolutely monotonic in $(-b, -a]$, i.e. if

$$(-1)^k f^{(k)}(s) \geqslant 0 \quad (a < s < b \; ; k = 0, 1, 2, \ldots),$$

and $f(s)$ is continuous at a .

Theorem 2.2: A necessary and sufficient condition that $f(s)$ should be completely monotonic in $0 \leq s < \infty$ is that

$$f(s) = \int_{0-}^{\infty} e^{-sx} \, dF(x),$$

where $F(x)$ is bounded and non-decreasing and the integral converges for $0 \leq x < \infty$.

By using the concept of complete monotonicity, we solve below, for the case of non-negative random variables, a problem as yet unsolved in the general case. This is the specification of a criterion for a function of a complex variable to be the characteristic function of a random variable, which is easily applicable to a large class of functions. For a discussion of the general problem, see Lukács [49] p. 59.

From Theorem 2.2, we can deduce immediately the following

Theorem 2.3: Necessary and sufficient conditions for the function $f(s)$ to be the Laplace-Stieltjes transform of the distribution function of a non-negative random variable are:

(a) $f(0) = 1$,

(b) $f(s)$ is completely monotonic in $0 \leq s < \infty$.

Proof: It follows from theorem 2.2 and condition (b) that $f(s)$ can be represented in the form

$$f(s) = \int_{0-}^{\infty} e^{-sx}\, dF(x),$$

where $F(x)$ is bounded and non-decreasing. It then follows from condition (a) that

$$\int_{0-}^{\infty} dF(x) = 1.$$

This ensures that $F(x)$ is the distribution function of a non-negative random variable.

We shall restrict ourselves in the sequel to additive processes for which $\alpha(s)$ admits, for $Re(s) \geq 0$, the representation

$$\alpha(s) = \int_{0}^{+\infty} (1 - e^{-sx})\, dM(x) \tag{2.3}$$

where $M(x)$ is a non-decreasing function such that $M(\infty) = 0$, and $\lim_{x \to 0} x M(x) = 0$.

Let us show that, in this case, $\Theta(t, s)$ is the Laplace-Stieltjes transform of a distribution function.

We have

$$-\alpha(s) \leq 0 ,$$

$$(-1)\left[-\alpha'(s)\right] = \int_{0-}^{\infty} x\, e^{-sx}\, dM(x) \geqslant 0 ,$$

$$(-1)^n\left[-\alpha^{(n)}(s)\right] = \int_{0-}^{\infty} x^n\, e^{-sx}\, dM(x) \geqslant 0.$$

Because $-\alpha(s) \leq 0$, $-\alpha(s)$ is not completely monotonic. On the other hand, we have, writing $\Theta_{ij}(t,s)$ for $\partial^{i+j}\Theta(t,s)/\partial t^i\,\partial s^j$,

$$\Theta(t,s) = e^{-\alpha(s)t} > 0 ,$$

$$\Theta_{01}(t,s) = -t\, e^{-\alpha(s)t}\alpha'(s) < 0 ,$$

$$\Theta_{02}(t,s) = t^2 e^{-\alpha(s)t}\left[\alpha'(s)\right]^2 - t\, e^{-\alpha(s)t}\alpha''(s) > 0,$$

— — — — — — — — — — —

Continuing in this way, we see that $\Theta(t,s)$ is completely monotonic. Moreover, $\alpha(0) = 0$, so that $\Theta(t,0) = 1$. It follows from theorem 2.3 that $\Theta(t,s)$ is the Laplace-Stieltjes transform of a distribution function, and the representation $\Theta(t,s) = e^{-\alpha(s)t}$ ensures that the underlying stochastic process is additive.

Let us note at this stage that the conditions imposed on $M(x)$ enable us to integrate (2.3) by parts.

In fact, we have

$$\alpha(s) = (1 - e^{-sx})\, M(x)\Big]_0^{\infty} - s\int_0^{\infty} e^{-sx}\, M(x)\, dx ,$$

i.e.

$$\alpha(s) = -s\int_0^\infty e^{-sx} M(x)\,dx. \tag{2.4}$$

We thus see that, given any function $M(x)$ satisfying the conditions

(a) $M(x)$ is non-decreasing,

(b) $M(+\infty) = 0$,

(c) $\lim_{x\to 0} x\,M(x) = 0$,

we obtain an additive, non-negative, stationary stochastic process $\xi(t)$ by taking as the Laplace-Stieltjes transform of its distribution function $e^{-\alpha(s)t}$, where

$$\alpha(s) = -s\int_0^\infty e^{-sx} M(x)\,dx.$$

Let us note that the semi-invariants I_n of $\xi(t)$ are all linear functions of the time. In fact, we have

$$I_n = (-i)^n \left\{ \frac{d^n}{ds^n} \log E\left[e^{is\xi(t)}\right]\right\}_{s=0},$$

$$= (-i)^n \left\{ \frac{d^n}{ds^n}\left[-\alpha(-is)\right]\right\}_{s=0} t,$$

$$= (-1)^{n+1}\alpha^{(n)}(0)\,t.$$

In particular, the mean and standard deviation of $\xi(t)$ are

$$E\left[\xi(t)\right] = \alpha'(0)\,t,$$

$$Var\left[\xi(t)\right] = -\alpha''(0)\,t.$$

In the sequel, we shall denote the mean value of $\xi(t)$ per unit time by ρ and its standard deviation per unit time by

σ and we shall frequently make use of the representation (which follows from the Darboux expansion of $\alpha(s)$)

$$\alpha(s) = \rho s - \frac{\sigma^2}{2} s^2 + O(s^3).$$

3. The relationship between $M(x)$ and the derivatives of $K(t,x)$.

Consider the relationship

$$s\int_0^\infty e^{-sx} K(t,x)\,dx = e^{-\alpha(s)t}, \tag{2.5}$$

and denote $\partial K(t,x)/\partial t$ by $K_{10}(t,x)$. Suppose that $K_{10}(t,x)$ is dominated for $0 \leq t \leq T$ by a function $D(x) \geq 0$ whose Laplace transform converges for some s. We then have

$$s\int_0^\infty e^{-sx} K_{10}(0,x)\,dx = \frac{\partial}{\partial t} e^{-\alpha(s)t}\Big|_{t=0} = -\alpha(s),$$

$$= s\int_0^\infty e^{-sx} M(x)\,dx.$$

It follows that, for almost all x,

$$K_{10}(0,x) = M(x). \tag{2.6}$$

This can also be written, using the fact that $K(0,x) = 1$,

$$M(x) = -\lim_{t \downarrow 0} \frac{1 - K(t,x)}{t} \tag{2.7}$$

More generally, if we can differentiate the left-hand side of (2.5) n times under the integral sign, we find that

$$s\int_0^\infty e^{-sx} K_{n0}(0,x)\,dx = \left[-\alpha(s)\right]^n, \tag{2.8}$$

where
$$K_{n0}(t,x) = \frac{\partial^n}{\partial t^n} K(t,x).$$

4. <u>The Poisson Process as the simplest type of stationary additive process.</u>

We shall now assume that the state space of $\xi(t)$ is the set of positive integers $\{0, 1, 2, \ldots \}$. This implies that almost all sample functions of the process are step functions. We shall write

$$p_n(t) = P\{\xi(t) = n\}, \quad n = 0, 1, 2, \ldots$$

$$q_n(t) = \sum_{k=n+1}^{\infty} p_k(t). \qquad (2.9)$$

Moreover, we shall assume that, as $t \to 0$, $q_1(t)$ is $o(t)$. This means that, in a small interval of time, the probability of an increase in $\xi(t)$ of more than one is $o(t)$. We first notice that $p_0(t)$ satisfies the functional equation

$$p_0(t+u) = p_0(t)\, p_0(u),$$

because in two consecutive intervals $(\tau, \tau+t)$, $(\tau+t, \tau+t+u)$, the probability of having no change of state is the product of the probabilities of having no change of state in each of the intervals. As we have $0 \le p_0(t) \le 1$, we conclude, using a reasoning similar to that of Theorem 2.1, that $p_0(t)$ is of the form $e^{-\lambda t}$, where λ is some non-negative number.

We now notice that, by the theorem of total probability,

$$p_n(t+u) = p_n(t)\,p_0(u) + p_{n-1}(t)\,p_1(u) + \dots + p_0(t)\,p_n(u),$$
$$(n \geqslant 1).$$

As $u \to 0$, we have

$$p_0(u) = e^{-\lambda u} = 1 - \lambda u + o(u),$$

$$p_1(u) = 1 - p_0(u) - q_1(u) = \lambda u + o(u),$$

$$\sum_{k=2}^{n} p_{n-k}(t)\,p_k(u) \leq \sum_{k=2}^{n} p_k(u) \leq q_1(u) = o(u).$$

It follows that

$$p_n(t+u) = p_n(t)(1-\lambda u) + p_{n-1}(t)\lambda u + o(u),$$

and

$$\frac{p_n(t+u) - p_n(t)}{u} = -\lambda\left[p_n(t) - p_{n-1}(u)\right] + O(u),$$

so that finally

$$p_n'(t) = -\lambda\left[p_n(t) - p_{n-1}(t)\right], \quad n \geqslant 1.$$

We can solve this differential-difference equation by generating functions. Putting

$$\sum_{n=0}^{\infty} p_n(t)\,s^n = P(t,s),$$

we find that

$$\frac{\partial P}{\partial t} = \lambda(s-1)\,P.$$

Integrating and remembering that

$$P(0,s) = p_0(0) = 1,$$

we find

$$P(t,s) = e^{\lambda(s-1)t}$$
$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} s^n,$$

DIAGRAM

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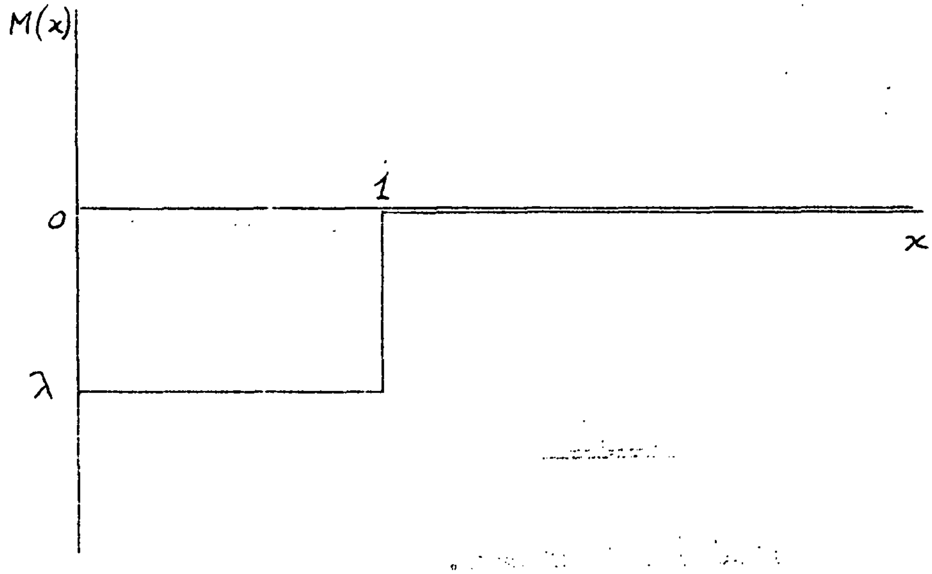

Figure 2.1 Graph of $M(x)$
for the Poisson Process

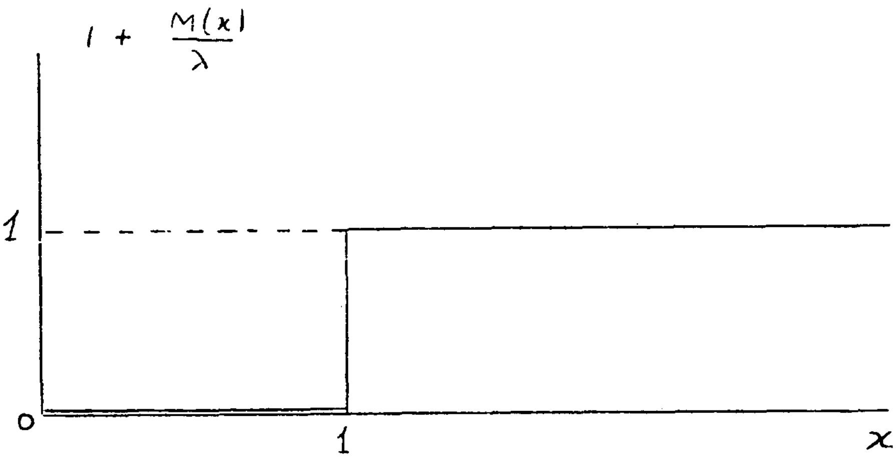

Figure 2.2 Graph of $1 + [M(x)/\lambda]$
for the Poisson Process

To face p. 35

so that $$p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}. \tag{2.10}$$

We thus see that, for an additive stationary process with state space $\{0, 1, 2, \ldots\}$, the unique condition $q_1(t) = o(t)$ ensures that the number of changes of state in an interval $(0,t)$ follows a Poisson law with parameter λt, where $\lambda = -\log p_0(1)$.

Let us also note that in the case of the Poisson process, the Laplace-Stieltjes transform of the distribution of $\xi(t)$, $\Theta(t,s)$, is given by

$$\Theta(t,s) = \sum_{n=0}^{\infty} e^{-sn} p_n(t) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t e^{-s})^n}{n!}$$
$$= e^{-\lambda t(1-e^{-s})},$$

so that $\alpha(s) = \lambda(1-e^{-s})$.

Remembering that

$$\int_0^\infty e^{-sx}[-M(x)]\,dx = \frac{\alpha(s)}{s},$$

we see that we can obtain $-M(x)$ by finding the inverse Laplace transform of $\lambda(1-e^{-s})/s$. We find

$$M(x) = \begin{cases} -\lambda & 0 < x < 1 \\ 0 & x > 1 \end{cases} \tag{2.11}$$

We note that $1 + [M(x)/\lambda]$ is the distribution function of a random variable which takes the value <u>one</u> almost surely. We thus see that $1 + [M(x)/\lambda]$ can be taken to represent the distribution of the length of jumps. We shall see in the sequel that such an interpretation can be extended to a wide class of additive stochastic processes.

5. <u>The compound Poisson process</u>.

We shall now relax the assumption that the state space is discrete, but we shall continue to assume that almost all sample functions are step functions having a finite number of discontinuities in any finite interval of time.

To analyse such a process, we associate with it a counting process, $N(t)$, which is equal to the number of jumps in $(0,t)$. This process is additive and stationary. We shall assume that it satisfies, for small t, the additional condition $P\{N(t)>1\} = o(t)$. It follows that $N(t)$ is a Poisson process. Let us assume that its parameter is λ. Moreover, let X represent the length of a jump. Because $\xi(t)$ is stationary and additive, jump lengths are independent random variables having all the same distribution. Let us write

$$P\{X \leq x\} = B(x).$$

Moreover, let us denote the nth convolution of $B(x)$ with itself by $B_n(x)$. Using the theorem of total probability, we find

$$K(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} B_n(x). \tag{2.12}$$

Expanding $e^{-\lambda t}$ in powers of t and multiplying out the two series, we find

$$K(t,x)=\sum_{n=0}^{\infty} a_n(x)\,\frac{t^n}{n!}\,,\quad t\geq 0\,, \tag{2.13}$$

where $\quad a_n(x)=(-1)^n\lambda^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k B_k(x)\,.$

Differentiating (2.13) n times and putting $t=0$, we find

$$K_{n0}(0,x)=a_n(x)=(-1)^n\lambda^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k B_k(x).$$

and in particular

$$M(x)=K_{10}(0,x)=-\lambda+\lambda\,B(x),$$

so that $\quad B(x)=1+[M(x)/\lambda]\,.$

The Laplace-Stieltjes transform of the distribution of $\xi(t)$, $\Theta(t,s)$, will be given by

$$\Theta(t,s)=\int_{0-}^{\infty}e^{-sx}d_xK(t,x)=\sum_{n=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_{0-}^{\infty}e^{-sx}dB_n(x)\,,$$

$$=\sum_{n=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}[\psi(s)]^n=\exp[-\lambda t\{1-\psi(s)\}]$$

where $\psi(s)$ is the Laplace-Stieltjes transform of $B(x)$,

$$\psi(s)=\int_{0}^{\infty}e^{-sx}dB(x)\,.$$

The term-by-term integratioh is easily justified by the fact that the total variation of $B_n(x)$ is unity. It follows that $\Theta(t,s)=\sum_{n=0}^{\infty}e_n(s)\,t^n/n!$, $t\geq 0$, where

$e_n(s)=\lambda^n[\psi(s)-1]^n=(-1)^n\lambda^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k[\psi(s)]^k$, so that

$$e_n(s) = s\int_0^\infty e^{-sx} K_{n0}(0,x)\,dx.$$

In this case, $\alpha(s)$ is given by the formula

$$\alpha(s) = \lambda\left[1 - \psi(s)\right],$$

$$= \lambda\int_0^\infty \left(1 - e^{-sx}\right) dB(x). \qquad (2.14)$$

Let us note at this juncture that if $B(x)$ is absolutely continuous, then so is $M(x)$, and if $B(x)$ is a step function, so is $M(x)$.

Looking now at the shape of $K(t,x)$, we note that

$$\lim_{x\to 0} K(t,x) = \lim_{s\to\infty} \Theta(t,s).$$

We shall assume that

$$\lim_{x\to 0} B(x) = P\{X = 0\} = 0 .$$

Then

$$\lim_{s\to\infty} \psi(s) = 0,$$

so that we finally have

$$\lim_{x\to 0} K(t,x) = e^{-\lambda t}.$$

Thus $K(t,x)$ will always have a jump at the origin, in the case of the Compound Poisson input.

Suppose finally that the service time distribution $B(x)$ has a continuous derivative $b(x)$. Then $B_n(x)$ has a continuous derivative $b_n(x)$. It follows immediately that $K(t,x)$ has a continuous partial derivative in x for all $x > 0$, given by

$$\frac{\partial}{\partial x} K(t,x) = k(t,x) = \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\, b_n(x),$$

and, for fixed t , has a jump of magnitude $K(t,0)=e^{-\lambda t}$ at $x=0$, so that we can write

$$K(t,x) = K(t,0) + \int_0^x k(t,y)\,dy .$$

6. The input process of the queue M/M/1.

Let us examine the input process of the queueing model given in section 6 of Chapter 1, in the special case when the arrival process is of Poisson type with parameter λ , and the service time $\mathcal{X}$ is exponentially distributed. We then have

$$B(x) = P\{\mathcal{X} \leq x\} = 1 - e^{-\mu x} .$$

In this case, $\mathcal{X}$ has a density function given by $\mu e^{-\mu x}$, and $\psi(s)$ is given by

$$\psi(s) = \int_0^\infty e^{-sx} . \mu e^{-\mu x} dx = \frac{\mu}{\mu+s} .$$

This queueing model is known as the queue M/M/1.

We have in this case

$$s\int_0^\infty e^{-sx} K(t,x)\,dx = \Theta(t,s) = \exp\left\{-\lambda t\left[1 - \frac{\mu}{\mu+s}\right]\right\} .$$

Let us write

$$\int_0^\infty e^{-sx} K(t,x)\,dx = e^{-\lambda t} . \frac{1}{s}\, e^{\frac{\lambda\mu t}{\mu+s}} ,$$

$$= e^{-\lambda t}\left[\frac{e^{\frac{\lambda\mu t}{\mu+s}}}{s+\mu} + \frac{\mu}{s} . \frac{e^{\frac{\lambda\mu t}{\mu+s}}}{s+\mu}\right] .$$

Using Erdelyi [19] p. 245, No. 35, we see that the Inverse Laplace Transform of $e^{\lambda\mu t/s}/s$ is $I_0(2\sqrt{\lambda\mu t x})$ so that the Inverse Laplace Transform of $e^{\lambda\mu t/(s+\mu)}/(s+\mu)$ is $e^{-\mu x} I_0(2\sqrt{\lambda\mu t x})$. We conclude that $K(t,x)$ is given by

$$K(t,x) = e^{-\lambda t}\left[e^{-\mu x} I_0(2\sqrt{\lambda\mu t x}) + \mu\int_0^x e^{-\mu y} I_0(2\sqrt{\lambda\mu t y})\,dy\right].$$

From this formula, we obtain, as expected

$$\lim_{x\to 0} K(t,x) = e^{-\lambda t},$$

as $I_0(0) = 1$.

Also, for $x > 0$, $t > 0$, $K(t,x)$ has a continuous derivative, given by

$$k(t,x) = \frac{\partial}{\partial x} K(t,x) = e^{-\lambda t}\left[-\mu e^{-\mu x} I_0 + e^{-\mu x}\frac{\partial I_0}{\partial x} + \mu e^{-\mu x} I_0\right],$$

$$= e^{-\lambda t - \mu x}\frac{\partial}{\partial x} I_0(2\sqrt{\lambda\mu t x}).$$

Using $I_0'(z) = I_1(z)$, we find

$$k(t,x) = e^{-\lambda t - \mu x}\sqrt{\frac{\lambda\mu t}{x}}\; I_1(2\sqrt{\lambda\mu t x}),$$

and we note that $k(t,0) = \lambda\mu t\, e^{-\lambda t}$.

7. <u>Bunched arrivals.</u>

Let us return to the process $\xi(t)$ with state space $\{0,1,2,\ldots\ldots\}$, but let us now replace the condition $q_1(t) = o(t)$ by the conditions

$$p_0(t) = 1 - \lambda t + o(t),$$

$$p_n(t) = \lambda_n t + o(t), \quad n = 1, 2, \ldots \tag{2.15}$$

We again use the equation

$$p_n(t+u) = \sum_{k=0}^{n} p_{n-k}(t)\, p_k(u), \quad n \geq 1,$$

which becomes

$$p_n(t+u) = p_n(t)(1-\lambda u) + \sum_{k=1}^{n} p_{n-k}(t)\lambda_k u + o(u).$$

Rearranging the terms and letting u tend to zero, we finally obtain

$$p_n'(t) = -\lambda p_n(t) + \sum_{k=1}^{n} p_{n-k}(t)\lambda_k .$$

Putting, as before

$$P(t,s) = \sum_{n=0}^{\infty} p_n(t) s^n,$$

and solving the differential-difference equation by generating functions, we find

$$\frac{\partial P}{\partial t} = -\lambda P + \left(\sum_{n=1}^{\infty} \lambda_n s^n\right) P. \tag{2.16}$$

Let us now write $L(s) = \lambda - \sum_{n=1}^{\infty} \lambda_n s^n$.

The solution of equation (2.16) becomes, using the fact that $P(0,s) = 1$,

$$P(t,s) = e^{-L(s)t}$$

We now note that for the $p_n(t)$ to form a probability distribution, we must have $P(t,1) = 1$, so that

$$\lambda = \sum_{m=1}^{\infty} \lambda_m . \tag{2.17}$$

The Laplace-Stieltjes transform of the distribution of $\xi(t)$ can be obtained in this case by replacing s in $P(t,s)$ by e^{-s}. Thus we have

$$\Theta(t,s) = \exp\left[\left\{-\lambda + \sum_{m=1}^{\infty} \lambda_m e^{-ms}\right\} t\right] ,$$

and

$$\alpha(s) = \lambda - \sum_{m=1}^{\infty} \lambda_m e^{-ms} .$$

We now notice that $\alpha(s)$ can be written in the form

$$\alpha(s) = \lambda \int_0^{\infty} (1 - e^{-sx})\, d\, B(x),$$

where $B(x) = \sum_{m=0}^{[x]} \lambda_m / \lambda$

Because of the relation (2.17), $B(x)$ is a genuine probability distribution.

We thus see that the process defined by condition (2.15) can be described as follows:

(a) the points at which there is a change of state form a Poisson process of parameter λ.

(b) the probability that, given that a change of state has occurred, the magnitude of the jump is m, is λ_m / λ.

If we regard the value of $\xi(t)$ as representing the number of arrivals in $(0,t)$, then the process described by equation (2.15) is a process of <u>bunched arrivals</u>.

8. Processes where the sample functions are not a.s. step functions with isolated discontinuities.

We have seen in section 5 that, in the case of a Compound Poisson process,

$$P\{\xi(t)=0\} = K(t,0) = e^{-\lambda t}.$$

We conclude that $\lim\limits_{t\to 0} P\{\xi(t)=0\} = 1$, i.e. for almost all sample functions, there is an interval to the right of any point of increase where $\xi(t)$ is constant.

Thus, in the case of the Compound Poisson process, almost all sample functions will have isolated points of increase.

Let us now consider the general form of the process where

$$\alpha(s) = \int_0^\infty (1-e^{-sx})\, dM(x).$$

If $\int_0^\infty dM(x)$ is finite and equals λ, then the process is a compound Poisson process, with $B(x) = 1 + [M(x)/\lambda]$.

If, however, $\int_0^\infty dM(x) = \infty$, then $\lim\limits_{s\to 0+} \alpha(s) = +\infty$, so that $P\{\xi(t)=0\}=0,\ t>0$, and in this case points of increase will not be isolated for almost all sample functions.

The question now arises whether a non-negative, stationary additive process can have continuous sample functions. The answer is given by a theorem of Doob [17] p. 420, which we quote here.

Theorem 2.4: Let $\{\xi(t)\ ;\ a \leq t \leq b\}$ be a centered process with independent increments and no fixed points of discontinuity. Then the following conditions are equivalent:

(a) $\xi(b) - \xi(a)$ is Gaussian ,

(b) every difference $\xi(t+u) - \xi(t)$ is Gaussian ,

(c) if the process is separable, almost all sample functions are continuous on $[a, b]$.

Thus processes having almost all sample functions continuous are necessarily Gaussian in character, i.e. they cannot be non-negative.

It follows that processes for which $\int_0^\infty dM(x) = +\infty$ have sample functions with an infinite number of discontinuities in any finite interval.

9. <u>The derivative of the sample functions of non-negative processes.</u>

We shall now show that, in spite of the fact that the set of discontinuities of the sample functions of the process is dense, the sample functions still have zero derivatives almost everywhere. To show this, we write

$$\int_0^\infty (1 - e^{-sx})\, dM(x) = \sum_{n=0}^\infty \int_{\frac{1}{n+1}}^{\frac{1}{n}} (1 - e^{-sx})\, dM(x).$$

Now $\exp\left[-\int_{\frac{1}{n+1}}^{\frac{1}{n}} (1 - e^{-sx})\, dM(x)\right]$ is the Laplace-Stieltjes transform of a Compound Poisson process. It follows that

$$\xi(t) = \sum_{n=0}^\infty \xi_n(t),$$

where the Laplace Stieltjes transform of $\xi_n(t)$ is

$\exp\left[-\int_{\frac{1}{n+1}}^{\frac{1}{n}} (1 - e^{-sx})\, dM(x)\right]$, and the series converges in probability. It follows that there exists a subsequence of partial sums which converges to $\xi(t)$ a.s., and because all the terms of the series are non-negative, the series itself must converge almost surely. Finally, we use the fact that the sample functions of the process are non-decreasing functions and apply Fubini's theorem on series of monotonic functions, (see Boas [8] p.139) which yields

$$\xi'(t) = \sum_{n=0}^{\infty} \xi_n'(t) \qquad \text{for almost all } t .$$

But as $\xi_n'(t) = 0$ for almost all t, the required result follows. Thus we can state

<u>Theorem 2.5</u>: For almost all sample functions of non-negative additive processes satisfying equation (2.3), the following property holds:

$$\underline{\xi'(t) = 0 \qquad \text{for almost all } t} .$$

We shall now discuss in detail two non-Poisson processes:

(a) the Gamma process,

(b) the Inverse Gaussian process.

For both processes, the distribution function $K(t,x)$ is absolutely continuous. We shall denote the density function of the process by $k(t,x)$. This is related to the distribution function by the relation

$$K(t,x) = \int_0^x k(t,y)\,dy .$$

10. <u>The Gamma process</u>.

The density function of this process is given by

$$k(t,x) = [\rho\,\Gamma(t)]^{-1} e^{-x/\rho} (x/\rho)^{t-1}. \tag{2.19}$$

The Laplace-Stieltjes transform of the distribution of the process is given by

$$\Theta(t,s) = \frac{1}{\rho\,\Gamma(t)} \int_0^\infty e^{-sx}\, e^{-\frac{x}{\rho}} \left(\frac{x}{\rho}\right)^{t-1} dx\,.$$

Let us put $y = x\left(s + \frac{1}{\rho}\right)$. Then

$$\Theta(t,s) = \left(\frac{1}{\rho}\right)^t \left(\frac{1}{s+\frac{1}{\rho}}\right)^t \frac{1}{\Gamma(t)} \int_0^\infty e^{-y}\, y^{t-1}\, dy\,,$$

i.e. $$\Theta(t,s) = \left(\frac{1}{1+\rho s}\right)^t. \tag{2.20}$$

It follows that

$$\alpha(s) = -\frac{1}{t} \log \Theta(t,s) = \log(1+\rho s). \tag{2.21}$$

We can obtain $-M(x)$ by finding the inverse Laplace transform of $\alpha(s)/s$ i.e. $[\log(1+\rho s)]/s$

Write

$$\frac{1}{s}\log(1+\rho s) = \frac{1}{s}\log\rho + \frac{1}{s}\log\left(\frac{1}{\rho}+s\right).$$

From Erdelyi [19] p. 251, item (5), we find

$$-M(x) = \log\rho + \log\frac{1}{\rho} - Ei\left(-\frac{x}{\rho}\right)$$

i.e.

$$M(x) = Ei\left(-\frac{x}{\rho}\right) \tag{2.22}$$

where $Ei(x)$ is defined by

$$- Ei(-x) = \int_x^\infty \frac{e^{-y}}{y}\, dy .$$

We can note immediately that

$$\int_0^\infty dM(x) = \lim_{x \to 0} Ei\left(-\frac{x}{\rho}\right) = -\int_0^\infty \frac{e^{-y}}{y}\, dy = \infty ,$$

so that the Gamma process is <u>not</u> a Compound Poisson process.

We can also obtain $M(x)$ directly from the formula

$$M(x) = -\lim_{t \to 0} \frac{1 - K(t, x)}{t} ,$$

$$= -\lim_{t \to 0} \frac{1}{\rho t \Gamma(t)} \int_x^\infty e^{-\frac{y}{\rho}} \left(\frac{y}{\rho}\right)^{t-1} dy .$$

We note that $\lim_{t \to 0} t\,\Gamma(t) = \lim_{t \to 0} \Gamma(t+1) = \Gamma(1) = 1$, so that

$$M(x) = -\frac{1}{\rho} \int_x^\infty e^{-\frac{y}{\rho}} \left(\frac{y}{\rho}\right)^{-1} dy .$$

Putting $\frac{y}{\rho} = z$, we finally find

$$M(x) = -\int_{\frac{x}{\rho}}^\infty \frac{e^{-z}}{z}\, dz = Ei\left(-\frac{x}{\rho}\right) ,$$

as before.

We can easily calculate the mean and variance of the process per unit time. We have

$$\frac{1}{t} E\left[\xi(t)\right] = \alpha'(0) = \frac{\rho}{1+\rho s}\bigg|_{s=0} = \rho \tag{2.23}$$

$$\frac{1}{t} Var\left[\xi(t)\right] = -\alpha''(0) = \frac{\rho^2}{(1+\rho s)^2}\bigg|_{s=0} = \rho^2 \tag{2.24}$$

11. <u>The Inverse Gaussian process.</u>

This process has only recently attracted the attention of statisticians. Its properties have been extensively studied by Tweedie [73]. However, the additive character of the process has not been pointed out explicitly by Tweedie.

The density function of the Inverse Gaussian process is given by

$$k(t,x) = \frac{t}{\sigma\sqrt{2\pi}}\left(\frac{\rho}{x}\right)^{\frac{3}{2}} \exp\left[-\frac{\rho}{2\sigma^2 x}(x-\rho t)^2\right]$$

We shall show in the sequel that ρ and σ^2 are respectively the mean and the variance per unit time of the process. This shows that the Inverse Gaussian process is much more flexible than the Gamma process, as it has two parameters as against one for the Gamma process.

We first obtain the Laplace-Stieltjes transform of the distribution. This is the ordinary Laplace transform of $k(t,x)$ with respect to x. To calculate it we first rewrite $k(t,x)$ as follows

$$k(t,x) = \frac{t\rho^{\frac{3}{2}}}{\sigma\sqrt{2\pi}}\, x^{-\frac{3}{2}} \exp\left[-\frac{\rho x}{2\sigma^2} + \frac{\rho^2 t}{\sigma^2} - \frac{\rho^3 t^2}{2\sigma^2 x}\right].$$

We now note that the Laplace transform of

$$x^{-\frac{3}{2}} \exp\left(-\frac{\rho^3 t^2}{2\sigma^2 x}\right) \tag{2.25}$$

is, using Erdelyi [19] p. 146, item (28),

$$2\pi^{\frac{1}{2}}\left(\frac{2\rho^3 t^2}{\sigma^2}\right)^{-\frac{1}{2}} \exp\left[-\left(\frac{2\rho^3 t^2}{\sigma^2}\right)^{\frac{1}{2}} s^{\frac{1}{2}}\right].$$

If we premultiply (2.25) by $\exp\left[-\frac{\rho x}{2\sigma^2}\right]$, the Laplace transform of the new function will be obtained by replacing s by $s + (\rho/2\sigma^2)$. Finally, introducing the remaining constant factor,

$$\frac{t\,\rho^{\frac{3}{2}}}{\sigma\sqrt{2\pi}} \exp\left[\frac{\rho^2 t}{\sigma^2}\right],$$

we obtain

$$\Theta(t,s) = \exp\left\{-\frac{\rho}{\sigma^2}\left(\sqrt{2\rho\sigma^2 s+\rho^2} - \rho\right)t\right\}.$$

This shows the additive character of the process, and we see that

$$\alpha(s) = \frac{\rho}{\sigma^2}\left(\sqrt{2\rho\sigma^2 s+\rho^2} - \rho\right). \tag{2.26}$$

We shall now calculate $M(x)$ for this process. We must invert $\alpha(s)/s$, i.e. find the inverse Laplace transform of

$$\frac{\rho}{\sigma^2}\left(\frac{\sqrt{2\rho\sigma^2 s+\rho^2}}{s} - \frac{\rho}{s}\right)$$

We rewrite this expression as

$$\left(\frac{\sigma\sqrt{2\rho}}{\sqrt{s+\frac{\rho}{2\sigma^2}}} + \frac{\rho^{\frac{3}{2}}}{\sigma\sqrt{2}\,s\sqrt{s+\frac{\rho}{2\sigma^2}}} - \frac{\rho}{s}\right)$$

Each term of this expression can be easily inverted, using Erdelyi [19] p. 176, item (4) and p. 235, item (21). The result is

$$-M(x) = \frac{1}{\sigma}\sqrt{\frac{2\rho^3}{\pi x}}\exp\left[-\frac{\rho x}{2\sigma^2}\right] + \frac{\rho^2}{\sigma^2}\,\mathrm{Erf}\sqrt{\frac{\rho x}{2\sigma^2}} - \frac{\rho^2}{\sigma^2}, \tag{2.27}$$

where $\mathrm{Erf}(x)$ is defined by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy.$$

Finally, we note that

$$m(x) = \frac{d}{dx} M(x) = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{\rho}{x}\right)^{\frac{3}{2}} \exp\left(-\frac{\rho x}{2\sigma^2}\right), \qquad (2.28)$$

so that $\alpha(s)$ can be expressed in the form

$$\alpha(s) = \int_0^\infty \left(1 - e^{-sx}\right) m(x)\, dx,$$

where $m(x)$ is given by formula (2.28).

We now calculate the mean and variance of $\xi(t)$ per unit time. We have

$$\frac{1}{t} E\left[\xi(t)\right] = \alpha'(0) = \frac{\rho}{\sigma^2} \cdot \frac{1}{2}\left(2\rho\sigma^2 s + \rho^2\right)^{-\frac{1}{2}} \cdot 2\rho\sigma^2 \Big|_{s=0} = \rho,$$

$$\frac{1}{t} Var\left[\xi(t)\right] = -\alpha''(0) = \frac{\rho}{\sigma^2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right)\left(2\rho\sigma^2 s + \rho^2\right)^{-\frac{3}{2}} \cdot \left(2\rho\sigma^2\right)^2 \Big|_{s=0} = \sigma^2,$$

as stated at the beginning of this section.

12. <u>The non-stationary additive process.</u>

If we drop assumption 3 from the assumptions of Section 1, we obtain a <u>non-stationary</u>, non-negative additive process. Put

$$\Theta(u, t; s) = E\left[e^{-s\{\xi(t) - \xi(u)\}}\right].$$

We then conclude from assumption 2 of Section 1 that, if

$u < w < t$,

$$\Theta(u, t; s) = \Theta(u, w; s)\ \Theta(w, t; s). \qquad (2.30)$$

The formula corresponding to (2.3) is

$$\log \Theta(u, t; s) = -\int_0^\infty \left(1 - e^{-sx}\right)\left[d_x M(t, x) - d_x M(u, x)\right], \qquad (2.31)$$

(see Doob [17] p. 418), where $M(t, x)$ is non-decreasing in both variables t and x, and $M(t,\infty)=0$, $M(0,x)=0$, for $x>0$. In the case of a stationary process, we have $M(t,x)=t\,M(1,x)$. One particular case is important in applications, namely that where $M(t,x)=\Lambda(t)\,M(x)$, $\Lambda(t)$ being a function of t which has a continuous positive derivative $\lambda(t)$, and $M(x)$ being a non-decreasing function of x such that $M(\infty)=0$.

In this particular case, the process can be made stationary by changing the way of measuring the time. Let us put

$$\tau = \Lambda(t).$$

Then equation (2.31) reduces to

$$\log \Theta(u,t;s) = -(\tau_2-\tau_1)\,\alpha(s),$$

where $\tau_2=\Lambda(t)$, $\tau_1=\Lambda(u)$, and

$$\alpha(s)=\int_0^\infty (1-e^{-sx})\,d\,M(x),$$

as before, The process is now stationary with respect to the new time scale.

One example is the non-stationary Poisson process where the probability of n arrivals in (u,t) is

$$e^{-[\Lambda(t)-\Lambda(u)]}\,\frac{[\Lambda(t)-\Lambda(u)]^n}{n!}. \tag{2.32}$$

Another is the non-stationary Compound Poisson process for which the condition $M(t,x)=\Lambda(t)\,M(x)$ is satisfied and $M(0)$ is finite and can therefore be assumed without loss of generality to be equal to one. The transformation $\tau = \Lambda(t)$ will reduce the process to a stationary one. In that case, therefore, the Compound Poisson process can be analysed into a non-stationary process of arrivals, such that the probability of n arrivals in (u, t) is given by formula (2.32), and upward jumps of magnitude X at each point of arrival, where the X 's are independent, <u>identically distributed</u> random variables, with distribution function given by

$$P\{X \leq x\} = 1 + M(x).$$

We obviously have in this case

$$\Theta(u, t; s) = \sum_{n=0}^{\infty} e^{-[\Lambda(t) - \Lambda(u)]} \frac{[\Lambda(t) - \Lambda(u)]^n}{n!} [\psi(s)]^n,$$

$$= \exp\left[-\{\Lambda(t) - \Lambda(u)\}\{1 - \psi(s)\}\right].$$

CHAPTER 3

THE TIME OF FIRST EMPTINESS AND ITS DISTRIBUTION

1. Definition, measurability and elementary properties.

In this and subsequent chapters, we shall be concerned with storage models where the input $\xi(t)$ is a non-negative, additive stochastic process of the type investigated in Chapter 2, and the planned output function is given by the formula $\eta(t)=t$. Let us notice immediately that $\eta(t)$ can also be considered as a (degenerate) non-negative stationary additive stochastic process.

The net planned input function, $\nu(t) = \xi(t) - t$, will also be in this case an additive stochastic process, which is stationary if $\xi(t)$ is stationary.

Let us note here that the time of first emptiness of the store is a variable which is not specifically related to the storage model under consideration. It does not depend on the assumptions made about the behaviour of the store after it has become empty. The time of first emptiness is simply the first passage time of the process $\nu(t)$ at the value $-z$, where z is the initial content of the store. We shall denote the first passage time by $\tau(z)$. More precisely, let Ω be the sample space of all random functions $\{\nu(t,\omega);\ \omega \in \Omega\}$. To every random function $\nu(t,\omega)$, and every non-negative number z, we make to correspond a number $\tau(z,\omega)$ as follows:

(a) if $\nu(t,\omega) > -z$ for all t, we put $\tau(z,\omega) = +\infty$.

(b) otherwise we put

$$\tau(z,\omega) = \inf\{t\,;\,\nu(t,\omega) \leq -z\}.$$

Let us note that the fact that $\nu(t,\omega)$ has no downward discontinuities ensures the validity of the equation $\nu[\tau(z,\omega);\omega] = -z$. This can be shown as follows:

There must exist by definition a sequence $\{t_n\}$ tending from above to τ, and such that $\nu(t_n,\omega) \leq -z$. It then follows that we must have $\nu(\tau,\omega) \leq -z$. On the other hand, if we had $\nu(\tau,\omega) < -z$, there would be a value of $t < \tau$ such that $\nu(t,\omega) < -z$, contradicting the fact that τ is the infimum of the t's satisfying $\nu(t,\omega) \leq -z$.

The above rules define $\tau(z,\omega)$ as a function on the sample space Ω. We shall now prove that it is a measureable function of ω. To achieve this, we must prove that the set $\{\omega\,;\,\tau(z,\omega) \leq t\}$ is measurable for every t.

Let us note to start with, that $\tau(z,\omega)$ can be redefined as follows: $\tau(z,\omega)$ is the smallest value of t for which the equation $\nu^*(t,\omega) = z$ holds, where $\nu^*(t,\omega)$ is defined as in Chapter 1 by the formula

$$\nu^*(t,\omega) = -\inf_{0 \leq u \leq t} \nu(u,\omega),$$

the only difference being that now ν^* is also a function of the sample space point ω. We also recall that ν^* is a non-decreasing function of t.

We note that, if t_0 is the smallest value of t

satisfying $\nu^*(t,\omega) = z$, then $\inf_{0\le u\le t} \nu(u,\omega) = -z$. If the value $-z$ were reached by $\nu(t,\omega)$ at a smaller value of t , say t_0' , then we would also have $\inf_{0\le u\le t_0'} \nu(u,\omega) = -z$, which contradicts the hypothesis that t_0 is the smallest value of t satisfying $\nu^*(t,\omega) = z$. Thus, we must have $\nu(u,\omega) > -z$ for all $u < t_0$. Moreover, because of the fact that $\nu(t,\omega)$ has no downward jumps, we have $\nu(t_0,\omega) = -z$. Thus we must have

$$t_0 = \inf\{t\,;\ \nu(t,\omega) \le -z\},$$

as required.

We can now show that the set $\{\omega\,;\ \tau(z,\omega) \le t\}$ is measurable. We must first note that as $\nu(t,\omega)$ is a function of bounded variation in t , it is separable, i.e. its value at any point of $[0, t]$ is a limit point of the set of its values on any denumerable set R that is dense in $[0, t]$.

Now the event $\{\omega\,;\ \tau(z,\omega) \le t\}$ is given by

$$\begin{aligned}\{\omega\,;\ \tau(z,\omega) \le t\} &= \{\omega\,;\ \nu^*(t) \ge z\} \\ &= \{\omega\,;\ \inf_{0\le u\le t} \nu(u) \le -z\} \\ &= \bigcup_{0\le u\le t} \{\omega\,;\ \nu(u) \le -z\} \\ &= \bigcup_{u\in R} \{\omega\,;\ \nu(u) \le -z\}.\end{aligned}$$

But, as each event $\{\omega ; \nu(\omega) \leq -z\}$ is measurable and the set R is denumerable, it follows that the set $\{\omega ; \tau(z,\omega) \leq t\}$ is measurable, and so $\tau(z,\omega)$ is a measurable function of ω.

We now recall that a random variable is any measurable function on the sample space Ω, whose range is the real line $(-\infty, +\infty)$ (c.f. Loève [47], p. 150). If the probability that the function is smaller than $+\infty$ is less than one, then the random variable is said to be improper, or defective (see Feller [21] p. 283).

In our case, we have proved that $\tau(z,\omega)$ is a measurable function on the sample space Ω. It is therefore a random variable in the sense indicated above. However, it is easy to see that this random variable may well be defective. For instance, if $\xi(t,\omega) = 2t$ a.s., then obviously, $P\{\tau(z,\omega) < +\infty\} = 0$, so that we have in fact $P\{\tau(z,\omega) = +\infty\} = 1$. We shall later show that a necessary and sufficient condition for $\tau(z,\omega)$ to be defective is $E[\xi(t,\omega)] > t$.

Finally, we note that the first passage time at zero, starting from z, is the same as the first passage time at y, starting from $z + y$.

2. <u>The L.S. transform of $\tau(z)$ in the case of a stationary input process.</u>

We shall now consider the case when the input process $\xi(t,\omega)$ is stationary. It immediately follows that

$\nu(t,\omega) = \xi(t,\omega) - t$, is also stationary. We shall write
$$E\left\{ e^{-s\nu(t,\omega)} \right\} = e^{\gamma(s)t},$$

$$P\{\tau(z) \le t\} = G(t,z).$$

We shall prove the following theorem:

Theorem 3.1: Under the above assumptions, the Laplace-Stieltjes transform

$$\Gamma(p,z) = E\left\{ e^{-p\tau(z,\omega)} \right\} = \int_{-\infty}^{+\infty} e^{-pt} d_t G(t,z),$$

is given by

$$\Gamma(p,z) = e^{-\theta(p)z},$$

where θ satisfies the equation

$$p = \gamma(\theta).$$

This equation will be called the characteristic equation of the process $\xi(t,\omega)$.

Proof: We use the remark, made at the end of the preceding chapter, that the first passage time at zero, starting from z , is the same as the first passage time at y , starting from $z+y$. We shall henceforth ignore ω -dependence, and write symbolically,

$$\tau(y+z) \doteq \tau(y) + \tau(z),$$

i.e., we can write $\tau(y+z)$ as the sum of two independent random variables, having the same distribution as $\tau(y,\omega)$, $\tau(z,\omega)$ respectively. It follows that $\tau(z)$ is infinitely divisible with respect to the parameter z and the Laplace-Stieltjes transform of its distribution, $\Gamma(p,z)$, is of the form $e^{-\theta(p)z}$. We now show that $\tau(z)$ also satisfies the equation

$$\tau(z) = z + \tau[\xi(z)]. \tag{3.1}$$

In fact, if the initial content of the store is z, then, after a period of time of length z, the initial content has been exhausted, and the new content is the input in the period $(0,z]$. Equation (3.1) is to be interpreted as follows: the distribution of the random variable, $\tau(z)$, given $\xi(z)$, is the same as that of $\tau[\xi(z)] + z$.

Using the theorem of total probability, we find

$$\begin{aligned} E\left[e^{-p\tau(z)}\right] &= E\left\{E\left[e^{-pz-p\tau[\xi(z)]} \mid \xi(z)\right]\right\}, \\ &= E\left\{e^{-pz-\theta\xi(z)}\right\} \\ &= e^{-(p+\theta)z} E\left\{e^{-\theta[\xi(z)-z]}\right\} \\ &= e^{-(p+\theta)z + z\gamma(\theta)}. \end{aligned}$$

But as

$$E\left[e^{-p\tau(z)}\right] = e^{-\theta z},$$

we must have

$$-\theta z = -(p+\theta)z + z\gamma(\theta),$$

i.e.

$$p = \gamma(\theta), \tag{3.2}$$

as required. This completes the proof of the theorem.

3. Some properties of $\theta(p)$ and the corresponding properties of $\tau(z)$.

We shall restrict our attention to real positive values of p in this section. We first notice that because of the relation

$$\Gamma(p,z) = \int_{-\infty}^{+\infty} e^{-pt}\, d_t G(t,z) = e^{-\theta(p)z},$$

there must correspond to every $p \geq 0$ a real value of θ. Moreover, because of the obvious relation

$$0 \leq e^{-\theta(p)z} \leq 1,$$

we must also have $\theta(p) \geq 0$.

Just as $\Gamma(p,z)$ can be continued analytically to the half-plane $\mathrm{Re}(p) > 0$, so can $\theta(p)$, so that $\theta(p)$ must be analytic in $\mathrm{Re}(p) > 0$.

Finally, we note that we must have

$$\lim_{p\to\infty} \Gamma(p,z) = \lim_{p\to\infty} \int_{0-}^{\infty} e^{-pt}\, d_t G(t,z) = P\{\tau(z)=0\} = 0$$

for all $z > 0$, as $\tau(z)$ cannot be smaller than z.

Thus we must have $\lim_{p\to\infty} e^{-\theta(p)z} = 0$ for all $z > 0$, and this implies that

$$\lim_{p\to\infty} \theta(p) = +\infty. \tag{3.3}$$

On the other hand, in order to determine whether $\tau(z)$ is a defective random variable or not, we consider the limit of

$\Gamma(p,z)$ when p tends to zero, for we have

$$\lim_{p\to 0}\Gamma(p,z)=\lim_{p\to 0}\int_0^\infty e^{-pt}\,dG(t,z)=\lim_{t\to\infty}G(t,z)=P\{\tau(z)<+\infty\}.$$

Thus, a necessary and sufficient condition for $\tau(z)$ to be defective is

$$\lim_{p\to 0}\Gamma(p,z)=\lim_{t\to\infty}G(t,z)<1 \tag{3.4}$$

Using the representation $\Gamma(p,z)=e^{-\theta(p)z}$, we see that condition (3.4) is equivalent to

$$\theta^*=\lim_{p\to 0}\theta(p)>0\ .$$

Now, because of the continuity of $\theta(p)$, θ^* must satisfy the equation

$$\gamma(\theta^*)=0\ . \tag{3.5}$$

We shall write

$$E\left[e^{-s\,\xi(t)}\right]=e^{-\alpha(s)t},$$

and we shall assume that $\alpha(s)$ can be expressed in the form

$$\alpha(s)=\int_0^{+\infty}(1-e^{-sx})\,dM(x),$$

where $M(x)$ is a non-decreasing function such that $\lim_{x\to\infty}M(x)=0$.
We shall further assume that $\lim_{x\to 0}x\,M(x)=0$.
The possibility and implications of such a representation have been discussed at length in Chapter 2.

Integrating by parts, we find that

$$\alpha(s)=s\int_0^\infty e^{-sx}M(x)\,dx=s\,\beta(s),\ \text{say.}$$

We shall also assume that $\alpha'(0)$ is finite, and consequently, as $\alpha'(0) = \lim_{s\to 0} \frac{\alpha(s)}{s} = \lim_{s\to 0} \int_0^\infty e^{-sx} M(x)\,dx$, the last limit will exist.

Finally, we note that in this case, the function which we had previously denoted by $\gamma(s)$ is now equal to $s - \alpha(s)$, so that equation (3.2) becomes

$$p = \theta - \alpha(\theta) . \tag{3.6}$$

Equation (3.5) can then be rewritten

$$\theta^* = \alpha(\theta^*) . \tag{3.7}$$

Thus θ^* is a fixed point of the continuous function $\alpha(\theta)$.

We now note that

$$\alpha'(s) = \int_0^\infty e^{-sx} x \, dM(x) > 0 ,$$

$$\alpha''(s) = -\int_0^\infty e^{-sx} x^2 \, dM(x) < 0 ,$$

so that $\alpha'(s)$ is a decreasing function.

Also

$$\lim_{s\to\infty} \alpha(s)/s = \lim_{s\to\infty} \int_0^\infty e^{-sx} M(x)\,dx = 0,$$

so that, for large s, we must have $\alpha(s) < s$.

Finally, we have $\alpha(0) = 0$.

We now consider two cases :

(a) $\alpha'(0) > 1$. Then $\alpha(s) > s$ in the neighbourhood of the origin, and as $\alpha(s) < s$ for large s, there must be at least one root of

(3.7) other than zero. Moreover, as $\alpha'(s)$ is monotonic decreasing, there can be at most one oxtromum of $\alpha(s)-s$ in $s>0$, so that there can be at most one non-zero root of $\alpha(s)-s=0$. Thus, if $\alpha'(0)>1$, there is exactly one real positive root θ^* of (3.7), and we must have

$$P\{\tau(z)<+\infty\}=e^{-\theta^* z},$$

$$P\{\tau(z)=+\infty\}=1-e^{-\theta^* z}.$$

(b) $\alpha'(0)\leq 1$. Then $\alpha'(s)\leq 1$ for all $s>1$, and as $\alpha(0)=0$, the only non-negative root of $\alpha(s)=s$ is $s=0$, so that $\theta(0)=0$, and we have

$$P\{\tau(z)<+\infty\}=1,$$

so that $\tau(z)$ is not a defective variable.

We thus have the following theorem:

<u>Theorem 3.2</u>: Under the conditions laid down at the beginning of this section, a necessary and sufficient condition for $\tau(z)$ to be a defective random variable is

$$\rho=\frac{1}{t}E[\xi(t)]=\alpha'(0)>1.$$

If this condition is realized, the equation $\alpha(\theta)=\theta$ has a unique positive solution θ^*, and we have

$$\underline{P\{\tau(z)=+\infty\}}=1-e^{-\theta^* z}.$$

Theorem 3.2 has a simple intuitive meaning. As ρ is the mean rate of input to the store, the meaning of the theorem is that the store will almost surely become empty in finite time if and only if the mean rate of input is smaller than or equal to the rate of output (i.e. unity).

We proceed to calculate the mean and variance of $\tau(z)$ when it is a proper random variable, i.e. when $\lim_{p \to 0} \theta(p) = \theta(0) = 0$. We then have

$$E[\tau(z)] = -\frac{\partial}{\partial p}\Gamma(p,z)\Big|_{p=0} = e^{-\theta(p)z}\,\theta'(p)\,z\Big|_{p=0} = \theta'(0)\,z .$$

Using now $-p + \theta - \alpha(\theta) = 0$, we find that $-1 + \theta'(p) - \alpha'[\theta(p)]\,\theta'(p) = 0$, so that

$$\theta'(p) = \frac{1}{1 - \alpha'[\theta(p)]},$$

and

$$\theta'(0) = \frac{1}{1 - \alpha'(0)} = \frac{1}{1-\rho} .$$

It follows that

$$E[\tau(z)] = \frac{z}{1-\rho} \qquad \text{if} \quad \rho < 1 .$$

If $\rho \geq 1$, we obviously must have

$$E[\tau(z)] = +\infty,$$

as $\tau(z)$ takes the value $+\infty$ with finite probability. For the variance, we find

$$\sigma^2[\tau(z)] = \frac{\partial^2}{\partial p^2}\Gamma(p,z)\Big|_{p=0} - [\theta'(0)]^2 = -\theta''(0)\,z .$$

But

$$\theta''(p) = \frac{\alpha''[\theta(p)]\ \theta'(p)}{\left\{1 - \alpha'[\theta(p)]\right\}^2} ,$$

so that

$$\theta''(0) = \frac{\alpha''(0)}{(1-\rho)^3}$$

But $-\alpha''(0) = \sigma^2$ is the variance per unit time of $\xi(t)$.

Thus we finally have

$$\sigma^2[\tau(z)] = \frac{\sigma^2 z}{(1-\rho)^3} .$$

Let us now consider the behaviour of $\tau(z)$ for large values of z. As $\tau(z)$ is a stochastic process with independent increments as a function of z, it follows from the strong law of large numbers (see Doob [17] p. 364) that, with probability one

$$\lim_{z\to\infty} \frac{\tau(z)}{z} = E\left[\frac{\tau(z)}{z}\right] .$$

When $\rho < 1$, we have

$$\lim_{z\to\infty} \frac{\tau(z)}{z} = \frac{1}{1-\rho}$$

and when $\rho \geq 1$, we have

$$\lim_{z\to\infty} \frac{\tau(z)}{z} = +\infty .$$

Applying the Central Limit theorem, we can also state that, if $\rho < 1$, the random variable

$$\frac{(1-\rho)^{\frac{1}{2}}[(1-\rho)\tau(z) - z]}{\sigma\sqrt{z}}$$

is asymptotically normally distributed as z tends to infinity.

In the case where $\rho = 1$, a more precise result can be obtained. Let us first note that the Laplace-Stieltjes transform of the distribution of $\tau(z)/z^2$ is given by

$$E\left[\exp\left\{-p\,\frac{\tau(z)}{z^2}\right\}\right] = \exp\left\{-\theta\left(\frac{p}{z^2}\right)z\right\}.$$

Let us calculate $\lim\limits_{z\to\infty} \theta(p/z^2)\,z$. We have

$$-\frac{p}{z^2} + \theta\left(\frac{p}{z^2}\right) - \alpha\left[\theta\left(\frac{p}{z^2}\right)\right] = 0 .$$

But
$$\alpha(s) = \rho s - \frac{\sigma^2 s^2}{2} + o(s^2) = s - \frac{\sigma^2 s^2}{2} + o(s^2).$$

It follows that

$$-\frac{p}{z^2} + \theta\left(\frac{p}{z^2}\right) - \theta\left(\frac{p}{z^2}\right) + \frac{\sigma^2}{2}\left[\theta\left(\frac{p}{z^2}\right)\right]^2 + o\left[\left\{\theta\left(\frac{p}{z^2}\right)\right\}^2\right] = 0$$

This can be written

$$\left[\theta\left(\frac{p}{z^2}\right)z\right]^2 = -\frac{2p}{\sigma^2} + o\left[\left\{\theta\left(\frac{p}{z^2}\right)\right\}^2\right].$$

Finally, letting z tend to infinity, we find

$$\lim_{z\to\infty} \theta\left(\frac{p}{z^2}\right)z = -\frac{\sqrt{2p}}{\sigma}.$$

It follows that

$$\lim_{z\to\infty} E\left[\exp\left\{-p\,\frac{\tau(z)}{z}\right\}\right] = e^{-\frac{\sqrt{2p}}{\sigma}}.$$

But the right-hand side of this expression is the Laplace transform of

$$\frac{1}{\sigma\sqrt{2\pi t^3}}\, e^{-\frac{1}{2\sigma^2 t}}. \tag{3.8}$$

We can thus conclude that the limiting distribution of $\tau(z)/z^2$ as z tends to infinity is given by (3.8), provided that a continuity theorem similar to that of Lévy (see Lukács [49] p. 54) for characteristic functions can be shown to hold for Laplace transforms. This we shall presently establish.

<u>Lemma</u>: Let $\{F_n(x)\}$ be the distribution functions of a sequence $\{\xi_n\}$ of non-negative random variables, $f_n(s)$ the corresponding Laplace-Stieltjes transforms. Then a necessary and sufficient condition for $\{F_n(x)\}$ to converge (weakly) to a limit distribution function $F(s)$ having $f(s)$ as its Laplace-Stieltjes transform, is that $f_n(s)$ converge to $f(s)$ for a set of points $\{s_i\}$ on the positive real axis such that $\sum_{i=0}^{\infty}(1/s_i)$ diverges

<u>Proof</u>: The necessity of the condition follows immediately from Helly's extended second theorem (see Lukács [49] p. 52). To prove its sufficiency, we first note that $\{F_n(x)\}$ contains a (weakly) convergent subsequence $\{F_{n_k}(x)\}$ by Helly's first theorem (ibid. p. 49). This subsequēnce will converge to some non-decreasing bounded function $F'(x)$. But then the Laplace-Stieltjes transform of $F'(x)$, $f'(s)$, coincides with $f(s)$ on the set $\{s_i\}$, and it follows from the uniqueness theorem for Laplace transforms given in Doetsch [14] p. 76, that $f(s) = f'(s)$ for all $s > 0$, and that $F(x) = F'(x)$. The argument just used applies however to every convergent subsequence of $\{F_n(x)\}$ and thus $\{F_n(x)\}$ must converge (weakly) to $F(x)$.

The preceding result can also be written as follows, by making a suitable change of variable:

When $\rho = 1$, the distribution of $\sigma^2 \tau(z)/z^2$ is given asymptotically for large z by the formula

$$P\left\{ \frac{\sigma^2 \tau(z)}{z^2} \leq t \right\} = \int_0^t \frac{e^{-\frac{1}{2u}}}{\sqrt{2\pi u^3}}\, du .$$

4. <u>The uniqueness of the solution of the equation $p = \gamma(s)$.</u>

Theorem 3.1 can be only of little use unless it can be shown that the equation $p = \gamma(s)$ has a unique solution $s = \theta(p)$ which is such that $\exp\{-\theta(p) z\}$ is the Laplace-Stieltjes transform of a distribution function.

We shall first prove a general theorem concerning the uniqueness of the solution of (3.2) which satisfies the conditions laid down in section 3, and we shall then proceed to prove stronger theorems which hold when the distribution function of $\xi(t)$ satisfies certain specified conditions.

<u>Theorem 3.3:</u> There exist two real positive numbers p_0 , σ_0 , such that equation (3.6) has exactly one root , θ, satisfying $Re(\theta) > \sigma_0$, for all real values of p satisfying $p > p_0$. Moreover, if $f(s)$ is a function of s which is analytic in $Re(s) > \sigma_0$, $f(\theta)$ is given by

$$f(\theta) = f(p) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dp^{n-1}} \left[f'(p) \{ \alpha(p) \}^n \right].$$

<u>Proof:</u> Let $s = \sigma + i\omega$. Then $\lim_{\sigma \to \infty} \beta(\sigma + i\omega) = 0$. Moreover, $|\beta(\sigma + i\omega)| \leq \beta(\sigma_0)$ for all $\sigma \geq \sigma_0$, and all ω . It follows that we can choose σ_0 such that

$\beta(s) < \mu < \frac{1}{2}$ for all s such that $\mathrm{Re}(s) > \sigma_0$. We then have, in the same region, $|\alpha(s)| < \mu |s|$. We now show that, if p is real and $|s-p| > \mu p/(1-\mu)$, we have $|s-p| > |\alpha(s)|$ for all s such that $\mathrm{Re}(s) > \sigma_0$.

In fact, we then have

$$|\alpha(s)| \leq \mu|s| = \mu|s-p+p| \leq \mu|s-p| + \mu p < |s-p|.$$

Finally, we note that if p satisfies the inequality $p > (1-\mu)\sigma_0/(1-2\mu)$, all points such that $|s-p| \leq \mu p/(1-\mu)$ will have an abscissa larger than σ_0. In fact, the point of the circle $|s-p| \leq \mu p/(1-\mu)$ with smallest abscissa will have an abscissa of $p - \mu p/(1-\mu) = (1-2\mu)p/(1-\mu)$, and this will be larger than σ_0 provided $p > (1-\mu)\sigma_0/(1-2\mu)$, as stated above.

Every point in $\mathrm{Re}(s) > \sigma_0$ can then be surrounded by a contour C in the same region containing the circle $|s-p| = \mu p/(1-\mu)$. On this contour, we shall have $|s-p| > |\alpha(s)|$, and by applying Rouché's theorem (see Stewart [69] p. 440), we conclude that the equation $s - p = \alpha(s)$ has only one root in $\mathrm{Re}(s) > \sigma_0$. Moreover, any function which is analytic in a region containing the contour C can be expanded by using Lagrange's theorem, (see Stewart [69] p. 440), yielding the required expansion.

We shall now assume that the input is of the Compound Poisson type, i.e. that we can write $\alpha(s)$ in the form

DIAGRAM

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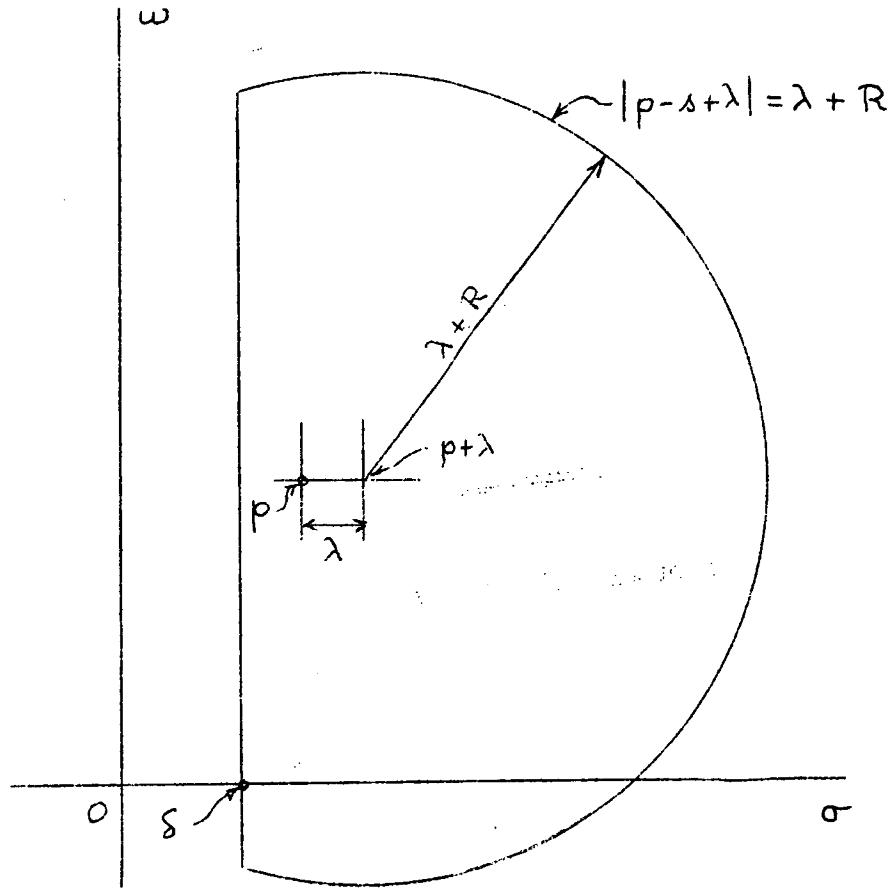

Figure 3.1

To face p. 69

$$\alpha(s) = \lambda\left[1 - \psi(s)\right],$$

(see Chapter 2, Section 5), where

$$\psi(s) = \int_0^\infty e^{-sx}\, dB(x),$$

$B(x)$ being a distribution function. It follows that for all s such that $\mathrm{Re}(s) \geq 0$,

$$|\psi(s)| \leq 1 .$$

The following theorem is due to Beneš [3] .

<u>Theorem 3.4</u>: If $\alpha(s)$ is of the above form, then the equation

$$p - s + \lambda\left[1 - \psi(s)\right] = 0 \tag{3.9}$$

has only one root θ in the region $\mathrm{Re}(s) > 0$ for all p such that $\mathrm{Re}(p) > 0$. Moreover, if $f(s)$ is analytic in $\mathrm{Re}(s) > 0$, then

$$f(\theta) = f(p+\lambda) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \frac{d^{n-1}}{dp^{n-1}}\left[f'(p+\lambda)\left\{\psi(p+\lambda)\right\}^n\right]. \tag{3.10}$$

<u>Proof</u>: Choose δ such that $0 < \delta < \mathrm{Re}(p)$ and R such that $R > \mathrm{Re}(p)$. Let the contour C be defined as follows:

For $\mathrm{Re}(s) > \delta$, C coincides with the circle $|p - s + \lambda| = \lambda + R$. For $\mathrm{Re}(s) = \delta$, C coincides with the vertical line $\mathrm{Re}(s) = \delta$. The contour C is shown in Figure 3.1.

Now on the circle we have

$$|p-s+\lambda| = \lambda + R > \lambda \geq \lambda|\psi(s)|.$$

On the line,

$$|p-s+\lambda| \geq Re(p) - \delta + \lambda > \lambda \geq \lambda|\psi(s)|.$$

It follows that on the contour C, $|p-s+\lambda| > \lambda|\psi(s)|$. Applying Rouché's theorem, we conclude that equation (3.5) has only one root to the right of $Re(s) > \delta$. Finally, as δ can be arbitrarily small, (3.9) must have only one root in $Re(s) > 0$. Equation (3.10) now follows immediately on using the Lagrange expansion.

In chapter 7, we shall need a stronger result than theorem 3.4, which will extend the region of uniqueness to the left of the origin. The required result is given in the following theorem.

Theorem 3.5: Let $\gamma(s) = s - \lambda[1-\psi(s)]$, and let σ_0 be a real number such that $\gamma(\sigma_0)$ is finite. Then the equation $p = \gamma(s)$ has only one root $\theta(p)$ in $Re(s) > \sigma_0$ for any p such that $Re(p) > \gamma(\sigma_0)$.

Moreover, let $\gamma(s)$ be analytic at the origin, and let $\gamma'(0)$ satisfy the condition $\gamma'(0) > 0$. Then there exists $\alpha < 0$ such that $\gamma(\alpha) < 0$.

Proof: If $Re(s) > \sigma_0$, we have

$$|\psi(s)| = \left|\int_0^\infty e^{-sx}\, dB(x)\right| \leq \int_0^\infty e^{-\sigma_0 x}\, dB(x) = \psi(\sigma_0).$$

Choose p such that $Re(p) > \gamma(\sigma_0) = \sigma_0 - \lambda[1 - \psi(\sigma_0)]$, and choose δ such that

$$\sigma_0 < \delta < Re(p) + \lambda - \lambda\,\psi(\sigma_0).$$

For $Re(s) > \delta$, take as the contour C the circle with centre $p+\lambda$ and radius $\lambda\psi(\sigma_0) + R$, given by the equation

$$|p - s + \lambda| = \lambda\,\psi(\sigma_0) + R.$$

For $Re(s) = \delta$, take the contour C to be the line $Re(s) = \delta$. On the circle, we have the inequality

$$|p - s + \lambda| = \lambda\,\psi(\sigma_0) + R > \lambda\,\psi(\sigma_0) \geq \lambda\,|\psi(s)|.$$

On the line, we have

$$|p - s + \lambda| \geq Re(p) - \delta + \lambda > \lambda\,\psi(\sigma_0) \geq \lambda\,|\psi(s)|.$$

It follows that the inequality $|p - s + \lambda| > \lambda\,|\psi(s)|$ is satisfied on the whole contour C, and by Rouché's theorem it follows that the equation $p - s + \lambda - \lambda\,\psi(s) = 0$ has one root inside the contour C, and as R is arbitrary, there is only one root to the right of the line $Re(s) = \delta$. As δ can be as near to σ_0 as we like, we finally conclude that the equation has only one root in $Re(s) > \sigma_0$.

Now, if $\gamma(s)$ is analytic at the origin, then so is $\psi(s)$. It follows that there exists $\alpha < 0$ such that $\psi(\alpha)$ is finite. Also the condition $\gamma'(0) > 0$ is equivalent to $-\lambda\,\psi'(0) < 1$. We now show that there exists $\alpha < 0$ such that $\alpha < \lambda[1 - \psi(\alpha)]$. In fact

$$\lambda[\psi(\alpha)-1] = \lambda\int_0^\infty (e^{-\alpha x}-1)\,dB(x) \le \lambda\int_0^\infty (-\alpha)\,x\,e^{-\alpha x}\,dB(x) = \lambda(-\alpha)[-\psi'(\alpha)] \quad \ldots (3.11)$$

Now, as $\psi(s)$, and therefore $\psi'(s)$, are analytic in the neighbourhood of the origin, and $-\lambda\psi'(0) < 1$, we can choose $|\alpha|$ small enough to have $-\lambda\psi'(\alpha) < 1$, and it follows from (3.11) that

$$\lambda[\psi(\alpha)-1] < -\alpha ,$$

i.e. $\gamma(\alpha) = \alpha - \lambda[1-\psi(\alpha)] < 0$.

Corollary 1: Under the conditions of theorem 3.5, there exist two negative numbers α and β such that the equation $p = \gamma(s)$ has one root in $Re(s) > \alpha$ for all p such that $Re(p) > \beta$.

Proof: It is sufficient to take α as in theorem 3.5, and $\beta = \gamma(\alpha)$.

Corollary 2: Under the conditions of theorem 3.4, the Lagrange expansion (3.10) will hold for all p such that $Re(p) > \beta$, and all functions $f(s)$ analytic in the region $Re(s) > \beta$, where β is as in Corollary 1.

We shall now obtain certain properties of $\theta(p)$ and $\psi[\theta(p)]$ which will prove useful in the sequel.

Theorem 3.6: If $\gamma(\sigma_c)$ is finite, then, in the region $Re(p) > \gamma(\sigma_0)$, the following properties hold :

(a) $|\psi(\theta)| \leq \psi(\sigma_0)$,

(b) $|\theta - p| \leq \lambda[1 + \psi(\sigma_0)]$, *i.e.* $\theta \sim p$ when $|p| \to \infty$,

(c) $Re(\theta) \geq Re(p) + \lambda - \lambda\,\psi(\sigma_0)$.

If, in addition, $Re(\sigma_0) < 0$ and $\gamma'(\sigma_0)$ is finite, then

(d) $[1 - \psi(\theta)]/\theta$ is uniformly bounded.

(e) $|1 + \lambda\,\psi'(\theta)| \geq 1 - |-\lambda\,\psi'(\sigma_0)|$.

<u>Proof:</u> We have, as $Re(\theta) \geq \sigma_0$,

$$|\psi(\theta)| = \left|\int_0^\infty e^{-\theta x}\, dB(x)\right| \leq \int_0^\infty e^{-\sigma_0 x}\, dB(x) = \psi(\sigma_0),$$

and, if $\psi'(\sigma_0)$ is finite,

$$|\psi'(\theta)| = \left|\int_0^\infty e^{-\theta x}\, x\, dB(x)\right| \leq \int_0^\infty e^{-\sigma_0 x}\, x\, dB(x) = \psi'(\sigma_0). \quad (3.12)$$

Also θ satisfies the equation

$$-p + \theta - \lambda[1 - \psi(\theta)] = 0. \quad (3.13)$$

It follows that $|\theta - p| = \lambda[1 - \psi(\theta)]$, and

$$|\theta - p| = \lambda|1 - \psi(\theta)| \leq \lambda[1 + |\psi(\theta)|] \leq \lambda[1 + \psi(\sigma_0)].$$

Also, it follows from $-p + \theta - \lambda + \lambda\,\psi(\theta) = 0$ that

$$-Re(p) + Re(\theta) - \lambda + \lambda\, Re[\psi(\theta)] = 0.$$

Therefore

$$\mathrm{Re}(\theta) = \mathrm{Re}(p) + \lambda - \lambda\,\mathrm{Re}[\psi(\theta)],$$

$$\geq \mathrm{Re}(p) + \lambda - \lambda\,|\psi(\theta)|$$

$$\geq \mathrm{Re}(p) + \lambda - \lambda\,\psi(\sigma_o).$$

Finally, it follows from (3.13) that if $\mathrm{Re}(\sigma_o) < 0$, then if $\theta = 0$, $p = 0$, i.e. the value of θ corresponding to $p = 0$ is $\theta = 0$. But it follows from the implicit function theorem that $\theta(p)$ is an analytic function of p in $\mathrm{Re}(p) > \sigma_o$. We deduce that, as p tends to zero, θ tends to zero, and

$$\lim_{p \to 0} \frac{1 - \psi(\theta)}{\theta} = -\psi'(0).$$

Also, as $\theta \neq 0$ for $p \neq 0$ because of the uniqueness of the solution of (3.13), it follows that $[1 - \psi(\theta)]/\theta$ is finite in $\mathrm{Re}(p) > \gamma(\sigma_o)$. But as $1 - \psi(\theta)$ is uniformly bounded and $\theta \sim p$, it follows that $[1 - \psi(\theta)]/\theta$ is uniformly bounded.

Finally, it follows from (3.12) that

$$|1 + \lambda\,\psi'(\theta)| \geq 1 - |-\lambda\,\psi'(\theta)| \geq 1 - |-\lambda\,\psi'(\sigma_o)|.$$

Corollary: If $|-\lambda\psi'(\sigma_o)| < 1$, it follows that $|1 + \lambda\psi'(\theta)| \geq k > 0$. This will be the case if $\sigma_o = \alpha$, where α is as chosen in theorem 3.5.

5. Some examples.

(a) A simple type of input which leads to a quadratic characteristic equation is a Compound Poisson input where the density of the arrival process is λ and the distribution function of the jumps is $B(x) = 1 - e^{-\mu x}$. We then have (see Chapter 2, section 6)

$$\psi(s) = \int_0^\infty e^{-sx} \mu e^{-\mu x} dx = \frac{\mu}{\mu + s} .$$

The characteristic equation of this process is

$$-p + s - \lambda + \frac{\lambda \mu}{\mu + s} = 0 .$$

This reduces to

$$s^2 - s(\lambda - \mu + p) - \mu p = 0$$

The equation has two roots :

$$s = \frac{1}{2}(\lambda - \mu + p) \pm \frac{1}{2}\sqrt{p^2 + 2(\lambda + \mu)p + (\lambda - \mu)^2} .$$

For real positive p there is obviously only one positive root :

$$\theta = \frac{1}{2}(\lambda - \mu + p) + \frac{1}{2}\sqrt{p^2 + 2(\lambda + \mu)p + (\lambda - \mu)^2} .$$

$$= \frac{1}{2}(\lambda - \mu + p) + \frac{1}{2}\sqrt{(p + \lambda + \mu)^2 - 4\lambda\mu} , \qquad (3.14)$$

which can be continued analytically to the whole plane, the two points $-(\sqrt{\lambda} + \sqrt{\mu})^2$ and $-(\sqrt{\lambda} - \sqrt{\mu})^2$ being branch points.

When $p \to 0$, we note that if $\lambda < \mu$, $\theta \to 0$.

However, if $\lambda \geqslant \mu$, $\theta \rightarrow (\lambda-\mu)$.

(b) Let us now consider the Gamma process. In this case, the characteristic equation is

$$-p+s-\log(1+\rho s)=0 \tag{3.15}$$

The equation is transcendental, and θ cannot be expressed in closed form as a function of p .

Let us, however, note in passing that in order for $\Theta(t,s)$ to have an argument lying between $-\pi$ and $+\pi$, the imaginary part of $\log(1+\rho s)$ must be between $-\pi$ and $+\pi$, i.e. the logarithm must take its principal value. It follows that, if β is the imaginary part of p , then θ must be in the strip $\beta-\pi<\mathrm{Im}(\theta)<\beta+\pi$.

(c) In the case of the Inverse Gaussian input, the characteristic equation is

$$-p+s-\frac{\rho}{\sigma^2}\left(\sqrt{2\rho\sigma^2 s+\rho^2}-\rho\right)=0 \tag{3.16}$$

This can be rewritten, after squaring, as

$$s^2-2s\left(p-\frac{\rho^2}{\sigma^2}+\frac{\rho^3}{\sigma^2}\right)+\left(\frac{\rho^2}{\sigma^2}-p\right)^2-\frac{\rho^4}{\sigma^4}=0.$$

Here again, we obtain a quadratic equation, whose roots are

$$s=p-\frac{\rho^2}{\sigma^2}(1-\rho)\pm\sqrt{\frac{\rho^4}{\sigma^4}(1-\rho)^2+\frac{2\rho^3}{\sigma^2}p} .$$

Restricting our attention to real values of p , we notice that the two roots will have opposite signs if

$$\left(\frac{\rho^2}{\sigma^2}-p\right)^2-\frac{\rho^4}{\sigma^4}<0 ,$$

i.e. if $$p^2 - \frac{2\rho^2}{\sigma^2} p < 0 .$$

But this will be true if $0 < p < \frac{2\rho^2}{\sigma^2}$. However, as $\theta(p)$ must necessarily be positive for all positive values of p, it follows that $\theta(p)$ can be only the largest of the two roots, namely,

$$\theta(p) = p - \frac{\rho^2}{\sigma^2}(1-\rho) + \sqrt{\frac{\rho^4}{\sigma^4}(1-\rho)^2 + \frac{2\rho^3}{\sigma^2} p} . \qquad (3.17)$$

Moreover, in order to eliminate the spurious roots introduced by squaring, we must impose the condition

$$-p + s + \frac{\rho^2}{\sigma^2} \geqslant 0 ,$$

i.e. $$s \geqslant p - \frac{\rho^2}{\sigma^2} .$$

This condition is obviously satisfied for $\theta(p)$ as given by equation (3.14). For the other root, we must have

$$p - \frac{\rho^2}{\sigma^2}(1-\rho) - \sqrt{\frac{\rho^4}{\sigma^4}(1-\rho)^2 + \frac{2\rho^3}{\sigma^2} p} \geqslant p - \frac{\rho^2}{\sigma^2} ,$$

i.e. $$\sqrt{\frac{\rho^4}{\sigma^4}(1-\rho)^2 + \frac{2\rho^3}{\sigma^2} p} \leq \frac{\rho^3}{\sigma^2} ,$$

so that we must have

$$p \leq \frac{\rho(1-2\rho)}{2\sigma^2}$$

for the second root to satisfy the original equation (3.16). We thus see that, for real values of p larger than $\rho(1-2\rho)/2\sigma^2$, equation (3.16) has only one root, given by (3.17). Equation (3.17) can then be used to continue

$\theta(p)$ analytically to the whole plane, except for the branch point

$$p = -\frac{\rho(1-\rho)^2}{2\sigma^2}.$$

Letting $p \to 0$, we find, as expected, that, if $\rho \le 1$, then $\theta(p) \to 0$.

If, however, $\rho > 1$, we find that

$$\lim_{p\to 0} \theta(p) = \frac{2\rho^2}{\sigma^2}(\rho - 1) > 0.$$

6. <u>The inversion of $\Gamma(p, z)$ when the input has a density function.</u>

<u>Theorem 3.7:</u> If $\xi(t)$ has a density function $k(t, x)$, and satisfies the conditions laid down in section 3, and if $\int_z^t e^{-pu}\frac{z}{u}k(u, u-z)\,du$ is of bounded variation as a function of z in some neighbourhood of z, then $\tau(z)$ has a density function, $g(t, x)$, which is given for almost all t by the formula

$$g(t,x) = \begin{cases} \dfrac{z}{t}k(t, t-z) & \text{if} \quad t \ge z, \\ 0 & \text{otherwise.} \end{cases}$$

<u>Proof:</u> We have

$$\int_0^\infty e^{-sx}k(t, x)\,dx = e^{-\alpha(s)t} \quad \text{for} \quad \mathrm{Re}(s) \ge 0,$$

where $\mathrm{Re}[\alpha(s)] \ge 0$. We deduce that

$$\int_0^\infty\int_0^\infty e^{-pt-sx}k(t,x)\,dx\,dt = \frac{1}{p+\alpha(s)}, \; \mathrm{Re}(p) > 0 \qquad (3.18)$$

Let us for the moment restrict s and p to real positive values, and let us change variables in (3.18) by replacing x

by $t - z$. We find that

$$\frac{1}{p+\alpha(s)} = \int_0^\infty \int_{-\infty}^t e^{-(p+s)t+sz} k(t, t-z)\, dz\, dt.$$

Write now p for $p+s$. We obtain

$$\frac{1}{p-s+\alpha(s)} = \int_0^\infty e^{-pt} \left\{ \int_{-\infty}^t e^{sz} k(t, t-z)\, dz \right\} dt.$$

Differentiate both sides with respect to s. We have

$$\frac{1-\alpha'(s)}{[p-s+\alpha(s)]^2} = \int_0^\infty e^{-pt} \left\{ \int_{-\infty}^t e^{sz} z\, k(t, t-z)\, dz \right\} dt, \tag{3.19}$$

where the double integral converges absolutely.

Integrate both sides of (3.19) with respect to p from p to infinity. We obtain

$$-\frac{1-\alpha'(s)}{p-s+\alpha(s)} = \int_0^\infty e^{-pt} \left\{ \int_{-\infty}^t e^{sz} \frac{z}{t} k(t, t-z)\, dz \right\} dt. \tag{3.20}$$

Let us put

$$g^*(t,z) = \begin{cases} \frac{z}{t} k(t, t-z) & \text{for } z \le t, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write (3.20) as

$$-\frac{1-\alpha'(s)}{p-s+\alpha(s)} = \int_0^\infty e^{-pt} \int_{-\infty}^{+\infty} e^{sz} g^*(t,z)\, dz\, dt.$$

As the double integral converges absolutely, we can use Fubini's theorem to interchange the integrals, thus obtaining

$$-\frac{1-\alpha'(s)}{p-s+\alpha(s)} = \int_{-\infty}^{+\infty} e^{sz} \left\{ \int_0^\infty e^{-pt} g^*(t,z)\, dt \right\} dz.$$

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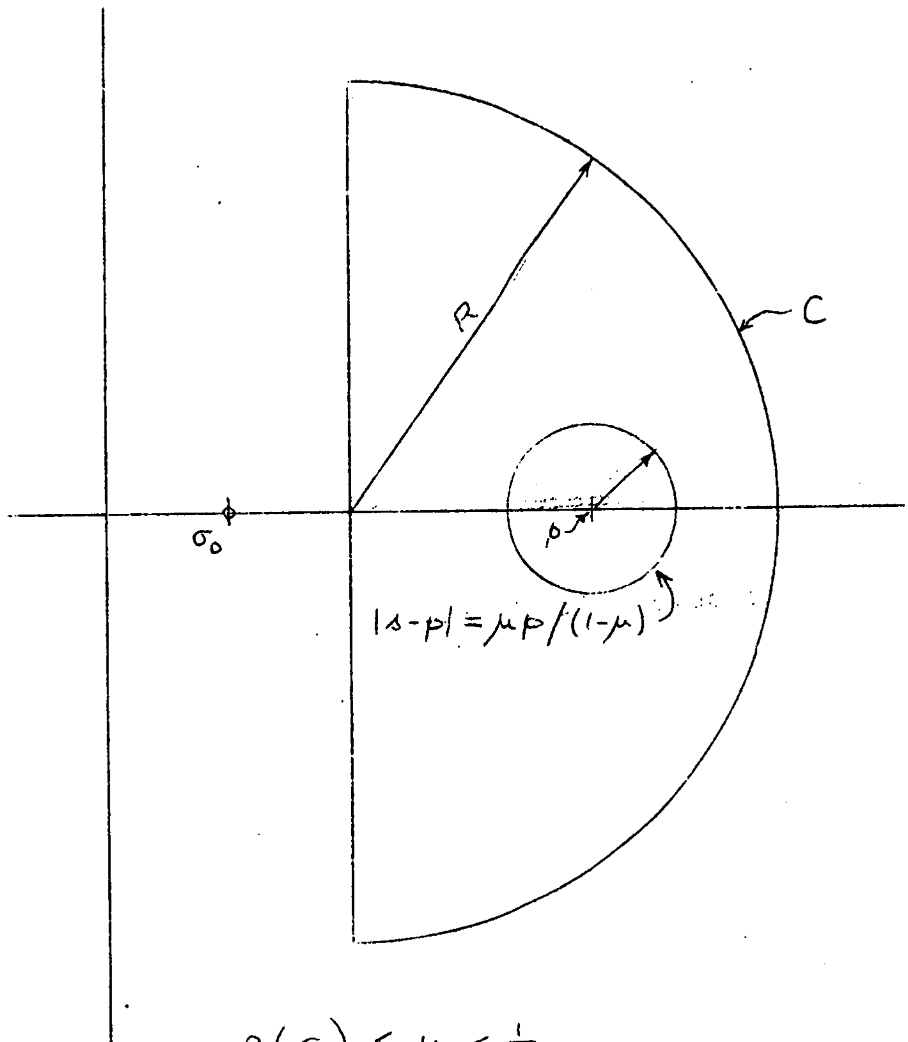

$$\beta(\sigma_c) \le \mu < \frac{1}{2}$$

$$p > \frac{(1-\mu)\sigma_o}{(1-2\mu)}$$

Figure 3.2

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As the integral converges for all positive values of p, and all values of s such that $Re(s) > 0$, the last equation holds for all s such that $Re(s) > 0$.

We shall now show that

$$\lim_{R\to\infty} -\frac{1}{2\pi i}\int_{c-iR}^{c+iR} \frac{[1-\alpha'(s)]}{p-s+\alpha(s)} e^{-sz} ds = e^{-\theta z} \quad , z > 0 \qquad (3.21)$$

if $\sigma_0 < c < \mu p/(1-\mu)$, where σ_0 and μ are as defined in theorem 3.3, $p > (1-\mu)\sigma_0/(1-2\mu)$, and θ is the unique root of $p - s + \alpha(s) = 0$ in $Re(s) > \sigma_0$. To prove this result we first show that

$$\frac{1}{2\pi i}\int_C \frac{[1-\alpha'(s)]}{p-s+\alpha(s)} e^{-sz} ds = e^{-\theta z} \quad , z > 0,$$

where C is a contour made up of the line $Re(s) = c$ and the right-hand half of the circle $|s-p| = R$ for large R. (see figure 3.2). It follows from theorem 3.3 that $p - s + \alpha(s)$ has only one zero inside C, namely $\theta(p)$, and the residue of the integrand is precisely $-e^{-\theta z}$. It remains to show that the contribution of the half circle to the integral tends to zero when $R \to \infty$. But this follows immediately from the fact that in $Re(s) > \sigma_0$, $|1-\alpha'(s)| \leq 1 + \alpha'(0)$, and is therefore bounded, and $|p-s+\alpha(s)| > |s| - p - \mu|s| > \frac{1}{2}|s| - p$, so that the integrand is $O(1/R)$ uniformly in $Re(s) > \sigma_0$.

Having established formula (3.21) we now use the standard inversion theorem for the bilateral Laplace Transform, (see for instance Widder [74] p. 241) to obtain the result that

$$\int_0^\infty e^{-pt} g^*(t,z)\,dt = e^{-\theta z} \quad , \quad z > 0 ,$$

for all sufficiently large positive p .

Finally, it follows from the uniqueness theorem for Laplace transforms (see for instance Doetsch [14] p. 74, Satz 4) that

$$g^*(t,z) = g(t,z) \qquad \text{for almost all } t \ .$$

This completes the proof of the theorem.

7. <u>The inversion of $\Gamma(p,z)$ in the case of a Compound Poisson input.</u>

In the case of a Compound Poisson input, $K(t,x)$ admits the expansion (see Chapter 2, section 5)

$$K(t,x) = U(x)\left[1 + t\,K_{10}(0,x) + \frac{t^2}{2!}K_{20}(0,x) + \dots\right], \qquad (3.22)$$

where the K_{m0} are given by

$$K_{m0}(0,x) = (-1)^m \lambda^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_k(x) .$$

It follows that

$$\left|K_{m0}(0,x)\right| \le \lambda^m \sum_{k=0}^{m} \binom{m}{k} = (2\lambda)^m ,$$

so that

$$\left|\sum_{m=0}^{N} \frac{t^m}{m!} K_{m0}(0,x)\right| \le \sum_{m=0}^{N} \frac{(2\lambda t)^m}{m!} \le e^{2\lambda t} \quad \text{for all } t \ge 0. \qquad (3.23)$$

Thus the partial sums of expansion (3.22) are uniformly dominated by $e^{2\lambda t}$.

<u>Theorem 3.8</u>: If $\xi(t)$ is a Compound Poisson input, the distribution function, $G(t,z)$, of $\tau(z)$ is given by the

formula

$$G(t,z)=\begin{cases}\frac{z}{t}K(t,t-z)-\int_z^t \frac{z}{u^2}\left[K_{10}(u,u-z)-K(u,u-z)\right]du & \text{if} \quad t \geq z \\ 0 & \text{otherwise.}\end{cases}$$

Proof: Taking the Laplace-Stieltjes transform of (3.22) term by term, and equating the coefficients of the powers of t, we find, as in section 3 of chapter 2, (formula 2.8), that

$$p\int_0^\infty e^{-px}K_{n0}(0,x)\,U(x)\,dx=(-1)^n[\alpha(p)]^n \quad , n=0,1,2,\dots$$

From this we deduce, using the usual rules for change of variable in Laplace transforms

$$e^{-pz}[\alpha(p)]^n=(-1)^n p\int_0^\infty e^{-pt}K_{n0}(0,t-z)\,U(t-z)\,dt,$$

and, denoting the Laplace-Stieltjes transform of $f(t)$ by $\mathcal{L}[f(t)]$, we can write

$$\frac{1}{n!}\frac{d^{n-1}}{dp^{n-1}}\left[e^{-pz}\{\alpha(p)\}^n\right]=-\frac{p}{n!}\int_0^\infty e^{-pt}t^{n-1}K_{n0}(0,t-z)\,U(t-z)\,dt$$

$$+\frac{n-1}{n!}\int_0^\infty e^{-pt}t^{n-2}K_{n0}(0,t-z)\,U(t-z)\,dt ,$$

$$=-\frac{p}{n!}\int_0^\infty e^{-pt}t^{n-1}K_{n0}(0,t-z)\,U(t-z)\,dt$$

$$+\frac{n-1}{n!}\,p\int_0^\infty e^{-pt}dt\int_0^t u^{n-2}K_{n0}(0,u-z)\,U(u-z)\,du ,$$

$$=\mathcal{L}\Big[\frac{t^{n-1}}{n!}K_{n0}(0,t-z)\,U(t-z)$$

$$+\frac{n-1}{n!}\int_0^t u^{n-2}K_{n0}(0,u-z)\,U(u-z)\,du\Big] \quad (3.24)$$

We now use the inequalities

$$\left|\sum_{n=0}^{N} \frac{t^{n-1}}{n!} K_{n0}(0,t-z)\right| \leq \frac{1}{t} e^{2\lambda t} U(t-z),$$

$$\left|\sum_{n=0}^{N} \frac{(n-1)}{n!}\int_0^t u^{n-2} K_{n0}(0,u-z)\, U(u-z)\, du\right| \leq \frac{1}{t} e^{2\lambda t} U(t-z),$$

which follow easily from (3.23). It follows that the sums involved in the inequalities are uniformly dominated by $\frac{1}{t} e^{2\lambda t} U(t-z)$, and this function in turn has a convergent Laplace-Stieltjes transform for all $p > 2\lambda$, $z > 0$.

Using now Lebesgue's dominated convergence theorem (see Loève [47] p. 125) we can sum equation (3.24) from $n = 1$ to $n = +\infty$, and we obtain

$$-z \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dp^{n-1}}\left[e^{-pz}\{\alpha(p)\}^{n}\right] = \mathcal{L}\Big[\frac{z}{t}\{K(t,t-z)-1\}U(t-z)$$
$$- \int_0^t \frac{z}{u} K_{10}(u,u-z)\, U(u-z)\, du$$
$$+ \int_0^t \frac{z}{u^2}\{K(u,u-z)-1\}\, U(u-z)\, du\Big].$$

Finally, we use the two identities

$$e^{-pz} = p\int_0^\infty e^{-pt}\, U(t-z)\, dt,$$

$$\int_0^t \frac{z}{u^2} U(u-z)\, du = \left(1 - \frac{z}{t}\right) U(t-z),$$

and we find, using the Lagrange expansion of $\Gamma(p,z)$ given in theorem 3.3,

$$\Gamma(p,z) = \mathcal{L}\Big[\frac{z}{t}K(t,t-z)U(t-z) - \int_0^t \frac{z}{u}K_{10}(u,u-z)U(u-z)\,du$$
$$+ \int_0^t \frac{z}{u^2}K(u,u-z)U(u-z)\,du\Big].$$

But as the Lagrange expansion holds for all $p > 0$, (see theorem 3.4), it follows from the uniqueness property of the Laplace-Stieltjes transform (see Widder [74] p. 63) that if $G(t,z)$ is the distribution function of $\tau(z)$, we have

$$G(t,z) = \frac{z}{t}K(t,t-z)U(t-z) - \int_0^t \frac{z}{u}K_{10}(u,u-z)U(u-z)\,du$$
$$+ \int_0^t \frac{z}{u^2}K(u,u-z)U(u-z)\,du.$$

This can be rewritten more simply

$$G(t,z) = \begin{cases} \frac{z}{t}K(t,t-z) - \int_z^t \frac{z}{u^2}\left[uK_{10}(u,u-z) - K(u,u-z)\right]du & \text{if } t \geq z \\ 0 & \text{otherwise} \end{cases} \quad (3.24)$$

<u>Corollary 1</u>: For fixed z, $G(t,z)$ vanishes for $t < z$, and has a jump of magnitude $K(z,0)$ at $t = z$.

<u>Proof</u>: The first assertion is obvious and the second follows immediately by letting $t \to z$ from above in (3.24).

<u>Corollary 2</u>: If, for fixed z, $K(t,x)$ has continuous derivatives in both t and x at the point $(t, t-z)$, and if we write

$$\frac{\partial}{\partial x} K(t,x) = k(t,x),$$

then at the point (t,z), $G(t,z)$ has a continuous partial derivative in t , given by

$$\frac{\partial}{\partial t} G(t,z) = g(t,z) = \frac{z}{t} k(t, t-z).$$

<u>Proof</u>: Differentiating both sides of (3.24), we obtain

$$g(t,z) = \frac{\partial}{\partial t} G(t,z) = -\frac{z}{t^2} K(t,t-z) + \frac{z}{t} K_{10}(t,t-z) + \frac{z}{t} k(t,t-z)$$

$$- \frac{z}{t} K_{10}(t,t-z) + \frac{z}{t^2} K(t,t-z) ,$$

$$= \frac{z}{t} k(t, t-z) .$$

Corollaries 1 and 2 enable us to visualise the general shape of $G(t,z)$ in the case of a compound Poisson input, when the service time distribution $B(x)$ has a continuous derivative, $b(x)$. In that case, as discussed in Chapter 2, section 5, $K(t,x)$ satisfies the conditions of Corollary 2 for all $t > z$. The shape of $G(t,z)$ for fixed z will then be as follows:

(a) For $t < z$, $G(t,z)$ vanishes.

(b) At $t = z$ there is a jump of magnitude $K(z,0) = e^{-\lambda z}$.

(c) For $t > 0$, $G(t,z)$ is a differentiable curve whose derivative is given by

$$g(t,z) = \frac{z}{t} k(t, t-z) = \frac{z}{t} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} b_n(t-z).$$

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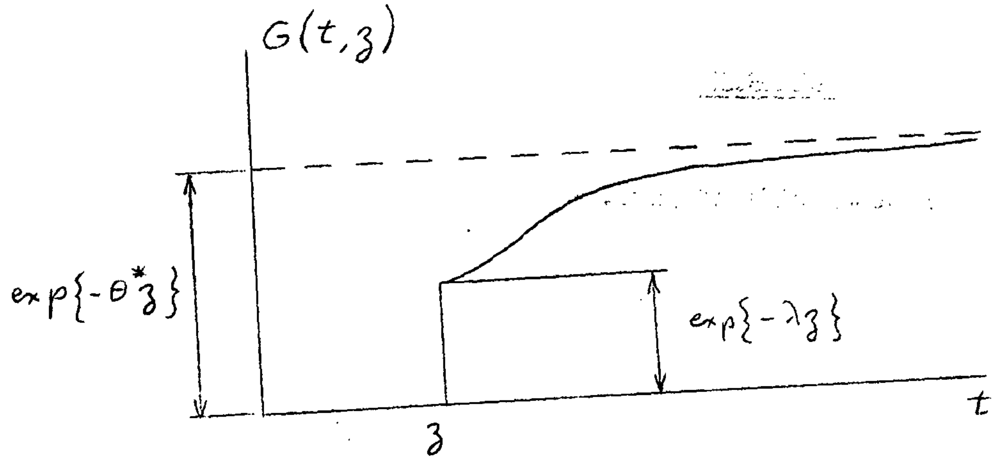

Figure 3.3

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In particular

$$\lim_{t \downarrow z} g(t, z) = k(z, 0) = 0,$$

so that the tangent to $G(t, z)$ at $t = z$ is horizontal. Moreover, it follows from the results of section 3 that

$$\lim_{t \to \infty} G(t, z) = P\{\tau(z) < +\infty\} = e^{-\theta^* z},$$

where θ^* is the largest real root of $\alpha(\theta) = \theta$.

The general shape of $G(t, z)$ in this case is shown in figure (3.3). Finally, we note that we can write in this case

$$G(t, z) = \begin{cases} K(z, 0) + \int_z^t g(u, z)\, du & \text{for } t \geq z, \\ 0 & \text{otherwise} \end{cases} \qquad (3.25)$$

where $g(t, z) = \frac{z}{t} k(t, t - z)$.

We then have

$$\Gamma(p, z) = \int_0^\infty e^{-pt} d_t G(t, z) = e^{-pt} K(z, 0) + \int_z^\infty e^{-pt} g(t, z)\, dt.$$

8. <u>The case of a discrete input</u>

Let us now assume that the input $\xi(t)$ takes only integral values. It is then clear that emptiness can occur only at times $z + n$, where $n = 0, 1, 2, \ldots$.

We shall write, as in Chapter 2, section 7,

$$P\{\xi(t) = n\} = p_n(t),$$

and we shall introduce the notation

$$P\{\tau(z) = z + n\} = q_n(z).$$

We shall assume that the $p_n(t)$ have continuous derivatives. We then have

$$K(t,x) = \sum_{k=0}^{[x]} p_k(t),$$

$$G(t,z) = \sum_{k=0}^{[t-z]} q_k(z).$$

Equation (3.24) now takes the form

$$\sum_{k=0}^{m} q_k(z) = \frac{z}{z+m}\sum_{k=0}^{m} p_k(z+m) - \int_{z}^{z+m} \frac{z}{u^2}\left[u\sum_{k=0}^{[u-z]} p_k'(u) - \sum_{k=0}^{[u-z]} p_k(u)\right] du.$$

Write $m-1$ for m and subtract. We find

$$q_m(z) = \frac{z}{z+m} p_m(z+m) + \sum_{k=0}^{m-1} z\left[\frac{p_k(z+m)}{z+m} - \frac{p_k(z+m-1)}{z+m-1}\right]$$
$$- \int_{z+m-1}^{z+m} \frac{z}{u^2}\left[u\sum_{k=0}^{[u-z]} p_k'(u) - \sum_{k=0}^{[u-z]} p_k(u)\right] du.$$

It is easily checked that the last two terms of the right-hand-side of this equation cancel out, for the last term can be written

$$-z\sum_{k=0}^{m-1}\int_{z+m-1}^{z+m} \frac{\partial}{\partial u}\left[\frac{1}{u} p_k(u)\right] du = -z\sum_{k=0}^{m}\left[\frac{p_k(z+m)}{z+m} - \frac{p_k(z+m-1)}{z+m-1}\right],$$

and so we are left with

$$q_m(z) = \frac{z}{z+m} p_m(z+m). \qquad (3.26)$$

9. Some expressions for the time of first emptiness

We shall now use the results of the preceding section to obtain the distribution function of the time of first emptiness for specific kinds of input.

(a) Consider a simple Poisson input. In this case

$$p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

So we have

$$P\{\tau(z) = z+n\} = q_n(z) = \frac{z\, e^{-\lambda(z+n)}}{z+n}\,\frac{[\lambda(z+n)]^n}{n!}, \qquad (3.27)$$

and, in particular

$$P\{\tau(z) = z\} = e^{-\lambda z}, \qquad (3.28)$$

as shown before in the general case. This will also hold for (b). Equation (3.27) is, as expected, the Borel-Tanner distribution.

(b) In the case of the queue M/M/1 (see section 6 of Chapter 2 and section 5 of this chapter), $G(t,z)$ has a continuous derivative in t for all $t > z$, and we have

$$g(t,z) = \frac{z}{t}\, e^{-\lambda t - \mu(t-z)}\sqrt{\frac{\lambda\mu t}{t-z}}\; I_1\left(2\sqrt{\lambda\mu t(t-z)}\right)$$

i.e.

$$g(t,z) = z\sqrt{\frac{\lambda\mu}{t(t-z)}}\; e^{-(\lambda+\mu)t+\mu z}\, I_1\left(2\sqrt{\lambda\mu t(t-z)}\right) \qquad (3.29)$$

$$\text{for } t \geq z.$$

(c) In the case of the queue M/G/1, when the service time distribution $B(x)$ has a density function $b(x)$, we have

$$g(t,z) = \sum_{n=0}^{\infty} e^{-\lambda t}\,\frac{\lambda^n t^{n-1}}{n!}\, z\, b_n(t-z),$$

where $b_n(x)$ is the n^{th} convolution of $b(x)$ with itself.

(d) In the case of the Gamma input

$$g(t,z) = \frac{z}{\rho\,\Gamma(t+1)}\, e^{-\frac{t-z}{\rho}} \left(\frac{t-z}{\rho}\right)^{t-1}, \quad t \geq z. \tag{3.30}$$

In that case

$$\lim_{t \downarrow z} g(t,z) = 0.$$

(e) In the case of the inverse Gaussian input

$$g(t,z) = \frac{z}{\sigma\sqrt{2\pi}} \left(\frac{\rho}{t-z}\right)^{\frac{3}{2}} \exp\left[-\frac{\rho\{(1-\rho)t-z\}^2}{2\sigma^2(t-z)}\right], \quad t \geq z, \tag{3.31}$$

and here also we can check that $\lim\limits_{t \downarrow z} g(t,z) = 0.$

In cases (b) and (e) we have explicit expressions for $\theta(p)$, and so we can check the correctness of expressions (3.29) and (3.31).

However in cases (a), (c) and (d), the characteristic equation of the process has no explicit solution, so that equations (3.27) and (3.30) can be established only as special cases of general theorems.

If we now consider the asymptotic behaviour of $g(t,z)$ as $t \to \infty$ for the various distributions considered, we obtain the following results:

(a) for the simple Poisson input, using Sterling's formula, we find that

$$q_n(z) \sim \frac{z}{\sqrt{2\pi n^3}}\, e^{-(\lambda-1-\log\lambda)n - \lambda z + z},$$

so that $q_n(z) \sim \dfrac{z}{\sqrt{2\pi n^3}}$ when $E[\xi(t)]/t = 1$, i.e., when $\lambda = 1$. We note that, in this case

$$\lim_{n \to \infty} q_n(z) = 0 \qquad \text{in all cases,}$$

as $\lambda - 1 - \log \lambda$ is never negative, and vanishes only for $\lambda = 1$.

(b) For the queue M/M/1

$$g(t,z) \sim \frac{z}{\sqrt{4\pi t^3}} (\lambda\mu)^{\frac{1}{4}} \exp\left[(\sqrt{\lambda t} - \sqrt{\mu(t-z)})^2 \right]$$

which reduces to

$$g(t,z) \sim \frac{z}{\sqrt{4\pi t^3}} (\lambda\mu)^{\frac{1}{4}} \qquad \text{for} \quad E[\xi(t)]/t = \frac{\lambda}{\mu} = 1 .$$

(c) For the Gamma input, using again Sterling's formula, we find

$$g(t,z) \sim \frac{z}{\sqrt{2\pi t^3}} \exp\left[-\left(\frac{1-\rho}{\rho} + \log \rho \right) t - z\left(\frac{1-\rho}{\rho} \right) - \left(\frac{z^2}{2} - z \right)\frac{1}{t} - \left(\frac{z^3}{3} - \frac{z^2}{2} \right)\frac{1}{t^2} - \cdots \right].$$

Let us now note that

$$\frac{1-\rho}{\rho} + \log \rho \begin{cases} > 0 & \text{when} \quad \rho \neq 1 , \\ = 0 & \text{when} \quad \rho = 1 . \end{cases}$$

Then, for $\rho \neq 1$ we can write

$$g(t,z) \sim \frac{z}{\sqrt{2\pi t^3}} \exp\left[-\left(\frac{1-\rho}{\rho} + \log \rho \right) t - z\left(\frac{1-\rho}{\rho} \right) \right]$$

while for $\rho = 1$ the formula reduces to

$$g(t,z) \sim \frac{z}{\sqrt{2\pi t^3}} \exp\left[-\left(\frac{z^2}{2} - z \right) \frac{1}{t} - \left(\frac{z^3}{3} - \frac{z^2}{2} \right) \frac{1}{t^2} - \cdots \right]$$

with the exponential term approaching one as $t \to \infty$, so that the formula can be written equivalently

$$g(t,z) \sim \frac{z}{\sqrt{2\pi t^3}} .$$

(d) For the Inverse Gaussian input, we find

$$g(t,z) \sim \frac{z}{\sigma\sqrt{2\pi t^3}}\, \rho^{\frac{3}{2}} \exp\left[-\frac{\rho(1-\rho)^2 t}{2\sigma^2}\right],$$

which reduces to

$$g(t,z) \sim \frac{z}{\sigma\sqrt{2\pi t^3}} \quad \text{for} \quad \rho = 1 .$$

10. <u>The uniqueness of the solution of Kendall's integral equation.</u>

As mentioned in the introduction, the formula

$$g(t,z) = \frac{z}{t}\, k(t, t-z)$$

was first given by Kendall [39]. But, as pointed out by Lloyd [46], Kendall only showed that $g(t,z)$ satisfied the integral equation

$$g(t,z) = \int_0^{t-z} g(t-z, y)\, k(z,y)\, dy \tag{3.32}$$

which follows immediately from equation (3.1) by using the theorem of total probability.

This equation, however, has a general solution depending on an arbitrary function. In fact, we have the following theorem:

<u>Theorem 3.9</u>: The integral equation (3.32) has the general solution

$$g(t,z) = \int_0^{t-z} k(t, t-z-x)\, dP(x) \tag{3.33}$$

where $P(x)$ is an arbitrary function of bounded variation. The particular solution

$$g(t,z) = \begin{cases} \dfrac{z}{t}\, k(t, t-z) & \text{if } t \geq z \\ 0 & \text{otherwise} \end{cases} \tag{3.34}$$

is obtained by taking $P(x) = U(x) - K_{10}(0, x)$, where $U(x)$ is Heaviside's unit function and

$$K_{10}(0, x) = \frac{\partial}{\partial t} K(t, x)\Big|_{t=0}.$$

<u>Proof</u>: Put $w = t - z$ and write $g(t, z) = h(t, t-z)$.

Equation (3.32) becomes

$$h(z+w, w) = \int_0^w h(w, w-y)\, k(z, y)\, dy \tag{3.35}$$

We shall solve the more general equation

$$h(z+w, x) = \int_0^x h(w, x-y)\, k(z, y)\, dy \tag{3.36}$$

which reduces to (3.35) by putting $x = w$.

Take the Laplace Transform of (3.36) and put

$$h^*(u, s) = \int_0^\infty e^{-sx} h(u, x)\, dx.$$

Let us also recall that

$$\int_0^\infty e^{-sx} k(t, x)\, dx = e^{-\alpha(s)t}.$$

Equation (3.36) then becomes

$$h^*(z+w, s) = h^*(w, s)\, e^{-\alpha(s)z} \tag{3.37}$$

Putting $w = 0$, we find

$$h^*(z, s) = h^*(0, s)\, e^{-\alpha(s)z}.$$

We now note that $h^*(0,s)$ can be taken as an arbitrary function of s , $\psi(s)$, which is the Laplace-Stieltjes transform of some function $P(x)$, i.e.

$$\psi(s) = \int_0^{\infty} e^{-sx}\, dP(x).$$

We see that in fact the solution

$$h^*(t,s) = \psi(s)\, e^{-\alpha(s)t} \qquad (3.38)$$

does satisfy equation (3.37).

Inverting (3.38) we find

$$h(t,x) = \int_{0-}^{x} k(t,x-y)\, dP(y),$$

or, reverting to $g(t,z)$,

$$g(t,z) = h(t,t-z) = \int_{0-}^{t-z} k(t,t-z-y)\, dP(y).$$

We can check directly that this solution satisfies (3.33). In fact, replacing in the right-hand side of (3.33), we find

$$\begin{aligned}
\int_0^{t-z} g(t-z,y)\, k(z,y)\, dy &= \int_0^{t-z} k(z,y) \int_{0-}^{t-z-y} k(t-z,t-z-y-x)\, dP(x)\, dy, \\
&= \int_{0-}^{t-z} dP(x) \int_0^{t-z-x} k(t-z,t-z-x-y)\, k(z,y)\, dy, \\
&= \int_{0-}^{t-z} k(t,t-z-x)\, dP(x) = g(t,z),
\end{aligned}$$

as required. We have used in the proof the equation

$$k(t,t-z-x) = \int_0^{t-z-x} k(t-z,t-z-x-y)\, k(z,y)\, dy,$$

which is a consequence of the property of the process $\xi(t)$

having independent increments.

It remains to obtain the form of $P(x)$ which will yield Kendall's formula (3.34).

We must solve the equation

$$\int_{0-}^{t-z} k(t, t-z-x)\, dP(x) = \frac{z}{t}\, k(t, t-z).$$

We first change variables, putting $t-z = u$. We obtain

$$\int_{0-}^{u} k(t, u-x)\, dP(x) = k(t,u) - \frac{u}{t}\, k(t,u).$$

Put $P(x) = U(x) - Q(x)$, where $U(x)$ is Heaviside's unit function. The equation reduces to

$$\int_{0-}^{u} k(t, u-x)\, dQ(x) = \frac{u}{t}\, k(t,u).$$

Take Laplace transforms with respect to u. This yields

$$\begin{aligned} e^{-\alpha(s)t}\int_0^\infty e^{-sx}\, dQ(x) &= \int_0^\infty e^{-su}\, \frac{u}{t}\, k(t,u)\, du \\ &= -\frac{1}{t}\, \frac{\partial}{\partial s}\int_0^\infty e^{-su}\, k(t,u)\, du \\ &= -\frac{1}{t}\, \frac{\partial}{\partial s}\, e^{-\alpha(s)t} \\ &= \alpha'(s)\, e^{-\alpha(s)t}, \end{aligned}$$

i.e. the Laplace-Stieltjes transform of $Q(x)$ is $\alpha'(s)$.

Finally, differentiating the relation

$$e^{-\alpha(s)t} = \int_0^\infty e^{-sx}\, k(t,x)\, dx,$$

first with respect to t and then with respect to s , and putting $t=0$, we find

$$\alpha'(s) = \int_0^\infty e^{-sx} k_{10}(t,x)\,dx \Big|_{t=0} = \int_0^\infty e^{-sx}\, dK_{10}(0,x),$$

so that we can take $Q(x)$ to be $K_{10}(0,x)$.

Our final result is therefore

$$P(x) = U(x) - K_{10}(0,x).$$

This completes the proof of the theorem.

11. The distribution of the busy period:

In the queueing interpretation of our storage model, the notion of length of busy period is important. This random variable will be denoted by $\hat{\lambda}$ and is defined as follows: Suppose that at an instant t when the store is empty there is an instantaneous input of magnitude X . It follows that the store will not be empty for a length of time of at least X . Let t' be the first point of time when the store is again empty. Then $\hat{\lambda}$ is defined by

$$\hat{\lambda} = t' - t.$$

In queueing terminology, $\hat{\lambda}$ is the length of time that the server remains continuously busy, between two idle periods.

Let us note, however, that the notion of busy period is not a well defined one in the case of inputs where the sample functions are not a.s. step functions.

We shall first calculate the Laplace-Stieltjes transform of the distribution of $\hat{\lambda}$. Let the distribution function of a jump X in the input be $B(x)$. Then, using the theorem of total probability, we obtain, using the fact that $\hat{\lambda} = \tau(z)$ if $X = z$,

$$E\left[e^{-p\hat{\lambda}}\right] = \int_0^\infty E\left[e^{-p\hat{\lambda}} \mid X = z\right] dB(z)$$
$$= \int_0^\infty E\left[e^{-p\tau(z)}\right] dB(z)$$
$$= \int_0^\infty e^{-\theta(p) z} dB(z)$$
$$= \psi\left[\theta(p)\right],$$

where $\psi(s)$ is the Laplace-Stieltjes transform of $B(x)$.

Let us now assume that $B(x)$ has a continuous derivative $b(x)$. Then $G(t,z)$ has the form given at the end of section 7. Using again the theorem of total probability, we find

$$P\{\hat{\lambda} \leq t\} = \int_0^t G(t,z)\, b(z)\, dz$$
$$= \int_0^t \left[e^{-\lambda z} + \int_z^t \frac{z}{u} k(u, u-z)\, du\right] b(z)\, dz$$
$$= \int_0^t e^{-\lambda z} b(z)\, dz + \int_0^t\int_0^u z\, k(u,u-z)\, b(z)\, dz\, \frac{du}{u}$$
$$= \int_0^t e^{-\lambda z} b(z)\, dz + \int_0^t \sum_{n=1}^\infty e^{-\lambda u} \frac{\lambda^n u^{n-1}}{n!} \int_0^u z\, b_n(u-z)\, b(z)\, dz\, du$$
$$= \int_0^t \sum_{n=0}^\infty e^{-\lambda u} \frac{(\lambda u)^n}{(n+1)!} b_{n+1}(u)\, du. \qquad (3.39)$$

Here we have used the identity

$$\int_0^t z\, b_n(t-z)\, b(z)\, dz = \frac{t\, b_{n+1}(t)}{n+1}.$$

This can be most easily proved by taking Laplace transforms, for we have

$$\int_0^\infty e^{-pt}dt \int_0^t z\, b_n(t-z)\, b(z)\, dz = \int_0^\infty e^{-pz} z\, b(z)\, dz \int_0^\infty e^{-pu} b_n(u)\, du$$

$$= \left[-\frac{\partial}{\partial p}\psi(p)\right]\left[\psi(p)\right]^n$$

$$= -\frac{1}{n+1}\frac{\partial}{\partial p}\left[\psi(p)\right]^{n+1}$$

$$= \frac{1}{n+1}\int_0^\infty e^{-pt}\, t\, b_{n+1}(t)\, dt.$$

From equation (3.39) we can conclude that $\hat{\lambda}$ has in the case considered an absolutely continuous distribution whose density function, which we shall denote by $\ell(t)$, is given by

$$\ell(t) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{(n+1)!}\, b_{n+1}(t).$$

As an example, let us calculate $\ell(t)$ in the case of the queue M/M/1. The Laplace transform of $\ell(t)$ will be given by

$$\int_0^\infty e^{-pt}\ell(t)dt = \psi[\theta(p)] = \frac{\mu}{\mu + \frac{1}{2}(p+\lambda-\mu) + \frac{1}{2}\sqrt{(p+\lambda+\mu)^2 - 4\lambda\mu}}$$

$$= \frac{2\mu}{p+\lambda+\mu + \sqrt{(p+\lambda+\mu)^2 - 4\lambda\mu}}.$$

Inverting the Laplace transform, by using Erdelyi [19] p. 237, formula No. 49, we find

$$\ell(t) = e^{-(\lambda+\mu)t} \sqrt{\frac{\mu}{\lambda}} \; \frac{I_1(2\sqrt{\lambda\mu}\, t)}{t} .$$

CHAPTER 4

THE DISTRIBUTION OF THE CONTENT

1. The fundamental equation for the Laplace transform of the content distribution.

In this chapter, we shall at first deal with a slightly more general model than the one discussed in the preceding chapter. We shall assume that the input $\xi(t)$ is a non-homogeneous additive process of the type discussed in section 12 of Chapter 2, and we shall write

$$E\left[e^{-s[\xi(t)-\xi(u)]}\right] = \Theta(u,t;s). \tag{4.1}$$

We now note that

$$E\left[e^{-s[\nu(t)-\nu(u)]}\right] = E\left[e^{-s[\xi(t)-t-\xi(u)+u]}\right] = e^{s(t-u)}\,\Theta(u,t;s).$$

In the homogeneous case, when $\Theta(u,t;s) = e^{-\alpha(s)(t-u)}$, we shall just write $\Theta(t,s)$ for $\Theta(0,t;s)$, and we shall then have

$$\Theta(u,t;s) = \Theta(t-u,s).$$

We shall assume, as in Chapter 3, that the initial content of the store is z, and we shall introduce the notations

$$W(t,x) = P\{\zeta(t) \leq x\},$$

$$\Omega(t,s) = E\left[e^{-s\zeta(t)}\right] = \int_{-\infty}^{+\infty} e^{-sx}\, d_x W(t,x), \quad \mathrm{Re}(s) \geq 0,$$

$$W^*(p,x) = \int_0^\infty e^{-pt} W(t,x)\,dt \quad , \quad Re(p) > 0 \, ,$$

$$\Omega^*(p,s) = \int_0^\infty e^{-pt} \Omega(t,s)\,dt \quad , \quad Re(p) > 0 \, .$$

We note that the transforms with respect to x are Laplace-Stieltjes, while the transforms with respect to t are ordinary Laplace transforms. The integrals necessarily all converge in the regions given because the total variation of $W(t,x)$ in x is unity, and we have $|W(t,x)| \leq 1$, $|\Omega(t,s)| \leq 1$.

We also note that since $\zeta(t) \geqslant 0$, we must have $W(t,x) = 0$ for $x < 0$ although $W(t,x)$ may have a jump for $x = 0$, of magnitude $W(t,0)$, so that

$$P\{\zeta(t) = 0\} = W(t,0).$$

The fundamental equation satisfied by $\Omega(t,s)$ is readily obtained by using formula (1.10) of Chapter 1, namely

$$e^{-s\zeta(t)} = e^{-sz - s\nu(t)} - s\int_0^t e^{-s[\nu(t)-\nu(u)]}\, U[-\zeta(u)]\, d\nu_-(u). \quad (4.2)$$

We first note that for the process $\nu(t)$, we must have, for almost all sample functions

$$\nu_-(t) = t \quad .$$

This follows from the fact that $\xi'(t) = 0$ for almost all

t, (see Theorem 2.5), so that the Stieltjes measure determined by $\xi(t)$ is singular with respect to the measure determined by $\eta(t) = t$, which is of course the usual Lebesgue measure. Secondly, we note that the random variables $e^{-s[\nu(t)-\nu(u)]}$, $U[-\zeta(u)]$ are independent, in view of the additivity of the process $\nu(t)$.

Finally, we note that

$$E\{U[-\zeta(t)]\} = P\{\zeta(t)=0\} = W(t,0),$$

for $U[-\zeta(t)]$ takes value 1 when $\zeta(t)=0$ with probability $W(t,0)$ and value 0 otherwise.

Taking now expectations on both sides of (4.2), we obtain

$$\Omega(t,s) = e^{-sz+st}\Theta(0,t;s) - s\int_0^t e^{s(t-u)}\Theta(u,t;s)W(u,0)\,du. \quad (4.3)$$

This is the fundamental equation for $\Omega(t,s)$. From it we deduce immediately

$$E[\zeta(t)] = -\frac{\partial}{\partial s}\Omega(t,s)\Big|_{s=0} = z - t + \rho(t) + \int_0^t W(u,0)\,du,$$

where $\rho(t) = E[\xi(t)]$.

Let us note that, if $\Theta(u,t;s)$ is of the form

$$\Theta(u,t;s) = e^{-\alpha(s)[\Lambda(t)-\Lambda(u)]},$$

then $\Omega(t,s)$ is given by

$$\Omega(t,s) = e^{-sz+st-\alpha(s)\Lambda(t)} - s\int_0^t e^{s(t-u)-\alpha(s)[\Lambda(t)-\Lambda(u)]}W(u,0)\,du,$$

and, in particular, if the input process is stationary, i.e. if $\Lambda(t) = \lambda t$, we have

$$\Omega(t,s) = e^{-sz+\lambda[s-\alpha(s)]t} - s\int_0^t e^{\lambda[s-\alpha(s)](t-u)} W(u,0)\,du.$$

2. <u>The inversion of the fundamental formula</u>

Let us write

$$P\{\xi(t)-\xi(u)\le x\} = K(u,t;x).$$

Then

$$P\{\nu(t)-\nu(u)\le x\} = P\{\xi(t)-\xi(u)\le x+t-u\}$$

$$= K(u,t;x+t-u).$$

We also find, on integrating by parts, that

$$\Omega(t,s) = s\int_{-\infty}^{+\infty} e^{-sx} W(t,x)\,dx = s^2\int_{-\infty}^{+\infty} e^{-sx}\left\{\int_{-\infty}^{x} W(t,y)\,dy\right\}dx,$$

and

$$e^{-s(z-t)}\Theta(0,t;s) = s\,e^{-s(z-t)}\int_{-\infty}^{+\infty} e^{-sx} K(0,t;x)\,dx$$

$$= s\int_{-\infty}^{+\infty} e^{-s(x+z-t)} K(0,t;x)\,dx$$

$$= s\int_{-\infty}^{+\infty} e^{-sx} K(0,t;x+t-z)\,dx$$

$$= s^2\int_{-\infty}^{+\infty} e^{-sx}\left\{\int_{-\infty}^{x} K(0,t;y+t-z)\,dy\right\}dx.$$

Finally,

$$e^{s(t-u)}\,\Theta(u,t;s) = s\,e^{s(t-u)}\int_{-\infty}^{+\infty} e^{-sx}\,K(u,t;x)\,dx$$
$$= s\int_{-\infty}^{+\infty} e^{-sx}\,K(u,t;x+t-u)\,dx .$$

Replacing in equation (4.3), we find

$$s^2\int_{-\infty}^{+\infty} e^{-sx}\left\{\int_{-\infty}^{x} W(t,y)\,dy\right\}dx = s^2\int_{-\infty}^{+\infty} e^{-sx}\left\{\int_{-\infty}^{x} K(0,t;y+t-z)\,dy\right\}dx$$
$$- s^2\int_0^t\left\{\int_{-\infty}^{+\infty} e^{-sx}K(u,t;x+t-u)\,dx\right\}W(u,0)\,du,\quad \mathrm{Re}(s)>0.$$

If we restrict s to the positive real axis, the integrand of the double integral is positive so that, by Fubini's theorem, we can interchange the order of integration. Finally, using the uniqueness theorem for the bilateral Laplace transform (see Widder [74] p. 243), we conclude that, for almost all x,

$$\int_{-\infty}^{x} W(t,y)\,dy = \int_{-\infty}^{x} K(0,t;y+t-z)\,dy - \int_0^t K(u,t;x+t-u)\,W(u,0)\,du.$$

But as each term of the equality is non-decreasing and continuous to the right, the equality holds for all x.

Finally, using the right-continuity of $W(t,x)$ and $K(0,t;x+t-z)$ in x, we obtain

$$W(t,x) = K(0,t;x+t-z) - \left(\frac{\partial}{\partial x}\right)^{+}\int_0^t K(u,t;x+t-u)\,W(u,0)\,du, \tag{4.4}$$

where $\left(\frac{\partial}{\partial x}\right)^{+}$ denotes a right-hand partial derivative with respect to x.

This is the general form of the formula for $W(t,x)$. We shall now obtain various other forms of the formula by imposing restrictions on $K(u,t;x)$.

Firstly, suppose that $K(u,t;x)$ can be written in the form

$$K(u,t;x) = K(u,t;o)\,U(x) + \int_{-\infty}^{x} k(u,t;y)\,U(y)\,dy.$$

We then have

$$\int_{-\infty}^{x} W(t,y)\,dy = \int_{-\infty}^{x} K(o,t;y+t-z)\,dy - \int_{0}^{t} K(u,t;o)\,U(x+t-u)\,W(u,o)\,du$$
$$- \int_{0}^{t} \left\{ \int_{-\infty}^{x} k(u,t;y+t-u)\,U(y+t-u)\,du \right\} W(u,o)\,du,$$

so that, for $x \geqslant 0$,

$$\int_{-\infty}^{x} W(t,y)\,dy = \int_{-\infty}^{x} K(o,t;y+t-z)\,dy - \int_{0}^{t} K(u,t;o)\,W(u,o)\,du$$
$$- \int_{-\infty}^{x} \left\{ \int_{0}^{t} k(u,t;y+t-u)\,U(y+t-u)\,W(u,o)\,du \right\} dy,$$

and finally

$$W(t,x) = K(o,t;x+t-z) - \int_{0}^{t} k(u,t;x+t-u)\,W(u,o)\,du. \quad (4.5)$$

Secondly, we shall write the last term of (4.3) in the form

$$s\int_{0}^{t} e^{s(t-u)}\,\Theta(u,t;s)\,W(u,o)\,du = \lim_{m\to\infty} \sum_{i=0}^{m-1} s\,e^{s(t-v_i)}\,\Theta(v_i',t;s)\,W(v_i',o)(u_{i+1}-u_i)$$

where $0 = u_0 < u_1 < \dots < u_m = t$ is a suitable dissection of the interval $(0,t)$, and $u_i \leq v_i \leq u_{i+1}$, $u_i \leq v_i' \leq u_{i+1}$.

But, by the mean value theorem, we can choose v_i so that

$$s(u_{i+1}-u_i)\,e^{-sv_i} = e^{-su_i} - e^{-su_{i+1}}.$$

DIAGRAM

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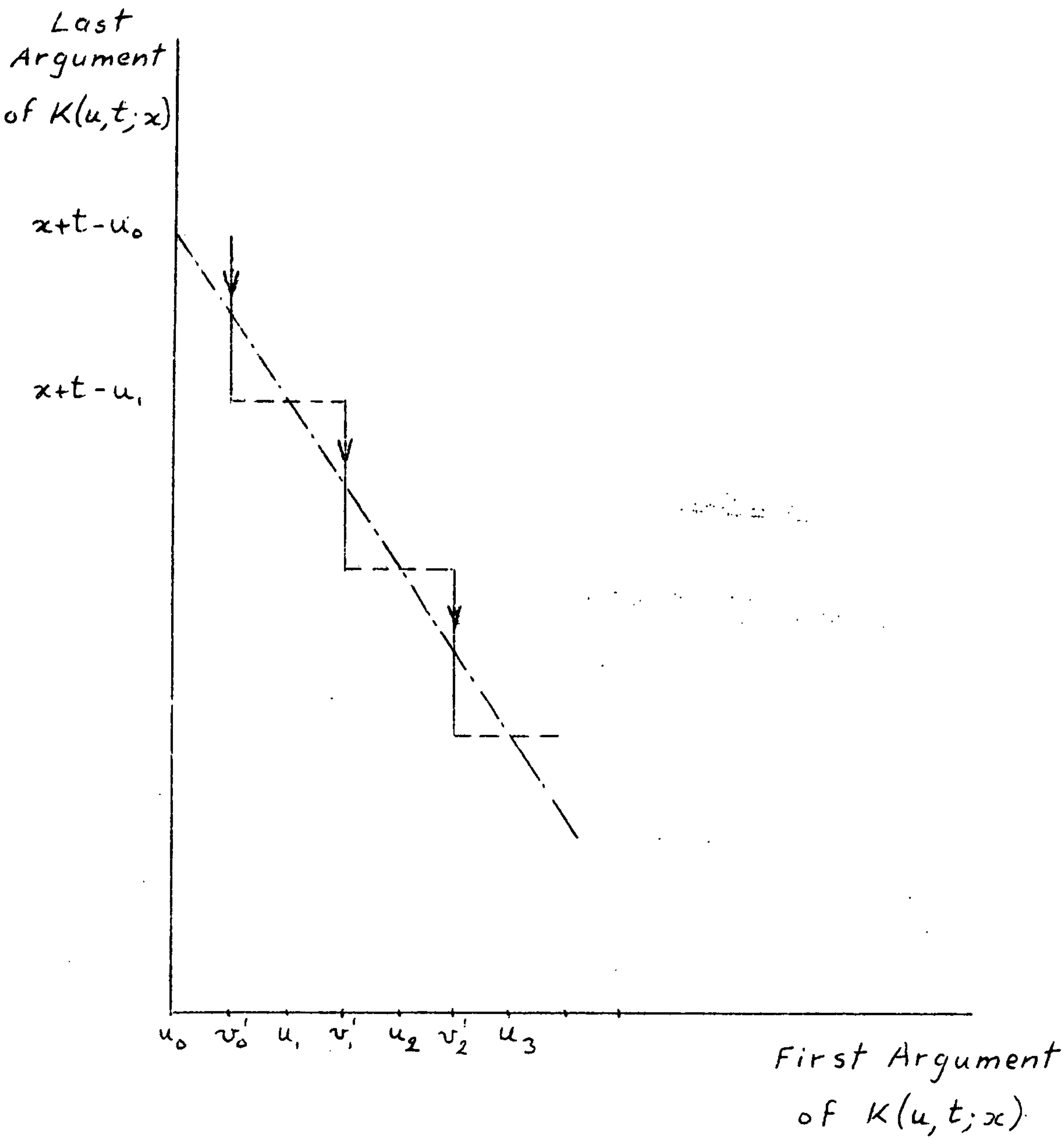

Figure 4.1

To face p.105

It follows that we have

$$s\int_0^t e^{s(t-u)}\Theta(u,t;s)W(u,0)\,du = \lim_{n\to\infty}\sum_{i=0}^{n-1} W(v_i',0)\left[e^{s(t-u_i)}\Theta(v_i',t;s)-e^{s(t-u_{i+1})}\Theta(v_i',t;s)\right].$$

Using now the relations

$$e^{s(t-u_i)}\Theta(v_i',t;s)=s\int_{-\infty}^{+\infty}e^{-sx}K(v_i',t;x+t-u_i)\,dx\,,$$

$$e^{s(t-u_{i+1})}\Theta(v_i',t;s)=s\int_{-\infty}^{+\infty}e^{-sx}K(v_i',t;x+t-u_{i+1})\,dx\,,$$

we find that

$$s\int_0^t e^{s(t-u)}\Theta(u,t;s)W(u,0)\,du = s\int_{-\infty}^{+\infty}e^{-sx}dx.$$

$$\left\{\lim_{n\to\infty}\sum_{i=0}^{n-1}W(v_i',0)\left[K(v_i',t;x+t-u_i)-K(v_i',t;x+t-u_{i+1})\right]\right\},$$

provided the interchange of the integration sign and the limit can be justified. This will be so if, for instance, the sum is uniformly bounded for all x .

We shall write, for short

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}W(v_i',0)\left[K(v_i',t;x+t-u_i)-K(v_i',t;x+t-u_{i+1})\right]$$

$$=\int_0^t W(u,0)\,d_u'K(u,t;x+t-u). \qquad (4.6)$$

We can then deduce from (4.3) the formula

$$W(t,x)=K(0,t;x+t-z)-\int_0^t W(u,0)\,d_u'K(u,t;x+t-u). \quad (4.7)$$

We should, however, bear in mind that the integral does not have its usual meaning, but must be interpreted according to formula (4.6), the dash on the d expressing the fact that the variation of $K(u,t\,;\,x+t-u)$ in u is only in the last argument. The path along which the approximating sum is taken is shown in figure (4.1).

Formulae (4.6), (4.7) can also be written, writing $t-u$ for u, in the form

$$W(t,x) = K(0,t\,;\,x+t-z) - \int_0^t W(t-u,0)\,k(t-u,t\,;\,x+u)\,du \qquad (4.8)$$

$$W(t,x) = K(0,t\,;\,x+t-z) - \int_0^t W(t-u,0)\,d'_u K(t-u,t\,;\,x+u)\,. \qquad (4.9)$$

Formula (4.9) is useful, in particular, when $\xi(t)$ can take only integral values. Let us write in this case, generalising the notation of Chapter 3, section 8,

$$K(u,t\,;\,x) = \sum_{k=0}^{[x]} p_k(u,t).$$

To evaluate the integral, we write

$$\int_0^t W(t-u,0)\,d'_u K(t-u,t\,;\,x+u)$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t-v'_i,0)\Big[K(t-v'_i,t\,;\,x+u_{i+1}) - K(t-v'_i,t\,;\,x+u_i)\Big].$$

We now note that in this case $K(u,t\,;\,x)$ increases only at integral values of x, irrespective of the values of u and t. So we have

$$\int_0^t W(t-u,0)\,d'_u K(t-u,t\,;\,x+u)$$

$$= \sum_{n=1}^{[x+t]-[x]} W(t-[x]-n+x,0)\,p_{[x]+n}(t-[x]-n+x,t),$$

and the approximating sums are clearly uniformly bounded in this case.

Replacing in (4.7), we find

$$w(t,x)=\sum_{m=0}^{[x+t-z]} p_m(0,t)-\sum_{m=1}^{[x+t]-[x]} W(t-[x]-m+x,0)\,p_{[x]+m}(t-[x]-m+x,t). \quad (4.10)$$

This can also be written, writing m for $[x]+m$,

$$W(t,x)=\sum_{m=0}^{[x+t-z]} p_m(0,t)-\sum_{m=[x]+1}^{[x+t]} W(t-m+x,0)\,p_m(t-m+x,t). \quad (4.11)$$

To conclude this section, we note that in the stationary case formulae(4.8) and (4.11) become

$$W(t,x)=K(t,x+t-z)-\int_0^t W(t-u,0)\,k(u,x+u)\,du, \quad (4.8)'$$

$$W(t,x)=\sum_{m=0}^{[x+t-z]} p_m(t)-\sum_{m=[x]+1}^{[x+t]} W(t-m+x,0)\,p_m(m-x). \quad (4.11)'$$

3. <u>The calculation of $W(t,0)$ in the stationary case</u>.

In the stationary case, i.e. when $\Theta(u,t;s)=\Theta(t-u,s)$, the integral in equation (4.3) becomes a convolution integral. In that case, if we take the Laplace transform of both sides with respect to t , we find

$$\Omega^*(p,s)=e^{-sz}\Theta^*(p-s,s)-s\,\Theta^*(p-s,s)\,W^*(p,0), \quad (4.12)$$

where

$$\Theta^*(p,s)=\int_0^\infty e^{-pt}\Theta(t,s)\,dt=\int_0^\infty e^{-pt}e^{-\alpha(s)t}\,dt=\frac{1}{p+\alpha(s)},$$

(c.f. equation (3.18)), the equation holding for $Re(p) > 0$, $Re(s) \geq 0$.

Replacing in (4.12), we obtain

$$\Omega^*(p,s) = \frac{e^{-sz} - s\,W^*(p,0)}{p - s + \alpha(s)}. \tag{4.13}$$

We now note that in the given region $|\Omega^*(p,s)| \leq 1$, and for every fixed p, $\Omega^*(p,s)$ is an analytic function of s.

Let now p_0 and σ_0 be as in theorem 3.3, and let us restrict p to real values such that $p > p_0$. Then, in $Re(s) > \sigma_0$, the equation $p - s + \alpha(s) = 0$ has exactly one root $\theta(p)$ (Theorem 3.3). It follows that, for $p > p_0$, the numerator of (4.13) must vanish for $s = \theta(p)$, so that we must have

$$W^*(p,0) = \frac{e^{-z\theta(p)}}{\theta(p)}. \tag{4.14}$$

We can then calculate $w(t,0)$ by inverting (4.14). Let us note that (4.14) can be written, putting as before $\Gamma(p,z) = e^{-\theta(p)z}$,

$$W^*(p,0) = \int_z^{+\infty} e^{-y\theta(p)}\,dy = \int_z^{+\infty} \Gamma(p,y)\,dy. \tag{4.15}$$

But we have shown in chapter 3 (theorem 3.1) that

$$\Gamma(p,z) = \int_{0-}^{\infty} e^{-pt}\,d_t G(t,z) = p\int_0^{\infty} e^{-pt}\,G(t,z)\,dt,$$

where the lower limit of integration can be taken as zero, as $G(t,z) = 0$ for $t < z$. Also, we have

$$W^*(p,0) = \int_0^\infty e^{-pt} W(t,0)\,dt = p\int_0^\infty e^{-pt}\left\{\int_0^t W(u,0)\,du\right\} dt,$$

so that (4.15) becomes

$$p\int_0^\infty e^{-pt}\left\{\int_0^t W(u,0)\,du\right\} dt = p\int_0^\infty e^{-pt}\left\{\int_z^{+\infty} G(t,y)\,dy\right\} dt, \qquad (4.16)$$

the interchange of the integrals being again justified by the use of Fubini's theorem. Thus, we must have

$$\int_0^t W(u,0)\,du = \int_z^{+\infty} G(t,y)\,dy = \int_z^t G(t,y)\,dy, \qquad (4.17)$$

as $G(t,z) = 0$ for $z > t$.

It follows from the well-known differentiation properties of the Lebesgue integral, that, for almost all t, we have

$$W(t,0) = \frac{\partial}{\partial t}\int_z^t G(t,y)\,dy. \qquad (4.18)$$

This is the general formula for $W(t,0)$.

Let us now, as in section 2, put some restrictions on the form of $G(t,z)$ and obtain various other forms of the formula.

Firstly, let us assume that $G(t,z)$ is of the form given in equation (3.25), namely

$$G(t,z) = \begin{cases} K(z,0) + \int_z^t g(u,z)\,du & \text{for } t \geq z, \\ 0 & \text{otherwise}. \end{cases}$$

Then, as shown in Chapter 3, section 7, we have

$$\Gamma(p,z) = e^{-pz} K(z,0) + \int_z^{+\infty} e^{-pt} g(t,z)\,dt,$$

and equation (4.15) becomes

$$\int_0^{\infty} e^{-pt} W(t,0)\,dt = \int_z^{+\infty}\left[e^{-py} K(y,0) + \int_y^{+\infty} e^{-pt} g(t,y)\,dt\right] dy$$

$$= \int_0^{\infty} e^{-pt} K(t,0)\,U(t-z)\,dt + \int_z^{\infty}\int_0^{\infty} e^{-pt} g(t,y)\,U(t-y)\,dt\,dy,$$

$$= \int_0^{\infty} e^{-pt}\left[K(t,0)\,U(t-z) + \int_z^{\infty} g(t,y)\,U(t-y)\,dy\right] dt,$$

so that, for almost all t again

$$W(t,0) = K(t,0)\,U(t-z) + \int_z^{\infty} g(t,y)\,U(t-y)\,dt.$$

This can be written

$$W(t,0) = \left\{\begin{array}{ll} K(t,0) + \int_z^{t} g(t,y)\,dy & \text{for } t \geq z \\ 0 & \text{for } t < z. \end{array}\right\} \quad (4.9)$$

Thus, for fixed z, $W(t,0)$ coincides, for almost all t, with a function which vanishes for $t < z$, and has a jump of height $K(t,0)$ at $t = z$.

Equation (4.9) can be written, for $t \geq z$,

$$W(t,0) = K(t,0) + \int_z^{t} \frac{y}{t}\, k(t, t-y)\,dy,$$

$$= K(t,0) + \int_0^{t-z} \frac{t-y}{t}\, k(t,y)\,dy,$$

$$= K(t,0) + \int_0^{t-z} k(t,y)\,dy - \frac{1}{t}\int_0^{t-z} y\, k(t,y)\,dy ,$$

i.e.

$$W(t,0) = \begin{cases} K(t,t-z) - \frac{1}{t}\int_0^{t-z} y\, k(t,y)\,dy & \text{for } t \geq z \\ 0 & \text{for } t < z \end{cases} \tag{4.10}$$

Further, integrating by parts, we obtain,for $t \geq z$,

$$W(t,0) = K(t,t-z) - \frac{1}{t}\Big[y\,K(t,y)\Big]_0^{t-z} + \frac{1}{t}\int_0^{t-z} K(t,y)\,dy$$

$$= \frac{z}{t} K(t,t-z) + \frac{1}{t}\int_0^{t-z} K(t,y)\,dy .$$

Thus

$$W(t,0) = \begin{cases} \frac{z}{t} K(t,t-z) + \frac{1}{t}\int_0^{t-z} K(t,y)\,dy & \text{for } t \geq z , \\ 0 & \text{for } t < z . \end{cases}$$

We can obtain a more precise definition of $W(t,0)$ if we assume that $\xi(t)$ has a density function, $k(t,x)$, which is a continuous function of both t and x . In that case, the conditions of theorem 3.7 will be obviously satisfied, and we shall have

$$g(t,z) = \frac{z}{t}\, k(t,t-z)\, U(t-z)$$

for almost all t .

Let now $W_0(t,0)$ be defined by $W_0(t,0) = P\{\zeta(t)=0 \mid z=0\}$.
By enumeration of the paths, we obtain the relation

$$W(t,0) = \int_0^{t-z} g(t-u,z)\, W_0(u,0)\,du , \tag{4.11}$$

where $g(t,z)$ is continuous for almost all t.

The continuity of $w(t,o)$ follows immediately from the form of (4.11).

It then follows that for all t, $w(t,o)$ is given by

$$w(t,o) = \begin{cases} \int_z^t g(t,y)\,dy & \text{for } t \geq z \\ 0 & \text{for } t < z. \end{cases}$$

We now turn to the case where $\xi(t)$ takes only integral values. Then $\tau(z)$ takes only integral values, and, as shown in Chapter 3, section 8, equation (3.26)

$$P\{\tau(z) = z+n\} = q_n(z) = \frac{z}{z+n}\, p_n(z+n).$$

In that case, we have

$$G(t,z) = \sum_{n=0}^{[t-z]} q_n(z).$$

We then have

$$\int_z^t G(t,y)\,dy = \int_z^t q_0(y)\,dy + \int_z^{t-1} q_1(y)\,dy + \dots + \int_z^{t-[t-z]} q_{[t-z]}(y)\,dy.$$

It follows that, for almost all t,

$$w(t,o) = \frac{\partial}{\partial t}\int_z^t G(t,y)\,dy = q_0(t) + q_1(t-1) + \dots + q_{[t-z]}\{t-[t-z]\}.$$

This can be written

$$w(t,o) = \sum_{n=0}^{(t-z]} q_n(t-n) \tag{4.12}$$

i.e. $$w(t,o) = \sum_{n=0}^{[t-z]} \frac{t-n}{t}\, p_n(t). \tag{4.13}$$

To conclude this section, let us note that sections 2 and 3 provide a complete solution, at least in theory, for the problem of the determination of $W(t,x)$ in the stationary case, in terms of $K(t,x)$, whenever $G(t,z)$ can be calculated, i.e. as shown in Chapter 3, whenever $\xi(t)$ is a Compound Poisson process, and whenever $\xi(t)$ has a density function satisfying the conditions of theorem 3.7. Then, by combining equations (4.4) and (4.18), we can obtain $W(t,x)$.

4. The asymptotic behaviour of the content in the stationary case

In section 3 of Chapter 3, we have considered the behaviour of $\tau(z)$ as t tends to ∞. In this section, we shall use the results obtained there to investigate the asymptotic behaviour of the distribution of $\zeta(t)$ as t tends to infinity. We first prove

Theorem 4.1: Let $E[\xi(t)] = \rho t$ and $Var[\xi(t)] = \sigma^2 t$, where $\sigma^2 < \infty$. Then

(a) if $\rho < 1$, $W(t,x)$ tends to a stationary distribution which is independent of z as $t \to \infty$.

(b) If, on the other hand, $\rho > 1$, we have, for every x

$$\lim_{t\to\infty} W(t,x) = 0.$$

Proof: We shall use the representation

$$\zeta(t) = \nu(t) + \max\left[z, \nu^*(t)\right],$$

which was obtained in Chapter 1 (formula 1.9). This can be written

$$\zeta(t) = \max\left[z + \nu(t),\ \nu(t) + \nu^*(t)\right],$$

so that

$$W(t;x) = P\left\{\zeta(t) \le x\right\} = P\left\{z + \nu(t) \le x\ ;\ \nu(t) + \nu^*(t) \le x\right\}.$$

Let us first consider case (a).

It then follows from Chebyshev's inequality that

$$P\left\{\nu(t) - (\rho - 1)t \geq \varepsilon\right\} \le \frac{\sigma^2 t}{\varepsilon^2}.$$

Take $t > z/(1-\rho)$ and put $\varepsilon = (1-\rho)t - z + x$. It follows that

$$P\left\{z + \nu(t) \geq x\right\} \le \frac{\sigma^2 t}{\left[z - x + (\rho - 1)t\right]^2},$$

and letting t tend to infinity, we find that

$$\lim_{t\to\infty} P\left\{z + \nu(t) \geq x\right\} = 0.$$

This implies

$$\lim_{t\to\infty} P\left\{z + \nu(t) < x\right\} = 1 \tag{4.14}$$

Consider now the distribution of $\nu(t) + \nu^*(t)$. Let us first note that we have

$$\nu(t) + \nu^*(t) = \nu(t) - \inf_{0\le u\le t} \nu(u)$$

$$= \sup_{0\le u\le t}\left[\nu(t) - \nu(u)\right].$$

But as $\nu(t)$ is a stationary process, the distribution of

$\sup\limits_{0\le u\le t}[\nu(t)-\nu(u)]$ is the same as that of $\sup\limits_{0\le u\le t}\nu(u)$.

We conclude that

$$\lim_{t\to\infty} P\left\{\sup_{0\le u\le t}[\nu(t)-\nu(u)]\le x\right\} = P\left\{\sup_{0\le u<\infty}\nu(u)\le x\right\}. \quad (4.15)$$

This is possibly not a proper distribution function.

Let now

$$A_t = \{\omega;\ z+\nu(t)\le x\},$$

$$A_t^c = \{\omega;\ z+\nu(t)>x\},$$

$$B_t = \{\omega;\ \nu(t)+\nu^*(t)\le x\}.$$

Then

$$P\{B_t\} = P\{B_t\cap A_t\} + P\{B_t\cap A_t^c\},$$

so that

$$\lim_{t\to\infty} W(t,x) = \lim_{t\to\infty} P\{A_t\cap B_t\} = \lim_{t\to\infty} P\{B_t\} - \lim_{t\to\infty} P\{B_t\cap A_t^c\}.$$

But

$$0\le \lim_{t\to\infty} P\{B_t\cap A_t^c\} \le \lim_{t\to\infty} P\{A_t^c\} = \lim_{t\to\infty} P\{z+\nu(t)>x\} = 0.$$

It follows that

$$\lim_{t\to\infty} W(t,x) = \lim_{t\to\infty} P\{B_t\} = P\left\{\sup_{0\le u<\infty}\nu(u)\le x\right\}. \quad (4.16)$$

This completes the proof of case (a).

Let us now turn to case (b). We can then write

$$P\{(\rho-1)t-\nu(t)\ge\varepsilon\} \le \frac{\sigma^2 t}{\varepsilon^2}. \quad (4.17)$$

Put $\varepsilon = \frac{1}{2}(\rho - 1)t + z$. Then (4.17) becomes

$$P\{(\rho-1)t - \nu(t) \geq \left(\frac{\rho-1}{2}\right)t + z\} \leq \frac{\sigma^2 t}{\left[\left(\frac{\rho-1}{2}\right)t + z\right]^2} .$$

This can be written

$$P\{\nu(t) + z \leq \frac{(\rho-1)}{2} t\} \leq \frac{\sigma^2 t}{\left[\left(\frac{\rho-1}{2}\right)t + z\right]^2} .$$

Take t large enough to have $(\rho-1)t/2 > x$, where x is an arbitrarily large number. Then

$$P\{\nu(t) + z \leq x\} \leq \frac{\sigma^2 t}{\left[\left(\frac{\rho-1}{2}\right)t + z\right]^2} ,$$

so that

$$\lim_{t\to\infty} P\{\nu(t) + z \leq x\} = 0$$

It follows that

$$0 \leq \lim_{t\to\infty} w(t,x) \leq \lim_{t\to\infty} P\{\nu(t) + z \leq x\} = 0 .$$

This completes the proof of case (b).

<u>Corollary</u>: (a) If $\rho < 1$, $\lim_{t\to\infty} \Omega(t,s)$ exists and is the Laplace-Stieltjes transform of a (possibly defective) probability distribution.

(b) If $\rho > 1$, $\lim_{t\to\infty} \Omega(t,s) = 0$ for all s such that $\mathcal{Re}(s) > 0$.

<u>Proof</u>: Part (a) follows immediately from a theorem of Gnedenko and Kolmogorov [31] p. 33, while part (b) follows immediately from case (b) of theorem (4.1).

We are now in a position to obtain a formula for the Laplace transform of the asymptotic distribution of $\zeta(t)$ in the case where $\rho < 1$. This is the generalised form of the celebrated Pollaczek-Khintchine formula.

<u>Theorem 4.2:</u> The Laplace transform of the asymptotic distribution of $\zeta(t)$, when $\rho < 1$, is given by

$$\bar{\Omega}(s) = \lim_{t\to\infty} \Omega(t,s) = \frac{s(1-\rho)}{s-\alpha(s)}.$$

<u>Proof:</u> To obtain $\bar{\Omega}(s)$, we shall use the well-known Abelian theorem for Laplace transforms (c.f. Doetsch[14], p. 458) which states that if $\lim_{t\to\infty} F(t)$ exists and if $f(p)$ is the Laplace transform of $F(t)$, then

$$\lim_{t\to\infty} F(t) = \lim_{p\to 0} p\,f(p).$$

Now the Laplace transform of $\Omega(t,s)$ with respect to t is given by formula (4.13) as

$$\Omega^*(p,s) = \frac{e^{-s z} - s W^*(p,0)}{p - s + \alpha(s)}.$$

It follows that

$$\bar{\Omega}(s) = \lim_{p\to 0} \frac{p e^{-s z} - s\,p W^*(p,0)}{p - s + \alpha(s)}$$

$$= \frac{s \lim_{p\to 0} p W^*(p,0)}{s - \alpha(s)} \qquad (4.18)$$

It remains to calculate $\lim_{p\to 0} p W^*(p,0)$. But, from section 3, we know that

$$W^*(p,0) = \frac{e^{-\theta(p) z}}{\theta(p)},$$

and from the results of section 3 of Chapter 3, we know that in this case $\lim_{p \to 0} \theta(p) = 0$. It follows that

$$\lim_{p \to 0} \frac{\theta(p)}{p} = \theta'(0) = \frac{1}{1-\rho} .$$

Thus we have

$$\lim_{p \to 0} p W^*(p,0) = \frac{\lim_{p \to 0} e^{-\theta(p)z}}{\lim_{p \to 0} [\theta(p)/p]} = 1-\rho .$$

Replacing in (4.18), we finally obtain

$$\bar{\Omega}(s) = \frac{s(1-\rho)}{s-\alpha(s)} .$$

Corollary 1: When $\rho < 1$, the asymptotic distribution of $\zeta(t)$ is not defective.

Proof: Let $\bar{W}(x)$ be the asymptotic distribution of $\zeta(t)$. Then it suffices to prove that

$$\int_{-\infty}^{+\infty} d\bar{W}(x) = \lim_{s \to 0} \int_{-\infty}^{+\infty} e^{-sx} d\bar{W}(x) = \lim_{s \to 0} \bar{\Omega}(s) = 1$$

But

$$\lim_{s \to 0} \bar{\Omega}(s) = \lim_{s \to 0} \frac{1-\rho}{1-\frac{\alpha(s)}{s}} = \frac{1-\rho}{1-\alpha'(0)} = 1 .$$

Corollary 2: $\bar{W}(0) = 1-\rho$.

Proof: Here we shall use the Abelian theorem for Laplace transforms (c.f. Doetsch [14] p. 475) which states that if $\lim_{x \to 0+} F(x)$ exists, and if $f(s)$ is the Laplace transform of $F(x)$, then $\lim_{x \to 0+} F(x) = \lim_{s \to \infty} s f(s)$. The application of the theorem is valid here, as $\bar{W}(0) = \lim_{x \to 0+} \bar{W}(x)$, by the right-continuity of distribution functions. Using the formula

$$\overline{\Omega}(s) = s\int_0^{+\infty} e^{-sx}\,\overline{W}(x)\,dx,$$

we conclude that

$$\overline{W}(0) = \lim_{s\to+\infty} \frac{1-\rho}{1-\frac{\alpha(s)}{s}} = 1-\rho.$$

as $|\alpha(s)|$ is bounded for $\mathrm{Re}(s) > 0$.

It remains to consider the case $\rho = 1$. In that case we have the following result

Theorem 4.3: If $\rho = 1$, then, as when $\rho > 1$,

$$\lim_{t\to\infty} W(t,x) = 0 \qquad \text{for every } x.$$

Proof: We obviously have

$$w(t,x) \le P\{\nu(t)+\nu^*(t) \le x\},$$

so that, for all values of ρ,

$$\overline{\lim_{t\to\infty}}\, W(t,x) \le \lim_{t\to\infty} P\{\nu(t)+\nu^*(t) \le x\} = P\{\sup_{0\le u<\infty} \nu(u) \le x\} \quad (4.19)$$

Let us put

$$\zeta_0(t) = \nu(t) + \nu^*(t).$$

$\zeta_0(t)$ is the content of the store when $z = 0$.

Using the same argument as in theorem 4.1, we see that the distribution of $\zeta_0(t)$ tends to a limit $\overline{W_0}(x)$, whose Laplace transform $\overline{\Omega}_0(s)$ will be given by

$$\overline{\Omega}_0(s) = \lim_{p\to 0} \frac{p[1 - sW^*(p,0)]}{p - s + \alpha(s)}.$$

When $\rho = 1$, we have still, as shown in Chapter 3, Section 3, $\lim_{p \to 0} \theta(p) = 0$, so that here again

$$\lim_{p \to 0} p\, W^*(p, 0) = 1 - \rho .$$

But, as $\rho = 1$, we have

$$\bar{\Omega}_0(s) = 0 \qquad \text{for all } s .$$

It follows from the unicity of the Laplace-Stieltjes transform that

$$\bar{W}_0(x) = 0 .$$

Finally, we conclude, from (4.19) that for any z, if $\rho = 1$,

$$\lim_{t \to \infty} W(t, x) = 0$$

for all x.

This completes the proof of the theorem.

We can now summarize the asymptotic behaviour of $\zeta(t)$ as t tends to infinity in the case of a stationary input in the following way:

(a) If $\rho < 1$, $W(t, x)$ tends (weakly) to a limit distribution $\bar{W}(x)$, which is independent of z, and whose Laplace-Stieltjes transform is given by

$$\bar{\Omega}(s) = \frac{s(1-\rho)}{s - \alpha(s)} .$$

(b) If $\rho \geq 1$, $W(t, x)$ tends to zero for all x. However, if $\rho = 1$, the first passage time $\tau(z)$ is a proper random variable, so that the store becomes empty with probability one in finite time, although the mean value of $\tau(z)$ is infinite. If $\rho > 1$, $\tau(z)$ is a defective random variable, and there is a probability of $1 - e^{-\theta^* z}$ of the store not becoming empty, where θ^* is the positive root of $\alpha(\theta) = \theta$.

Note: From the Pollaczek - Khintchine formula, we can easily obtain the moments of the limiting distribution $\overline{W}(x)$. Here we shall only give the formula for the mean of $\overline{W}(x)$. We use the fact that

$$\alpha(s) = \rho s - \frac{\sigma^2}{2} s^2 + O(s^3),$$

so that

$$\beta(s) = \frac{\alpha(s)}{s} = \rho - \frac{\sigma^2}{2} s + O(s^2).$$

Now

$$\bar{\Omega}(s) = \frac{1-\rho}{1-\beta(s)},$$

so that

$$\bar{\Omega}'(a) = -\frac{(1-\rho)\beta'(0)}{[1-\beta(0)]^2}$$

$$= -\frac{(1-\rho)\sigma^2}{2(1-\rho)^2}$$

It follows that

$$\int_0^\infty x \, d\overline{W}(x) = -\bar{\Omega}'(0) = \frac{\sigma^2}{2(1-\rho)}.$$

5. <u>The inversion of the Pollaczek-Khintchine formula.</u>

Let us assume that $\int_{-\infty}^{+\infty} e^{-sx} d\overline{W}(x)$ converges for some negative value of s. This will be so if $\overline{\Omega}(s)$ is analytic at the origin, by Theorem 5b of Widder [74] p. 58. Equivalently $\beta(s) = \int_0^\infty e^{-sx} M(x)\, dx$ must be analytic at the origin. In the case of the Compound Poisson process, it suffices to assume that $\psi(s)$ is analytic at the origin. We then note that if $\alpha'(0) = \rho < 1$, there is a real $c > 0$, such that for $-c \leq s < 0$, we have $\alpha(s) > s$, and the above integral converges. It follows that, for s in the given range,

$$\lim_{x \to \infty} \int_x^{+\infty} e^{-sx}\, d\overline{W}(x) = 0,$$

so that

$$\lim_{x \to \infty} e^{-sx}\left[1 - \overline{W}(x)\right] \leq \lim_{x \to \infty} \int_x^{+\infty} e^{-sx}\, d\overline{W}(x) = 0.$$

Thus $1 - \overline{W}(x)$ is $o(e^{-cx})$ as $x \to \infty$. Similarly, using the analyticity of $\Theta(t, s) = \exp\left[-\alpha(s)\, t\right]$ for $Re(s) > -c$, we find that $1 - K(t, x)$ is $o(e^{-cx})$ as $x \to \infty$.

Integrating by parts we find

$$\frac{1}{s}\int_{-\infty}^{+\infty} e^{-sx} d\overline{W}(x) = -\frac{1}{s}\, e^{-sx}\left[1 - \overline{W}(x)\right]\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-sx}\left[1 - \overline{W}(x)\right] dx$$

$$= -\int_{-\infty}^{+\infty} e^{-sx}\left[1 - \overline{W}(x)\right] dx. \tag{4.19}$$

Integrating again by parts, we find

$$-\int_{-\infty}^{+\infty} e^{-sx}[1-\overline{w}(x)]\,dx = e^{-sx}\int_{x}^{+\infty}[1-\overline{w}(x)]dx\Big|_{-\infty}^{+\infty} + s\int_{-\infty}^{+\infty} e^{-sx}\Big[\int_{x}^{+\infty}[1-\overline{w}(y)]\,dy\Big]\,dx$$

$$= s\int_{-\infty}^{+\infty} e^{-sx}\Big[\int_{x}^{+\infty}[1-\overline{w}(y)]\,dy\Big]\,dx,$$

as $e^{-sx}\int_{x}^{+\infty}[1-\overline{w}(y)]\,dy \leq e^{-sx}\int_{x}^{\infty} e^{-cx}dx = \dfrac{e^{-(c+s)x}}{c}$,

which tends to zero as x tends to infinity. Thus

$$\frac{1}{s}\int_{-\infty}^{+\infty} e^{-sx}\,d\overline{w}(x) = s\int_{-\infty}^{+\infty} e^{-sx}\Big[\int_{x}^{+\infty}[1-\overline{w}(y)]\,dy\Big]\,dx.$$

On the other hand, in a similar way to (4.19),

$$\frac{1-\rho}{s-\alpha(s)} = -(1-\rho)\int_{0}^{\infty} e^{-[\alpha(s)-s]t}\,dt$$

$$=(1-\rho)\int_{0}^{\infty}\Big\{s\int_{-\infty}^{+\infty} e^{-sx}[1-K(t,t+x)]\,dx\Big\}\,dt.$$

Interchanging the order of integration, we find

$$\frac{1-\rho}{s-\alpha(s)} = (1-\rho)\,s\int_{-\infty}^{+\infty} e^{-sx}\Big\{\int_{0}^{\infty}[1-K(t,t+x)]\,dt\Big\}\,dx.$$

Writing now the Pollazcek-Khintchine formula in the form

$$\frac{1}{s}\int_{-\infty}^{+\infty} e^{-sx}\,d\overline{w}(x) = \frac{1-\rho}{s-\alpha(s)},$$

We obtain, using the above results and the uniqueness theorem for Laplace transforms,

$$\int_{x}^{+\infty}[1-\overline{w}(y)]\,dy = (1-\rho)\int_{0}^{\infty}[1-K(t,t+x)]\,dt \tag{4.20}$$

Applying right-hand differentiation to both sides, we obtain

$$-\big[1-\overline{w}(x)\big] = (1-\rho)\Big(\frac{\partial}{\partial x}\Big)^{+}\int_{0}^{\infty}[1-K(t,t+x)]\,dt,$$

so that finally

$$\overline{w}(x) = 1 + (1-\rho)\left(\frac{\partial}{\partial x}\right)^{+}\int_0^{\infty}[1 - K(t, t+x)]\,dt. \tag{4.21}$$

In this form, the formula can be easily seen to be the limiting form of equation (4.4) as t tends to infinity. However, it does not seem possible to obtain it directly from (4.4) without using the argument of section 5.

Following a procedure similar to that of section 5, we can also obtain two special forms of equation (4.21).

When $K(t,x)$ is of the form

$$K(t,x) = K(t,0) + \int_0^x k(t,y)\,dy,$$

equation (4.21) becomes

$$\overline{w}(x) = 1 - (1-\rho)\int_0^{\infty} k(t, t+x)\,dt \tag{4.22}$$

On the other hand, when $\xi(t)$ is discrete, we obtain

$$\overline{w}(x) = 1 - (1-\rho)\sum_{n=[x]+1}^{\infty} p_n(n-x). \tag{4.23}$$

Finally we can obtain the result corresponding to equation (4.7) as follows: We use again the identity

$$e^{st_{i+1}} - e^{st_i} = s\, e^{st_i'}(t_{i+1} - t_i),$$

where $t_i < t_i' < t_{i+1}$. We then have

$$-s\int_{-\infty}^{+\infty} e^{-sx}[1-\overline{w}(x)]\,dx = (1-\rho)\, s\int_0^{\infty} e^{-[\alpha(s)-s]t}\,dt$$

$$= (1-\rho)\, s \lim_{n\to\infty}\sum_{i=0}^{n-1} e^{st_i'}\, e^{-\alpha(s)t_i'}(t_{i+1}-t_i),$$

$$= (1-\rho) \lim_{n\to\infty} \sum_{i=0}^{n-1} \left[e^{st_{i+1} - \alpha(s)t_i'} - e^{st_i - \alpha(s)t_i'} \right]$$

$$= s\int_{-\infty}^{+\infty} e^{-sx}(1-\rho)\lim_{n\to\infty}\sum_{i=0}^{n-1}\left[K(t_i', t_{i+1}+x) - K(t_i', t_i+x)\right]dx.$$

We deduce that

$$\overline{W}(x) = 1-(1-\rho)\int_0^{\infty} d_t' \, K(t, t+x), \qquad (4.24)$$

where the integral is to be interpreted as

$$\int_0^{\infty} d_t' \, K(t, t+x) = \lim_{n\to\infty}\sum_{i=0}^{n-1}\left[K(t_i', t_{i+1}+x) - K(t_i', t_i+x)\right].$$

Another inversion formula can be obtained as follows:

Let us write

$$\overline{\Omega}(s) = \frac{(1-\rho)s}{s-\alpha(s)} = \sum_{n=0}^{\infty} (1-\rho)\rho^{n}\left[\frac{\alpha(s)}{\rho s}\right]^{n}. \qquad (4.25)$$

Now

$$\frac{\alpha(s)}{\rho s} = \int_0^{\infty} e^{-sx}\left[\frac{M(x)}{\rho}\right]dx,$$

and

$$\int_0^{\infty} \frac{M(x)}{\rho}\,dx = \lim_{s\to 0}\frac{\alpha(s)}{\rho s} = \frac{\alpha'(0)}{\rho} = 1.$$

Thus $M(x)/\rho$ is the density function of a non-negative random variable φ, the Laplace-Stieltjes transform of whose distribution is $\alpha(s)/\rho s$. It follows that $[\alpha(s)/\rho s]^n$ is the Laplace-Stieltjes transform of the distribution of the sum of n independent random variables, each distributed like φ. Formula (4.25) can then be interpreted as follows: The limiting distribution $\overline{W}(x)$ is the distribution of the sum of a random number N of independent, identically distributed random variables φ , with density function $M(x)/\rho$, where N has a geometric distribution of parameter ρ , i.e.

$$P\{N=n\}=(1-\rho)\rho^{n}.$$

6. The asymptotic behaviour of $\overline{W}(x)$.

Let $\overline{\zeta}$ be the random variable corresponding to $\overline{W}(x)$. The Pollaczek-Khintchine formula can be written

$$E\left[e^{-s\overline{\zeta}}\right]=\frac{s(1-\rho)}{s-\alpha(s)}\,.$$

It follows that

$$E\left[e^{-s(1-\rho)\overline{\zeta}}\right]=\frac{s(1-\rho)^{2}}{s(1-\rho)-\alpha\left[s(1-\rho)\right]}\,. \qquad (4.25)$$

Let us again use the result, quoted in section 3 of Chapter 3,

$$\alpha(s)=\rho s-\frac{\sigma^{2}s^{2}}{2}+o(s^{2}),$$

as s tends to zero.

Formula (4.25) now becomes

$$E\left[e^{-s(1-\rho)\overline{\zeta}}\right]=\frac{s(1-\rho)^{2}}{s(1-\rho)-\rho(1-\rho)s+\frac{\sigma^{2}}{2}(1-\rho)^{2}s^{2}+o\left[(1-\rho)^{2}s^{2}\right]},$$

so that, letting $\rho\to1$, we find

$$\lim_{\rho\to1}E\left[e^{-s(1-\rho)\overline{\zeta}}\right]=\frac{1}{1+\frac{\sigma^{2}}{2}s}=\int_{0}^{\infty}e^{-sx}\,\frac{2}{\sigma^{2}}\,e^{-\frac{2x}{\sigma^{2}}}\,dx\,.$$

We now use again the Lemma which was proved in section 3 of Chapter 3, and conclude that the distribution of the random variable $(1-\rho)\overline{\zeta}$ tends, as ρ tends to one, to a limit distribution. Thus we have

<u>Theorem 4.4</u>: The distribution of the random variable tends, as ρ tends to one, to an absolutely continuous limit distribution whose density function is $2e^{-\frac{2x}{\sigma^2}}/\sigma^2$.

We shall now obtain an asymptotic result relating to the behaviour of $\overline{W}(x)$ for large x which holds for all $\rho < 1$.

We shall assume that $\overline{W}(x_1)$ can be written in the form

$$\overline{W}(x) = (1-\rho) + \int_0^x \overline{\omega}(y)\,dy .$$

Then

$$\int_{0-}^{\infty} e^{-sx}\,d\overline{W}(x) = (1-\rho) + \int_0^{\infty} e^{-sx}\,\overline{\omega}(x)\,dx .$$

It follows that

$$\int_0^{\infty} e^{-sx}\,\overline{\omega}(x)\,dx = \frac{(1-\rho)s}{s-\alpha(s)} - (1-\rho) = (1-\rho)\frac{\alpha(s)}{s-\alpha(s)} .$$

We now note that the equation $s - \alpha(s) = 0$ has no roots in $\mathrm{Re}(s) > 0$. The root with largest real part is zero. However we have

$$\lim_{s\to 0} \frac{\alpha(s)}{s-\alpha(s)} = \frac{\alpha'(0)}{1-\alpha'(0)} = \frac{\rho}{1-\rho} ,$$

so that $\alpha(s)/[s-\alpha(s)]$ is analytic in $\mathrm{Re}(s) > 0$, and is regular at $s = 0$. To obtain the asymptotic behaviour of $\overline{\omega}(x)$, we shall make use of a theorem of Doetsch [14] p. 488, which reads as follows:

Let $f(s)$ be analytic in the strip $\sigma_1 \leq \mathrm{Re}(s) \leq c$ except at the point $s_0 = \sigma_0 + i\omega_0$, where the function $f(s)$ has an isolated singularity. Let the principal part of the Laurent expansion of $f(s)$ about s_0 be

$$\frac{a_1}{s - s_0} + \dots + \frac{a_k}{(s - s_0)^k} + \dots$$

Let $f(\sigma + i\omega) \to 0$ uniformly with respect to σ in the strip when $\omega \to \pm\infty$. Let the integral

$$\int_{-\infty}^{+\infty} e^{ix\omega} f(\sigma_0 + i\omega)\, d\omega$$

be uniformly convergent for $x \geq X > 0$. Then the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xs} f(s)\, ds = F(x)$$

is uniformly convergent for $x \geq X$, and we have the asymptotic formula

$$F(x) = e^{s_0 x}\left(a_1 + \frac{a_2}{1!}x + \dots + \frac{a_k}{(k-1)!}x^{k-1} + \dots\right) + o\left(e^{\sigma_1 t}\right),$$

as $x \to +\infty$.

We can make use of this theorem as follows:

First we note that if $\varpi(x)$ is continuous at x and of bounded variation in some neighbourhood of x, then

$$\varpi(x) = \text{P.V.}\int_{c-i\infty}^{c+i\infty} e^{xs}(1-\rho)\frac{\alpha(s)}{s - \alpha(s)}\, ds, \quad \text{for } c \geq 0.$$

If we now assume that the equation $s - \alpha(s) = 0$ has, in the half plane $\mathrm{Re}(s) \geq -\sigma$, only the simple root $-\sigma_0 < 0$ apart from the obvious root $s = 0$, and if $(1-\rho)\alpha(s)/[s - \alpha(s)]$

DIAGRAM

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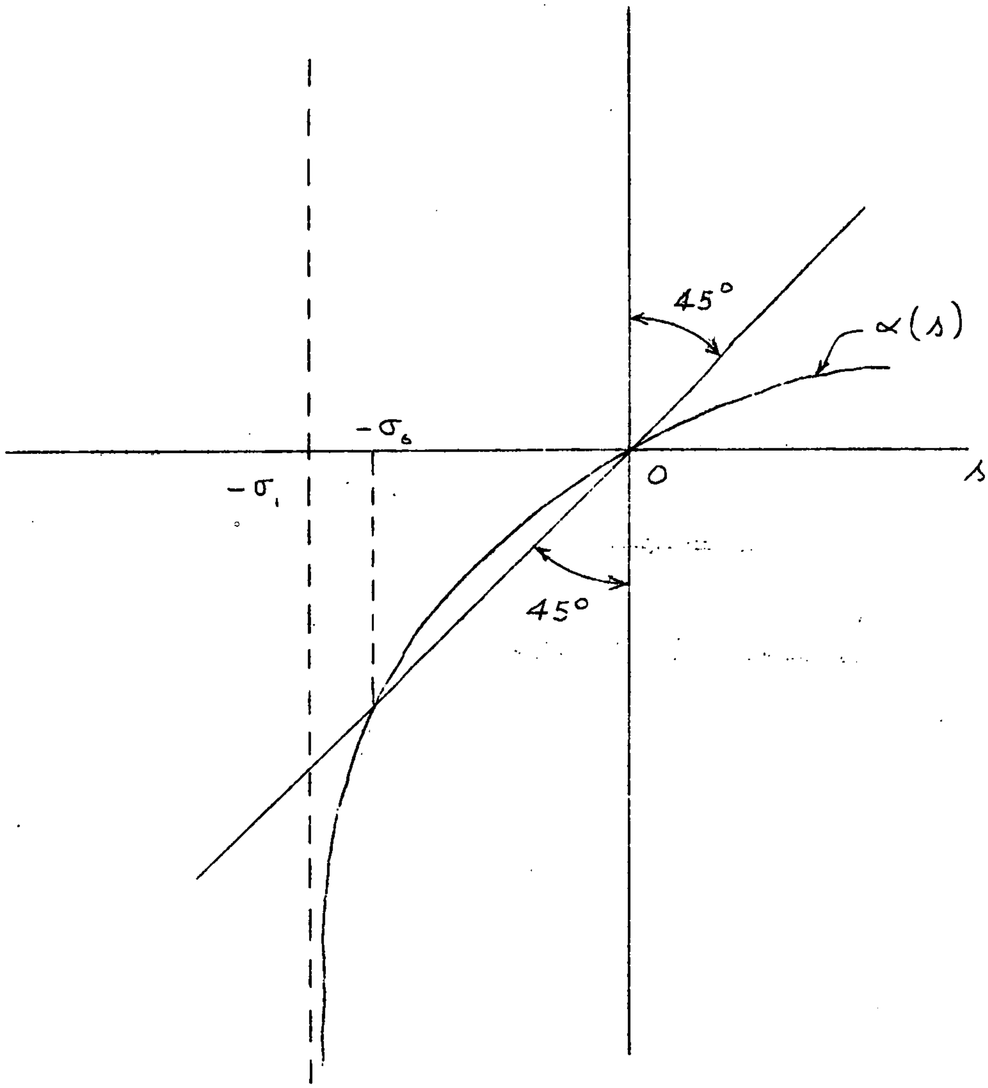

Figure 4.2

To face p. 129

satisfies the conditions laid down in the theorem quoted above, we shall have the asymptotic relation

$$\bar{\omega}(x) = a_1 e^{-\sigma_0 x} + o\left(e^{-\sigma_1 x}\right)$$

where a_1 is the residue of $(1-\rho)\alpha(s)/[s - \alpha(s)]$ at $-\sigma_0$. But this residue is obviously equal to

$$(1-\rho)\frac{\alpha(-\sigma_0)}{1-\alpha'(-\sigma_0)},$$

so that finally, remembering that $\alpha(-\sigma_0) = -\sigma_0$, we have the formula

$$\bar{\omega}(x) \sim (1-\rho)\frac{\sigma_0 e^{-\sigma_0 x}}{1-\alpha'(-\sigma_0)}. \qquad (4.26)$$

It remains to make plausible the assumption that $s - \alpha(s)$ has, apart from zero, a negative root in some interval to the left of the origin.

We first note that, as $\rho < 1$, the slope of $\alpha(s)$ at the origin is less than one. Also $\alpha(s)$ is of the form

$$\alpha(s) = s\int_0^\infty e^{-sx} M(x)\, dx,$$

so that it is plausible that for some negative value of s, say $-\sigma_1$, $\alpha(s)$ will tend to $-\infty$. Then there must be a root of $s - \alpha(s) = 0$ in the interval $(-\sigma_1, 0)$, as shown in figure (4.1).

CHAPTER 5

SOME EXAMPLES

1. The content distribution for various initial contents.

Let us denote by $W_z(t,x)$ the distribution of the content $\zeta(t)$ at time t when the initial content is z, i.e.

$$W_z(t,x) = P\{\zeta(t) \le x \mid \zeta(0) = z\}. \tag{5.1}$$

We shall only consider the case of a stationary input. By enumeration of the paths, we get the relation

$$W_z(t,0) = \int_{z-}^{t} W_0(t-u,0)\, d_u G(u,z). \tag{5.2}$$

Thus, it is necessary to calculate only $W_0(t,0)$, as $W_z(t,0)$ can then be calculated by a quadrature if $G(t,z)$ is known, and we know that explicit expressions for $G(t,z)$ are in fact available, as shown in Chapter 3.

Having calculated $W_z(t,0)$, we can easily calculate $W_z(t,x)$ by using the formula

$$W_z(t,x) = K(t,\, x+t-z) - \int_0^t W_z(u,0)\, d'_u K(t-u,\, x+t-u) \tag{5.3}$$

2. <u>The waiting time for the queue M/M/1.</u>

We shall obtain $W_0(t, 0)$ for the queue M/M/1 in this section by inverting $W_0^*(p, 0)$. As was shown in Chapter 4,

$$W_0^*(p, 0) = \left.\frac{e^{-\theta(p)z}}{\theta(p)}\right|_{z=0} = \frac{1}{\theta(p)},$$

and

$$\theta(p) = \frac{1}{2}(\lambda - \mu + p) + \frac{1}{2}\sqrt{(p + \lambda + \mu)^2 - 4\lambda\mu}.$$

We shall write $\sqrt{\frac{\mu}{\lambda}} = a$, $2\sqrt{\lambda\mu} = b$, $\lambda + \mu = c$.

Then we can write

$$\theta(p) = \frac{1}{2}\left[-ab + (p + c) + \sqrt{(p + c)^2 - b^2}\right]$$

$$= \frac{1}{2}(-ab + R),$$

where

$$R = p + c + \sqrt{(p + c)^2 - b^2}$$

Using Erdelyi [19] p. 237, formula 49, we see that the inverse Laplace transform of R^{-n} is

$$e^{-ct} \cdot n b^{-n} t^{-1} I_n(bt).$$

Now

$$W_o^*(p,o) = \frac{1}{\theta(p)} = \frac{2}{-ab+R} = \frac{2}{R} \cdot \frac{1}{1-\frac{ab}{R}},$$

$$= \frac{2}{R}\sum_{n=0}^{\infty}\left(\frac{ab}{R}\right)^n = 2\sum_{n=1}^{\infty}\frac{(ab)^{n-1}}{R^n}.$$

Inverting, we find

$$W_o(t,o) = e^{-ct}\sum_{n=1}^{\infty} a^{n-1}\left[\frac{2n}{bt} I_n(bt)\right]. \qquad (5.4)$$

We now use the formula

$$\frac{2n}{bt} I_n(bt) = I_{n-1}(bt) - I_{n+1}(bt).$$

Then (5.4) becomes

$$W_o(t,o) = e^{-ct}\sum_{n=1}^{\infty} a^{n-1}\left[I_{n-1}(bt) - I_{n+1}(bt)\right],$$

or

$$W_o(t,o) = e^{-ct}\left[I_0 + aI_1 + \left(1-\frac{1}{a^2}\right)\sum_{n=2}^{\infty} a^n I_n\right], \qquad (5.5)$$

where I_n stands for $I_n(bt)$. We shall write this in the form

$$W_o(t,o) = e^{-ct}\sum_{n=0}^{\infty} a_n I_n(bt),$$

where $a_0 = 1$; $a_1 = a$; $a_n = \left(1-\frac{1}{a^2}\right)a^n, \quad n > 1$.

We can now calculate $W_o(t,x)$, using the formula (see

Section 2 of Chapter 4)

$$W_0(t,x) = K(t, x+t) + \int_0^t W_0(t-u,0)\, k(u,u+x)\, du, \quad (5.6)$$

Replacing $K(t,x)$, $W_0(t,0)$, $k(t,x)$ by their values,

we find

$$W_0(t,x) = e^{-ct}\left\{ e^{-\mu x} I_0\left[b\sqrt{t(x+t)}\right] + \mu\, e^{\mu t} \int_0^{x+t} e^{-\mu y} I_0(by)\, dy \right\}$$
$$+ \sum_{n=0}^{\infty} a_n e^{-\mu x} \int_0^t e^{-cu} \frac{b}{2}\left(1+\frac{x}{u}\right)^{-\frac{1}{2}} I_n\left[b(t-u)\right] I_1\left[b\sqrt{u(x+u)}\right] du \quad (5.7)$$

A simpler formula is that for the mean value of the waiting

time. We have

$$E\left[\zeta(t)\right] = (\rho - 1)t + \int_0^t W_0(u,0)\, du,$$
$$= \left(\frac{1}{a^2} - 1\right) t + e^{-ct} \sum_{n=0}^{\infty} a_n \int_0^t I_n(bu)\, du.$$

Finally, the Laplace-Stieltjes transform of the limit distribution, when $\lambda < \mu$, is

$$\bar{\Omega}(s) = \frac{(1-\rho)s}{s - \alpha(s)} = \frac{(\mu-\lambda)(s+\mu)}{\mu(s+\mu-\lambda)}.$$

This can be written

$$\bar{\Omega}(s) = \left(1 - \frac{\lambda}{\mu}\right) + \left(1 - \frac{\lambda}{\mu}\right) \frac{\lambda}{s+\mu-\lambda}.$$

Inverting $\bar{\Omega}(s)$, we find

$$\bar{W}(x) = 1 - \frac{\lambda}{\mu} e^{-(\mu-\lambda)x}.$$

Thus the limit distribution of the waiting time, given that it is not zero, is of negative exponential type.

3. The content of the dam with simple Poisson input.

This model, like the preceding one, has both a queueing interpretation, namely the queue with Poisson arrivals and unit service time, and a dam interpretation, namely a dam with unit inputs which are fed in at instants of time which are Poisson distributed, and a release of one unit per unit time.

We shall first calculate $W_0(t,0)$. We have

$$W_0(t,0) = \sum_{n=0}^{[t]} \frac{t-n}{t} p_n(t)$$

$$= \sum_{n=0}^{[t]} \left(\frac{t-n}{t}\right) \frac{e^{-\lambda t}}{n!} (\lambda t)^n.$$

This can be written

$$W_0(t,0) = e^{-\lambda t}\left[\sum_{n=0}^{[t]} \frac{(\lambda t)^n}{n!} - \lambda \sum_{n=0}^{[t-1]} \frac{(\lambda t)^n}{n!}\right],$$

or, if we write $[t] = N$,

$$W_0(t,0) = e^{-\lambda t}\left[(1-\lambda)\sum_{n=0}^{N-1} \frac{(\lambda t)^n}{n!} + \frac{(\lambda t)^N}{N!}\right]. \qquad (5.8)$$

We can now calculate $W_0(t,x)$. We find, writing $[t+x] = N$, $[x] = M$,

$$W_0(t,x) = \sum_{n=0}^{N} p_n(t) - \sum_{n=M+1}^{N} p_n(n-x)\, W_0(t-n+x,0)$$

$$= e^{-\lambda t}\sum_{n=0}^{N}\frac{(\lambda t)^n}{n!} - \sum_{n=M+1}^{N} e^{-\lambda(n-x)} \frac{[\lambda(n-x)]^n}{n!}.$$

$$\left\{e^{-\lambda(t-n+x)}\left[(1-\lambda)\sum_{k=0}^{N-n-1} \frac{[\lambda(t-n+x)]^k}{k!} + \frac{[\lambda(t-n+x)]^{N-n}}{(N-n)!}\right]\right\},$$

that is,

$$W_0(t,x) = e^{-\lambda t}\sum_{n=0}^{N}\frac{(\lambda t)^n}{n!} - (1-\lambda)e^{-\lambda t}\sum_{n=M+1}^{N}\sum_{k=0}^{N-n-1}\frac{\lambda^{n+k}(n-x)^n(t-n-x)^k}{k!\,n!}$$

$$-\lambda^N e^{-\lambda t}\sum_{n=M+1}^{N}\frac{(n-x)^n(t-n-x)^{N-n}}{n!(N-n)!} \qquad (5.9)$$

Let us now calculate the mean value of the content. We have

$$E[\zeta(t)] = (\rho-1)t + \int_0^t W(u,0)\,du,$$

$$= (\rho-1)t + \int_0^t \sum_{n=0}^{[u]}\left(\frac{u-n}{u}\right)p_n(u)\,du,$$

$$= (\rho-1)t + \sum_{n=0}^{[t]}\int_n^t \left(\frac{u-n}{u}\right)p_n(u)\,du,$$

$$= (\rho-1)t + \sum_{n=0}^{[t]}\frac{\lambda^n}{n!}\left[\int_n^t e^{-\lambda u}u^n\,du - n\int_n^t e^{-\lambda u}u^{n-1}\,du\right],$$

so that

$$E[\zeta(t)] = (\rho-1)t + \sum_{n=0}^{[t]}\frac{1}{n!}\left[\frac{1}{\lambda}\left\{\gamma(n+1,\lambda t) - \gamma(n+1,\lambda n)\right\}\right.$$

$$\left. - n\left\{\gamma(n,\lambda t) - \gamma(n,\lambda n)\right\}\right], \qquad (5.10)$$

where $\gamma(a,x)$ is the incomplete Gamma function, given by

$$\gamma(a,x) = \int_0^x e^{-t}\,t^{a-1}\,dt.$$

Finally, the steady-state distribution is given by

$$\overline{W}(x) = 1 - (1-\rho)\sum_{n=[x]+1}^{\infty} p_n(n-x),$$

i.e.

$$\overline{W}(x) = 1 - (1-\lambda)e^{\lambda x}\sum_{n=[x]+1}^{\infty} e^{-\lambda n}\frac{[\lambda(n-x)]^n}{n!}. \qquad (5.11)$$

Noting that $\overline{W}(x) = 0$ for $x < 0$, we can rewrite (5.11) as

$$\overline{W}(x) = (1-\lambda)e^{\lambda x}\sum_{n=0}^{[x]} e^{-\lambda n}\frac{[\lambda(n-x)]^n}{n!}.$$

4. The content of the dam with Gamma input.

In Chapter 3, we have already obtained the distribution of the time of first emptiness for the dam with Gamma input. We found that $\tau(z)$ had a density function given by

$$g(t,z) = \frac{z}{\rho\,\Gamma(t+1)}\left(\frac{t-z}{\rho}\right)^{t-1} e^{-\frac{t-z}{\rho}}$$

We shall now calculate $W_0(t,0)$, using the formula (Chapter 4, Section 3)

$$W_0(t,0) = K(t,t) - \frac{1}{t}\int_0^t y\; k(t,y)\,dy\,.$$

We obtain

$$W_0(t,0) = \frac{\gamma\left(t,\frac{t}{\rho}\right)}{\Gamma(t)} - \frac{\rho\,\gamma\left(t+1,\frac{t}{\rho}\right)}{\Gamma(t+1)} \qquad (5.12)$$

where $\gamma(t,x)$ is the Incomplete Gamma function defined in Section 3.

Using now the recurrence formula

$$\gamma(t+1,x) = t\,\gamma(t,x) - x^t e^{-x},$$

we find

$$W_0(t,0) = (1-\rho)\frac{\gamma\left(t,\frac{t}{\rho}\right)}{\Gamma(t)} - \frac{1}{\Gamma(t+1)}\,e^{-\frac{t}{\rho}}\left(\frac{t}{\rho}\right)^t. \qquad (5.13)$$

From this we deduce the formula for $W_0(t,x)$:

$$W_0(t,x) = \frac{1}{\rho\Gamma(t)}\int_0^{t+x} e^{-\frac{y}{\rho}}\left(\frac{y}{\rho}\right)^{t-1}dy + (1-\rho)\int_0^t \frac{\gamma\left(t-u,\frac{t-u}{\rho}\right)}{\rho\,\Gamma(t-u)\,\Gamma(u)}\,e^{-\frac{u+x}{\rho}}\left(\frac{u+x}{\rho}\right)^{u-1}du$$

$$+\,e^{-\frac{t+x}{\rho}}\int_0^t \frac{1}{\rho\,\Gamma(t-u+1)\,\Gamma(u)}\left(\frac{t-u}{\rho}\right)^{t-u}\left(\frac{u+x}{\rho}\right)^{u-1}du\,. \qquad (5.14)$$

The mean value of the content is given by

$$E[\zeta(t)] = -(1-\rho)t + (1-\rho)\int_0^t \frac{\gamma\left(u,\frac{u}{\rho}\right)}{\Gamma(u)}\,du - \int_0^t \frac{e^{-\frac{u}{\rho}}}{\Gamma(u+1)}\left(\frac{u}{\rho}\right)^u du\,. \qquad (5.15)$$

Finally, the limiting distribution of the content is given by

$$\overline{W}(x) = 1 - \frac{(1-\rho)}{\rho}\int_0^\infty \frac{e^{-\frac{t+x}{\rho}}}{\Gamma(t)}\left(\frac{t+x}{\rho}\right)^{t-1} dt. \qquad (5.16)$$

5. The content of the dam with Inverse Gaussian input.

As mentioned in the introduction, this process was constructed by the author so as to have the following two properties:

(1) The characteristic equation of a store with this type of input should be a quadratic.

(2) The Laplace transform of $W_0(t,0)$, $\frac{1}{\theta(p)}$, should be invertible in closed form.

As shown in Section 5 of Chapter 3, formula (3.17), we have

$$\theta(p) = p - \frac{\rho^2}{\sigma^2}(1-\rho) + \sqrt{\frac{\rho^4}{\sigma^4}(1-\rho)^2 + 2\frac{\rho^3}{\sigma^2}p} .$$

Let us write

$$\frac{\rho^2}{\sigma^2}(1-\rho) = b ,$$

$$\frac{2\rho^3}{\sigma^2} = \lambda^2 .$$

Then

$$\theta(p) = p - b + \sqrt{b^2 + p\lambda^2}$$

We shall first obtain $g(t,z)$ by inverting $\Gamma(p,z) = e^{-\theta(p)z}$, using formula (1) p.245 of Erdelyi [19]. We find

$$g(t,z) = \frac{\lambda z (t-z)^{-\frac{3}{2}}}{2\sqrt{\pi}} \exp\left[bz - \frac{b^2}{\lambda^2}(t-z) - \frac{1}{4}\frac{\lambda^2 z^2}{(t-z)}\right] U(t-z). \quad (5.17)$$

Replacing λ and b by their values and rearranging, we obtain

$$g(t,z)=\begin{cases}\dfrac{z}{\sigma\sqrt{2\pi}}\left(\dfrac{\rho}{t-z}\right)^{\frac{3}{2}}\exp\left\{-\dfrac{\rho}{2\sigma^2(t-z)}\left[\rho t-(t-z)\right]^2\right\} & \text{if } t\geq z\\ 0 & \text{if } t<z,\end{cases}$$

thus establishing directly in this case the correctness of Kendall's formula $g(t,z)=\frac{z}{t}\,k(t,t-z)$.

We shall now obtain $W_0(t,0)$ by inverting $\dfrac{1}{\theta(p)}$.

We first write

$$\frac{1}{\theta(p)}=\frac{1}{p-b+\sqrt{b^2+\lambda^2 p}}=\frac{p-b-\sqrt{b^2+\lambda^2 p}}{p\,(p-2a\lambda)}$$

where $a=(2b+\lambda^2)/2\lambda=\sqrt{\rho}/\sigma\sqrt{2}$.

Then

$$\frac{1}{\theta(p)}=\frac{1}{2a}\left(\frac{a-\frac{\lambda}{2}}{p}+\frac{a+\frac{\lambda}{2}}{p-2a\lambda}\right)-\frac{1}{2a}\,\frac{\sqrt{p+\left(a-\frac{\lambda}{2}\right)^2}}{(p-2a\lambda)}+\frac{1}{2a}\,\frac{\sqrt{p+\left(a-\frac{\lambda}{2}\right)^2}}{p}.$$

This expression can now be inverted term by term, and rearranging the results, we find, using formula (22) p. 235 of Erdelyi [19],

$$W_0(t,0)=\left(\frac{2a-\lambda}{2a}\right)\left\{1-\frac{1}{2}\operatorname{Erfc}\left[\left(\frac{2a-\lambda}{2}\right)\sqrt{t}\right]\right\}+\left(\frac{2a+\lambda}{4a}\right)e^{2a\lambda t}\operatorname{Erfc}\left[\left(\frac{2a+\lambda}{2}\right)\sqrt{t}\right],$$

where $\operatorname{Erfc}(x)$ is defined by

$$\operatorname{Erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\,dy.$$

Reverting to ρ and σ, we find

$$W_0(t,0)=(1-\rho)\left\{1-\frac{1}{2}\operatorname{Erfc}\left[\left(\frac{1-\rho}{\sigma}\right)\sqrt{\frac{\rho t}{2}}\right]\right\}+\left(\frac{1+\rho}{2}\right)\operatorname{Erfc}\left[\left(\frac{1+\rho}{\sigma}\right)\sqrt{\frac{\rho t}{2}}\right].\quad(5.18)$$

From this expression, we can see immediately that, if $\rho < 1$, $\lim_{t\to\infty} W_0(t,0) = 1-\rho$, but that, if $\rho \geq 1$,

$$\lim_{t\to\infty} W_0(t,0) = 0 .$$

There is little point in writing out the formula for $W_0(t,x)$ for this model, as no simplifications ensue, but we can find explicit expressions for $E[\zeta(t)]$ and $\Omega_0(t,s)$.

We need the two formulae

$$\int_0^t e^{-\alpha u}\,\mathrm{Erfc}\,\beta\sqrt{u}\,du = \frac{\beta}{\alpha\sqrt{\beta^2+\alpha}}\left(1-\mathrm{Erfc}\sqrt{(\beta^2+\alpha)t}\right)+\frac{1}{\alpha}\left(1-e^{-\alpha t}\,\mathrm{Erfc}\,\beta\sqrt{t}\right)$$

$$\int_0^t \mathrm{Erfc}\,\beta\sqrt{u}\,du = \frac{1}{2\beta^2}-\frac{1}{\beta}\sqrt{\frac{t}{\pi}}\,e^{-\beta^2 t}+\left(t-\frac{1}{2\beta^2}\right)\mathrm{Erfc}\,\beta\sqrt{t},$$

which can easily be established by replacing $\mathrm{Erfc}\,\beta\sqrt{t}$ by its integral representation and interchanging integrals.

Using the formula

$$E[\zeta(t)] = (\rho-1)t + \int_0^t W_0(u,0)\,du,$$

and rearranging terms, we obtain

$$\begin{aligned} E[\zeta(t)] = {} & \frac{\sigma^2}{2(1-\rho)} + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\frac{t}{\rho}}\exp\left[-\frac{(1-\rho)^2\rho t}{2\sigma^2}\right] \\ & -\left[\left(\frac{1-\rho}{2}\right)t + \frac{\sigma^2\rho(1+\rho^2)}{4\rho^3(1-\rho)}\right]\mathrm{Erfc}\left[\left(\frac{1-\rho}{\sigma}\right)\sqrt{\frac{\rho t}{2}}\right] \\ & + \frac{\sigma^2(1+\rho)}{4\rho^2}\exp\left[\frac{2\rho^2 t}{\sigma^2}\right]\mathrm{Erfc}\left[\left(\frac{1+\rho}{\sigma}\right)\sqrt{\frac{\rho t}{2}}\right]. \end{aligned} \qquad (5.19)$$

Here again we see immediately that, if $\rho < 1$, then

$$\lim_{t\to\infty} E[\zeta(t)] = \frac{\sigma^2}{2(1-\rho)},$$

while if $\rho \geq 1$, $\lim_{t\to\infty} E[\zeta(t)] = +\infty$, as expected from the general theory.

Using now the formula

$$\Omega_0(t,s) = e^{[s-\alpha(s)]t} - s\int_0^t e^{[s-\alpha(s)](t-u)} W_0(u,0)\,du,$$

and writing

$$A = s - \alpha(s), \quad B = \left(\frac{1-\rho}{\sigma}\right)\sqrt{\frac{\rho}{2}}, \quad C = \left(\frac{1+\rho}{\sigma}\right)\sqrt{\frac{\rho}{2}},$$

we find

$$\Omega_0(t,s) = e^{At} + \frac{s(1-\rho)}{A}\left(1 - e^{At}\right)$$

$$- \frac{s(1-\rho)}{2} e^{At}\left[\frac{B}{A\sqrt{B^2+A}}\left(1 - \operatorname{Erfc}\sqrt{(B^2+A)t}\right) + \frac{1}{A}\left(1 - e^{-At}\operatorname{Erfc} B\sqrt{t}\right)\right]$$

$$+ \frac{s(1+\rho)}{2} e^{At}\left[\frac{C}{A\sqrt{C^2+A}}\left(1 - \operatorname{Erfc}\sqrt{(C^2+A)t}\right) + \frac{1}{A}\left(1 - e^{-At}\operatorname{Erfc} C\sqrt{t}\right)\right]. \tag{5.20}$$

It is easy to see that, for sufficiently small purely imaginary values of s, $\operatorname{Re}(A)$ is negative. In fact, as

$$\alpha(s) = \rho s - \frac{\sigma^2 s^2}{2} + O(s^3),$$

we have

$$A(i\omega) = (1-\rho)i\omega - \frac{\sigma^2\omega^2}{2} + O(\omega^3).$$

Letting now t tend to infinity in (5.20), we see that, for all A such that $\operatorname{Re}(A) < 0$, if $\rho < 1$,

$$\lim_{t\to\infty} \Omega_0(t,s) = \frac{s(1-\rho)}{s-\alpha(s)},$$

the Pollaczek-Khintchine formula.

Finally, we shall obtain an explicit formula for the density function $\overline{\omega}(x)$ of the limiting distribution, by inverting the Pollaczek-Khintchine formula. We note that

$$\frac{s(1-\rho)}{s-\alpha(s)} = (1-\rho)\left[1 + \frac{\alpha(s)}{s-\alpha(s)}\right].$$

Now $$\alpha(s) = \lambda\left(\sqrt{s+a^2} - a\right),$$

where λ and a are as defined above. Then

$$\frac{\alpha(s)}{s-\alpha(s)} = \frac{1}{[s/\alpha(s)]-1} = \frac{1}{[(\sqrt{s+a^2}+a)/\lambda]-1} = \lambda\frac{\sqrt{s+a^2}-(a-\lambda)}{s+2a\lambda-\lambda^2}$$

$$= \lambda\frac{\sqrt{s+a^2}}{s+(2a\lambda-\lambda^2)} - \frac{a\lambda-\lambda^2}{s+(2a\lambda-\lambda^2)},$$

and the last expression can be readily inverted, using Erdelyi [19] p. 235, formula (22), giving

$$\frac{\lambda}{\sqrt{\pi x}}e^{-a^2x} + \lambda(a-\lambda)e^{-(2a\lambda-\lambda^2)x}\,\mathrm{Erf}\left[|a-\lambda|\sqrt{x}\right]$$
$$- (a\lambda-\lambda^2)e^{-(2a\lambda-\lambda^2)x}.$$

Replacing λ and a by their values and rearranging terms, we finally find

$$\overline{W}(x) = (1-\rho) + \int_0^x \overline{\omega}(y)\,dy,$$

where

$$\overline{\omega}(x) = \frac{2\rho(1-\rho)}{\sigma}\sqrt{\frac{\rho}{2\pi x}}\,e^{-\frac{\rho x}{2\sigma^2}} - \frac{\rho^2}{\sigma^2}(1-\rho)(1-2\rho)\cdot$$
$$e^{-\frac{2\rho^2}{\sigma^2}(1-\rho)x}\,\mathrm{Erfc}\left[\frac{|1-2\rho|}{\sigma}\sqrt{\frac{\rho x}{2}}\right].$$

CHAPTER 6

THE INTEGRO-DIFFERENTIAL EQUATION OF TAKÁCS

1. Preliminary remarks.

The investigation of the store content distribution, $W(t,x)$, was first performed by Takács who wrote down an integro-differential equation for the process of the same general type as those obtained by Feller [20]. However, the derivation of the integro-differential equation necessitated certain assumptions about the continuity and differentiability of $W(t,x)$ as a function of both variables t and x. Takács himself [70] p. 108, hinted at a method for showing the continuity of $W(t,x)$, but did not pursue the matter further. In the subsequent literature,, the integro-differential equation technique has been used repeatedly, and the assumptions made justified by appeal to the general theory of Markov processes in continuous time as given e.g. by Doob [17] p. 261.

Unfortunately, the general theory cannot be applied to the case under consideration, because Doob's condition (2.1) p. 257 is not satisfied in this case. In fact, the processes to which the general theory applies have sample functions which are almost surely step functions, while the process $\zeta(t)$ has a slope of -1 whenever $\zeta(t) > 0$.

Moreover, it is easy to show that $W(t,x)$ need not be a continuous function of x for $x>0$, as stated by some authors. In fact, we have seen in Chapter 5 that, for a stationary input,

$$W(t,x)=K(t,t+x-z)-\left(\frac{\partial}{\partial x}\right)^{+}\int_{0}^{t} w(t-u,0)\,K(u,u+x)\,du\,.$$

If $t<z$, we have $w(t-u,0)=0$ for all u such that $0\leq u\leq t$, so that

$$W(t,x)=K(t,t+x-z) \tag{6.1}$$

Thus, if the input has a discrete distribution, $W(t,x)$ will also have a discrete distribution, at least for some values of t. Also, depending on the properties of $K(t,x)$, $\frac{\partial W}{\partial x}$ may not exist for some values of the pair (t,x).

To overcome these difficulties, the simplest way is to proceed as in Chapter 4, where an equation for the Laplace-Stieltjes transform $\Omega(t,s)$ of $W(t,x)$ was obtained directly from the properties of the sample functions. However, the problem of the continuity and differentiability of $W(t,x)$ still remains to be solved.

In this Chapter, we shall confine ourselves to Compound Poisson inputs, stationary or not. We shall obtain sufficient conditions for the continuity and differentiability of $W(t,x)$, and we shall show that, when these conditions are satisfied, $W(t,x)$ does in fact satisfy the Takács integro-differential equation.

2. Conditions for the continuity and differentiability of $W(t,x)$

We shall first obtain a difference equation for $W(t,x)$. We shall assume that the input $\xi(t)$ is a Compound Poisson process with density of arrivals $\lambda(t)$, and jumps χ of distribution $B(x)$. Then

Theorem 6.1. $W(t,x)$ satisfies, for $t \geq 0$, $x \geq 0$, $h > 0$, the difference equation

$$W(t+h,x) - W(t,x+h) = \lambda(t)h\left[V(t,x) - W(t,x)\right] + o(h), \qquad (6.2)$$

where

$$V(t,x) = \int_{0-}^{x+} B(x-y)\, d_y W(t,y) = \int_{0-}^{x+} W(t,x-y)\, dB(y).$$

Proof: Let A_i be the event of i arrivals in $(t, t+h)$, $i = 0, 1, 2, \ldots$

Then
$$P(A_0) = 1 - \lambda(t)h + o(h),$$

$$P(A_1) = \lambda(t)h + o(h),$$

$$1 - P(A_0) - P(A_1) = o(h).$$

We first consider the case of no arrivals in $(t, t+h)$.

Then
$$P\{\zeta(t+h) \leq x \mid A_0\} = W(t, x+h), \qquad x \geq 0,$$
since $\zeta(t+h) \leq x$ if and only if $\zeta(t) \leq x+h$ in this case.

Consider now the case of the event A_1, and let $\zeta(t) = y \leq x$. Then, using the obvious inequality

$$\zeta(t) + \chi - h \leq \zeta(t+h) \leq \zeta(t) + \chi,$$

we find

$$P\{\mathcal{X} \leq x-y\} \leq P\{\zeta(t+h) \leq x \mid A_1, \zeta(t)=y\} \leq P\{\mathcal{X} \leq x+h-y\}.$$

From this we deduce, using the theorem of total probability,

$$V(t,x) \leq P\{\zeta(t+h) \mid A_1\} \leq V(t, x+h).$$

This can be written, using the right continuity of $V(t,x)$ in x,

$$P\{\zeta(t+h) \leq x \mid A_1\} = V(t,x) + O(h).$$

Finally, using again the theorem of total probability, we find

$$W(t+h, x) = [1-\lambda(t)h]\,W(t, x+h) + \lambda(t)h\,V(t,x) + o(h).$$

Rearranging, we obtain (6.2) . This completes the proof of the theorem.

Writing now $t-h$ for t and $x-h$ for x, we obtain

$$W(t-h,x) - W(t,x-h) = -\lambda(t-h)h\left[V(t-h,x-h) - W(t-h,x-h)\right] + o(h). \quad (6.3)$$

This formula is valid for $h>0$, $x \geq h$, $t \geq h$.

<u>Lemma 6.1</u>:

$$W(t+0,x) = W(t,x), \quad (6.4)$$

$$W(t-0,x) = W(t,x-0). \quad (6.5)$$

<u>Proof</u>: Let h tend to zero in formulae (6.2) and (6.3), and use the right continuity of $W(t,x)$ in x .

Lemma 6.2 : $W(t,x)$ is Riemann-integrable in t .

Proof: It follows from Lemma 6.1 that all the discontinuities of $W(t,x)$ as a function of t are ordinary, and this implies Riemann integrability (see Hobson [36] p. 439).

Lemma 6.3 : $W(t-u, x+u)$ is a continuous function of u for $x \geq 0$, $t \geq 0$, $-x \leq u \leq t$.

Proof: In formula (6.2), replace t by $t-u$ and x by $x+u-h$. We obtain

$$W(t-u+h, x+u-h) - W(t-u, x+u)$$

$$= \lambda(t-u)\, h\left[V(t-u, x+u-h) - W(t-u, x+u-h)\right] + o(h),$$

for $-x+h \leq u \leq t$.

Similarly, writing $t-u-h$ for t and $x+u$ for x in (6.2), we obtain

$$W(t-u-h, x+u+h) - W(t-u, x+u)$$

$$= -\lambda(t-u)\, h\left[V(t-u-h, x+u) - W(t-u-h, x+u)\right] + o(h)$$

for $-x \leq u \leq t-h$.

The result follows by letting h tend to zero.

Lemma 6.4 : $V(t-u, x+u)$ is a continuous function of u .

Proof: This follows from the continuity of $W(t-u, x+u)$ in u by using the formula

$$V(t,x) = \int_{0-}^{x+} W(t, x-y)\, dB(y).$$

We note that this can be written

$$V(t,x) = \int_{-\infty}^{+\infty} W(t, x-y)\, d\,B(y),$$

as $W(t,x-y)=0$ for $y > x$ and $B(y)=0$ for $y<0$.
We then have

$$V(t-u-\ell, x+u-\ell) = \int_{-\infty}^{+\infty} W(t-u-\ell, x+u+\ell-y)\, dB(y). \qquad (6.6)$$

Using now Lebesgue's Dominated Convergence theorem, (see Loève [47] p. 125) we obtain the result.

<u>Lemma 6.5</u>: $W(t,x)$ satisfies the equation

$$W(t,x) = e^{-\Lambda(t)}\Big[U(x-z+t)$$

$$+ \int_0^t \lambda(t-u)\, e^{\Lambda(t-u)}\, V(t-u, x+u)\, du\Big] \qquad (6.7)$$

where $\Lambda(t) = \int_0^t \lambda(u)\,du$, $U(x)$ is the Heaviside unit function, and z is the initial content of the store.

<u>Proof:</u> We note that the event $\zeta(t) \le x$ can occur in two exhaustive and mutually exclusive ways:

(a) There is no arrival in $(0,t)$ and $z-t \le x$. The probability of this event is $e^{-\Lambda(t)}\, U(x-z+t)$.

(b) The last arrival occurs at $t-u$ and $\zeta(t-u)+Y-u \le x$. The probability of this event is

$$\int_0^t V(t-u, x+u)\,\lambda(t-u)\, e^{-\Lambda(t)+\Lambda(t-u)}\, du.$$

Adding these two probabilities, we obtain the result.

<u>Corollary</u>: $W(t,x)$ vanishes for $x+t<z$ and has a discontinuity of height $e^{-\Lambda(t)}$ as a function of either x or t at each point of the line $x+t=z$.

We are now in a position to discuss the continuity and differentiability of $W(t,x)$.

Theorem 6.2. If $B(x)$ is absolutely continuous, then $W(t,x)$ is a continuous function of both t and x in the region $x+t \geq z$.

Proof: We first notice that if $B(x)$ is absolutely continuous, $V(t,x)$ is continuous in x for all t and x. This follows from standard properties of the convolution operation (see Lukács [49] p. 45).

Let now

$$I(t,x) = \int_0^t V(t-u, x+u)\lambda(t-u)\, e^{\Lambda(t-u)}\, du.$$

Then, for $x+t \geq z$,

$$W(t,x) = e^{-\Lambda(t)}\left[1 + I(t,x)\right], \tag{6.8}$$

and $$I(t, x+h) = \int_0^t V(t-u, x+u+h)\lambda(t-u)\, e^{\Lambda(t-u)}\, du.$$

The integrand obviously satisfies the conditions of Lebesgue's Dominated Convergence theorem, and we conclude that $\lim_{h\to 0} I(t,x+h) = I(t,x)$. $I(t,x)$ is therefore continuous in x. The continuity of $W(t,x)$ in x follows from (6.8), and its continuity in t from Lemma 6.1.

Theorem 6.3. If $B(x)$ has a bounded derivative for all x, then $W(t,x)$ has a bounded derivative in both t and x for $t+x \geq z$, and satisfies the integro-differential equation

$$\frac{\partial w(t,x)}{\partial t} = \frac{\partial w(t,x)}{\partial x} + \lambda(t)\left[V(t,x) - w(t,x)\right].$$

Proof: Applying Lebesgue's Dominated Convergence theorem successively to

$$\frac{V(t,x+h)-V(t,x)}{h} = \int_{-\infty}^{+\infty} \frac{B(x+h-y)-B(x-y)}{h}\, d_y w(t,y),$$

and

$$\frac{I(t,x+h)-I(t,x)}{h} = \int_0^t \frac{V(t-u,x+u+h)-V(t-u,x+u)}{h}\lambda(t-u)e^{\Lambda(t-u)}\,du,$$

we conclude that first $V(t,x)$ and then $I(t,x)$ have bounded derivatives in x. The existence of $\frac{\partial w}{\partial x}$ in $t+x \geq z$ then follows from (6.8).

From formulae (6.2) and (6.3) we now deduce, for $h>0$,

$$\frac{w(t+h,x)-w(t,x)}{h} = \frac{w(t,x+h)-w(t,x)}{h} + \lambda(t)\left[V(t,x)-w(t,x)\right] + O(h),$$

$$\frac{w(t-h,x)-w(t,x)}{(-h)} = \frac{w(t,x-h)-w(t,x)}{(-h)} + \lambda(t-h)\left[V(t-h,x-h)-w(t-h,x-h)\right] + O(h).$$

Letting $h \downarrow 0$, we finally find

$$\lim_{h\downarrow 0}\frac{w(t+h,x)-w(t,x)}{h} = \lim_{h\downarrow 0}\frac{w(t-h,x)-w(t,x)}{(-h)}$$

$$= \frac{\partial w(t,x)}{\partial x} + \lambda(t)\left[V(t,x)-w(t,x)\right],$$

the continuity of $V(t,x)$ in t following from the application of Lebesgue's Dominated Convergence theorem to

$$v(t+\ell, x) = \int_{-\infty}^{+\infty} w(t+\ell, x-y)\, dB(y).$$

This completes the proof of the theorem.

CHAPTER 7

THE CASE OF A NON-STATIONARY COMPOUND POISSON INPUT

1. Introduction.

In Chapter 4, a complete solution was given to the problem of the determination of the store content distribution $W(t,x)$ in terms of the input distribution $K(t,x)$, when the input is stationary. It was also shown that, in that case, $W(t,x)$ tends weakly, when $t \to \infty$, to a limit distribution.

Such a complete solution is not available for the case of a non-stationary input. In this case, the probability of emptiness, $W(t,0)$, satisfies a Volterra integral equation of the second kind. This equation can be obtained by putting $x=0$ in equation (4.5) of Chapter 4. We find that

$$W(t,0) = K(0,t;t-\zeta) - \int_0^t k(u,t;t-u)\,W(u,0)\,du \; . \tag{7.1}$$

However, this equation is not suitable for obtaining an explicit method of solution for $W(t,0)$. Instead, we shall obtain a Volterra integral equation of the <u>first</u> kind directly from equation (4.3) of Chapter 4, which we shall solve by Laplace transforms. Because of analytic difficulties, we shall confine ourselves to Compound Poisson inputs, and we shall put restrictions on $\lambda(t)$ and the jump distribution $B(x)$. Particular attention will be given to the case where the density of arrivals $\lambda(t)$ is a periodic function of time. The main result of the investigation is that the probability of emptiness $W(t,0)$ and the Laplace transform of the content distribution $\Omega(t,s)$ are then both asymptotically periodic in t. We shall assume that $\lambda(t)$ is a continuous function of t and we shall write

$$\Lambda(t) = \int_0^t \lambda(u)\,du = t - \omega\,h(t),$$

where $h(t)$ is a periodic function of time whose mean value is zero. It will then be shown that both $W(t,0)$ and $\Omega(t,s)$ can be expanded in a power series in ω, and a method for calculating explicitly the asymptotic values of the leading terms will be obtained. Various mathematical results needed in the investigation will be proved in Section 4.

Throughout this Chapter, we shall say that a function $f(t)$, $t \geq 0$, is an L^2 function, or belongs to the L^2 class, if $\int_0^\infty e^{-2pt}|f(t)|^2\,dt$ converges for some real value p_0 of p. It is well known that this implies the absolute convergence of $\int_0^\infty e^{-pt} f(t)\,dt$, $\int_0^\infty e^{-2pt}[f(t)]^2 dt$ for all p such that $\mathrm{Re}(p) \geq p_0$.

We shall assume that the jump distribution $B(x) = P\{X \leq x\}$ is absolutely continuous and has a density function $b(x)$. Moreover, we shall assume that $b(x)$ is an L^2 function, and we shall write

$$Q(s) = \int_0^\infty e^{-2sx}\,[b(x)]^2\,dx.$$

$Q(0)$ will be assumed to be finite, i.e. we shall assume that $\int_0^\infty [\ell(x)]^2 dx < \infty$. We shall also assume that the Laplace-Stieltjes transform of $B(x)$, $\psi(s)$, is analytic in $Re(s) > \sigma_0$, where σ_0 is some <u>negative</u> number, and is $O(1/|s|)$ in that region. Finally, let us note that, when the mean value of $h(t)$ is zero, the mean density of arrivals is unity, so that the mean input per unit time will be $\rho = -\psi'(0)$. It is intuitively clear that in this case the content will tend to infinity when $t \to \infty$ unless we assume $\rho < 1$. This is the assumption that we shall make throughout this chapter.

If $\omega = 0$, then $\Lambda(t) = t$, $\lambda(t) = 1$, and the input process is stationary. In this case, the characteristic equation of the process will be $p - s + 1 - \psi(s) = 0$. We shall write $\gamma(s) = s - 1 + \psi(s)$, so that the characteristic equation will be $p - \gamma(s) = 0$. As we have assumed that $\psi(s)$ is analytic at the origin, and that $\rho < 1$, it follows from Theorem 3.5 of Chapter 3 that there exists a real number $\alpha < 0$ such that $\gamma(\alpha) < 0$, and moreover that $\gamma(s) - p$ has only one zero, $\theta(p)$, in $Re(s) > \alpha$ for any p such that $Re(p) > \gamma(\alpha)$. We shall make an extensive use of this result in the sequel.

2. Remarks on the Poisson process with periodic parameter.

In many practical storage problems, it is expected that the probability of arrivals will vary periodically. For example, in the queueing realisation of the storage model, e.g. in restaurants or at service stations, arrivals are more probable at rush hours than at slack periods, and rush hours are repeated day after day. In the dam realisation, it is expected that the rate of flow of the water into the dam will vary periodically.

A theoretical model for Poisson arrivals with periodic density of arrivals can be constructed as follows:

Suppose that arrivals come from a large number N of independent sources, each having a Poisson output with parameter λ. The arrivals will be of Poisson type with parameter λN. If we assume that the number of sources is a function of time, $N(t)$, the resulting arrival process will be a non-stationary Poisson process with parameter $\lambda N(t)$. Finally, if we assume that the number of sources varies periodically, we obtain a Poisson process with periodic parameter. Such a process has been used as a model, for instance by Bliss [7] p. 25 to represent births in a hospital. In this Chapter, only functions $\lambda(t)$ which can be expanded into a Fourier series with a finite number of terms are considered, as this simplifies the treatment considerably. This imposes only a mild restriction on the practical applicability of the model, while excluding discontinuous functions $\lambda(t)$, which seem to fall outside the scope of the approach of this Chapter.

3. An outline of the approach. We shall start with the equation

$$\Omega(t,s) = e^{-sz+st}\,\Theta(0,t;s) - s\int_0^t e^{s(t-u)}\,\Theta(u,t;s)\,W(u,0)\,du \;, \qquad (7.2)$$

which was proved in Section 1 of Chapter 4.

As shown in Chapter 2, Section 12 , we shall have in the case of a Compound Poisson input

$$\Theta(u,t;s) = \exp\left[-\left\{\Lambda(t)-\Lambda(u)\right\}\left\{1-\psi(s)\right\}\right].$$

We shall now make use of the fact that $\Omega(t,s)$ is an analytic function of s in $Re(s)>0$, and that in that region $|\Omega(t,s)|\leq 1$. Let us apply on both sides of equation (7.2) the operator

$$P.V.\ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{ds}{s^2} \;, \qquad x>0.$$

It then follows from an immediate application of Cauchy's theorem that

$$P.V.\ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\Omega(t,s)\,\frac{ds}{s^2} = 0.$$

If we can justify the interchange of the integrals, equation (7.2) will then reduce to the Volterra equation of the first kind

$$\int_0^t R(t,u)\,W(u,0)\,du = r(t) \;, \qquad (7.3)$$

where

$$R(t,u) = \text{P.V.}\ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{s(t-u)}\,\Theta(u,t;s)\,\frac{ds}{s}\ ,\ x>0 \qquad (7.4)$$

$$r(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{-s z+st}\,\Theta(0,t;s)\,\frac{ds}{s^2}\ ,\ x>0. \qquad (7.5)$$

We note that $r(t)$ is not a principal value, as the integral converges absolutely.

Equation (7.3) can be transformed into a Volterra equation of the second kind by differentiation with respect to t .

It follows immediately from equation (7.4) that

$$R(t,t) = \text{P.V.}\ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{ds}{s} = 1.$$

Thus we must have

$$w(t,0) + \int_0^t P(t,u)\,w(u,0)\,du = r'(t), \qquad (7.6)$$

where

$$P(t,u) = \frac{\partial}{\partial t}R(t,u)\ , \qquad (7.7)$$

$$r'(t) = \frac{d}{dt}\,r(t). \qquad (7.8)$$

In the two integral equations (7.3) and (7.6), we shall put $\Lambda(t) = t - \omega h(t)$, where ω is a complex number, and we shall attempt to show that the functions (7.4),(7.5), (7.7) and (7.8) all become analytic functions of ω , and can therefore be expanded into power series in ω .
We are thus led to consider various analytical properties of the above-mentioned functions. These properties are studied in

the next section.

To emphasise functional dependence on w, we shall write

$$R = R(t, u; w),$$

$$r = r(t; w),$$

$$P = P(t, u; w),$$

$$r' = r'(t; w).$$

Finally, let us note that R and P need be defined only for $0 \leq u \leq t$.

4. Some analytical properties of the functions P, R, r, r'.

Theorem 7.1.

$$\left| \int_{x-iM}^{x+iM} \frac{e^{ts}}{s} \, ds \right| \leq 2\pi(1 + 2e^{tx}) \quad \text{for all real } x \neq 0,$$

and all real t.

Proof: We first note that we may assume $t \geq 0$, for if $t < 0$, we can change the variable to $-s$, and the integral becomes $\int_{-x-iM}^{-x+iM} [e^{(-t)s}/s] \, ds$.

Let now Γ denote the contour which bounds the region $Re(s) < x$, $-iM < Im(s) < iM$. It is easily verified that

$$\int_{\Gamma} \frac{e^{ts}}{s} \, ds = \begin{cases} 2\pi i & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

In both cases
$$\left|\int_\Gamma \frac{e^{ts}}{s}\,ds\right| \le 2\pi ,$$
so that
$$2\pi \ge \left|\int_\Gamma \frac{e^{ts}}{s}ds\right| = \left|-\int_{-\infty}^{x} \frac{e^{t(y+iM)}}{y+iM}\,dy + \int_{-\infty}^{x} \frac{e^{t(y-iM)}}{y-iM}\,dy + \int_{x-iM}^{x+iM} \frac{e^{ts}}{s}\,ds\right|$$
$$\ge \left|\int_{x-iM}^{x+iM} \frac{e^{ts}}{s}ds\right| - \left|\int_{-\infty}^{x} \frac{e^{t(y+iM)}}{y+iM}\,dy\right| - \left|\int_{-\infty}^{x} \frac{e^{t(y-iM)}}{y-iM}\,dy\right| .$$
We conclude that
$$\left|\int_{x-iM}^{x+iM} \frac{e^{ts}}{s}ds\right| \le 2\pi + 2\left|\int_{-\infty}^{x} \frac{e^{ty}}{M}dy\right| = 2\pi + \frac{2}{M}\,\frac{e^{tx}}{t} . \qquad (7.9)$$
Using now the elementary inequality $|e^{iy}-1| \le |y|$,

where y is real, we see that
$$\left|\int_{x-iM}^{x+iM} \frac{e^{ts}-e^{tx}}{s}ds\right| \le t e^{tx}\int_{-M}^{+M} dy = 2Mt\,e^{tx} . \qquad (7.10)$$
But
$$\left|\int_{x-iM}^{x+iM} \frac{ds}{s}\right| = \left|2i \arg(x+iM)\right| < 2\pi \qquad (7.11)$$

It follows from (7.10) and (7.11) that

$$\left|\int_{x-iM}^{x+iM} \frac{e^{ts}}{s}ds\right| < (2\pi + 2Mt)\,e^{tx} . \qquad (7.12)$$

Comparing now (7.9) and (7.12), we obtain

$$\left|\int_{x-iM}^{x+iM} \frac{e^{ts}}{s}ds\right| \le 2\pi + 2\,e^{tx} \min_{Mt}\left(\frac{1}{Mt}, \pi + Mt\right)$$

It is, however, easy to see that $\min_x\left(\frac{1}{x}, \pi + x\right) < 2\pi$.

The theorem follows.

Theorem 7.2.

$$\left|\int_{x-iM}^{x+iM} \psi(s)\,\frac{e^{ts}}{s}\,ds\right| \le 2\pi\left[1+2\,e^{tx}\,\psi(x)\right], \quad x>0.$$

Proof:

$$\left|\int_{x-iM}^{x+iM} \psi(s)\,\frac{e^{ts}}{s}\,ds\right| = \left|\int_0^\infty b(u)\left\{\int_{x-iM}^{x+iM} e^{(t-u)s}\,\frac{ds}{s}\right\}du\right|$$

$$\le 2\pi\int_0^\infty b(u)\left[1+2\,e^{(t-u)x}\right]du$$

$$= 2\pi\left[1+2\,e^{tx}\,\psi(x)\right].$$

Theorem 7.3. Let $\gamma(s) = s-[1-\psi(s)]$. Then, for $x>0$, $t \geqslant 0$, $M>0$, we have

$$\left|\int_{x-iM}^{x+iM} \psi(s)\exp\{\gamma(s)t\}\,\frac{ds}{s}\right| \le 2\pi\,(1+t)\,e^{-t} + \left[\frac{t^2 Q(x)}{2x}\exp\{\gamma(s)t\} + 2\pi\,e^{t(x-1)}\left\{1+t\,\psi(x)\right\}\right]\psi(x). \qquad (7.13)$$

Proof:

$$\left|\int_{x-iM}^{x+iM} \psi(s)\exp\{\gamma(s)t\}\,\frac{ds}{s}\right| = \left|\int_0^\infty b(u)\,du\left\{\int_{x-iM}^{x+iM}\exp\left[s(t-u)-t+t\,\psi(s)\right]\frac{ds}{s}\right\}\right|$$

$$\le e^{-t}\int_0^\infty b(u)\,du\left\{\left|\int_{x-iM}^{x+iM} e^{(t-u)s}\left[e^{t\psi(s)}-1-t\,\psi(s)\right]\frac{ds}{s}\right|\right.$$

$$\left. + \left|\int_{x-iM}^{x+iM} e^{(t-u)s}\,\frac{ds}{s}\right| + t\left|\int_{x-iM}^{x+iM} \psi(s)\,e^{(t-u)s}\,\frac{ds}{s}\right|\right\}.$$

But, from Parseval's identity, we see that

$$\int_{-\infty}^{+\infty} |\psi(x+iy)|^2\,dy = \int_0^\infty e^{-2xu}\left[b(u)\right]^2 du = Q(x).$$

It follows that

$$\left|\int_{x-iM}^{x+iM} e^{(t-u)s}\left[e^{t\psi(s)}-1-t\psi(s)\right]\frac{ds}{s}\right| \le \frac{1}{2}\int_{-\infty}^{+\infty} e^{(t-u)x} t^2 |\psi(x+iy)|^2 e^{t|\psi(x+iy)|}\frac{dy}{x}$$

$$\le \frac{t^2 e^{(t-u)x+t\psi(x)}}{2x}\int_{-\infty}^{+\infty}|\psi(x+iy)|^2 dy$$

$$= \frac{t^2}{2x} Q(x)\, e^{(t-u)x+t\psi(x)} .$$

Also

$$\left|\int_{x-iM}^{x+iM} e^{(t-u)s}\frac{ds}{s}\right| \le 2\pi\left\{1+2e^{(t-u)s}\right\}$$

and

$$\left|\int_{x-iM}^{x+iM}\psi(s)e^{(t-u)s}\frac{ds}{s}\right| \le 2\pi\left[1+2e^{(t-u)x}\psi(x)\right] .$$

Replacing in the original inequality and integrating, we obtain the result.

Corollary:

$$\left|\int_{x-iM}^{x+iM}[1-\psi(s)]^n \exp\{\gamma(s)t\}\frac{ds}{s}\right| \le 2\pi(n+1)(1+t)e^{-t}+\left[\frac{t^2 Q(x)}{2x}\exp\{\gamma(s)t\}\right.$$

$$\left.+\,4\pi e^{t(x-1)}\{1+t\psi(x)\}\right][1+\psi(x)]^n . \qquad (7.14)$$

Proof: We expand $[1-\psi(s)]^n$ by the Binomial Theorem and note that $[\psi(s)]^n$ is the Laplace transform of the n-th convolution of $b(t)$ with itself. The result follows from an application of Theorem 7.3.

Theorem 7.4 .

$$\left|\int_{x-iM}^{x+iM}\psi(s)\exp\{\gamma(s)t\}\frac{ds}{s^2}\right| \le \frac{\pi\,\psi(x)}{x}\exp\{\gamma(x)t\} . \qquad (7.15)$$

Proof: On the line $Re(s) = x$, we have

$$Re\left\{\gamma(s)t\right\} = Re\left\{\left[s-\left\{1-\psi(s)\right\}\right]t\right\} = \left[x-1+Re\left\{\psi(s)\right\}\right]t$$

$$\leq \left\{x-1+\psi(x)\right\}t = \gamma(x)t.$$

It follows that

$$\left|\int_{x-iM}^{x+iM}\psi(s)\exp\left\{\gamma(s)t\right\}\frac{ds}{s^2}\right| \leq \psi(x)\exp\left\{\gamma(x)t\right\}\int_{-M}^{+M}\frac{dy}{x^2+y^2},$$

and the last integral is smaller than

$$\int_{-\infty}^{+\infty}\frac{dy}{x^2+y^2} = \frac{\pi}{x}$$

Corollary:

$$\left|\int_{x-iM}^{x+iM}e^{-sz}\left[1-\psi(s)\right]^m\exp\left\{\gamma(s)t\right\}\frac{ds}{s^2}\right| \leq \frac{\pi}{x}e^{-xz}\left[1+\psi(x)\right]^m\exp\left\{\gamma(x)t\right\}. \quad (7.16)$$

Proof: This follows from Theorem 4 by expanding as in the Corollary of Theorem 7.3.

Theorem 7.5. Let

$$R(t,u;w) = \frac{1}{2\pi i}\,\mathrm{P.V.}\int_{x-i\infty}^{x+i\infty}\exp\left\{(t-u)s-\left[(t-u)\right.\right.$$

$$\left.\left.-w\left\{h(t)-h(u)\right\}\right]\left[1-\psi(s)\right]\right\}\frac{ds}{s}, \quad x>0,$$

$$r(t;w) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}e^{-sz}\exp\left\{ts-\left[t-w\,h(t)\right].\right.$$

$$\left.\left[1-\psi(s)\right]\right\}\frac{ds}{s^2}, \quad x>0 .$$

Then $P(t,u;\omega)=\dfrac{\partial R(t,u;\omega)}{\partial t}$ and $r'(t;\omega)=\dfrac{\partial r(t;\omega)}{\partial t}$ exist and are analytic functions of ω and continuous functions of t.

Proof: Put

$$\alpha = t-u > 0,$$

$$\beta = (t-u) - \omega\{b(t)-b(u)\} = (t-u) - \omega\, q(t,u)$$

where $$q(t,u) = b(t) - b(u).$$

Then

$$R(t,u;\omega) = e^{-\beta}[m(\alpha,\beta) + n(\alpha,\beta)],$$

where $$m(\alpha,\beta) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{\alpha s}\left[e^{\beta\psi(s)} - 1 - \beta\psi(s)\right]\frac{ds}{s},$$

and $$n(\alpha,\beta) = \frac{1}{2\pi i}\,\mathrm{P.V.}\int_{x-i\infty}^{x+i\infty} e^{\alpha s}\frac{ds}{s} + \frac{\beta}{2\pi i}\,\mathrm{P.V.}\int_{x-i\infty}^{x+i\infty}\psi(s)\,e^{\alpha s}\frac{ds}{s} = 1+\beta B(\alpha).$$

It is sufficient to show that $\partial^2 P/\partial\omega\,\partial t$ exists.

We note however that

$$-\frac{\partial^2 R}{\partial\omega\,\partial t} = q(t,u)\frac{\partial^2 R}{\partial\beta\,\partial\alpha} + q(t,u)\left[1-\omega\frac{\partial q(t,u)}{\partial t}\right]\frac{\partial^2 R}{\partial\beta^2} + \frac{\partial R}{\partial\beta}\frac{\partial q(t,u)}{\partial t}.$$

It is therefore sufficient to show that $\partial^2 P/\partial\beta\,\partial\alpha$, $\partial^2 R/\partial\beta^2$ exist. As $n(\alpha,\beta)$ has obviously all the required derivatives, we must show that the same applies to $m(\alpha,\beta)$. But we have

$$\left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM}\frac{\partial^2}{\partial\beta^2}\left\{e^{\alpha s}\left[e^{\beta\psi(s)} - 1 - \beta\psi(s)\right]\right\}\frac{ds}{s}\right| = \left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM} e^{\alpha s}\left[\psi(s)\right]^2 e^{\beta\psi(s)}\frac{ds}{s}\right|$$

$$\leq \frac{Q(x)}{2\pi x}\, e^{\alpha x + |\beta|},$$

and

$$\left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM}\frac{\partial^2}{\partial\alpha\partial\beta}\left\{e^{\alpha s}\left[e^{\beta\psi(s)}-1-\beta\psi(s)\right]\frac{ds}{s}\right|=\left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM}\alpha e^{\alpha s}\psi(s)\left\{e^{\beta\psi(s)}-1\right\}\frac{ds}{s}\right|$$
$$\leq \frac{\alpha|\beta|}{2\pi x}Q(x)\,e^{\alpha x+|\beta|}.$$

It follows from a standard application of uniform convergence theorems that $\partial^2 m/\partial\beta^2$ and $\partial^2 m/\partial\beta\partial\alpha$ both exist and are continuous in α, β. We conclude that $\partial^2 R/\partial\omega\partial t$ exists, and is continuous in t, u, ω. A completely similar proof yields the result for $r(t;\omega)$.

<u>Corollary:</u> The functions

$$R_m(t)=\frac{1}{m!}\,\text{P.V.}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\left[1-\psi(s)\right]^m\exp\{\gamma(s)t\}\frac{ds}{s},\ x>0,\quad(7.17)$$

$$r_m(t)=\frac{1}{m!}\,\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}e^{-s\zeta}\left[1-\psi(s)\right]^m\exp\{\gamma(s)t\}\frac{ds}{s^2},\ x>0,\quad(7.18)$$

have continuous derivatives $R'_m(t)$ and $r'_m(t)$, and all four functions belong to the L^2 class.

<u>Proof:</u> The proof of differentiability follows exactly the same lines as those of Theorem 7.5.

Also the inequalities of Theorem 7.5, suitably transformed, show that $R_m(t)$, $r_m(t)$, $R'_m(t)$, $r'_m(t)$ are dominated by functions of t which belong to the L^2 class. The last part of the Corollary follows.

5. <u>Derivation of a power series for $W(t,0)$.</u> The results of the preceding section enable us to justify easily the interchange of integrals performed in Section 3. Thus we

have the following theorem, under the assumptions made in Section 1, namely:

(a) $\lambda(t)$ is a continuous function of t,

(b) the service time distribution has a density function $b(x)$ which fulfills the condition $\int_0^\infty [b(x)]^2 dx < \infty$,

(c) $\psi(s)$ is analytic in $\mathrm{Re}(s) > \sigma_0$, where $\sigma_0 < 0$, and is $O(1/|s|)$ in that region.

Theorem 7.6. $W(t,0)$ is the unique continuous solution of the Volterra equation of the first kind

$$\int_0^t R(t,u)\, W(u,0) = r(t), \qquad (7.3)$$

where $R(t,u)$ and $r(t)$ are defined by (7.4) and (7.5)

Proof: That $W(t,0)$ satisfies (7.3) follows from the results of Sections 3 and 4. The uniqueness follows from the fact that differentiation reduces (7.3) to the Volterra equation of the second kind

$$W(t,0) + \int_0^t P(t,u)\, W(u,0)\, du = r'(t), \qquad (7.6)$$

where $P(t,u)$ and $r'(t)$ are defined by (7.7) and (7.8), and the well-known results (see for instance Tricomi [72] p. 12) on the uniqueness of the solution of the Volterra equation of the second kind.

As foreshadowed in Section 2, we now put $\Lambda(t) = t - \omega h(t)$, where ω is a parameter. It follows that $h(t)$ is a differentiable function of t, and that its derivative is continuous. Only values of ω such that $\omega h(t)$ is real and $\omega h'(t) < 1$ for all t have relevance to the storage problem, as $\Lambda(t)$ must be real and non-decreasing. However, we shall allow ω to take complex values in the process of obtaining an explicit solution for $W(t,0)$. With the introduction of the parameter ω, the functions R, r, P, r' all become, as mentioned before, functions of the complex variable ω.

<u>Lemma 7.1</u>: For all complex ω,

(a)

$$R(t,u;\omega) = \sum_{n=0}^{\infty} \omega^n \left[q(t,u)\right]^n R_n(t-u), \tag{7.19}$$

where

$$q(t,u) = h(t) - h(u), \tag{7.20}$$

$$R_n(t) = \frac{1}{n!} \text{ P.V. } \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \left[1-\psi(s)\right]^n \exp\left[\gamma(s)t\right] \frac{ds}{s}, \quad x > 0,$$

(b)

$$r(t;\omega) = \sum_{n=0}^{\infty} \omega^n \left[h(t)\right]^n r_n(t), \tag{7.21}$$

where

$$r_n(t) = \frac{1}{n!} \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \left[\exp\{-zs\}\right]\left[1-\psi(s)\right]^n \exp\left[\gamma(s)t\right] \frac{ds}{s^2}, \quad x > 0.$$

<u>Proof:</u> We can write

$$R(t,u;\omega) = \text{P.V. } \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \left\{\exp\left[(t-u)\gamma(s)\right]\right\} \cdot \left[\sum_{n=0}^{\infty} \frac{1}{n!} \omega^n \left\{q(t,u)\right\}^n \left\{1-\psi(s)\right\}^n\right] \frac{ds}{s}, \quad x > 0.$$

Now for fixed ω, t, u, the series converges uniformly on the line $Re(s) = x > 0$, for there $|1-\psi(s)| \leq 2$. If we first consider the integral with the limits $x - iM$, $x + iM$, we can interchange the integration and summation signs. The general term of the series will now become

$$\omega^n \left[q(t,u)\right]^n \frac{1}{n!} \frac{1}{2\pi i} \int_{x-iM}^{x+iM} \left[1-\psi(s)\right]^n \exp\left[\gamma(s)(t-u)\right] \frac{ds}{s} .$$

But it has been shown in Section 4 (Corollary to Theorem 7.3) that the integral is dominated by an expression which is independent of M. Moreover, if we replace the integral in each term of the series by the corresponding dominating expression, the resulting series is absolutely convergent. It follows from Lebesgue's Dominated Convergence theorem that we can let M tend to infinity under the summation sign, and part (a) of the theorem follows.
Part (b) is proved in a similar way, using the Corollary to Theorem 7.4.

The introduction of ω makes $W(t,0)$, the unique continuous solution of the integral equation (7.3), a function of ω. We shall also write henceforth $W(t,0;\omega)$ for $W(t,0)$.

<u>Lemma 7.2</u>: $W(t,0;\omega)$ is an analytic function of ω.

<u>Proof:</u> From Theorem 7.5, it follows immediately that

$$\int_0^T \int_0^T \left[P(t,u;\omega)\right]^2 du\, dt < \infty ,$$

$$\int_0^T [r'(t;\omega)]^2 dt < \infty$$

for arbitrary finite $T > 0$.

We shall now use the fact that the solution of the integral equation (7.6) can be written down in the form

$$W(t,0;\omega) = r'(t;\omega) - \int_0^t H(t,u;\omega)r'(u;\omega)\,du ,$$

where the resolvent kernel $H(t,u;\omega)$ is given by

$$-H(t,u;\omega) = \sum_{n=0}^{\infty} (-1)^n P_{n+1}(t,u;\omega). \qquad (7.22)$$

The iterated kernels P_n are defined recursively by

$$P_1(t,u;\omega) = P(t,u;\omega)$$

$$P_{n+1}(t,u;\omega) = \int_u^t P(t,v;\omega)P_n(v,u;\omega)\,dv \quad , n \geq 1 .$$

As $P(t,u;\omega)$ is an analytic function of ω , the P_n are all analytic functions of ω . Also, for arbitrary M , the series for the resolvent kernel (7.22) is majorized inside $|\omega| \leq M$ by an absolutely convergent series whose convergence is independent of ω .(see Tricomi [72] p. 12) It is therefore uniformly convergent in $|\omega| \leq M$, and $H(t,u;\omega)$ is therefore itself an analytic function of ω , and is continuous in t and u .

It follows that $W(t,o;\omega)$ is an analytic function of ω, and is continuous in t.

Lemma 7.3: $W(t,o;\omega) = \sum_{n=0}^{\infty} \omega^n F_n(t)$, the series converging for all ω uniformly in $o \leq t \leq T$. Moreover, each $F_n(t)$ is a continuous function of t.

Proof: This follows immediately from the boundedness of $W(t,o;\omega)$ in $0 \leq t \leq T$, $|\omega| \leq M$, and the form of the remainder in the Taylor expansion of $W(t,o;\omega)$.

Theorem 7.7. The functions $F_n(t)$ can be calculated by recurrence as the solutions of the convolution-type integral equations

$$\int_0^t R_0(t-u)\,F_n(u)\,du = \left[h(t)\right]^n r_n(t)$$
$$-\int_0^t \left[\sum_{k=1}^{n} \left\{q(t,u)\right\}^k R_k(t-u)\,F_{n-k}(u)\right] du. \qquad (7.23)$$

Proof: Replace $R(t,u;\omega)$, $W(t,o;\omega)$, $r(t,\omega)$ by their power expansions in the integral equation

$$\int_0^t R(t,u;\omega)\,W(t,o;\omega)\,du = r(t;\omega)$$

and equate the coefficients. This gives us

$$\int_0^t \left[\sum_{k=0}^{n} \left\{q(t,u)\right\}^k R_k(t-u)\,F_{n-k}(u)\right] du = \left\{h(t)\right\}^n r_n(t).$$

Finally, we transfer all the terms of the left-hand side except the first to the right-hand side.

6. Some theorems on Laplace transforms. In Section 7, we shall proceed to solve the integral equations (7.23) by means of Laplace transforms. For this purpose, we shall need some theorems on integral equations of the convolution type and their solution by Laplace transforms.

All the functions we shall use will belong to the class L^2 defined in Section 1, i.e. for any function $f(t)$, $\int_0^\infty e^{-2pt}|f(t)|^2 dt$ will converge for some $p > 0$.

Theorem 7.8. If $F(t) \varepsilon L^2$ and $G(t)$ is uniformly bounded in $t > 0$, then $F(t)G(t) \varepsilon L^2$.

Proof: Let $|G(t)| \leq M$. Then

$$\int_0^\infty e^{-2pt}|F(t)G(t)|^2 dt \leq M^2 \int_0^\infty e^{-2pt}|F(t)|^2 dt.$$

Theorem 7.9. If $F(t)$ and $G(t)$ belong to L^2, then

$$H(t) = F * G(t) = \int_0^t F(t-u)G(u)\, du \qquad \text{belongs to } L^2.$$

Proof: By the Schwartz inequality

$$e^{-2pt}|H(t)|^2 = \left|\int_0^t e^{-p(t-u)}F(t-u)\, e^{-pu}\, G(u)\, du\right|^2$$

$$\leq \int_0^t e^{-2pu}|F(u)|^2 du \,.\int_0^t e^{-2pv}|G(v)|^2 dv \,,$$

$$\leq \int_0^\infty e^{-2pu}|F(u)|^2 du \,.\int_0^\infty e^{-2pv}|G(v)|^2 dv \,.$$

It follows that $e^{-2pt}|H(t)|^2$ is uniformly bounded for sufficiently large p, so that the integral

$$\int_0^\infty e^{-2(p+q)t}|H(t)|^2 dt$$

will converge for any $q > 0$.

This completes the proof.

Theorem 7.10. Let $K(t)$, $G(t)$ have continuous derivatives $K'(t)$, $G'(t)$, which are L^2 functions. Let $G(0) = 0$ and $K(0) \neq 0$. Then the Volterra equation of the first kind in $F(t)$:

$$\int_0^t K(t-u)F(u)\,du = G(t) \qquad (7.24)$$

has a unique continuous solution which is an L^2 function.

Moreover, its Laplace transform $F^*(p) = \int_0^\infty e^{-pt}F(t)\,dt$ is given by

$$F^*(p) = \frac{G^*(p)}{K^*(p)}$$

where $K^*(p) = \int_0^\infty e^{-pt}K(t)\,dt$, $G^*(p) = \int_0^\infty e^{-pt}G(t)\,dt$.

Proof: The first part follows immediately from the usual differentiation technique of transforming a Volterra integral equation of the first kind into an integral equation of the second kind, and then applying a theorem of Doetsch [15] p. 143 Satz 7.

The second part follows by taking Laplace transforms on both sides of equation 7.24, and using the convolution theorem of Doetsch [14] p. 123.

7. <u>The Laplace transforms of the coefficients in the power series for $W(t, s)$.</u> We shall now solve equations (7.23) by means of Laplace transforms. To do this, we require the Laplace transforms of the $R_m(t)$ and the $r_m(t)$. The required expressions are given in the following theorem.

<u>Theorem 7.11</u>. Let $R_m^*(p)$, $r_m^*(p)$ be the Laplace transforms of $R_m(t)$, $r_m(t)$. Then, if α is as defined at the end of Section 1 , we have

$$R_m^*(p) = \frac{1}{m!}\,\frac{[1-\psi(\theta)]^m}{\theta[1+\psi'(\theta)]}\,, \quad \mathrm{Re}(p) > \gamma(\alpha), \qquad (7.25)$$

$$r_m^*(p) = \frac{1}{m!}\,\frac{e^{-\theta z}[1-\psi(\theta)]^m}{\theta^2[1+\psi'(\theta)]}\,, \quad \mathrm{Re}(p) > \gamma(\alpha), \qquad (7.26)$$

where θ is the unique root in $\mathrm{Re}(s) > \alpha$ of $\gamma(s) - p = 0$.

<u>Proof:</u> The uniqueness of θ in $\mathrm{Re}(s)>\alpha$ for $\mathrm{Re}(p)>\gamma(\alpha)$ has already been discussed at the end of Section 1 .

Also, it has been shown in Section 4 that the functions

$$\frac{1}{2\pi i}\int_{x-iM}^{x+iM} [1-\psi(s)]^m \exp\{\gamma(s)t\}\,\frac{ds}{s}\,, \quad x > 0\,,$$

$$\frac{1}{2\pi i}\int_{x-iM}^{x+iM} e^{-sz}[1-\psi(s)]^m \exp\{\gamma(s)t\}\,\frac{ds}{s^2}\,, \quad x > 0\,,$$

are dominated by functions of t and x which are independent of M and are of the class L^2 . It follows from standard theorems on the inversion of the order of integration that the integrals $\int_{x-iM}^{x+iM} ds$ and $\int_0^t dt$ can be interchanged. An application of Lebesgue's Dominated Convergence theorem then justifies the interchange of the integrals $\int_{x-i\infty}^{x+i\infty} ds$ and $\int_0^\infty dt$, provided that the integral in t converges absolutely. We therefore have

$$R_m^*(p) = \text{P.V.}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{1}{m!}\frac{ds}{s}\left[1-\psi(s)\right]^m \int_0^\infty \exp\left[t\{\gamma(s)-p\}\right] dt.$$

For the integral in t to converge absolutely, the coefficient of t in the exponential must have a negative real part on the line $\mathrm{Re}(s) = x$. Putting $s = x+iy$, we have

$$\mathrm{Re}\left[\gamma(s)-p\right] = \mathrm{Re}\left[s-1+\psi(s)-p\right] = -\mathrm{Re}(p)+x-1+\int_0^\infty e^{-xt}(\cos yt)\, b(t)\,dt.$$

This will be negative if

$$\mathrm{Re}(p) > x-1+\int_0^\infty e^{-xt}(\cos yt)\, b(t)\,dt,$$

and a fortiori if

$$\mathrm{Re}(p) > x-1+\int_0^\infty e^{-xt}\, b(t)\,dt = x-1+\psi(x) = \gamma(x).$$

If this is satisfied, we shall have

$$R_m^*(p) = \text{P.V.}\frac{1}{m!}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{[1-\psi(s)]^m}{s\,[p-\gamma(s)]}\,ds.$$

To calculate this integral, we consider the integral around the closed contour consisting of the arc of the circle $C: |s| = R$ for $\mathrm{Re}(s) > x$ and the line $\mathrm{Re}(s) = x$.

We note that for $\mathrm{Re}(s)>0$, $|\psi(s)| \leq 1$, and therefore

$$|p-\gamma(s)| = |p-s+1-\psi(s)| > |s|-|p|-1-|\psi(s)| \geq |s|-M,$$

where M is some constant. It follows that the contribution of the circle to the integral tends to zero when R tends to infinity, for

$$\left| \int_C \frac{[1-\psi(s)]^n \, ds}{s[p-\gamma(s)]} \right| < \frac{\pi R}{R(R-M)} .$$

On the other hand, it follows from Theorem 3.5 of Chapter 3 that $p - \gamma(s)$ will have only one zero in $Re(s) > x > 0$ if $Re(p) > \gamma(x)$, namely $s = \theta$. It follows that

$$R_n^*(p) = \frac{1}{n!} \frac{[1-\psi(\theta)]^n}{\theta[1+\psi'(\theta)]}$$

for $Re(p) > \gamma(x)$.

We now use the fact that $\theta(p)$ is defined for all p such that $Re(p) > \gamma(\alpha)$. Thus $R_n^*(p)$, which was defined till now only in $Re(p) > x > 0$, and is an analytic function of p in that region, can be continued analytically to the region $Re(p) > \gamma(\alpha)$. This completes the proof of (7.25). A similar proof yields (7.26).

<u>Corollary:</u> The functions

$$R_0^*(p) = \frac{1}{\theta[1+\psi'(\theta)]} \qquad , \; r_1^*(p) = \frac{e^{-\theta z}[1-\psi(\theta)]}{\theta^2[1+\psi'(\theta)]}$$

have one simple pole in $Re(p) > \gamma(\alpha)$, namely at $p = 0$. The functions $R_n^*(p), (n > 0)$ and $r_n^*(p), (n > 1)$ are analytic and uniformly bounded in the same region. Moreover, for all $n \geq 0$, $R_n^*(p)$ is $O(1/|p|)$ and $r_n^*(p)$ is $O(1/|p|^2)$ in the region.

<u>Proof</u>: The Corollary follows immediately from the theorem and the results of Theorem 3.6 of Chapter 3.

Theorem 7.12. Suppose that $h(t)$, $h'(t)$ are uniformly bounded for all $t \geq 0$, and define $h_n^*(p)$ by

$$h_n^*(p) = \int_0^\infty e^{-pt}\left[h(t)\right]^n dt . \tag{7.27}$$

Then

$$\int_0^t e^{-pt}\left[h(t)\right]^n r_n(t)\,dt = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h_n^*(p-s)\, r_n^*(s)\,ds \tag{7.28}$$

for some $x > 0$ and all p such that $Re(p) > x$.

Proof: The result follows from a theorem of Doetsch [14] p. 258, Satz 2, because $r_n(t)$ is an L^2 function (see the Corollary to Theorem 7.5).

Theorem 7.13. Under the same assumptions for $h(t)$ as in Theorem 7.12, the Laplace transform of

$$J_n(t) = \int_0^t \left[\sum_{k=1}^{n}\left\{q(t,u)\right\}^k R_k(t-u)\,F_{n-k}(u)\right] du \tag{7.29}$$

is

$$\sum_{k=1}^{n}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h_{k-r}^*(p-s)\Big\{R_k^*(s)$$

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h_r^*(s-\sigma)\,F_{n-k}^*(\sigma)\,d\sigma\Big\}\,ds , \tag{7.30}$$

where $F_n^*(p)$ is the Laplace transform of $F_n(t)$, $Re(p) > x$ and x is sufficiently large.

Proof: Replacing $q(t,u)$ by its value $h(t) - h(u)$ in (7.29), we find that

$$J_n(t) = \sum_{k=1}^{n}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\left[h(t)\right]^{k-r}\int_0^t R_k(t-u)\left[h(u)\right]^r F_{n-k}(u)\,du .$$

We now apply repeatedly the theorem of Doetsch quoted in the proof of Theorem 7.12 and Satz 4 of Doetsch [14] p. 123 . The result will follow, provided we can justify the use of the theorems.

This justification follows easily from the fact that both quoted theorems hold for L^2 functions and from the results of Section 6 . In fact, assuming that the F_i, $(i = 0, 1, \ldots, n-1)$, are L^2 functions, it follows easily from the fact that $k(t)$, $k'(t)$, $R_k(t)$, $R_k'(t)$ are L^2 functions that the derivative of the right-hand side of the n-th equation (7.23) is an L^2 function. As $R_0'(t)$ is also an L^2 function, it follows from Theorem 7.10 of Section 6 that $F_n(t)$ is an L^2 function. An induction argument then shows that all the functions involved in the system of equations (7.23) are L^2 functions.

<u>Corollary:</u> Under the assumptions of Theorem 7.12 , the Laplace transforms $F_n^*(p)$ of the $F_n(t)$ are given by the recurrence formulae

$$F_n^*(p) = \frac{1}{R_0^*(p)}\left[\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} k_n^*(p-\sigma)\, r_n^*(\sigma)\, d\sigma \right.$$
$$\left. -\sum_{k=1}^{n}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} k_{k-r}^*(p-s)\left\{R_k^*(s)\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} k_r^*(s-\sigma)\, F_{n-k}^*(\sigma)\, d\sigma\right\} ds\right] \qquad (7.31)$$

for sufficiently large x , $Re(p) > x$.

8. <u>The asymptotic behaviour of the $F_n(t)$ when $k(t)$ is periodic</u>.

We shall now consider in detail the special case when

$h(t)$ can be expanded in a Fourier series with a finite number of non-zero coefficients. We shall write

$$h(t) = \sum_{n=1}^{N} a_n \sin(n\omega t + \varphi_n).$$

In this case, $h(t)$ obviously satisfies the conditions of Theorem 7.12 . In this case, the $F_n^*(p)$ are given for $\mathrm{Re}(p)$ sufficiently large by equation (7.31). We continue them analytically wherever this is possible and denote in the sequel by $F_n^*(p)$ the analytic functions thus defined.

<u>Lemma 7.4</u>: Let $F_1(p)$, $F_2(p)$ be meromorphic functions which can be written in the form

$$F_1(p) = \sum_{n=-N}^{+N} \frac{c_n}{p - in\omega} \quad , \quad F_2(p) = \sum_{n=-M}^{+M} \frac{\psi_n(p)}{p - in\omega}$$

where the $\psi_n(p)$ are analytic functions which are uniformly bounded in $\mathrm{Re}(p) > \sigma_0$, where $\sigma_0 < 0$. Then

$$G(p) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F_1(p-\sigma) F_2(\sigma)\, d\sigma \; , \quad \mathrm{Re}(p) > x > 0,$$

exists and can be written in the form

$$G(p) = \sum_{n=-(N+M)}^{N+M} \frac{\Gamma_n(p)}{p - in\omega}$$

where the $\Gamma_n(p)$ are analytic functions which are uniformly bounded in $\mathrm{Re}(p) > \sigma_0$. Further, if the $\psi_n(p)$ are $O(1/|p|^k)$ as $|p| \to \infty$ in $\mathrm{Re}(p) > \sigma_0$, then so are the $\Gamma_n(p)$.

<u>Proof</u>: We have

$$G(p) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\left[\sum_n \frac{c_n}{p-\sigma-inw}\right]\left[\sum_m \frac{\psi_m(\sigma)}{\sigma-imw}\right]d\sigma,$$

$$= \sum_n \sum_m \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{c_n\psi_m(\sigma)}{(p-\sigma-inw)(\sigma-imw)}d\sigma, \quad Re(p) > x > 0.$$

The poles of the integrand to the right of $Re(\sigma) = x$ are at $\sigma = p - inw$. The residue of the (n,m)th term is

$$-\frac{c_n\psi_m(p-inw)}{p-i(n+m)w}.$$

We notice that the integrand is $O(1/|\sigma|^2)$ to the right of $Re(\sigma) = x$ and therefore the integral is equal to minus the sum of the residues to the right of $Re(\sigma) = x$.

Put now $k = m+n$ and collect all terms whose denominator is $p - ikw$. The result is

$$\frac{\sum_n c_n\psi_{k-n}(p-inw)}{p-ikw} = \frac{\Gamma_k(p)}{p-ikw},$$

where

$$\Gamma_k(p) = \sum_n c_n\psi_{k-n}. \tag{7.32}$$

We note that as all $\psi_n(p)$ are uniformly bounded in $Re(p) > \sigma_0$, it follows that all the $\Gamma_k(p)$ are uniformly bounded in $Re(p) > \sigma_0$.

Finally,

$$G(p) = \sum_{n=-(N+M)}^{N+M}\frac{\Gamma_n(p)}{p-inw}.$$

This proves the first part of the theorem. The second part follows immediately from equation (7.32).

Corollary: The residue of $G(p)$ at $p = i n \omega$ is

$$\Gamma_n(in\omega) = \sum_k c_k \, \Psi_{n-k}\left[(n-k)i\omega\right] .$$

Lemma 7.5: The Laplace transform of $h(t) = \sum_{n=1}^{N} a_n \sin(n\omega t + \varphi_n)$ is of the form

$$\sum_{n=-N}^{+N} \frac{c_n}{p - in\omega} .$$

Proof: It is easily seen that $h(t)$ can be written in the form $h(t) = \sum_{n=-N}^{+N} c_n e^{in\omega t}$, and it follows that

$$\int_0^\infty e^{-pt} h(t)\, dt = \int_0^\infty e^{-pt}\left[\sum_{n=-N}^{+N} c_n e^{in\omega t}\right] dt$$

$$= \sum_{n=-N}^{+N} \frac{c_n}{p - in\omega} , \quad Re(p) > 0 .$$

Corollary: The Laplace transform of $\left[h(t)\right]^k$ is of the form

$$\sum_{n=-kN}^{+kN} \frac{c_{kn}}{p - in\omega} .$$

Proof: This follows by induction, using

$$\left(\sum_{n=-N}^{+N} c_n e^{in\omega t}\right)\left(\sum_{m=-M}^{+M} c'_m e^{im\omega t}\right) = \sum_{n=-(N+M)}^{N+M} c''_n e^{in\omega t}$$

We are now in a position to prove the following theorem:

<u>Theorem 7.14</u>. If $k(t) = \sum_{n=1}^{N} a_n \sin(n\omega t + \varphi_n)$, the functions $F_n^*(p), (n = 0, 1, \dots)$ can be written in the form of the finite sum:

$$F_n^*(p) = \sum_k \frac{\psi_{kn}(p)}{p - ik\omega},$$

where the ψ_{kn} are analytic functions which are uniformly bounded in $\mathrm{Re}(p) > \gamma(\alpha)$ and are $O(1/|p|)$ as $|p|$ tends to infinity in that region.

<u>Proof</u>: We proceed by induction.

The first equation of the set of equations (7.31) reads

$$F_0^*(p) = \frac{r_0^*(p)}{R_0^*(p)}.$$

Replacing $r_0^*(p)$ and $R_0^*(p)$ by their values from Theorem 7.11, we obtain

$$F_0^*(p) = \frac{e^{-\theta(p)z}}{\theta(p)}.$$

This is identical with equation (4.14) of Chapter 4.

We continue $F_0^*(p)$ analytically to the half-plane $\mathrm{Re}(p) > \gamma(\alpha)$. In that region, $e^{-\theta(p)z}$ is uniformly bounded. We conclude that the only pole in this region is where $\theta(p) = 0$, i.e. at $p = 0$. As

$$\lim_{p \to 0} \frac{\theta(p)}{p} = \theta'(0) = \frac{1}{1 + \psi'(0)} = \frac{1}{1 - \rho},$$

this pole is simple. It is also clear that, as $|p| \to \infty$, $F_0^*(p)$ is $O(1/|p|)$.

Suppose now that the $F_k^*(p)$, $(k = 0, 1, \ldots, n-1)$, satisfy the conditions of the theorem. We first note that

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h_m^*(p-\sigma) r_m^*(\sigma)\, d\sigma = \sum_{r=-mN}^{mN} \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{c_{mr}}{(p-\sigma-ir\omega)}\left[\frac{1}{m!}\frac{e^{-\theta(\sigma)z}[1-\psi\{\theta(\sigma)\}]^m}{[\theta(\sigma)]^2[1+\psi'\{\theta(\sigma)\}]}\right] d\sigma$$

$$= \sum_{r=-mN}^{mN} \frac{c_{mr}}{m!}\, \frac{e^{-\theta(\sigma_r)z}[1-\psi\{\theta(\sigma_r)\}]^m}{[\theta(\sigma_r)]^2[1+\psi'\{\theta(\sigma_r)\}]},$$

where $\sigma_r = p - ir\omega$, and this expression has all its poles at the points $p = ir\omega$ and can be written in the form $\sum_r \frac{\phi_r(p)}{p - ir\omega}$, where the $\phi_r(p)$ are $O(1/|p|)$ in $Re(p) > \gamma(\alpha)$ (because of the factor $1/[\theta(\sigma_r)]^2$). Applying now Lemmas 7.4 and 7.5 to the second term of the right-hand side of (7.31) and using the results of the Corollary to Theorem 7.11 (Section 7), we conclude that this term satisfies the conditions of the theorem and is $O(1/|p|^2)$ in $Re(p) > \gamma(\alpha)$. Finally, we notice that $\frac{1}{R_0^*(p)} = \theta(p)[1 + \psi'\{\theta(p)\}]$ has no poles in $Re(p) > \gamma(\alpha)$ and is $O(|p|)$ in that region.

We can now conclude from equations (7.31) that $F_m^*(p)$ satisfies the conditions of the theorem.

We shall now prove the key theorem of this chapter, which gives the asymptotic behaviour of the $F_m(t)$ as t tends to infinity.

Theorem 7.15 . The functions $F_n(t)$ admit the asymptotic expansion

$$F_n(t) \sim \sum_k \left(A_{kn} \cos k\omega t + B_{kn} \sin k\omega t\right).$$

Proof: As the Laplace transform of $F_n(t)$ converges absolutely for all $p > 0$, and as $F_n(t)$ is continuous for all $t > 0$, we have

$$F_n(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt} F_n^*(p)\,dp = \sum_k \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt} \frac{\psi_{kn}(p)}{p-ik\omega}\,dp.$$

We first show that the integral converges uniformly in $t \geq T > 0$ for any fixed x such that either $x > 0$ or $\gamma(\alpha) < x < 0$. In fact, we have, as $|\psi_{kn}(p)| \leq \frac{M_{kn}}{|p|}$ in $Re(p) > \gamma(\alpha)$,

$$\left|\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt} \frac{\psi_{kn}(p)}{p-ik\omega}\,dp\right| = \frac{1}{2\pi}\left|\int_{-\infty}^{+\infty} e^{xt} e^{iyt} \frac{\psi_{kn}(x+iy)}{x+i(y-k\omega)}\,dy\right|$$
$$\leq \frac{e^{xt}}{\pi}\int_{-\infty}^{+\infty} \frac{M_{kn}\,dy}{(|x|+|y|)(|x|+|y-k\omega|)},$$

and the latter integral converges absolutely and independently of t .

We shall now use the Abelian theorem for the complex inversion integral given in Doetsch [14] p. 488, which has already been used in restricted form in Section 6 of Chapter 4 . We note that the poles of $F_n^*(p)$ are at the points $\pm in\omega$ and occur in pairs. Also all the conditions of the theorem are clearly satisfied. It follows that

$$F_n(t) \sim \sum_k c_{kn} e^{ik\omega t}$$

where the c_{kn} are constants. This can obviously be rewritten as

$$F_n(t) \sim \sum_k \left(A_{kn} \cos k\omega t + B_{kn} \sin k\omega t\right).$$

9. <u>The asymptotic behaviour of the Laplace transform of the content distribution.</u> We shall now investigate the asymptotic behaviour of the Laplace transform of the content distribution, using formula (7.2), which now reads

$$\Omega(t,s;w) = \sum_{n=0}^{\infty} w^n S_n(t)$$
$$- s\int_0^t \left[\sum_{n=0}^{\infty} w^n \{q(t,u)\}^n T_n(t-u)\right]\left[\sum_{n=0}^{\infty} w^n F_n(u)\right]du,$$

where

$$S_n(t) = \frac{[1-\psi(s)]^n}{n!}[k(t)]^n e^{-sz+t\gamma(s)},$$

$$T_n(t) = \frac{1}{n!}[1-\psi(s)]^n e^{t\gamma(s)}.$$

All the series being uniformly and absolutely convergent for $0 \leq u \leq t$, we can multiply out and integrate term by term, obtaining

$$\Omega(t,s;w) = \sum_{n=0}^{\infty} \Omega_n(t,s) w^n,$$

where

$$\Omega_m(t,s) = S_m(t) - s\sum_{k=0}^{m}\int_0^t [q(t,u)]^k \, T_k(t-u) F_{m-k}(u)\, du$$

$$= S_m(t) - s\sum_{k=0}^{m}\sum_{r=0}^{k}(-1)^r\binom{k}{r}[h(t)]^{k-r}\int_0^t T_k(t-u)\,[h(u)]^r F_{m-k}(u)\,du.$$

Theorem 7.16. As t tends to infinity, we have

$$\Omega_m(t,s) \sim \sum_k \left(D_{km}\cos k\omega t + E_{km}\sin k\omega t\right)$$

for $s = i v$, where v is a real number such that $0 < |v| \leq \delta$ for some $\delta > 0$.

Proof: The Laplace transforms of $S_m(t)$ and $T_m(t)$ are given respectively by

$$S_m^*(p) = \frac{[1-\psi(s)]^m}{m!} e^{-sz} \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\left[\frac{1}{p-\sigma-\gamma(s)}\right]\left[\sum_k \frac{c_{nk}}{\sigma - ik\omega}\right] d\sigma$$

$$= \frac{[1-\psi(s)]^n}{m!} e^{-sz}\sum_k \frac{c_{nk}}{p-\gamma(s)-ik\omega},$$

$$T_m^*(p) = \frac{1}{m!}\,\frac{[1-\psi(s)]^m}{p-\gamma(s)}.$$

The Laplace transform of $\Omega_m(t,s)$ is

$$\Omega_m^*(p,s) = S_m^*(p) - s\sum_{k=0}^{m}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h_{k-r}^*(p-\sigma)\Big\{T_k^*(\sigma)$$

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h_r^*(\sigma-\nu)\, F_{m-k}(\nu)\,d\nu\Big\}\, d\sigma.$$

We now use the fact that the $F_r^*(p)$, $(r = 1, 2, \ldots, m)$ are of the form

$$F_r^*(p) = \sum_k \frac{\psi_{kr}(p)}{p - ik\omega},$$

that the $h_m^*(p)$ are of the form

$$h_m^*(p) = \sum_k \frac{c_{mk}}{p - ik\omega},$$

that $T_m^*(p)$ has only one pole at $p = \gamma(s)$ and that $S_m^*(p)$ has poles at $\gamma(s) + ik\omega$.

Now from the representation

$$\psi(s) = 1 - \rho s + \frac{\sigma^2}{2} s^2 + O(s^3),$$

we conclude that

$$\gamma(s) = s - 1 + \psi(s) = (1-\rho) s + \frac{\sigma^2}{2} s^2 + O(s^3),$$

so that if $s = iv$, and v is sufficiently small, we shall have, for $0 < |v| \leq \delta$, $\mathrm{Re}[\gamma(s)] < 0$.
Applying again the Abelian theorem for the complex inversion integral given by Doetsch [14] p. 488, we deduce that $S_m(t)$ tends to zero exponentially as t tends to infinity, and that the second term is asymptotic to an expression of the form

$$\sum_k \left(D_{km} \cos k\omega t + E_{km} \sin k\omega t \right).$$

<u>Corollary</u>: The Laplace transform of the content distribution, $\Omega(t,s;\omega)$, is asymptotically periodic for $s = iv$.

<u>Proof</u>: For the values of s considered, the series $\sum_n \Omega_n(t,s)\omega^n$ converges uniformly in $t \geq 0$. It follows that

$$\Omega^*(p,s;\omega) = \int_0^\infty e^{-pt}\,\Omega(t,s;\omega)\,dt = \sum_n \Omega_n^*(p,s)\omega^n.$$

This shows that $\Omega^*(p,s;\omega)$, as a function of p, has all its poles with largest real parts on the imaginary axis. The Corollary follows immediately by using a slight extension of Doetsch's ([14] p. 488) Abelian theorem.

10. <u>Explicit expressions for the leading terms in the case of a simple harmonic input</u>. Consider the special case $h(t) = \sin \omega t$. In this case, $h^*(p) = \dfrac{\omega}{p^2+\omega^2}$, and we find, for $n = 0$,

$$F_0^*(p) = \frac{r_0^*(p)}{R_0^*(p)} = \frac{e^{-z\theta_p}}{\theta_p},$$

where θ_p represents the unique solution in θ of the equation

$$p - \gamma(\theta) = p - \theta + 1 - \psi(\theta) = 0.$$

For $n = 1$, we find, after some cancellation,

$$F_1^*(p) = \frac{R_1^*(p)}{R_0^*(p)}\,\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} h^*(p-s)\,F_0^*(s)\,ds,$$

i.e.

$$F_1^*(p) = [1-\psi(\theta_p)] \cdot \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{\omega\, e^{-\theta_s z}}{[(p-s)^2+\omega^2]\,\theta_s}\, ds\ , \quad Re(p) > x > 0,$$

where θ_s is the root of $s - \gamma(\theta) = 0$.

Evaluating the contour integral and writing $p + i\omega = q$, $p - i\omega = r$, we find

$$F_1^*(p) = \frac{1-\psi(\theta_p)}{2i}\left[\frac{e^{-\theta_r z}}{\theta_r} - \frac{e^{-\theta_q z}}{\theta_q}\right]$$

$$= \frac{\theta_p - p}{2i}\left[\frac{e^{-\theta_r z}}{\theta_r} - \frac{e^{-\theta_q z}}{\theta_q}\right].$$

The function $F_0^*(p)$ has only one simple pole in $Re(p) > \gamma(\alpha)$, at $p = 0$, and the residue there is $\frac{1}{\theta'(0)} = 1 - \rho$.

Similarly, $F_1^*(p)$ has two poles, at $p = \pm i\omega$, and the residues are

$$-\frac{\theta(-i\omega) + i\omega}{2i}(1-\rho) \qquad \text{at} \quad p = -i\omega\ ,$$

$$\frac{\theta(i\omega) - i\omega}{2i}(1-\rho) \qquad \text{at} \quad p = +i\omega\ .$$

Using Doetsch's asymptotic theorem, we find that

$$F_0(t) \sim (1-\rho)\ ,$$

$$F_1(t) \sim \frac{1-\rho}{2i}\left[-\{\theta(-i\omega)+i\omega\}e^{-i\omega t} + \{\theta(i\omega)-i\omega\}e^{i\omega t}\right].$$

Put $\theta(i\omega) = x + iy$.

Then it is easy to check that $\theta(-i\omega) = x - iy$.

Replacing in the expression for $F_1(t)$, we find

$$F_1(t) \sim (1-\rho)\left[x \sin \omega t - (y - \omega) \cos \omega t \right].$$

The asymptotic expression for $F_0(t)$ is, of course, the limiting value of $W(t, 0)$ for stationary input previously obtained in Chapter 4.

Consider now the first terms of the expansion for $\Omega(t, s)$. We find

$$\Omega_0^*(p, s) = \frac{e^{-zs}}{p - \gamma(s)} - \frac{e^{-\theta_p z}}{[p - \gamma(s)]\theta_p},$$

$$\Omega_1^*(p, s) = \frac{e^{-zs}[1 - \psi(s)]}{2i}\left[\frac{1}{r - \gamma(s)} - \frac{1}{q - \gamma(s)}\right]$$
$$+ \frac{s(\theta_p - p)}{[p - \gamma(s)]}\left[\frac{1}{2i}\left\{\frac{e^{-\theta_q z}}{\theta_q} - \frac{e^{-\theta_r z}}{\theta_r}\right\}\right]$$
$$+ \frac{s[1 - \psi(s)]}{2i}\left[\frac{1}{\{q - \gamma(s)\}}\frac{e^{-\theta_q z}}{\theta_q} - \frac{1}{\{r - \gamma(s)\}}\frac{e^{-\theta_r z}}{\theta_r}\right]$$
$$- \frac{s[1 - \psi(s)]}{[p - \gamma(s)]}\left[\frac{1}{2i}\left\{\frac{e^{-\theta_q z}}{\theta_q} - \frac{e^{-\theta_r z}}{\theta_r}\right\}\right].$$

From this expression, we can deduce, putting as before $\theta(i\omega) = x + iy$,

$$\Omega_0(t, s) \sim \frac{s(1-\rho)}{\gamma(s)} = \frac{(1-\rho)s}{s - 1 + \psi(s)},$$

which is the Pollaczek-Khintchine formula previously obtained in Chapter 4 , and

$$\Omega_1(t,s) \sim s(1-\rho)\left[\left\{\frac{\gamma x+\omega(\omega-y)}{\gamma^2+\omega^2}+\frac{s(1-\psi)}{\gamma}-\frac{(1-\psi)\gamma}{\gamma^2+\omega^2}\right\}\sin\omega t\right.$$
$$\left.+\left\{\frac{\omega x-\gamma(\omega-y)}{\gamma^2+\omega^2}-\frac{(1-\psi)\omega}{\gamma^2+\omega^2}\right\}\cos\omega t\right],$$

where $\psi = \psi(s)$, $\gamma = \gamma(s) = s-1+\psi(s)$.

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