TRANSIENT ELECTROMAGNETIC WAVES
APPLIED TO
MINERAL EXPLORATION

by

Terence John LEE

T. Lee

graduating 1986

Thesis presented in fulfilment
of the degree D.Sc.
University of Tasmania
March 1985

1. *Introduction*

The effective use of transient electromagnetic waves for mineral exploration depends on a proper understanding of how these waves propagate in and are effected by different geological materials and structures. A study of these effects forms the basis of the work presented here.

Transient electromagnetic waves propagate in the ground at a rate that is determined by the electrical conductivity and the magnetic permeability of the ground. The direction in which the maximum current density propagates is determined by the type of source that is used to generate the waves (15, 18,19). For realistic host rock conductivities and measurement times, the velocity of propagation is quite fast. For this reason the effect of loop height of the transmitter is unimportant except for the earliest of times (16). However, some care should be exercised in interpreting the depths to conductors by means of the direction of the observed magnetic fields since the currents, even in a uniform ground, can behave as though they derive from sources at depth (17). For the situations that are generally met with in practice, displacement currents have a negligible effect.

Drastic changes to the appearance of the transient decay curves can be caused by geological materials whose conductivity or magnetic permeability varies with frequency. For a ground with a frequency dependent conductivity that may be described by means of the Cole-Cole Model, changes in sign of the transient may be observed (11,12). These effects may also be seen in the case where the conductivity of a structure, within an otherwise uniform ground, varies in the manner of the Cole-Cole Model (4). These studies show that the transient electromagnetic method of prospecting is sensitive to induced polarization effects, and this may account for some of the anomalous transients that have been encountered by the field geophysicist (4,12).

A magnetic permeability that varies with frequency is sufficient to explain all the observations that have been made in regard to surveys carried out over areas where superparamagnetic minerals are present on the earth's surface (1,2). A weakly magnetic ground, however, produces a transient that is quite similar to those observed over a non magnetic ground (2).

The effect of a conducting host enclosing a conducting structure is, for some stage of the transient, to produce a transient that can not be described by a simple exponential function (4,5,6,9,13). For the case of a conducting host the final form of the transient may be modelled by assuming that the electric field within the conductor can be approximated by the inducing electric field (6). One study showed that the resonant responses of the conductor rapidly decay and that the final form of the transient is determined by the conducting host rock enclosing the conductor (5).

It has been usual to model the transients by Fourier transformation of the frequency response of the structure. A consequence of this is that it is frequently desirable to have closed form expressions for a number of frequently occurring integrals. Some of these integrals have now been evaluated (7,10).

An alternative approach is to view the equation, for the transient, as being made up of a number of terms that are related to the singularities in the function describing the spectrum of the transient (14). This approach readily separates out the resonant from the non resonant response of the structure (5,13). It frequently leads to quite simple expressions, that are suitable for practical purposes (1,2,5,6,8). Also, the alternative perspective leads to quite different questions being asked about the transient (14).

Finally, it can be demonstrated that the effect of the host rock is to make the inversion of transient electromagnetic data quite difficult and that supplementary information is needed. This is true for layered and non layered structures (3,8).

2. *Acknowledgements*

In all cases, the help the author has had in the preparation of the papers, presented here, can be found in the acknowlegement section of the respective papers. This omits the contribution of Dr Roger Lewis who co-authored several of the papers.

There is no doubt in the other author's mind that Dr Lewis is, perhaps, the best geophysical programmer that Australia has produced to date. The difficulty in writing the code needed for the papers which he co-authored bears ample witness to this fact. Dr Lewis wrote this code. In these cases both authors shared in the selection of the cases to be computed and helped check one anothers work. The formulation of the problems remained the concern of the present writer. None of the work has previously been submitted for any degree.

3. References

(1) Lee, T., 1984; The effect of a superparamagnetic layer on the transient electromagnetic response of a ground. Geophysical Prospecting, Vol. 32, pp 480-496.

(2) Lee, T., 1984: The transient electromagnetic response of a magnetic or a superparamagnetic ground. Geophysics, Vol. 49. pp 854-860.

(3) Lee, T., 1984: Inversion of transient electromagnetic data from a spherical conductor. I.E.E.E. Trans., vol. GE-22 January, 1984, pp 14-20.

(4) Lewis, R., Lee, T., 1984: The detection of induced polarization with a transient electromagnetic system. I.E.E.E. Trans., Vol. GE-22 January, 1984, pp. 69-80.

(5) Lee, T., 1983: The transient electromagnetic response of a conducting sphere in an imperfectly conducting halfspace. Geophysical Prospecting, Vol. 31, pp 766-781.

(6) Lee, T., 1982: Asymptotic expansions for transient electromagnetic fields. Geophysics, vol. 47, pp.38-46.

(7) Lee, T., 1982: The two-dimensional Green's function for electromagnetic scattering. ASEG Bull, Vol. 13, pp 2-4.

(8) Lee, T., 1981: The effect of displacement currents on time domain electromagnetic fields, ASEG Bull., Vol. 12, pp. 34-36.

(9) Lewis, R., Lee, T., 1981: The effect of host rock on transient electromagnetic fields. ASEG Bull., vol. 12, pp. 5-12.

(10) Lee, T., 1981: Evaluation of a class of integrals occurring in mode matched solutions for electrical and electromagnetic modelling. Geophys. J.R. Astr. Soc., Vol. 66, pp. 455-460.

(11) Lee, T., 1981: The Cole Cole model in time domain I.P. Geophysics, Vol. 46, pp. 932-933.

(12) Lee, T., 1981: Transient electromagnetic response of a polarizable ground. Geophysics, Vol. 46, pp. 1037-1041.

(13) Lee, T., 1981: Transient electromagnetic response of a sphere in a layered medium. Pure and Applied Geophysics, Vol. 119, pp. 309-338.

(14) Lee, T., 1979: Transient electromagnetic waves applied to prospecting. Proc. IEEE, Vol. 67, pp. 1016-1021.

(15) Lewis, R., Lee, T., 1978: The transient electric fields about a loop on a halfspace. ASEG Bull., Vol. 9, pp. 173-177.

(16) Lee, T., 1978: The effect of Loop Height in transient electromagnetic modelling or prospecting, ASEG Bull., Vol. 9,. pp. 34-35.

(17) Lee, T., 1978: Depth estimation with PEM - a cautionary note, ASEG Bull., Vol. 9, pp. 168-169.

(18) Lee, T., 1977: Estimating of depth to conductors by the use of electromagnetic transients. Geophysical Prospecting, Vol. 25, pp. 61-75.

ıphysical Prospecting 32, 480–496, 1984.

ꟷIE EFFECT OF A SUPERPARAMAGNETIC LAYER ON THE TRANSIENT ELECTROMAGNETIC RESPONSE OF A GROUND*

T. LEE**

ABSTRACT

, T. 1984, The Effect of a Superparamagnetic Layer on the Transient Electromagnetic ɕonse of a Ground, Geophysical Prospecting 32, 480–496.

A thin superparamagnetic layer on the earth's surface greatly affects the transient electro-ʒnetic response of a conducting ground. The effect of the layer is most evident for single-ɔ transient electromagnetic data where transient voltages decay as $1/t$ Even when a ırate transmitter and receiver are used, the effect of the superparamagnetic layer is still nounced. In this case the effect of the $1/t$ term in the equation is much less. More ınant now is a $1/t^2$ term. The effect of the superparamagnetism can readily be seen in the lytical expressions for the apparent resistivities. If the presence of the superparamagnetic ɜr is not recognized, then the apparent resistivities decrease with time rather than ıroach the true value of the host rock.

1. INTRODUCTION

·e effect of superparamagnetic minerals on the transient response of various geo-.ical structures has been observed in a large amount of TEM data collected in ny different areas of Australia by CSIRO and a number of institutes and private npanies (Buselli 1982).

Buselli has analyzed the effect in some detail. He lists the following essential tures:

At the later stages of the transient the apparent resistivities of the ground decrease with time in areas where they would be expected to increase.

For coincident-loop transient data the voltages decay as $1/t$ at the later stages of the transient.

Received March 1983, revisions July 1983.
G.P.O. Box 1984, Canberra City A.C.T. 2601, Australia.

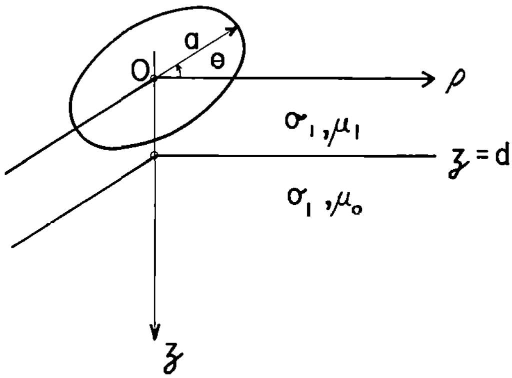


Fig 1 The geometry of the model.

The effect is due to superparamagnetic minerals in a thin layer at the earth's surface.

. The $1/t$ dependence of the transient can be dramatically reduced by employing a separate receiving loop. Moreover, the reduction occurs even when the separate receiving loop is only slightly separated from the transmitting loop.

At the time that Buselli published his paper, Weidelt (1982) argued that one hould pay much more attention to the frequency dependence of the electrical onductivity and the magnetic permeability of the ground. Previously Lee (1981a) ad shown that if the conductivity of the ground were to vary with frequency, then nomalous transients could be produced. The purpose of this paper is to show that ll of the features noted by Buselli with regard to what is now called the superaramagnetic effect can be explained if the magnetic permeability of the ground is llowed to vary with frequency.

The actual model of the ground is shown in fig. 1. This figure shows a twoiyered ground, both layers of which have the same conductivity σ_1. The magnetic ermeability of this ground is set equal to the permeability of free space (μ_0) everyhere except in the top layer where it is a function of frequency. The precise nature f μ_1 (the magnetic permeability of the top layer) is discussed in the next section.

The ground is assumed to be excited by a step current of height I_0 flowing in ircular loop of radius a on the earth's surface. Initially we shall be concerned with ie calculation of the transient electric field (**E**) at (ρ, θ) (fig. 1). Once this quantity as been found it is a simple matter to calculate the voltage in any receiving loop, or the voltage V is simply

$$V = \int_1 \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{1}$$

Here, d**l** denotes an element of the receiving loop and the integral is understood ɔ be taken along the entire circumference of the receiving loop.

2. The Susceptibility of the Superparamagnetic Layer

From equation (15.15) of Chikazumi and Charap (1978) one sees that if a constar magnetic field is suddenly created at zero time ($t = 0$) in the vicinity of some supe paramagnetic material, then magnetization $I_n(t)$ of the material can be described b the equation

$$I_n(t) = I_{n0}\left[1 - \frac{1}{\ln(\tau_2/\tau_1)}\left(\int_{t/\tau_2}^{\infty} \frac{\exp(-y)}{y}\,\mathrm{d}y - \int_{t/\tau_1}^{\infty} \frac{\exp(-y)}{y}\,\mathrm{d}y\right)\right] \qquad (2$$

Here τ_1 and τ_2 are the lower and upper time constants of the material respe tively and I_{n0} is the final value of the magnetization of the material.

Taking the Laplace transform of (2), one has that

$$\int_0^{\infty} \exp(-pt) I_n(t)\,\mathrm{d}t = \overline{I_n} \qquad (3$$

$$\overline{I_n} = \frac{I_{n0}}{p}\left[1 - \frac{1}{\ln(\tau_2/\tau_1)}\ln\left(\frac{1+\tau_2 p}{1+\tau_1 p}\right)\right]. \qquad (4$$

This result can be obtained by first integrating by parts and then using resu No. 5.4 of Oberhettinger and Badii (1973). If we allow for the spectrum of the ste function source, then the transfer function S for the magnetization is given by

$$S = \overline{I_n}\,p. \qquad ($$

Suppose now that in the presence of a magnetic field H the magnetization of th rocks is such that they are essentially nonmagnetic. In this case the quanti $I_{n0}/H = \chi_0$ (the susceptibility of the rocks) is quite small, and

$$S = \chi_0 H\left[1 - \frac{1}{\ln(\tau_2/\tau_1)}\ln\left(\frac{1+\tau_2 p}{1+\tau_1 p}\right)\right]. \qquad ($$

Thus the spectrum of the susceptibility χ of the rocks is simply S/H. If in th equation p is replaced by $\mathrm{i}\omega$ to represent the spectrum for an $\exp(\mathrm{i}\omega t)$ dependenc one has

$$\chi = \chi_0\left[1 - \frac{1}{\ln(\tau_2/\tau_1)}\cdot\ln\left(\frac{\tau_2\mathrm{i}\omega + 1}{\tau_1\mathrm{i}\omega + 1}\right)\right], \qquad (7$$

and the permeability of the material μ_1 is given by

$$\mu_1 = \mu_0[1 + \chi]. \qquad (8$$

It is important to realize that this equation can be simplified. For the probler we are now interested in, the time range is much greater than the shortest relaxatio time but less than the longest relaxation time. That is to say, the quantity τ_1 is sma relative to τ_2. Therefore,

$$\mu_1 = \mu_0\left[1 + \chi_0\left(1 - \frac{\ln(\mathrm{i}\omega\tau_2)}{\ln(\tau_2/\tau_1)}\right)\right]. \qquad (9$$

Notice that in this last equation the absolute value of the quantity

$$\chi_0\left(1 - \frac{\ln (i\omega\tau_2)}{\ln (\tau_2/\tau_1)}\right)$$

small since χ_0 is small and ω is comparable to $1/\tau_2$.

3. The Transient Electromagnetic Response of the Ground

ıppose that a large circular loop of radius r lies at $z = -h$, $h > 0$ above a two-yered ground, the upper layer of which has a thickness d, permeability μ_1, and nductivity σ, and the lower layer a conductivity of σ and a permeability μ_0. If a rrent of I exp $(i\omega t)$ flows in the loop, then the electric field $\bar{E}_1$, observed at the rface of the ground at a distance ρ from the center of the loop, is given by

$$\bar{E}_1(i\omega) = -i\omega\mu_0 rI \int_0^\infty \frac{Z^1 \exp(-\lambda z)}{Z^1 + Z_0} J_1(\lambda r) J_1(\lambda\rho)\, d\lambda \tag{10}$$

ee Morrison, Phillips and O'Brien 1969, p. 87.)

Here Z^1 and Z^2 are quantities that were first introduced by Wait (1962). They tisfy the relationships

$$Z^1 = Z_1 \frac{Z^2 + Z_1 \tanh(n_1 d)}{Z_1 + Z^2 \tanh(n_1 d)}, \tag{11}$$

$$Z^2 = Z_2 = \frac{-i\omega\mu_0}{n_2}$$

$$Z_0 = -\frac{i\omega\mu_0}{\lambda},$$

$$Z_1 = -\frac{i\omega\mu_0}{n_1},$$

ıere

$$n_1 = \sqrt{(\lambda^2 + i\omega\sigma_1\mu_1)},$$

$$n_2 = \sqrt{(\lambda^2 + i\omega\sigma_1\mu_0)}.$$

Notice that any effect of displacement currents has been ignored since the dielec-ɔ constants of all the media have been set to zero.

If a step current of height I_0 flows in the loop, a transient electromagnetic field (t), is observed. This field can be described by

$$E_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t)\bar{E}_1(\omega)\, d\omega. \tag{12}$$

Since d is small, $\bar{E}_1(\omega)$ can be approximated by expanding $Z^1/(Z_0 + Z^1)$ as
Taylor series in terms of d. From this we learn that

$$\frac{Z^1}{Z_0 + Z^1} = \frac{\lambda}{n_2 + \lambda} + \frac{d\lambda^3}{(\lambda + n_2)^2} \frac{(\mu_1^2 - \mu_0^2)}{\mu_0 \mu_1}. \quad (1$$

Since μ_1 is almost equal to μ_0 one can further simplify the expression to

$$\frac{Z^1}{Z_0 + Z^1} = \frac{\lambda}{n_2 + \lambda} + \frac{2\lambda^3 \chi d}{(\lambda + n_2)^2}. \quad (1$$

In this last expression

$$\chi = \chi_0\left(1 - \frac{\ln (\mathrm{i}\omega\tau_2)}{\ln (\tau_2/\tau_1)}\right). \quad (1$$

In what follows we have to consider the behavior of E at the wire itself becau
the superparamagnetic effect is greatest for the one-loop system. That is to say,
must allow for the finite radius b of the wire. If we assume that the current
uniformly distributed throughout the cross-section of the wire, then

$$E = \int_A \frac{E_1}{\pi b^2}\, \mathrm{d}A, \quad ($$

where A is the cross-sectional area of the wire.

With these remarks in mind one separates the contributions to the total elect
field into primary and secondary contributions. To this end one writes

$$\begin{aligned} E(t) = \frac{-I_0 \mu_0}{2\pi} \int_{-\infty}^{\infty} \exp (\mathrm{i}\omega t) \Bigg[& \int_0^{\infty} \int_A \frac{\lambda r \exp (-\lambda z)}{(n_2 + \lambda)} \cdot \frac{\mathrm{J}_1(\lambda r)\mathrm{J}_1(\lambda\rho)}{\pi b^2}\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\rho. \\ & + \int_0^{\infty} \int_A \frac{\chi r \lambda \exp (-\lambda z)}{2} \cdot \frac{\mathrm{J}_1(\lambda r)\mathrm{J}_1(\lambda\rho) d}{\pi b^2}\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\lambda \\ & + \int_A \int_0^{\infty} \left(\frac{2\lambda^3\chi}{(\lambda + n_2)^2} - \frac{\lambda\chi}{2}\right) \exp (-\lambda z) \frac{r \mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda r) d}{\pi b^2}\, \mathrm{d}\lambda\, \mathrm{d}z\, \mathrm{d}r \Bigg] \mathrm{d}\omega. \quad (\end{aligned}$$

The first two terms in (17) can be simplified immediately because they a
respectively, the response of a uniform ground (Lee and Lewis, 1974) and a singu
term which has arisen because of the superparamagnetic layer. If the wire is allow
to lie on the ground, then (16) can be reduced to

$$\begin{aligned} E(t) = E_p & - \frac{I_0 \mu_0 a d \Delta(\rho)}{4\pi} \int_{-\infty}^{\infty} \exp (\mathrm{i}\omega t)\chi\, \mathrm{d}\omega \\ & - \frac{a\mu_0 I_0 d}{2\pi} \int_{-\infty}^{\infty} \exp (\mathrm{i}\omega t) \int_0^{\infty} \chi\lambda \mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda a)\left[\frac{2\lambda^2}{(\lambda + n_2)^2} - \frac{1}{2}\right] \mathrm{d}\lambda\, \mathrm{d}\omega, \end{aligned}$$

where

$$\Delta(\rho) = \frac{1}{a} \int_0^{\infty} \int_A \lambda \frac{\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda r)}{\pi b^2} \exp (-\lambda z) r\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\lambda. \quad ($$

Notice that in (18) the integral over the cross-sectional area of the wire for the ondary term is approximated. This was made possible because for that integral integrand varies slowly over the wire since b is very small in comparison to a. re a is the distance from the center of the wire loop to the middle of the wire ckness; that is to say, it is the average radius of all the infinitesimal wire loops. cordingly, r and z were set equal to a and zero, respectively. By this means the egral was approximated by πb^2 which cancelled against the πb^2 in the denomina- . The quantity $\Delta(\rho)$ is evaluated in the Appendix.

The integral in the second term of (18) can be readily evaluated by contour egration for $t > 0$. From this we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\omega t)\chi \, d\omega = \frac{+\chi_0}{\ln(\tau_2/\tau_1)} \cdot \frac{1}{t}, \qquad t > 0. \tag{19}$$

nsequently

$$E(t) = E_p - \frac{I_0 \mu_0 a}{2t} \frac{\chi_0 d\Delta(\rho)}{\ln(\tau_2/\tau_1)} + E_{sp} + E_s, \tag{20}$$

ere

$$E_{sp} = \frac{-I_0 \mu_0 a \chi_0 d}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t) \int_0^{\infty} \lambda J_1(\lambda\rho) J_1(\lambda a)\left[\frac{2\lambda^2}{(\lambda + n_2)^2} - \frac{1}{2}\right] d\lambda \, d\omega \tag{21}$$

l

$$E_s = \frac{I_0 d \mu_0 a \chi_0}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\omega t)\ln(i\omega\tau_2)}{\ln(\tau_2/\tau_1)} \int_0^{\infty} \lambda J_1(\lambda\rho) J_1(\lambda a)\left[\frac{2\lambda^2}{(\lambda + n_2)^2} - \frac{1}{2}\right] d\lambda \, d\omega. \tag{22}$$

Notice that at the later stages

$$E_p \approx -\frac{\mu_0 \rho I_0}{40\sqrt{\pi}\, at}\left(\frac{\sigma_1 \mu_0 a^2}{t}\right)^{3/2} \tag{23}$$

Lee and Lewis 1974).

The first thing to notice at this stage of the analysis is that even though the top er of the ground is quite thin, it has nevertheless given rise to a very intense local d in the vicinity of the transmitting wires. The second term in (20) can be erpreted as a current proportional to $1/t$ flowing in the transmitting loop. From selli's observations one knows that this current can dominate the transient tage. We shall return to this point later.

Terms involving $t^{-7/2}$, $t^{-9/2}$, etc., have been ignored in (23). The behavior of the d term for the later stages of the transient is easy to find. All one has to do is to erchange the order of integration and integrate around the branch cut in the per ω-plane. This cut extends from $\omega = i\lambda^2/(\sigma_1\mu_0)$ to $i\infty$ (fig. 2). The integral with pect to λ is easy to evaluate once the Bessel functions are expanded in powers of This procedure will become much clearer after we consider the evaluation of E_s).

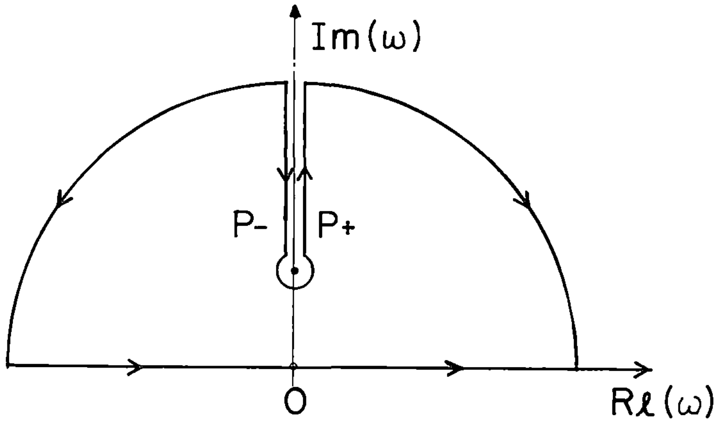


Fig. 2 The path for the contour integral for the second term of equation (20).

By this means one finds that at the later stages

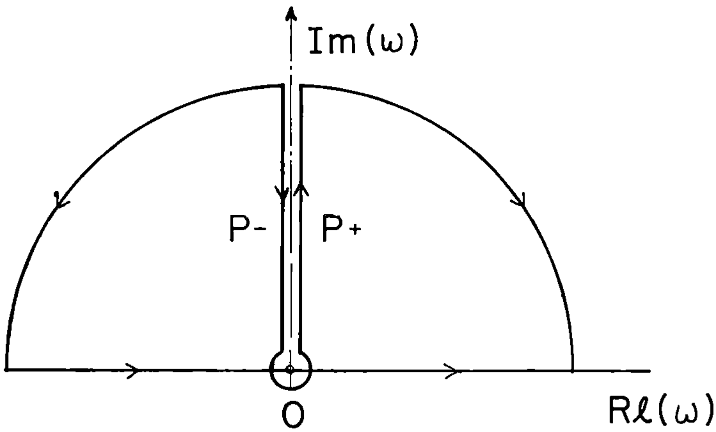
$$E_{sp} \approx \frac{-I_0 \mu_0 \chi_0}{t} \frac{5}{128} \frac{\rho}{a^2}\left[\frac{\sigma_1 \mu_0 a^2}{t}\right]^2 d.$$

Terms involving t^{-4}, t^{-5}, etc., have been ignored in (24).

The expression for E_s may be reduced by first interchanging the order of integ tion and then evaluating the integral over ω by contour integration. To this end notices that the quantity n_2 implies a branch cut along the positive imaginary ω-a from the point $i\lambda^2/(\sigma_1\mu_0)$ to infinity and that the term log $(i\omega)$ also implies a bra cut along all of the positive imaginary ω-axis. Because of Jordan's theorem integral of the spectrum of E_s about infinitely large, quarter circular arcs in upper ω-plane is zero. Consequently the integral over ω along the real axis may replaced by an integral about the positive imaginary ω-axis (fig. 3). It is conven to split the integral into two parts, the first part from 0 to $i\lambda^2/(\sigma_1\mu_0)$ and the sec part from $i\lambda^2/(\sigma_1\mu_0)$ to $i\infty$ (fig. 2). If one sets $\omega = \exp(i\pi/2)R$ on P^+ and $\omega =$ $(-3\pi i/2)R$ on P^-, where P^+ and P^- are paths to the right and the left of imaginary ω-axis, then one finds that

$$E_s = \frac{dI_0 a\mu_0 \chi_0}{\ln(\tau_2/\tau_1)} \int_0^\infty J_1(\lambda\rho)J_1(\lambda a)$$
$$\times \left\{\int_0^{\lambda^2/(\sigma_1\mu_0)} - 2\pi\left[\frac{\lambda^3 \exp(-Rt)}{(\lambda+n_2)^2} - \frac{1}{4}\right] dR + 2\int_{\lambda^2/(\sigma_1\mu_0)}^\infty \frac{\exp(-Rt)}{(\sigma_1\mu_0 R)^2}\right.$$
$$\left.\times (2\lambda \ln(\tau_2 R)\sqrt{(R\sigma_1\mu_0 - \lambda^2)} - \pi(2\lambda^2 - R\sigma_1\mu_0))\, dR\right\} d\lambda.$$

To proceed, one writes $R = \lambda^2/(\sigma_1\mu_0)$ and splits the first integral with respect into two parts, the first part from 0 to ε and the second from ε to 1. The reason this is to allow us to evaluate the integral with respect to λ after interchanging



g. 3. The path for the contour integral for the remaining integration over ω in equation 0).

·der of integration for all the integrals with respect to s, except for that one that is :tween 0 and ε which may be evaluated now because ε is small. In fact

$$
\begin{aligned}
E_s = {} & \frac{dI_0 a\mu_0 \chi_0}{\ln(\tau_2/\tau_1)} \int_0^\infty \frac{J_1(\lambda\rho)J_1(\lambda a)}{2\pi} \\
& \times \left\{ \frac{-2\pi\lambda^3}{\sigma_1\mu_0} \int_0^\varepsilon \exp(-st\lambda^2/(\sigma_1\mu_0)) \left[\left(\frac{1}{1+\sqrt{(1-s)}} \right)^2 - \frac{1}{4} \right] ds - \frac{2\pi\lambda^3}{\sigma_1\mu_0} \right. \\
& \times \int_\varepsilon^1 \exp(-st\lambda^2/(\sigma_1\mu_0)) \left[\frac{1}{(1+\sqrt{(1-s)})^2} - \frac{1}{4} \right] ds \\
& \left. + \frac{2\lambda^3}{\sigma_1\mu_0} \int_1^\infty \frac{\exp(-st\lambda^2/(\sigma_1\mu_0))}{s^2} \left(2\sqrt{(s-1)} \ln\left(\frac{s\lambda^2\tau_2}{\sigma_1\mu_0} \right) - \pi(2-s) \right) ds \right\} d
\end{aligned}
\tag{26}
$$

To proceed, one notices that all integrals over λ are of the form

$$
\int_0^\infty \lambda^3 J_1(\lambda a) J_1(\lambda\rho) \exp(-\lambda^2 ts/(\sigma_1\mu_0)) F(\lambda)\, d\lambda.
$$

ere $F(\lambda)$ is either 1 or $\ln(\lambda^2 s\tau_2/(\sigma_1\mu_0))$.

By means of the addition theorems for the Bessel functions, one finds that

$$
\begin{aligned}
& \int_0^\infty J_1(\lambda a)J_1(\lambda\rho)F(\lambda) \exp\left(-\frac{\lambda^2 ts}{\sigma_1\mu_0} \right) \lambda^3\, d\lambda \\
& \qquad = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} J_0(\lambda R)\lambda^3 \exp\left(-\frac{\lambda^2 ts}{\sigma_1\mu_0} \right) F(\lambda) \cos(\theta)\, d\theta\, d\lambda,
\end{aligned}
\tag{27}
$$

here $R^2 = a^2 + \rho^2 - 2a\rho\cos\theta$.

This result can be expressed as

$$\frac{-\sigma\mu_0}{t2\pi}\frac{\partial}{\partial s}\int_0^{2\pi}\cos(\theta)\int_0^{\infty}\mathrm{J}_0(\lambda R)\lambda\exp\left(-\frac{\lambda^2 ts}{\sigma_1\mu_0}\right)F(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\theta.$$

To proceed, one first evaluates the integral from 0 to ε and then the integral ov λ after first interchanging the order of integration where necessary. The integral wi respect to s from 0 to ε is easily evaluated after noting that for small s

$$\frac{1}{[1+\sqrt{(1-s)}]^2}-\frac{1}{4}\approx\frac{s}{8}. \qquad (2$$

Also the integrations with respect to λ are easily evaluated by means of sor results (Nos. 1.705 and 1.706) of Wheelan (1968).

After neglecting terms smaller than ε^2 and employing Wheelan's results, one ha

$$\begin{aligned}
E_s = \frac{I_0 a\mu_0\chi_0}{\ln(\tau_2/\tau_1)}\, d\Bigg[&\frac{\pi\sigma_1\mu_0}{8t^2}\int_0^{2\pi}\frac{\exp(-R^2\sigma_1\mu_0/(4t))}{2\pi}\\
&\times\cos(\theta)\,\mathrm{d}\theta+\int_0^{\infty}\frac{\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda a)}{2\pi}\\
&\times\Bigg[\frac{6\pi}{4t}\lambda\exp\left(-\frac{t\lambda^2}{\sigma_1\mu_0}\right)-\frac{2\pi\varepsilon\lambda}{8t}\exp(-\varepsilon t\lambda^2/(\sigma_1\mu_0))\\
&-\frac{2\pi\lambda}{t}\int_\varepsilon^1\frac{1}{\sqrt{(1-s)}}\cdot\frac{\exp\left(-\frac{st\lambda^2}{\sigma_1\mu_0}\right)}{[\sqrt{(1-s)}+1]^3}\,\mathrm{d}s\Bigg]\,\mathrm{d}\lambda\\
&+\int_0^{2\pi}\frac{\cos(\theta)}{2\pi}\Bigg[\frac{2}{\sigma_1\mu_0}\int_1^{\infty}\frac{2\sqrt{(s-1)}}{s^2}\left(\frac{\sigma_1\mu_0}{4st}\right)^2\\
&\times\Bigg\{8\exp\left(-\frac{\sigma_1\mu_0 R^2}{4st}\right)\left(\frac{\sigma_1\mu_0 R^2}{4st}-1\right)\left(\mathrm{Ei}\left(\frac{\sigma_1\mu_0 R^2}{4st}\right)-\ln\left(\frac{\sigma_1\mu_0 R^2}{4st}\right)\right.\\
&\left.-8+16\exp\left(-\frac{\sigma_1\mu_0 R^2}{4st}\right)\right)+64\ln\left(\frac{\tau_2}{t}\right)\\
&\times\left(\exp\left(-\frac{\sigma_1\mu_0 R^2}{st}\right)\left(1-\frac{\sigma_1\mu_0 R^2}{st}\right)\right)\Bigg\}\,\mathrm{d}s\\
&+\frac{2\pi}{t}\int_1^{\infty}\frac{(2-s)}{s^2}\frac{\partial}{\partial s}\exp(-\sigma_1\mu_0 R^2/(4st))\left(\frac{\sigma_1\mu_0}{2ts}\right)\mathrm{d}s\Bigg]\,\mathrm{d}\theta\Bigg].
\end{aligned} \qquad (2$$

In (29) Ei(x) is the exponential integral see (Abramowitz and Stegun 1965, p. 2 No. 5-1-2). The remaining integral over s may be reduced by first subtracti (adding) the term 1/8 to the integral and then integrating with respect to λ af nging the order of integration. After these manipulations one finds that

$$E_s = \frac{I_0 a \mu_0 \chi_0 d}{\ln(\tau_2/\tau_1)} \left[\frac{\pi}{8t^2} \cdot \frac{\sigma_1 \mu_0}{1} \int_0^{2\pi} \frac{\exp(-R^2 \sigma_1 \mu_0/(4t))}{2\pi} \cos(\theta)\, d\theta \right.$$

$$\times \int_0^{2\pi} \cos(\theta) \left[\frac{3}{8t^2} \cdot \frac{\sigma_1 \mu_0}{1} \right.$$

$$\left. \times \exp(-\sigma_1 \mu R^2/(4t)) - \frac{\sigma_1 \mu_0}{16t} \exp(-R^2 \sigma_1 \mu_0/(4\varepsilon t)) \right] d\theta$$

$$+ \int_0^\infty \frac{\sigma_1 \mu_0 \pi}{4\lambda t} \left[\frac{J_1(\lambda a) J_1(\lambda \rho)}{2\pi} (\exp(-t\lambda^2/(\sigma_1 \mu_0)) - 1) \right] d\lambda$$

$$- \frac{\pi \sigma_1 \mu_0}{t^2} \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{(1-s)}} \left(\frac{1}{(\sqrt{(1-s)}+1)^3} - \frac{1}{8} \right)$$

$$\times \exp(-R^2 \sigma_1 \mu_0/(4ts)) \frac{\cos(\theta)}{2\pi s}\, ds\, d\theta + \int_0^{2\pi} \frac{\cos(\theta)}{2\pi}$$

$$\times \left[\frac{2}{\sigma_1 \mu_0} \int_0^\infty \frac{2\sqrt{(s-1)}}{s^2} \left(\frac{\sigma_1 \mu_0}{4st} \right)^2 \left\{ 8 \exp(-\sigma_1 \mu_0 R^2/(4st)) \left(\frac{\sigma_1 \mu_0 R^2}{4st} - 1 \right) \right. \right.$$

$$\times \left(\mathrm{Ei}\left(\frac{\sigma_1 \mu_0 R^2}{4st} \right) - \ln\left(\frac{\sigma_1 \mu_0 R^2}{4st} \right) - 8 + 16 \exp(-\sigma_1 \mu_0 R^2/(4st)) \right)$$

$$\left. + 64 \ln\left(\frac{\tau_2}{t} \right) \left(\exp(-\sigma_1 \mu_0 R^2/(st)) \left(1 - \frac{\sigma_1 \mu_0 R^2}{st} \right) \right\} ds$$

$$\left. \left. + \frac{2\pi}{t} \int_1^\infty \frac{(2-s)}{s^2} \frac{\partial}{\partial s} \exp(-\sigma_1 \mu_0 R^2/(4st)) \left(\frac{\sigma_1 \mu_0}{2ts} \right) ds \right] d\theta \right]. \tag{30}$$

By this stage the reader will be glad to learn that (30) can be simplified. Firstly, : allows ε to approach zero. Secondly, the single integrals involving θ are approx- ıted by expanding the exponential integral as a power series and then integrating n by term. Thirdly, the integral involving Bessel functions and the exponential ction is approximated by expanding the Bessel functions as a power series in λ then integrating term by term. The remaining integral involving a Bessel func- ı is evaluated by means of result No. 11.204 of Wheelan (1968). Fourthly, the ;gral with respect to s between 0 and 1 is evaluated by making the substitution $1/\cosh^2(\psi)$ and then following the procedure of Lee (1981b). Fifthly, the infinite ;gral with respect to s is simplified by expanding the function within the outer- st pair of brackets as a power series in inverse powers of s. This is quite straight- ward for the exponential integral may be expanded by means of result No. 5.1.10 Abramowitz and Stegun (1965, p. 229). By this means one finds that once terms of less than $(\sigma_1 \mu R^2/t)^4 \ln(t/\tau_2)$ are neglected, one finds that

$$E_s = \frac{I_0 a\mu_0 \chi_0 d}{\ln(\tau_2/\tau_1)}\left[\frac{\pi\sigma_1\mu_0\rho}{at^2}\left(\frac{\sigma_1\mu_0 a^2}{4t}\right) - \frac{\sigma_1\mu_0}{16t^2} G\left(\frac{\rho}{a}, \frac{a}{\rho}\right)\right.$$

$$- \frac{\sigma_1\mu_0}{2t^2}\int_0^{2\pi} \cos\theta \exp(-R^2\sigma_1\mu_0/(8t))$$

$$\times \left[\frac{1}{8}\left(3K_1\left(\frac{R^2\sigma_1\mu_0}{8t}\right) + 3K_2\left(\frac{R^2\sigma_1\mu_0}{8t}\right) + K_3\left(\frac{R^2\sigma_1\mu_0}{8t}\right)\right)\right.$$

$$\left. - \exp(-R^2\sigma_1\mu_0/(8t))\left(8\left(\frac{8t}{R^2\sigma_1\mu_0}\right)^3 + 7\left(\frac{8t}{R^2\sigma_1\mu_0}\right)^2 + 3\left(\frac{8t}{R^2\sigma_1\mu_0}\right)\right)\right] d\theta$$

$$+ \int_0^{2\pi} \frac{\cos\theta}{2\pi}\left[\frac{\sigma_1\mu_0}{4t^2}\int_1^\infty \frac{\sqrt{(s-1)}}{s^4}\left(8\left((2\gamma+31)\frac{\sigma_1\mu_0 R^2}{4st} - 8 - \gamma\right)\right.\right.$$

$$\left. + 64\ln\left(\frac{\tau_2}{t}\right)\left(1 - \frac{2\sigma_1\mu_0 R^2}{st} + \frac{3}{2s^2}\left(\frac{\sigma_1\mu_0 R}{t}\right)^2\right)\right) ds$$

$$\left.\left. + \frac{2\pi}{t}\int_1^\infty \frac{(2-s)}{s^2}\left[\frac{-\sigma_1\mu_0}{2ts^2} + \frac{\sigma_1\mu_0}{2ts^3}\cdot\frac{2\sigma_1\mu_0 R^2}{4t}\right] ds\right] d\theta\right].$$

In (31) γ is Euler's constant which has the approximate value of 0.57 (Abramowitz and Stegun 1965, p. 255). The function $G(\rho/a, a/\rho)$ is equal to ρ/a $\rho < a$ or is equal to a/ρ for $\rho > a$. To proceed, one expands the exponential Bessel functions into their respective series form (Abramowitz and Stegun 1965, 69 and 377). If the integrals with respect to s are evaluated by elementary means terms smaller than $\ln(t)/t^4$ are neglected in the series representation for the exponential or Bessel functions, one has

$$E_s \approx \frac{I_0 a\mu_0 \chi_0 d}{\ln(\tau_2/\tau_1)}\left[\frac{\pi\sigma_1\mu_0\rho}{at^2}\left(\frac{\sigma_1\mu_0 a^2}{4t}\right) - \frac{\sigma_1\mu_0}{16t^2} G(\rho/a, a/\rho)\right.$$

$$- \frac{\sigma_1\mu_0}{2t^2}\int_0^{2\pi} \cos\theta\left(\frac{29}{48} - \frac{\sigma_1\mu_0}{t}\cdot\frac{R^2}{2}\cdot\left(\frac{359}{48} - 3\gamma\right)\right.$$

$$\left. + \frac{3}{8}\left(\frac{\sigma_1\mu_0 R^2}{4t} - \frac{3}{32}\left(\frac{\sigma_1\mu_0 R^2}{t}\right)^2\right)\ln\left(\frac{\sigma_1\mu_0 R^2}{4t}\right)\right) d\theta$$

$$+ \int_0^{2\pi} \frac{\cos\theta}{2\pi}\cdot\left\{\frac{\sigma_1\mu_0}{4t^2}\left\{8\left(\frac{(2\gamma+31)\sigma_1\mu_0 R^2}{4t}\cdot\frac{5\pi}{128} - (8+\gamma)\cdot\frac{\pi}{16}\right)\right.\right.$$

$$\left. + 64\ln\left(\frac{\tau_2}{t}\right)\left(\frac{\pi}{16} - \frac{2\sigma_1\mu_0 R^2}{t}\cdot\frac{5\pi}{128} + \frac{3}{2}\left(\frac{\sigma_1\mu_0 R^2}{t}\right)^2\cdot\frac{7\pi}{256}\right)\right\}$$

$$\left.\left. + \frac{2\pi}{t}\left(\frac{-\sigma_1\mu_0}{12t} + \frac{\sigma_1\mu_0}{12t}\cdot\frac{2\sigma_1\mu_0 R^2}{4t}\right)\right\} d\theta\right].$$

quation (32) may be reduced by integrating with respect to θ. This may be ily done either by elementary means or by using result Nos. 4.2214 and 4.3976 radshteyn and Ryzhik (1980). By this means we find that

$$
\begin{aligned}
E_s \approx\ & \frac{-I_0\, a\mu_0\, \chi_0\, d}{\ln(\tau_2/\tau_1)}\, \frac{\sigma_1\mu_0}{16t^2}\, G(\rho/a,\, a/\rho) \\
& - \frac{I_0\, \mu_0\, \chi_0\, d}{\ln(\tau_2/\tau_1)} \cdot \frac{\pi\sigma_1\mu_0\,\rho}{t^2}\left(\frac{a^2\sigma_1\mu_0}{4t}\right) \\
& \times \left[\frac{1059}{64} - \frac{187}{32}\,\gamma + \frac{3}{16}\, G^2\!\left(\frac{\rho}{a}, \frac{a}{\rho}\right) - \frac{3}{8}\left(\frac{a^2+\rho^2}{a^2}\right)\frac{a}{\rho}\right. \\
& \times G\!\left(\frac{\rho}{a}, \frac{a}{\rho}\right) - \frac{3}{4}\, H\!\left(0, \ln\left(\frac{\rho}{a}\right)\right) \\
& + \ln(\tau_2/t)\left(-5 + \frac{21}{4}\left(\frac{a^2+\rho^2}{a^2}\right)\left(\frac{\sigma_1\mu_0\, a^2}{4t}\right)\right) \\
& \left. + \ln\left(\frac{\sigma_1\mu_0\, a^2}{4t}\right)\left(\frac{-3}{8} + \frac{15}{8}\left(\frac{a^2+\rho^2}{a^2}\right)\left(\frac{\sigma_1\mu_0\, a^2}{4t}\right)\right)\right].
\end{aligned} \tag{33}
$$

n (33), the function

$$
\begin{aligned}
H(0, \ln(\rho/a)) &= 0, \qquad \rho \leqslant a, \\
&= \ln(\rho/a), \qquad \rho > a.
\end{aligned} \tag{34}
$$

ollecting the results from (20), (21), (24), and (33) we have

$$
\begin{aligned}
E \approx\ & \frac{-I_0\, \mu_0}{2t}\, \frac{\chi_0\, d\Delta(\rho)a}{\ln(\tau_2/\tau_1)} - \frac{I_0\, a\mu_0\, \chi_0\, d}{\ln(\tau_2/\tau_1)} \cdot \frac{\sigma_1\mu_0}{16t^2} \cdot G(\rho/a,\, a/\rho) \\
& - \frac{\rho\mu_0\, I_0}{40\sqrt{\pi}\, at}\left(\frac{\sigma_1\mu_0\, a^2}{t}\right)^{3/2} - \frac{I_0\, \mu_0\, \chi_0\, d}{\ln(\tau_2/\tau_1)}\, \frac{\pi\sigma_1\, \mu_0\,\rho}{t^2}\left(\frac{a^2\sigma_1\mu_0}{4t}\right) \\
& \times \left[\frac{1059}{64} - \frac{187}{32}\,\gamma + \frac{3}{16}\, G^2(\rho/a,\, a/\rho)\right. \\
& - \frac{3}{8}\left(\frac{a^2+\rho^2}{a^2}\right)\frac{a}{\rho}\, G(\rho/a,\, a/\rho) - \frac{3}{4}\, H(0, \ln(\rho/a)) \\
& + \ln(\tau_2/t)\left(-5 + \frac{21}{4}\left(\frac{a^2+\rho^2}{a^2}\right)\left(\frac{\sigma_1\mu_0\, a^2}{4t}\right)\right) \\
& \left. + \ln\left(\frac{\sigma_1\mu_0\, a^2}{4t}\right)\left(\frac{-3}{8} + \frac{15}{8}\left(\frac{a^2+\rho^2}{a^2}\right)\left(\frac{\sigma_1\mu_0\, a^2}{4t}\right)\right)\right].
\end{aligned} \tag{35}
$$

4. Discussion

From fig. 4 and (1) one sees that the voltage V induced in a receiving loop is

$$V = \int_1 E \cos(\psi) \, \mathrm{d}l.$$

Here, ψ is the angle between the direction of the electric field and the element of the receiving loop. To be more specific and to gain a better idea c meaning of (35), it is convenient to examine a number of limiting cases.

The first case to consider is if the transmitting loop coincides with the rece loop. In that case the voltage induced in the receiver is:

$$V \approx \frac{-I_0 \mu_0 a}{bt} \frac{\chi_0 d}{\ln(\tau_2/\tau_1)} - \frac{-I_0 a \mu_0 \chi_0 d}{\ln(\tau_2/\tau_1)} \frac{\sigma_1 \mu_0}{16t^2} \cdot (2\pi a) - \frac{\sqrt{\pi}\, a \mu_0}{20t} I_0 \left(\frac{\sigma_1 \mu_0 a^2}{t} \right)^{3/}$$

Equation (37) shows why the superparamagnetic effect is so serious. Wh happening is that, although χ_0 is small, so too is b. The result is that the first te (37) will be the dominant term for sufficiently small b. Accordingly, the vc decays as $1/t$. Notice also that from a plot of V against t on log paper, it mig possible to estimate the important quantity $\chi_0 d/\ln(\tau_2/\tau_1)$.

We show below that this quantity is required if we are to estimate the app resistivity of the ground. The point is made, therefore, that even though coinci loop time-domain data obtained over a superparamagnetic layer are only slightly affected by the conductivity of the ground, they still contain inform that can be used to help interpret noncoincident loop transient electromag data.

Now suppose that the transmitting loop is separate from the receiving loo the wires of the transmitting and receiving loops do not cross each other. In case, the observed voltage is described by (38). For the case where the wires do each other, (38) must be modified by using the exact expression for $\Delta(\rho)$ i

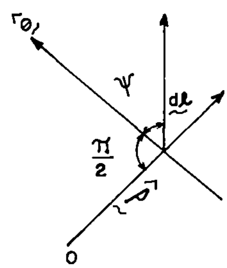


Fig. 4 Geometry for the element of the loop of wire

inity of the crossed wires (see (A3) of the appendix):

$$V \approx \frac{-I_0 \mu_0}{2t} \frac{\chi_0 db}{\ln (\tau_2/\tau_1)} \int_1 \frac{4}{3\pi^2} \sqrt{\left(\frac{a}{\rho}\right)} \cdot \frac{1}{(\rho - a)^2}$$

$$\times \left[\frac{(\rho^2 + a^2)}{2\rho a} Q_{-1/2}\left(\frac{\rho^2 + a^2}{2a\rho}\right) - Q_{1/2}\left(\frac{\rho^2 + a^2}{2a\rho}\right)\right] \cos (\psi) \, dl$$

$$- \frac{I_0 \mu_0 d\chi_0}{\ln (\tau_2/\tau_1)} \frac{\sigma_1 \mu_0}{16t^2} \int_1 G(\rho/a, a/\rho) \cos (\psi) \, dl$$

$$- \frac{I_0 \mu_0}{40\sqrt{\pi} at} \left(\frac{\sigma_1 \mu_0 a^2}{t}\right)^{3/2} \int_1 \rho \cos (\psi) \, dl. \tag{38}$$

Notice that for this case the presence of the term b in the numerator has stically reduced the contribution of the first term. However, even though the atour of the receiving loop might be adjusted to minimize that term and to ximize the secondary terms for some particular times, nevertheless it is still truc t the transient eventually decays as $1/t$.

It is interesting to calculate the apparent resistivity ρa_1 $(= 1/\sigma_1)$ of the ground the method of Lee (1977). Lee (1977) showed that the apparent resistivity of the und ρ_a at the later stages of the transient is given approximately by

$$\rho_a = \frac{\mu_0 a^2}{t^{5/3}} \left(\frac{I_0 \sqrt{\pi} a \mu_0}{20V}\right)^{2/3}. \tag{39}$$

If we apply this result to (38) and allow t to be large, we find that

$$\rho_a \propto 1/t. \tag{40}$$

From (40) one sees that the apparent resistivities ultimately do not approach the e resistivity of the host rock. Rather, the estimated resistivity will decrease as one s data from the later stages of the transient.

The results presented above show that some care is needed when interpreting nsient electromagnetic data over a superparamagnetic ground. They also support observations made by Buselli (1982) and provide some insight into the nature of transient. It is also clear that any further analysis will require some accurate asurements of the susceptibility of the various superparamagnetic soils. This task awaits the experimentalists.

In the absence of any satisfactory rock measurements, the geophysicist must try stimate from his field data alone the conductivity and the magnetic properties of ground. We have already noticed that if coincident loop data are plotted on a -log graph paper, the curve corresponding to the later stages obeys $V) =$ constant $- \ln (t)$. In this case, the constant can be found from the intercept he graph with the $\ln V$-axis. Accordingly, an estimate for $\chi_0 d/\ln (\tau_2/\tau_1)$ can be

found. Once this is known, one can estimate from (38) the resistivity of the grou
This estimate might be compared with that from an alternative method.

Suppose the transient voltages for a noncoincident loop geometry are known a series of times. At the later stages the data are described by (38). Let the volta V_1 and V_2 be known at times t_1 and t_2. Accordingly, one has two sets of equati for the unknowns. These equations can be used to calculate σ and $\chi_0 d/\ln(\tau_2/\tau_1)$.

Alternatively, one can form the equation

$$t^2 V \approx \frac{-4I_0\mu_0\chi_0 dbt}{3\pi^2 \ln(\tau_2/\tau_1)} \int_l \sqrt{\left(\frac{a}{\rho}\right)}\left(\frac{(\rho^2+a^2)}{2\rho a} Q_{-1/2}\left(\frac{\rho^2+a^2}{2\rho a}\right) - Q_{1/2}\left(\frac{\rho^2+a^2}{2a\rho}\right)\right) \times \frac{\cos(\psi)\, dl}{(\rho-a)^2} \frac{-I_0\mu_0 d\chi_0\mu_0\sigma}{16\ln(\tau_2/\tau_1)} \int_l G(\rho/a, a/\rho)\cos(\psi)\, dl. \quad ($$

If this data set is fitted to (41) in a least-squares sense, it will be found that fr the slope of the line one can estimate the quantity $\chi_0 d/\ln(\tau_2/\tau_1)$, and once quantity is known the conductivity or the resistivity can be determined from constant term in (41).

APPENDIX

If a is the average radius of the wire loop (i.e., the distance from the center of current loop to the middle of the wire thickness), and if $\Delta(\rho)$ is calculated for average height b above the ground (i.e., the wire lies on the ground), then

$$\pi b^2 \Delta(\rho) = \frac{1}{a}\int_0^\infty \int_{-b}^{b} (a+s) \int_{-\sqrt{(b^2-s^2)}}^{\sqrt{(b^2-s^2)}} \lambda J_1(\lambda\rho) J_1(\lambda(a+s)) \times \exp[-\lambda(b-z)]\, dz\, ds\, d\lambda \quad ($$

$$= \frac{1}{a}\int_0^\infty \int_{-b}^{b} (a+s) J_1(\lambda\rho) J_1(\lambda(a+s)) \times \exp[-\lambda(b-\sqrt{(b^2-s^2)})] - \exp[-\lambda(b+\sqrt{(b^2-s^2)})]\, ds\, d\lambda \quad ($$

$$= \frac{1}{\pi a}\int_{-b}^{b} \sqrt{\left(\frac{a+s}{\rho}\right)}\left(Q_{1/2}\left(\frac{(b-\sqrt{(b^2-s^2)})^2+\rho^2+(a+s)^2}{2\rho(a+s)}\right) - Q_{1/2}\left(\frac{(b+\sqrt{(b^2-s^2)})^2+\rho^2+(a+s)^2}{2\rho(a+s)}\right)\right) ds$$

(see result No. 560.02 of Byrd and Friedman 1954).

If $a = \rho$ then both the arguments of the two ring functions are close to unity consequently the expression for $\Delta(\rho)$ can be approximated by using the asympt form of the ring functions near their singularities (Magnus, Oberheltinger and S 1966, p. 196).

Accordingly, for this case

$$\pi b^2 \Delta(\rho) = \frac{1}{\pi a} \int_0^b \ln\left(\frac{b + \sqrt{(b^2 - s^2)}}{b - \sqrt{(b^2 - s^2)}}\right) ds \tag{A4}$$

$$\approx \frac{1}{\pi a} \cdot b\pi \tag{A5}$$

$$\therefore \quad \Delta(\rho) = \frac{1}{\pi ab}. \tag{A6}$$

However, if $|a - \rho| \gg s$, then (A3) can be approximated by expanding the differ- of the ring functions as a Taylor series about the point $(r^2 + (a+s)^2)/$ $a + s))$.

By this means one finds that for $|a - \rho| \gg s$

$$\Delta(\rho) = \frac{-1}{\pi ab^2} \int_{-b}^{b} \sqrt{\left(\frac{a+s}{\rho}\right)} \frac{(b^2 - s^2)}{(\rho - a - s)^2} \left[\frac{\rho^2 + (a+s)^2}{2\rho(a+s)} \times Q_{1/2}\left(\frac{\rho^2 + (a+s)^2}{2\rho(a+s)}\right) - Q_{-1/2}\left(\frac{\rho^2 + (a+s)^2}{2\rho(a+s)}\right)\right] ds, \tag{A7}$$

pproximately

$$\Delta(\rho) \approx \frac{4b}{3\pi^2 a} \sqrt{\left(\frac{a}{\rho}\right)} \frac{1}{(\rho - a)^2} \left[\left(\frac{\rho^2 + a^2}{2\rho a}\right) Q_{-1/2}\left(\frac{\rho^2 + a^2}{2a\rho}\right) - Q_{1/2}\left(\frac{\rho^2 + a^2}{2a\rho}\right)\right]. \tag{A8}$$

Acknowledgments

author has benefitted from discussions with a number of people at CSIRO. se include Drs Dave Clark, Brian Embleton, G. Buselli, and Harry Brown. Mrs yn Harbour displayed great patience in typing the manuscript.

References

MOWITZ, M. and STEGUN, I.A. (eds) 1965, Handbook of Mathematical Functions, Dover, New York

LLI, G. 1982, The effect of near surface superparamagnetic material on electromagnetic transients, Geophysics 47, 1315–1324.

, P.F. and FRIEDMAN, M.D. 1954, Handbook of Elliptic Integrals for Engineers and Physicists, Springer-Verlag, Berlin, Heidelberg, New York.

AZUMI, S. and CHARAP, S.H. 1978, Physics of Magnetism, Robert E Krieger, Huntington, New York.

DSHTEYN, J.S. and RYZHIK, I.M. 1980 Tables of Integrals, Series and Products, Academic Press, New York, London.

T 1977, Estimation of depth to conductors by the use of electromagnetic transients, Geophysical Prospecting 25, 61–75.

LEE, T 1981a, Transient electromagnetic response of a polarizable ground, Geophysics 1037–1041

LEE, T.J 1981b, Evaluation of a class of integrals occurring in mode matching solutions electrical and electromagnetic modelling, Geophysical Journal of the Royal Astronom Society 66, 455–460.

LEE, T. and LEWIS, R 1974 Transient EM response of a large loop on a layered grou Geophysical Prospecting 22, 430–444.

MAGNUS, W., OBERHETTINGER, F and SONI, R P. 1966, Formulas and Theorems for Special Functions of Mathematical Physics, Springer-Verlag, Berlin, Heidelberg, N York.

MORRISON, H F, PHILLIPS, R.J. and O'BRIEN, D.P. 1969, Quantitative interpretation of t sient electromagnetic fields over a layered halfspace, Geophysical Prospecting 17, 82–

OBERHETTINGER, F. and BADII, L. 1973, Tables of LaPlace Transforms, Springer-Ve Berlin, Heidelberg, New York.

WAIT, J.R. 1962, Theory of magnetotelluric fields, Journal of National Bureau of Stand Radio Propagation 66D 5, 509–541

WEIDELT, P 1982, Response characteristics of coincident loop transient electromagn systems, Geophysics 47, 1325–1330.

WHEELAN, A D 1968, Tables of Summable Series and Integrals Involving Bessel Functi Holden Day, San Francisco.

GEOPHYSICS, VOL 49, NO 7 (JULY 1984), P 854–860, 3 FIGS, 1 TABLE

The transient electromagnetic response of a magnetic or superparamagnetic ground

T. Lee*

ABSTRACT

The effect of superparamagnetic minerals on the transient response of a uniform ground can be modeled by allowing the permeability of the ground μ to vary with frequency ω as

$$\mu = \mu_0\left[1 + \chi_0\left(1 - \frac{\ln(i\omega\tau_2)}{\ln(\tau_2/\tau_1)}\right)\right].$$

Here τ_1 and τ_2 are the upper and lower time constants for the superparamagnetic minerals and χ_0 is the direct current value of the susceptibility.

For single-loop data it is found that the voltage will decay as $1/t$, provided that

$$\frac{10\chi_0 \ln(2a/b)}{\ln(\tau_2/\tau_1)} \gg \sqrt{\pi}\left(\frac{a^2\mu_0\sigma}{t}\right)^{3/2} \quad \text{and} \quad \left(\frac{\sigma\mu_0 a^2}{t}\right) \ll 1.$$

Here, a is the radius of the wire loop and b is the radius of the wire, t represents time and μ_0 is the permeability of free space Even if a separate transmitter and receiver are used, the transient will still be anomalous For this case the $1/t$ term in the equations is less important, and more prevalent now is the $1/t^2$ term These results show that a uniform ground behaves in a similar way to a ground which only has a thin superparamagnetic layer. A difference is that whereas the amplitude of the $1/t$ term could be drastically reduced by using a separate receiver, this is not the case for a uniform ground

A magnetic ground for late times will decay as $1/t^{2.5}$. However, if the conductivity of the ground is estimated from apparent conductivities it will be found that the value of the conductivity will be incorrect by a factor that is related to the susceptibility χ_0 of the ground For a weakly magnetic ground the estimated conductivity σ_1 is related to the true value of the conductivity $\sigma_1 = \sigma[1 + 19\chi_0/14]^{2/3}$.

INTRODUCTION

As Lee (1983) showed, the presence of superparamagnetic minerals on the transient electromagnetic (EM) response of a ground is to cause the transient to decay as $1/t$ A consequence of this is that the apparent resistivities decrease rather than increase with time While Lee's results were obtained by analytical methods, they are in accord with those of Buselli (1982) who reported similar effects from some of his field data. The results are a special case of those of Weidelt (1982) who showed that anomalous results could be obtained if either the conductivity or the permeability of the ground were to vary with frequency. Lee (1981) showed that the effect of allowing the conductivity of a uniform ground to vary with frequency was to produce transients that changed sign with time. The purpose of this paper is to examine the transient EM response of a uniform superparamagnetic ground

Previously Lee (1983) considered the case of a thin superparamagnetic layer overlying a conductive ground In that paper it was shown that even if the layer were quite thin, it still had a pronounced effect on the observed transient. It was shown there that it might be possible to allow for the effect if the field data were fitted to the formulas in that paper. The purpose of this paper is to highlight some of the subtleties of the transients caused by superparamagnetic minerals. We will show that for the case of a half-space of superparamagnetic material the equations to be fitted to the data are different from those already described One important difference will be that for this case the transients cannot be expected to be altered as much by using a separate transmitting loop A conclusion following from these investigations will be that in the case where superparamagnetic minerals are present in the ground then it is necessary to allow for the permeability of the ground to vary with frequency before attempting to interpret the transient EM data.

Another conclusion is that in the later stages a magnetic ground will decay in a way similar to that of a nonmagnetic ground. However, if the conductivity of the ground is estimated from the later stages of the transient, then it will be found to be in error by a complicated factor that depends upon the permeability of the ground

For this analysis, the ground is excited by a step current of strength I_0, flowing in a circular loop of radius a on the earth's surface First I shall be concerned with the calculation of the transient electric field (**E**), then the voltage V induced in a receiving loop. This voltage is described by

$$\mathbf{V} = \int_{\ell} \mathbf{E} \cdot \mathbf{d}\ell. \tag{1}$$

Manuscript received by the Editor April 12, 1983; revised manuscript received September 15, 1983
*Bureau of Mineral Resources, G P O. Box 378, Canberra, A C.T 2601 Australia
This paper was prepared by an agency of the Australian government.

Here, $\mathbf{d}\ell$ is an element of the receiving loop and the integral is taken about the entire length of the receiving loop.

TRANSIENT EM RESPONSE OF A SUPERPARAMAGNETIC ~~SPHERE~~ Ground

Suppose that a large circular loop of radius r lies at $z = -h$, $h \geq 0$, above a uniform superparamagnetic ground of conductivity σ and permeability μ_1. If a current $Ie^{i\omega t}$ flows in the loop then the electric field $\bar{\mathbf{E}}_1$ (which only has an azimuthal component $\bar{\mathbf{E}}_1$), observed at the surface of the ground at a distance ρ from the center of the loop will be given by the equation (Morrison et al., 1969, p. 87).

$$\bar{E}_1(i\omega) = -i\omega\mu_0 r \int_0^\infty \frac{e^{-\lambda h}\lambda\mu_1 Ie^{i\omega t}}{\lambda\mu_1 + n_1\mu_0} J_1(\lambda r)J_1(\lambda\rho)\, d\lambda, \quad (2)$$

where

$$n_1 = \sqrt{\lambda^2 + i\omega\sigma\mu_1}$$

Also from Lee (1983, eq. 7)

$$\mu_1 = \mu_0\left\{1 + \chi_0\left[1 - \ln\frac{(i\omega\tau_2 + 1)/(i\omega\tau_1 + 1)}{\ln(\tau_2/\tau_1)}\right]\right\}. \quad (3)$$

Notice that when $\omega = 0$

$$\mu_1 = \mu_0[1 + \chi_0]. \quad (4)$$

Accordingly, χ_0 is the direct current value of the susceptibility. τ_1 and τ_2 are the lower and upper time constants of the superparamagnetic ground, respectively.

For this study I will be concerned with those times for which the superparamagnetic effect is visible. Accordingly I assume that $|\tau_2\omega| \gg 1$ and presuppose that $|\tau_1\omega| \ll 1$. In these cases the terms $i\omega\tau_2 + 1$ and $i\omega\tau_1 + 1$ in equation (3) can be approximated by $i\omega\tau_2$ and $i\omega\tau_1$, respectively. One then finds that

$$\mu_1 \approx \mu_0\left[1 + \chi_0\left(1 - \frac{\ln(i\omega\tau_2)}{\ln(\tau_2/\tau_1)}\right)\right]. \quad (5)$$

Notice that equation (5) reduces to equation (4) if $\tau_1 \to 0$. Thus one expects that in the final result the expression should reduce to the transient response of a magnetic ground if τ_1 is allowed to approach zero I return to this point later

If a step current of strength I_0 flows in the loop, then a transient EM field $E_1(t)$ will be observed. This field can be described by

$$E_1(t) = \frac{I_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{i\omega}\bar{E}_1(i\omega)\, d\omega \quad (6)$$

Since $|\mu_1|$ is almost equal to μ_0, equation (2) can be reduced by expanding it as a Taylor series in terms of $(\mu_1 - \mu_0)/\mu_0$

$$E_1(t) = -\frac{I_0\mu_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\int_0^\infty re^{-\lambda h}J_1(\lambda r)J_1(\lambda\rho)\left[\frac{\lambda}{n_0 + \lambda} + \frac{(\mu_1 - \mu_0)}{\mu_0}\left[\frac{\lambda}{n_0 + \lambda} - \frac{\lambda}{2n_0} + \frac{\lambda^3}{n_0(\lambda + n_0)^2}\right]\right] d\lambda\, d\omega, \quad (7)$$

where

$$n_0 = \sqrt{\lambda^2 + i\omega\mu_0\sigma}.$$

In what follows I consider the behavior of E_1 at the wire itself. This is because the superparamagnetic effect is most commonly observed in single loop transient EM data Accordingly, we must allow for the finite radius b of the wire. If one assumes that the current is uniformly distributed throughout the cross-section of the wire, then

$$E = \int_A \frac{E_1}{\pi b^2}\, dA, \quad (8)$$

where A is the cross-sectional area of the wire.

With these remarks in mind, one separates the contributions to the total electric field into primary and secondary contributions. To this end one writes

$$\begin{aligned} E(t) = -\frac{I_0\mu_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\Bigg[&\int_0^\infty\int_A \frac{r\lambda e^{-\lambda h}}{n_0 + \lambda}\frac{J_1(\rho\lambda)J_1(\lambda r)}{\pi b^2}\, dr\, dh\, d\lambda \\ &+ \int_0^\infty\int_A \frac{e^{-\lambda h}(\mu_1 - \mu_0)r}{4\mu_0\pi b^2} J_1(\lambda\rho)J_1(\lambda r)dr\, dh\, d\lambda \\ &+ \int_0^\infty\int_A \frac{(\mu_1 - \mu_0)e^{-\lambda h}}{\pi b^2\mu_0} r\left[\frac{\lambda}{n_0 + \lambda} - \frac{\lambda}{2n_0} + \frac{\lambda^3}{n_0(n_0 + \lambda)^2} - \frac{1}{4}\right] J_1(\lambda\rho)J_1(\lambda r)dr\, dh\, d\lambda\Bigg] d\omega. \quad (9) \end{aligned}$$

Notice that a factor 1/4 has been added and subtracted from equation (9). The reason for this is to ensure that the last integral in the expression will be convergent even for the case $\rho = r$ and $h = 0$

The first two terms in equation (9) can be simplified immediately because they are, respectively, the response of a uniform ground (Lee and Lewis, 1974) and a singular term which has arisen because of the superparamagnetic layer If the wire is allowed to lie on the ground, then equation (9) can be reduced to

$$\begin{aligned} E(t) = E_p &- \frac{I_0\mu_0 a\Delta(\rho)}{8\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left(\frac{\mu_1 - \mu_0}{\mu_0}\right) d\omega \\ &- \frac{I_0 a\mu_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\int_0^\infty \frac{\mu_1 - \mu_0}{\mu_0} J_1(\lambda\rho)J_1(\lambda a)\cdot \\ &\cdot\left[\frac{\lambda}{n_0 + \lambda} - \frac{\lambda}{2n_0} + \frac{\lambda^3}{n_0(n_0 + \lambda)^2} - \frac{1}{4}\right] d\lambda\, d\omega, \quad (10) \end{aligned}$$

where

$$\Delta(\rho) = \frac{1}{a}\int_0^\infty\int_A \frac{rJ_1(\lambda\rho)J_1(\lambda r)e^{-\lambda h}}{\pi b^2}\, dr\, dh\, d\lambda.$$

Notice that when equation (10) was written, the integral over the cross-sectional area of the wire for the secondary term was approximated This was made possible because, for that integral, the integrand was slowly varying over the wire because b is very small in comparison with a. Here a is the distance from the center of the wire loop to the middle of the wire thickness. That is to say, it is the average radius of all the infinitesimal wire loops. Accordingly, r and h were set equal to a and zero, respectively By this means the integral was approximated by πb^2 which cancelled the πb^2 in the denominator. The quantity $\Delta(\rho)$ is evaluated in Appendix A

The integral in the second term of equation (10) can be readily evaluated by contour integration for $t > 0$. By this

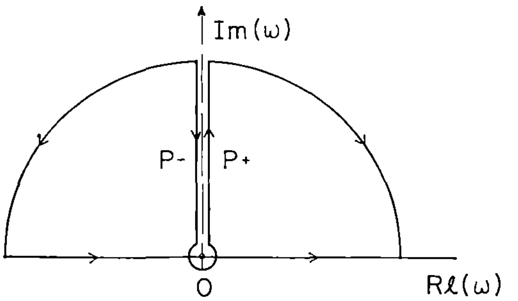


FIG. 1. The path for the contour integral for equation (15).

means one learns that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\frac{\mu_1-\mu_0}{\mu_0}\,d\omega=\frac{\chi_0}{t\ln(\tau_2/\tau_1)},\qquad t>0. \tag{11}$$

Consequently

$$E(t)=E_p-\frac{I_0\mu_0 a}{4t}\,\frac{\chi_0\Delta(\rho)}{\ln(\tau_2/\tau_1)}+E_s, \tag{12}$$

where

$$E_s=-\frac{I_0 a\mu_0}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\int_0^{\infty} J_1(\lambda\rho)J_1(\lambda r)\,\frac{(\mu_1-\mu_0)}{\mu_0}\cdot\left[\frac{\lambda}{n_0+\lambda}-\frac{\lambda}{2n_0}+\frac{\lambda^3}{n_0(n_0+\lambda)^2}-\frac{1}{4}\right]d\lambda\,d\omega. \tag{13}$$

To proceed, first notice that (Abramowitz and Stegun, 1965, formula 9 179)

$$\begin{aligned}\int_0^{\infty} J_1(\lambda\rho)J_1(\lambda a)\left(\frac{\lambda}{n_0+\lambda}-\frac{\lambda}{2n_0}+\frac{\lambda^3}{n_0(n_0+\lambda)^2}-\frac{1}{4}\right)d\lambda \\ =\int_0^{2\pi}\frac{\cos\theta}{2\pi}\int_0^{\infty} J_0(\lambda R)\left\{\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\left(\frac{\lambda}{n_0}e^{-n_0 z}-e^{-\lambda z}\right)\right. \\ -\frac{\lambda}{2n_0}-\frac{1}{4}+\frac{1}{k_0}\frac{\partial}{\partial k_0}\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} \\ \left.\cdot\left[\frac{\partial^2}{\partial z^2}\left(e^{-\lambda z}-\frac{\lambda e^{-n_0 z}}{n_0}\right)+\frac{k_0^2\lambda e^{-n_0 z}}{n_0}\right]\right\}d\lambda\,d\theta.\end{aligned} \tag{14}$$

Here z is a dummy variable which is set equal to zero once the differentiations are carried out.

Also

$$R=\sqrt{\rho^2+a^2-2a\rho\cos\theta},$$

and

$$k_0^2=i\omega\sigma\mu_0.$$

Since the integral with respect to λ is absolutely convergent, the order of the differentiation and integration may be interchanged. If this is done, then the integral may be evaluated by using elementary results or Sommerfeld's integral. If the differentations are carried out after the integral has been expanded as a power series in k_0, then one finds that

$$\begin{aligned}\int_0^{\infty} J_1(\lambda\rho)J_1(\lambda a)\left(\frac{\lambda}{\lambda+n_0}-\frac{\lambda}{2n_0}+\frac{\lambda^3}{n_0(\lambda+n_0)^2}-\frac{1}{4}\right)d\lambda \\ =\int_0^{2\pi}\frac{\cos\theta}{2\pi}\left[\sum_{n=3}^{\infty}\frac{(-1)^n(n-1)k_0\,n(Rk_0)^{n-3}}{n!}\right. \\ -3k_0\sum_{n=5}^{\infty}\frac{(-1)^n(n-1)(n-2)(k_0R)^{n-5}}{n!} \\ \left.-\frac{1}{2R}\sum_{n=1}^{\infty}\frac{(-1)^n(k_0R)^n}{n!}+\frac{1}{k_0^2}\sum_{n=3}^{\infty}\frac{(-1)^n k_0^n(n-1)R^{n-3}}{n!}\right]d\theta.\end{aligned} \tag{15}$$

It is now a simple matter to evaluate the integral with respect to ω. To this end one deforms the contour integral into the one about the positive imaginary ω axis. The contributions to the integral from the large quarter-circles vanish because of Jordan's lemma (see Figure 1). If one sets $\omega=se^{i\pi/2}$ and $\omega=se^{-3\pi i/2}$ on P^+ and P^-, respectively, then one learns that

$$\begin{aligned}E_s=-\frac{I_0 a\mu_0}{2\pi}\frac{\chi_0}{2\pi}\int_0^{2\pi}\cos\theta\left[\sum_{n=3}^{\infty}\frac{(-1)^n(n-1)nR^{n-3}}{n!}I(n-2)\right. \\ -3\sum_{n=5}^{\infty}\frac{(-1)^n(n-1)(n-2)}{n!}R^{n-5}I(n-4) \\ \left.-\frac{1}{2R}\sum_{n=1}^{\infty}\frac{(-1)^nR^n}{n!}I(n)+\sum_{n=3}^{\infty}\frac{(-1)^nR^{n-3}}{n!}I(n-2)\right]d\theta,\end{aligned} \tag{16}$$

where if n is even $(=2m)$

$$\begin{aligned}I(n)&=\frac{2(\sigma\mu_0)^{n/2}\pi}{\ln(\tau_2/\tau_1)}\int_0^{\infty}(-1)^{n/2}e^{-st}s^{n/2}\,ds \\ &=\frac{2\pi(-1)^{n/2}(\sigma\mu_0)^{n/2}}{t^{n/2+1}}\,\frac{\Gamma(n/2+1)}{\ln(\tau_2/\tau_1)},\end{aligned} \tag{17}$$

or if n is odd $(=2m+1)$

$$I(n)=2(\sigma\mu_0)^{n/2}(-1)^{(n+1)/2}\int_0^{\infty}s^{n/2}e^{-st}\left[1-\frac{\ln(s\tau_2)}{\ln(\tau_2/\tau_1)}\right]ds$$

and (18)

$$\begin{aligned}=2(\sigma\mu_0)^{n/2}(-1)^{(n+1)/2}\left[\int_0^{\infty}e^{-st}s^{n/2}\left[1-\frac{\ln(\tau_2)}{\ln(\tau_2/\tau_1)}\right]ds\right. \\ \left.-\frac{\partial}{\partial v}\int_0^{\infty}\frac{s^v}{\ln(\tau_2/\tau_1)}e^{-st}\,ds\right].\end{aligned} \tag{19}$$

Here v is a dummy variable which is set equal to $n/2$ once the differentiation is carried out.

By this means one learns that for n odd

$$I(n)=2(\sigma\mu_0)^{n/2}(-1)^{(n+1)/2}\,\frac{\Gamma(n/2+1)}{t^{n/2+1}}\left[1-\frac{\psi(n/2+1)}{\ln(\tau_2/\tau_1)}+\frac{\ln(t/\tau_2)}{\ln(\tau_2/\tau_1)}\right], \tag{20}$$

where $\psi(z)$ is the digamma function (Abramowitz and Stegun, 1965, p 258, no 6.3.7).

In order to clarify, it is interesting to calculate the first few terms of E_s. When one does this, one finds that

$$E_s(t) \approx -\frac{I_0 a\mu_0 \chi_0}{2\pi}\int_0^{2\pi} \cos(\theta)\left\{\frac{-R\pi\sigma\mu_0}{4\ln(\tau_2/\tau_1)t^2}\right.$$
$$-\frac{19R^2(\sigma\mu_0)^{3/2}\Gamma(5/2)}{210\pi t^{5/2}}\left[1-\frac{\psi(5/2)}{\ln(\tau_2/\tau_1)}+\frac{\ln(t/\tau_2)}{\ln(\tau_2/\tau_1)}\right]$$
$$+\frac{7R^3\pi(\sigma\mu_0)^2}{144\ln(\tau_2/\tau_1)t^3}$$
$$\left.+\frac{R^5}{1032}(\sigma\mu_0)^{5/2}\frac{\sqrt{\pi}}{t^{7/2}}\left[1-\frac{\psi(7/2)}{\ln(\tau_2/\tau_1)}+\frac{\ln(t/\tau_2)}{\ln(\tau_2/\tau_1)}\right]\right\} d\theta. \tag{21}$$

When

$$\frac{\sigma\mu_0 R^2}{t} \ll 1$$

then

$$E_p \approx -\frac{I_0\mu_0\rho}{40\sqrt{\pi}ta}\left(\frac{a^2\sigma\mu_0}{t}\right)^{3/2} \tag{22}$$

and one has, at the later stages of the transient, the result

$$E(t) \approx -\frac{I_0 a\mu_0}{4t}\frac{\chi_0\Delta(\rho)}{\ln(\tau_2/\tau_1)}$$
$$-\frac{I_0\mu_0\chi_0}{12t\ln(\tau_2/\tau_1)}\frac{\sigma\mu_0 a(a+\rho)}{t}\left\{\left(\frac{\rho^2+a^2}{2a\rho}\right)\right.$$
$$\cdot Q_{1/2}\left(\frac{\rho^2+a^2}{2a\rho}\right)-\frac{1}{2}\left[\left(\frac{\rho^2+a^2}{2a\rho}\right)^2+1\right]$$
$$\left.\cdot Q_{-1/2}\left(\frac{\rho^2+a^2}{2a\rho}\right)\right\}\frac{\sqrt{4\rho a}}{\rho+a}-\frac{19}{560}\frac{I_0\mu_0\chi_0\rho}{at}$$
$$\left(\frac{\sigma\mu_0 a^2}{t}\right)^{3/2}\left[1-\frac{\psi(5/2)}{\ln(\tau_2/\tau_1)}+\frac{\ln(t/\tau_2)}{\ln(\tau_2/\tau_1)}\right]$$
$$-\frac{I_0\mu_0\rho}{40\sqrt{\pi}at}\left(\frac{a^2\sigma\mu_0}{t}\right)^{3/2}. \tag{23}$$

The reader should notice that the integrals in equation (21) have been evaluated in terms of ring functions so as to follow the notation of Lee (1983). The necessary details can be found in Lee (1983) and Hobson (1955).

TRANSIENT EM RESPONSE OF A MAGNETIC GROUND

For the case of a magnetic ground where μ_1 is independent of frequency, one has from equations (2), (6), and (8) that

$$\mathbf{E} = -\frac{I_0\mu_0}{2\pi}\int_A\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\pi b^2}\int_0^{\infty}\frac{e^{-\lambda h}\lambda\mu_1 r}{\lambda\mu_1+n_1\mu_0}\cdot$$
$$\cdot J_1(\lambda\rho)J_1(\lambda r)\,d\lambda\,d\omega\,dh\,dz. \tag{24}$$

To proceed, one first interchanges the order of integration and transforms the integral with respect to ω to one about part of the positive imaginary ω axis (see Figure 2). Notice that the integrals about the large quarter-circles vanish as a result of Jordan's lemma. When one writes $\omega = (\lambda^2 s/\sigma\mu_1)e^{i\pi/2}$ and $\omega = (\lambda^2 s/\sigma\mu_1)e^{-3\pi i/2}$ for ω on the paths P^+ and P^-, respectively, then one has after integrating about the branch point the result

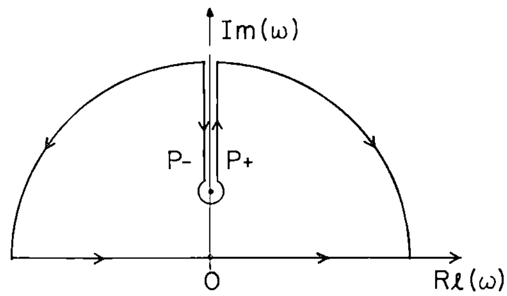


FIG 2. The path for the contour integral for equation (24).

$$\mathbf{E} = -\frac{I_0\mu_0}{\pi}\int_A\int_0^{\infty}\frac{e^{-\lambda h}}{\pi b^2}\int_1^{\infty}\frac{\mu_1\mu_0\sqrt{(s-1)}\exp(-\lambda^2 ts/(\sigma\mu_1))}{(\mu_1^2-\mu_0^2)+\mu_0^2 s}$$
$$\cdot\frac{\lambda^2 r J_1(\lambda r)}{\sigma\mu_1}J_1(\lambda\rho)\,ds\,d\lambda\,dh\,dr. \tag{25}$$

Now notice that the integrand for the integral over the cross-sectional area of the wire is slowly varying because for $t > 0$ the integrals with respect to s or λ are absolutely convergent. Accordingly, r and h can be set equal to a and zero, respectively. By this means the integral can be approximated by πb^2. This quantity will cancel the πb^2 in the denominator, and accordingly one has

$$\mathbf{E} = -\frac{I_0\mu_0 a}{\pi}\int_1^{\infty}\int_0^{\infty}\frac{\sqrt{(s-1)}\mu_0\mu_1\exp[-\lambda^2 ts/(\sigma\mu_1)]}{(\mu_1^2-\mu_0^2)+\mu_0^2 s}$$
$$\cdot\lambda^2\frac{J_1(\lambda a)J_1(\lambda\rho)\,d\lambda\,ds}{\sigma\mu_1} \tag{26}$$

The λ integral may be evaluated by expanding the Bessel functions as a power series in λ and then integrating term by term. Employing the result for the power series of the product of two Bessel functions that has been given by Erdelyi et al. (1953, p 10), there results

$$\mathbf{E} = -\frac{I_0\mu_0\rho}{\pi ta}\sum_{m=0}^{\infty}\left(\frac{\sigma\mu_1 a^2}{4t}\right)^{m+3/2} I_m\sum_{n=0}^{m}\left(\frac{\rho}{a}\right)^m\cdot$$
$$\cdot\frac{\Gamma(m+5/2)}{n!(n+1)!(m-n)!(m-n+1)!}, \tag{27}$$

in which

$$I_m = \int_1^{\infty}\frac{\sqrt{s-1}\mu_0\mu_1}{(\mu_1^2-\mu_0^2)+\mu_0^2 s}\frac{ds}{s^{2m+5/2}}. \tag{28}$$

Notice that at the later stages of the transient

$$\mathbf{E} \approx -\frac{I_0\mu_0}{\pi ta}\rho\left(\frac{\sigma\mu_1 a^2}{4t}\right)^{3/2} I_1\frac{3\sqrt{\pi}}{4}. \tag{29}$$

In general I_m must be evaluated numerically. However, it is possible to find an approximation for $\mathbf{E}$ at the later stages if one assumes that

$$\mu_1 = \mu_0[1+\chi_0] \tag{30}$$

and also that χ_0 is small.

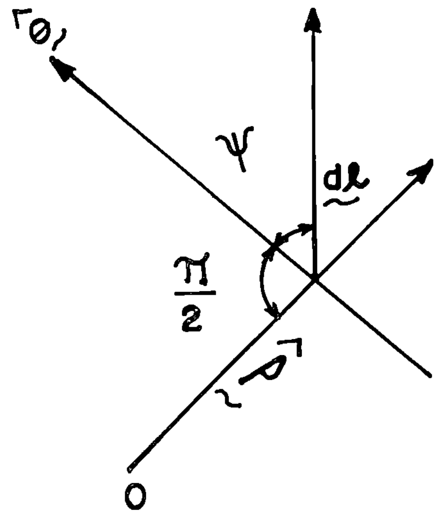


FIG. 3. Geometry for the element of the loop of wire.

When one expands equation (29) as a Taylor series about the point $\chi_0 = 0$, then one learns that

$$\mathbf{E} = -\frac{I_0 \mu_0 \rho}{40\sqrt{\pi}at}\left(\frac{a^2\sigma\mu_0}{t}\right)^{3/2} - \frac{19 I_0 \chi_0 \rho}{560\sqrt{\pi}at}\left(\frac{a^2\sigma\mu_0}{t}\right)^{3/2}. \tag{31}$$

Now notice that equation (31) could have been obtained by allowing $\tau_1 \rightarrow 0$ in equations (3) and (23) given previously By this means, I have provided a check on the solutions.

When one examines equation (27) or (31), one sees that eventually the electric field will decay as $t^{-5/2}$. This form of decay is the same as that for a nonmagnetic ground. I will return to this point later.

DISCUSSION

From equation (1), one has that the voltage V induced in a receiving loop is given by

$$V = \int_{\ell} \mathbf{E} \cos(\psi)\, \mathbf{d}\ell \tag{32}$$

Here, ψ is the angle between the direction of the electric field and the line element of the receiving loop (see Figure 3). A better idea of the meaning of equation (16) can be had by examining a number of limiting cases.

The simplest limiting case which one can consider is that for a magnetic ground. If one were to examine equation (31), for example, then one would see that the equation can be written in the form

$$\mathbf{E} \approx -\frac{I_0 \mu_0 \rho}{40\sqrt{\pi}at}\left(\frac{a^2\sigma_1\mu_0}{t}\right)^{3/2}, \tag{33}$$

where

$$\sigma_1 = \sigma\left[1 + \frac{19\chi_0}{14}\right]^{2/3}. \tag{34}$$

One sees, therefore, that a weakly magnetic ground will eventually behave as a nonmagnetic ground of a different conductivity. The same conclusion would have followed if I had used equation (29) instead of equation (31).

In marked contrast to this, however, is the behavior of a superparamagnetic ground. If one considers the case for which the transmitting loop coincides with the receiving loop, then the voltage induced in the receiver at the later stages will be

$$\begin{aligned} V \approx & -\frac{I_0 a\mu_0}{2t}\frac{\chi_0 \ln(2a/b)}{\ln(\tau_2/\tau_1)} \\ & -\frac{2I_0 a\mu_0 \chi_0 \pi}{3t \ln(\tau_2/\tau_1)}\left(\frac{\sigma\mu_0 a^2}{t}\right) \\ & -\frac{19}{280}\frac{I_0\mu_0\chi_0}{t}\pi a\left(\frac{a^2\sigma_1\mu_0}{t}\right)^{3/2} \cdot \\ & \cdot\left[1 - \frac{\psi(5/2)}{\ln(\tau_2/\tau_1)} + \frac{\ln(t/\tau_2)}{\ln(\tau_2/\tau_1)}\right] \\ & -\frac{I_0 a\mu_0}{20t}\sqrt{\pi}\left(\frac{a^2\mu_0\sigma}{t}\right)^{3/2} \end{aligned} \tag{35}$$

Equation (35) shows that although χ_0 is small, the term $\ln(2a/b)$ need not be This term will be the dominant one for sufficiently small b and large a. Consequently, the voltage decays like $1/t$ Results are given in Table 1 for two different values of χ_0. Those results illustrate the last point. They also show the percentage contribution to the total voltage that the first term makes.

Suppose that a transmitting loop is separate from the receiving loop and the wires of the transmitting and receiving loops do not cross each other. For this case, the observed voltage will be described by equation (36) below. [For the case where the wires do cross each other, equation (36) must be modified by using the exact expression for $\Delta(\rho)$ in the vicinity of the crossed wires; see equation (A-2) of the Appendix.] Then

$$\begin{aligned} V \approx & -\frac{I_0 a\mu_0}{4t}\frac{\chi_0}{\ln(\tau_2/\tau_1)}\int_{\ell}\frac{\cos\psi}{\pi\sqrt{a\rho}}Q_{1/2}\left(\frac{a^2}{2a\rho}\right)d\ell \\ & -\frac{I_0\mu_0\chi_0}{6\ln(\tau_2/\tau_2)}\left(\frac{\sigma\mu_0 a}{t^2}\right) \cdot \\ & \cdot\int_1 \cos(\psi)\sqrt{4a\rho}\left\{\left(\frac{\rho^2+a^2}{2a\rho}\right)Q_{1/2}\left(\frac{\rho^2+a^2}{2a\rho}\right)\right. \\ & \left. -\frac{1}{2}\left[\left(\frac{\rho^2+a^2}{2a\rho}\right)^2 + 1\right]Q_{-1/2}\left(\frac{\rho^2+a^2}{2a\rho}\right)\right\}d\ell \\ & -\frac{19}{560}\frac{I_0\mu_0\chi_0}{at}\left(\frac{\sigma_1\mu_0 a^2}{t}\right)^{3/2} \cdot \\ & \left[1 - \frac{\psi(5/2)}{\ln(\tau_2/\tau_1)} + \frac{\ln(t/\tau_2)}{\ln(\tau_2/\tau_1)}\right] \\ & \cdot\int_{\ell}\rho\cos(\psi)\,d\ell - \frac{I_0\mu_0}{40\sqrt{\pi}at}\left(\frac{a^2\sigma\mu_0}{t}\right)^{3/2}\int_1 \rho\cos(\psi)\,d\ell. \end{aligned} \tag{36}$$

Notice the absence of the term $\ln(2a/b)$ has reduced the contribution of the first term. Nevertheless, it is still true that the transient will eventually decay as $1/t$.

Table 1.

Time (ms)	$\chi_0 = 0.1$ V	$\chi_0 = 0.1$ $(T_1/V) \times 100$	$\chi_0 = 0.01$ V	$\chi_0 = 0.01$ $(T_1/V) \times 100$
.1	-6.05×10^{-2}	74.3	-1.48×10^{-2}	30.3
.2	-2.57×12^{-2}	87.5	-4.13×10^{-3}	54.5
.3	-1.63×10^{-2}	92.0	-2.19×10^{-3}	68.3
.4	-1.19×10^{-2}	94.2	-1.47×10^{-3}	76.6
.5	-9.42×10^{-3}	95.5	-1.10×10^{-3}	81.8
.6	-7.78×10^{-3}	96.4	-8.78×10^{-4}	85.4
.7	-6.63×10^{-3}	97.0	-7.31×10^{-4}	87.9
.8	-5.78×10^{-3}	97.4	-6.26×10^{-4}	89.8
.9	-5.12×10^{-3}	97.7	-5.48×10^{-4}	91.3
1.0	-4.59×10^{-3}	98.0	-4.87×10^{-4}	92.4
2.0	-2.27×10^{-3}	99.1	-2.32×10^{-4}	97.0
3.0	-1.51×10^{-3}	99.4	-1.52×10^{-4}	98.3
4.0	-1.13×10^{-3}	99.6	-1.14×10^{-4}	98.8
5.0	-9.03×10^{-4}	99.7	-9.08×10^{-5}	99.1
6.0	-7.52×10^{-4}	99.7	-7.55×10^{-5}	99.3
7.0	-6.44×10^{-4}	99.8	-6.47×10^{-5}	99.4
8.0	-5.64×10^{-4}	99.8	-5.65×10^{-5}	99.5
9.0	-5.01×10^{-4}	99.8	-5.02×10^{-5}	99.6
10.0	-4.51×10^{-4}	99.8	-4.52×10^{-5}	99.6

Lee (1977) showed that the apparent resistivity of a conducting ground, ρ_a $(=1/\sigma)$, at the later stages of the transient is given approximately by

$$\rho_a \approx \frac{\mu_0 a^2}{t^{5/3}} \left(\frac{I_0 \sqrt{\pi} a \sigma \mu_0}{20V} \right)^{2/3} \tag{37}$$

For large t, one has that

$$\rho_a \propto 1/t. \tag{38}$$

This equation shows that the apparent resistivities will decrease with time.

One sees from this discussion that the usual methods of interpreting the response of a uniform ground give unsatisfactory results for the case where the ground is superparamegnetic All one can do is to try the methods which Lee (1983) advocated to interpret the results over a thin superparamagnetic layer Meanwhile, there is a need for some rock properties to be measured so that one can gain an idea of the quantities τ_1, τ_2, and χ_0.

It is interesting to compare the transient EM response of a uniform ground with the response of a ground where only the top layer is superparamagnetic.

In the case where the superparamagnetic material was confined to a thin top layer of the ground of thickness d, one had that

$$V \approx -\frac{I_0 a \mu_0}{bt} \frac{\chi_0 d}{\ln(\tau_2/\tau_1)} - \frac{I_0 a \mu_0 \chi_0 d \sigma \mu_0}{\ln(\tau_2/\tau_1) 8t^2} \pi a - \frac{\sqrt{\pi} a \mu_0 I_0}{20t} \left(\frac{\sigma \mu_0 a^2}{t} \right)^{3/2} \tag{39}$$

Further, when separate transmitting and receiving loops are used one has from equation (38) of Lee (1983) that

$$V \approx -\frac{I_0 \mu_0}{2t} \frac{\chi_0 db}{\ln(\tau_2/\tau_1)} \int_\ell \frac{4}{3\pi^2} \sqrt{\frac{a}{\rho}} \frac{1}{(\rho - a)^2} \cdot \left[\left(\frac{\rho^2 + a^2}{2a\rho} \right) Q_{-1/2}\left(\frac{\rho^2 + a^2}{2a\rho} \right) - Q_{1/2}\left(\frac{\rho^2 + a^2}{2a\rho} \right) \right] \cos\psi \, d\ell$$
$$- \frac{I_0 \mu_0 d \chi_0 \sigma \mu_0}{\ln(\tau_2/\tau_1) 16t^2} \int_\ell G(\rho/a, a/\rho) \cos(\psi) \, d\ell$$
$$- \frac{I_0 a \mu_0}{40\sqrt{\pi a t}} \left(\frac{\sigma \mu_0 a^2}{t} \right)^{3/2} \int_\ell \rho \cos(\psi) \, d\ell. \tag{40}$$

Here $G(\rho/a, a/\rho)$ is equal to ρ/a, for $\rho < a$, or equal to a/ρ for $\rho > a$.

When one compares equations (39) and (40) with equations (35) and (36), the following points can be made. First, the effect of the radius of the wire on the transients is very different in both cases. In the case of a superparamagnetic layer, one sees that the effect of separating the transmitter and receiver is to transfer the term b from the denominator to the numerator, and accordingly the effect of the top layer of superparamagnetic material is greatly reduced.

In the case of a superparamagnetic half-space, however, the effect of the wire thickness is quite different. For in that case the loop radius appears as $\ln(2a/b)$ where the transmitting loop coincides with the receiving loop. On the other hand, in the case where the transmitter is separate from the receiver, the radius of the wire does not enter into the equation at all. Taken together, these last two remarks show that for the case of a superparamagnetic ground the effect of separating the transmitting and receiving loops is much less than in the case where the superparamagnetic effect arises because of a thin layer of superparamagnetic material

The second point to be made is that it may not always be possible to allow for the effect of superparamagnetic materials when interpreting transient EM data. This conclusion is conditional on more work being done on this problem. In order for this work to be done, however, it is essential that the susceptibility and the time constants of various superparamagnetic materials be measured. It is only then that more detailed calculations can be made.

Finally, I remark that when the results presented here are combined with those of Lee (1981), one sees that, if one allows for the electrical properties of the ground to vary with frequency, then by a study of the theory of transient electromagnetics one is led in quite a natural way to the study of mineral discrimination by means of the spectral characteristics of the electrical properties of the various minerals.

ACKNOWLEDGMENTS

The author has benefitted from discussions with a number of people at C.S.I.R.O These include Dr. Dave Clark, Dr. Brian Embleton, Dr. G. Buselli, and Harry Brown. Once again, Robyn Harbour displayed patience in typing my manuscript.

REFERENCES

Abramowitz, M., and Stegun, I. A, Eds., 1965, Handbook of mathematical functions: New York, Dover Publications, p 1046.

Buselli, G., 1982, The effect of near surface superparamagnetic material on electromagnetic transients Geophysics, v. 47, p. 1315–1324

Erdelyi, A., Magnus, W, Oberhettinger, F., and Tricomi, F. G., 1953, Higher transcendental functions, v. II: New York, McGraw-Hill Book Co, p. 392.

Gradshteyn, J. S., and Ryzhik, I M., 1980, Tables of integrals, series and products: New York, Academic Press Inc., p. 1160.

Hobson, E W., 1955, The theory of spherical and ellipsoidal harmonics· Chelsea Publ. Co, p. 500

Lee, T., 1977, Estimation of depth to conductors by the use of electromagnetic transients· Geophys. Prosp, v 25, p. 61–75.

——— 1981, Transient electromagnetic response of a polarizable ground: Geophysics, v 46, p. 1037–1041.

——— 1983, The effect of a superparamagnetic layer on the transient electromagnetic response of a ground; Geophys Prosp, (in press)

Lee, T, and Lewis, R., 1974, Transient E.M. response of a large loop on a layered ground: Geophys. Prosp, v. 22, p. 430–444

Magnus, W., Oberhettinger, F., and Soni, R. P., 1966, Formulas and theorems for the special functions of mathematical physics: Berlin, Springer-Verlag, p 508.

Morrison, H. F., Phillips, R. J, and O'Brien, D. P, 1969, Quantitative interpretation of transient electromagnetic fields over a layered half-space· Geophys. Prosp., v. 17, p 82–101.

Weidelt, P., 1982, Response characteristics of coincident loop transient electromagnetic systems: Geophysics, v. 47, p. 1325–1330.

Wheelan, A. D., 1968, Tables of summable series and integrals involving Bessel functions· San Francisco, Holden-Day, p. 125

APPENDIX

If a is the average radius of the wire loop (i.e., the distance from the center of the current loop to the middle of the wire thickness) and if $\Delta(\rho)$ is calculated for filaments of average height b above the ground (i.e., the wire lies on the ground), then

$$\pi b^2 \Delta(\rho) = \frac{1}{a}\int_{-b}^{b}\int_{-\sqrt{b^2-s^2}}^{\sqrt{b^2-s^2}} (a+s)\cdot \cdot \int_0^\infty e^{-b\lambda - z\lambda} J_1[\lambda(a+s)] J_1(\lambda\rho)\, dz\, ds\, d\lambda \tag{A-1}$$

and

$$= \frac{1}{a}\int_{-b}^{b}\int_{-\sqrt{b^2-s^2}}^{\sqrt{b^2-s^2}} \frac{(a+s)}{\pi\sqrt{(a+s)\rho}}\cdot \cdot Q_{1/2}\left[\frac{(b+z)^2+\rho^2+(a+s)^2}{2\rho(a+s)}\right] dz\, ds \tag{A-2}$$

(see result no. 11.401 of Wheelan, 1968).

If $a = \rho$, then the argument of the ring function is close to unity, and consequently the expression for $\Delta(\rho)$ can be approximated by using the asymptotic form of the ring functions near their singularities (see Magnus et al. 1960, p 196).

Accordingly for this case

$$\pi b^2 \Delta(\rho) = \frac{1}{a}\int_{-b}^{b}\int_{-\sqrt{b^2-s^2}}^{\sqrt{b^2-s^2}} -\frac{1}{2\pi}\frac{(a+s)}{\sqrt{(a+s)\rho}}\cdot \cdot \ln\left[\frac{(b+z)^2+s^2}{4a(a+s)}\right] dz\, ds \tag{A-3}$$

$$\approx -\frac{1}{2\pi a}\int_{-b}^{b}\int_{-\sqrt{b^2-s^2}}^{\sqrt{b^2-s^2}} \cdot \cdot \ln\left[\frac{(b+z)^2+s^2}{4a(a+s)}\right] dz\, ds$$

$$= -\frac{1}{2\pi a}\int_{-b}^{b} r \int_0^{2\pi} \cdot \cdot \ln\left(\frac{r^2+2br\cos\theta+b^2}{4a^2}\right) d\theta\, dr \tag{A-4}$$

$$= -\frac{1}{2\pi a}\pi b^2 \ln\left(\frac{b^2}{4a^2}\right) \tag{A-5}$$

(see result no. 4.2214 of Gradshteyn and Ryzhik, 1980). Therefore

$$\Delta(\rho) = \ln(2a/b)/(\pi a). \tag{A-6}$$

However, if $|a-\rho| \gg b$, then equation (A-2) can be approximated as

$$\pi b^2 \Delta(\rho) = \frac{1}{a}\int_{-b}^{b}\int_{-\sqrt{b^2-s^2}}^{\sqrt{b^2-s^2}} \frac{a}{\pi\sqrt{a\rho}} Q_{1/2}\left(\frac{\rho^2+a^2}{2\rho a}\right) dz\, ds \tag{A-7}$$

whence

$$\Delta(\rho) = \frac{1}{\pi\sqrt{a\rho}} Q_{1/2}\left(\frac{\rho^2+a^2}{2\rho a}\right) \tag{A-8}$$

INVERSION OF TRANSIENT ELECTROMAGNETIC DATA FROM A SPHERICAL CONDUCTOR

Terry J. Lee

Reprinted from IEEE Transactions on Geoscience and Remote Sensing, Vol. GE-22, No. 1, January 1984

Inversion of Transient Electromagnetic Data From a Spherical Conductor

TERRY J. LEE

Abstract—This paper considers the problem of inverting single loop transient electromagnetic data from a spherical conductor.

First, analytical expressions are presented for the transient voltage, and then the various partial derivatives of this voltage with respect to the parameters of the ground are found. Secondly, it is shown how all these expressions can be used with existing inversion schemes, such as the Marquardt or ridge regression scheme for the inversion of data. Finally, results are presented to show that, even when the transient data is contaminated by 20-percent random noise, good estimates for the earth parameters can be found. In this case, however, only 4 parameter combinations are well resolved. Further, when there is only 5-percent noise present in the data, it is still not possible to resolve more than 5 sets of parameters.

I. Introduction

For at least the last ten years there has been a growing interest in the inversion of electrical and electromagnetic data by a method known as ridge regression. As far back as 1974, Ware *et al.* [11] discussed the inversion of horizontal loop sounding data. More recently Petrick *et al.* have described a method of inverting three-dimensional resistivity data using alpha centers [2]. Basing their analysis on a previously published algorithm by Jupp and Vozoff [3], Jupp and Vozoff [4] have described their results for the inversion of two-dimensional magnetotelluric data. It comes as a surprise to find that, despite such a rich literature devoted to the inversion of steady-state electromagnetic data and resistivity data, there is practically nothing written on the inversion of time domain data. Transient electromagnetic data from a layered ground has been inverted by Rodriguez-Ovejero [5]. However, the results of this Master of Science thesis have still to be published. In any case, nonlayered cases are the most frequently occurring situations. There is a need then to be able to invert data from discreet ore bodies that are approximately spherical.

The purpose of this paper is to describe the results we have obtained by using the inversion subroutines of Dr. D.L.B. Jupp to invert time domain electromagnetic data from a buried spherical conductor. The theoretical basis of Dr. D.L.B. Jupp's programs have been described in a paper by Jupp and Vozoff [3]. The transient electromagnetic data to be inverted was generated by solving the integral equation for the electrical fields about a spherical conductor in a poorly conducting halfspace. The solution of the appropriate integral equation has been described by Lee [6]. The actual geometry is shown in Fig. 1.

Manuscript received September 30, 1982, revised June 8, 1983.

The author is with Geopeko, Division of Peko-Wallsend, Limited, Gordon, N.S.W., 2070, Australia.

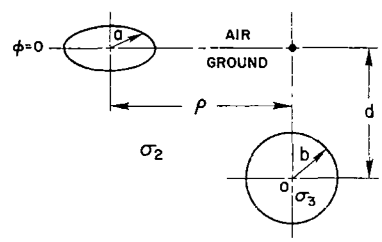


Fig. 1. Loop traversing a buried sphere.

II. Analysis

Suppose that a sphere of conductivity σ_3 radius b and permeability μ_0 lies at a depth d in a uniform ground of conductivity σ_2 and permeability μ_0. When a step current of height I_0 flows in a loop of radius a and at a distance ρ from the vertical axis of the sphere, then a transient voltage $V(t)$ will be measured. This voltage may be considered to be made up of two others: $Vp(t)$, the voltage of the uniform ground, that is the voltage that would be measured when there was no spherical conductor present, and $Vs(t)$, a voltage that arises because of the presence of the conducting sphere. Expressions for these voltages have been found previously (see Lee [6]).

From equations (29), (41), (43), and (50) of Lee [6], one has that

$$V(t) = Vp(t) + Vs(t)$$

where

$$Vp = \frac{a\mu_0 I_0 \sqrt{\pi}}{t} \sqrt{\left(\frac{a^2\sigma_2\mu_0}{4t}\right)} \sum_{j=0}^{\infty} \cdot \frac{2(-1)^j(2j+2)!}{(2j+5)\,j!\,(j+1)!\,(j+2)!} \left(\frac{a^2\sigma_2\mu_0}{4t}\right)^{j+1}$$

$$\approx \frac{a\mu_0 I_0 \sqrt{\pi}}{t} \sqrt{\left(\frac{\sigma_2\mu_0 a^2}{4t}\right)} \left(\frac{2}{5}\left(\frac{a^2\sigma_2\mu_0}{4t}\right) - \frac{4}{7}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^2 + \frac{5}{9}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^3 \cdots\right.$$

$$Vs = \frac{3\mu_0 a^2 I_0}{4} \int_c^i e^{-iwt} \left\{\frac{1}{k_2^3}\,[T_{11}^2 + T_{01}^2]\,\frac{WM_1}{WM_1^1} - \frac{1}{k_2}\,\frac{WN_1}{WN_1^1}\,T_{1b}^2\right\} dw. \tag{1}$$

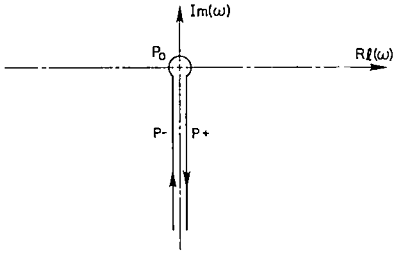


Fig. 2. Path for the contour integral.

Here c denotes that the integral is taken about a path shown in Fig. 2.

$$WM_n = b\,k_3\,j_{n+1}(bk_3)\,j_n(bk_2) - (bk_2)\,j_n(bk_3)\,j_{n+1}(bk_2)$$

$$WM_n^1 = bk_3 j_{n+1}(bk_3)\,h_n^1(bk_2) - (bk_2)\,j_n(bk_3)\,h_{n-1}^1(bk_2)$$

$$WN_n = \frac{1}{2n+1}\left[WM_{n-1}(n+1) + nWM_{n+1}\right]$$

$$WM_n^1 = \frac{1}{2n+1}\left[WM_{n-1}^1(n+1) + n\,WM_{n+1}^1\right]$$

$$k_2^2 = iw\sigma_2\mu_0, \qquad k_3^2 = iw\sigma_3\mu_0$$

$$T_{11} = \int_0^\infty \frac{2iJ_1(\lambda\rho)\,J_1(\lambda a)\,\lambda h_1\,e^{ih_1 d}}{(h_0 + h_i)\,i}\,d\lambda$$

$$T_{01} = \int_0^\infty \frac{2\lambda^2 J_0(\lambda\rho)\,J_1(\lambda a)}{i(h_1 + h_0)}\,e^{ih_1 d}\,d\lambda$$

$$T_{1b} = \int_0^\infty \frac{2\lambda J_1(\lambda\rho)\,J_1(\lambda a)}{(h_1 + h_0)}\,e^{ih_1 d}\,d\lambda$$

$$h_1 = \sqrt{k_2^2 - \lambda^2}$$

$$h_0 = i\lambda\,.$$

In writing down these equations displacement currents have been neglected.

This expression for $V(t)$ was found by the solution of an integral equation and is only valid for a poorly conducting ground. As the solution stands, a considerable amount of computing is required in order to compute answers from it. Because of this, Lee [7] sought to find an alternative expression for this voltage. Such a solution has been found, but subject to the restriction that the conductivity of the ground is quite small. So as to highlight some of the characteristics of the equations used here, some of this previous analysis will be repeated here. This should allow the reader to follow better the subsequent developments and to allow him to appreciate the limitations to be placed on the data to be inverted.

To proceed, one has to find approximations to the integrals in (3), which are valid for a poorly conducting ground. To this end one notices that

$$T_{11} = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\,\frac{\partial}{\partial d}\,U\,d\theta$$

$$T_{01} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial a}\,U\,d\theta$$

$$T_{1b} = \frac{i}{2\pi}\int_0^{2\pi} U\cos\theta\,d\theta$$

where

$$U = \int_0^\infty \frac{2J_0(\lambda P)\,\lambda e^{ih_1 d}}{(i\lambda + h_1)\,i}\,d\lambda$$

and

$$P = \sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}\,.$$

The function U can now be approximated by noting that

$$U = -\frac{2}{k_2^2}\left[\frac{\partial^2}{\partial d^2}\int_0^\infty \frac{\lambda}{ih_1}\,e^{ih_1 d}\,J_0(\lambda P)\,d\lambda + \left(k_2^2\frac{\partial}{\partial d} + \frac{\partial^3}{\partial d^3}\right)\int_0^\infty \frac{e^{ih_1 d}}{ih_1}\,J_0(\lambda P)\,d\lambda\right]$$

since

$$ih_1 = \pm i\sqrt{k_2^2 - \lambda^2} \qquad k_2 > \lambda \quad \text{on } P\pm$$
$$= -\sqrt{\lambda^2 - k_2^2} \qquad k_2 < \lambda \quad \text{on } P\pm$$

respectively the integrals can be readily evaluated by using results 2.54, 6.48, and 6.50 of Oberhettinger [8]. By this means one finds that

$$U = +\frac{2}{k_2^2}\frac{\partial^2}{\partial d^2}\frac{e^{ik_2 R}}{R} - \frac{2}{k_2^2} \cdot \left[k_2^2\frac{\partial}{\partial d} + \frac{\partial^3}{\partial d^3}\right]\left[\frac{\pi}{2}J_0\left(\frac{k_2}{2}(R-d)\right)Y_0\left(\frac{k_2}{2}(R+d)\right) \mp \frac{i\pi}{2}J_0\left(\frac{k_2}{2}(R-d)\right)J_0\left(\frac{k_2}{2}(R+d)\right)\right]$$

on $P\pm$, respectively.

Expanding the Bessel functions and neglecting the higher order terms in both the real and the imaginary terms one has that

$$U \approx -\left[\frac{1}{R} - \frac{k_2^2 d}{8}\ln\left(\frac{k_2}{4}(R+d)\right) \pm \frac{2}{3}ik_2 \mp i\frac{\pi}{16}k_2^2 d \mp \frac{ik_2^3}{15}(R^2 + 2d^2)\right].$$

Therefore

$$T_{01}^2 \approx \left[I_1^2 \pm \frac{4i}{15}k_2^3 a^3 I_1\right]\frac{1}{a^4}$$

where

$$I_1 = -\frac{1}{2\pi}\frac{\partial}{\partial a}\int_0^{2\pi}\frac{a^2}{R}\,d\theta$$

$$= a^2 \int_0^\infty e^{-\lambda d} \lambda J_1(\lambda a) J_0(\lambda \rho)\, d\lambda$$

$$= \frac{a^3}{(a^2 + d^2)^{3/2}}, \quad \rho = 0$$

$$= a^2 \left[\frac{k^3(a^2 - \rho^2 - d^2) E(k)}{8\pi(a\rho)^{3/2} a(1 - k^2)} + \frac{k K(k)}{2\pi(a\rho)^{1/2} a} \right], \quad \rho \neq 0$$

and $k^2 = 4a\rho/(d^2 + (a+\rho)^2)$, Luke [9] $K(k)$ and $E(k)$ are the complete ellipitic integrals of the first and second kinds.

$$T_{11}^2 \approx \frac{1}{a^4} I_2^2$$

$$I_2 = a^2 \int_0^\infty e^{-\lambda d} J_1(\lambda \rho) J_1(\lambda a)\, \lambda d\lambda$$

$$= 0, \quad \rho = 0$$

$$= \frac{dka^2}{2\pi(a\rho)^{3/2}(1 - k^2)} \{(1 - \tfrac{1}{2}k^2) E(k) - (1 - k^2) K(k)\}$$

$$\rho \neq 0.$$

Luke [9]:

$$T_{1b}^2 \approx -\frac{1}{a^2} [I_3^2 + 2iI_3 k_2^3 \rho a^2/15]$$

$$I_3 = a \int_0^\infty J_1(\lambda \rho) J_1(\lambda a) e^{-\lambda d} d\lambda$$

$$= 0, \quad \rho = 0$$

$$= \frac{2a}{\pi k (a\rho)^{1/2}} [(1 - \tfrac{1}{2}k^2) - E(k)], \quad \rho \neq 0.$$

The remaining quantities that must be computed are the WM_1^1, WM_1, WN_1^1, and WN_1 factors. Along P^+ and P^- the WM_1^1 and WN_1^1 terms are easily split into real and imaginary parts by replacing the spherical Hankel functions h_n^1 by $j_n \pm iy_n$ on $P\pm$, respectively. Since it has been shown that there are no poles in the spectrum when $\sigma_2 = 0$, (see p. 325 [6]). We may expand the factors WM_1/WM_1^1 and WN_1/WN_1^1 in power of $k_2 b$. This can be done once the Bessel functions involving these quantities are also expanded in powers of $k_2 b$.

The relevant expansions of WM_1/WM_1^1 and WN_1/WN_1^1 are

$$\frac{WM_1}{WM_1^1} \approx \frac{(k_2 b)^3 j_2(k_3 b)}{(k_2 b)^3 j_2(k_3 b) + 3i\left(j_0(k_3 b) - \frac{(k_2 b)^2}{k_3 b} j_1(k_3 b)\right)}$$

$$\frac{WN_1}{WN_1^1} \approx \frac{2(k_2 b)^3}{2(k_2 b)^2 + 3i} \approx -\frac{2}{3} i (k_2 b)^3 + \frac{4(k_2 b)^5}{9}.$$

Notice that when $\sigma_2 = 0$ the contour of Fig. 2 is replaced by an integral about the poles of the integrand which are at $(k_3 b) = n\pi$, $n = 1, 2, \cdots$. Accordingly this resonance response constitutes the first term in the asymptotic expansion. The other terms still involve integrals about the negative w axis.

Combining the results that we have obtained so far yields (after writing $w = -is$ in the integrals along the branch cut) the approximate expression for Vs.

$$Vs = 3 \frac{\mu_0 a^2 I_0}{2} \Bigg\{ \oint \frac{e^{-iwt}}{1} \frac{b^3}{6a^4} \frac{j_2(b\sqrt{iw\sigma_3 \mu_0})}{j_0(b\sqrt{iw\sigma_3\mu_0})}$$

$$\cdot (I_1^2 + I_2^2)\, dw + \frac{4}{15} \fint_0^\infty \frac{(k_2 b)^3 a^3 I_1}{a^4} \frac{j_1(k_3 b)}{3 j_0(k_3 b)} e^{-st} ds$$

$$+ \frac{4}{45} \int_0^\infty I_3 b^3 \rho k_2^5 e^{-st} ds + \frac{4}{9} \int_0^\infty I_3^2 \frac{b^5}{a^2} k_2^4 e^{-st} ds$$

$$+ \frac{b^3 Rl}{a^4}$$

$$\cdot \left(\int_0^\infty \frac{j_2(k_3 b)((I_1^2 + I_2^2) e^{-st})\, ds}{(k_2 b)^3 j_2(k_3 b) + 3i\, j_0(k_3 b) - \frac{(k_2 b)^2}{(k_3 b)} j_1(k_3 b)} \right) \Bigg\}.$$

Here k_j has been redefined as $k_j = \sqrt{(\sigma_j \mu_0 b^2 s)}$. The bar on the integral denotes the Cauchy principal value and $Rl\,(x)$ denotes that only the real part of x is added to Vs.

The integrals in the equation above have been integrated by Lee [7]. If we set $\beta^2 = \sigma_3 \mu_0 b^2$, $\alpha^2 = \sigma_2/\sigma_3$ we find the following expression for Vs.

$$Vs \approx \frac{3\mu_0 b^3 I_0 \pi}{a^2 \beta^2} (I_1^2 + I_2^2) D + \frac{a^2}{b^2} \frac{\sqrt{\pi}}{4} \frac{\rho \mu_0 I_0}{t^{7/2}} (\sigma_2 \mu_0 b^2)^{5/2} I_3$$

$$+ \frac{2}{5} \alpha^3 a \mu_0 I_0 I_1 J + \frac{3\mu_0 b^3}{2a^2} C(I_1^2 + I_2^2)$$

$$+ \frac{4}{3} \frac{\mu_0 I_0 b}{t^3} I_3^2 (\sigma_2 \mu_0 b^2)^2$$

where

$$D = \sum_{n=1}^\infty e^{-n^2 \pi^2 t/\beta^2}, \quad \beta^2 = \sigma_3 \mu_0 b^2, \quad \alpha^2 = \sigma_2/\sigma_3.$$

$$J = -2\sqrt{\pi}\beta \sum_{n=1}^\infty \frac{e^{-\beta^2 n^2/t}}{n} \left[\frac{3\beta^2 n^2}{2t^{5/2}} - \frac{n^4 \beta^4}{t^{7/2}} \right]$$

$$+ \sqrt{\pi}\beta/(2t^{3/2}) - \Gamma(\tfrac{5}{2})\beta^3/(3t^{5/2})$$

$$C = \alpha^3 \left[\left(\frac{2t}{2} - \frac{2}{3} \right) J - \frac{\beta^3 \sqrt{\pi}}{12 t^{5/2}} \right].$$

It was also shown in Lee [6] that alternative expressions for J for β^2/t small are

$$J \approx \frac{\sqrt{\pi}\beta^5}{t^{7/2}} \left[\frac{1}{24} + \frac{\beta^2}{72t} + \frac{\beta^4}{160t^2} + \frac{\beta^3}{288t^3} + \frac{\beta^8 691}{302400 t^4} + \frac{\beta^{10}}{552 t^5} \right].$$

See (A9) of Lee [6].

The interpretation of these equations is that the solution consists of three parts. The first is that due to the transients in a uniform ground. The second term is the resonant response of the sphere, and the third term has been interpreted as a series of waves that cater for the interaction of the resonant currents and those of the uniform ground. Further discussion of these equations can be found in Lee [7].

Notice that the derivation of the equations requires that if l is a typical length, then $k_2 l$ must be small. This is equivalent to requiring that $\sigma_2 \mu_0 l^2/t$ be small. See Lee [7] for further discussion on this matter. Subject ot this restriction, the expression can be used as a starting point to develop the remaining necessary equations to allow transient data to be inverted.

As shown by Jupp and Vozoff [3] and other writers on inversion techniques, all the computer routines start from some initial set of values that are assumed for the parameters, and then try to step in the direction of the parameter space that will best minimize the error between the calculated voltages and the measured voltages. In order to find the direction for this step, it is necessary to compute the partial derivatives with respect to the various parameters of the model. Further, as shown by Jupp and Vozoff [3], these derivatives are needed in order to gain a better estimate for the parameters. For these reasons, expressions must be found for the partial derivatives of $V(t)$ with respect to the various parameters of the model. As shown below, these derivatives may be conveniently written in terms of the derivatives of the quantities I_1, I_2, I_3, C, J, and D with respect to the various parameters. Further, in the case of the functions C, J, and D, it is convenient first to calculate the derivative of these quantities with respect to α and β.

After many straightforward but tedious differentiations, one finds that

$$\frac{\partial J}{\partial \beta} = \frac{\sqrt{\pi}}{2t^{3/2}} - \frac{\sqrt{\pi}}{4}\frac{3\beta^2}{t^{5/2}}$$

$$- 2\pi \cdot \sum_{n=1}^{\infty} \frac{e^{-\beta^2 n^2/t}}{n}\left[\frac{9\beta^2 n^2}{2t^{5/2}} - \frac{8\beta^4 n^4}{t^{7/2}} + \frac{2n^6\beta^6}{t^{9/2}}\right]$$

or

$$\frac{\partial J}{\partial \beta} \approx \frac{\sqrt{\pi}\beta^4}{t^{7/2}}\left[\frac{5}{24} + \frac{7}{72}\frac{\beta^2}{t^2} + \frac{9}{160}\frac{\beta^4}{t^2} + \frac{11}{288}\frac{\beta^6}{t^3}\right.$$

$$\left. + \frac{8983}{302400}\frac{\beta^8}{t^4} + \frac{15\beta^{10}}{552t^5}\right]$$

$$\frac{\partial C}{\partial \beta} = \alpha^3\left(\left(\frac{2t}{\beta^2} - \frac{2}{3}\right)\frac{\partial J}{\partial \beta} - \frac{4t}{\beta^3}J - \frac{3\beta^2\sqrt{\pi}}{12t^{5/2}}\right)$$

$$\frac{\partial C}{\partial \alpha} = \frac{3C}{\alpha}$$

$$\frac{\partial Vp}{\partial \sigma_2} = \frac{a\mu_0 I_0\sqrt{\pi}}{t}\left(\frac{a^2\mu_0}{4t}\right) \cdot$$

$$\sum_{j=0}^{\infty} \frac{2(-1)^j(2j+2)!\,(2j+1.5)}{(2j+5)j!\,(j+1)!\,(j+2)!}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^{j+0.5}$$

$$\approx \frac{a\mu_0 I_0\sqrt{\pi}}{t}\left(\frac{a^2\mu_0}{4t}\right)\sqrt{\left(\frac{(a^2\sigma_2\mu_0)}{4t}\right)}$$

$$\cdot\left(\frac{3}{5} - 2\left(\frac{a^2\sigma_2\mu_0}{4t}\right) + \frac{55}{18}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^2\right.$$

$$\left. - \frac{35}{11}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^3 \cdots\right)$$

$$\frac{\partial D}{\partial \beta} = \sum_{n=1}^{\infty}\left(e^{-n^2\pi^2 t/\beta^2}\left(\frac{2n^2\pi^2 t}{\beta^3}\right)\right)$$

$$\frac{\partial I_1}{\partial d} = -\frac{1}{a}I_4$$

$$\frac{\partial I_1}{\partial \rho} = -\frac{1}{a}I_5$$

$$\frac{\partial I_2}{\partial d} = -\frac{1}{a}I_5$$

$$\frac{\partial I_2}{\partial \rho} = \frac{1}{a}I_4 - \frac{1}{\rho}I_2$$

$$\frac{\partial I_3}{\partial d} = -\frac{1}{a}I_2$$

$$\frac{\partial I_3}{\partial \rho} = \frac{1}{a}I_1 - \frac{1}{\rho}I_3$$

where

$$I_4 = \frac{3a^4 d}{(a^2+d^2)^{5/2}}, \qquad \rho = 0$$

$$= \left[\frac{2k^3a^3dE(k)}{16a(1-k^2)(a\rho)^{3/2}} + \frac{kda^3}{((a+\rho)^2+d^2)}\right.$$

$$\cdot\left[\frac{3k^2(a^2-\rho^2-d^2)E(k)}{16a(1-k^2)(a\rho)^{3/2}}\right.$$

$$+ \frac{k^3(a^2-\rho^2-d^2)(E(k)-K(k))}{16ak(1-k^2)(a\rho)^{3/2}}$$

$$+ \frac{2k^4(a^2-\rho^2-d^2)(E(k))}{16a(1-k^2)^2(a\rho)^{3/2}} + \frac{K(k)}{4(a\rho)^{1/2}a}$$

$$\left.\left. + \frac{k}{4a(a\rho)^{1/2}}\left(\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\right)\right]\right]\frac{2}{\pi}$$

$$
\begin{aligned}
I_5 &= 0, \qquad \rho = 0\\
&= \frac{a^3 k}{2\pi(a\rho)^{3/2}}\left(K(k) - \frac{E(k)(1-\frac{1}{2}k^2)}{1-k^2}\right)\\
&\quad - \frac{3d^2}{\pi a^2 \rho}\,\frac{E(k)}{((\rho+a)^2+d^2)^{3/2}(1-k^2)}\\
&\quad + \frac{3d^2a^2}{\pi\rho}(a^2+\rho^2+d^2)\\
&\quad \cdot \left[\frac{2(2-k^2)E(k)-(1-k^2)K(k)}{((a+\rho)^2+d^2)^{5/2}(3(1-k^2)^2)}\right].
\end{aligned}
$$

Combining the results of all of the above equations, one learns that

$$
\begin{aligned}
\frac{\partial V(t)}{\partial \sigma_2} &= \frac{\partial V\rho(t)}{\partial \sigma_2} + \frac{5a^2}{b^2}\frac{\sqrt{\pi}}{8\sigma_2}\frac{I_3\rho\mu_0 I_0}{t^{7/2}}(\sigma_2\mu_0 b^2)^{5/2}\\
&\quad + \frac{9}{4}\frac{I_0\mu_0 b^3}{a^2\sqrt{\sigma_2}}(I_1^2+I_2^2)\,\alpha^3\,\frac{\partial C}{\partial \alpha}\\
&\quad + \frac{3}{5}\frac{\alpha^3}{\sigma_2}a\mu_0 I_0 I_1 J + \frac{a^2}{b^2}\frac{\sqrt{\pi}}{4}\frac{I_3\rho\mu_0}{t^{7/2}}\cdot\frac{5}{2}\cdot\frac{1}{\sigma_2}\\
&\quad \cdot(\sigma_2\mu_0 b^2)^{5/2} + \frac{25}{8}\sqrt{\pi}\frac{\mu_0 I_0 b}{t^{7/2}}\frac{I_3^2}{\sigma_2}(\sigma_2\mu_0 b^2)^{5/2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V(t)}{\partial \sigma_3} &= -\frac{3\mu_0 b I_0 \pi}{\mu_0\sigma_3^2 a^2}(I_1^2+I_2^2)\cdot D\\
&\quad + \frac{3\mu_0 b^3 I_0\pi}{a^2\beta^2}(I_1^2+I_2^2)\frac{\mu_0 b^2}{2\beta}\frac{\partial D}{\partial \beta}\\
&\quad - \frac{3}{5\sigma_3}\alpha^3 a\mu_0 I_0 I_1 J + \frac{1}{5}\alpha^3 a\mu_0\frac{I_1\mu_0 b^2}{\beta}\frac{\partial J}{\partial \beta} + \frac{3}{2}\frac{\mu_0 b^3}{a}\\
&\quad \cdot(I_1^2+I_2^2)\left(\frac{\mu_0 b^2}{2\beta}\frac{\partial C}{\partial \beta} - \frac{\alpha}{2\sigma_3}\frac{\partial C}{\partial \alpha}\right)I_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V(t)}{\partial b} &= \frac{3\mu_0 I_0\pi}{a^2\mu_0\sigma_3}(I_1^2+I_2^2)\cdot D + \frac{3\mu_0 b^3 I_0\pi}{a^2\beta^2}(I_1^2+I_2^2)\cdot\frac{\beta}{b}\frac{\partial D}{\partial \beta}\\
&\quad + \frac{3}{4}\sqrt{\pi}\frac{I_3}{b^3}\frac{\rho\mu_0 I_0}{t^{7/2}}(\sigma_2\mu_0 b^2)^{5/2} + \frac{2}{5}\alpha^3 a\mu_0 I_0 I_1\frac{\beta}{b}\\
&\quad \cdot\frac{\partial J}{\partial b} + \left(\frac{9}{4}\frac{\mu_0 b^2}{a^2}C + \frac{3}{2}\frac{\mu_0 b^2}{a^2}\frac{\beta}{b}\frac{\partial C}{\partial \beta}\right)(I_1^2+I_2^2)\\
&\quad + \frac{20\mu_0 I_0}{3t^3}I_3^2(\sigma_2\mu_0 b^2)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V(t)}{\partial d} &= \frac{6\mu_0 b^3 I_0}{a^2\beta^2}\left[I_1\frac{\partial I_1}{\partial d} + I_2\frac{\partial I_2}{\partial d}\right]\pi D + \frac{a^2}{b^2}\frac{\sqrt{\pi}}{4}\frac{\rho\mu_0 I_0}{t^{7/2}}\\
&\quad \cdot(\sigma_2\mu_0 b^2)^{5/2}\frac{\partial I_3}{\partial d} + \frac{2}{5}\alpha^3 a\mu_0 I_0 J\cdot\frac{\partial I_1}{\partial d}\\
&\quad + \frac{3\mu_0 b^3 C}{a^2}\left(I_1\frac{\partial I_1}{\partial d} + I_2\frac{\partial I_2}{\partial d}\right)\\
&\quad + \frac{8}{3}\frac{\mu_0 I_0 b}{t^3}(\sigma_2\mu_0 b^2)^2 I_3\frac{\partial I_3}{\partial d}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V(t)}{\partial \rho} &= \frac{6\mu_0 b^3 I_0}{a^2\beta^2}\left[I_1\frac{\partial I_1}{\partial \rho} + I_2\frac{\partial I_2}{\partial \rho}\right]\pi D\\
&\quad + \frac{a^2\sqrt{\pi}}{4b^2}\frac{I_3\mu_0}{t^{7/2}}(\sigma_2\mu_0 b^2)^{5/2}\\
&\quad + \frac{a^2}{b^2}\frac{\sqrt{\pi}}{4}\rho\frac{\partial I_3}{\partial \rho}\frac{\mu_0 I_0}{t^{7/2}}(\sigma_2\mu_0 b^2)^{5/2} + \frac{2}{5}\alpha^3 a\mu_0 I_0 J\\
&\quad \cdot\frac{\partial I_1}{\partial \rho} + \frac{3\mu_0 b^3}{a^2}C\left(I_1\frac{\partial I_1}{\partial \rho} + I_2\frac{\partial I_2}{\partial \rho}\right) + \frac{8}{3}\frac{\mu_0 I_0 b}{t^3}\\
&\quad \cdot I_3(\sigma_2\mu_0 b^2)^2\frac{\partial I_3}{\partial \rho}
\end{aligned}
$$

Suppose that the voltages are known at the coordinates (x, y) of a rectangular grid on the ground and the coordinates of the point directly above the sphere are (x_0, y_0), then

$$\rho = \sqrt{((x-x_0)^2 + (y-y_0)^2)}$$

and

$$
\begin{aligned}
\frac{\partial V(t)}{\partial x_0} &= \frac{\partial V(t)}{\partial \rho}\frac{(x_0-x)}{\rho}\\
\frac{\partial V(t)}{\partial y_0} &= \frac{\partial V(t)}{\partial \rho}\frac{(y_0-y)}{\rho}.
\end{aligned}
$$

Once the formulas for the derivatives have been found, it is possible to consider the problem of "inverting" the voltages so as to obtain the parameters of the assumed model of the ground. In this case the actual parameters are $\sigma_2, \sigma_3, b, d, x_0$, and y_0.

This analysis is completed by ensuring that the inversion problem discussed here is well posed. To this end, two considerations must be borne in mind. The first of these is that the data is about the same. The second is that some consideration has to be given to the relative sizes of the parameters. Because the relative range of the parameters is quite large, we have chosen to use the logarithms of the actual parameters. This also forces the parameters to be positive, for the exponential function is always greater than or equal to zero.

Further, since the transient voltages can vary through several decades, we have decided not to use the actual voltages but the logarithm of the voltages. The problem therefore is to find which values of $\log(\sigma_2)$, $\log(\sigma_3)$, $\log(b)$, $\log(d)$, $\log(x_0)$, and $\log(y_0)$ will minimize S where

$$S = \sum_{i=1}^{N}(\log(V_i/V_{id}))^2.$$

Here V_i is the calculated voltage and V_{id} is the measured voltage or that voltage that would be obtained if measurements were made over the spherical conductor described above. The subscript i denotes the ith value of the voltage which may be due to a measurement made at a new position or a different time. There are N such voltages. Notice that the coordinate system used has to be placed well away from the center of the anomaly so that only positive values of x_0 and y_0 are used.

The inversion subroutines, written by Dr. D.L.B. Jupp and described by Jupp and Vozoff [3], require the derivatives of S with respect to the various parameters. If q_j represents any one of the parameters, $\sigma_2, \sigma_3, b, d, x_0$, or y_0, then

$$\frac{\partial S}{\partial \log q_j} = 2 \sum_{i=1}^{N} \log (V_i/V_{id}) \cdot \frac{q_j}{V_i} \frac{\partial V_i}{\partial q_j} .$$

All the derivatives can be calculated with the aid of the results given above.

III. Results

Dr. D.L.B. Jupp's program allows for a number of different methods of inversion. The special option used for this study ensured that the inversions were performed using the normal Marquardt method. As discussed by Jupp and Vozoff [3], the inversion scheme regroups the parameters into another set $\boldsymbol{p}$, say in such a way that the regrouped set is related to the original set $\boldsymbol{x}$, say by the equation

$$\boldsymbol{P} = S_1 V^T \boldsymbol{x}.$$

Here S_1 is the largest singular value of the generalized inverse of the Jacobian matrix. See Jupp and Vozoff [3] equation (2.21). The matrix V is a matrix that rotates the parameter space. For all the cases studied here a standard model was used. This model consisted of a set of data to which a percentage of random noise could be added and a model which was specified by a set of parameters for which estimates were available. The actual value of the parameters, the initial value of the parameters given to the computer program and the final values of the parameters after inversion for the different computer runs are shown in Table I. The distinguishing quantity for each of the runs was the level of random Gaussian noise relative to different percent levels in the data. The level of that noise is also shown in the table.

The data to be inverted consisted of the voltages that would be measured by a circular loop in the vicinity of the sphere. A given noise level was added to eacn voltage. The acutal position of the loop center was controlled by a rectangle of the coordinates (x, y). The coordinates of the respective vertices were (100, 400), (300, 400), (300, 150), and (100, 150), respectively.

The voltage was assumed to be measured within and on the sides of the rectangle at points that defined a fifty-meter square grid. At these stations the voltages were assumed to be known for times of $\frac{1}{2}$, 1, 2, 3, and 4 ms. It is important to recall when looking at the results that the original parameters $\boldsymbol{q}$ are related to the $\boldsymbol{x}$ parameter by

$$x_i = \log(q_i), \quad i = 1, 6.$$

Tables II, III, and IV summarize the results of the calculations. These tables show the actual V matrix after inversion, the relative sizes of the singular values of the Jacobian, and the damping factors of the various parameters. See equation (2.7.1) of Jupp and Vozoff [3]. The important thing to notice about the damping factors and the singular values is their relative sizes.

As shown by Jupp and Vozoff [3] the parameters may be classed into two groups: those that are well resolved and those that are poorly resolved. Whether or not a parameter is well resolved depends upon the noise level of the data and the normalized singular values of the Jacobian. These quantities allow a damping factor to be assigned to a given parameter or combination of actual parameters. For the inversion techniques described here, the parameters are well resolved if their damping factors are well above 0.5 and are poorly resolved if their

TABLE I
The Initial and Final Parameters for the Assumed Model for Various Amounts of Noise in the Data

(The radius of the loop is 50 m.)

Parameter	True Value of Parameter	Estimates of Parameter	Final Values for 5 0% Noise	Final Values for 10% Noise	Final Values for 20% Noise
σ_2	0 01	0 005	0 01	0 01	0 01
σ_3	5 0	2 50	4 90	4 60	4.88
d	100 0	60 0	100 53	102 39	102 70
b	50 0	30 0	50 47	51.72	51 16
x_0	225 0	190 0	225 05	224 22	222.21
y_0	275 0	290 0	274 64	275 79	273 77

TABLE II
Singular Values, Damping Factors, and the V Matrix for the Final Model

(The data had 5-percent noise present.)

Normalized singular values of Jacobian							
		1 0	0 901	0 733	0 241	0 214	0 0327
Damping factors (10 0 per cent level)							
		0 99	0 988	0 982	0 854	0 821	0 0964
Parameter space eigenvectors (V matrix)							
		$\log(q_1)$	$\log(q_2)$	$\log(q_3)$	$\log(q_4)$	$\log(q_5)$	$\log(q_6)$
1	$\log(\sigma_2)$	0 18	-0 000	-0 001	0 862	-0 478	-0 119
2	$\log(\sigma_3)$	0 150	-0 001	0 000	0 052	0 359	-0 920
3	log (d)	-0 558	0 003	0 001	0 468	0 658	0 192
4	log (b)	0 808	-0 004	-0 001	0 187	0.458	0 321
5	$\log(x_0)$	0 002	-0 000	1 000	-0 001	0 000	0 000
6.	$\log(y_0)$	-0 005	-1 000	-0 000	0 001	-0.000	-0 000

TABLE III
Singular Values, Damping Factors, and the V Matrix for the Final Model

(The data had 10-percent noise present.)

Normalized singular values of Jacobian							
		1 0	0 920	0 754	0 247	0 214	0 0331
Damping factors (5 0 per cent level)							
		0 998	0 997	0 996	0 961	0 948	0 304
Parameter space eigenvectors (V matrix)							
		$\log(q_1)$	$\log(q_2)$	$\log(q_3)$	$\log(q_4)$	$\log(q_5)$	$\log(q_6)$
1	$\log(\sigma_2)$	-0 120	-0 000	0 000	-0 885	-0 435	-0 116
2	$\log(\sigma_3)$	-0 148	0 000	-0 000	-0 036	0 359	-0 921
3	log(d)	0 558	-0 002	0 000	-0 436	0 680	0 193
4	log(b)	-0 808	0 002	-0 000	-0 163	0 468	0 319
5	$\log(x_0)$	0 000	-0 000	-1 000	-0 000	0 000	-0 000
6	$\log(y_0)$	-0 003	-1 000	0 000	0 000	0 000	0 000

TABLE IV
Singular Values, Dumping Factors, and the V Matrix for the Final Model

(The data had 20-percent noise present.)

Normalized single values of Jacobian							
		1 0	0 906	0 835	0 246	0 214	0 0328
Damping factors (20 0 per cent level)							
		0 962	0 954	0 931	0 603	0 533	0.0262
Parameter space eigenvectors (V matrix)							
		$\log(q_1)$	$\log(q_2)$	$\log(q_3)$	$\log(q_4)$	$\log(q_5)$	$\log(q_6)$
1	$\log(\sigma_2)$	0 121	0 000	0 003	0 900	-0 403	-0 115
2	$\log(\sigma_3)$	0 143	0 001	-0 001	0 023	0 357	-0 923
3	log(d)	-0 569	0 005	0 005	0 409	0 688	0 189
4	log(b)	0 801	0.006	-0 004	0 150	0 486	0 316
5	$\log(x_0)$	0 006	-0 000	1 000	-0 004	-0 000	0 000
6	$\log(y_0)$	0 008	1 000	0.000	-0 002	-0 000	0 000

damping factors are 0.5 or less. An inspection of Tables II, III, and IV shows that only 5 parameters are well resolved for the model with 5 percent noise and only 4 parameters are resolved by the time the noise level has reached 20 percent. This is despite the fact that good estimates have been found for the parameters of the ground. The suggestion is made, therefore, that for the time range considered here other geophysical data would be required in order to satisfactorily interpret transient data from a spherical conductor. Further, the fact that only 5 parameters have been well resolved, even for low noise data, can be interpreted by saying that the background conductivity of the ground must be considered in inverting transient data. The conclusion follows once one notices that once this quantity has been fixed then there are 5 well-resolved parameters among the 5 remaining unknown quantities.

The table for the V matrix allows the interpreter to say something of the way in which the parameters of the model are grouped. Just as Jupp and Vozoff [3] have interpreted the output of their program for the inversion of resistivity data in terms of lengths and conductivities, we may also do the same. It should be borne in mind, however, that, although the V matrix can provide some insight into the problem of parameter groupings, there is need for experience by a variety of users before any general rule can be laid down.

In looking at the V matrix one sees that the first parameter can be interpreted as an increase in the size of the sphere offsetting a greater depth of burial. The second and third parameters are the horizontal coordinates for the center of the sphere and do not group significantly with any of the other parameters. This result could have been expected because of the symmetry of the circular conductor. Also these parameters have the property that they are invariant with regard to the other parameters with respect to translations. Interestingly, these parameters would be the best resolved by inspection from the field data. The fourth parameter of the table is easier to interpret for it is approximately $\sigma_2^2 d$ and suggests that an increase in background conductivity could be confused with a depth of burial. A more interesting parameter grouping is that for the fifth parameter of Table II. That parameter is a combination of all the parameters of the model, except the x and y coordinates of the sphere. Consequently, it is difficult to interpret that parameter. This result might also have been anticipated. The sixth parameter is interesting because its predominant terms suggest that one should look for a parameter like b/σ_3^3. This quantity is quite different from the β^2 quantity we have frequently seen in our equations and its ultimate interpretation must await further analytical as well as numerical investigations of the problem examined in this paper. Notice that an increase in the noise level of the data does not substantially alter this discussion.

The conclusion is that commonly-used inversion schemes can be used for the inversion of time domain electromagnetic data. Further, such schemes result in some parameter combinations which are not easy to interpret. In any case, single loop transient electromagnetic data obtained at the later stages of the transient appears to be insufficient to fix all the parameters of a buried spherical conductor.

ACKNOWLEDGMENT

The author would like to thank Dr. D.L.B. Jupp of C.S.I.R.O. Land Use Research, Canberra, Australia, for the use of his subroutines and for several informative conversations. Miss F. Buckley helped with the programming.

REFERENCES

[1] S. H. Ward, J. Ryu, W. E. Glenn, G. W. Hohmann, A. Dey, and B. D. Smith, "Electromagnetic methods in conductive terranes," *Geoexploration*, vol 12, pp. 121-183, 1974.

[2] W. R Petrick, Wm. R. Sill, and S. H. Ward, "Three-dimensional resistivity inversion using alpha centres," *Geophys.*, vol. 46, pp. 1148-1162, 1981.

[3] D.L.B. Jupp and K. Vozoff, "Stable iterative methods for the inversion of geophysical data," *Geophys J. Roy Astron Soc.*, vol 42, pp. 957-976, 1975

[4] —, "Two-dimensional magnetotelluric inversion," *Geophys J. Roy. Astron. Soc.*, vol. 50, pp. 333-352, 1977.

[5] J. C. Rodriguez-Ovejero, "Inversion of TDEM data with a catalog interpolation," unpublished M.Sc. thesis, Colorado School of Mines, Golden, CO.

[6] T. Lee, "Transient electromagnetic response of a sphere in a layered medium," *Pageoph.*, vol. 119, pp. 310-338, 1980/81.

[7] —, "The transient electromagnetic response of a conducting sphere in an imperfectly conducting halfspace," *Geophysical Prospecting*, to be published.

[8] F. Oberhettinger and L. Baddii, *Tables of Laplace Transforms.* New York: Springer, 1973, p. 427.

[9] Y L. Luke, *Integrals of Bessel Functions.* New York: McGraw-Hill, 1962, p. 419

*

Terry J. Lee, for a biography please see p. 80 of this issue.

The Detection of Induced Polarization with a Transient Electromagnetic System

ROGER J. G. LEWIS AND TERRY J LEE

Abstract—The study of two-dimensional TEM models consisting of a ·linder in a conductive host using measured rock properties show that ·EM surveys are capable of detecting induced polarization (IP) effects. case history showing a comparable type of response is presented. ie IP effects manifest themselves in extreme cases as negatives in the sponse and in other cases as anomalously high apparent resistivities. ie models used allow a study of the variation of responses produced varying the Cole-Cole constant for the cylinder properties. The sults suggest that the different parameters produced distinct effects the response. If the rock properties reported in the literature are cor- ct, it appears quite feasible to observe IP effects via TEM using an grounded system. There may also occur a set of negatives in the re- onse of a cylinder which are associated with high-conductivity con- ists and geometrical effects. In the cases studied these two groups readily distinguished. IP effects might be interpreted from a family zero responses (when these occur), apparent resistivity/time sections by stripping away the host rock response and comparison with a uni- m polarizable ground. There is a great deal of geometrical informa- n about the scattering body at early times our of reach of many ntemporary instruments.

I. Introduction

NDUCED polarization (IP) surveys are widely used in the exploration for sulphide mineralization The conventional IP ethod and the magnetic induced polarization method (MIP) th require that current be passed into the ground via at least

Manuscript received January 14, 1983; revised June 27, 1983 This rk was partially supported by a grant from Peko-Wallsend Limited

R. J. G. Lewis is with the Geology Department, University of Tasmia, Hobart, 70001, Australia.

T J. Lee is with Geopeko, Division of Peko-Wallsend Operations, nited, Gordon, N S.W , 2072, Australia

two current electrodes. The traditional IP method requires the use of other grounded electrodes for potential measurement, though in MIP these are replaced by a sensitivite magnetometer and the electrode geometry is selected to maximize electric flux collection by a conductive body [1]. More recently similar techniques have been described for the measurement of resistivity by Edwards *et al* [2]. The possibility of eliminating the physical connections with the ground completely and the possibility of achieving an airborne IP system using entirely electromagentic methods has been recognized for over a decade [3], [4]. We show that in at least some cased IP effects are readily detectable in TEM results and indicate certain characteristics of the TEM responses of polarizable bodies that appear to be of diagnostic value.

Hohmann *et al.* [3] studied theoretical models at frequencies below 30 Hz and made field measurements in the range 15-1500 Hz in an effort to evaluate the potential of an inductive IP system. Only amplitude measurements were used and it was shown that the observations could be fitted by simple structures made up of nonpolarizable materials. The conclusion from this study was that amplitude measurements alone were unlikely to be of use in determining the presence of an IP response but the question of the use of phase measurements was left open. Since that time, phase measurements with standard electrical IP have been widely used [5] and further detailed measurements of the phase responses of *in situ* mineralization have become available [6].

Among the results of Hohmann *et al.* [3] is an interesting observation that at the Northern Lights test site the resitivity

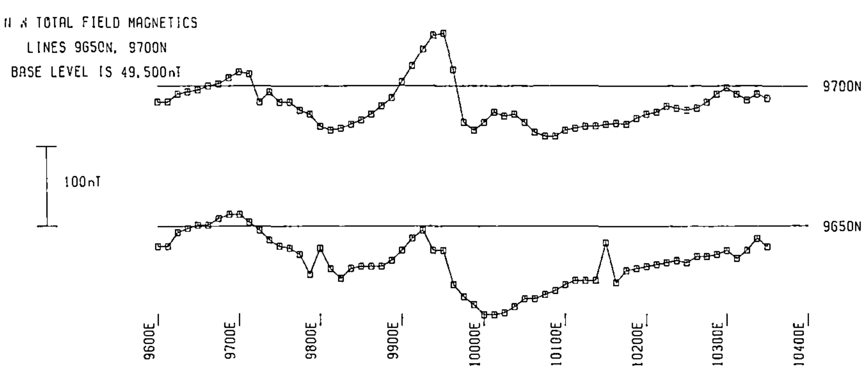


Fig. 1 Total magnetic intensity profiles across NKW. The anomaly between 9900E and 10000E appears to be associated with the mineralization.

obtained by conventional and inductive methods differed significantly This may be an indication of polarization effects in light of the following discussion on apparent resistivities as measured with TEM.

There have been suggestions in the literature for some time that IP effects may manifest themselves as anomalous negative responses in transient electromagnetic (TEM) measurements [26]. In this sense, negative is taken to mean in the opposite sign to the TEM response of a single loop system on a uniform earth of real conductivity [7]. The fact that induced polarization effects could cause the transient response of a dipole on a uniform ground to change sign was pointed out by Bhattacharyya [8]. Bhattacharyya [9] also gave a discussion of the significance of displacement currents whose effect are also equivalent to a complex conductivity.

This possibility for uniform and layered grounds has been discussed by Morrison *et al.* [10] who showed that negative TEM responses could arise in layered structure responses where extreme IP parameters were present. Rathor [11] considered the effects of IP properties of uniform grounds on various geophysical measurements and concluded that the effects of polarization should be discernable in the field.

It should be noted that the works discussed above have used conductivity spectra which differ significantly from the Cole-Cole model used here.

Lee [12], [13] investigated the effect of polarizable materials upon TEM responses of a sphere in free space using the measured properties of rocks as a basis of discussion. This work showed that the sign of the response would change from normal to negative with the passing of time. While this result is of interest its application to real cases must be considered doubtful in view of the effect of a conducting host rock upon TEM responses [14].

The response of a uniform halfspace of polarizable material has been studied by Lee [15]. These results also show a negative response at late times and are especially interesting as some light is shed on the subleties of the mathematical basis for the anomalous TEM effects. Further, these results show that the late time decay for a polarizable halfspace is at a quite different rate than that of a ground of real conductivity.

In addition, actual TEM surveys have often produced anomalous negative TEM responses where it is known or suspected that polarizable media are present, e.g., Spies [16].

More recently, Weidelt [17] has shown that single loop transient data will be of the same sign for arbitrary structure provided that the conductivity and magnetic permeability ar frequency independent.

II. A Case History

To illustrate the type of effect occasionally observed i actual surveys and long suspected of being due to the presenc of polarizable material, we present a brief case history from a area in northern Australia. Here several drill holes (SD-13 SD-9, etc.) over a negative TEM anomaly have revealed sulphid mineralisation [27].

The local geology consists of three distinct rock types. Th oldest is a mineralized quartz staurolite schists overlain by fine-grained biotite schist. These basement rocks are expose through a tectonic window in fine-grained porphyritic aci flow rocks or shallow intrusives. The surface exposures in clude ironstones and gossans with boxworks. Below the sur face the gossans contain some cerussite.

The rocks have undergone at least three deformations an the sulphides occur as anastomosing veins and stringers, onl some of which area parallel to the main cleavage. The sulphide typically make up about 5 percent of the rock and those sam pled by shallow drilling are predominantly pyrite.

The mineralization in this case is associated with a total fiel magnetic anomaly of up to 100 nT peak amplitude. It i better defined on line 9700N, the line to the north of tha where the other geophysical measurements were made (Se Fig. 1). The cause of this anomaly is not clear at present.

An ordinary induced polarization survey reveals resistivit and chargeability anomalies in the region of the magneti anomaly (Fig. 2). The maximum chargeability observed in time slice with mean delay of 780 ms is 3.2 percent. The ap parent resistivities are in the range 329 to 20 $\Omega \cdot$ m.

A TEM survey was conducted on line 9650N using a 100-r coincident loop and the SIROTEM instrument (Buselli an O'Neil [18]). Again, the effects of the mineralization ar quite apparent from the results in Fig. 3 which shows all th observations. The effects of a grounded fence at 9700E ar quite apparent and distinct from the effects of the mineral zation further east. The most interesting feature of the TEM survey is that the response starts off normally at early time but becomes negative between channels 3 and 4 (between 1

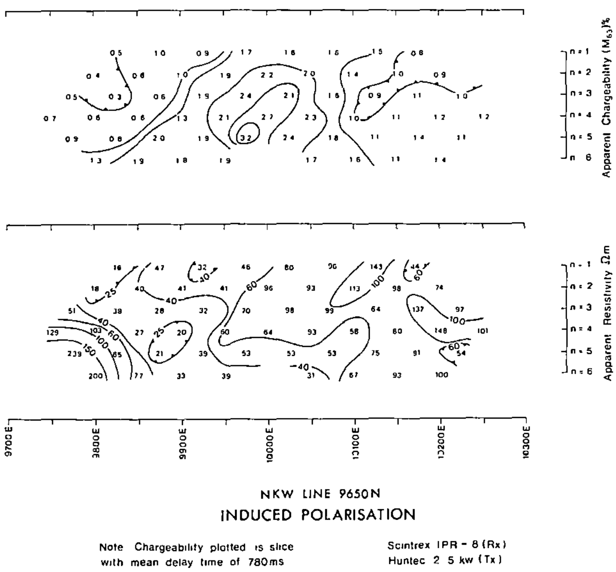


Fig. 2. IP and resistivity profiles from NKW. The chargeability high is at about the position of the magnetic anomaly. The lowest apparent resistivities lie further west. Collectively the chargeability high and resistivity low cover the area of anomalous TEM response.

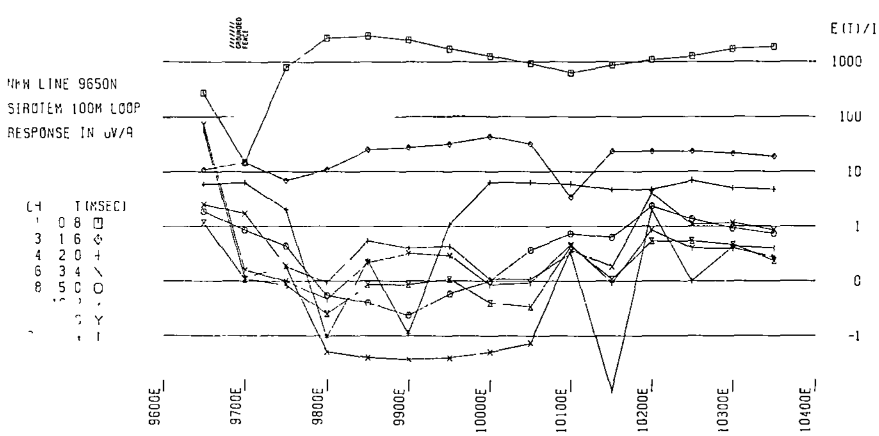


Fig. 3. TEM results from NKW. Note that the early and late responses are essentially normal and greater than the mid-time responses.

d 2.0-ms mean delay) and reaches a maximum negative re- onse at channel 3 (3.4-ms mean delay) From that time on e response becomes progressively more positive until at the est observation time (channel 20, 33.4 ms) the response is te normal.

As will be seen later, the response at 2 ms closely resembles t expected from a cylinder in that there is a central positive h flanking negatives.

It is manifestly apparent that the response is generally *less* r the mineralization than elsewhere and that the decay over ange of times is more rapid than in the regions away from mineralization. Both observations are at variance with the nple notion of slow decays over conductors.

n the following sections we show that some features of this onse are obtained in theoretical two-dimensional models h cylindrical polarizable bodies in a halfspace of real con- ctivity. It is, however, not an exact comparison for two sons. The first is that the field case given is at least $2\frac{1}{2}$-di- nsional (i.e., the loop used is not two-dimensional though the body may be) and secondly we are unable to compare the actual rock properties with those in the numerical models due to lack of measurements. Indeed, the paucity of detailed property measurements on Australian rocks will be a major problem in extending work of this type.

III. Numerical Models

We have used the theory for a mode-matched solution described by Howard [19] to calculate the steady-state electromagnetic response of a cylinder in a halfspace. The resulting frequency domain data was transformed to the time domain by the use of cubic splines [20]. The calculations allow all the conductivities to be complex so that IP effects can be included. The necessity for a frequency-dependent conductivity to be complex is discussed in Fuller and Ward [21] and Weidelt [17]. In essence it is a requirement for causality. Other calculations of combined IP-EM effects using complex conductivities have been presented by Hohmann [22]. In the results presented here the halfspace resistivity is always real. We have

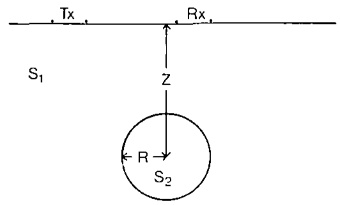


Fig. 4. The basic geometry for all the models discussed here S1 and S2 are the conductivities which may be arbitrary complex functions of frequency.

COLE-COLE MODEL

$$\rho = \rho_0 \left(1 - m\left(1 - \frac{1}{1+(i\omega\tau)^c}\right)\right)$$

$$\sigma = \sigma_0 \left(1+(i\omega\tau)^c\right)/\left(1+\alpha(i\omega\tau)^c\right)$$

ρ_0/σ_0 = DC values

m = Chargeability

α = 1-m

τ = Time Constant

c = Frequency Dependence

ρ/σ = Resistivity/Conductivity

KIDD CREEK

(Pelton et al, 1978)

σ_0 = .064 s^{-1}, τ = 0308 s^{-1}, c = 306,

α = 089, m = 911

Fig. 5. The basic formulae of the Cole-Cole model and the values used for the Kidd Creek parameters

used the well-known Cole-Cole model [23] to describe the electrical properties of the cylinder. We have used as a starting point the actual rock properties measured by Pelton *et al.* [6] at the Kidd Creek deposit which may, in the absence of more extensive data, be taken as respresentative of the properties of the "massive sulphide" class of deposits. As it is of interest to determine which of the four Cole-Cole parameters are responsible for particular features in the TEM response, we have studied cases where a single parameter is varied systematically from the original Kidd Creek value. We also study certain other cases which shed light on the requirements for detecting the IP effects in a TEM survey and also certain real conductivity cases which are useful for comparative purposes.

IV. Results

Fig. 4 shows the basic model used for all the cases presented here. The transmitting loop (TX) is modeled by a step current flowing in two parallel wires, but in opposite directions. The receiving loop (RX) is similarly modeled The actual quantity calculated is volts per ampere per meter.

We now turn to the cases using measured induced polarization parameters or limiting cases. The properties of the Cole-Cole model and the measured values for the Kidd Creek deposit are set out in equation (2) of Lee [15] (Fig. 5). First consider the TEM response of the structure where the cylinder has the real conductivity given by the zero frequency value of the Kidd Creek parameters.

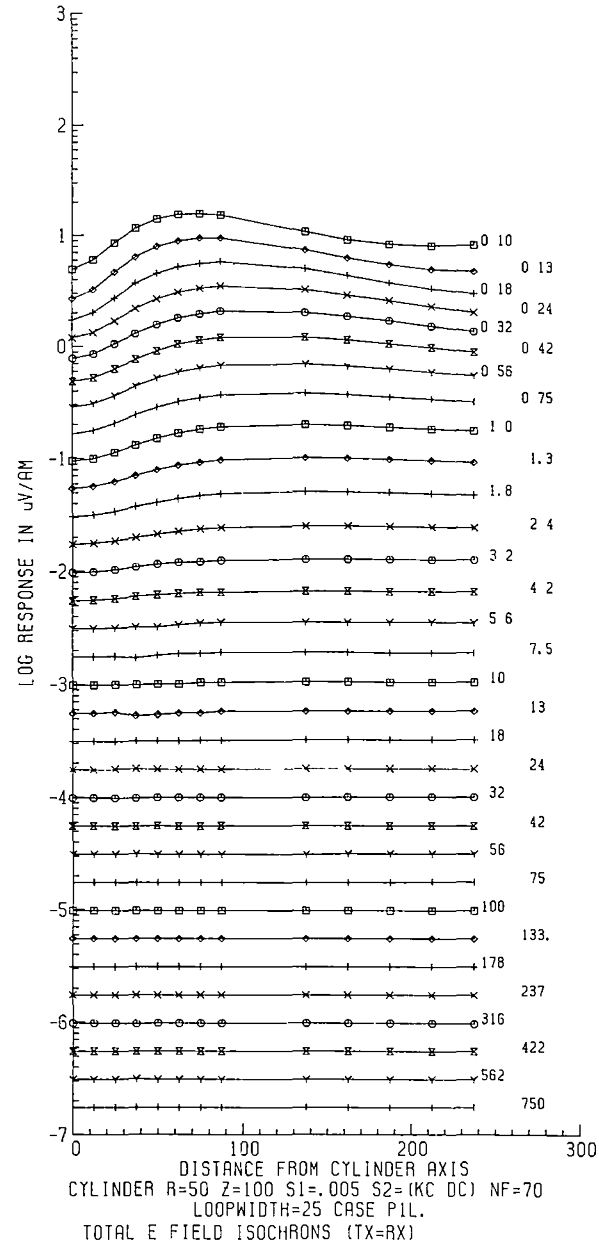
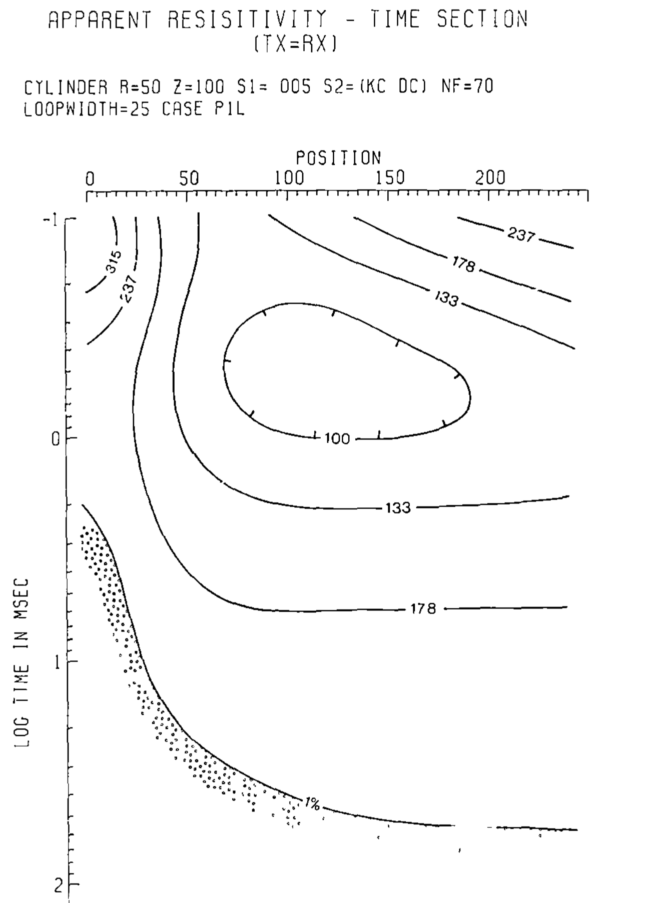


Fig. 6 The response obtained with the standard model using the zero frequency Kidd Creek conductivity for the cylinder (6.4 × 10^{-2} s/m) Here, as in all subsequent models the loop width is 25 m, the cylinder radius is 50 m and the center of the cylinder is at a depth of 100 m The background has a real conductivity of 5 × 10^{-3} s/m. Here as elsewhere *NF* refers to the number of frequency domain calculations that were used to produce the transient

Fig 6 shows a typical response where the conductivity contrast is low, in this case 15.67 The response has a minimum over the cylinder and a maximum on either side Although more type cases should be calculated, it appears that a maximum occurs when the loop is a little less than the cylinder depth, removed from the cylinder axis and at moderate times The corresponding apparent resistivity time section is shown in Fig 7. Below the line labeled 1 percent, the apparent resistivity is within that tolerance of the background resistivity The simplicity and form of the apparent resistivity is remarkable.

We next turn to the case when the conductivity of the cylinder is that obtained from the Kidd Creek parameters a



ᶦig. 7. The apparent resistivity-time section corresponding to the case shown in Fig. 11. This may be taken as representative of the behavior of a real conductivity cylinder Below the shaded line the apparent resistivities are within 1 percent of the true resistivity of the halfspace.

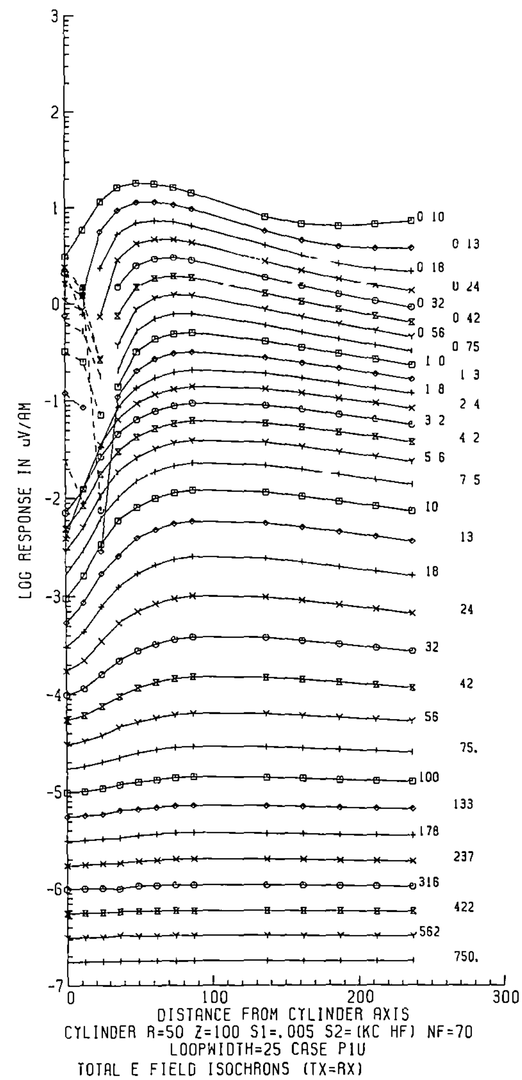


Fig. 8. The response obtained when the cylinder has the high frequency limit of the Kidd Creek parameters ($\sigma = 0.77$ s/m). The higher conductivity contrast produces negatives at early times above the cylinder.

the high-frequency limit (Fig. 8). The resistivity is appreciably ɔwer due to the high chargeability and the general form of the esponse is as in the previous case with the exception of the rea close to the cylinder axis where negatives occur at early ımes. This negative is produced purely by geometrical and esistivity contrast features. This does not contradict the reılts of Weidelt [17] whose proof that negatives cannot arise .om nonpolarizable materials depends upon the transmitter eing of finite dimensions. This is not the case here. The ccurrence of these types of negatives does not limit the useılness of the model because we will show that they have difːrent characteristics than those associated with a polarizable ound.

The use of the actual Kidd Creek properties produces a specɥcular change in the isochrons (Fig. 9). The decay is abnorɥally rapid in the region off the cylinder and indeed goes ːgative in the time range 5.6 to 18 ms. At later times the reɥonse is of normal polarity. It is thus apparent that we have produced the gross from of the TEM survey in the case story outlined above. However, there is no central normal ɥne as seen in the cylinder cases so the comparison is not ɥmpletely valid

The waveforms display a combination of the features of ɥth the upper and lower frequency limit cases as well as the IP effects. In particular the early time negatives seen in the upper frequency limit case are present here. Fig. 10 shows the corresponding apparent resistivity time section. There is a fundamental difference from that of the nonpolarizable bodies in that a second lobe of high (and sometimes, as here negative) apparent resistivities appears. The upper part of the plot has the same general form as that for a nonpolarizable body but it is compressed into earlier time and somewhat deformed by the early time response immediately above the cylinder. The bi-lobed form of the apparent resistivity time section seems, from the models studied so far, to be quite diagnostic of the occurrence of IP effects. It is also clear that these effects are very persistent and extend to times too late to be observed by current instruments. That this is so is hardly surprising. We are entering the time range where ordinary IP responses are observed and despite the tendency to regard IP and EM/TEM as distinct topics, they are fundamentally of the same nature.

It is instructive to examine the region of negatives a little more closely. In Fig. 11 we plot the time of passage through

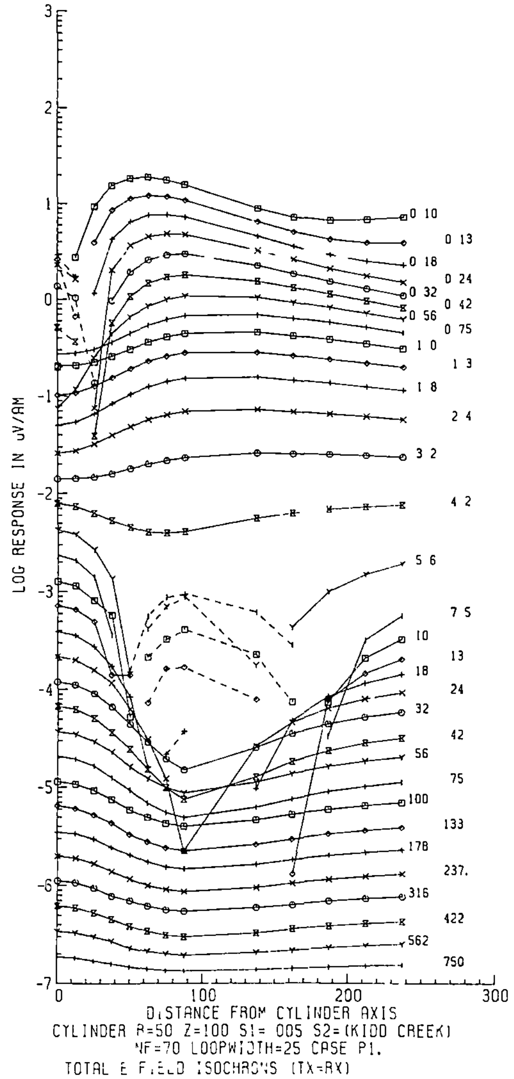


Fig. 9. The response when the cylinder has the Kidd Creek properties. The zone of negatives (dashed isochrons) reproduces in gross form the results from NKW. In addition every feature found to be indicative of effects appears in this figure.

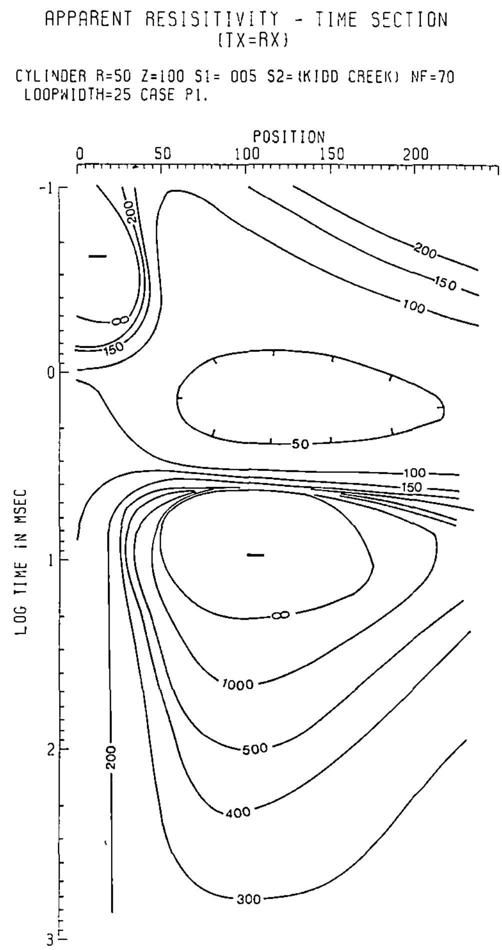


Fig. 10. The apparent resistivity-time section corresponding to Fig. 14. The occurrence of the late time negative resistivities seems diagnostic of large IP effects. The bi-lobed form off the cylinder axis is typical of all the IP cases, though the central negative is often absent.

zero against the round trip distance from the loop to the cylinder and back. Fig. 11 shows the coincident transmit/receive results and there are clearly two populations. A horizontal line connects consecutive zero pairs on the same waveform or decay curve. The population on the left-hand side are the zeros associated with geometrical and conductivity contrast effects over the cylinder at early times. The right-hand population are the zero-crossings due to IP effects. In all the cases we have studied, these populations are quite distinct. Fig. 12 shows the same plot for the permutations of 13 transmit/receive combinations when we remove the coincident restriction. Clearly the same general "footprint" pattern is enhanced. In fact, the outlines exactly coincide. However, in the noncoincident loop cases there are also sign changes due entirely to the two loop geometry. It is apparent that the shape of the "footprint" may be diagnostic in cases where the response changes sign.

While is it clear that IP effects can produce the type of effect sought, it is also interesting to systematically vary the Cole-Cole parameters one at a time to see in what way the IP effects

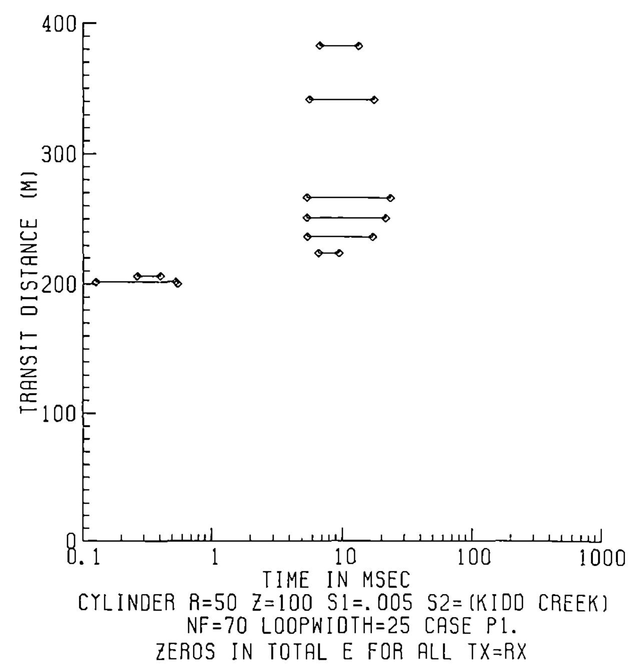
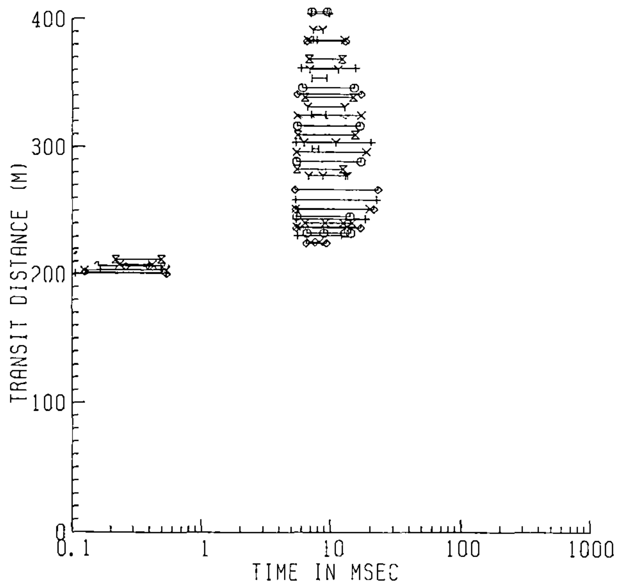


Fig. 11. The footprint for the Kidd Creek case using only coincident transmit-receive loops. There are no zeros due to loop geometry in this case. Points joined by a line are consecutive zeros in the same decay curve. There are two distinct zero populations. The lat time population is associated with IP effects.



ig. 12. This corresponds to the results in Fig. 17 but includes all the zeros from the permutations of 13 loops used as transmitters or receivers. The outline of the IP zero group is the same as obtained in the previous case. However, there are also zeros due to loop geometry.

hange. If we hold the dc conductivity constant there are hree parameters to vary Some results of the variations are as ollows

1. *Variation of c (Frequency Dependence)*

Fig. 13 shows the response when $c = 0.25$. The reduction in has removed the negatives associated with IP effects but the sochrons are still deformed downwards. In some cases the extent of the deformation is about two orders of magnitude. Apparently with this model the threshold for negative transitions occurs for a value of c between 0.25 and 0.3

Fig 14 shows the result of increasing c to 0.5. The negative one extends to a much later time (42 ms) than in the original nodel. As with all the cases showing IP negatives, the negative alued isochrons are convex upward in the off axis zone while he positive isochrons following the negative have the opposite onvexity.

. *Variation of m (Chargeability)*

Since the chargeability m may be more related to conventional IP measurements than the other parameters, we study our cases with different values of m.

Increasing m to 0 95 further increases the extent of the negative response region (Fig. 15).

Decreasing m to 0.85 again changes the response significantly. here is a zone of anomalously rapid decays but only a very nall region where the response is actually negative (Fig. 16).

At $m = 0.7$ the response (Fig. 17) lacks the IP negatives altogether but there are still signs of abnormally rapid decay here the convexity of the isochrons changes in the off axis one. Also, there are effects corresponding to a resistivity low ver the cylinder extending to very late times. These characteristics seem to be characteristics of a polarizable cylinder.

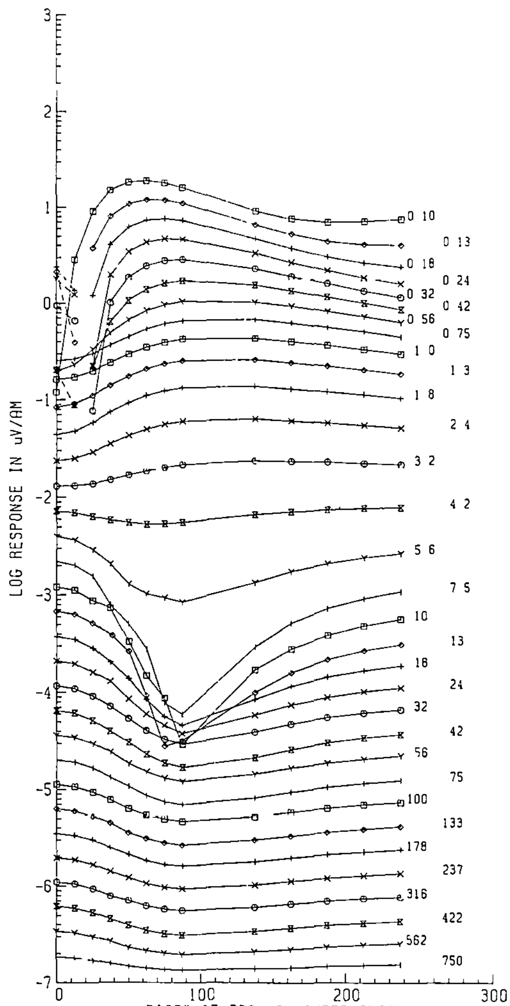


Fig 13. The response obtained by reducing c to 0.25 but otherwise using Kidd Creek parameters There are no negatives due to IP effects but instead a zone of extremely rapid decays.

The response with $m = 0.5$ appears in many ways "normal" but there are again anomalous effects over the cylinder corresponding to an apparent resistivity low extending to very late times (Fig. 18). Thus while the more spectacular effects of the polarization are absent its presence can still be deduced.

C. *Variation of Tau (Time Constant)*

The time constant in a loose sense controls the duration of the IP effects. This is certainly true in the case of ordinary IP surveys (Pelton *et al* [6]). In the Cole-Cole formulae, tau is always raised to the power c, so we use very large variations to produce appreciable effects.

Increasing tau by a factor of 100 produces the results in Fig. 19. The effects are dramatic. The response goes negative at about 10 ms and is only back to normal at 750 ms

Decreasing tau by a factor of 0.01 might be expected to dramatically reduce the duration of IP effects. Indeed, the response as shown in Fig. 20 is without the IP family of negatives.

It is clear that apart from the continuation of disturbances to very late times, several of the responses of less polarizable

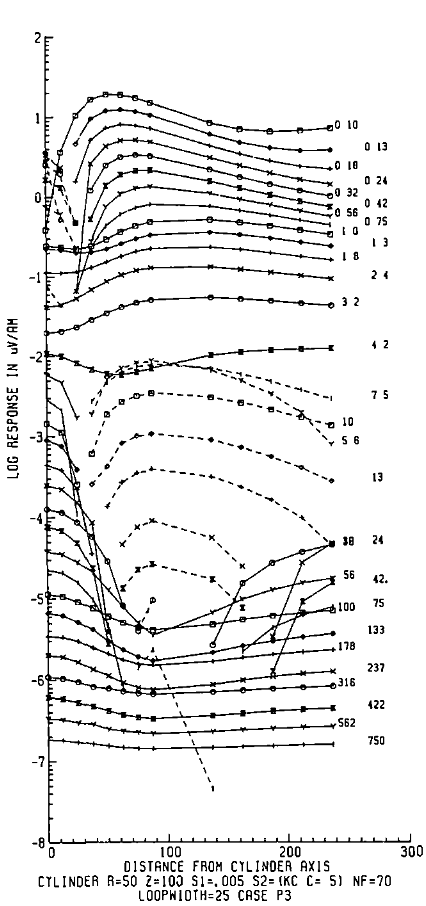


Fig. 14. The response obtained by using $c = 0.50$. The size of the negative response region is greatly increased over that obtained in the original Kidd Creek case.

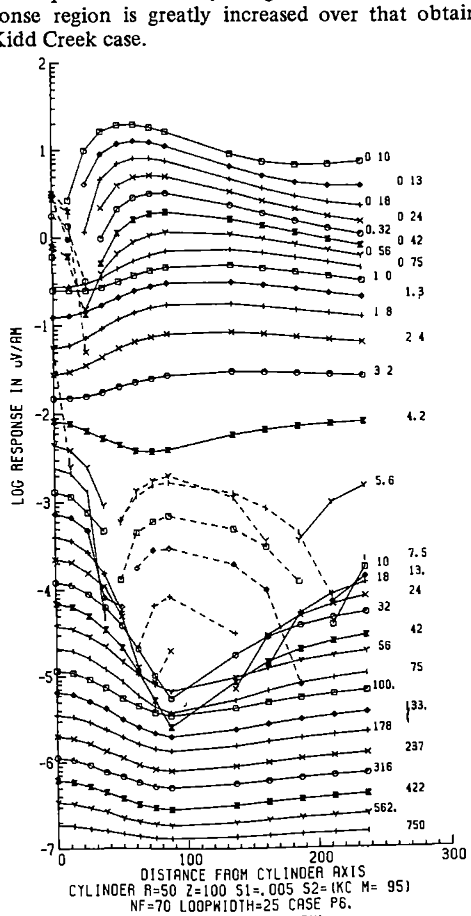


Fig. 15. The response obtained by increasing the chargeability to 0.95 (above the Kidd Creek value). Again the negative area is increased.

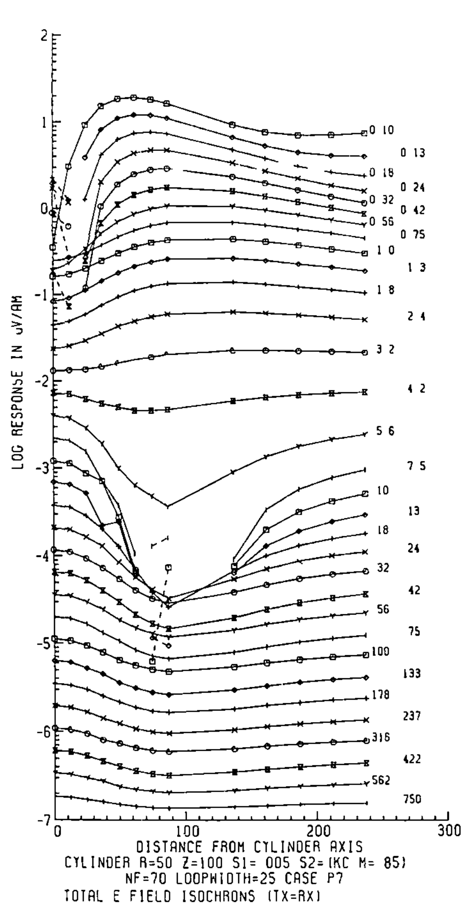


Fig. 16. Reducing the chargeability to 0.85 produces this response. While there is a broad area of depression the response only just goes negative.

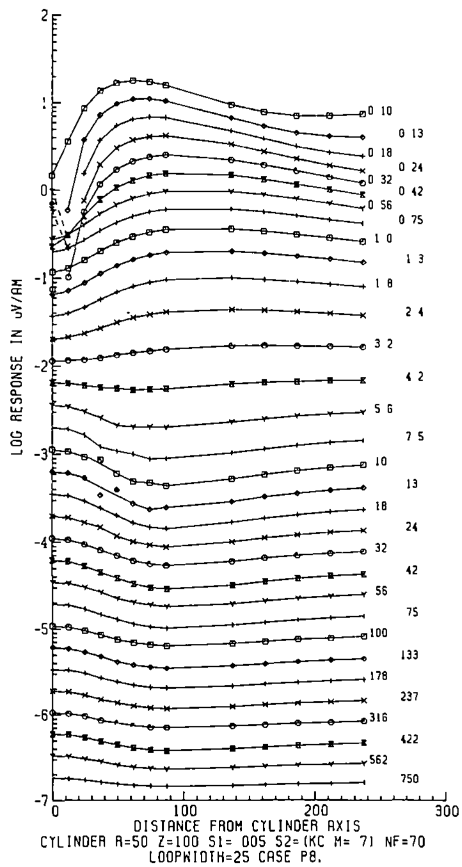


Fig. 17. The response when m is reduced to 0.7. While there are no IP negatives there are effects persisting until the latest times over the cylinder and in the off axis zone.

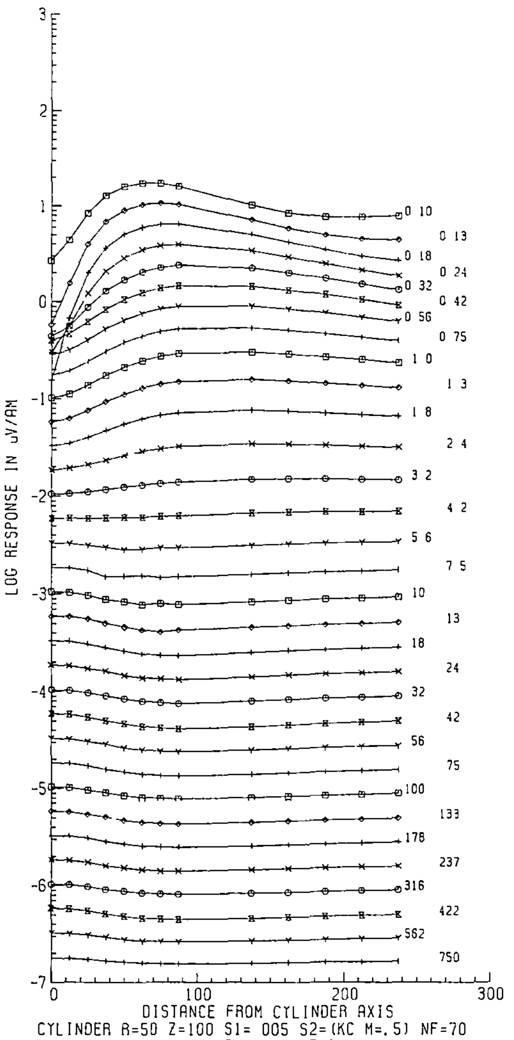


Fig. 18. The response for $m = 0.5$. The long persisting effects evident in all the polarizable models are still present.

bodies appear, in waveform, to be rather "normal". That may be so but the IP effects are clearly visible in the apparent resistivity time sections. To illustrate this we include Fig. 21 which is the section obtained when the chargeability is reduced to 0.5. The apparent resistivity section again has the characteristic bi-lobed form clearly revealing the IP properties of the cylinder. It may well be that such effects are present in many field observations but unless they are sought out they would be easily overlooked, particularly in view of the rather narrow time window available on most instruments. This would typically cover just over a quarter of the section used here

It is apparent from the above that IP effects are readily detected by TEM and the question naturally arises. How can one extract the IP parameters from the results in real surveys? Undoubtedly in the cases extreme enough to produce negatives, a technique based on the analysis of the "footprint" may be viable. This observation is based on the fact that for all the cases studied here, variations on the characteristics of the footprint shown in Fig. 12 were found [25]. Equally, the apparent resistivity section for polarizable bodies is of a quite diagnostic form and might be used. However, in keeping with conventional IP methods one would like to define an apparent

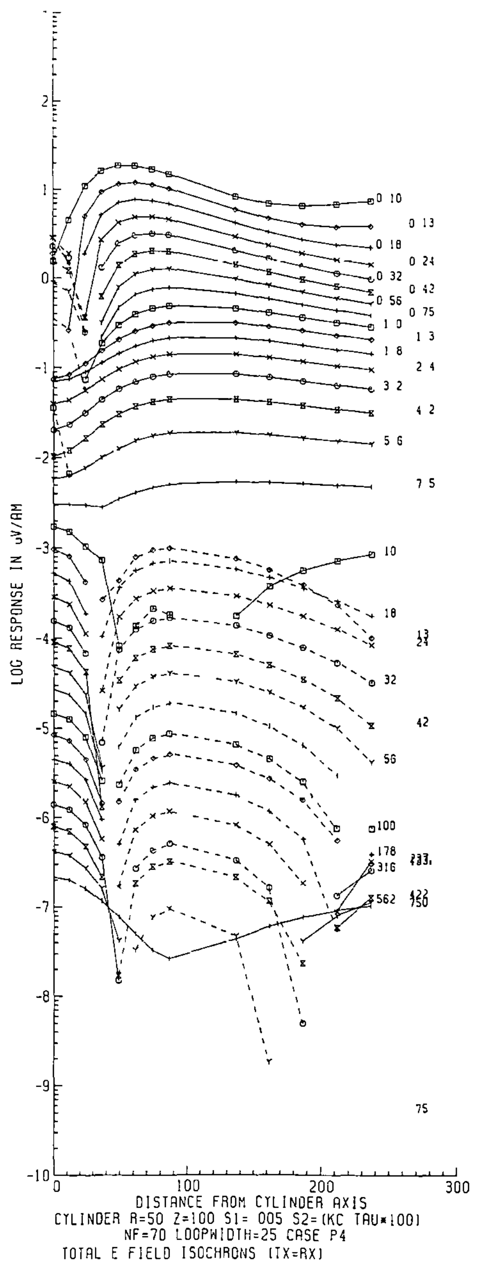


Fig. 19. Increasing tau by a factor of 100 drastically affects the later time response At the last time (750 ms) the response has returned to normal. The onset of the negative region seems to be somewhat delayed in comparison with other cases.

chargeability using comparisons with a uniform ground in the same way that one defines apparent resistivity. The transient response of a uniform polarizable ground is described by Lee [15]. An apt comparison is the case where the ground has the Kidd Creek properties (Fig. 22). The chief features are the extra rapid early decay, a change in sign and then an abnormally slow decay. Reducing the chargeability drastically shows what may be a more typical situation (Fig 23) The negative transition is shifted towards the end of the operational range of some instruments and may pass undetected in noise. Applications of standard interpretation techniques, such as the determination of apparent resistivity (Lee and Lewis [20]), will produce errors of up to about an order of magnitude.

The fundamental difficulty in attempting to derive an apparent chargeability is the difference in waveforms between

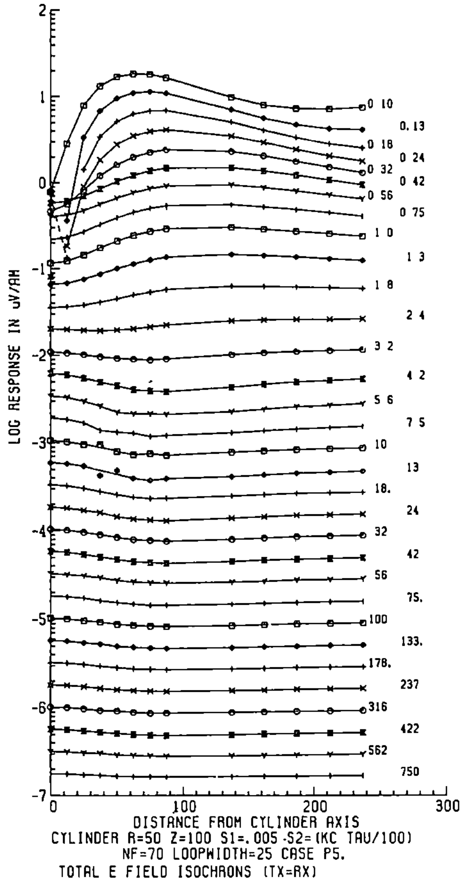


Fig. 20. With tau reduced by a factor of 100 there are still very small late effects over the cylinder and in the off axis zone.

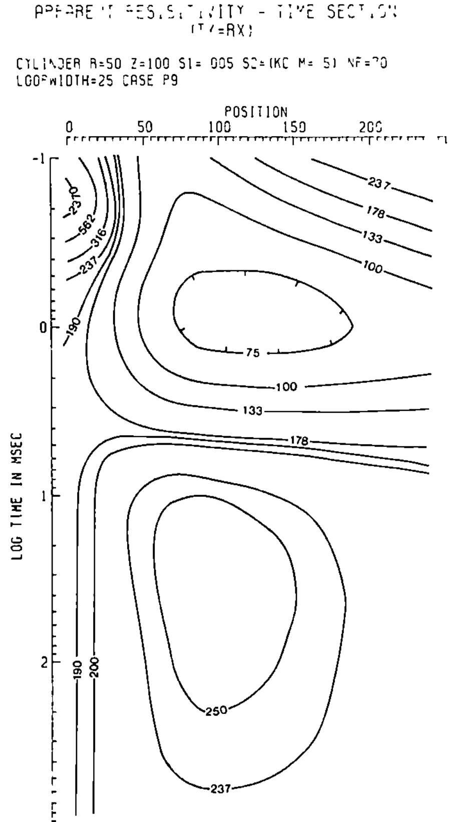


Fig. 21. The apparent resistivity time section for the case $m = 0.5$. The resistivity low over the cylinder and the lower lobe of high resistivities clearly reveal the IP effects. However, most instruments are capable of spanning less than half the time range shown here.

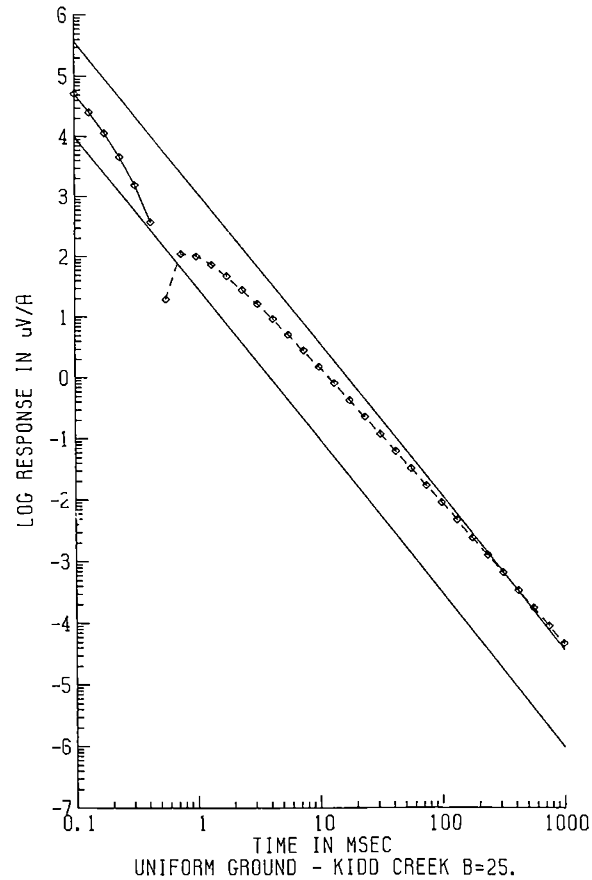


Fig. 22. The TEM response of a uniform ground with the Kidd Creek properties. The response is for a circular loop of radius 25 m. Note the early change of sign and the extraordinarily slow negative decay at late times. The asymptotes are for grounds with the dc and infinite tau conductivity values.

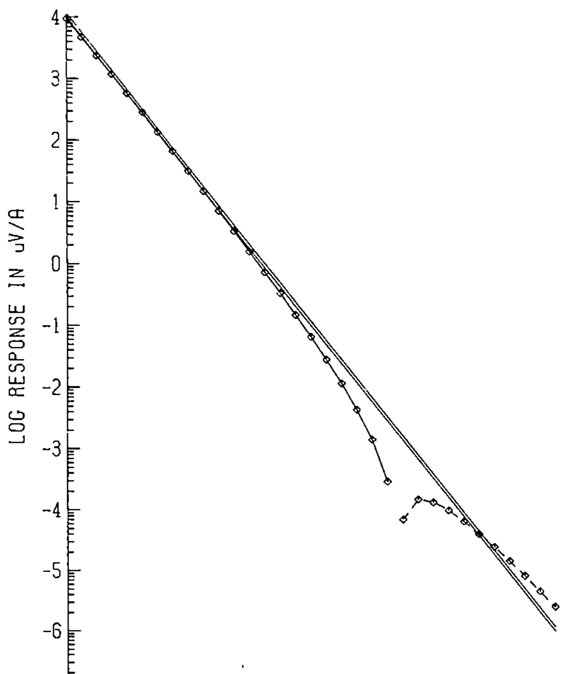


Fig. 23. Reducing the chargeability of the uniform ground to 0.1 move the change in sign into the time range where it could easily be lost in noise and then anomalously rapid decay might pass unremarked even though the observed voltages are misleading by about an order of magnitude.

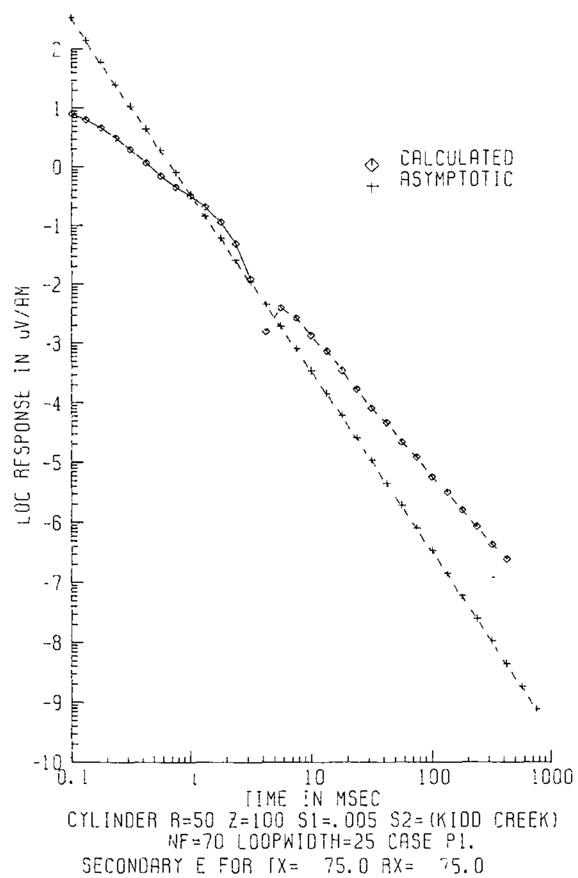


Fig. 24. The secondary E waveform from the cylinder with the Kidd Creek properties. It is more comparable with the uniform polarizable ground response than the directly observable quantity and might form the basis of an interpretation scheme where "apparent" parameters are extracted. In particular note the abnormally slow decay as compared with the asymptotic value calculated assuming only modes 0 and 1 exist in the body current system.

.he body and ground responses. The ground produces a single ;ign transition, the body an ephemeral transition or no transi- ion at all. The two are basically not comparable. However, f one strips off the effects of the conductive host and looks ıt the secondary field due to the body alone then waveforms)f a similar character are obtained (Fig. 24) and this might orm an appropriate basis of comparison. Indeed it would ·eem that a complete set of apparent Cole-Cole parameters ıight be produced.

V. Conclusions

For the model used here there are two groups of negatives /hich may occur in the TEM response of a cylinder. One is ssociated with high conductivity contrasts and geometrical ıctors; the other is clearly shown by the models to be asso- ıated with properties of the cylinder.

The common form of the early parts of all the apparent re- tivity time sections for cylindrical bodies suggests that this is iagnostic of the body geometry It also suggests that by con- ɜntrating on late times only, instrument designers are remov- ıg a potentially very important class of features from the ;ach of the user.

It must be assumed that sulphide mineralization will show IP ffects in view of the extensive use of conventional IP surveys. he case history presented strongly suggests that such effects ·e also important in TEM surveys as the gross observed effects can be reproduced using models with polarization effects. Indeed, the models presented suggest that IP effects are readily detected by TEM measurements and that failing to take account of polarization effects may result in interpretations which are orders of magnitude in error. In particular, estimates of apparent resistivity are likely to be far too high.

In light of the results presented it seems fair to say that at least there is an urgent need for new interpretation techniques which can cope with the problems revealed.

The applicability of two-dimensional EM models to the real world is well known to be fraught with some difficulties as effects shown by three-dimensional bodies are absent. Accordingly there is also great incentive to produce practical 3-D TEM modeling techniques.

Finally we agree with the remarks made by Dr. J. R. Wait that ultimately it will be found necessary to untangle IP and EM responses [26]. Also, the reader should be aware that the Cole-Cole model is not the only one for as far back as 1954 Dr. Wait suggested that the inverse loss tangent factor could also be used [28]. Some discussion of dispersive complex conductivities and permittivities can be found in chapter five of a book edited by Dr. L. B. Felsen [29].

Acknowledgment

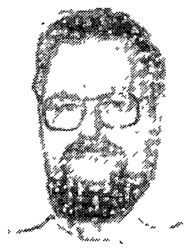
The authors wish to thank the University of Tasmania Computer Center for providing facilities for the numerical calculations. Dr. A. V. Hershey of Dahlgren, VA, kindly provided the computer programs for the Bessel functions, *I* and *K*.

References

[1] H. O. Seigel, "The magnetic induced polarization methods," *Geophys.*, vol. 39, pp. 321-339, 1974.
[2] R. N Edwards, H. Lee, and M. N. Nabighian, "On the theory of magnetometric resistivity (MMR) methods," *Geophys.*, vol. 43, pp. 1176-1203, 1978.
[3] G. W. Hohmann, P. R. Kintzinger, G. D. Van Voorhis, and S. H. Ward, "Evaluation of induced electrical polarization with an inductive system," *Geophys.*, vol. 35, pp. 901-915, 1970.
[4] C. A. Dias, "A non-grounded method for measuring induced electrical polarization and conductivity," Ph.D. dissertation, University of California, Berkeley, CA, 1968.
[5] P G. Hallof and W. H. Pelton, "Sepctral IP survey Elura deposit (Line 50 800N) Cobar N.S.W.," *Bull. Aust. Soc Explor. Geophys.*, vol. 11, pp. 196-200, 1980.
[6] W. H. Pelton, S. H. Ward, P. G Hallof, W. R. Sill, and P. H. Nelson, "Mineral discrimination and removal of inductive coupling with multifrequency IP," *Geophys.*, vol. 43, pp. 588-609, 1978.
[7] R. J. G. Lewis and T. J. Lee, "The transient electric fields about a loop on a halfspace," *Bull. Aust. Soc. Explor. Geophys.*, vol. 9, pp. 173-177, 1978.
[8] B K Bhattacharyya, "Electromagnetic fields of a small loop antenna on the surface of a polarizable medium," *Geophys.*, vol. 29, pp 814-831, 1963.
[9] ——, "Electromagnetic fields of a transient magnetic dipole on the earth's surface," *Geophys.*, vol. 24, pp. 89-108, 1959.
[10] H. F. Morrison, R. J Phillips, and D. P. O'Brien, "Quantitative interpretation of transient electromagnetic fields over a layered halfspace," *Geophysical Prospecting*, vol. 17. pp. 82-101, 1969.
[11] B S Rathor, "Transient electromagnetic field of a polarizable halfspace due to various current pulses," *Geophysical Prospecting*, vol. 26, pp 337-351, 1978.
[12] T. J. Lee, "Transient electromagnetic response of a sphere in a layered medium," Unpublished Ph.D. dissertation.
[13] ——, "Sign reversals in the transient method of electrical prospecing (one loop version)," *Geophysical Prospecting*, vol 23, pp. 453-462, 1975
[14] R. J. G. Lewis and T J. Lee, "The effect of host rock on transient electromagnetic fields," *Bull Aust. Soc Explor. Geophys.*, vol. 12, pp. 5-12, 1981

[15] T. J. Lee, "Transient electromagnetic response of a polarizable ground," *Geophys.*, vol. 46, pp. 1037-1041, 1981.
[16] B. R. Spies, "Transient electromagnetic tests, N.T. and Queensland 1972," *Bur. Min. Res. Austr. Rec.*, 74/126, 1974.
[17] P. Weidelt, "Response characteristics of coincident loop transient electromagnetic systems," *Geophys.*, vol. 47, pp. 1325-1330, 1982.
[18] G. Buselli and B. O'Neill, "SIROTEM· A new portable instrument for multichannel transient electromagnetic measurements," *Bull Aust. Soc Explor. Geophys*, vol. 8, pp. 82-87, 1977.
[19] A. Q. Howard, "The electromagnetic fields of a subterranean cylindrical inhomogeneity excited by a line source," *Geophys.*, vol. 37, pp. 975-984, 1972.
[20] T. J. Lee and R. J. G. Lewis, "Transient EM response of a large loop on a layered ground," *Geophysical Prospecting*, vol. 22, pp. 430-444, 1974.
[21] B. D. Fuller and S. H. Ward, "Linear system description of the electrical parameters of rocks," *IEEE Trans. Geosci. Electron*, vol. GE-8, pp. 7-18, 1969.
[22] G. W. Hohmann, "Three-dimensional induced polarization and electromagnetic modelling," *Geophys.*, vol. 40, pp. 309-324, 1975.
[23] K. S. Cole and R. H. Cole, "Dispersion and absorption in dielectrics," *J Chem Phys.*, vol. 9, p. 341, 1941.
[24] T. J. Lee, "Asymptotic expansion for transient electromagnetic fields," *Geophys.*, vol. 47, pp. 38-46, 1982.
[25] R. J. G. Lewis and T. J. Lee, "TEM with cylinders-1," University of Tasmania Occasional Papers in Geophysics, 1981.
[26] J. R. Wait, *Geo-Electromagnetism.* New York. Academic Press, 1982.
[27] P. Maher, Ed., "Big zinc, silver deposit," *Australian Financial Rev.*, p. 21, May 5, 1983.
[28] J. R. Wait, "Mutual coupling of loops lying on the ground," *Geophys.*, vol. 19, pp. 290-296, 1954.
[29] L. B. Felsen, Ed., *Transient EM Fields.* New York: Springer, 1976.

*



Roger J. G. Lewis received the B.Sc. degree in geophysics from the University of Tasmania, Hobart, Australia in 1965 and the Ph.D. degree from Princeton University, Princeton, NJ in 1970.

He was at Macquarie University until 1974 where T. J. Lee was pursuing the Ph.D degree. They have worked together sporadically on transient EM problems since then. Since 1974 he has been with the Geology Department, University of Tasmania, where his main interest is exploration geophysics, particularly EM and seismology. His other interests include horticulture, CAD in circuit design, and property development.

*

Terry J. Lee is currently completing the B.A. Hons (History of Fine Art) degree at Sydney University, Sydney, Australia.

At Dr. J. R. Wait's suggestion, he visited CIRES, Boulder, CO as a Visiting Fellow. At CIRE he pursued his interest in time domain electromagnetics. Upon his return to Australia he continued that work, some of it being done at the University of Tasmania, Hobart, Australia.

Geophysical Prospecting 31, 766–781, 1983.

THE TRANSIENT ELECTROMAGNETIC RESPONSE OF A CONDUCTING SPHERE IN AN IMPERFECTLY CONDUCTING HALF-SPACE*

T. LEE**

ABSTRACT

LEE, T. 1983, The Transient Electromagnetic Response of a Conducting Sphere in an Imperfectly Conducting Half-Space, Geophysical Prospecting 31, 766–781.

The presence of a conducting environment about a spherical ore body must be considered when calculating the transient electromagnetic response of the ore body due to a step current flowing in a large circular loop at the earth's surface. Failure to do this can easily lead to errors in excess of 10% in numerical calculations. Moreover, there is only a limited time interval in which the response of the spherical conductor is easily seen.

In a poorly conducting ground the resonance response of the sphere is the first to be excited Later, however, the non-resonance or wave-type response is excited. These waves destructively interfere and finally the response of the sphere decays with time as $t^{-7/2}$.

For a range of times and depths the best loop for detecting the sphere has about the same radius as the sphere.

INTRODUCTION

The purpose of this paper is to extend the results of two earlier papers. In the first a solution was found for the transient electromagnetic response of a sphere in a layered ground to a step current that flows in a large loop at the surface of the ground (Lee 1981). In the second paper Lee (1982) examined the later stages of the transient response of various conductors.

One criticism of the first paper was that the equations were quite complex and were not suitable for programming on a minicomputer. Although the equations of the second paper were quite simple they only described what was happening to the transient at the later stages. The equations presented here are designed for a computer program to be run on a minicomputer and at the same time describe the transient for the times commonly used in prospecting.

* Received June 1982, revisions January 1983.
** Peko-Wallsend Operations Ltd, PO Box 217, Gordon, NSW 2072, Australia. CSIRO, Div. of Mineral Physics, POB 136, North Ryde, NSW 2113, Australia.

The approach to the analysis begins by noticing that the transient response of a ɔhere in air has been known for a long time. Moreover, the solution given by amenetski (1969) and Wait (1951) has a very simple form. The idea is to seek a ɔlution for a poorly conducting environment which will reduce to the simple form r the case of a sphere in air.

It should be pointed out that the solution given here does not require that the ɔhere be placed in a conducting whole space, as was the case with Kaufman (1981). urther, the solution given here allows the loop to traverse the sphere. Consequent- , the following analysis allows the geophysicist to compute profiles of the transient sponse of the sphere.

Analysis

he analysis presented here is aimed at finding a good approximation for the ansient electromagnetic response of a conducting sphere in a conducting half- ɔace (fig. 1). The ground in which the sphere is embedded is excited by a step current ɔwing in a large circular loop that lies on the surface of the ground. The following ıalysis commences where Lee (1981) ended. The particular feature of the analysis resented below is that approximations are found for the equations of the more ɜneral analysis that was presented previously. By this means one is able to carry ut all the required integrations analytically.

The overriding concept in the derivation is to obtain an expression for the ansient voltage which will be valid when $\sigma_2 \mu_0 l^2/t$ is about 0.2 or less. Here σ_2 is ıe conductivity of the host rock, μ_0 the permeability of free space, t is time and l is ıe distance from the center of the sphere to the center of the receiving loop. No striction is placed on the size of the spherical conductor. Before commencing the ıalysis it is worth noting that (1) which follows was produced by mode-matching. characteristic feature of modes is that they have different spatial characteristics. ccordingly, we retain some terms in one particular mode that would be neglected

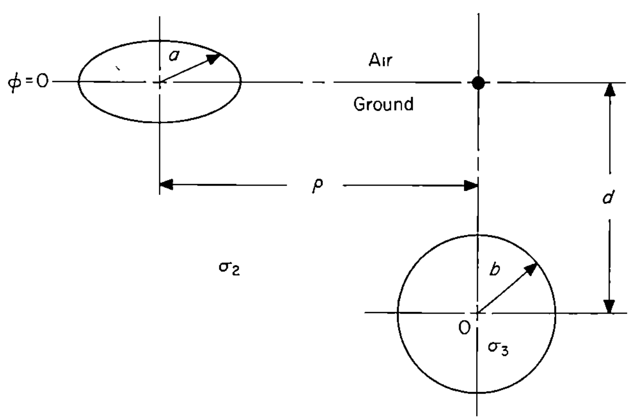


Fig 1. Loop traversing a buried sphere.

in others as being too small. The idea, then, is not just to ensure that the calculatio is accurate for transients in some places but also to retain as much accuracy ε possible, even when the position of measurement changes.

Suppose that a sphere of conductivity σ_3, radius b, and permeability μ_0 lies at depth d in a uniform ground of conductivity σ_2. When a step current of strength I flows in a loop of radius a and at a distance ρ from the vertical axis of the spher then a transient voltage $V(t)$ will be measured. This voltage may be considered to b made up of two parts: $V_p(t)$, the voltage of the uniform ground, that is the voltag that would be measured when there was no spherical conductor present, and $V_s(t)$, voltage that arises because of the presence of the conducting sphere. Expressions fc these voltages have been found previously.

From (29) of Lee (1981) one has

$$V(t) = V_p(t) + V_s(r),$$

where

$$V_p = \frac{a\mu_0 I_0\sqrt{\pi}}{t}\frac{\sqrt{(a^2\sigma_2\mu_0)}}{\sqrt{4t}}\sum_{j=0}^{\infty}\frac{2(-1)^j(2j+2)!}{(2j+5)j!(j+1)!(j+2)!}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^{j+1}$$

$$\approx \frac{a\mu_0 I_0\sqrt{\pi}}{t}\frac{\sqrt{(\sigma_2\mu_0 a^2)}}{\sqrt{4t}}\left(\frac{2}{5}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)-\frac{4}{7}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^2+\frac{5}{9}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^3\cdots\right),$$

$$V_s = \frac{3\mu_0 a^2 I_0}{4}\,\mathrm{i}\int_c \exp(-\mathrm{i}\omega t)\left\{\frac{1}{k_2^3}\left[T_{11}^2+T_{01}^2\right]\frac{\mathrm{WM}_1}{\mathrm{WM}_1^1}-\frac{1}{k_2}\frac{\mathrm{WN}_1}{\mathrm{WN}_1^1}T_{1b}2\right\}d\omega \quad (1$$

(here c indicates that the integral is taken about a path shown in fig. 2),

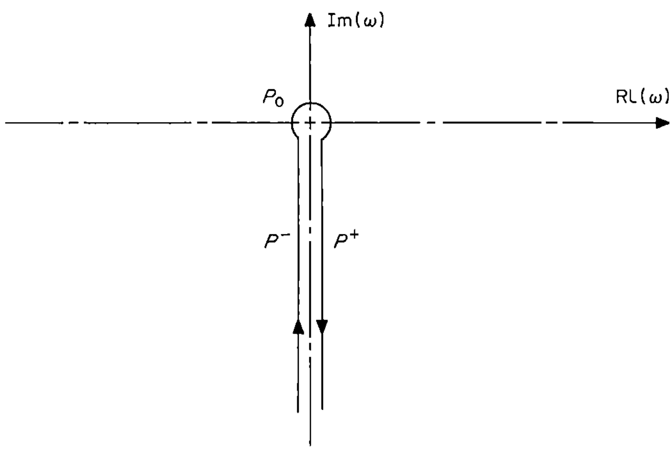


Fig 2. Path for the contour integral.

$$\mathrm{WM}_n = bk_3\,\mathrm{j}_{n+1}(bk_3)\mathrm{j}_n(bk_2) - (bk_2)\mathrm{j}_n(bk_3)\mathrm{J}_{n+1}(bk_2),$$

$$\mathrm{WM}_n^1 = bk_3\,\mathrm{j}_{n+1}(bk_3)h_n^1(bk_2) - (bk_2)\mathrm{j}_n(bk_3)h_{n+1}^1(bk_2),$$

$$\mathrm{WN}_n = \frac{1}{2n+1}\,[\mathrm{WM}_{n-1}(n+1) + n\mathrm{WM}_{n+1}],$$

$$\mathrm{WM}_n^1 = \frac{1}{2n+1}\,[\mathrm{WM}_{n-1}^1(n+1) + n\mathrm{WM}_{n+1}^1], \tag{2}$$

$$k_2^2 = \mathrm{i}\omega\sigma_2\,\mu_0,$$

$$k_3^2 = \mathrm{i}\omega\sigma_3\,\mu_0,$$

$$T_{11} = \int_0^\infty \frac{2\mathrm{i}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda a)\lambda h_1 \exp(\mathrm{i}h_1 d)\,\mathrm{d}\lambda}{(h_0 + h_1)\mathrm{i}},$$

$$T_{01} = \int_0^\infty \frac{2\lambda^2\mathrm{J}_0(\lambda\rho)\mathrm{J}_1(\lambda a)}{\mathrm{i}(h_1 + h_0)} \exp(\mathrm{i}h_1 d)\,\mathrm{d}\lambda,$$

$$T_{1b} = \int_0^\infty \frac{2\lambda\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda a)}{(h_1 + h_0)} \exp(\mathrm{i}h_1 d)\,\mathrm{d}\lambda,$$

$$h_1 = \sqrt{(k_2^2 - \lambda^2)},$$

$$h_0 = \mathrm{i}\lambda. \tag{3}$$

In writing down these equations displacement currents have been neglected.

To proceed, one first has to find approximations to the integrals in (3). To this end one notices that

$$T_{11} = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\,\frac{\partial}{\partial d}\,U\,\mathrm{d}\theta,$$

$$T_{01} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\mu}{\partial a}\,\mathrm{d}\theta,$$

$$T_{1b} = \frac{\mathrm{i}}{2\pi}\int_0^{2\pi} U\cos\theta\,\mathrm{d}\theta, \tag{4}$$

where

$$U = \int_0^\infty \frac{2\mathrm{J}_0(\lambda P)\lambda\exp(\mathrm{i}h_1 d)}{(\mathrm{i}\lambda + h_1)\mathrm{i}}\,\mathrm{d}\lambda,$$

and

$$P = \sqrt{(\rho^2 + a^2 - 2a\rho\cos\theta)}.$$

The function U can now be approximated by noting that

$$U = -\frac{2}{k_2^2}\left[\frac{\partial^2}{\partial d^2}\int_0^\infty \frac{\lambda}{ih_1}\exp(ih_1 d)J_0(\lambda P)\,d\lambda + \left(k_2^2\frac{\partial}{\partial d}+\frac{\partial^3}{\partial d^3}\right)\int_0^\infty \frac{\exp(ih_1 d)}{ih_1}J_0(\lambda P)\,d\lambda\right]. \quad (5$$

Since

$$ih_1 = \pm i\sqrt{(k_2^2-\lambda^2)} \qquad \text{for} \quad k_2 > \lambda \text{ on } P\pm$$
$$= -\sqrt{(\lambda^2-k_2^2)} \qquad \text{for} \quad k_2 < \lambda \text{ on } P\pm,$$

respectively, the integrals can be readily evaluated by using results 2.54, 6.48, an 6.50 of Oberhettinger (1972). By this means one finds that

$$U = +\frac{2}{k_2^2}\frac{\partial^2}{\partial d^2}\frac{\exp(ik_2 R)}{R} - \frac{2}{k_2^2}\left[k_2^2\frac{\partial}{\partial d}+\frac{\partial^3}{\partial d^3}\right]\left[\frac{\pi}{2}J_0\left(\frac{k_2}{2}(R-d)\right)Y_0\left(\frac{k_2}{2}(R+d)\right) \mp \frac{i\pi}{2}J_0\left(\frac{k_2}{2}(R-d)\right)J_0\left(\frac{k_2}{2}(R+d)\right)\right] \quad (6$$

on $P\pm$, respectively.

Expanding the Bessel functions and neglecting the higher order terms in both th real and the imaginary terms one has

$$U \approx -\left[\frac{1}{R}-\frac{k_2^2 d}{8}\ln\left(\frac{k_2}{4}(R+d)\right) \pm \tfrac{2}{3}ik_2 \mp i\frac{\pi}{16}k_2^2 d \mp \frac{ik_2^3}{15}(R^2+2d^2)\right]. \quad (7$$

Therefore

$$T_{01}^2 \approx \left[I_1^2 \pm \frac{4i}{15}k_2^3 a^3 I_1\right]\frac{1}{a^4}$$

where

$$I_1 = -\frac{1}{2\pi}\frac{\partial}{\partial a}\int_0^{2\pi}\frac{a^2}{R}\,d\theta$$
$$= a^2\int_0^\infty \exp(-\lambda d)\lambda J_1(\lambda a)J_0(\lambda\rho)\,d\lambda$$
$$= \frac{a^3}{(a^2+d^2)^{3/2}}, \qquad \text{for} \quad \rho = 0$$
$$= a^2\left[\frac{k^3(a^2-\rho^2-d^2)E(k)}{8\pi(a\rho)^{3/2}a(1-k^2)} + \frac{kK(k)}{2\pi(a\rho)^{1/2}a}\right], \qquad \text{for} \quad \rho \neq 0 \quad (8$$

d

$$k^2 = 4a\rho/(d^2 + (a+\rho)^2)$$

uke 1962, p. 316, No. 21), K(k) and E(k) are the complete elliptic integrals of the st and second kinds;

$$T^2_{11} \approx \frac{1}{a^4} I_2^2,$$

$$\begin{aligned} I_2 &= a^2 \int_0^\infty \exp(-\lambda d) \mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda a)\lambda \, \mathrm{d}\lambda \\ &= 0, \qquad \text{for} \quad \rho = 0 \\ &= \frac{\mathrm{d}ka^2}{2\pi(a\rho)^{3/2}(1-k^2)} \{(1 - \tfrac{1}{2}k^2)\mathrm{E}(k) - (1-k^2)\mathrm{K}(k)\}, \qquad \rho \neq 0 \end{aligned} \tag{9}$$

e Luke 1962, p. 316, No. 20),

$$T^2_{1b} \approx -\frac{1}{a^2}[I_3^2 + 2\mathrm{i} I_3 k_2^3 \rho a^2/15],$$

$$\begin{aligned} I_3 &= a \int_0^\infty \mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda a) \exp(-\lambda d) \, \mathrm{d}\lambda \\ &= 0, \qquad \text{for} \quad \rho = 0 \\ &= \frac{2a}{\pi k(a\rho)^{1/2}} [(1 - \tfrac{1}{2}k^2)\mathrm{K}(k) - \mathrm{E}(k)], \qquad \text{for} \quad \rho \neq 0. \end{aligned} \tag{10}$$

The remaining quantities that must be computed are the factors WM_1^1, WM_1, N_1^1, and WN_1. Along P^+ and P^- the terms WM_1^1 and WN_1^1 are easily split into al and imaginary parts by replacing the spherical Hankel functions h_n^1 by $\mathrm{j}_n^{\pm}\mathrm{i}y_n$ on $^{\pm}$, respectively. Since it has been shown that there are no poles in the spectrum ien $\sigma_2 = 0$ (see Lee 1981, p. 325), we may expand factors $\mathrm{WM}_1/\mathrm{WM}_1^1$ and $\mathrm{N}_1/\mathrm{WN}_1^1$ in powers of $k_2 b$. This can be done once the Bessel functions involving ese quantities are also expanded in powers of $k_2 b$.

The relevant expansions of $\mathrm{WM}_1/\mathrm{WM}_1^1$ and $\mathrm{WN}_1/\mathrm{WN}_1^1$ are

$$\frac{\mathrm{WM}_1}{\mathrm{WM}_1^1} \approx \frac{(k_2 b)^3 \mathrm{j}_2(k_3 b)}{(k_2 b)^3 \mathrm{j}_2(k_3 b) + 3\mathrm{i}\left(\mathrm{j}_0(k_3 b) - \dfrac{(k_2 b)^2}{k_3 b} \mathrm{j}_1(k_3 b)\right)},$$

$$\frac{\mathrm{WN}_1}{\mathrm{WN}_1^1} \approx \frac{2(k_2 b)^3}{2(k_2 b)^2 + 3\mathrm{i}} \approx -\tfrac{2}{3}\mathrm{i}(k_2 b)^3 + \frac{4(k_2 b)^5}{9}. \tag{11}$$

otice that for $\sigma_2 = 0$ the contour of fig. 2 is replaced by an integral about the poles the integrand at $(k_3 b) = n\pi$, $n = 1, 2, \ldots$ Accordingly, this response constitutes the st term in the asymptotic expansion. The other terms still involve integrals about e negative ω-axis.

Combining the results that we have obtained so far yields (after writing $\omega = -$ in the integrals along the branch cut) the approximate expression

$$V_s = 3\,\frac{\mu_0 a^2 I_0}{2}\left\{\oint \frac{\exp(-\mathrm{i}\omega t)}{l}\,\frac{b^3}{6a^4}\,\frac{\mathrm{j}_2(b\sqrt{(\mathrm{i}\omega\sigma_3\mu_0)})}{\mathrm{j}_0(b\sqrt{(\mathrm{i}\omega\sigma_3\mu_0)})}\,(I_1^2 + I_2^2)\,\mathrm{d}\omega \right.$$
$$+ \frac{4}{15}\int_0^\infty \frac{(k_2 b)^3 a^3 I_1}{a^4}\,\frac{\mathrm{j}_1(k_3 b)}{3\mathrm{j}_0(k_3 b)}\exp(-st)\,\mathrm{d}s$$
$$+ \frac{4}{45}\int_0^\infty I_3 b^3 \rho k_2^5 \exp(-st)\,\mathrm{d}s + \frac{4}{9}\int_0^\infty I_3^2\,\frac{b^5}{a^2}\,k_2^4\exp(-st)\,\mathrm{d}s$$
$$\left. + \frac{b^3 \mathrm{R}_1}{a^4}\left(\int_0^\infty \frac{\mathrm{j}_2(k_3 b)((I_1^2 + I_2^2)\exp(-st)\,\mathrm{d}s}{(k_2 b)^3 \mathrm{j}_2(k_3 b) + 3\mathrm{i}\left(\mathrm{j}_0(k_3 b) - \dfrac{(k_2 b)^2}{k_3 b}\,\mathrm{J}_1(k_3 b)\right)}\right)\right\}. \quad (12$$

Here k_j has been redefined as $k_j = \sqrt{(\sigma_j \mu_0 b^2 s)}$. The bar on the integral denotes th Cauchy principal value and $\mathrm{R}_1(x)$ denotes that only the real part of x is added to V_s

The integrals in (13) may now be integrated using the results given in th appendix. The terms involving k_2^5 may be integrated with the aid of the definition c the gamma function. The first term in (13) is easily integrated by noting that th poles are at $(k_3 b) = n\pi$, $n = 1, 2, \ldots$ If we set $\beta^2 = \sigma_3 \mu_0 b^2$, $\alpha^2 = \sigma_2/\sigma_3$, we have

$$J = -2\sqrt{(\pi)}\beta \sum_{n=1}^{\infty} \frac{\exp(-\beta^2 n^2/t)}{n}\left[\frac{3\beta^2 n^2}{2t^{5/2}} - \frac{n^4 \beta^4}{t^{7/2}}\right]$$
$$+ \sqrt{(\pi)}\beta/(2t^{3/2}) - \Gamma(5/2)\beta^3/(3t^{5/2}) \quad (1$$

and

$$C = \alpha^3\left[\left(\frac{2t}{\beta^2} - \frac{2}{3}\right)J - \frac{\beta^3\sqrt{\pi}}{12t^{5/2}}\right], \quad (1$$

then we obtain

$$V_s \approx \frac{3\mu_0 b^3 I_0 \pi}{a^2 \beta^2}(I_1^2 + I_2^2)\sum_{n=1}^{\infty}\exp(-n^2\pi^2 t/\beta^2) + \frac{a^2}{b^2}\,\frac{\sqrt{\pi}}{4}\,\frac{\rho u_0 I_0}{t^{7/2}}(\sigma_2 \mu_0 b^2)^{5/2} I_3$$
$$+ \tfrac{2}{5}\alpha^3 a \mu_0 I_0 I_1 J + \frac{2}{3}\,\frac{\mu_0 b^3}{a^2}\,C(I_1^2 + I_2^2)$$
$$+ \frac{4}{3}\,\frac{\mu_0 I_0 b}{t^{6/2}}\,I_3^2(\sigma_2 \mu_0 b^2)^{4/2}. \quad (1$$

Futher details concerning the functions C and J can be found in the appendi There the reader will find alternative expressions which are more useful for numer cal purposes when β^2/t is small. The details of the integrations that produced thes quantities are also in this appendix.

GEOPHYSICS OF THE PROBLEM

.he interesting thing about the final formula is that it enables one to say something ɔout the physics of the problem. Such statements generally cannot be made from ırely numerical solutions. To understand the situation, notice that when the sphere in air no currents can cross its boundary. This is not the case when the sphere is a conducting environment. The terms, apart from the first, of (17) arise because of .e conducting environment. In these terms, the function J plays a crucial part. To ıderstand the nature of J it is necessary first to look briefly at a simple model.

It is well known that a plane electromagnetic wave striking a two-layered ·ound results in a transient that could be considered as a series of waves that are ;flected back and forth between the layers (see Kunetz 1972, or Lee 1974). If E ·presents the transient electromagnetic field then

$$E = \frac{2H_0\mu_0}{\sqrt{(\pi\sigma_1 t)}}\left[1 + 2\sum_{n=1}^{\infty} r^n \exp\left(-n^2h^2\sigma_1\mu_0/t\right)\right], \tag{16}$$

here

$$r = (\sqrt{(\sigma_1)} - \sqrt{(\sigma_2)})/(\sqrt{(\sigma_1)} + \sqrt{(\sigma_2)}).$$

:ere H_0 is the amplitude of the magnetic field which suddenly drops to zero ʜhereby causing the waves), t is time, μ_0 is the magnetic permeability, σ_1 is the ɔnductivity of the top layer and σ_2 the conductivity of the basement, r is a reflec- ·on factor, and h is the thickness of the top layer. Notice that this expression is uite similar to that for J. The suggested interpretation of J, therefore, is a series of aves that propagate about the sphere. Interestingly, the series for J can be trans- ·rmed to another by the Poisson transform. The transformed series converges ·pidly and the leading term of the series decays as $t^{-7/2}$. This is the same transform :at was applied to (18). In that case it was shown that the electric field decayed like $^{-1/2}$ (See Lee 1977). When the alternative expression for J (i.e., equation (A.9)) is ·serted into the final expression for V_s one finds that at the later stages the term V_s ecays like $t^{-7/2}$.

Lee (1982) showed that this form of the decay can be found if it is assumed that . the later stages the field within the conductor is the applied field. What is ıppening, then, is that the waves are destructively interfering and collapsing to a ːld that is related to the primary field in a very simple manner. Because these aves are all over the sphere and destructively interfering, I have called them ·Carmel waves" (after my 18-month-old daughter who sometimes shows similar ehavior in our house). At the same time the natural resonances are also decaying. /hat is happening is that the resonant response at first dominates the response of ıe sphere, but with the passage of time the nonresonant response dominates the ansient. This last point will be substantiated when we come to compute the rela- ve errors that arise in calculations which neglect the effect of the conducting host ɔck.

Numerical Checks

Before presenting some computed results it is worth looking at some checks thε have been made of the solution. These checks compare computed results from tŀ equations above with model measurements and other calculations.

The first point that has to be made is that such comparisons are essentially a exercise in trying to reconcile the irreconcilable. The theoretician is well aware of tŀ vagaries of the pontifical computer. On the other hand, any good experimenter wi tell you that one does not just push the buttons. Both are well aware of the almo uncanny insight that is required to produce reliable results. Consequently, i examining the following results, one should be aware that all measurements an experiments are subject to error. This problem is made more acute in our case whe we compare one result with another.

To be more specific, a 10% error in calculation seemingly can become a 20% error on comparison with other results because they too had a 10% error c opposite sign.

Figure 3 shows this last point very well. In that figure we see that the resul from the equations given here are close to either a modeled or a calculated resul There is an urgent need, then, for more results from scale-model experiments to b published so that better checks can be placed on the numerical results.

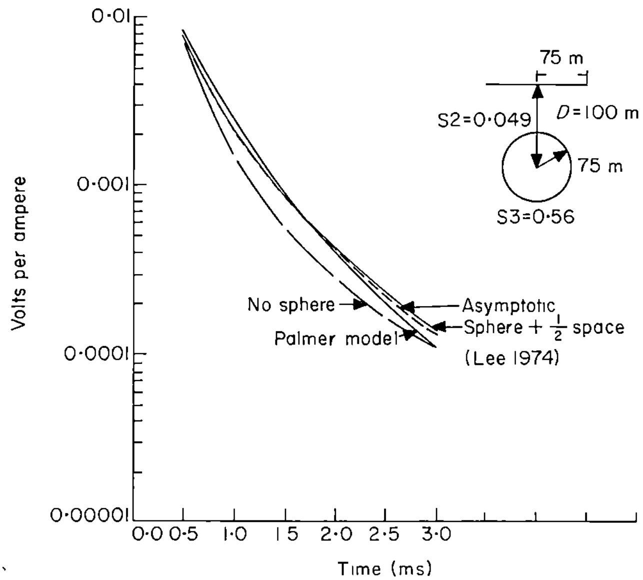


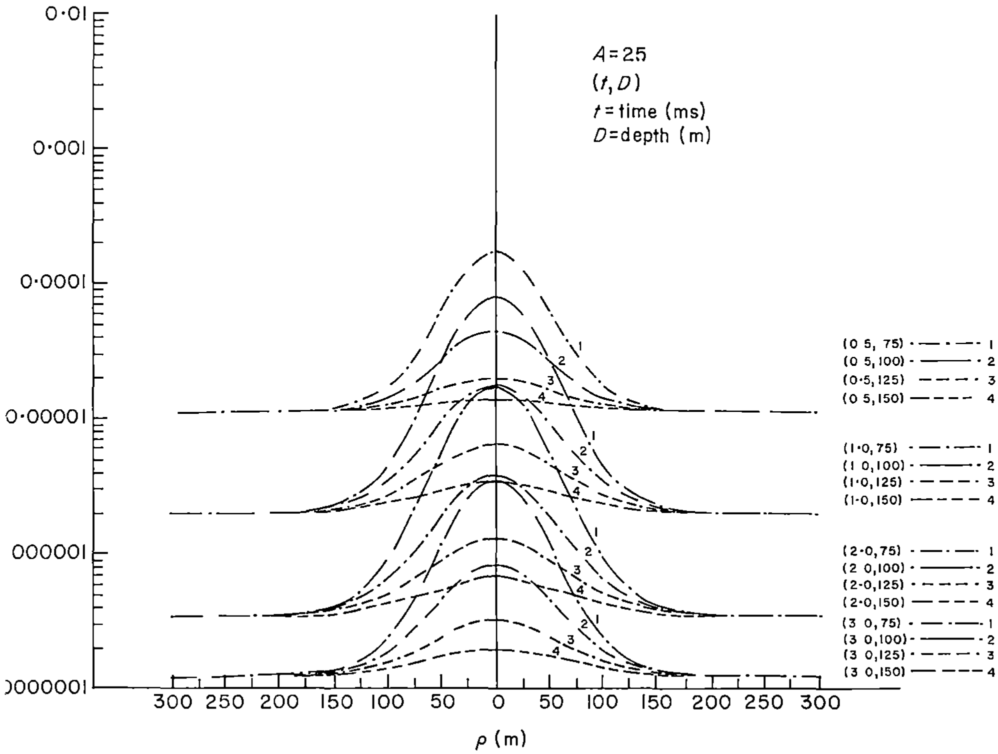
Fig. 3. Comparison of the results from the asymptotic solution with those of Palmer (1974 and Lee (1974).

Calculations and Conclusions

he first set of calculations involves looking at the response of a buried sphere for ɜrious depths and transmitter loop sizes. These results are shown in figs 4, 5, and 6. he conclusion is that the best response of the sphere is obtained at the earlier times nd when the loop size is about the same size as the sphere.

It was remarked in the Introduction that it is common to model the transient ɜsponse of a spherical ore body by a sphere in air. That is, all the terms of (17) are ,nored except those that arise because of natural resonances. The effect of the host ɔck is totally ignored.

Strictly speaking, there is always a response from the country rock in the field ıeasurements. All the prospector can do is to measure the transient for those times ɔr which the country rock response is small in comparison to that of the ore body. is interesting, then, to compute the percentage error incurred by those calcu- tions that totally neglect this effect. The practical geophysicist might remove the ackground response by subtracting V_p from V and then calculating the relative ror. For this reason we have decided to compute the relative error with and ithout the V_p term.



ıg 4 Profiles for times of 1, 2, and 3 ms of a 25 m loop traversing a buried sphere. The ıdius of the sphere is 50 m and the depth 75, 100, 125, and 150 m, respectively. The conduc- vity of the sphere is 2 S/m and that of the ground 0.01 S/m.

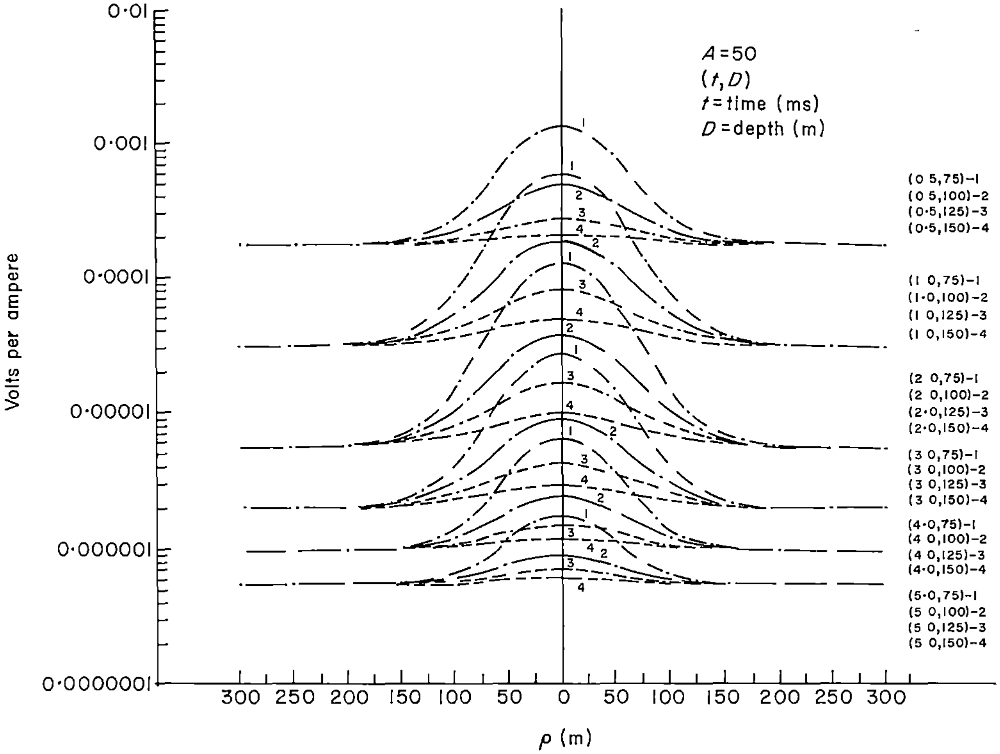


Fig. 5. Profiles for times of 1.0, 2.0, 3 0, 4.0, and 5 0 ms of a 50 m loop traversing a burie sphere. The radius of the sphere is 50 m and the depth 75, 100, 125, and 150 m, respectively. The conductivity of the sphere is 2 S/m and that of the ground 0.01 S/m.

Results for these calculations are shown in figs 7 and 8. The model used for thes calculations is also shown in these figures. Notice that the errors can easily excee 10% and that there is only a limited time range where the errors are quite small Moreover, at the later stages of the transient one sees that the effect of the host rock is quite significant, even when V_p has been removed from V. By this stage the resonant response makes only a small contribution to the transient. This is becaus at this stage the ore-body response decays like $t^{-7/2}$ and can be less than the hos rock response which decays like $t^{-5/2}$.

The conclusion to be drawn from the results presented here is that the nature o the transient electromagnetic response of a sphere in a conducting environment car be quite different from that of a similar sphere in free space. Moreover, these differences in the nature of the response can be significant even when the back ground is only very poorly conducting. The consequence of this is that the transient should be measured for those times for which the ore-body response is largest. The theory presented here allows the geophysicist to know in advance what these times

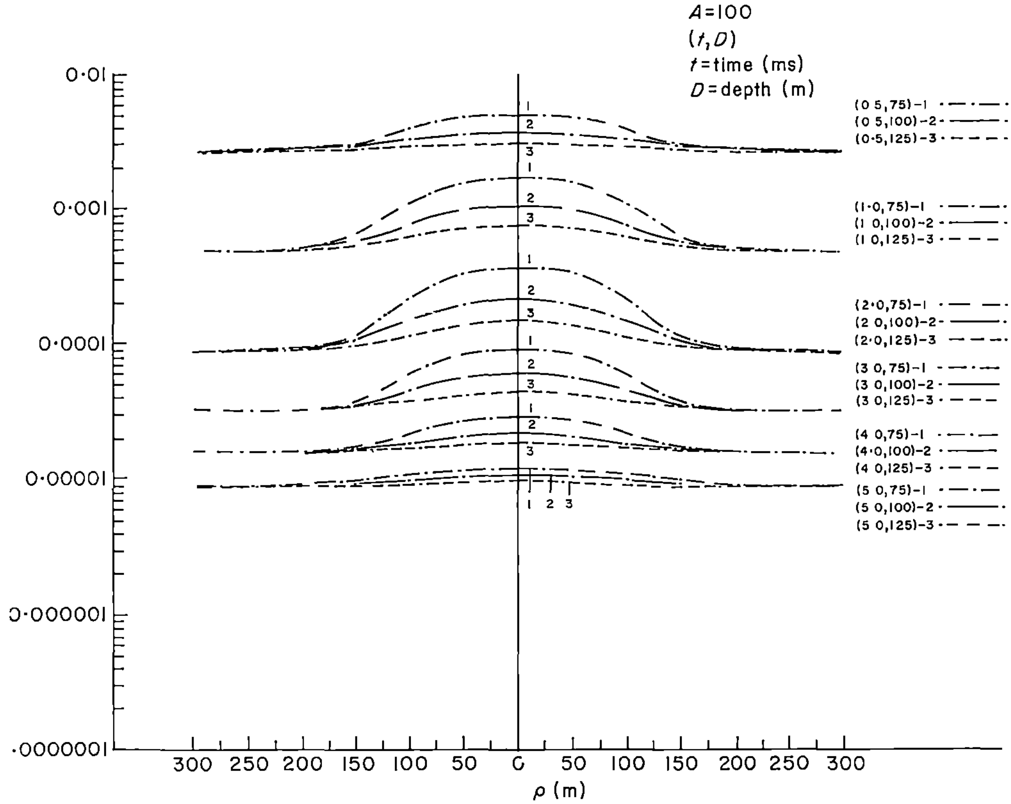


Fig. 6. Profiles for times of 1.0, 2.0, 3.0, 4.0, and 5.0 ms of a 100 m loop traversing a buried sphere The radius of the sphere is 50 m and the depth 75, 100, and 125 m, respectively. The conductivity of the sphere is 2 S/m and that of the ground 0.01 S/m.

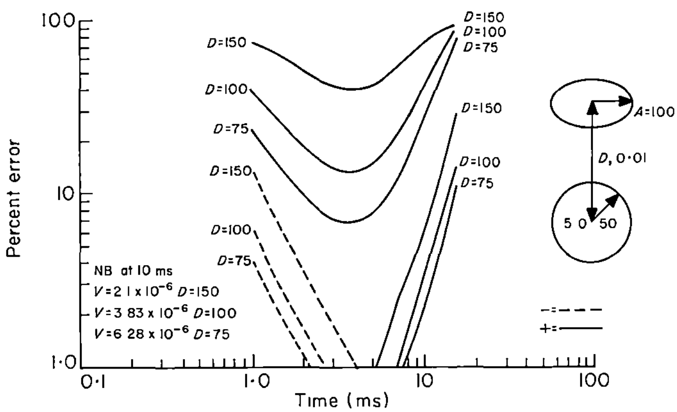


Fig. 7. The relative error incurred in the calculation of the transient electromagnetic response by neglecting the effect of the host rock. The upper set of curves is for the case when the effect of the half-space V_p is included in the calculations. The lower set of curves is for the case when the effect of the half-space V_p has been first subtracted from V before the error has been calculated. The various parameters used for the model are shown in the figure.

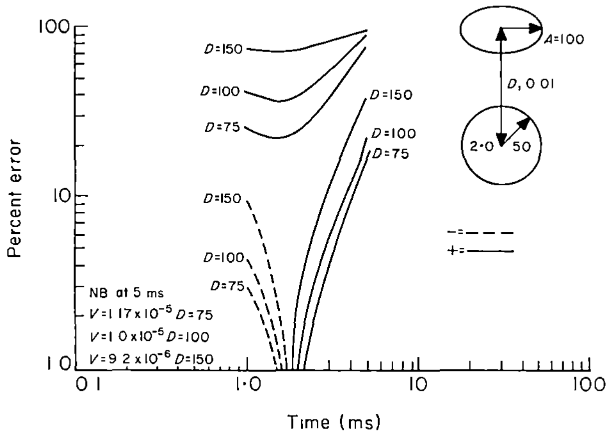


Fig. 8 Some further calculations of the relative errors Here the conductivity of the spher has been reduced to 2 S/m

are. It would appear, besides, that it is a good idea when prospecting to use a loop size that is about the same size as the ore body being sought. This choice of loop size appears to be independent of the depth of burial of the ore body.

Appendix

1. *The integral J*

$$J = \fint_0^\infty \frac{x^3 j_2(x) \exp(-\omega t)\, d\omega}{3 j_0(x)}, \tag{A.1}$$

$$x = \beta\sqrt{\omega}.$$

The bar on the integral denotes the Cauchy principal value.

Since

$$j_2(x) = \frac{3 j_1(x)}{x} - j_0(x)$$

one finds that

$$J = \fint_0^\infty \frac{x^2 j_1(x)}{j_0(x)} \exp(-\omega t)\, d\omega - \int_0^\infty \frac{x^3 \exp(-\omega t)\, d\omega}{3}. \tag{A.2}$$

Integrating the first integral by parts and the second by noting the definition of the gamma function reduces J to the form given in (3):

$$J = \beta^3 \left\{ \frac{2}{\beta^2} \int_0^\infty (\ln(|2 \sin(x)|) - \ln(2x))(\tfrac{3}{2}\omega^{1/2} - t\omega^{3/2}) \exp(-\omega t)\, d\omega - \frac{\Gamma(5/2)}{3t^{5/2}} \right\} \tag{A.3}$$

The term $\ln(2x)$ in the above integral can be evaluated by using result No. 6.36 of Oberhettinger and Badii (1973, p. 53).

Therefore

$$J = 2\beta \int_0^\infty \ln(|2\sin(x)|)\left(\tfrac{3}{2}\omega^{1/2} - t\omega^{3/2}\right)\exp(-\omega t)\,\mathrm{d}\omega + \frac{\beta\sqrt{\pi}}{2t^{3/2}} - \frac{\beta^3\Gamma(5/2)}{3t^{5/2}}. \tag{A.4}$$

The remaining integral can be evaluated by first expanding $\ln(2|\sin(x)|)$ as a Fourier series (result 1.14 of Oberhettinger 1973, p. 8 and then integrating term by term with the aid of result No. 7.78 of Oberhettinger and Badii (1973, p. 66). By this means one finds that

$$J = -2\beta\sqrt{\pi}\sum_{n=1}^{\infty}\frac{\exp(-\beta^2 n^2/t)}{n}\left[\frac{3}{2}\frac{\beta^2 n^2}{t^{5/2}} - \frac{n^4\beta^4}{t^{7/2}}\right] + \frac{\beta\sqrt{\pi}}{2t^{3/2}} - \frac{\Gamma(5/2)}{3t^{5/2}}\beta^3. \tag{A.5}$$

Notice that the series in (A.5) converges quite rapidly for "large" β^2/t, but only slowly for "small" β^2/t. It is possible to find an alternative expression for J which is preferable for "small" β^2/t. That expression can be derived from (A.3) which may be written as

$$J = -2\beta\int_0^\infty \frac{\partial}{\partial\omega}\ln\left(\left|\frac{\sin x}{x}\right|\right)\exp(-\omega t)\omega^{3/2}\,\mathrm{d}\omega - \frac{\beta^3\Gamma(5/2)}{3t^{5/2}}. \tag{A.6}$$

However,

$$\ln\left(\left|\frac{\sin x}{x}\right|\right) = \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}\mathrm{B}_{2n}}{n(2n)!}\beta^{2n}\omega^n. \tag{A.7}$$

Here B_{2n} are the Bernoulli numbers (see Abramowitz and Stegun 1975, p. 75, No. 4.3.71).

Therefore,

$$J = -2\beta\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n-1}\mathrm{B}_{2n}\beta^{2n}\frac{\Gamma(n+3/2)}{t^{n+3/2}}, \tag{A.8}$$

whence, for small β^2/t, one has

$$J \approx \frac{\sqrt{(\pi)}\beta^5}{t^{7/2}}\left[\frac{1}{24} + \frac{\beta^2}{72t} + \frac{\beta^4}{160t^2} + \frac{\beta^6}{288t^3} + \frac{96\beta^8}{302\,400t^4} + \frac{\beta^{10}}{522t^5}\right]. \tag{A.9}$$

We remark in passing that (A.8) may be deduced from (A.5) by the application of the Poisson transform (Titchmarsh 1937, p. 60).

2. *The integral C*

$$C \approx \mathrm{R}_1\left(\int_0^\infty \frac{\mathrm{j}_2(x)\exp(-st)\,\mathrm{d}s}{\alpha^3 x^3\mathrm{j}_2(x) + 3\mathrm{i}(\mathrm{j}_0(x) - \alpha^2 x\mathrm{j}_1(x))}\right). \tag{A.10}$$

Here

$$x = \beta\sqrt{s},$$

$$\alpha^2 = \sigma_2/\sigma_3 .$$

Equation (10) may be approximated by neglecting those terms that contai powers of α that are greater than the third. Whence

$$C \approx \fint_0^\infty \frac{\alpha^3 j_2(x)^2 x^3 \exp(-st)\, ds}{9 j_0(x)(j_0(x) - 2\alpha^2 x j_1(x))}. \tag{A.11}$$

The bar on the integral in (A.11) denotes that the Cauchy principal value of th integral is understood.

Since

$$j_2(x) = \left(\frac{3}{x} j_1(x) - j_0(x)\right),$$

we get

$$C = \int_0^\infty \frac{\alpha^3 j_2(x) x^2 j_1(x) \exp(-st)\, ds}{3 j_0(x)(j_0(x) - 2\alpha^2 x j_1(x))} - \frac{\alpha^3}{9}\int_0^\infty \frac{x^3 j_2(x) \exp(-st)\, ds}{j_0(x)} \tag{A.12}$$

$$\approx \frac{1}{3 j_0(x)} \frac{2}{\beta^2} [x^3 \alpha^3 j_2(x) \exp(-st)] \Big|_0^\infty$$

$$- \fint_0^\infty \frac{2}{\beta^2} \frac{\alpha^3}{3 j_0(x)} \left[\frac{\partial}{\partial s} x^3 j_2(x) \exp(-st)\right] ds - \frac{J\alpha^3}{3} \tag{A.13}$$

$$= \frac{-\alpha^3}{3} \fint_0^\infty \frac{x^2 j_1(x)}{j_0(x)} \exp(-st)\, ds + \alpha^3\left(\frac{2t}{\beta^2} - \frac{1}{3}\right) J. \tag{A.14}$$

Equation (A.14) may be further reduced by the aid of (A.2). One finds that

$$C = \alpha^3\left[\left(\frac{2t}{\beta^2} - \frac{2}{3}\right) J - \frac{\beta^3\sqrt{\pi}}{12 t^{5/2}}\right]. \tag{A.15}$$

Notice that for small $\beta^{2/t}$ (A.15) can be further reduced by the aid of (A.9).

By this means one finds that

$$C \approx \frac{\alpha^3 \beta^7 \sqrt{\pi}}{t^{9/2}}\left[\frac{7}{2160} + \frac{\beta^2}{t} \cdot \frac{1}{360} + \frac{32}{14\,175}\left(\frac{\beta^2}{t}\right)^2 + \frac{\beta^6}{t^3} \frac{2423}{1\,159\,200}\right]. \tag{A.16}$$

References

Abramowitz, M and Stegun, I.A. 1965, Handbook of Mathematical Functions, p. 1046 Dover, New York.

Kamenetski, F M. 1969, The elements of the theory of time domain mining electro prospecting, Prikladia Geofizika 57, 137–154 (in Russian).

KAUFMAN, A. 1981, The influence of currents induced in the host rock on electromagnetic response of a spheroid directly beneath a loop, Geophysics 46, 1121–1136.

KUNETZ, G. 1972, Processing and interpretation of magnetotelluric soundings, Geophysics 37, 1005–1021

LEE, T 1974, Transient electromagnetic response of a sphere in a layered medium, Unpublished PhD thesis, Macquarie University, Australia.

LEE, T. 1977, Estimation of depth to conductors by the use of electromagnetic transients, Geophysical Prospecting 25, 61–75.

LEE, T. 1981, Transient electromagnetic response of a sphere in a layered medium, Pageoph 119, 309–338

LEE, T 1982, Asymptotic expansions for transient electromagnetic fields, Geophysics 47, 38–46.

LUKE, Y.L. 1962, Integrals of Bessel Functions, McGraw-Hill, New York.

OBERHETTINGER, F. 1972, Tables of Bessel Transforms, Springer-Verlag, Berlin, Heidelberg, New York.

OBERHETTINGER, F. 1973, Fourier Expansions: A Collection of Formulas, Academic Press, New York, London.

OBERHETTINGER, F and BADII, L. 1973, Tables of Laplace Transforms, Springer-Verlag, Berlin, Heidelberg, New York.

PALMER, T.M. 1974, TEM modelling—sphere in a half space Unpublished report for L.A. Richardson and Associates, Pty Ltd, Chatswood, NSW, Australia.

TITCHMARSH, E.C. 1937, An Introduction to the Theory of Fourier Integrals, Clarendon Press, Oxford

WAIT, J.R. 1951, A conducting sphere in a time varying magnetic field, Geophysics 16, 666–672.

Asymptotic expansions for transient electromagnetic fields

T. Lee

GEOPHYSICS, VOL 47, NO. 1 (JANUARY 1982); P 38-46, 4 FIGS , 3 TABLES

‹symptotic expansions for transient electromagnetic fields

. Lee*

ABSTRACT

Asymptotic expansions may be derived for transient electromagnetic (EM) fields. The expansions are valid when $\sigma\mu_0 l^2/t$ is less than about 0 1 Here l, σ, μ_0, and t are the respective lengths, conductivities, permeabilities of free space and time

Cases for which asymptotic expansions are presented include (1) layered grounds, (2) axisymmetric structure, and (3) two-dimensional (2-D) structures.

In all cases the transient voltage eventually approaches that of the host medium alone, the ratio of anomalous response to the half-space response being proportional to $1/t^{\nu}$ Here ν is equal to 0 5 for layered structures and 1 0 for 2-D or 3-D structures.

INTRODUCTION

There are few results available for the transient electromagnetic M) response of layered structures either with or without inmogeneities In a review of the literature, Lee (1979) argued ıt the treatment of the problem is often very simplistic. The ,ult was that many of the published results were confusing ‹cause they often disregarded the singularities in the spectrum ıen deriving approximations for it at the low frequencies. Lee ›79) also noted that instruments now exist which will measure ; response at very late stages of the transient. That is, those ;truments concentrate on the very low-frequency (VLF) con- ‹it of the spectrum Here we show that by giving careful con- "eration to the singularities of the spectrum, it is possible to ‹tain asymptotic expansions for the transient EM processes

These expansions will show that at the very late stages, the nsient is dominated by the ground without the heterogeneity. r earlier times the transient is sensitive to any heterogeneity the ground The purpose of this paper is to present a way of :ermining how the heterogeneity affects the transient before it ■ost in the uniform ground response

This study supports the results of Singh (1973) who argued that : effect of the host rock on the transient must be considered if ·ious errors are to be avoided Singh (1973) also noted that, for ‹phere in a uniformly conducting space, the transient was con- lled by a branch cut type of singularity in the spectrum of the nsient We also show that this is the most important singularity the spectrum produced by a wide range of geometries.

Kamenetski (1969) also considered the problem of deriving asymptotic expansions for the transient EM response of various structures. However, our study goes much further than Kamenetski's (1969) study. In particular we give expressions for the transient response of layered structures where the basement is conducting. These expressions show that Kamenetski's (1969) model of a conducting overburden is a very poor one. Kamenetski (1969) also considered the later stages of the transient response of a spherical conductor However, he only gave expressions for a conductor in free space As Singh (1973) showed, these expressions are no longer valid when the conductor is in a conducting environment

Although we restrict our attention to those systems that employ large loops, it should be noted that the results can be extended in a straightforward manner to other systems. First we consider the transient response of a layered structure. We first rederive a result from Kamenetski (1969). However, our treatment is more rigorous in that we first show how the analytic properties of the spectrum have been changed and then give an exact expression for the model. Then we show how this expression may be reduced by further approximations to that expression from Kamenetski (1969). Following this, we consider the problem where the conductivity of the basement is nonzero. There it will be shown that the previous model of a conductive overburden is a very poor one The following two sections treat the problem of finding the later stages of the transient response of conductors in a conductive environment. The results of these sections are checked against other numerical solutions

LAYERED STRUCTURES

Morrison et al (1969) gave an expression for the voltage $\overline{V}$ measured by a loop of radius a at a height h above a medium of n layers It was assumed that a uniform current of $I(\omega)e^{i\omega t}$ flowed in the loop When displacement currents are neglected, the expression becomes

$$\overline{V} = -i\omega\mu_0 a^2 2\pi I \int_0^{\infty} \frac{Z^1 e^{-2\lambda h}}{Z^1 + Z_0} J_0^2(\lambda a)\, d\lambda. \tag{1}$$

Here Z^1 is found from the backward recurrence relation due to Wait (1962)

$$Z^n = Z_n,$$

and

‹inuscript received by the Editor September 2, 1980, revised manuscript received April 13, 1981
·ormerly Cooperative Institute for Research in Environmental Sciences/NOAA, University of Colorado, presently, P O Box 386, Epping, N S.W 2121, stralia

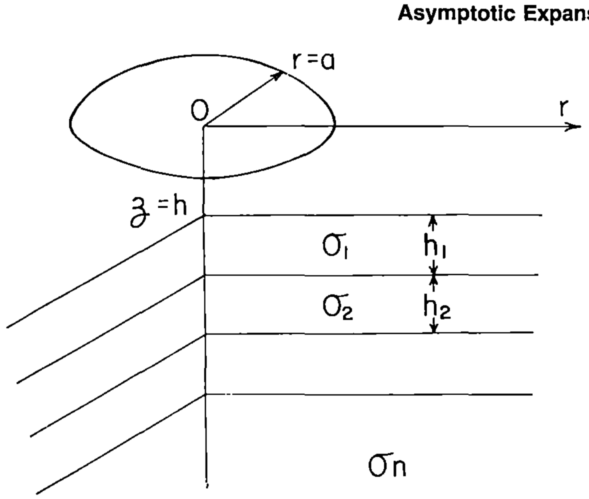


FIG. 1. Geometry for the layered ground.

$$Z^j = Z_j \frac{Z^{j+1} + Z_j \tanh n_j h_j}{Z_j + Z^{j+1} \tanh n_j h_j}, \; Z_j = -\frac{i\omega\mu_0}{n_j},$$

where $j = 0, 1, \ldots\; n$, $n_j = \sqrt{(\lambda^2 + i\omega\sigma_j\mu_0)}$ and σ_j and h_j are the conductivity and thickness of the jth layer (see Figure 1) The recurrence relation begins at the nth or basal layer. For simplicity, all permeabilities have been set to μ_0, the permeability of free space.

If the spectrum of $I(\omega)$ is $I_0/(i\omega)$, that is, if a step current flows in the loop, then the transient voltage $V(t)$ is

$$V(t) = \frac{a^2 I \mu_0}{i} \int_0^\infty \int_{-i\infty}^{i\infty} \frac{Z^1 e^{-2\lambda h}}{Z^1 + Z_0} J_1^2(\lambda a) e^{pt} dp\, d\lambda \qquad (2)$$

Here we have replaced $i\omega$ by p The inner integral in equation (2) can be evaluated by contour integration once the singularities of $Z^1/(Z^1 + Z_0)$ are known

An inspection of the recurrence relation given in equation (1) shows that there will be no branch cuts in the p-plane provided the conductivity of the bottom layer is zero Alternatively, if the conductivity of the bottom layer is not zero, there is a branch cut because of the term $n_n = \sqrt{\lambda^2 + p\sigma_n\mu_0}$. This is the only branch cut. This can be seen by dividing the numerator and the denominator by Z_j and then noting that in the recurrence relation the branch cut contributions can only arise because of the Z^{j+1} term. Only one Z^{j+1} term requires the introduction of a branch cut in order to define all the others; it occurs when $j + 1$ equals n This last observation verifies the stated result. One should note, however, that poles must be allowed for in all cases

In the following section we give an asymptotic expansion for the case where the conductivity of the bottom layer (σ_n) is finite, but here we consider the response of a slab If we take these together we will see the effect of our gross simplification.

Conducting slab in free space

For a loop resting on a slab of thickness d and conductivity σ, the transient voltage from equation (2) is

$$V(t) = \frac{\mu_0 I a^2}{2\sigma\mu_0 d^3} \int_0^\infty \int_{-i\infty}^{i\infty} \frac{\partial}{\partial\tau} \frac{J_1(z\alpha)^2}{i} \cdot$$

$$\cdot \frac{\sinh m_1 e^{s\tau}\, ds\, dz}{(m_1^2 + z^2)\sinh m_1 + 2 m_1 z \cosh m_1} \qquad (3)$$

Here we have rescaled the particular case of equation (2) by the substitutions $\tau = t/\sigma\mu_0 d^2$, $m_1^2 = s + z^2$ and $\alpha = a/d$, $z = \lambda d$, and $s = \sigma\mu_0 d^2 p$ Clearly, there are no branch cuts This can be seen by noticing that the numerator and the denominator can both be expanded in odd powers of m_1 On cancelling an m_1 term, one finds an expression in even powers of m_1. Since there are only poles, the s-integration can be evaluated by contour integration once the locations of the poles are known

Jaeger (1959) showed that the roots of tan $(\alpha x) = 2xz/(z^2 - x^2)$ are all real Writing $m_1 = i x$, the equation for the poles becomes

$$-(z^2 - x^2)\sin(x) + 2zx\cos(x) = 0 \qquad (4$$

Comparing equation (4) with that of Jaeger (1959), we see tha x must be real. Further, an asymptotic expansion for these zero can be found by McMahon's method (McMahon, 1894) Thi method yields

$$x_n = \beta + 2z/\beta - \left(\frac{2}{3}z^3 + 4z^2\right) \Big/ \beta^3 + \left(\frac{2}{5}z^5 + \frac{16}{3}z^4 + 16z^3\right) \Big/ \beta^5,\; n = 0,$$

and

$$x_0 \approx \sqrt{z}\sqrt{2 - z/3 - z^2/90} \approx \sqrt{2z}[1 - 1/(12z)] \qquad (5$$

Here $\beta = n\pi$, $n = 1, 2$.

Equation (5) can be checked against Jaeger's (1959) table. Th results for $z = 0.2$ are shown in Table 1 The table shows tha provided z is less than 0.2, only the first two terms of equatio (4) are needed to specify the position of the poles

If S_n is the value of s at the nth pole, then

$$V(t) = \frac{\mu_0 I a^2 \cdot 2\pi}{2\sigma\mu_0 d^3} \int_0^\infty [J_1(z\alpha)]^2 \cdot$$

$$\cdot\, e^{-z^2\tau} \sum_{n=0}^{\infty} \frac{(S_n - z)^2 4zS_n e^{s\tau} dz}{2S_n z^2 + 2zS_n - 2z^3 - S_n^2 - z^4}.$$

For large τ,

$$V(t) \approx -\frac{\mu I_0 a^2 \cdot 2\pi}{2\sigma\mu_0 d^3} \int_0^\infty [J_1(z\alpha)]^2 z\, e^{-2z\tau} dz$$

$$= \frac{-I}{\sigma_0 d} \cdot 3\pi \left[\frac{\sigma\mu_0 da}{2t}\right]^4 \qquad (6$$

This is the result given by Kamenetski (1968), who derived it b ignoring the singularities of the spectrum In the next sectio we show how this procedure leads to errors in estimation later stages of the transient response of a conductive overburde The errors arise because a branch cut in the spectrum was r placed by a series of poles We will also see that the anomalo field decays as t^{-3} and not t^{-4}

Layered structures—Conducting base

We now return to equation (2) and attempt to approximate t integral with respect to p. This time we will not assume that t basement is nonconducting The model being considered then a layered ground.

The p integral in equation (2) can be evaluated by deformi path of ıntegratıon about the branch poınt at $p = -\lambda^2/$ $:\mu_0) = p_0$ and around the poles ın the left hand sıde of complex p-plane (Fıgure 2). This yıelds

$$V(t) = I_1 + I_2.$$

e I_1 ıs the contrıbution from the sum of the integrals about poles and I_2 ıs the integral about the branch poınt (Fıgure 1) ..ınce $\bar{V}$ ıs causal and $V(t)$ ıs real, then ıf there ıs a pole at $\cdot\, p_n$, there ıs also one at the complex conjugate of p_n um, 1976, p. 134). Notıce also that the real part of p_n must less than zero so as to have decayıng transients at the later ;es Thıs ıs ın contrast to the contrıbutıon from the branch cut ch can result ın a zero argument in the exponentıal term For , reason the following asymptotıc expansıon is derıved usıng ·y the contrıbution from I_2 That is, for large t,

$$V(t) \approx I_2. \tag{8}$$

:ently Mahmoud et al (1979) evaluated the transıent EM fields a vertical magnetıc dipole on a two-layer earth by contour :gration They gave numerıcal evıdence whıch showed that he later stages ıt was the ıntegral about the branch cut that e the most ımportant contrıbutıon.

Vrıtıng $s = \lambda^2/(\sigma_n\mu_0) + p,\ h = 0$ in equation (2) yıelds

$$'(t) = -\frac{I_0 a^2 \mu_0}{\imath}\int_0^\infty \int_c \frac{e^{-\lambda^2 t/(\sigma_n\mu_0)+st} Z^1(s)}{Z^1(s) + Z_0(s)} \cdot \cdot J_1(\lambda a)^2 ds\, d\lambda \tag{8}$$

'e c denotes the path of ıntegratıon that begıns at minus ınfinıty ɔw the negatıve s-axis, passes around the orıgin, and returns nınus infinity above the negatıve s-axıs Thus the effect of the ubstıtutıon has been to translate the branch poınt p_0 of the lane to the orıgın of the s-plane.

ınce $Z^1(s)/[Z^1(s) + Z_0(s)]$ is bounded, ıt follows from ation (8) that the domınant contrıbutıon to $V(t)$ for large t ıes from small values of λ and s For thıs reason we next and $Z^1(s)/[Z^1(s) + Z_0(s)]$ for small values of s and λ After roxımating tan (x) by x, we learn that

$$\frac{Z^1(s)}{(s) + Z_0(s)} \approx \frac{1}{\lambda + \sqrt{(s\sigma_n\mu_0 + \lambda^2)}d + 2s\sigma_n\mu_0\alpha - 2\lambda^2\alpha}, \tag{9}$$

equatıon (9),

$$d = \sum_{\imath=1}^{n-1} h_\imath,$$

$$\alpha = \frac{1}{2}\sum_{\imath=1}^{n-1}\frac{h_\imath\sigma_\imath}{\sigma_n}$$

th thıs approxımation one obtaıns an approxımatıon for). In fact,

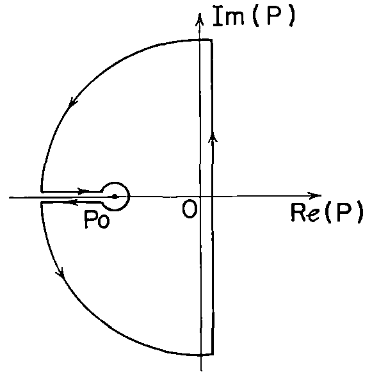


FIG. 2. Path for contour integratıon Any poles are ıgnored

Table 1. Comparison of the roots of $(z^2 - x^2)\sin x + 2zx\cos x = 0$, which are estimated from equation (5), with those published by Jaeger (1959).

n	x_n Jaeger	x_n estımated
0	0 6222	0 6219
1	3 2640	3 2689
2	6 3462	6 3468
3	9 4670	9 4672
4	12.5982	12 5982
5	15 7334	15 7334

The first term in the asymptotıc expansıon of $V(t)$ can be found as follows First, write $R = \lambda x$, then expand the resultıng x ıntegral ın powers of x tımes $1/(1 + x^2)$. Second, expand the Bessel functions ın powers of a and ıntegrate the λ ıntegral by usıng the results of Lee and Lewıs [1979, equation (13)] Thıs shows that

$$V(t) \approx -\frac{a^4\mu_0 I_0\sqrt{\pi}}{\sigma_n\mu_0 2_0\tau^{5/2}}\left[1 - \frac{5}{4}(d - 2\alpha)\sqrt{\frac{\pi}{\tau}}\right], \tag{12}$$

$\tau = t/(\sigma_n\mu_0)$. For a two-layer ground, the equatıon agrees wıth the one given by Kaufman [1979, equatıon (37)], when hıs mısprınt had been corrected.

Because of the approximatıons, equatıon (12) ıs only valıd when every $l^2\sigma\mu_0/t$ is much less than 1. Here l are the varıous lengths ın the model and σ the varıous conductıvıtıes As a check on the accuracy of equation (12), we have computed the transient response of a layered structure in two ways first,

$$V(t) \approx -\frac{a^2\mu_0 I}{\imath}\int_0^\infty\int_c \frac{e^{-\lambda^2 t/(\sigma_n\mu_0)+st}J_1(\lambda a)^2}{\lambda + \sqrt{(s\sigma_n\mu_0 + \lambda^2)}d + 2\alpha s\sigma_n\mu_0 - 2\lambda^2\alpha}ds\, d\lambda \tag{10}$$

itıng $s = R^2 e^{\imath\pi}/(\sigma_n\mu_0)$ and $s = R^2 e^{-\imath\pi}/(\sigma_n\mu_0)$ on the ɔer and lower paths of the c ıntegral yields

$$V(t) = \frac{a^2\mu_0 I_0}{\sigma_n\mu_0\imath}\int_0^\infty\int_0^\infty e^{-\lambda^2 t/(\sigma_n\mu_0) - R^2 t/(\sigma_n\mu_0)} J_1(\lambda a)^2 \cdot$$

$$\cdot\left[\frac{2R}{\lambda + \imath R + \lambda^2 d - 2\alpha R^2 - 2\lambda^2\alpha} - \frac{2R}{\lambda - \imath R + \lambda^2 d - 2\alpha R^2 - 2\lambda^2\alpha}\right] dR\, d\lambda \tag{11}$$

$$= \frac{-4a^2\mu_0 I_0}{\sigma_n\mu_0}\int_0^\infty\int_0^\infty \frac{e^{-\lambda^2 t/(\sigma_n\mu_0) - R^2 t/(\sigma_n\mu_0)}J_1(\lambda a)^2 R^2 dR\, d\lambda}{(\lambda + \lambda^2 d - 2\alpha R^2 - 2\lambda^2\alpha)^2 + R^2}$$

Table 2. Comparison of the calculated transient voltages for a layered ground. The asymptotic value was obtained by using equation (12). The other values were obtained by using the results of Lee and Lewis (1974).

t (msec)	Asymptotic $\times 10^3$ (volts/amp)	Calculated $\times 10^3$ (volts/amp)
0.1	365.0	92.7
0.2	45.7	25.0
0.3	13.5	9.58
0.4	5.71	4.58
0.5	2.92	2.52
0.6	1.69	1.52
0.7	1.06	0.984
0.8	0.713	0.672
0.9	0.501	0.477
1.0	0.365	0.350

by using the exact method as given by Lee and Lewis (1974), and second, by using equation (12). The results are given in Table 2.

Table 2 is for a three-layer ground with the top and basal layers of conductivity 0.025 S/m. The conductivity of the middle layer is ~~1.0~~ 0.1 S/m. The thickness of the top and middle layers are 10 and 5 m, respectively. The radius of the loop is 100 m. Equation (12) consists of two terms, the first of which is the asymptotic expansion of the transient EM field for a loop on a half-space whose conductivity is the same as the basal layer. For the model chosen this conductivity is the same as the top layer. In this case, the secondary term which Lee and Lewis (1974) calculated is the second term in equation (12). This is the quantity computed in Table 2. The conclusion from the table is that equation (12) is correct under the conditions stated. The slight decrease in accuracy at the later stages is due to a slight inaccuracy in the compute program. Another point to be noted is that a two-layer groun with a thin top layer and resistive basement will be difficult t calculate by the method of Lee and Lewis (1974) because th secondary term of equation (12) is almost Lee and Lewis's pri mary term but of opposite sign. This is why curve 1 of Figure of that paper diverges from curve 5 at the later stages, i.e., ther is an increase in errors. However, formula (12) provides means of overcoming this difficulty.

A comparison of equation (12) with equation (6) shows th effect of approximating a conductive overburden by a conductin slab. The equations are quite different. The result given in equa tion (12) should be used. Equations (6) and (12), then, show th errors can easily arise in asymptotic representation of transie processes when insufficient care is paid to the singularities i the continued spectrum.

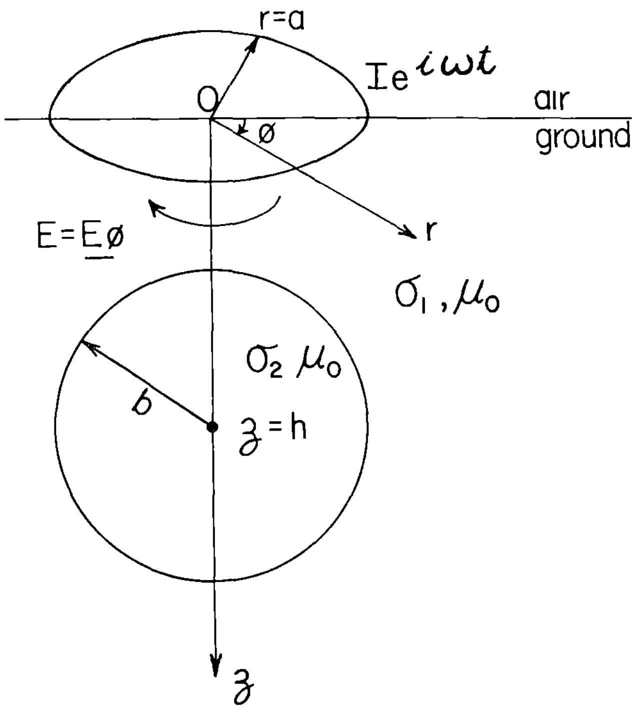


FIG 3. Geometry for the axisymmetric structures. The particular case shown here is for a sphere.

AXISYMMETRIC STRUCTURES

The integral equation describing the electric field $\overline{E}(r_1$ produced by a current of $I(\omega)e^{i\omega t}$ flowing in a loop of radius a on half-space in which there is an axial symmetric conductor who axis coincides with the axis of the loop is

$$\overline{E} = \overline{E}^i + \frac{k_2^2 - k_1^2}{i\omega\mu_0} \int_{V^1} \overline{E}\,\overline{G}\,dV^1 \qquad (1$$

Here

$$\overline{G} = -\frac{i\omega\mu_0}{2}\int_0^\infty J_1(\lambda r)J_1(\lambda r^1)\cdot$$
$$\cdot\left[e^{-n_1|z-z^1|} + \frac{n_1-\lambda}{n_1+\lambda}e^{-n_1(z+z^1)}\right]\cdot\frac{\lambda}{n_1}d\lambda,$$

$$\overline{E}^i = -\frac{i\omega\mu_0 Ia}{2}\int_0^\infty J_1(\lambda r)J_1(\lambda a)e^{-n_1 z}\left[1+\frac{n_1-\lambda}{n_1+\lambda}\right]\frac{\lambda}{n_1}\,c$$

and

$n_1 = \sqrt{\lambda^2 + i\omega\mu_0\sigma_1}$, $k_j^2 = i\omega\sigma_j\mu_0$

[Lee, 1975, equation (4)]. The geometry is shown in Figure 3

In these equations $\overline{E}^i(r, z)$ is the incident electric field a the cylindrical coordinates (r, z, ϕ) are arranged such that t origin is at the center of the loop and the z-axis is directed p pendicular to the ground's surface, positive within the grour The primes on V, r, and z denote the variables of integration o volume V of the inhomogeneity. The conductivity of the homogeneity is σ_2. All permeabilities have been set equal μ_0, the permeability of free space. Note from Figure 3 t because of the axial symmetry of the geometry, there is o an angular component for the Green's function and the elect field. Consequently, the ϕ integration in equation (13) may readily evaluated to yield a factor $2\pi r^1$.

The function $r^1 G$ has a simple interpretation. It represe the electric field produced by a horizontal loop of radius The depth to the center of the loop is z^1, and a current of c ampere flows in the loop.

The previous section shows that in order to derive use asymptotic expansions, it is necessary to pay particular attent to the singularities of the spectrum. An inspection of equat (13) suggests attempting to solve it for small k_j^2 by a N mann series. We also have seen that branch points in spectrum of $\overline{E}$ are crucial in determining the asymptotic expansi Provided this singularity is preserved, we may try radical appro ations of the functions being integrated In a previous study, ee (1975) found that in the mode matched solution, for the rticular case where the body is a sphere, the off-diagonal ements of the scattering matrix could be neglected. In other ords, for the time domain problem discussed here the secondary rm in the Green's function could be neglected These observa- ons suggest that within the integration over the structure we place $\overline{E}$ by $\overline{E}_1^i$ and $\overline{G}$ by $\overline{G}_1$ where

$$\overline{G}_1 = -\frac{i\omega\mu_0}{2}\int_0^\infty J_1(\lambda r)J_1(\lambda r')e^{-n_1|z-z'|}\frac{\lambda}{n_1}\,d\lambda,$$

$$= -\frac{i\omega\mu_0}{4\pi}\int_0^{2\pi}\frac{e^{-k_1R_1}\cos\phi_1}{R_1}\,d\phi_1,$$

$$\overline{E}_1^i = -\frac{i\omega\mu_0 al}{2}\int_0^\infty \frac{\lambda}{n_1}J_1(\lambda r')J_1(\lambda_a)e^{-n_1 z}d\lambda,$$

$$= -\frac{i\omega\mu_0 al}{4\pi}\int_0^{2\pi}\frac{\cos\phi_2 e^{-k_1R_2}}{R_2}\,d\phi_2,$$

$$R_1 = \sqrt{(z-z')^2 + r'^2 + r^2 - 2rr'\cos\phi_1},$$

d

$$R_2 = \sqrt{(z^2 + r'^2 + a^2 - 2ar'\cos\phi_2)}.$$

iese approximations still retain the branch cut in the spectrum $\overline{E}$ in the complex ω-plane This cut runs along the positive iaginery ω-axis.

If a current of $I_0e^{i\omega t}/(i\omega)$ flows in the loop, then the re- lting transient electric field is approximately $E(t)$ where

$$E(t) = E^i(t) + \frac{(\sigma_2-\sigma_1)}{2\pi i}r'\int_A\int_{-i\infty}^{i\infty}\frac{\overline{E}_1^i(r',z')G_1(r',z')}{p}\cdot e^{pt}dp\,dA \qquad (14)$$

re $p = i\omega$.

The integral in equation (14) may be simplified by integrating out the branch cut in the same way as equation (10) After aluating this integral, one finds that

$$E(t) = E^i(t) + \frac{\mu_0 I_0(\sigma_2-\sigma_1)}{16\pi^2\sigma_1}\int_A\int_0^{2\pi}\int_0^{2\pi}F(t)\cos\phi_1\cdot\cos\phi_2 d\phi_2 dA',$$

d

$$F(t) = \frac{r'a(R_1+R_2)}{R_1R_2\,4\sqrt{\pi}}\frac{1}{\sigma_1\mu_0\tau^{5/2}}\left(\frac{(R_1+R_2)^2}{2\tau} - 3\right)\cdot e^{-(R_1+R_2)^2/(4\tau)}, \qquad (15)$$

d $\tau = t/(\sigma_1\mu_0)$

Because $(R_1^2 + R_2^2)/(4\tau)$ is much less than unity at the er stages

$$F(t) \approx \frac{r'a}{R_1R_2}\left[\frac{(R_1+R_2)\tau^{-5/2}}{4\sqrt{\pi}\,\sigma_1\mu_0}\right]\left[\frac{5}{4}\left(\frac{(R_1+R_2)^2}{\tau}\right) - 3\right] \qquad (16)$$

nsequently,

$$t) \approx E^i(t) - \frac{I_0\mu_0(\sigma_2-\sigma_1)}{16\pi^2\sigma_1^2}\int_{A'}\int_0^{2\pi}\frac{5r'a\sqrt{\pi}}{8\tau^{7/2}}\cdot\left(\frac{ar'}{R_1} + \frac{rr'}{R_2}\right)\cos\phi\,d\phi\,dA \qquad (17)$$

In equation (17), the ϕ integrations have been simplified and in R_1 and R_2 we have set $\phi_1 = \phi_2 = \phi$ Consequently,

$$E(t) \approx E^i - \frac{\mu_0 I_0(\sigma_2-\sigma_1)}{16\pi^2\sigma_1}\int_{A'}\frac{5r'a\pi^{3/2}}{4\tau^{7/2}}H(r,z)\,dA',$$

where

$$H(r,z) = \frac{2ar'}{\pi}\frac{1}{\sqrt{(z-z')^2 + (r+r')^2}}\cdot K\left[2\sqrt{\frac{rr'}{(z-z')^2 + (r+r')^2}}\right] + \frac{2rr'}{\pi}\frac{1}{\sqrt{z'^2 + (r+a)^2}}K\left[2\sqrt{\frac{ar'}{z'^2 + (r+r')^2}}\right]. \qquad (18)$$

Here $K(k)$ is the complete elliptic integral of the first kind of modulus k. This function may be calculated easily by the de- scending Landen's transformation [see Abramowitz and Stegun, 1965, p. 597]. To calculate the magnetic fields, the derivative of the elliptic integral is required. Note that

$$\frac{d}{dk}K(k) = [E(k) - (1 - k_1^2)K(k)]/[k(1-k^2)].$$

Here $E(k)$ is the elliptic integral of the second kind.

Few results are available for the transient response of structures of the type discussed above. One structure which has been studied is a sphere (Lee, 1975). It is these results which we must use to check equation (18)

If the sphere is of radius b and is at depth h beneath the loop, then equation (18) reduces to

$$E = E^i - \frac{I_0a(\sigma_2-\sigma_1)ar\,b^2}{\tau^{7/2}\sigma_1^2\sqrt{\pi}\,96}\left[\left(\frac{b}{P_1}\right)^3 + \left(\frac{b}{P_2}\right)^3\right], \qquad (19)$$

where

$$P_1 = \sqrt{r^2 + (z-h)^2},$$

and

$$P_2 = \sqrt{a^2 + h^2}$$

The easiest way to derive equation (19) is to expand $1/R_1$ and $1/R_2$ of equation (17) in terms of Legendre functions and then

Table 3. Comparison of the calculated voltages for a spherical conductor in half space. The V_a was obtained by using equation (18) and V_c was obtained by a mode-matching solution (Lee, 1975).

t (msec)	V_c(volts/amp $\times 10^3$)	V_a(volts/amp $\times 10^3$)
1	2.81	2 42
1 5	0 89	0 87
2 0	0 40	0 43
2 5	0.22	0 23
3 0	0 13	0.14
3.5	0 09	0 094

effect the integrations by using the addition formulas for Legendre functions.

Equation (19) is the first term of an asymptotic expansion which has previously been given by Thio [1977, equation (7.3-19)]. Thio obtained his series by finding the asymptotic expansion of an exact solution to equation (13) For the case of a coincident loop prospecting system, the transient voltage is approximated by

$$V(t) = V_1(t) - \frac{Ia(\sigma_2 - \sigma_1)\sqrt{\pi}\, a^4 b^5}{24\, \tau^{7/2} \sigma_1^2 P_2^3}. \qquad (20)$$

This equation may be used to check equation (18)

Table 3 shows the result of calculating the transient response of a sphere in two different ways. The first way is to use the method of Lee (1975), and the second way is to use equation (20) For this calculation the first term in the approximation to $V_1^i(t)$ was also used. That is,

$$V_1(t) = \frac{-aI_0 \mu_0 \sqrt{\pi}}{20t} \left(\frac{\sigma_1 \mu_0 a^2}{t} \right)^{3/2} \qquad (21)$$

[Lee and Lewis, 1974, equation (14)]. For the model, the radius of the loop and the sphere was 75 m and the depth to the center of the sphere is 100 m The conductivities of the ground and the sphere are 0.049 and 0.56 S/m, respectively. As Table 3 shows, results derived from equation (18) can be in excellent agreement with those of other less approximate methods provided $\sigma_1 \mu_0 l^2/t$ is much less than unity. For example, when $t = 3.5$ msec, the percentage error is about 4 percent and $\sigma_1 \mu_0 l^2/t$ is about 0.1

TWO-DIMENSIONAL STRUCTURES

The method given in the previous section can be used to obtain asymptotic expansions for the transient response of two-dimensional (2-D) structures. The idea is to approximate the

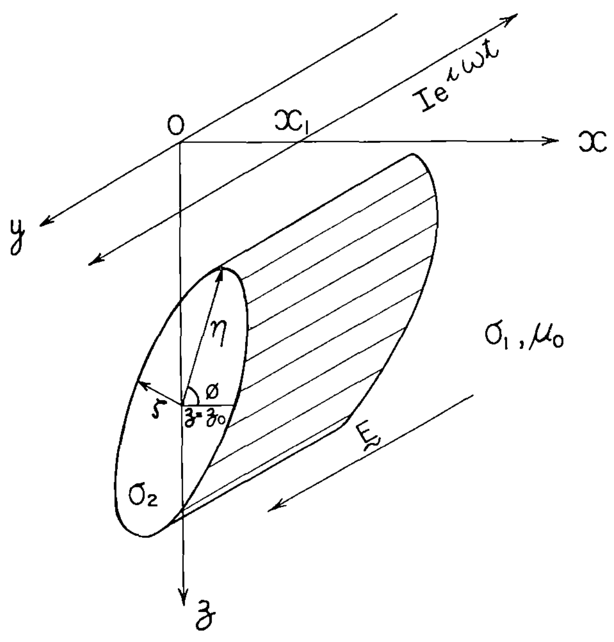


FIG. 4. Geometry for the 2-D structures The particular case shown here is for an elliptic cylinder.

sending and receiving loops by a pair of wires The integr equation to be solved is

$$\overline{E} = \overline{E}^i + (k_2^2 - k_1^2) \int_A \overline{E}\, \overline{G}\, dA', \qquad (2.$$

where

$$\overline{E}^i = -\frac{i\omega\mu_0 I}{\pi} \int_0^\infty \frac{\cos\lambda(x - x_1) e^{-n_1 z}}{n_1 + \lambda} d\lambda,$$

$$\overline{G} = -\frac{1}{2\pi} \int_0^\infty \left[\frac{n_1 - \lambda}{n_1 + \lambda} e^{-n_1(z+z^1)} + e^{-n_1|z-z^1|} \right] \cdot \frac{\cos\lambda(x - x^1)}{n_1} d\lambda,$$

$$n_1 = \sqrt{\lambda^2 + i\omega\mu_0\sigma_1},$$

and

$$k_j^2 = i\omega\mu_0\sigma_j,$$

[Hohmann (1971), Wait (1962)]. Again the z-axis is directed ir the ground. The conductivities of the ground and inhomogene are σ_1 and σ_2, respectively All the permeabilities have be set to μ_0, and displacement currents have been neglected Equ tion (22) is for a simple line source which is on the earth's su face at $(x_1, 0)$ (see Figure 4 which shows an elliptical conducto

Howard (1972) solved equation (22) for the particular ca where the conductor is a cylinder. The method used was mo matching, and one result was that the scattering matrix w diagonally dominant. Consequently, we can use an argume analogous to that used in the previous section to approxima the unknown field by

$$\overline{E} = \overline{E}^i + (k_2^2 - k_1^2) \int_A \overline{E}_1 \overline{G}_1 dA,$$

$$\overline{E}_1 = -\frac{i\omega\mu_0 I_0}{2\pi} K_0(k_1 R_1), \qquad (2$$

$$\overline{G}_1 = -\frac{1}{2\pi} K_0(k_1 R),$$

where

$$R_1 = \sqrt{z'^2 + (x_1 - x)^2},$$

and

$$R = \sqrt{(z' - z)^2 + (x - x')^2}.$$

We have neglected the secondary terms in the Green's functi and in the approximation of $\overline{E}$ in the integral. A further simp fication was evaluating the integrals involved in the primary ter in terms of the modified Bessel function K_0

Suppose that the signal is transmitted by two wires which at $(x_2, 0)$ and $(x_3, 0)$ and received as the difference in electric field $\overline{E}$ between (x_1, z_1) and (x_4, z_4) Note that the ceived voltage is simply $\overline{\Delta E}\, M$, where M is the length of receiving wires If the spectrum of $I(\omega)$ is $I_0/(i\omega)$, then the tr sient signal at the later stages $\overline{\Delta E}(t)$ can be found by cont integration. Again we have to integrate about a cut along positive imaginary ω-axis. The details are quite straightforw and the results are given in equation (24)

$$\overline{\Delta E}(t) = \overline{\Delta E}^i(t) + \frac{(\sigma_2 - \sigma_1)\mu_0^2 I_0}{4\pi^2} \int_{A'} H(R_1, R_2)$$

$$- H(R_1, R_3) - H(R_4, R_2) + H(R_4, R_3)\,dA' \qquad (24)$$

Here

$$R_i = \sqrt{(x' - x_i)^2 + (z' - z_i)^2},\ i = 1, 2, 3, 4,$$

$$z_1 = z_3 = 0,$$

$$H(R_i, R_j) = \frac{\pi}{2}\int_0^\infty z^3 e^{-z^2 t}[J_0(R_i^z\sqrt{\sigma_1\mu_0})Y_0(R_j^z\sqrt{\sigma_1\mu_0}) + J_0(R_j^z\sqrt{\sigma_1\mu_0})Y_0(R_i^z\sqrt{\sigma_1\mu_0})]dz$$

$$= + \frac{\partial}{\partial t}\frac{1}{2t}e^{-(R_i^2+R_j^2)\sigma_1\mu_0/(4t)}K_0[R_iR_j\sigma_1\mu_0/(2t)]$$

Equation (24) can be simplified for a finite structure by using approximations for the exponential and Bessel functions of small argument These approximations reduce equation (24) to

$$\Delta E(t) \approx \Delta E'(t) - \frac{(\sigma_2 - \sigma_1)\mu_0 I_0}{32\pi^2 t^3}\int_A \sigma_1\mu_0^2[(R_2^2 - R_3^2)\cdot \ln(R_1/R_4) + (R_1^2 - R_4^2 - R_4^2)\ln(R_2/R_3)]dA' \qquad (25)$$

Equation (25) can be simplified for one useful geologic model, an elliptic cylinder

When all the wires are at the earth's surface,

$$R_i^2 - R_j^2 = \alpha_{ij} + \beta_{ij}x',$$

where

$$\alpha_{ij} = x_i^2 - x_j^2,$$

and

$$\beta_{ij} = -2(x_i - x_j). \qquad (26)$$

Suppose that the center of the elliptic cylinder is at $(0, z_0)$ and that its major and minor axes are η and ξ, respectively In addition, the major axis dips to the left at an angle ϕ (see Figure 4 where only one wire is shown) For convenience we choose a new system of coordinates (x, z) that are aligned with the major and minor axes of the cylinder In terms of these coordinates,

$$x' = x\cos\phi + z\sin\phi,$$
$$z_0 - z' = x\sin\phi - z\cos\phi \qquad (27)$$

If we put $\gamma_{ij} = \beta_{ij}\cos\phi$ and $\delta_{ij} = -\beta_{ij}\sin\phi$, then

$$R_i^2 - R_j^2 = \alpha_{ij} + \gamma_{ijx} - \delta_{ijz}. \qquad (28)$$

Now choose a third set of coordinates so that the surface of the ellipse is easy to define. We transform (x, z) to elliptic cylinder coordinates by the substitutions $x = b\cosh\mu\cos\theta$, and $z = -b\sinh\mu\sin\theta$. The surface of the elliptic cylinder is now at $\mu = \mu^*$ and the points (x_i', z_i') become (μ_i, θ_i).

With these changes equation (25) becomes

$$\Delta E(t) = \Delta E'(t) - \frac{(\sigma_2 - \sigma_1)}{16\pi^2 t^3}\mu_0^3 I_0(\sigma_1 b^2)\int_0^{2\pi}\int_0^{\mu*}[\cosh^2\mu - \cos^2\theta]\cdot$$
$$\cdot\Bigg\{[\alpha_{23} + b\gamma_{23}\cosh\mu\cos\theta + b\delta_{23}\sinh\mu\sin\theta]\Bigg[\mu_1 - \mu_4 + 2\sum_{n=1}^\infty \frac{\cos n\theta}{n}\cdot$$
$$\cdot(e^{-n\mu_4}\cos n\theta_4 - e^{-n\mu_1}\cos n\theta_1)\cosh n\mu + \frac{\sin n\theta}{n}\cdot$$
$$\cdot(e^{-n\mu_4}\cos n\theta_4 - e^{-n\mu_1}\sin n\theta_1)\sinh n\mu\Bigg]$$
$$+ [\gamma_{14} + b\gamma_{14}\cosh\mu\cos\theta + b\delta_{14}\sinh\mu\sin\theta]\Bigg[\mu_2 - \mu_3 + 2\sum_{n=1}^\infty\frac{\cos n\theta}{n}\cdot$$
$$\cdot(e^{-n\mu_3}\cos n\theta_3 - e^{-n\mu_2}\cos n\theta_2)\cosh n\mu + \frac{\sin n\theta}{n}\cdot$$
$$\cdot(e^{-n\mu_3}\sin n\theta_3 - e^{-n\mu_2}\sin n\theta_2)\sinh n\mu\Bigg]\Bigg\}d\mu\,d\theta. \qquad (29a)$$

Multiplying the bracketed terms and integrating yields

$$\Delta E = \Delta E' - \frac{(\sigma_2 - \sigma_1)}{8\pi t^3}\mu_0^3 I_0 b^2\sigma_1[I_1^{23}(\mu_1 - \mu_4)$$
$$+ I_2^{23}(e^{-\mu_4}\cos\theta_4 - e^{-\mu_1}\cos\theta_1)$$
$$+ I_3^{23}(e^{-\mu_4}\sin\theta_4 - e^{-\mu_1}\sin\theta_1)$$
$$+ I_4^{23}(e^{-2\mu_4}\cos 2\theta_4 - e^{-2\mu_1}\cos 2\theta_1)/2$$
$$+ I_5^{23}(e^{-2\mu_4}\cos 3\theta_4 - e^{-2\mu_1}\cos 3\theta_1)/3$$
$$+ I_6^{23}(e^{-3\mu_4}\sin 3\theta_4 - e^{-3\mu_1}\sin 3\theta_1)/3$$
$$+ I_1^{14}(\mu_2 - \mu_3) + I_2^{14}(e^{-\mu_3}\cos\theta_3 - e^{-\mu_2}\cos\theta_2)$$
$$+ I_3^{14}(e^{-\mu_3}\sin\theta_3 - e^{-\mu_2}\sin\theta_2)$$
$$+ I_4^{14}(e^{-2\mu_3}\cos 2\theta_3 - e^{-2\mu_2}\cos 2\theta_2)/2$$
$$+ I_5^{14}(e^{-3\mu_3}\cos 3\theta_3 - e^{-3\mu_2}\cos 3\theta_2)/3$$
$$+ I_6^{14}(e^{-3\mu_3}\sin 3\theta_3 - e^{-3\mu_2}\sin 3\theta)/3], \qquad (29b)$$

where

$$I_1^{ij} = \frac{\alpha_{ij}}{2}\sinh\mu^*\cosh\mu^* = \frac{\alpha_{ij}}{2b^2}\xi\eta,$$

$$I_2^{ij} = \frac{b\gamma_{ij}}{8}(\sinh\mu^*\cosh\mu^* + \sinh^3\mu^*\cosh\mu^* + \cosh^3\mu^*\sinh\mu^*)$$
$$= \frac{b\gamma_{ij}}{8}\left(\frac{\eta\xi}{b^2} + \frac{\xi^3\eta}{b^4} + \frac{\eta^3\xi}{b^4}\right),$$

$$I_3^{ij} = \frac{b\delta_{ij}}{8}(\sinh^3\mu^*\cosh\mu^* + \cosh^3\mu^*\sinh\mu^* - \sinh\mu^*\cosh\mu^*)$$
$$= \frac{b\delta_{ij}}{8}\left(\frac{\xi^3\eta}{b^4} + \frac{\eta^3\xi}{b^4} - \xi\eta\right),$$

$$I_4^{ij} = -I_1^{ij},$$

$$I_5^{ij} = -I_2^{ij},$$

and

$$I_6^{ij} = -I_3^{ij}$$

For computing purposes, note that

$$\cos\theta_i = \frac{1}{2b}\left(\sqrt{(x_i + b\cos\phi)^2 + (z_0 + b\sin\phi)^2}\right.$$
$$\left. - \sqrt{(x_i - b\cos\phi)^2 + (z_0 - b\sin\phi)^2}\right),$$
$$\cosh\mu_i = \frac{1}{2b}\left(\sqrt{(x_i + b\cos\theta)^2 + (z_0 + b\sin\phi)^2}\right.$$
$$\left. + \sqrt{(x_i - b\cos\phi)^2 + (z_0 - b\sin\phi)^2}\right),$$

and

$$b = \sqrt{\eta^2 - \xi^2}. \tag{30}$$

It is interesting to reduce the ellipse to a circle This reduction can be achieved by allowing ξ to approach η This happens when μ^* becomes large If the radius is to remain finite, then b must also become small. If a is the radius of the circle, then for small b and large μ^* we obtain

$$I_1^y = \frac{\alpha_y}{2}\frac{a^2}{b^2},$$
$$I_2 = \frac{b\gamma_y}{4}\frac{a^4}{b^4},$$
$$I_3 = \frac{b\delta_y}{4}\frac{a^4}{b^4},$$
$$e^{-\mu} = \frac{b}{\sqrt{x^2 + z^2}}, \quad \cos\theta = \frac{x}{\sqrt{x^2 + z^2}} \tag{31}$$

With these results one finds that

$$\Delta E \approx \Delta E^i - \frac{(\sigma_2 - \sigma_1) I_0 \mu_0^3 \sigma_1}{8\pi t^3}\left[\frac{\alpha_{23} a^2}{2}\ln\sqrt{\frac{x_1^2 + y_1^2}{x_4^2 + y_4^2}} + \frac{la^4}{4}\right.\cdot$$
$$\cdot\left(\frac{x_4}{x_4^2 + y_4^2} - \frac{x_1}{x_1^2 + y_1^2}\right) + \frac{\alpha_{14} a^2}{2}\ln\sqrt{\frac{x_2^2 + y_2^2}{x_3^2 + y_3^2}}$$
$$\left. + \frac{la^4}{4}\left(\frac{x_3}{x_3^2 + y_3^2} - \frac{x_2}{x_2^2 + y_2^2}\right)\right]. \tag{32}$$

Here l is the distance between the transmitting wires; it is also the distance between the receiving wires. Equation (32) is what we would obtain if we had derived the first term of an asymptotic expansion by integrating about the same branch cut but using Howard's (1972) exact expression for the electric field

To evaluate the equations of this section, one requires $\overline{\Delta E}^i(t)$ to be known This quantity can easily be obtained from the results of Lewis and Lee (1980). In fact,

$$\overline{\Delta E}^i(t) = \overline{E}_1^i(t, x_1, x_2) - \overline{E}_1^i(t, x_1, x_3) - \overline{E}_1^i(t, x_4, x_2)$$
$$+ \overline{E}_1^i(t, x_4, x_3),$$
$$E_1^i(t, x_i, x_j) = -\frac{I_0 \sigma_0 \mu_0}{4t\sigma_1\pi}\left[4t/(\sigma_1\mu_0(x_i - x_j)^2)\right]$$
$$\left[1 - e^{-\sigma_1\mu_0(x_i - x_j)^2/(4t)}\right]. \tag{33}$$

DISCUSSION

The main results of this paper are in equations (6), (12), (18) (25) and (29). Some of these equations were reduced to particula cases, and numerical results derived from these cases wer favorably compared with the results derived from more exac methods Good agreement was obtained when $l^2\sigma\,\mu_0/t$ wa much less than unity. In the examples given, this quantity wa about 0.1. While the Neumann series solution can be used t solve the 3-D problem formally, this was not done because n results were available to check the approximations.

Some particular features of the equations should be noted. I all cases the asymptotic expansion for the transient voltage consists of two terms, the first of these terms is the same as woul be obtained if it were assumed that the ground was a half-space The conductivity of this half-space is the same as the half-spac in the original problem A feature of the second term is that i decays more rapidly than the first term Consequently, th frequently heard statement that the later stages of the transien are due to the underlying conductor is misleading Note that ver late stages are not suitable for determining the parameters o the buried conductors

For the case of layered structures, one finds that the late stages depend upon only two parameters These are the con ductivity of the lower half-space and the parameter

$$\sum_{i=1}^{n-1} h_i(1 - \sigma_i/\sigma_n).$$

Here h_i is the thickness of the ith layer, σ_i is the con ductivity of the ith layer, and δ_n is the conductivity of th bottom layer. Thus, many structures are equivalent at the late stages

In deriving the results of this paper great attention had to b paid to the singularities in the spectrum of the transient EM field The example of approximating a layered structure by conducting slab shows that the approximation breaks down be cause the approximate spectrum has quite different singularitie from the original spectrum.

The results show that in order to be able to derive all the param eters for a geologic model, one must be able to calculate th field at the earlier stages However, the above results can b used to help verify the results of more exact formulations.

ACKNOWLEDGMENTS

I thank Dr. James R Wait and Dr Misac Nabighian for critically reading the manuscript

REFERENCES

Abramowitz, M , and Stegun, I A , 1965, Handbook of mathematica functions with formulas, graphs and mathematical tables New York Dover, p 1046

Baum, C E , 1976, The singularity expansion method, *in* Transien electromagnetic fields L B Felsen, Ed , Springer-Verlag, Berlin p 129–179

Hohmann, G. W., 1971, Electromagnetic scattering by conductors i the earth by a line source of current Geophysics, v 36, p 101–131

Howard, A Q , 1972, The electromagnetic fields of a subterranean cy lindrical inhomogeneity excited by a line source. Geophysics, v 37 p 975–984

Jaeger, J C., 1959, Conduction of heat in solids. Oxford, Oxfor Clarendon, p. 510.

Kamenetski, F M., 1968, Transient processes using combined loop

for a two layer section with a non-conducting base Izv Vozov, sect , Geology Prosp , no 6
—— 1969, The elements of the theory of time domain mining electroprospecting Prikla Geofiz, v 57, p 137–154 (in Russian)
aufman, A , 1979, harmonic and transient fields on the surface of a two layered medium· Geophysics, v 44, p 1208–1217.
ee, T , 1975, Transient electromagnetic response of a sphere in a layered medium. Geophy Prosp , v 23, p 492–512.
—— 1979, Transient electromagnetic waves applied to prospecting Proc IEEE , v 67, p 1016–1021
ee, T , and Lewis, R., 1974, Transient EM response of a large loop on a layered ground. Geophys Prosp , v 22, p. 430–444
ewis, R , and Lee, T , 1980, The effect of host rock on transient electromagnetic fields ASEG Bull , in press.
Mahmoud, S F , Botros, A Z , and Wait, J R , 1979, Transient electromagnetic fields of a vertical magnetic dipole on a two layered earth Proc IEEE, v. 67, p 1022–1029.
McMahon, J., 1894, On the roots of the Bessel and certain related functions Annals of Math., v 9, p 23–30
Morrison, H F , Phillips, R. J , and O'Brien, D P , 1969, Quantitative interpretation of transient electromagnetic fields over a layered half space. Geophys Prosp., v 17, p 82–101
Singh, S K., 1973, Electromagnetic response of a conducting sphere embedded in a conductive medium Geophysics, v 38, p 384–393.
Thio, Y T., 1977, Riemann boundary value problems in electromagnetism Unpublished Ph D thesis, Monash University, Clayton, Australia, p 257
Wait, J. R., 1962, Electromagnetic waves in stratified media. New York, Pergamon Press

Paper

The Two-dimensional Green's Function for Electromagnetic Scattering

T. J. Lee

Geopeko
P.O. Box 217
Gordon, N.S.W. 2072

Key words electromagnetic, scattering, modelling, Green's function, two-dimensional

Abstract

The usual integral form for the two-dimensional Green's function (a cosine transform) is evaluated in closed form. This alternative form is particularly useful when calculations have to be performed on a mini-computer.

1. Introduction

Because the number of mining companies owning mini-computers has increased, there is a corresponding increase in interest in the modelling of electromagnetic fields. A particular scheme for modelling the scattering of electromagnetic fields has been described by Hohmann (1971). The basis of the scheme is the solution of an integral equation, and to that end a Green's function must be evaluated many times. A particular feature of the Green's function is that it is the cosine transform of a function which depends on the vertical distance of a buried line source. In fact, the Green's function is also used to describe the electromagnetic fields about a line source. Because of the oscillating nature of the cosine function, the numerical integrations are slow and often inaccurate. The inaccuracies arise because it is usual to integrate between successive zeros of the cosine function and then to add all the terms together. The point is that when such a calculation is performed on a mini-computer, errors can arise because of numerical 'round-off'. The purpose of this paper is to present closed form expressions for the electric and magnetic Green's functions. By such means it is possible to avoid the problems mentioned above.

2. Electric Green's Function

The function used by Hohmann is $G(x^1,z^1;x,z)$ and it is defined by eqn 1:

$$G = \frac{-i\omega\mu_0}{2\pi}\int_0^\infty \left[\frac{(n_1-n_0)e^{-n_1(z+z^1)}}{(n_1+n_0)} + e^{-n_1|z-z^1|}\right]\frac{\cos(\lambda(x-x^1))d\lambda}{n_1} \qquad (1)$$

Here an $e^{i\omega t}$ time dependence has been assumed; the point (x^1,z^1) represents the source point while the point (x,z) denotes the position at which the function is to be evaluated. See Fig. 1.

$$n_1 = \sqrt{(\lambda^2+k_1^2)}, \qquad n_0 = \sqrt{(\lambda^2+k_0^2)}$$

$$k_1^2 = i\omega\mu_0\sigma_1 - \omega^2\mu_0\epsilon_1$$

$$k_0^2 = -\omega^2\mu_0\epsilon_0$$

The conductivity and permittivity of the ground are σ_1 and ϵ_1 respectively. The permittivity of the air is ϵ_0 and all permeabilities have been assumed to be the same as for free space, μ_0.

When displacement currents are neglected, eqn (1) may be reduced to:

$$G = \frac{-i\omega\mu_0\{K_0(k_1R)+I\}}{2\pi} \qquad (2)$$

$$\text{where } I = \int_0^\infty \left(\frac{n_1-\lambda}{n_1+\lambda}\right)\frac{e^{-n_1(z+z^1)}}{n_1}\cos(\lambda(x-x^1))d\lambda$$

$$\text{and} \quad R = \sqrt{((z-z^1)^2 + (x-x^1)^2)}$$

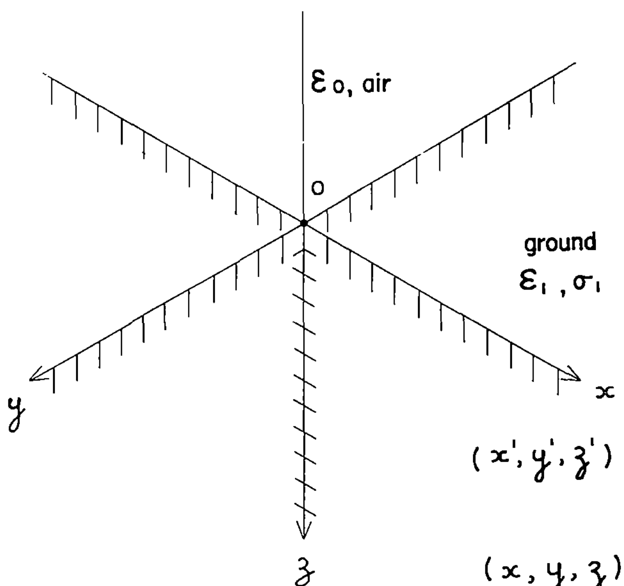


FIGURE 1
Geometry for the Green's Function

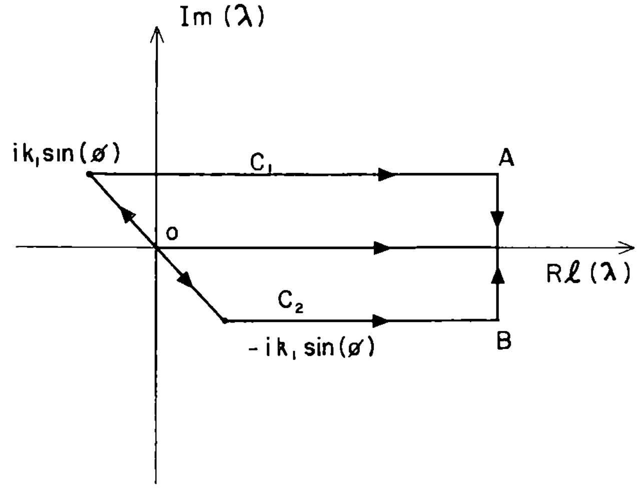


FIGURE 2
Path for the contour integrals

$K_0(z)$ is the modified Bessel function of argument z.

The remainder of this section is concerned with the evaluation of the term denoted by I.

To proceed, one notices that the term I is the sum of two terms denoted by I^+ and I^- where

$$I^+ = \frac{1}{2}\int_0^\infty \left(\frac{n_1 - \lambda}{n_1 + \lambda}\right) \frac{e^{-n_1(z+z^1)+i\lambda(x-x^1)}}{n_1} d\lambda$$

and

$$I^- = \frac{1}{2}\int_0^\infty \left(\frac{n_1 - \lambda}{n_1 + \lambda}\right) \frac{e^{-n_1(z+z^1)-i\lambda(x-x^1)}}{n_1} d\lambda \qquad (3)$$

In making this choice it is assumed that $(x-x^1) \geqslant 0$. In the event that this is not the case, the two terms are interchanged. Consequently it is possible to assume that the term $(x-x^1)$ is positive.

Notice that I^+ and I^- are analytic in the upper and lower λ planes with branch points at $\lambda = +ik_1$ and $-ik_1$ respectively. Also, the stationary points of

$$(z+z^1)n_1 \pm \lambda(x-x^1)i$$

are at

$$\lambda = \pm ik_1 \sin(\phi)$$

Here

$$(z+z^1) = P\cos(\phi);$$
$$(x-x^1) = P\sin(\phi)$$

and

$$P = \sqrt{((x-x^1)^2 + (z+z^1)^2)}$$
$$\phi = \arctan(|x-x^1|/(z+z^1))$$

These observations suggest that the paths of integration for I^+ or I^- be deformed to the paths C_1 and C_2 respectively.

To proceed, one divides the path of each integral into two parts. The first part is to $\pm ik_1 \sin(\phi)$ respectively, and the second part is to infinity via the points A and B respectively. The variable of integration is now changed in all of these integrals. In the integrals between 0 and $\pm ik_1 \sin(\phi)$ one sets $\lambda = \pm ik_1 \sin(\alpha)$ respectively, and in the integrals between $\pm ik_1 \sin(\phi)$ and infinity one writes $\lambda = \pm k_1 \sinh(\theta)$ respectively. Next, notice that the integral along AB vanishes because of Jordan's theorem.

By this means one finds that

$$I^+ + I^- = I = \int_0^\phi e^{-Pk_1\cos(\alpha-\phi)} \sin(2\alpha)d\alpha + \int_0^\infty e^{-Pk_1\cosh(\theta)-2\theta} \cosh(2\phi)d\theta \qquad (4)$$

The expression for I can now be reduced by straightforward integration to:

$$I = \cos(2\phi)\left[K_2(Pk_1) - \frac{2e^{-Pk_1\cos(\phi)}(1+Pk_1\cos(\phi))}{(Pk_1)^2}\right]$$
$$+\sin(2\phi)\int_0^\phi e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha \qquad (5)$$

Here we have made use of the integral representation for the modified Bessel function, i.e.

$$K_2(z) = \int_0^\infty e^{-z\cosh(\theta)}\cosh(2\theta)d\theta \qquad (6)$$

See Abramowitz & Stegun (1964, p. 376, No. 9-6-24).

The remaining integral may be expressed in terms of modified incomplete Struve functions of orders zero and one. The other quantities needed are modified and incomplete Bessel functions also of zero and unit order. A full discussion of this matter can be found in the book by Agrest and Maksimov (1971, p. 289). Since numerical results are required, however, it is easier to use eqn 5.

In terms of the initial rectangular co-ordinates one finds that the expression for the electric Green's function is:

$$G = \frac{-i\omega\mu_0}{2\pi}\Bigg\{K_0(k_1 R) +$$
$$\frac{(z+z^1)^2 - (x-x^1)^2}{P^2}\left[K_2(Pk_1) - \frac{2e^{-k_1(z+z^1)}(1+(z+z_1)k_1)}{(Pk_1)^2}\right]$$
$$+ \frac{2|x-x^1|(z+z^1)}{P^2}\left[\int_0^{\arctan\left(\frac{|x-x^1|}{(z+z^1)}\right)} e^{-pk_1\cos(\alpha)}\cos(2\alpha)d\alpha\right]\Bigg\} \qquad (7)$$

3. The Magnetic Green's Function

Once the electric field is known it is a simple matter to calculate the associated magnetic fields. To this end we require the derivative of G with respect to x or z. In fact, if GH_z and GH_x are respectively the vertical and horizontal Green's function for the magnetic fields, then

$$GH_x = \frac{1}{i\omega\mu_0} \cdot \frac{\partial G}{\partial z}$$

and

$$GH_z = \frac{-1}{i\omega\mu_0} \frac{\partial G}{\partial x} \qquad (8)$$

fter some tedious but not unpleasant work one finds that:

$$H_x = -\frac{1}{2\pi}\left\{ -\frac{(z-z^1)k_1}{R} K_1(k,R) + \right.$$

$$e^{-k_1(z+z^1)}\left[\frac{((z+z^1)^2-(x-x^1)^2)}{P^2}\left\{\frac{4k_1^2(z+z^1)}{(Pk_1)^4}(1+k_1(z+z^1))\right.\right.$$

$$\left.\frac{2k_1^2(z+z^1)}{(Pk_1)^2}\right\} - \frac{8(z+z^1)(x-x^1)^2}{P^4}\left.\left(\frac{1+k_1(z+z^1)}{(Pk_1)^2}\right)\right]$$

$$K_0(Pk_1)\left\{\frac{3(z+z^1)}{P^4}\left(2(x-x^1)^2-(z+z^1)^2\right)\right\}$$

$$K_1(k_1P)\left\{\frac{8(z+z^1)(x-x^1)^2}{k_1R^5} - \frac{((z-z^1)^2-(x-x^1)^2)k_1(z+z^1)}{P^3}\left(1+\frac{4}{(Pk_1)^2}\right)\right\}$$

$$\left(\frac{2|x-x^1|}{P^2} - \frac{4|x-x^1|(z+z^1)^2}{R^4}\right)\int_0^{\arctan\left(\frac{|x-x^1|}{(z+z^1)}\right)} e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha$$

$$2\frac{|x-x^1|(z+z^1)^2}{P^3}\int_0^{\arctan\left(\frac{|x-x^1|}{(z+z^1)}\right)} k_1 e^{-k_1P\cos(\alpha)}\cos(\alpha)\cos(2\alpha)d\alpha$$

$$\left.-\frac{2(x-x^1)^2(z+z^1)}{P^6}\left((z+z^1)^2-(x-x^1)^2\right)e^{-k_1(z+z^1)}\right\} \quad (9)$$

and $(x \gtrless x_1)$

$$iH_z = \pm\frac{1}{2\pi}\left\{\frac{\mp(x-x^1)k_1}{R}K_1(k_1R) + \right.$$

$$e^{-k_1(z+z^1)}\left(1+k_1(z+z^1)\right)\left[\frac{4(x-x^1)k_1^2}{(Pk_1)^4P^2}\left((z+z^1)^2-(x-x^1)^2\right)\right.$$

$$\left.\frac{8(x-x^1)(z+z_1)^2}{P^4(Pk_1)^2}\right] \frac{+2e^{-k_1(z+z^1)}(x-x^1)\left((z+z^1)^2-(x-x^1)^2\right)(z+z^1)^2}{P^6}$$

$$K_1(Pk_1)\left[\frac{8(x-x^1)(z+z^1)^2}{P^5k_1} + \frac{((z+z^1)^2-(x-x^1)^2)k_1(x-x^1)}{P^3}\right.$$

$$\left.\left(1+\frac{4}{(Pk_1)^2}\right)\right]$$

$$K_0(Pk_1)\left[\frac{2(x-x^1)\left((z+z^1)^2-(x-x^1)^2\right)}{P^4} + \frac{4(x-x^1)(z+z^1)^2}{P^4}\right]$$

$$\left(\frac{2(z+z^1)}{P^2} - \frac{4(x-x^1)^2(z+z^1)}{P^4}\right)\int_0^{\arctan\left(\frac{(x-x^1)}{(z+z^1)}\right)} e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha$$

$$\left.\frac{2(x-x^1)^2(z+z^1)k_1}{P^3}\int_0^{\arctan\left(\frac{(x-x^1)}{(z+z^1)}\right)} e^{-Pk_1\cos\alpha}\cos(\alpha)\cos(2\alpha)d\alpha\right\} \quad (10)$$

4. Discussion

The expressions given here have uses other than the evaluation of the Green's function. Thus, the electromagnetic fields about line sources on or in a uniform ground can be expressed in terms of the Green's functions that have just been described. More specifically, if a current of $I_0e^{i\omega t}$ flows in a wire, then the electric field E is just

$$E = I_0e^{i\omega t}G \quad (11)$$

Associated with the electric field are two components of a magnetic field H_x and H_z. These quantities are simply

$$H_x = I_0e^{i\omega t}GH_x$$

and

$$H_z = I_0e^{i\omega t}GH_z \quad (12)$$

For a discussion of results alternative to those given here, the reader is referred to Wait & Spies (1971). There the interested reader will find a table of G for various values of x and z.

References

Abramowitz, M. & Stegun, J. A. (1964), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables,* Dover, New York.

Agrest, M. M. & Maksimov, M. S (1971), *Theory of Incomplete Cylindrical Functions and their Applications,* Springer Verlag, New York

Hohmann, G. W. (1971), 'Electromagnetic scattering by conductors in the earth near a line source of current', *Geophysics* **36,** 101-31.

Wait, J. R. & Spies, K. P. (1971), 'Subsurface electromagnetic fields of a line source on a conducting halfspace', *Radio Science* **6,** 781-86

(Received 22 August 1981)

The Effect of Displacement Currents on Time Domain Electromagnetic Fields

Terry Lee

Geopeko
P.O. Box 217, Gordon, N.S.W. 2072 *and*

Division of Mineral Physics
Commonwealth Scientific and Industrial Research Organization
P.O. Box 136, North Ryde, N.S.W. 2113

stract

: usual practice of neglecting displacement currents in calculation of electromagnetic transients is critically -mined for the case of a uniform ground. For this case it hown that the percentage error involved in the calcula- 1 by the neglect of the displacement currents is approxi- tely (1500/4) arctan $[\epsilon_1/(\sigma_1 t)]$, where t denotes time σ_1 and ϵ_1 denote the conductivity and permittivity of ground respectively.

ntroduction

:ently there has been a growing interest in the possibility letecting induced polarization effects in transient electro- gnetic method (TEM) data. In a review of the literature · (1979) noted that there were few theoretical results t might provide an insight into this problem. More re- tly Spies (1980) published results of a TEM survey in :ensland. Spies concluded that induced polarization cts might account for the negative responses which were erved in some of the transient data of that survey.

er Lee (1981) showed that induced polarization effects e present in the transient response of a ground whose ductivity function could be described by a 'Cole-Cole' del (Pelton et al. 1978). In all these results the effects lisplacement currents were ignored. The purpose of the ent paper is to show that that procedure is valid except some extreme cases.

he Transient Response

pose that a large loop of radius a lies on a uniform und of conductivity σ_1 and permittivity ϵ_1. When a uni- n current of spectrum $-I_0 e^{-i\omega t}/(i\omega)$ flows in the loop lectric field $\bar{E}(\omega)$ will be observed.

rison, Phillips & O'Brien (1969 p. 87, eqn 22) have wn that at a distance r from the centre of the loop, $\bar{E}(\omega)$ be written as:

$$) = \frac{I_0 a\mu_0}{2\pi}\int_0^{2\pi}\int_0^{\infty} J_0(\lambda R)\,\frac{\lambda}{n_0}\,\frac{n_0\, e^{-i\omega t}}{n_0+n_1}\cos(\theta)\,d\lambda\,d\theta \quad (1)$$

ice that we have used the addition theorem of the Bessel ctions to group the functions into a single term.

In the above equation:

$$R = \sqrt{r^2 + a^2 - 2ar\cos(\theta)}$$
$$n_j = \sqrt{\lambda^2 - k_j^2}$$
$$k_0^2 = \omega^2\mu_0\epsilon_0$$

and

$$k_1^2 = \omega^2\mu_0\epsilon_1 + i\omega\mu_0\sigma_1$$

An assumption made is that the permeabilities of the air and ground are the same as the permeability of free space, μ_0. The notation ϵ_0 is for the permittivity of the air; ω denotes angular frequency, and t time.

Equation (1) may be conveniently written as:

$$\bar{E}(\omega) = \frac{-a\mu_0 I_0 e^{-i\omega t}}{(k_0-k_1)(k_0+k_1)}\int_0^{2\pi}\int_0^{\infty}\frac{(\lambda n_0 - \lambda n_1)J_0(\lambda R)\,d\lambda\cos(\theta)\,d\theta}{2\pi} \quad (2)$$

With the help of Sommerfield's integral one can reduce eqn (2) to eqn (3):

$$\bar{E}(\omega) = \frac{-a\mu_0 I_0 e^{-i\omega t}}{(k_0-k_1)(k_0+k_1)}\int_0^{2\pi}\frac{\partial^2}{\partial z^2}\left(\frac{e^{ik_0P}}{P} - \frac{e^{ik_1P}}{P}\right)\frac{\cos(\theta)\,d\theta}{2\pi} \quad (3)$$

Here $P = \sqrt{R^2 + z^2}$ and z is set equal to zero once the differentiation is carried out.

The transient electric field E(t) may be found by taking the Fourier transform:

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\bar{E}(\omega)\,d\omega$$

This transform may be evaluated by contour integration once a suitable path is selected in the lower complex ω plane.

Notice that there is no pole at $k_0 = k_1$, because the exponential terms are also zero there. Also, there is not a pole at $k_0 = -k_1$. Thus it remains only to consider the branch points of $\bar{E}(\omega)$ in the lower complex ω plane. In the following we shall assume that E(t) is to be evaluated for times greater than $\sqrt{\epsilon_1\mu_0}P$ and $\sqrt{\epsilon_0\mu_0}P$. This assumption is because we are interested only in those times for which the electric field has had time to propagate to all the points on the loop.

Since $k_1 = \sqrt{\omega^2\mu_0\epsilon_1 + i\omega\mu_0\sigma_1}$ one sees that there are branch points at $\omega = 0$ and $\omega = -i\sigma_1/\epsilon_1$. The spectrum of

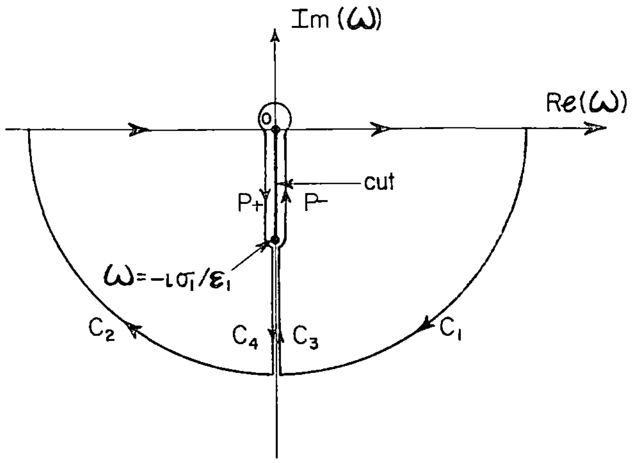


FIGURE 1
Path for the contour integral.

$\bar{E}(\omega)$ is an analytic function of ω provided we make a cut along the ray from $\omega = 0$ to $\omega = -i\sigma_1/\epsilon_1$. We have ignored the point at infinity because it is a removable singularity (see Sveshnikov & Tikhonov 1973, p. 18).

Notice that the spectrum of $\bar{E}(\omega)$ vanishes exponentially for ω tending to infinity because of the exponential functions. Moreover, the integrals of $\bar{E}(\omega)$ along the arcs C_1 and C_2 of Fig. 1 vanish because of Jordan's lemma (see Sveshnikov & Tikhonov 1973, p. 130). Notice, also, that the integrals along the paths C_3 and C_4 of Fig. 1 cancel each other as the paths of integration approach the negative ω axis. Also, the integral about the origin of that figure vanishes as the path of integration approaches the origin because the spectrum of $\bar{E}(\omega)$ is finite there. Thus by Cauchy's theorem the only contribution to the contour integral is the integral about the branch cut from $\omega = 0$ to $\omega = -i\sigma_1/\epsilon_1$.

To proceed one writes: $\omega = qe^{-i\pi/2}$ on P−
and $\omega = qe^{i\pi+i\pi/2}$ on P+
and learns that $k_1 = \pm\sqrt{q\mu_0\sigma_1 - q^2\epsilon_1\mu_0}$ on P− and P+ respectively.

Notice that the integrals about the ends of the contours vanish and so after adding the integrals along each side of the cut together, one has from Cauchy's theorem that:

$$E(t) = \frac{aI_0\mu_0}{2\pi^2}\int_0^{2\pi}\frac{\partial^2}{\partial z^2}\frac{\cos(\theta)}{P}\int_0^{\sigma_1/\epsilon_1}\frac{e^{-qt}\sin(P\sqrt{q\mu_0\sigma_1 - q^2\mu_0\epsilon_1})}{q^2\mu_0(\epsilon_1-\epsilon_0) - q\mu_0\sigma_1}\,dq\,d\theta \tag{4}$$

The integration with respect to θ may be performed by first interchanging the order of integration, expanding the sine function as a power series and then differentiating with respect to z at z = 0. The resulting θ integral may be evaluated by using the double-angle formulae for the trigonometric functions. These manipulations lead to the expression for E(t) given in eqn (5):

$$E(t) = \frac{aI_0\mu_0}{\mu_0\pi}\int_0^{\sigma_1/\epsilon_1}\frac{e^{-qt}}{q^2(\epsilon_1-\epsilon_0)-q\sigma_1}\sum_{s=0}^{\infty}\frac{(-1)^s a^{2s+2}(q\mu_0\sigma_1 - q^2\mu_0\epsilon_1)^{\frac{2s+5}{2}}}{(2s+5)\,(2s+3)s!\,(s+2)!}\,dq \tag{5}$$

A convenient check can be placed on this result by setti $\epsilon_1 = \epsilon_0 = 0$. For this case $E(t) = E_P(t)$ where:

$$E_P(t) = -\frac{aI_0}{\sigma_1\pi}\int_0^{\infty}\frac{e^{-qt}}{q}\sum_{s=0}^{\infty}\frac{(-1)^s a^{2s+2}(q\mu_0\sigma_1)^{\frac{2s+5}{2}}}{(2s+5)\,(2s+3)s!\,(s+2)!}\,dq$$

This last expression can be integrated term by term by usi the definition of the gamma function. This integrati yields:

$$E_P(t) = -\frac{aI_0\mu_0\sqrt{\pi}}{2\pi t}\sqrt{\frac{\sigma_1\mu_0}{t}}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+2)!}{(2s+5)\,(s+1)!s!\,(s+2)!}\left(\frac{a^2\sigma_1\mu_0}{4t}\right)^{s+1}$$

Equation (7) is identical to that given by Lee & Le (1974, eqn 11).

3. Effects of Displacement Currents

Equation (5) may be transformed into a more convenie form by the following substitutions:

$$qt = \tan\theta$$
$$\phi = \arctan\,[\epsilon_1/(\sigma_1 t)]$$
$$\alpha = (\epsilon_1 - \epsilon_0)/\epsilon_1$$

One now finds that eqn (5) can be written as:

$$E(t) = \frac{I_0}{\sigma_1\pi a^2}\int_0^{\frac{\pi}{2}-\phi}\frac{e^{-\tan\theta}\sec^2\theta}{\alpha\tan\phi - \cot\theta}\cdot\sum_{s=0}^{\infty}(-1)^s\left(\frac{a^2\mu_0\sigma_1}{t}\right)^{\frac{2s+5}{2}}\frac{\tan^{2s+3}\theta}{(2s+5)\,(2s+3)s!\,(s+2)!}\left(\frac{\cos(\theta+\phi)}{\cos\phi\,\sin\theta}\right)^{\frac{2s+5}{2}}d\theta$$

Notice that when $\phi = 0$:

$$E(t) = E_P(t) \quad ($$

For the problem we are considering we may suppose t $\epsilon_1 \gg \epsilon_0$, in which case α can be taken to be unity.

Since we are interested only in the effects of displacem currents we may restrict our attention to the case wh σ_1 is small. For that case, for all the times used in p pecting, eqn (7) may be approximated by the first term.

Consequently: $E(t) \approx (1 - \frac{\phi z^2\partial^2}{\partial z^2})\,E_P(t)$ where $z = 1/t$ (

and therefore: $E(t) \approx [1 - (15/4)\phi]\;E_P(t)$ (

Since the voltage induced in a receiving loop which coincident with the transmitting loop is just $2\pi a\,E(t)$ follows that:

$$V(t) \approx [1 - (15/4)\phi]\;V_P(t) \quad ($$

Here $V_P(t)$ denotes the transient voltage when displacem currents are ignored. Thus if ϕ is small the percentage e χ that would result in the transient electric field (or m sured voltage) due to the neglect of the displacement rents is approximated by:

$$\chi = (1500/4)\arctan\,[\epsilon_1/(\sigma_1 t)] \quad ($$

Since $\epsilon_0 = 8.854 \times 10^{-12}$ F.m^{-1}, one finds that for the case where ϵ_1 is 100 times ϵ_0 and σ, a and t have values of 10^{-3} S.m^{-1}, 50 m and 10^{-4} s respectively, the resultant percentage error is about 4%.

4. Discussion

Equation (14) shows that only at the very early stages of the transient response of a highly resistive rock would the effect of displacement currents be seen.

Instruments with the capability of measuring transients at times less than 0.1 ms have a potential application for mapping rocks with a large $\epsilon_1/(\sigma_1 t)$: for example, coal seams.

Acknowledgments

The author would like to acknowledge Dr A. P. Raiche of the Commonwealth Scientific and Industrial Research Organization, North Ryde, N.S.W. for suggesting the problem and for pointing out that the analysis requires that $t > P\sqrt{\mu_0 \epsilon_1}$.

References

Lee, T. J. (1979), Transient electromagnetic waves applied to prospecting, *Proc. IEEE* **67**, 1016-21.

Lee, T. (1981), Transient electromagnetic response of a polarizable ground, *Geophysics* (in press).

Lee, T. & Lewis, R. (1974), 'Transient EM response of a large loop on a layered ground', *Geophys. Prosp.* **22**, 430-44.

Morrison, M. F., Phillips, R. J. & O'Brien, D. P. (1969), 'Quantitative interpretation of transient electromagnetic fields over a layered halfspace', *Geophys. Prosp.* **17**, 82-101.

Pelton, W. H., Rijo, L. & Swift, C. M. (1978), 'Inversion of two-dimensional resistivity and induced-polarization data', *Geophysics* **44**, 788-803.

Spies, B. R. (1980), 'A field occurrence of sign reversals with the transient electromagnetic method', *Geophys. Prosp.* **28**, 620-632.

Sveshnikov, A. & Tikhonov, A. (1973), *The Theory of a Complex Variable.* Translated from the Russian by G. Yankovsky, MIR Publishers, Moscow.

Paper

The Effect of Host Rock on Transient Electromagnetic Fields

R. Lewis
Geology Department
University of Tasmania, G.P.O. Box 252C, Hobart, Tas. 7001

T. Lee
Peko-Wallsend Operations Ltd
Peko Geoscience Division
P.O. Box 217, Gordon, N.S.W. 2072

Abstract

The presence of a conducting material about an orebody modifies the transient electromagnetic fields in two ways. Firstly at the early stages by strongly directing the primary field and secondly at the later stages by swamping the target field. These effects must be recognised when interpreting field data, in view of the numerical studies of two-dimensional structures. The effects may be advantageously used in designing field measurements.

1. Introduction

The transient electromagnetic (TEM) methods differ in one fundamental respect from the frequency domain methods. The frequency domain systems create steady-state oscillating fields while transient systems involve the propagation of travelling fields localised in well defined areas. In a uniform ground before interaction with scatterers the source fields are usually in the form of deformed images of the source which become more blurred with time as illustrated in Lewis and Lee (1978) and in Nabighian (1979). Consequently it is informative to examine the fields radiated from various transient sources in an endeavour to understand the first stage of the TEM process.

It is well known that in other exploration techniques the use of special source configurations is advantageous. This is so in the use of appropriate electrode arrays in resistivity and induced polarisation methods and has been the subject of intense studies, e.g. Pratt and Whiteley (1974). The use of source arrays in seismic exploration, a method showing many of the propagating features of TEM, is also of considerable antiquity. In this connection it is interesting to note other common features such as the use of ray tracing techniques which are currently under renewed study (Kan and Clay 1979) following the early work of Yost (1952).

At the present time there is some uncertainty concerning the way a conducting orebody decays in a conducting half-space. McCracken (1979) has tried to summarise the various views. McCracken wrote that if a target has a slower decay (of eddy currents) then the transient of the target is on its own at later stages. Alternatively if the target decays faster than the overburden then a frequency domain method would be superior because the target spectrum shifts to the higher frequencies.

Spies (1978) in an attempt to clarify the situation carried out a series of model studies of a conductor in a conducting half-space. Spies' diagram showed that initially the transient from the half-space decayed more rapidly without the body, but at later stages, the various transients arising from whether or not there was a conducting cylinder present could not be distinguished from each other. Thus the situation described by McCracken does not always apply.

Previously Thio (1977) had also attempted to resolve the situation by numerically modelling a sphere in a conducting half-space. As part of his study, Thio derived an asymptotic expansion for the later stages and calculated the ratio of the transient voltage, due to the half-space, to the secondary transient arising from the sphere in the half-space. The results were checked by comparing them with those of Lee (1974). Thio's results showed that, for purely electrical conductivity, the target part of the transient decayed much more rapidly. For a magnetic as well as a conducting sphere, the ratio was asymptotic to a constant, which was small when the sphere was several radii deep.

Some of the above confusion arises because there is a tendency to assume *a priori* the final form of the transient. The advantages and disadvantages of this approach have been argued by Wait, Spies and Hjelt (1971). The argument revolves around whether or not the later stages of a transient can be described by a series of decaying exponentials. Kaufman (1978) attempted to resolve these arguments when he argued that when the surrounding medium was an insulator the later stages could be described by a series of exponentials. Since a series of exponentials corresponds to a series of poles in the time domain Kaufman is saying that the singularities of the transfer function of the ground does not have a branch cut type of singularity.

The difficulty for the practical application of these results is that in the actual situation in the field the surrounding material is not an insulator. Moreover, when Thio calculated the later stages of the response of a buried sphere, he ignored all contributions from poles and only considered the contribution from the branch cut, a process that led to an accurate estimation of the later stages of the transient and a decay form as inverse powers of time. This is the same type form of decay that Lee and Lewis (1974) had previously found for layered structures.

There are two important problems in the analysis of transient electromagnetic results and they are the problem of controlling the sources for TEM and resolving the debate over the type of the response to be expected at later stages.

The purpose of this paper is to indicate possible avenues of approach to both of these problems.

2. The Directed Electric Field

(a) Field from a wire

Consider a Cartesian co-ordinate system (x,y,z) in which a uniform half-space of magnetic permeability μ_0 occupies the space $y>0$. A source wire is assumed to lie along the z axis. Wait (1962) has shown that if a current $Ie^{i\omega t}$ flows in the wire then the electric field at (x,z) is given by

$$E(\omega) = \frac{-i\omega\mu_0 I}{\pi}\int_0^\infty \frac{\cos\lambda x}{n+\lambda}e^{-nz}\,d\lambda \qquad (1)$$

where $n = \sqrt{\lambda^2 + i\omega\mu_0\sigma}$.

If L^{-1} denotes the inverse Laplace transform then the transient electric field due to a step current in the wire is

$E(t) = L^{-1}\ (E(\omega)/i\omega)$

Using result 5·95 of Oberhettinger and Badii (1973) this may be reduced

$$E(t) = \frac{-\mu_0 I}{\pi}\int_0^\infty \cos\lambda x\, e^{-\lambda^2 t/\sigma\mu_0}\left[\sqrt{\sigma\mu_0/t}\ e^{-z^2\sigma\mu_0/4t} - e^{\lambda z+\lambda^2 t/\sigma\mu_0}\operatorname{erfc}(\lambda\sqrt{t/\sigma\mu_0} + z\sqrt{\sigma\mu_0/4t})\right]d\lambda \qquad (2)$$

Equation 2 can be simplified by putting

$$H = z\sqrt{\sigma\mu_0/4t}, \qquad X = x\sqrt{\sigma\mu_0/4t},$$
$$\tau = \sigma\mu_0/4t, \qquad R^2 = X^2+H^2 \qquad (3)$$

and evaluating the integral. After some tedious algebra eqn (2) reduces to

$$E(t) = \frac{-I\tau}{\pi\sigma}\Big[e^{-R^2}(2+1/R^2-2X^2/R^2-2X^2/R^4+p(X)HX(4/R^2+4/R^4)/\sqrt{\pi}) -2He^{-H^2}/(\sqrt{\pi}R^2) - \operatorname{erfc}(H)\,(H^2-X^2)/R^4\Big] \qquad (4)$$

where $p(x)$ is Dawson's integral:–

$$p(x) = \int_0^x e^{t^2}dt \qquad (5)$$

which is available as a standard function on some computers and so need not be evaluated by integration.

A related result where the current is an impulse rather than a step is given by Wait (1971).

The result is readily generalised for multiple line sources by appropriately summing the fields of each source.

Figure 1 shows the distribution of $-E(t)\pi\sigma/(I\tau)$ obtained by evaluating eqn (4). It is apparent that the electric field is concentrated in a roughly elliptical zone which propagates vertically downward with the passage of time. It may be noted in passing that the maximum value on the surface of the ground is always at the source but that in the ground the maximum value always occurs directly below the wire.

Figure 2 shows the same data recast in the form of a nomogram useful for calculating waveforms. The quantity plotted here is $E\pi r^2\sigma/I$ where $r^2 = x^2 + z^2$. The nomogram is used as follows. Firstly evaluate r^2 and draw a line across the diagram from the origin with a slope given by x/z. Knowing the conductivity and the required time τ, X and Z are evaluated and $E\pi r^2\sigma/I$ is read off the diagram at (X,Z). As

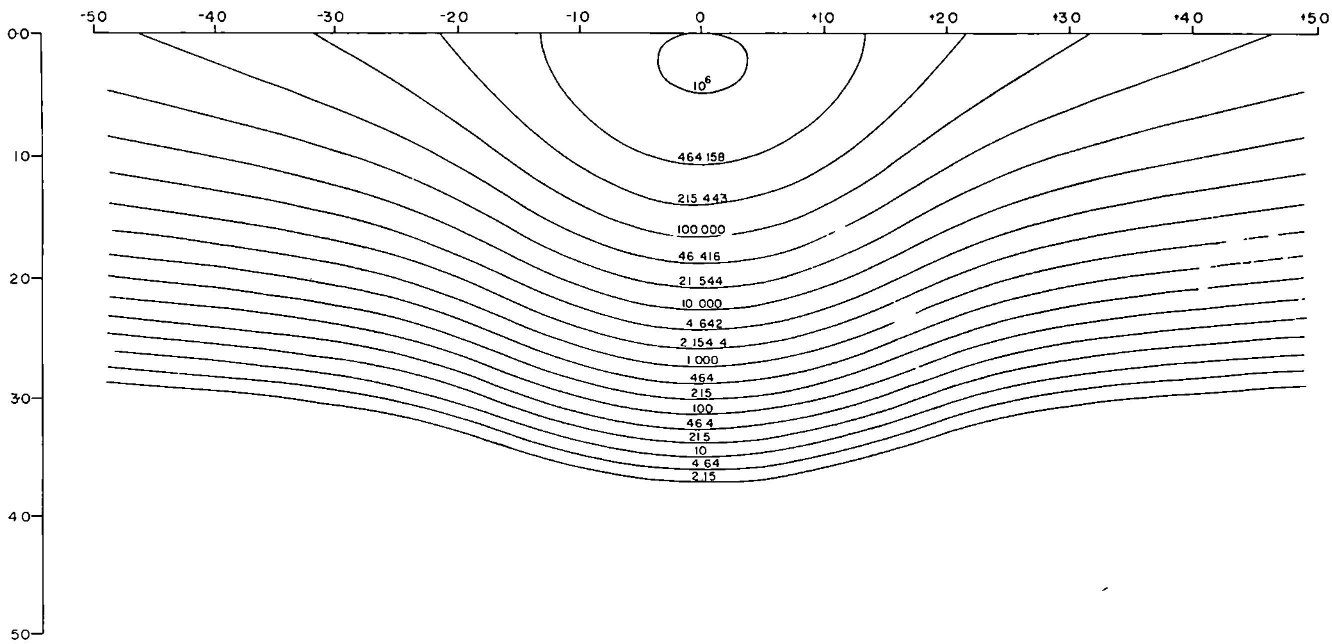


FIGURE 1

The distribution of $-E(t)\pi\sigma/(I\tau)\times10^6$ for a line source at the origin. The axes are labelled in terms of the scaled coordinates (X,H).

time increases the point (X,Z) due to a fixed (x,z) moves along the line constructed previously. At early times the appropriate point is at a great distance from the origin; at infinite time it is the origin. As presented the nomogram is best suited to studying early and moderate times. It may be readily recast so that the late-time part is expanded if required.

The nomogram illustrates another point about a line source. It follows that along any line drawn through the source the same basic waveform is found but it is scaled in amplitude and distorted in time, i.e. by replotting the time scale the same waveform is recovered.

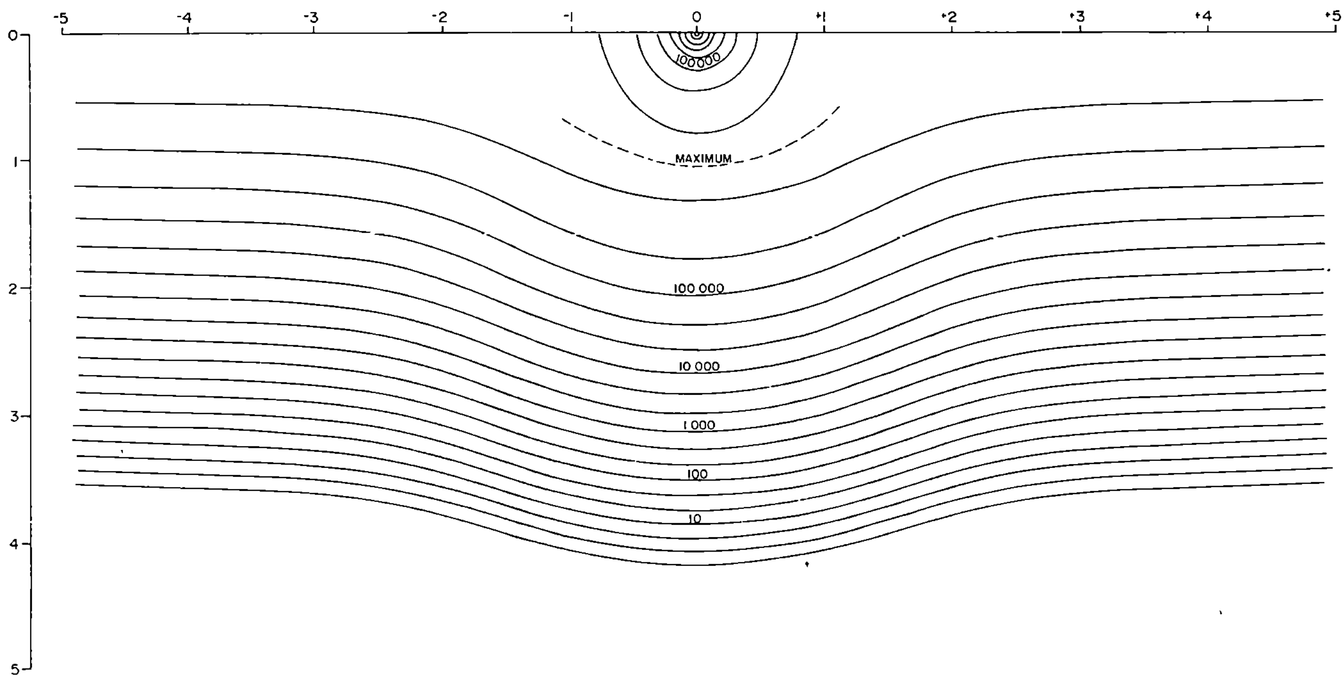


FIGURE 2

A nomogram for calculating incident field waveforms from a line source with a step current. Using the scaling described in the text the contours here are multiplied by 10°.

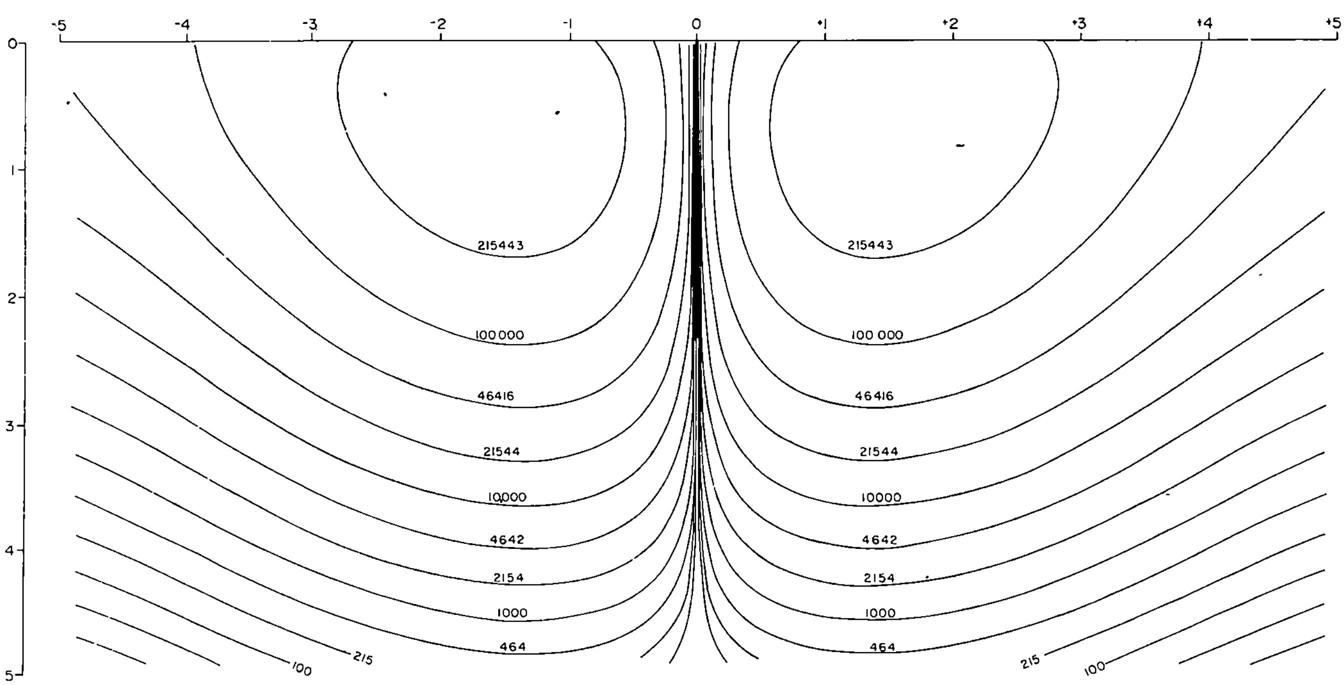


FIGURE 3

The same quantity as in Fig. 1 for a loop source with the loop sides at X = ± 0.5. The value of τ is 0.316. Note the striking difference from Fig. 1 and the resemblance to the corresponding results for a circular loop given in Lewis and Lee (1978).

(b) Directed waves

It is apparent from Fig. 1 that there is a fundamental difference in the type of field distribution obtained from a circular loop as given in Lewis and Lee (1978). However, a single line source with no return circuit is not physically realisable so we examine in Fig. 3 the distribution of field for a pair of lines carrying oppositely directed currents which represent the long sides of a rectangular loop. It is immediately apparent that a similar pattern to that for a circular loop is obtained. This result is important in that it demonstrates that this pattern is *characteristic of loops* and arises because of the interference of the fields radiated from opposite sides of the loop. Since this is essentially a property of the source geometry we examine the possibility of obtaining other field distributions from more complex sources. We restrict this discussion to two simple realisable sources but point out that there are many other possibilities based on similar considerations. The two source circuits are shown in Fig. 4. That in the top of Fig. 4 is just two loops side by side with two different currents. That in the lower figure is a three-wire system with current partitioned between two of the wires. We plot the locus of maximum electric field in the right-hand half of the half-space in each case but note that the distribution in the other half is not symmetric.

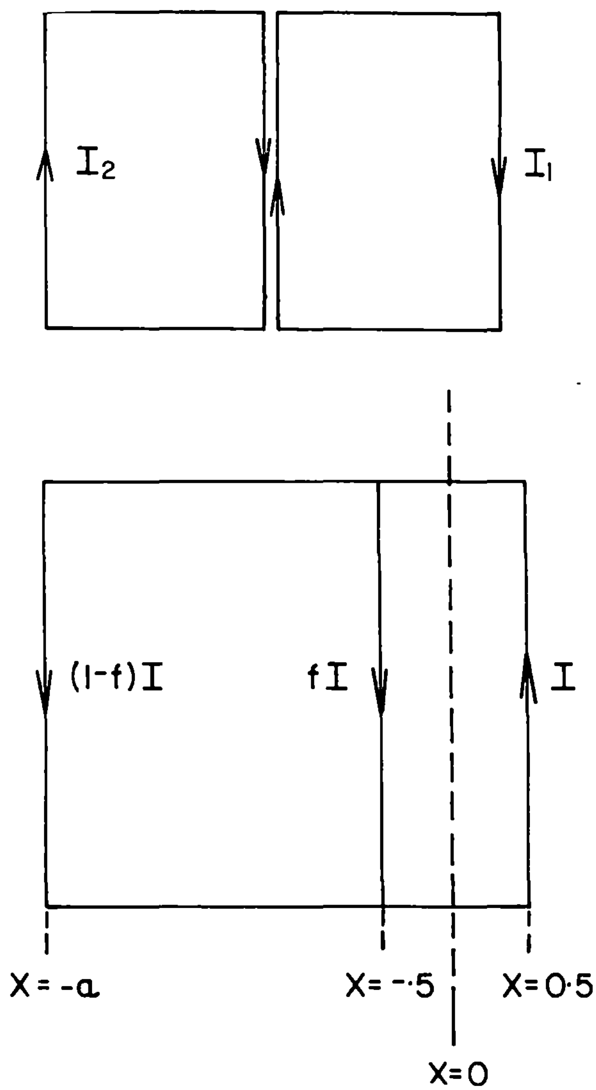


FIGURE 4

The source geometries used to study directed fields. The lower configuration may also be realised as two loops of different sizes.

Figure 5 depicts the results for the upper circuit of Fig. 4 and shows that while the locus can be varied by changing the currents in the loops this is not an efficient method of steering the field in that the range of control available is minimal.

Figures 6 to 12 show the same result for the three-wire system. If a, the loop width ratio shown in the lower part of Fig. 4 is reasonably large then quite effective steering can be obtained and the instrumentation required is quite simple. It is, however, important to realise that the steered concentration of electric field is quite diffuse. Whether the variation in excitation produced is of diagnostic value in practical situations is a question which remains and requires further model or field studies.

3. Late Stage Response with a Conductor

The discussion of the previous work on the effect of the host rock on electromagnetic transients showed that geophysicists often think that the response of the orebody is quite separate from that of the surrounding medium. In fact, the two are inextricably coupled. This can be seen by considering the singularity expansion of the primary and secondary fields. We first treat the axi-symmetric problem of a sphere that is excited by a coaxial loop. Later we consider the case of a non-axially symmetric problem of a cylinder that is excited by a line source.

The electric field E, at time t, at a depth z, and distance r, from a loop of radius a, in a ground of conductivity σ_1, and permeability μ_0 is given by:

$$E = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\mu_0 a I \int_0^{\infty} J_1(\lambda a) J_1(\lambda r)\frac{\lambda}{n_1+\lambda}\, e^{-n_1 z} d\lambda d\omega \qquad (6)$$

where $n_1 = \sqrt{\lambda^2 + i\omega\sigma_1\mu_0}$ (Lee 1974).

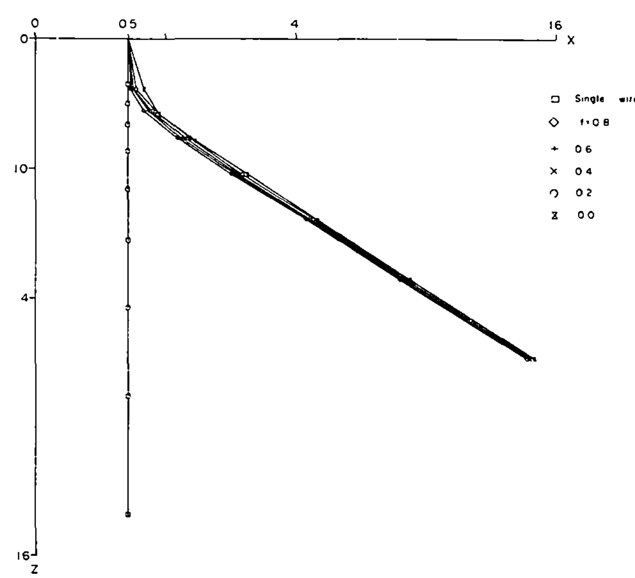


FIGURE 5

The locii of maximum e.m.f. produced by the upper circuit of Fig. 4. In comparison with the results in Figs 6 to 11 the control is minimal.

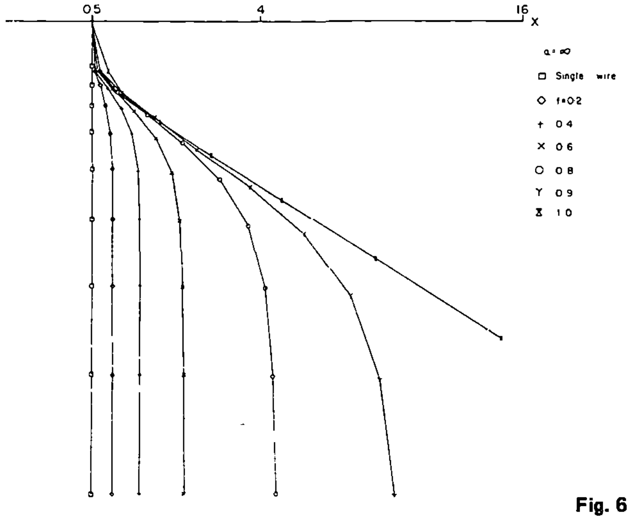


Fig. 6

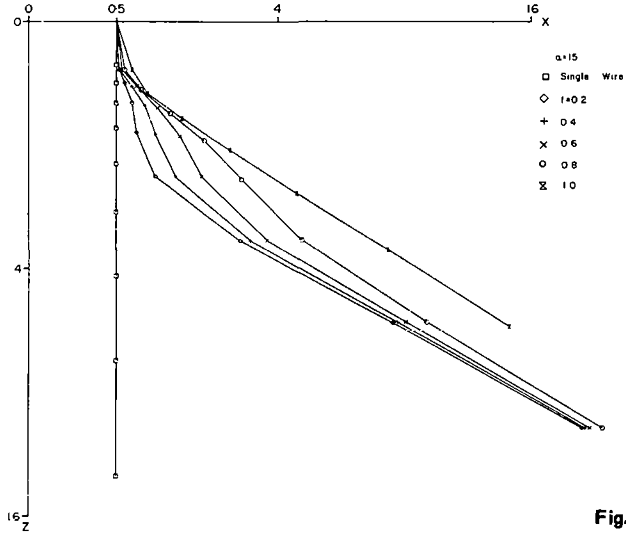


Fig. 9

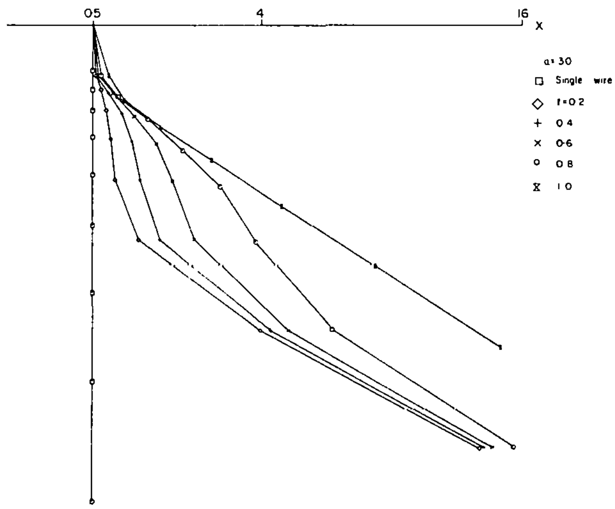


Fig. 7

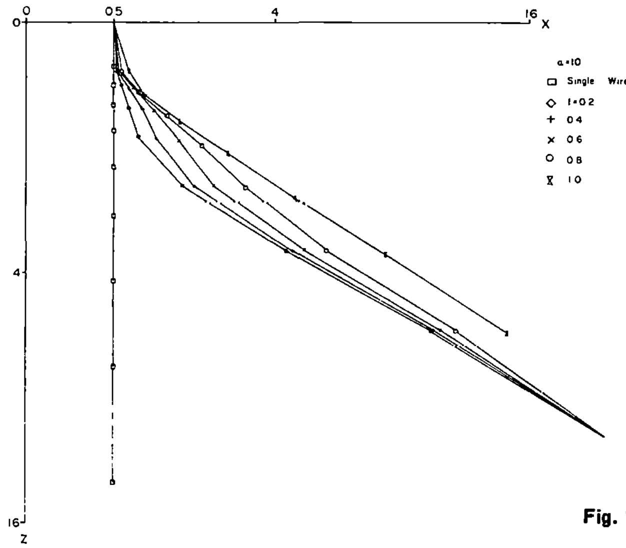


Fig. 10

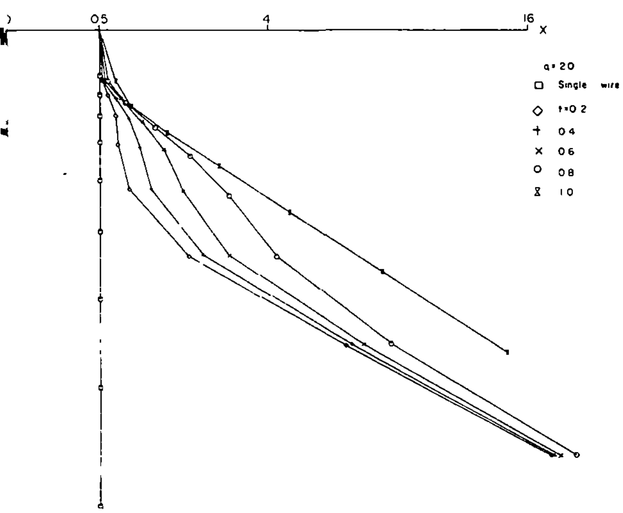


Fig. 8

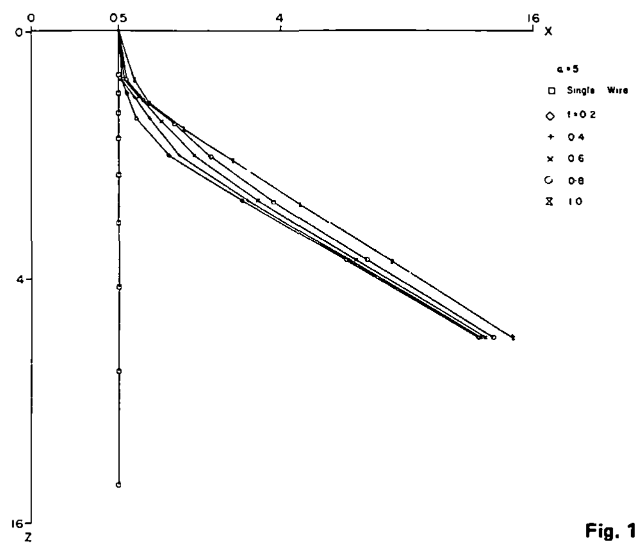


Fig. 11

GURES 6–11

ocii of maximum e.m.f. produced by the lower circuit in Fig. 4. s a increases, the possible deviation increases, until the limiting se in Fig. 6 is reached.

Here we have assumed that the field arises from a step current of amplitude I flowing in the loop.

Writing $S = \lambda^2 + i\omega\sigma_1\mu_0$ in eqn (6) yields:

$$\frac{-I\mu_0 a_I}{2\pi\sigma_1\mu_0}\int_0^\infty e^{-\lambda^2 t/(\sigma_1\mu_0)} J_1(\lambda r)\, J_1(\lambda a)\ \cdot \int_{-i\infty}^{i\infty} \frac{e^{-sz+ts/(\sigma_1\mu_0)}\,ds\, d\lambda}{\lambda+\sqrt{S}} \tag{7}$$

Writing $S = i\omega$ yields:

$$\frac{-I\mu_0 a_I}{2\pi i\sigma_1\mu_0}\int_0^\infty e^{-\lambda^2 t/(\sigma_1\mu_0)} J_1(\lambda r) J_1(\lambda a)\cdot \int_{-i\infty}^{i\infty} \frac{e^{-i\omega z+i\omega t/(\sigma_1\mu_0)}}{\lambda+\sqrt{i\omega}}\, d\omega\, d\lambda \tag{8}$$

The inner integral in eqn (8) can be evaluated by closing the path of the integration by a semi-circle about the upper half plane. This semi-circle is indented so that it passes to the left and the right of the branch cut from the origin along the imaginary axis. See Fig. 12. This is the only singularity and since the contributions on the arcs are zero by Jordan's theorem, then it follows that the transient can be considered as the manifestation of the integral about the branch cut. Writing $\sqrt{i\omega} = ue^{-i\pi/2}$ on P^- and $\sqrt{i\omega} = ue^{i\pi/2}$ on P^+ we find that the above integral becomes:

$$\frac{I\mu_0 a}{\sigma_1\mu_0}\int_0^\infty J_1(\lambda a)\, J_1(\lambda r)\lambda e^{-\lambda^2 t/(\sigma_1\mu_0)}\ \cdot \frac{2}{\pi}\int_0^\infty e^{-u^2 t/(\sigma_1\mu_0)}\, \frac{u(\lambda \sin\lambda u + u\cos\lambda u)}{\lambda^2+u^2}\, du\, d\lambda$$

Therefore

$$E = \frac{I\mu_0 a}{\sigma_1\mu_0}\int_0^\infty J_1(\lambda r)\, J_1(\lambda a)\lambda\, e^{-\lambda^2 t/(\sigma_1\mu_0)}\left[\sqrt{\frac{\sigma_1\mu_0}{\pi t}}\, e^{-z^2\sigma_1\mu_0/4t} - \lambda e^{\lambda z+\lambda^2 t/(\sigma_1\mu_0)}\,\mathrm{erfc}\left(\lambda\sqrt{\frac{t}{\sigma_1\mu_0}} + z\sqrt{\frac{\sigma_1\mu_0}{4t}}\right)\right] d\lambda$$

Here we have used the results of Erdelyi et al. (1954, p. 74 no. 26 and p. 15 no. 15).

Equation (9) readily reduces to

$$E(t) = \frac{aI}{2\sigma\sqrt{\pi}}\left(\frac{\sigma\mu_o}{t}\right)^{3/2} e^{-(z^2+a^2+r^2)\frac{\sigma\mu_o}{4t}}\ I_1\left(\frac{ar\sigma\mu_o}{2t}\right) - \frac{aI}{\sigma}\int_0^\infty \lambda^2 J_1(\lambda a)\, J_1(\lambda r)\, e^{\lambda z}\,\mathrm{erfc}\left(\tfrac{1}{2}z\sqrt{\frac{\sigma\mu_o}{t}} + \lambda\sqrt{\frac{t}{\sigma\mu_o}}\right) d\lambda \tag{10}$$

a result previously given by Lewis and Lee (1978).

When we examine the singularity expansion of the secondary field of a sphere of radius b, conductivity σ_2, permeability μ_0, at a depth of h beneath the loop described above we find that the singularities consist of not only the branc cut but also poles.

The secondary field, E_s, is approximately given by:

$$E_s = \int_{-\infty}^{\infty} -\frac{i\pi}{2}\frac{A_1}{\sqrt{R}} H^1_{3/2}(k_1 R)\, b\, P^1_1(\cos\theta)\ \cdot \left[k_1 J_{3/2}(k_2 b)\, J_{5/2}(k_1 b) - k_2 J_{5/2}(k_2 b)\, J_{3/2}(k_1 b)\right] e^{i\omega t} d\omega \tag{11}$$

Where

$$A_1 = \frac{-2\mu I a\int_0^\infty J_1(\lambda a)\, P^1_1\left(\frac{in_1}{k_1}\right)\frac{\lambda e^{-n_1 h}}{n_1+\lambda} d\lambda}{i\pi b\sqrt{\frac{2k_1}{\pi}}\cdot\frac{2}{3}\left[k_1 J_{3/2}(k_2 b) H^1_{5/2}(k_1 b) - k_2 J_{5/2}(k_2 b) H^1_{3/2}(k_1 b)\right.}$$

$k_j = \sqrt{i\omega\sigma_j\mu_0}$, $R = \sqrt{h^2+a^2}$ (Lee 1974).

To evaluate this expansion in terms of integrals about th singularities one needs to integrate about a branch cut fro the origin on and about the various poles that lie on th imaginary axis when $k_1 b = 0$ (Singh 1973). Since this equ tion is the same as Thio (1977) later obtained, we kno what the dominant contribution is, about the branch cu Thio (1977) obtained a transient that decayed with an i verse power law with time.

If one restricts oneself to the poles and assumes an infinite resistive half-space then the expression simplifies to:

$$E_s = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{i\omega t}\frac{I\mu_0}{4}\left(\frac{ab}{R^2}\right)^3 \frac{I_{5/2}(b\sqrt{S\sigma_2\mu_0})}{I_{1/2}(b\sqrt{s\sigma_2\mu_0})}\, ds$$

$$= \frac{3I}{2b^2\sigma_2}\left(\frac{ba}{R^2}\right)^3 \sum_1^\infty e^{-t\pi^2 n^2/(\sigma_2\mu_0 b^2)} \tag{12}$$

$s = i\omega$ Lee (1974)

The field decays exponentially.

These equations show that by concentrating on specific si gularities all sorts of responses can be derived for the sing physical situation and that much more care should be pa to interpreting the equations.

Returning to eqn (11) we see that the branch cut contrib tions arose because of the Hankel functions H_γ and th contributions from the poles arose from the Bessel fur tions, J_γ. The Hankel functions represent radiated fielc while the usual Bessel functions define the radial curre within the sphere.

Equation (11) expresses the coupling between the interr currents of the sphere and the resulting radiated currents

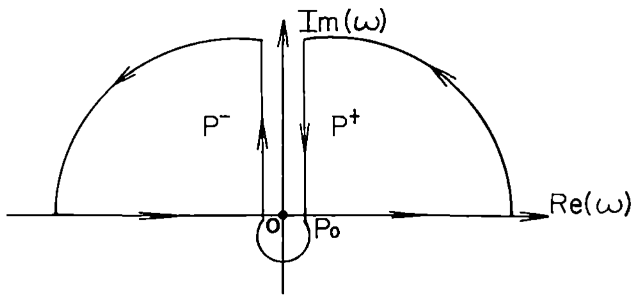


FIGURE 12

The path of integration used in studying the branch cut contributior

ıe surrounding half-space. The term in the numerator of A the first mode of the incident electric field. We have noзed previously that such fields decay as $1/t^{\gamma}$. The physical tuation is that the surrounding conducting material is able ·ı supply energy to the conductor long after the electric ɜld would have decayed were the conductor in an insulaŋg environment. The result is to prolong the life of the ɜcaying fields in the conductor. Unfortunately the fact ·at both the sphere response and the half-space response e both controlled by the same singularity in the spectrum sults in the conductor being difficult to detect (Lee 1978).

ıe only possible escape from this situation is to shape the cident field spectrum so that any 'pole' responses are enınced, but this may require the admittance of a generalised ɜquency. In the case of an insulating host the appropriate equencies are $i\omega = -\pi^2 n^2/(b^2 \sigma_2/\mu_0)$ (Singh 1973).

ıe above discussion of the later stages of the transient ɜctromagnetic response of a buried axial conductor sumes axial symmetry. We now show that similar effects · those described above can be found in transients from ⱱo dimensional structures.

ɜe and Lewis (1981) have studied the transient response a buried horizontal cylinder that has been excited by a ıe source. As part of this study they introduced a transfer nction, Z(t), that was related to the incident electric field, (t), and the anomalous secondary electric $E_s(t)$ field by e convolution

$$(t) = \int_0^t Z(\tau)\ E^i(t-\tau)\, d\tau \qquad (13)$$

is evaluated at the centre of the cylinder.

may be shown that the time-domain form of the transfer nction is given by

$$) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, \frac{2\pi k_2(1-\sigma_1/\sigma_2) I_1(k_2 a) G\, d\omega}{a k_1 I_0(k_2 a) K_1(k_1 a) + k_2 a I_1(k_2 a) K_0(k_1 a) + G_s I_1(k_2 a) 2\pi k_2 a}$$

ee Lee and Lewis (1981). (14)

ı this equation $k_j^2 = i\sigma_j \omega \mu_0 \qquad (j=1,2)$

here σ_1 is the conductivity of the ground, and σ_2 is the ɔnductivity of the cylinder. It is assumed that the cylinder at depth h and of radius a and further that G, G_s are the reens functions given by

$$_s = \frac{1}{2\pi}\left(K_0(\beta) + 2K_1(\beta)/\beta - 2(1+\beta)e^{-\beta}/\beta^2\right)$$

$(x_1 - x_0)$

$$= \frac{1}{2\pi}\int_0^{\infty}\left\{\frac{n_1-\lambda}{n_1+\lambda}e^{-n_1(y+h)} + e^{-n_1|y-h|}\right\}\frac{\cos\lambda(\cancel{x})}{n_1}\, d\lambda$$

$= 2hk_1$, $n_1^2 = \lambda^2 + k_1^2$ and x,y are the horizontal and verзal distances of the cylinder centre from the line source.

nce we are interested in the expression for the later part the transient the kernel in eqn (14) may be approximated / neglecting the secondary part of the Greens function G_s. ıe integral may then be evaluated using the contour in g. 12. Here we have neglected the contributions due to ɔles with the rationale that the branch cut yields responses terms of inverse powers of time, while the poles yield ‹ponential decays. It follows that the branch cut contriɹtion is dominant at late times.

With a little algebra and the use of relations given by Abramowitz and Stegun (1965 p. 375 and p. 358) to express the modified Bessel functions in terms of unmodified functions the transfer function reduces to

$$Z(t) \simeq \frac{2}{\pi}\int_0^{\infty} u e^{-u^2 t}(1-\sigma_1/\sigma_2)\frac{T(u)}{S_1^2 + S_2^2}\, du \qquad (15)$$

where

$$T(u) = -(-S_2 J_0(\sqrt{\sigma_1 \mu_1}\, uR) + S_1 Y_0(\sqrt{\sigma_1\mu_0}\, uR))J_1(\sqrt{\sigma_2\mu_0}\, ua)$$

$$S_1 = \sqrt{\sigma_2}\, J_1(\sqrt{\sigma_2\mu_0}\, au)J_0(\sqrt{\sigma_1\mu_0}\, au) - \sqrt{\sigma_1}\, J_0(\sqrt{\sigma_2\mu_0}\, au)J_1(\sqrt{\sigma_1\mu_0}\, au)$$

$$S_2 = \sqrt{\sigma_2}\, J_1(\sqrt{\sigma_2\mu_0}\, au)Y_0(\sqrt{\sigma_1\mu_0}\, au) - \sqrt{\sigma_1}\, J_0(\sqrt{\sigma_2\mu_0}\, au)Y_1(\sqrt{\sigma_1\mu_0}\, au)$$

Equation (15) may be evaluated numerically. This has been done for an example and the results are given in Table 1. The results of numerically transforming a mode-matched cylinder solution are given for comparison. The results given there are for a cylinder of radius 25 m and depth 100 m. The conductivity of the half-space is 0.01 $S.m^{-1}$ and all the permeabilities have been set equal to μ_0. The cylinder was excited by a step current flowing in a line source at the Earth's surface and offset 100 m from the axis of the cylinder.

TABLE 1

A comparison of the results for the transfer function of the cylinder calculated by two different methods. The Zm results were obtained by means of transforming the spectrum of a mode-matched solution while the Zm was obtained by evaluating the contour integral approximation to that spectrum. The contour integral takes account only of the effects of the branch cut; poles are ignored. The columns give sets of data for different receiver positions.

	0 m		50 m		100 m		200 m	
t (msec)	+Za	+Zm	+Za	+Zm	+Za	+Zm	+Za	+Zm
1 0	102 82	125 36	91 03	114 72	66 11	92 08	18 80	47.65
1 78	70 65	81 65	65 57	77 0	54 65	67 01	32 5	46 76
3 16	34 10	39 61	32 5	38 03	28 99	34 63	21 58	27 60
5 62	11 70	13 91	11 36	13 54	10 62	12 74	8 99	11 07
10	3 16	3 73	3 12	3 67	3 01	3 55	2 77	3 29
17 8	83	93	.82	92	81	90	78	87
31 6	23	25	23	25	23	025	22	24
56 2	068	071	069	071	068	071	067	070
100	021	021	021	021	021	021	021	021

Comparison of the results shows clearly that at the late stage of decay the branch cut integral is an adequate representation of the transfer function. It should be noted that the same branch cut is the only singularity in the expression for the incident field due to the line source. These results indicate that the results discussed above for the sphere have their counterparts for other structures.

4. Conclusions

We have demonstrated the importance of source geometry on TEM incident fields and indicated some simple methods of controlling incident field patterns. We are currently studying analogous problems with a conducting overburden and the results will appear elsewhere.

The cases studied are not exhaustive and there are a number of other interesting possibilities. While the multiwire configuration in Fig. 3 has been found to be not particularly efficacious as a steering source it may be possible to use such loops with currents phased in time to produce the well known "phased array" type source.

Another series of possibilities arise if one is able to control the current waveforms in the wires. Possibilities here include the pre-filtering of the fields to produce a specific waveform at a given point in the Earth and there are also locus steering possibilities.

We have also shown that the behaviour of the later stages of electromagnetic transients is best understood in terms of their singularity expansion. Those results show that at the later stages the anomalous fields are swamped by the primary field because both the primary and the secondary fields have the same sort of singularities. Much more attention should be given to the earlier stages where it is known that different singularities are important.

The occurrence of the same singularities in the two fields has unfortunate consequences for inversion which are discussed in Lee (1978).

In the light of the decay properties of transients discussed above, it is interesting to examine the development of TEM instrumentation in recent years and the prospects for further developments into the 1980s.

The instruments have been designed to measure the transient response at ever increasing times using in large part the classical stacking technique implemented with modern microprocessors. Such techniques for increasing sensitivity and measuring at later times are fairly obvious approaches for an instrument builder interested in enhancing signal to noise ratios.

A feature of the newer instruments is their capability to measure the secondary transients to at least 100 ms. The Newmont system, which has been described by Barnett et al. (1978) measures out to 100 ms while the SIROTEM system goes further by measuring out to nearly 200 ms (Buselli and O'Neill 1977).

The examples presented here indicate that there is a very fundamental limitation to the use of late time measurements and it may well be that current instruments are close to reaching that limit. Indeed, the data of Spies (1980) indicate that the limit may have been passed already for many practical situations.

Acknowledgments

We are indebted to Peko-Wallsend Operations Ltd (Peko Geoscience Division) for continued support, in particular for travel expenses making the co-operative effort possible.

References

ABRAMOWITZ, M. and STEGUN, I. A., 1965. Handbook o Mathematical Functions. Dover Publications, New York.

BARNETT, C. T., DAVIDSON, M. J., McLAUGHLIN, G. H. an NABIGHIAN, M. N., 1978 Exploration with the Newmont EM system. Paper presented at the 40th Annual EAEG Meeting, Jun 1978, Dublin, Ireland

BUSELLI, G. and O'NEILL, B., 1977. SIROTEM, a new portabl instrument for multi-channel transient electromagnetic measure ments. *Bull. Aust. Soc. Explor. Geophys.*, **8**, 82-87.

ERDELYI, A., MAGNUS, W., OBERHETTINGER, F. and TR COMI, F. G., 1954. Tables of Integral Transforms, Vol. 1. McGraw Hill, New York.

KAN, T.-K. and CLAY, C. S., 1979. Hybrid ray approximation i electromagnetic sounding. *Geophysics*, **44**, 1846-1861.

KAUFMAN, A., 1978. Frequency and transient responses of electro magnetic fields created by currents in confined conductors *Geo physics*, **43**, 1002-1010.

LEE, T., 1974. Transient electromagnetic response of a sphere in layered medium. Unpublished Ph.D. Thesis, Macquarie University New South Wales, Australia.

LEE, T., 1978. Transient electromagnetic waves applied to pros pecting. Invited paper presented at the 19th General Assembly c URSI, August, 1978, Helsinki, Sweden.

LEE, T. and LEWIS, R., 1974. Transient EM response of a larg loop on a layered ground. *Geophys. Prosp.*, **22**, 430-444.

LEE, T. and LEWIS, R., 1981. An analysis of the late stages of th transient electromagnetic response of a horizontal cylinder in a cor ducting halfspace (in preparation).

LEWIS, R. and LEE, T., 1978. The transient electric fields about loop on a half-space. *Bull. Aust. Soc. Explor. Geophys.*, **9**, 173-17

McCRACKEN, K., 1979. Unpublished memo to Australian Minera Industries Research Association SIROTEM Group.

NABIGHIAN, M. N., 1979. Quasi-static transient response of a cor ducting half-space — an approximate representation. *Geophysic* **44**, 1700-1705.

OBERHETTINGER, F. and BADII, L., 1973. Tables of Laplac Transforms. Springer-Verlag, Berlin.

PRATT, D. A. and WHITELEY, R. J., 1974. Computer simulatio and evaluation of electrode array responses in resistivity and I.F prospecting. *Bull. Aust. Soc. Explor. Geophys.*, **5**, 65-87.

SINGH, S. K., 1973. Electromagnetic transient response of a cor ducting sphere embedded in a conducting medium *Geophysics*, **3** 864-893.

SPIES, B R., 1978. Scale models of the TEM method Proceedin of a Workshop on Modelling of Electric and Electromagneti Methods, May 17-19, The Lawrence Berkeley Laboratory, Unive sity of California, Berkeley

SPIES, B. R., 1980. Interpretation and design of time domain E surveys in areas of conductive overburden. *Bull. Aust. Soc. Explo. Geophys.*, **11**, 272-281.

THIO, Y. T., 1977. Riemann boundary value problems in electr magnetism. Unpublished Ph.D. Thesis, Monash University, Clayto Australia.

WAIT, J. R., 1960. Propagation of electromagnetic pulses in a hom geneous conducting earth. *App. Sc. Res.*, **B8**, 213-253

WAIT, J R., 1962. Electromagnetic Waves in Stratified Medi Oxford, England.

WAIT, J R., 1971 Transient excitation of the earth by a line sour of current. *Proc. IEEE (Letters)*, **59**, 1287-1288.

WAIT, J. R., SPIES, K P. and HJELT, S.-E., 1971 On "Quas static transient response of a conducting permeable sphere" *Ge physics*, **36**, 615-617.

YOST, W. J, 1952. The interpretation of electromagnetic reflectio data in geophysical exploration — Part 1, general theory *Geophysic* **17**, 89-106.

Evaluation of a class of integrals occurring in mode matching solutions for electrical and electromagnetic modelling

T. J. Lee

REPRINTED FROM

Geophys. J. R. astr. Soc. (1981) **66**, 455–460

PUBLISHED BY

BLACKWELL SCIENTIFIC PUBLICATIONS

OXFORD LONDON EDINBURGH BOSTON MELBOURNE

Geophys. J. R. astr. Soc. (1981) **66**, 455–460

Research note

Evaluation of a class of integrals occurring in mode matching solutions for electrical and electromagnetic modelling

T. J. Lee* *Cooperative Institute for Research in Environmental Sciences, University of Colorado, Boulder, Colorado 80309, USA*

Received 1980 October 13; in original form 1980 July 16

Summary. Many integrals that go to make up the matrix for mode matching solutions of electrical or electromagnetic problems may be evaluated in closed form.

Examples where this is possible include the electrical problems of finding the potential about a point electrode that is in a layered ground that includes either a spherical or cylindrical conductor. The results given here greatly reduce the amount of computation that is required for the calculations proposed by Lee.

The integrals that arise analogous to the electromagnetic scattering of a cylinder in a half space may also be evaluated. For the case of a spherical scatter it is only possible to evaluate the integrals for the case where the source is a large coaxial loop.

1 Introduction

The solution of integral equations by mode matching is a powerful tool by which the more familiar boundary value problems may be extended to cater for more realistic geological geometries. The basis of the method is the solution of a set of simultaneous equations for the coefficients of a set of functions that have been chosen to represent the unknown field quantity across the heterogeneity.

Geometries which have been treated by this method have all involved either spheres or cylinders in a layered ground. Howard (1972) has solved the integral equation by this method for a horizontal cylinder in a half space that has been excited by a line source at the Earth's surface. The axisymmetric problem of a spherical conductor in a layered medium and beneath a large loop has been studied by Lee (1975b). More recently Lee (1981) has solved the equations for an arbitrary non-grounded source. Solutions for the corresponding potential problems of a sphere or cylinder in a layered medium are also known (see Lee 1975a).

An unfortunate feature of these solutions is that the matrix that must be inverted to yield the required coefficients is composed of integrals. These integrals relate the various multiples induced into the conductors to the corresponding multiples of the source. They therefore

*Now at Geopeko, PO Box 217, Gordon, NSW 2072, Australia.

represent the coefficients in the multiple expansions of the more familiar Sommerfeld integrals. A particularly unfortunate feature of these integrals is that they are frequently difficult to evaluate numerically. The purpose of this paper is to show how many of these integrals may be evaluated in closed form. Indeed almost all the integrals that are required for the calculations outlined in the paper by Lee (1975a) may be evaluated in closed form.

2 Spherical structure in a layered medium – potential problem

The simplest integrals arise for the potential about a point electrode at the surface of a layered ground that has a spherical conductor at a depth h. Suppose that the sphere has a radius b and conductivity σ_3. When the ground is a half space the $\gamma\mu$ element of the matrix is proportional to

$$^1I_{mn}^{\gamma\mu}$$

where

$$^1I_{mn}^{\gamma\mu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda^{m+n-1} \exp(-2\lambda h) \exp[i(\mu-\gamma)\psi]\, d\alpha\, d\beta \tag{1}$$

where $\lambda = \sqrt{\alpha^2+\beta^2}$ and $\psi = \arctan(\beta/\alpha)$. See Lee (1975a, equation 28).

The integral in equation (1) may be evaluated after changing the variables of integration to ψ and λ.

One finds, then, that

$$^1I_{mn}^{\gamma\mu} = 2\pi\delta_{\gamma\mu}\int_0^{\infty} \lambda^{m+n} \exp(-2\lambda h)\, d\lambda = \frac{2\pi\delta_{\gamma\mu}}{(2h)^{m+n+1}}\,\Gamma(m+n+1). \tag{2}$$

Here

$$\begin{aligned} \delta_{\gamma\mu} &= 1 \qquad m = n \\ &= 0 \qquad m \neq n. \end{aligned}$$

Now suppose that a conductive layer of thickness t and conductivity σ_1 exists at the surface of the ground. For this case the $\gamma\mu$ element of the matrix is proportional to $^2I_{mn}^{\gamma\mu}$ where:

$$^2I_{mn}^{\gamma\mu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp(-2\lambda h)\lambda^{m+n}[1-k_1\exp(2\lambda t)]\exp[i(\gamma-\mu)\psi]}{\lambda[1-k_1\exp(-2\lambda t)]}\, d\alpha\, d\beta \tag{3}$$

(Lee 1975, equation 30)

Here $k_1 = (\sigma_2-\sigma_1)/(\sigma_1+\sigma_2)$. σ_2 is the conductivity of the host rock. The other symbols have their former meanings (see Lee 1975, equation 30).

Since $k_1\exp(-2\lambda t)$ is always less than 1 we can write

$$^2I_{mn}^{\gamma\mu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-2\lambda h)\lambda^{m+n-1}\exp[i(\gamma-\mu)\psi][1-k_1\exp(2\lambda t)] \times \sum_{j=0}^{\infty} k_1^j \exp(-2t\lambda j)\, d\alpha\, d\beta. \tag{4}$$

The integral in equation (4) may be evaluated by integrating term by term and using equation (1).

One finds that

$$ {}^2I_{mn}^{\gamma\mu} = 2\pi\delta_{\gamma\mu} \sum_{j=0}^{\infty} \left\{ \frac{\Gamma(m+n+1)k_1^j}{(2h+2jt)^{m+n+1}} - \frac{k_1^{j+1}\Gamma(m+n+1)}{[2h+2(j-1)t]^{m+n+1}} \right\}. \tag{5} $$

3 Cylindrical structure in a layered medium – potential problem

Suppose that a horizontal cylinder has a radius b and conductivity σ_3 and is situated in a layered ground such that its axis is at a depth h. When the ground is a half space the integrals in the matrix are proportional to ${}^1I_{mn}$ where

$$ {}^1I_{mn} = \int_{-\infty}^{\infty} \frac{\exp(-2\lambda h)\cdot\exp[i(n-m)\psi]}{\lambda}\, d\beta. \tag{6} $$

Here

$\cos(\psi) = -\lambda/\alpha$

$\sin(\psi) = i\beta/\alpha.$ (Lee 1975, equation 21).

Writing $\beta = \alpha \sinh(\phi)$ in equation (6) one finds that

$$ {}^1I_{mn} = \int_{-\infty}^{\infty} \exp[-2\alpha h\cosh(\phi)]\,[-\cosh(\phi) + i\sinh(\phi)]^{n-m}\, d\phi $$

$$ = 2\int_0^{\infty} \exp[-2\alpha h\cosh(\phi)]\,(-1)^{n-m}\cosh[(n-m)\phi]\, d\phi. \tag{7} $$

Therefore

$$ {}^1I_{mn} = 2(-1)^{n-m}K_{n-m}(2\alpha h). \tag{8} $$

Here $K_n(z)$ is the modified Bessel function of order n and argument z (see Abramowitz & Stegun 1965, p. 376).

When there is a conductive overburden present whose conductivity is σ_1 and thickness is t then the integrals in the matrix are proportional to ${}^2I_{mn}$ where

$$ {}^2I_{mn} = \int_{-\infty}^{\infty} \frac{[1-k_1\exp(2\lambda t)]}{\lambda[1-k_1\exp(-2\lambda t)]} \exp(-2\lambda h)\cdot\exp[(n-m)i\psi]\, d\beta \tag{9} $$

(Lee 1975a, equation 23).

Here $k_1 = (\sigma_2 - \sigma_1)/(\sigma_1 + \sigma_2)$. σ_2 is the conductivity of the host rock. All the other symbols have their former meanings.

The integral in equation (9) may be evaluated by first expanding the term $1/[1 - k_1 \exp(-2\lambda t)]$ by the binomial theorem and then integrating term by term by using equation (8). One finds, then, that

$$ {}^2I_{mn} = 2(-1)^{n-m} \sum_{j=0}^{\infty} [\![k_1^j K_{n-m}(2\alpha h + 2j\alpha t) - k_1^{j+1} K_{n-m}[2\alpha h + 2(j-1)\alpha t]]\!]. \tag{10} $$

4 Electromagnetic response of a cylinder in a half space

Howard (1972) shows that when the electromagnetic response of a buried horizontal cylinder is sought by a mode matching technique, then the mn element of the matrix for the coefficients contains elements that are proportional to ${}^{eo}I_{mn}$ where

$$
{}^{e}_{o}I_{mn} = \int_0^\infty \frac{\exp(-2n_1 z_0)}{n_1} \frac{(n_1-\lambda)}{(n_1+\lambda)} g^{m+n}_{\substack{e\\o}} \, d\lambda
$$

$$
g^{m+n}_{\substack{e\\o}} = \left[\left(\frac{\lambda+n_1}{k_1}\right)^{m+n} \pm \left(\frac{n_1-\lambda}{k_1}\right)^{m+n}\right] \tag{11}
$$

$$
n_1 = \sqrt{\lambda^2 + k_1^2}
$$

$$
k_1^2 = i\omega\sigma_2\mu_0.
$$

Here z_0 is the depth to the axis of the cylinder and σ_2 and μ_0 are the conductivity and permeability of the ground respectively and ω is the angular frequency. First consider the case for ${}^{e}I_{mn}$.

The first step in the evaluation of equation (11) for ${}^{e}I_{mn}$ is to write $\lambda = k_1 \sinh\phi$. After noting that the contour of integration may be changed to lie along the positive real ϕ axis one finds that

$$
{}^{e}I_{mn} = 2 \int_0^\infty \exp(-2k_1 z_0 \cosh\phi) \exp(-2\phi) \cosh[(m+n)\phi] \, d\phi. \tag{12}
$$

To evaluate the integral in equation (12) one notices that

$$
\begin{aligned}
\exp(-2\phi)\cosh[(m+n)\phi] &= \frac{1}{2}[\![\cosh[(m+n-2)\phi] + \cosh[(m+n+2)\phi]]\!] \\
&\quad - 2\sinh\phi\cosh\phi \left[2^{m+n-1}\cosh^{m+n}(\phi) + (m+n)\sum_{j=1}^{[(m+n)/2]}\right. \\
&\quad \left. \times (-1)^j \frac{1}{j}\binom{m+n-j-1}{j-1} 2^{m+n-2j-1}\cosh(\phi)^{m+n-2j}\right].
\end{aligned} \tag{13}
$$

In equation (13) $E(x)$ means the integer part of x (see Gradshteyn & Ryzhik 1965, 1.331–4, p. 27). The integral can now be evaluated by using the integral representation for the modified Bessel function. Therefore

$$
\begin{aligned}
{}^{e}I_{mn} &= K_{m+n-2}(2k_1 z_0) + K_{m+n+2}(2k_1 z_0) - 4\left[2^{m+n-1}\,\Gamma(m+n+2, 2k_1 z_0) + (m+n)\right. \\
&\quad \left. \times \sum_{j=1}^{E[(m+n)/2]} (-1)^j \frac{2^{m+n-2j-1}}{j}\binom{m+n-j-1}{j-1}\Gamma(m+n-2j+2, 2k_1 z_0)\right]
\end{aligned} \tag{14}
$$

where $\Gamma(m,x) = \sum_{k=0}^{m-1}\binom{m-1}{k}\frac{k!\exp(-x)}{x^{k+1}}$.

${}^{o}I_{mn}$ may be evaluated in the same way. One finds that

$$
\begin{aligned}
{}^{o}I_{mn} &= 2\sum_{j=0}^{E[(m+n-1)/2]} (-1)^j \binom{m+n-j-1}{j} 2^{m+n-2j-1} \\
&\quad \times (2\Gamma(m+n-2j+2, 2k_1 z_0) - \Gamma(m+n-2j, 2k_1 z_0)) - [K_{m+n+2}(2k_1 z_0) \\
&\quad - K_{m+n-2}(2k_1 z_0)].
\end{aligned} \tag{15}
$$

5 Electromagnetic response of a sphere in a half space

The integrals that result from a mode matching solution for the electromagnetic scattering problem of a sphere in a half space in general cannot be evaluated. This is despite the fact that the integrals can be transformed from infinite integrals to finite ones (Lee 1981). It is possible to evaluate only the integrals for the particular case of a loop that is coaxial with the sphere. Although no general formula can be found the procedure for obtaining particular results is quite clear. This is illustrated by the example given below. The *mn* element of the matrix in this case is proportional to I_{mn}

$$I_{mn} = \int_0^\infty P_n^1\left(\frac{in_1}{k}\right)P_m^1\left(\frac{in_1}{k}\right)\frac{\lambda}{n_1}\left(\frac{n_1-\lambda}{n_1+\lambda}\right)\exp(-2n_1 h)\,d\lambda \tag{16}$$

where

$n_1 = \sqrt{\lambda^2 - k^2}$

$k^2 = -i\omega\sigma_2\mu_0$

$P_n^1(z) = \sqrt{1-z^2}\,\dfrac{d}{dz}\,[Pn(z)]$ (Lee 1975b, equation 13).

In equation (15) σ_2 and μ_0 are the conductivity and permeability of the ground respectively, h is the depth to the centre of the sphere and ω is the angular frequency. The impedement to writing down a general expression for I_{mn} is the lack of a suitable formula for the product of the two associated Legendre functions. Any given integral, however, can be handled as is done in the following example. Suppose that $m = 2$ and $n = 3$. For this case I_{mn} becomes I_{23} where

$$I_{23} = \frac{9}{2}\int_0^\infty\left[6\left(\frac{in_1}{k}\right)^3 - \left(\frac{in_1}{k}\right) - 5\left(\frac{in_1}{k}\right)^5\right]\frac{\lambda}{n_1}\left(\frac{n_1-\lambda}{n_1+\lambda}\right)\exp(-2hn_1)\,d\lambda. \tag{17}$$

This integral is very similar to the type that resulted from Howard's (1972) analysis of a cylindrical conductor. It can be treated in the same way. Since

$k = i\sqrt{i\omega\sigma_2\mu_0} = i\gamma, \qquad \mathrm{Re}(\gamma) > 0$

an alternative expression for I_{23} can be found by setting $\lambda = \gamma\sinh(\phi)$ in the integral. After noting that the path of integration may be changed to lie along the positive ϕ-axis one obtains

$$I_{23} = \frac{9}{2}\int_0^\infty [6\cosh^3(\phi) - \cosh(\phi) - 5\cosh^5(\phi)]\exp(-2\phi)\gamma\sinh(\phi) \times \exp[-2\gamma h\cosh(\phi)]\,d\phi.$$

The integral in equation (18) is easily evaluated when the integrand is reduced with the aid of the multiple angle formulas for the hyperbolic functions (see Gradshteyn & Ryzhik 1965, No. 1.3241–6).

One finds that

$$I_{23} = \frac{9}{2}\gamma\int_0^\infty \sinh(\phi)\exp[-2\gamma h\cosh(\phi)][\cosh(\phi) - 8\cosh^3(\phi) + 17\cosh^5(\phi) - 10\cosh^7(\phi)]\,d\phi - \frac{9}{128}\gamma\int_0^\infty \exp[-2\gamma h\cosh(\phi)][37 - 65\cosh(\phi) + 38\cosh(2(\phi)) - 15\cosh(3\phi) + 11\cosh(5\phi) - 6\cosh(6\phi) + 5\cosh(7\phi) - 5\cosh(8\phi)]\,d\phi. \tag{19}$$

The first of these integrals may be evaluated by elementary means and the second by using the integral representation for the modified Bessel function. Therefore

$$I_{23} = \frac{9}{2}\gamma[\Gamma(2, 2\gamma h) - 8\,\Gamma(4, 2\gamma h) + 17\,\Gamma(6, 2\gamma h) - 10\,\Gamma(8, 2\gamma h)] - \frac{9}{128}\gamma\,[37K_0(2\gamma h) - 65K_1(2\gamma h) + 38K_2(2\gamma h) - 15K_3(2\gamma h) + 11K_5(2\gamma h) - 6K_6(2\gamma h) + 5K_7(2\gamma h) - 5K_8(2\gamma h)]. \tag{20}$$

Discussion

The results given in this paper show that many of the integrals that occur in mode matching solutions for various cases reported in the literature can be found in closed form. The only special functions that are needed are the various modified Bessel functions. These functions can easily be evaluated by using the approximations that have been described by Abramowitz & Stegun (1965, pp. 358 *et seq.*).

Another application for the results given here is that they allow one to check more general numerical schemes that are required for the evaluation of the remaining integrals.

References

Abramowitz, M. & Stegun, I. A., 1965. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables,* Dover, New York, 1046 pp.

Gradshteyn, I. S. & Ryzhik, I. M., 1965. *Tables of Integrals, Series and Products,* 4th edn, Academic Press, New York, 1086 pp.

Howard, A. Q., 1972. Electromagnetic fields of a subterranean cylindrical inhomogeneity excited by a line source, *Geophysics,* **37,** 975–984.

Lee, T. J., 1975a. An integral equation and its solution for some two and three dimensional problems in resistivity and induced polarization, *Geophys. J. R. astr. Soc.,* **42,** 81–95. Unfortunately, this paper contains numerous misprints; copies of a revised paper are available on request.

Lee, T. J., 1975b. Transient electromagnetic response of a sphere in a layered medium, *Geophys. Prospec.,* **23,** 492–512.

Lee, T. J., 1981. Transient electromagnetic response of a sphere in a layered medium, *Pure appl. Geophys.,* in press.

PRINTED IN ENGLAND BY
ADLARD AND SON LTD, DORKING

The Cole-Cole model in time domain induced polarization

T. Lee

GEOPHYSICS, VOL 46, NO 6 (JUNE 1981), P 860–868, 6 FIGS

GEOPHYSICS, VOL 46, NO 6 (JUNE 1981), P 932–933, 1 FIG

Short Note
The Cole-Cole model in time domain induced polarization

T. Lee*

Recently Pelton et al (1978) used a Cole-Cole relaxation model to simulate the transient voltages that are observed during an induced-polarization survey. These authors took the impedance of the equivalent circuit $Z(\omega)$ to be

$$Z(\omega) = R_0 \left\{1 - m \left[1 - \frac{1}{1 + (i\,\omega\,\tau)^c}\right]\right\}. \tag{1}$$

They then gave the expression for the transient voltage $V_1(t)$ as

$$V_1(t) = m\,R_0 I_0 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n\,c + 1)} \left(\frac{t}{\tau}\right)^{nc}\right]. \tag{2}$$

In equation (2), I_0 was misprinted as $1/I_0$. In these equations, $m = 1/(1 + R_1/R_0)$ and R_1, R_0 and τ are constants to be determined for the given model. I_0 is the height of the step current that will flow in the transmitter A disadvantage of equation (2) is that it is only slowly convergent for large t/τ. Pelton et al (1978) used a τ which ranged from 10^{-4} to 10^{-2}. The purpose of this note is to provide an alternative expression for $V_1(t)$ that is valid only at the later stages but which does not have this disadvantage. The trivial case of $c = 1\,0$ is ignored.

From equation (1), the voltage $V(t)$ is

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{I_0 Z(\omega)}{i\,\omega}\, e^{i\omega t}\, d\omega \tag{3}$$

The integral in equation (3) may be evaluated by contour integration about the branch cut along the imaginary ω-axis (see Figure 1) There are no poles in the upper half of the complex ω-plane. One notices that for $t > 0$, the contribution from the quarter circles is negligible because of Jordan's theorem.

The integral about the origin 0, I_1 is defined by the path P_0 In fact,

$$I_1 = \frac{1}{2\pi} \int_{P_0} \frac{e^{i\omega t} I_0 Z(\omega)}{i\,\omega}\, d\omega \tag{4}$$

Writing $\omega = r\,e^{i\theta}$ in this integral and allowing r to become small, one finds that

$$I_1 = -R_0 I_0 \tag{5}$$

The contribution from the paths $P+$ and $P-$ may be obtained by writing $\omega = e^{i\,\pi/2} R$ on $P+$ and $\omega = e^{-3\pi/2\,i} R$ on $P-$ When this is done, we find that

$$\frac{1}{2\pi} \int_{P+} \frac{I_0 Z(\omega)\, e^{i\omega t}}{i\,\omega}\, d\omega + \frac{1}{2\pi} \int_{P-} \frac{I_0 Z(\omega)\, e^{i\omega t}}{i\,\omega}\, d\omega$$
$$= \frac{1}{2\pi} \int_0^{\infty} \frac{I_0 m\, R_0}{R} \cdot$$
$$\cdot\, e^{-Rt} \left[\frac{\tau^c R^c\, 2 \sin \pi\, c}{(1 + \cos \pi\, c\, \tau^c R^c)^2 + R^{2c} \tau^{2c} \sin^2 \pi\, c}\right] dR \tag{6}$$

By Cauchy's theorem and equations (3), (5), and (6) one now finds that

$$V(t) = I_0 R_0 - \frac{I_0}{\pi} \int_0^{\infty} \frac{m\, R_0 e^{-Rt} \tau^c R^c \sin \pi\, c\, dR}{[(1 + \cos \pi\, c\, \tau^c R^c)^2 + R^{2c} \tau^{2c} \sin^2 \pi\, c] R} \tag{7}$$

The interpretation of equation (7) is that the time-dependent term is the amount by which the voltage is reduced from the steady-state value of $I_0 R_0$

Thus the transient voltage $V_1(t)$ of Pelton et al (1978) is

$$V_1(t) = \frac{I_0}{\pi} \int_0^{\infty} \frac{m\, R_0 e^{-xt/\tau} x^c \sin \pi\, c}{x[(1 + x^c \cos \pi\, c)^2 + x^{2c} \sin^2 \pi\, c]}\, dx \tag{8}$$

To see how $V_1(t)$ behaves for large t/τ, one only needs to derive an asymptotic expansion for the integral in equation (8) This expansion is found by expanding the function $x^c/[(1 + x^{2c} + 2x^c \cos \pi\, c)]$ as a power series in x^c and then integrating term by term. Since

$$x^c/(1 + x^{2c} + 2x^c \cos \pi\, c)$$
$$\approx x^c[1 - (2 \cos \pi\, c) x^c + (4 \cos^2 \pi\, c - 1) x^{2c}], \tag{9}$$

$$V(t) \approx \frac{I_0 m\, R_0}{\pi} \sin \pi\, c \left[\Gamma(c) \left(\frac{\tau}{t}\right)^c - 2\,\Gamma(2c) \cos \pi\, c \left(\frac{\tau}{t}\right)^{2c} + (4 \cos^2 \pi\, c - 1)\,\Gamma(3c) \left(\frac{\tau}{t}\right)^{3c}\right] \tag{10}$$

Equation (10) clearly shows that the decay (growth) is much

Manuscript received by the Editor March 7, 1980, revised manuscript received October 14, 1980
* Formerly Cooperative Institute for Research in Environmental Sciences/NOAA, Boulder, CO, presently GEOPEKO, P O Box 217, Gordon, N S W 2072, Australia
0016-8033/81/0601—932$03 00

slower, at the later stages, than an exponential. It also provides a means of estimating that decay.

Pelton et al (1978) also used a modified form of $Z(\omega)$ Specifically, they took

$$Z(\omega) = R_0\left[1 - \frac{m_1}{1 + (i\,\omega\,\tau_1)^{c_1}}\right]\frac{1}{[1 + (i\,\omega\,\tau_2)^{c_2}]}. \tag{11}$$

For this model we concur with the remark made by Pelton et al that the Cole-Cole model is only a very idealized representation of the mineralized rock. Nevertheless, the model has been useful, and for this reason it is discussed here. Once again one finds that the integral about the origin yields

$$-R_0 I_0 = \frac{I_0}{2\pi}\int_{P_0}\frac{Z(\omega)}{i\,\omega}\,d\omega. \tag{12}$$

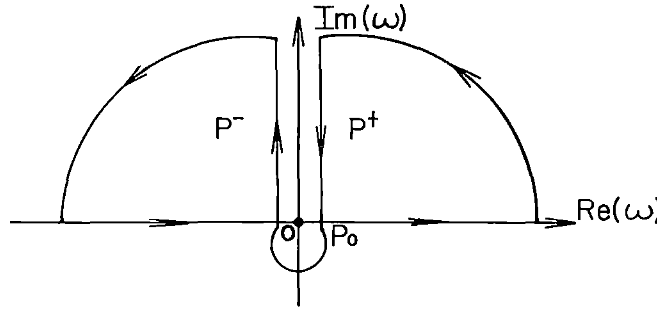


FIG. 1. Path for contour integration.

The integral about the branch cut is given in equation (13).

$$\frac{1}{2\pi}\int_{P+}\frac{I_0 Z(\omega)}{i\,\omega}e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{P-}\frac{I_0 Z(\omega)}{i\,\omega}d\omega$$

$$= -\frac{I_0R_0}{2\pi}\int_0^\infty \frac{e^{-Rt}}{Rt}\left\{\left[1-\frac{m_1}{1+(R\,e^{i\pi}\tau_1)^{c_1}}\right]\left[\frac{1}{1+(R\,e^{i\pi}\tau_2)^{c_2}}\right] - \left[1-\frac{m_1}{1+(R\,e^{-i\pi}\tau_1)^{c_1}}\right]\left[\frac{1}{1+(R\,e^{-i\pi}\tau_2)^{c_2}}\right]\right\}dR$$

$$= \frac{I_0R_0}{\pi}\int_0^\infty\frac{e^{-Rt}}{R}\left\{\frac{\sin\pi\,c_2(R\tau_2)^{c_2}}{[1+(\tau_2R)^{2c_2}+2(\tau_2R)^{c_2}\cos\pi\,c_2]} - \frac{m_1[(R\tau_1)^{c_1}\sin\pi\,c_1 + (R\tau_2)^{c_2}\sin\pi\,c_2]}{[1+(R\tau_1)^{2c_1}+2(R\tau_1)^{c_1}\cos\pi\,c_1][1+(R\tau_2)^{2c_2}+2(R\tau_2)^{c_2}\cos\pi\,c_2]}\right\}dR \tag{13}$$

In this case, when both t/τ_1 and t/τ_2 are large,

$$V_1(t) \approx \frac{I_0R_0}{\pi}\left\{\Gamma(c_2)\sin\pi\,c_2\left(\frac{\tau_2}{t}\right)^{c_2} - m_1\left[\left(\frac{\tau_1}{t}\right)^{c_1}\sin\pi\,c_1\Gamma(c_1) + \left(\frac{\tau_2}{t}\right)^{c_2}\sin\pi\,c_2\Gamma(c_2)\right]\right\}. \tag{14}$$

ACKNOWLEDGMENT

We thank Roger Deakin of Peko Geoscience for suggesting the problem.

REFERENCE

Pelton, W. H., Ward, S. H , Hallof, P G , Sill, W P , and Nelson, P H , 1978, Mineral discrimination and removal of inductive coupling with multifrequency I P Geophysics, v 43, p. 588–609

Transient electromagnetic response of a polarizable ground

T. Lee

GEOPHYSICS, VOL 46, NO 7 (JULY 1981), P 1037–1041, 4 FIGS.

HYSICS, VOL 46, NO 7 (JULY 1981), P 1037–1041, 4 FIGS

ansient electromagnetic response of a polarizable ground

Lee*

ABSTRACT

When a uniform ground has a conductivity which may be escribed by a Cole-Cole relaxation model with a positive me constant, then the transient response of such a ground ill show evidence of induced polarization (IP) effects. The effects cause the transient initially to decay quite rapidly nd to reverse polarity After this reversal the transient ecays much more slowly, the decay at this stage being bout the same rate as a nonpolarizable ground.

INTRODUCTION

ere is some question as to whether or not electromagnetic) methods are able to detect induced-polarization (IP) omena Hohmann et al (1970) used a model that allowed for ffects in their EM field data The data used were for a field m that measured frequencies in the range 15–1500 Hz They luded that IP effects could not be detected in the data

eoretical studies by Morrison et al (1969) and Lee (1975) ved that IP effects could be detected in transient EM data, pro- d the conductivity of the ground was allowed to be complex to have an appreciable phase angle.

ore recently, Rathor (1978) showed that if the conductivity e ground was a function of the square root of the frequency of source, then IP effects would be seen in the corresponding ients.

ies (1974) produced field data from a transient EM survey in ensland, Australia, which show that the transients went tive at the early stages Spies believed that the sign change have been due to the presence of IP effects Elsewhere, Lee 9) argued that the theoretical understanding of these effects hampered by a lack of reliable rock measurements. Mean- e, Pelton et al (1978) carried out IP surveys over a number of h American sulfide deposits. In these experiments, Pelton et al 8) used a wide frequency range and fitted their data to a model used a Cole-Cole model to represent the complex conductivity e ground. These last authors used field data to study the com- conductivity of the ground because they felt that such data more reliable than laboratory measurements of rock samples his means they hoped to determine whether or not one could ct the presence of various minerals by the use of the IP method rospecting.

y purpose is to use the same data as Pelton et al (1978) to show IP effects can be seen in transient EM data

THEORY

Suppose that a large loop of radius b lies on the surface of a uniform ground of conductivity σ and permeability μ. When a uniform current whose spectrum is $I_0 e^{i\omega t}/(i\omega)$ flows in the loop, then a transient voltage $V(t)$ will be observed because of the ground. $V(t)$ may be written as in equation (1) below

$$V(t) = \frac{\pi b^2 I_0 \mu_0}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} J_1^2(\lambda b) \frac{n_1 - \lambda}{n_1 + \lambda} \, d\lambda \, e^{i\omega t} d\omega, \quad (1)$$

where

$$n_1 = \sqrt{\lambda^2 + i\omega\mu\sigma}$$

Equation (1) may easily be derived from equation (3) of Lee and Lewis (1974).

In equation (1), t represents time after the step current of height I_0 flows in the loop and μ and μ_0 are the permeability of the ground and air, respectively The conductivity σ may be complex. Recently, Pelton et al (1978) used a Cole-Cole relaxation model to study the effects of different types of mineralization on observed IP data. With this model, one finds from Pelton et al [1978, equation (1)] that

$$i\omega\mu\sigma = \frac{\sigma_0 \mu i\omega[1 + (\tau i\omega)^c]}{1 + \alpha(\tau i\omega)^c} \quad (2)$$

Here τ is a time constant that determines the length of time the IP phenomenon lasts. c is a parameter that is frequently between 0 1 and 0 6. σ_0 is the conductivity of the ground at the very low frequencies, that is, when the ground is purely conductive. $\alpha = 1 - m$ where m is the chargeability For further discussion of these parameters see Pelton et al (1978, p. 590) In order to effect some of the approximations below, we will assume that c is less than or equal to 0.5.

To evaluate the integrals in equation (1), one first evaluates the integral of the Bessel functions Using the addition theorem for Bessel functions, one finds that

$$V(t) = \frac{b^2 \mu_0 I_0}{4\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} e^{i\omega t} J_0(\lambda R) \cdot \left(\frac{n_1 - \lambda}{n_1 + \lambda}\right) \cos\theta \, d\lambda \, d\theta \, d\omega \quad (3)$$

Here, $R = \sqrt{2b^2(1 - \cos\theta)}$ The innermost integral is now readily evaluated by using Sommerfeld's integral. So

$$V(t) = \frac{b^2 \mu_0 I_0}{4\pi} \int_{-\infty}^{\infty} e^{i\omega t} \int_0^{2\pi} \frac{\cos\theta}{R} \left[1 - \frac{2}{k_1^2 R^2}\right.$$

uscript received by the Editor April 2, 1980, revised manuscript received December 1, 1980
merly Cooperative Institute for Research in Environmental Sciences/NOAA, University of Colorado, Boulder, presently GEOPEKO, P O Box 217, lon, N S W 2072, Australia

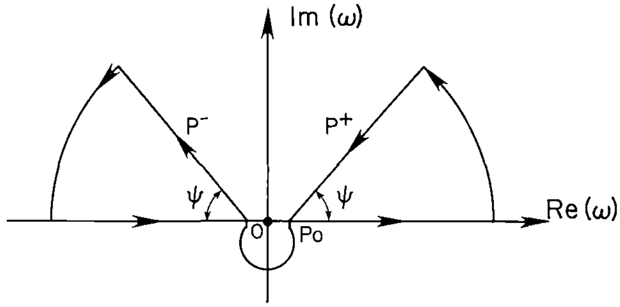


FIG. 1. Path for contour integration

$$+ \frac{2}{k_1^2 R^2}(1 + k_1 R)e^{-k_1 R}\Bigg] \, d\theta \, d\omega. \tag{4}$$

Here, $k_1 = \sqrt{\iota \omega \sigma \mu}$.

The integral with respect to θ may be evaluated by first expanding the exponential integral as a power series and then integrating the resultant absolutely convergent series term by term One finds that

$$V(t) = b\mu_0 I_0 \int_{-\infty}^{\infty} e^{\iota\omega t} \sum_{j=0}^{\infty} (a_j + b_j)\, d\omega$$

where

$$a_j = -2(k_1 b)^{2j+3}(2j + 2!)/[(2j + 5)!\, j!\,(j + 1)!],$$
$$b_j = (k_1 b)^{2j+2} 2^{4j+2} 2^{4j+2} j!\, j!/[(2j + 4)!\,(2j)!\,\pi]. \tag{5}$$

The remaining integral is evaluated by contour integration.

To evaluate the integral with respect to ω by contour integration, one must first examine the singularities of k_1. To ensure that the spectrum of $V(t)$ is single valued in the ω plane, it is necessary to introduce a branch cut in the complex ω plane. This cut extends from the origin to infinity along the positive imaginary axis. Because c is less than unity, this is the only singularity. We then approximate the integral along the real ω axis by three parts (see Figure 1). In the following analysis, we shall allow the radius of the small circle to approach zero.

The integral in equation (5) can be replaced by one around the branch cut (Figure 1). The contributions from the arcs are negligible because of the exponential in the integrand Moreover, the contribution from P_0 vanishes as the radius of that circle becomes infinitesimal. This is because the spectrum of $V(t)$ becomes zero as the modulus of k_1 decreases to zero. The approximate path along the real ω axis thus approaches the original path When Cauchy's theorem is applied to each of the lobes, one finds that the integral in equation (5) is equal to the negative sum of the integrals along the paths P^+ and P^- Since

$$\omega = R\, e^{\iota\psi} \quad \text{on} \quad P^+,$$

and

$$\omega = R\, e^{-\iota\psi - \iota\pi} \quad \text{on} \quad P^-,$$

one finds that

$$k_1^2 = F^2(R)\, e^{\pm\iota\left(\frac{\pi}{2} + \phi + \psi\right)}$$

Here the positive value of the phase of k_1 is taken on P^+ and the negative value is taken on P^- Also ψ must be such that $\psi + \phi \le \pi/2$. Moreover, $F(R)$ and φ are found to be

$$F(R) = \sqrt{\{(R\sigma_0\mu_0)[(1 + \gamma^2)(1 + \alpha^2\gamma^2) + 4\alpha\delta^2 + 2\gamma(1 + \alpha)(1 + \alpha\gamma\delta)]\}/(1 + \alpha^2\gamma^2 + 2\alpha\delta)},$$

where

$$\phi = \arctan[\eta/(1 + \delta)] - \arctan[\alpha\eta/(1 + \alpha\delta)],$$
$$\gamma = (\tau R)^c,$$
$$\delta = \gamma \cos[(\pi/2 + \psi)c],$$
$$\eta = \gamma \sin[(\pi/2 + \psi)c].$$

Notice that for $c < 0.5$,

$$|k_1| < \sqrt{R\sigma_0\mu}.$$

We return to this point later.

Combining equations (5) and (6) one finds that

$$V(t) = 2b\mu_0 I_0 \int_0^{\infty} e^{Rt\cos(\psi + \pi/2)} \Bigg[\sum_{j=0}^{\infty} \cdot \Big\{4\cos(\alpha_j)[bF(R)]^{2j+3}(2j + 2)!/[(2j + 5)!\, j!\,(j + 1)!] - 2\cos(\beta_j)[b\, F(R)]^{2j+2} 2^{4j+2} j!\, j!/[\pi(2j + 4)!\,(2j)!]\Big\}\Bigg] dR$$

where

$$\alpha_j = R\sin(\pi/2 + \psi) + \psi + (2j + 3)(\pi/2 + \psi + \phi$$

and

$$\beta_j = R\sin(\pi/2 + \psi) + \psi + (2j + 2)(\pi/2 + \psi + \phi$$

Equation (7) is not as complicated as it first appears. To see we note that as

$$|\cos(\theta)| \le 1$$

and

$$|F(R)b| < \sqrt{R\sigma_0\mu},$$

then

$$V(t) \le \frac{b^2\mu_0 I_0 2}{b} \int_0^{\infty} e^{+Rt\cos(\pi/2 + \psi)} \cdot \Bigg[\sum_{j=0}^{\infty} \frac{4(b\sqrt{R\sigma_0\mu})^{2j+3}(2j + 2)!}{(2j + 5)!\, j!\,(j + 1)!} + \frac{2}{\pi}\frac{(b\sqrt{R\sigma_0\mu})^{2j+2} 2^{4j} j!\, j!\,(2j + 3)}{(2j + 4)!\,(2j)!}\Bigg] dR$$

$$= \frac{8b\mu_0 I_0}{t\sin\psi} \sum_{j=0}^{\infty} \frac{b[\sqrt{(\sigma_0\mu/(\sin\psi\, t)}]^{2j+3}(2j + 2}{(2j + 5)!\, j!\,(j + 1)!} \cdot \Gamma[1 + (2j + 3)/2] + \frac{2}{\pi}(b\sqrt{\sigma_0\mu/(\sin\psi\, t)})^{2j+2} \frac{2^{4j} j!\,(j + 1)!}{(2j + 4)!\,(2j)!} \cdot \Gamma(1 + j + 2)$$

Equation (8) shows that when $b^2\mu\sigma/t$ is about 0.1 or less ψ is $\pi/4$, then only the first term in equation (7) is require calculate $V(t)$.

EXAMPLES

·s which do not show IP effects

.ere are three cases for which equation (7) can easily be ıated. These cases occur when $\phi = 0$. This happens when either zero, infinite, or when c is zero. When τ is zero, then 0 and $F(R) = \sqrt{\sigma_0 \mu}R$.

·r this case, one finds that

$$t) = -\frac{b\mu_0 I_0 \sqrt{\pi}}{t}\sqrt{\frac{b^2\sigma_0\mu}{4t}}\sum_{j=0}^{\infty}\cdot$$

$$\cdot\frac{(-1)^j(2j+2)!2}{(2j+5)j!(j+1)!(j+2)!}\left(\frac{b^2\sigma_0\mu}{4t}\right)^{j+1} \quad (9)$$

ןuation (9) has been given previously by Lee and Lewis 4, equation (13)]. It represents the transient EM response of ·ound of conductivity σ_0. When τ is infinite, $F(R) = $ $\sigma_0\mu/\alpha$ and

$$(t) = -\frac{b\mu_0 I\sqrt{\pi}}{t}\sqrt{\frac{b^2\sigma_0\mu}{\alpha 4t}}\sum_{j=0}^{\infty}\cdot$$

$$\cdot\frac{(2j+2)!(-1)^j 2}{(2j+5)j!(j+1)!(j+2)!}\left(\frac{b^2\sigma_0\mu}{4\alpha t}\right)^{j+1} \quad (10)$$

equation shows that the ground behaves as though its con- ·ivity were σ_0/α.

milarly, when c is zero, the ground behaves as though it were uniform conductivity $2\sigma_0/(1+\alpha)$.

:s which show IP effects

ıe most important features of equation (8) are the terms that : from the even powers of $(k_1 b)$ in equation (5). As shown 'e, when τ is zero these terms do not contribute to the transient. n τ is not zero, they make an important contribution This is use τ determines whether or not k_1^2 is analytic along the posi- imaginary ω-axis.

feature of these terms is that their coefficients are of opposite to the coefficients from the odd powers of $(k_1 b)$ In addition, ın $(k_1 b)$ are small they are the dominant terms in equation (8). .his basis one would expect to find that the transient EM re- se of the ground changes sign. The point at which the sign ge occurs is determined by τ in particular.

:lton et al (1978) found that c most commonly had a value of ıt 0.25 but that τ could vary considerably. For this reason, :xamples selected here are for a large range of τ. In all cases :adius of the loop is 25 m.

he first example is based on the Lornex (dry porphyry) copper ɔsit of British Columbia. This deposit is apparently typical ther deposits in British Columbia Here the total concentra- of sulfides is low, and they commonly occur in disseminated ı.

,gure 2 shows the calculated transient voltage for the param- ; derived by Pelton et al (1978) for this deposit This is curve ber 3. In this case τ, σ_0, c, and α were 1.0×10^{-4} sec, $\times 10^{-3}$ S/m, 0.16 and 0.54, respectively. We have also ed the uniform ground response for conductivities $\sigma_0/(\alpha)$ σ_0. These are curves 1 and 2, respectively.

lso shown is the transient that would be obtained by neglecting :ven powers of $(k_1 b)$ in equation (5). As might be expected, curve lies between curves 1 and 2, despite the fact that it ·oaches curves 1 and 2 at the early and late stages, respectively re is very little evidence for IP effects in this transient.

ı Figure 2 the most dramatic curve is curve 3. The most striking feature of this curve is its rapid decay at the early stages and its reversal of sign at about 1 msec. From about 2 msec on, it decays at a rate that is similar to the response of a uniform ground. Unless one was aware of the possibility of negative responses, one might not notice the change in sign of the readings or attribute them to a polarity error in connecting the coil

In Figure 3 we have repeated the calculations of Figure 2 for the "copper cities" (dry porphyry) copper deposit of Arizona. In this case we assumed that the second time constant is effectively zero and have taken τ, σ_0, c and α to be 6.9×10^{-3} sec, 6.45×10^{-3} S/m, 0.28 and 0.58, respectively. Although mineralization for the deposit is similar to the previous example, τ is almost two orders of magnitude greater. The calculated transients show the same features that were seen in Figure 2, the only difference being that the transition from positive to negative transient occurs at about 0.4 msec.

In general, the larger τ^c is the earlier the change of sign occurs.

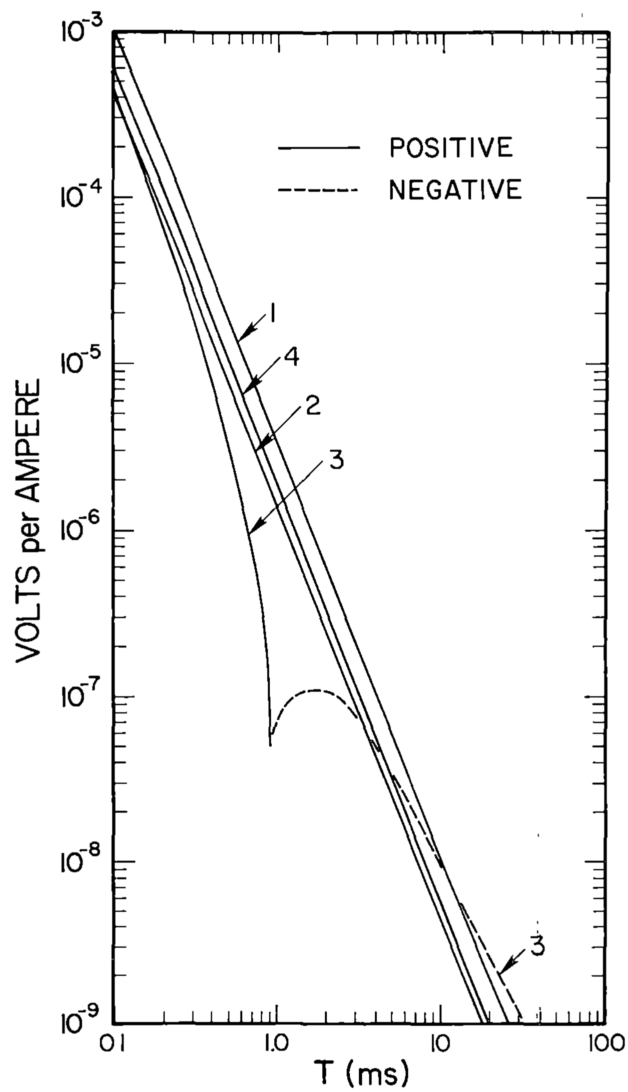


FIG. 2. Transient response of the Lornex copper deposit (curve 3) Curves 1 and 2 are for a uniform ground of conductivity $\sigma_0/(\alpha)$ and σ_0, respectively. Curve 4 shows the effect of neglecting the even powers of $(k_1 b)$ in the calculation of the transient

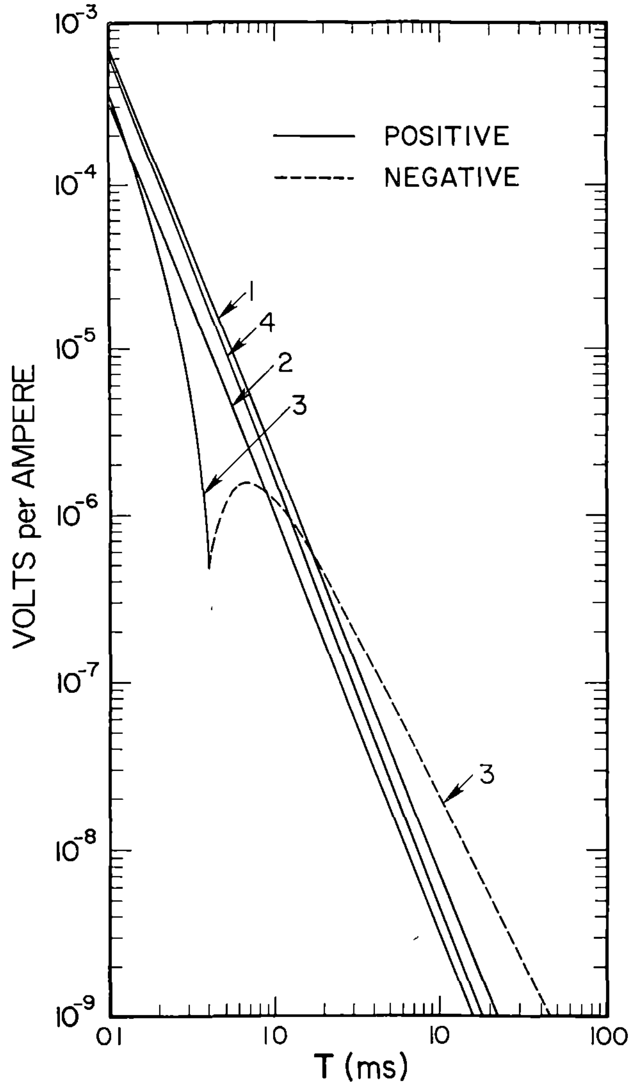


FIG 3 Transient response of the Copper Cities copper deposit (curve 3) Curves 1 and 2 are for a uniform ground of conductivity $\sigma_0/(\alpha)$ and σ_0, respectively Curve 4 shows the effect of neglecting the even powers of $(k_1 b)$ in the calculation of the transient.

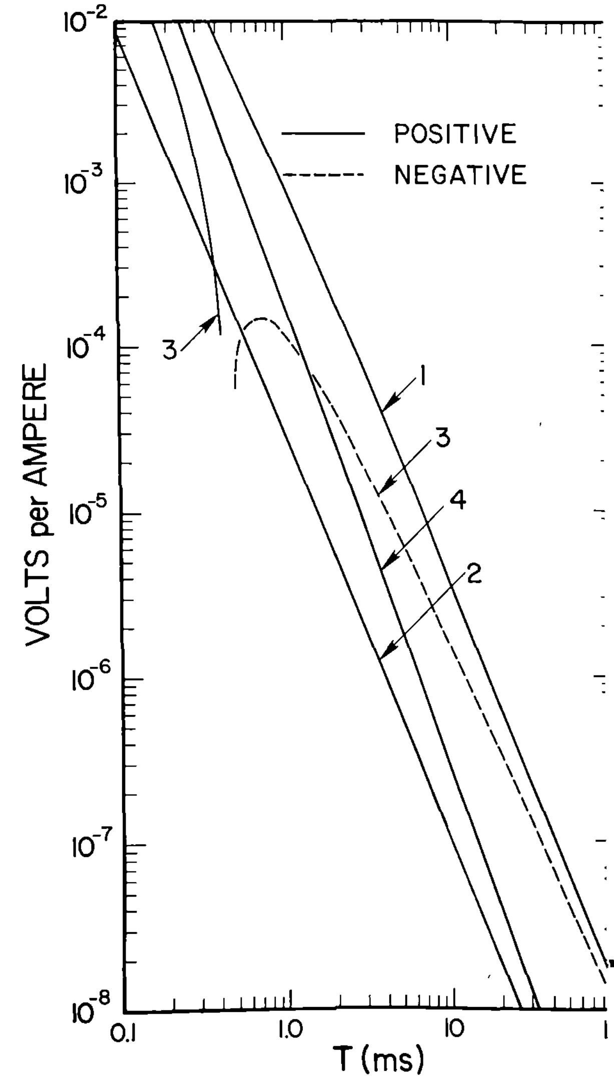


FIG. 4. Transient response of the Kidd Creek massive su deposit (curve 3). Curves 1 and 2 are for a uniform grour conductivity $\sigma_0/(\alpha)$ and σ_0, respectively. Curve 4 shows the e of neglecting the even powers of $(k_1 b)$ in the calculation o transient.

Figure 4 shows a similar calculation repeated for the case of the Kidd Creek massive sulfide deposit of Ontario. In this case, measurements were made in the vicinity of massive sulfides and the values obtained for τ, σ_0, c, and α were 3 08 $\times$ 10^{-2} sec, 6 4 $\times$ 10^{-2} S/m, 0.306 and 0.089, respectively. The results obtained on the basis of these parameters are shown in Figure 4 Features of the curves are similar to those previously described.

DISCUSSION

The results given in Figures 2–4 show that the transient EM method of prospecting is capable of detecting IP effects. Specifically, such transients can be detected for grounds that are composed of materials which have parameters that were measured by Pelton on small samples.

The most striking expression of the IP effect is the change in sign of the transient which occurs at the earlier stages.

Analysis shows that this change in sign occurs because o branch cut singularity in the expression for the frequency pendence of the conductivity.

It is tempting to attempt an explanation of the features o transient in terms of the equivalent circuit that was used by P et al (1978) to fit their field data. We have not done this bec the equivalent circuit is only an idealized representation o mineralized rock. Nevertheless, the results presented above sh provide an impetus for the development of a more realistic m for the electrochemistry of various rock types. It would also b interest to have transient soundings over many of the areas w Pelton et al (1978) carried out their experiments

Some care should be taken with this model because, as we seen, the resultant transients are very sensitive to the singula in the expression for the conductivity as a function of freque Failure to observe this restriction would lead to transients tha quite different from any of those that have been observed.

REFERENCES

ınn, G W., Kıntzınger, P. R , Van Voorhıs, G P , and Ward, ·I., 1970, Evaluatıon of the measurement of induced electrıcal rızatıon wıth an ınductıve system. Geophysıcs, 35, p. 901–918, , dıscussıons in Geophysıcs, v 36, p. 427–429

' J , and Lewıs, R J G , 1974, Transient EM response of a large on a layered ground. Geophys Prosp , v. 22, p 430–444

ı' J , 1975, Sıgn reversals ın the transıent method of electrıcal pectıng (one loop versıon)· Geophys Prosp , v 23, p 453–462

- 1979, Transient electromagnetıc waves applıed to prospectıng. Ǝ Trans and Proc., v 67, 1016–1021.

Morrıson, H F , Phıllıps, R J., and O'Brıen, D P ,1969, Quantıtatıve ınterpretatıon of transıent electromagnetıc fields over a layered half-space Geophys Prosp , v 17, p 82–101

Pelton, W H , Ward, S H , Hallof, P. G , Sıll, W R , and Nelson, P H , 1978, Mıneral dıscrımınation and removal of ınductıve couplıng wıth multı-frequency IP. Geophysıcs, v 43, p 588–609

Rathor, B S , 1978, Transıent electromagnetıc field of a polarızable half-space due to varıous current pulses Geophys Prosp , v 26, p 337–351

Spıes, B R., 1974, Transıent electromagnetıc tests, N T and Queensland 1972. Bur Mın Res , Austr Rec

Reprint from the Review
PURE AND APPLIED GEOPHYSICS (PAGEOPH)
Formerly 'Geofisica pura e applicata'

Vol. 119 No. 2 (1981) Pages 309–338

Transient Electromagnetic Response of a Sphere in a Layered Medium

by

T. J. Lee

BIRKHÄUSER VERLAG BASEL

PAGEOPH, Vol. 119 (1980/81)

0033–4553/81/002309–30$01.50 + 0.20/0

Transient Electromagnetic Response of a Sphere in a Layered Medium

By T. J. LEE[1])

Abstract – The integral equation for the electromagnetic response of a sphere in a layered medium may be solved as follows. First, the unknown time harmonic electric field in the sphere is expanded in spherical vector waves. Secondly, the coefficients for these wave functions are found by a set of equations. The equations are found by multiplying the integral equation throughout by each wave function and integrating over the spherical conductor.

Once the unknown coefficients have been determined, then the transient response may be found by taking the inverse Fourier transform. In carrying out the Fourier transform one learns that for most of the time range used in prospecting, only the lowest order vector wave function is significant. A study of the singularities of the spectrum of the transient shows that, for the time range considered, only a single branch cut is significant. There are no pole type responses. That is, the field does not decay exponentially. Previous studies of a sphere in free space reported only pole type responses. That is, at the later stages, the field decays exponentially. This study shows that, in order to model satisfactorily the effect of the host rock on transient electromagnetic fields, the sphere must be placed in layered ground.

1. Introduction

Recently a mode matching method for the solution of integral equations has been employed to solve a variety of integral equations that are of interest to the geophysicist. For example, HOWARD (1972) has used the method to compute the electric and magnetic fields about a buried cylindrical structure that is excited by a line source of current. Later, LEE (1975) used the method to compute the transient response of a spherical structure in a layered medium that was excited by a large co-axial loop. LEE (1975) has also used the method to obtain solutions for the integral equation describing the potential about a point electrode in which there is buried either spherical or cylindrical structures.

The purpose of this paper is to show how the same methods can be used to solve the very much more difficult problem of a sphere in a layered medium that has been excited by an arbitrary electromagnetic source at the surface.

There are few results available in the literature that can be used to provide a check on the results given here. D'YAKONOV (1959) has solved the problem of finding the

[1]) Cooperative Institute for Research in Environmental Sciences (CIRES) University of Colorado at Boulder, Boulder, Colorado 80309, USA. Present address: P.O Box 386, Epping, N.S.W. 2121, Australia.

electromagnetic response of a sphere in a half space. Unfortunately he did not give any numerical results. Later KRISTENSSON (1979) extended Waterman's T matrix formulations to treat the electromagnetic response of a conductor in a half space. Unfortunately, his numerical results for a spherical obstacle assume that all the materials were lossless. Moreover, a recent review by YAGHJIAN (1980) has questioned many of the other published results.

In the next section we show how the vector wave functions which were introduced by TAI (1971) can provide a suitable set of modes for our mode matched solution. In addition, we also use many of his results for the expansion of the dyadic Green's function (TAI, 1971, 1973). These results have greatly reduced the amount of tedious algebra.

In part three of this paper we show how to transform our solution to the time domain so as to obtain the transient response. The particular case considered is a sphere in a half space. Finally, we compare our solution with some scale model experiments of PALMER (1974). Elsewhere we will present detailed numerical results for particular cases of the geometry considered here.

2. *The integral equation*

Consider the geometry as shown in Fig. 1 and let σ_i and ε_i denote the conductivity and permittivity of the ith region. It will be assumed that the permeability, μ, is equal to the permeability of free space everywhere.

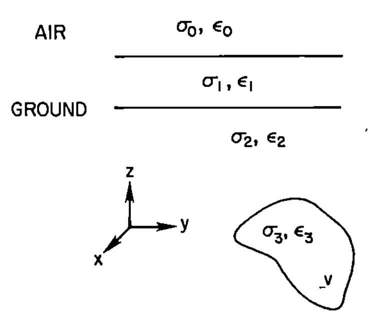


Figure 1
Geometry for the integral equation.

For an $e^{-i\omega t}$ time dependence HOHMANN (1971) has shown that the electric field, $\bar{E}$, in the various regions is:

$$\bar{E} = \bar{E}^i + (k_3^2 - k_2^2) \int_v \bar{\bar{G}}(R|R') \cdot \bar{E}\, dv' \tag{1}$$

Here $\bar{E}^i$ is the electric field for the same distribution of sources when $\sigma_3 = \sigma_2$ and $\varepsilon_2 = \varepsilon_3$. V is the volume of region 3.

$k_j^2 = \omega^2\mu\varepsilon_j + i\mu\omega\sigma_j$ and $\bar{\bar{G}}$ is the dyadic Green's function for the layered region and is a solution of the equation.

$$\nabla \times \nabla \times \bar{\bar{G}}(R|R') - k_j^2\bar{\bar{G}}(R|R') = \bar{\bar{I}}\,\delta(R - R') \tag{2}$$

for $j = 0$, 1 or 2. See Appendix, eqs (10) and (11). Now $\bar{E}$ is a solution of the equation

$$\nabla \times \nabla \times \bar{E} - k_3^2\bar{E} = 0. \tag{3}$$

To solve this equation for the geometry shown in Fig. 2 we set

$$\bar{E} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} A_{\substack{e\\o}}mn\overline{SM}_{\substack{e\\o}}mn(k_3) + B_{\substack{e\\o}}mn\overline{SN}_{\substack{e\\o}}mn(k_3). \tag{4}$$

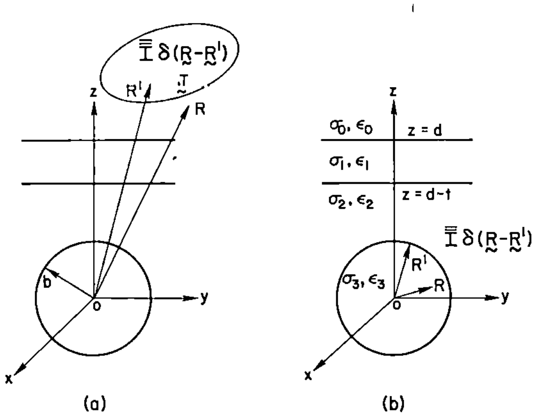


Figure 2

Geometry for the sphere: (a) Geometry for the incident field; (b) Geometry for Green's functions.

The constants $A_{\substack{e\\o}}mn$ and $B_{\substack{e\\o}}mn$, which are the coefficients of the vector wave functions, are to be determined. Notice that $\bar{\bar{G}}_0$ in spherical coordinates is given by:

$$\bar{\bar{G}}_0(\bar{R}|\bar{R}') = \frac{ik_2}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_{0m})\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\cdot$$

$$\begin{cases} \overline{SM}^{(1)}_{\substack{e\\o}}mn(k_2)\overline{SM}'_{\substack{e\\o}}mn(k_2) + \overline{SN}^{(1)}_{\substack{e\\o}}mn(k_2)\overline{SN}'_{\substack{e\\o}}mn(k_2), & R > R' \\ \overline{SM}_{\substack{e\\o}}mn(k_2)\overline{SM}'^{(1)}_{\substack{e\\o}}mn(k_2) + \overline{SN}_{\substack{e\\o}}mn(k_2)\overline{SN}^{(1)\prime}_{\substack{e\\o}}mn(k_2), & R < R' \end{cases}$$

$$-\,[\hat{R}\hat{R}\,\delta(\bar{R}-\bar{R}')]/k_2^2. \qquad \text{(See Appendix, eq. (12))} \tag{5}$$

Here $\overline{SM}^{(1)}_{\substack{e\\o}}mn(k_2)$ means that the vector wave function is defined with respect to the spherical Hankel function of the first kind, i.e. $h_n^1(k_2R)$ replaces $j_n(k_2R)$.

$\overline{SN}^{(1)}\underset{0}{e}mn(k_2 R)$ is similarly defined

$$\delta_{0m} = 1 \qquad m = 0$$
$$= 0 \qquad m \neq 0. \qquad \text{(See Appendix, eq. (12))}$$

Let

$$WM\underset{0}{e}mn(j_n(k_3 b), h_n^{(1)}(k_2 b)) = (bk_2 J_\nu(k_3 b)H_\nu^{1\prime}(k_2 b) - bk_3 J_\nu'(k_3 b)H_\nu^1(k_2 b))\frac{\pi}{2}\sqrt{\frac{1}{k_3 k_2}}$$

where $\gamma = n + 1/2$ and

$$WN\underset{0}{e}mn(j_n(k_3 b), h_n^{(1)}(k_2 b)) = [WM\underset{0}{e}mn - 1(j_{n-1}(k_3 b), h_{n-1}^1(k_2 b))(n+1)$$
$$+ nWM\underset{0}{e}mn+1(j_{n+1}(k_3 b), h_{n+1}^1(k_2 b))]\frac{1}{2n+1}. \qquad (6)$$

Next notice that

$$\overline{\overline{G}}(R \mid R') = \overline{\overline{G}}_0(R \mid R') + \overline{\overline{G}}_s(R \mid R'). \qquad (7)$$

Here $\overline{\overline{G}}_0(R \mid R')$ is the primary term in Green's function and $\overline{\overline{G}}_s$ is the secondary term. $\overline{\overline{G}}_s$ is variously defined depending upon the number of layers for a two-layered medium; it is the $\overline{\overline{G}}_{22}(R \mid R')$ of eqn. (11) otherwise it is the $\overline{\overline{G}}_{s1}(R \mid R')$ of eq. (10) in the Appendix. A comparison of the results in the Appendix shows that both $\overline{\overline{G}}_{s1}$ and $\overline{\overline{G}}_{22}$ are similar in construction. The only differences are the coefficients for the wave functions. Because of this if one solves for the coefficient in eq. (3) for a uniform ground then the corresponding coefficients for a two-layered ground may be written down immediately, or vice versa.

A set of equations for the coefficients in eq. (3) may be found as follows. First, substitute eq. (3) into eq. (1) and use the results in the Appendix (eqs (6 (ii, iii) and (7)) to effect the integrations.

When careful consideration is given to the integration of the Hankel functions one finds that

$$0 = \bar{E}^i + \sum_{n=1}^{\infty}\sum_{m=0}^{n}(k_3^2 - k_2^2)\int_V \overline{\overline{G}}_s(R \mid R')\cdot(A\underset{0}{e}mn\overline{SM}'\underset{0}{e}mn(k_3) + B\underset{0}{e}mn\overline{SN}'\underset{0}{e}mn(k_3))\,dv'$$
$$+ ik_2 A\underset{0}{e}mn WM\underset{0}{e}mn(j_n(k_3 b), h_n^1(k_2 b))\overline{SM}\underset{0}{e}mn(k_2 R)$$
$$+ ik_2 B\underset{0}{e}mn WN\underset{0}{e}mn(j_n(k_3 b), h_n^1(k_2 b))\overline{SN}\underset{0}{e}mn(k_2 R). \qquad (8)$$

The remaining integral can be evaluated by first expanding the Green's function in terms of spherical waves.

From eqs (10) and (11) in the Appendix we find that

$$\overline{\overline{G}}_s = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{r=0}^{\infty}\frac{2-\delta_{0m}}{\lambda h_2}[a_2\overline{M}\underset{0}{e}r\lambda(-h_2)\overline{M}'\underset{0}{e}r\lambda(h_2) + b_2\overline{N}\underset{0}{e}r\lambda(-h_2)\overline{N}'\underset{0}{e}r\lambda(h_2)]. \qquad (9)$$

In this equation a_2 and b_2 must be chosen in accordance with the number of layers. The $\bar{M}$ and $\bar{N}$ wave functions can be expanded in terms of spherical wave functions by using eq. (9 (i, ii)) in the Appendix. Therefore

$$\bar{\bar{G}}_s = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{r=0}^\infty \frac{2-\delta_{0m}}{\lambda h_2}$$

$$\Bigg[a_2 \overline{M}\genfrac{}{}{0pt}{}{e}{o}r\lambda(-h_2)\cdot\Bigg(\sum_{s=r}^\infty (\mp k_2\, r)\, [i^{s-1}\underset{s(s+1)}{(2s+1)}P_s^r(\cos\psi)\,(s-r)\,!/(s+r)!\{(-1)^s\,\overline{SN}'\genfrac{}{}{0pt}{}{o}{e}rs\}]$$

$$+ k_2 \sum_{s=r}^\infty \frac{i^{s-1}(2s+1)}{2s+1} P_s^r(\cos\psi)\,(s-r)!/(s+r)!\left\{\left(\frac{s-r+1}{s+1}\right)\overline{SM}'\genfrac{}{}{0pt}{}{e}{o}r(s+1)\right.$$

$$\left.+\frac{r+s}{s}\,\overline{SM}'\genfrac{}{}{0pt}{}{e}{o}r(s-1)\right\}(-1)^s\Bigg)$$

$$+ b_2 \overline{N}\genfrac{}{}{0pt}{}{e}{o}r\lambda(-h_2)\left(\mp k_2\, r \sum_{s=r}^\infty \left(\frac{i^{s-1}(2s+1)}{s(s+1)} P_s^r(\cos\psi)\,\frac{(s-r)!}{(s+r)!}\cdot(-1)^s\cdot\overline{SM}'\genfrac{}{}{0pt}{}{o}{e}rs\right)\right.$$

$$\left. + k_2 \sum_{s=r}^\infty i^{s-1}(-1)^s \frac{P_s^r(\cos\psi)\,(s-r)!}{(s+r)!}\left[\left(\frac{s-r+1}{s+1}\right)\overline{SN}'\genfrac{}{}{0pt}{}{e}{o}r(s+1)+\frac{s+r}{s}\,\overline{SN}'\genfrac{}{}{0pt}{}{e}{o}(s-1)\right]\right)\Bigg] \tag{10}$$

The integration in eq. (8) can now be performed by using the orthogonal conditions for the $\overline{SN}$ and $\overline{SM}$ functions. These are results (6 (i)), (6 (ii)) and (6 (iii)) in the Appendix.

For convenience we write

$$CM\genfrac{}{}{0pt}{}{e}{o}mn = \frac{2(1+\delta_{0m})n}{2n+1}\pi(n+1)\frac{(n+m)!}{(n-m)!}\cdot A\genfrac{}{}{0pt}{}{e}{o}mn(k_3^2-k_2^2)\int_0^b R^2 j_n(k_2 R) j_n(k_3 R)\, dR, \qquad n \geq 0$$

$$= 0, \qquad n < 0$$

and

$$CN\genfrac{}{}{0pt}{}{e}{o}mn = \frac{2(1+\delta_{0m})\pi n(n+1)}{(2n+1)}\frac{(n+m)!}{(n-m)!}\cdot B\genfrac{}{}{0pt}{}{e}{o}mn(k_3^2-k_2^2)\int_0^b \frac{R^2}{2n+1}$$

$$\times\,((n+1)j_{n-1}(k_2 R) j_{n-1}(k_3 R) + n j_{n+1}(k_2 R) j_{n+1}(k_3 R))\, dR, \qquad n \geq 0$$

$$= 0 \qquad n < 0 \tag{11}$$

One now finds that:

$$
\begin{aligned}
0 = \bar{E}^i &+ \frac{ik_2}{1} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (A\underset{o}{e}mnWM\underset{o}{e}mn(j_n(k_3 b), h_n^1(k_2 b)) \cdot \overline{SM}\underset{o}{e}mn(k_2 R) + \\
&+ B\underset{o}{e}mnWN\underset{o}{e}(j_n(k_3 b), h_n^1(k_2 b)) \cdot \overline{SN}\underset{o}{e}mn(k_2 R)) \\
&+ \frac{i}{4\pi} \int_0^{\infty} d\lambda \sum_{r=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} \left[a_2 \bar{M}\underset{o}{e}r\lambda(-h_2) \cdot \left(\sum_{s=r}^{\infty} (\mp k_2 r) \right.\right. \\
&\times [i^{s-1} \underbrace{(2s+1)}_{s(s+1)} P_s^r(\cos\psi)(s-r)!/(s+r)! \, \{CN\underset{e}{o}rs\}] \\
&+ k_2 \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{(2s+1)} P_s^r(\cos\psi)(s-r)!/(s+r)! \left\{ \frac{(s-r+1)}{(s+1)} CM\underset{o}{e}r(s+1) \right. \\
&\left.\left. + \frac{r+s}{s} CM\underset{o}{e}r(s-1) \right\} \right) + b_2 \bar{N}\underset{o}{e}r\lambda(-h_2) \left(\mp k_2 r \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{s(s+1)} \right. \\
&\times P_s^r(\cos\psi) \cdot \frac{(s-r)!}{(s+r)!} (CM\underset{e}{o}rs) + k_2 \sum_{s=r}^{\infty} i^{s-1} P_s^r \frac{(\cos\psi)(s-r)!}{(s+r)!} \\
&\times \left[\left(\frac{s-r+1}{s+1} \right) CN\underset{o}{e}r(s+1) + \frac{s+r}{s} CN\underset{o}{e}r(s-1) \right] \Bigg) \Bigg]. \qquad (12^*)
\end{aligned}
$$

To obtain a set of equations for the $A\underset{o}{e}mn$ and the $B\underset{o}{e}mn$ one first expands $\bar{E}^i$, $M\underset{o}{e}r\lambda(-h_2)$ and $N\underset{o}{e}r\lambda(-h_2)$ of eq. (12) in terms of spherical wave functions and then forms the scalar product of the resultant equation with $(k_3^2 - k_2^2)\overline{SN}\underset{o}{e}mn(k_3)$ and $(k_3^2 - k_2^2)\overline{SM}\underset{o}{e}mn(k_3)$ and then integrates over the sphere of radius b.

$$\bar{E}^i = i\omega\mu \iiint \bar{\bar{G}}^i(R \,|\, R')\bar{J}(R')\, dv'. \qquad (13)$$

Here $G^i(RR')$ is a Green's dyadic and $\bar{J}(R')$ is the current density throughout the volume v' which confines the source energizing apparatus.

$\bar{G}^i(R \,|\, R')$ is defined in the Appendix. For a halfspace it is G_1^i of eq. 10 and for a layered ground it is G_2^i of eq. (11). Notice that in these equations only the coefficients of the wave functions have changed.

From the Appendix then:

$$
\begin{aligned}
\bar{E}^i = i\omega\mu \iiint \frac{i}{4\pi} \int_o^{\infty} d\lambda \sum_{u=0}^{\infty} \left(\frac{2-\delta_{0m}}{\lambda h_0} \right) &[\cdot \bar{cJ} \cdot \overline{M'}\underset{o}{e}u\lambda(h_0)\overline{M}\underset{o}{e}u\lambda(-h_2) \\
&+ d\bar{J}_0 \cdot \bar{N}'\underset{o}{e}u\lambda(h_0)\bar{N}\underset{o}{e}u\lambda(-h_2)]\, dv'. \qquad (14)
\end{aligned}
$$

$* \; \frac{r+s}{s} CM\underset{o}{e}rs - 1 = \frac{r+s}{s} CN\underset{o}{e}rs - 1 = 0 \quad$ for $\quad r = s = 0.$

Looking at eq. (4) one notices that there are four sets of constants. They are *Aemn*, *Aomn*, *Bemn* and *Bomn*.

Forming the scalar products one finds the following sets of equations. For the *Aemn* one obtains.

$$-i\omega\mu \iiint dv' \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{r=0}^\infty \frac{2-\delta_{0m}}{\lambda h_0}\left[c\bar{J}\cdot\bar{M}'er\lambda(h_0)\sum_{s=r}^\infty\right.$$

$$+(k_2)i^{s-1}\frac{(2s+1)}{(2s+1)}P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}$$

$$\times\left\{\left(\frac{s-r+1}{s+1}\right)DM_{er(s+1)}\frac{\delta mr\,\delta n(s+1)(-1)^{s+r+1}}{1}\right.$$

$$\left.+\frac{(r+s)}{s}(-1)^{s-1+r}DM_{er(s-1)}\,\delta mr\,\delta n(s-1)\right\}$$

$$+d\bar{J}\cdot\bar{N}_0'm\lambda(h_0)i^{n-1}(2n+1)P_n^m(\cos\psi)\frac{(n-m)!}{(n+m)!}$$

$$\left.\times\left\{\frac{(2-\delta_0)}{\lambda h_0}\frac{k_2 m}{n(n+1)}DM_{emn}(-1)^{m+n}\right\}\right]=\frac{ik_2}{1}WMmn(j_n(k_3 b),h_n^1(k_2 b))\cdot CM_{emn}$$

$$+\frac{i}{4\pi}\int_0^\infty d\lambda\sum_{r=0}^\infty\frac{2-\delta_{0m}}{\lambda h_0}\left[a_2\left(\sum_{s=r}^\infty(-k_2 r)\left[\frac{i^{s-1}(2s+1)}{s(s+1)}\cdot P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}(CN_{ors})\right]\right.\right.$$

$$+k_2\sum_{s=r}^\infty\left(\frac{i^{s-1}(2s+1)}{(2s+1)}P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}\left\{\frac{(s-r+1)}{(s+1)}CM_{er(s+1)}\right.\right.$$

$$\left.\left.\left.+\frac{(r+s)}{s}CM_{er(s-1)}\right\}\right)\right)\left(\sum_{s=r}^\infty-(k_2)i^{s-1}\frac{(2s+1)}{2s+1}P_s^r(\cos\psi)\cdot\frac{(s-r)!}{(s+r)!}\right.$$

$$\times\left\{\frac{(s-r+1)}{s+1}DM_{er(s+1)}\quad\delta mr\,\delta n(s+1)(-1)^{s+1+r}\right.$$

$$\left.\left.+\frac{(r+s)(-1)^{s-1+r}}{s}DM_{er(s-1)}\,\delta mr\,\delta n(s-1)\right\}\right)$$

$$+b_2\left(+k_2 r\sum_{s=r}^\infty\frac{i^{s-1}(2s+1)}{s(s+1)}P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}(CMers)+k_2\sum_{s=r}^\infty\right.$$

$$\left.i^{s-1}P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}\left[\left(\frac{s-r+1}{s+1}\right)CN_{or(s+1)}+\frac{s+r}{s}CN_{or(s-1)}\right]\right)$$

$$\left.\times\left(i^{n-1}\frac{(2n+1)}{n(n+1)}P_n^m(\cos\psi)\frac{(n-m)!}{(n+m)!}mk_2\frac{CMemn(-1)^{n+m}}{1}\right)\right].\qquad(15)$$

For the *Aomn* one obtains

$$-i\omega\mu \iiint dv' \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{r=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}\, c\bar{J}\cdot \overline{M}'_{or}\lambda(h_0) \sum_{s=r}^{\infty}$$

$$+ k_2\, i^{s-1} \frac{(2s+1)}{(2s+1)} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!}$$

$$\times \left\{ \frac{(s-r+1)}{s+1} DM_{or(s+1)}\, \delta mr\, \delta n(s+1)(-1)^{s+r+1} \right.$$

$$\left. + \frac{(r+s)}{s}(-1)^{s-1+r} DM_{or(s-1)}\, \delta mr\, \delta n(s-1) \right\}$$

$$- d\bar{J}\cdot \overline{N}' em\lambda(h_0) i^{n-1}(2n+1) P_n^m(\cos\psi) \frac{(n-m)!}{(n+m)!} \left\{ \frac{k_2 m}{n(n+1)} DM_{omn}(-1)^{m+n} \right\}$$

$$\times \frac{(2-\delta_{0m})}{\lambda h_0} = \frac{ik_2}{1} WMomn(j_n(k_3 b), h_n^1(k_2 b)) CM_{omn}$$

$$+ \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{r=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} \left(a_2 \left(\sum_{s=r}^{\infty} \left[(+k_2 r) i^{s-1} \underset{s(s+1)}{\underline{(2s+1)}} \cdot P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} (CN_{ers}) \right] \right.\right.$$

$$+ k_2 \sum_{s=r}^{\infty} \left(\frac{i^{s-1}(2s+1)}{2s+1} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} \left\{ \frac{(s-r+1)}{(s+1)} CM_{or(s+1)} \right.\right.$$

$$\left.\left.\left.\left. + \frac{(r+s)}{s} CM_{or(s-1)} \right\}\right)\right)\left(\sum_{s=r}^{\infty} (k_2)\, i^{s-1} \frac{(2s+1)}{2s+1} P_s^r(\cos\psi) \cdot \frac{(s-r)!}{(s+r)!} \right.$$

$$\times \left\{ \frac{(s-r+1)}{s+1} DM_{or(s+1)}\, \delta mr\, \delta n(s+1)(-1)^{s+1+r} \right.$$

$$\left.\left. + \frac{(r+s)(-1)^{s-1+r}}{s} DM_{or(s-1)}\, \delta mr\, \delta n(s-1) \right\}\right)$$

$$+ b_2 \left(-k_2 r \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{s(s+1)} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} \left(CM_{0rs} \right.\right.$$

$$\left.\left. + k_2 \sum_{s=r}^{\infty} i^{s-1} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} \left[\frac{(s-r+1)}{(s+1)} CN_{er(s+1)} + \frac{s+r}{s} CN_{er(s-1)} \right] \right)\right)$$

$$\times \left((-1) i^{n-1} m(2n+1) P_n^m(\cos\psi) \frac{(n-m)!}{(n+m)!} k_2 \frac{DM_{omn}(-1)^{n+m}}{n(n+1)} \right)\Bigg) \qquad (16)$$

For the *Bemn* one obtains

$$+ i\omega\mu \iiint dv' \frac{i}{4\pi}\int_o^\infty d\lambda \sum_{r=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}$$

$$\times \left[c\bar{J}\cdot\bar{M}'_{0m}\,\lambda(h_0) i^{n-1}(2n+1)\,P_n^m(\cos\psi)\,\frac{(n-m)!}{(n+m)!}\left\{\frac{+k_2\,m}{n(n+1)}\,DN_{emn}(-1)^{n+m}\right\}\delta mr\right.$$

$$+ d\bar{J}\cdot\bar{N}'_{er}\,\lambda(h_0)\sum_{s=r}^{\infty} k_2\, i^{s-1}\,P_s^r(\cos\psi)\,\frac{(s-r)!}{(s+r)!}$$

$$\times\left\{\frac{(s-r+1)}{s+1}\,DN_{er(s+1)}\,\delta mr\,\delta n(s+1)(-1)^{s+1+r}\right.$$

$$\left.\left. + \frac{r+s}{s}(-1)^{s-1+r}\,DN_{er(s-1)}\,\delta mr\,\delta n(s-1)\right\}\right]$$

$$= \left[\frac{ik_2}{1}\,WN_{emn}\,(j_n(k_3\,b),\,h_n^1(k_2\,b))\right]CN_{emn} + \frac{i}{4\pi}\int_0^\infty d\lambda\sum_{r=0}^{\infty}\frac{2-\delta_{0m}}{\lambda h_2}\left[a_2\left(\sum_{s=r}^{\infty}(k_2\,r)\right.\right.$$

$$\times\left[i^{s-1}\underset{s(s+1)}{(2s+1)}\cdot P_s^r(\cos\psi)\,\frac{(s-r)!}{(s+r)!}\,(CN_{ers})\right]$$

$$+ k_2\sum_{s=r}^{\infty}\frac{i^{s-1}(2s+1)}{2s+1}\,P_s^r(\cos\psi)\,\frac{(s-r)!}{(s+r)!}\left\{\frac{(s-r+1)}{(s+1)}\,CM_{or(s+1)}\right.$$

$$\left.\left. + \frac{r+s}{s}\,CM_{or(s-1)}\right\}\right)\left(i^{n-1}(2n+1)\,P_n^m(\cos\psi)\,\frac{(n-m)!}{(n+m)!}\cdot\frac{mk_2\,DN_{emn}(-1)^{n+m}}{n(n+1)}\right)$$

$$+ b_2\left(-k_2\,r\sum_{s=r}^{\infty}\frac{i^{s-1}(2s+1)}{s(s+1)}\,P_s^r(\cos\psi)\,\frac{(s-r)!}{(s+r)!}\,(CM_{ors})\right.$$

$$\left. + k_2\sum_{s=r}^{\infty} i^{s-1}\,P_s^r(\cos\psi)\,\frac{(s-r)!}{(s+r)!}\left(\frac{(s-r+1)}{s+1}\,CN_{er(s+1)} + \frac{s+r}{s}\,CN_{e(s-1)}\right)\right)$$

$$\times\left(\sum_{s=r}^{\infty} k_2\, i^{s-1}\,\frac{(2s+1)}{2s+1}\,P_s^r(\cos\psi)\cdot\frac{(s-r)!}{(s+r)!}\right.$$

$$\times\left\{\frac{(s-r+1)}{s+1}\,DN_{er(s+1)}\,\delta mr\,\delta n(s+1)(-1)^{s+1+r}\right.$$

$$\left.\left.\left. + \frac{(r+s)(-1)^{s-1+r}}{s}\,DN_{er(s-1)}\,\delta mr\,\delta n(s-1)\right\}\right)\right] \tag{17}$$

For the *Bomn* one obtains

$$+ i\omega\mu \iiint dv' \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{r=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}$$

$$\times \Bigg[-c\bar{J}\cdot \overline{M}_e'\, m\lambda(h_0)\, i^{n-1}(2n+1)P_n^m(\cos\psi)\frac{(n-m)!}{(n+m)!}$$

$$\times \left\{ \delta mr \frac{k_2 m}{n(n+1)} DN_{omn}(-1)^{m+n} \right\}$$

$$+ d\bar{J}\cdot N_{or}'\lambda(h_0) \sum_{s=r}^{\infty} k_2\, i^{s-1}\frac{2s+1}{2s+1} P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}$$

$$\times \left\{ \frac{(s-r+1)}{(s+1)} DN_{or(s+1)}\cdot \delta mr\, \delta n(s+1)(-1)^{s+1+r} \right.$$

$$\left. + \frac{(r+s)}{s}(-1)^{s-1+r} DN_{or(s-1)}\, \delta mr\, \delta n(s-1) \right\} \Bigg]$$

$$= \frac{ik_2}{1} WN_{omn}(j_n(k_3 b), h_n^1(k_2 b))CN_{omn} + \frac{i}{4\pi}\int_o^\infty d\lambda \sum_{r=0}^{\infty}\frac{2-\delta_{0m}}{\lambda h_2}\Bigg[a_2\Bigg(\sum_{s=r}^{\infty} -(k_2 r)$$

$$\times \left[i^{s-1}\underset{s(s+1)}{\underline{(2s+1)}}\cdot(-1)^{s+r} P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}(CN_{ors})\right] + k_2\sum_{s=r}^{\infty}$$

$$\times (-1)^{s+r}\, i^{s-1}\frac{(2s+1)}{2s+1} P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}\left\{\frac{(s-r+1)}{(s+1)} CM_{er(s+1)}\right.$$

$$\left. + \frac{(r+s)}{s} CM_{er(s-1)}\right\}\Bigg)\Bigg((-1)i^{n-1}(2n+1)P_n^m(\cos\psi)\cdot(-1)^{s+m}$$

$$\times \frac{(n-m)!}{(n+m)!}\frac{k_2 m DN_{omn}(-1)^{n+m}}{n(n+1)}\Bigg)$$

$$+ b_2\Bigg(k_2 r \sum_{s=r}^{\infty}\frac{i^{s-1}(2s+1)}{s(s+1)} P_s^r(\cos\psi)\frac{(s-r)!}{(s+r)!}((-1)^{s+r} CM_{ers})$$

$$+ k_2\sum_{s=r}^{\infty} i^{s-1} P_s^r(\cos\psi)(-1)^s\frac{(s-r)!}{(s+r)!}\left(\left(\frac{s-r+1}{s+1}\right) CN_{er(s+1)} + \frac{(s+r)}{s} CN_{er(s-1)}\right)\Bigg)$$

$$\times \left(\sum_{s=r}^{\infty} k_2\, i^{s-1}\frac{(2s+1)}{2s+1} P_s^r(\cos\psi)\cdot\frac{(s-r)!}{(s+r)!}\right.$$

$$\times \left\{ \left(\frac{s-r+1}{s+1} \right) DN_{or(s+1)}\, \delta mr\, \delta n(s+1)(-1)^{s+1+r} \right.$$

$$\left. + \frac{(r+s)(-1)^{s-1+r}}{s} DN_{or(s-1)}\, \delta mr\, \delta n(s+1) \right\} \Bigg) \Bigg] \tag{18}$$

Here

$$DN\underset{o}{e}mn = CN\underset{o}{e}mn / B\underset{o}{e}mn$$
$$DM\underset{o}{e}mn = CM\underset{o}{e}mn / A\underset{o}{e}mn$$
$$\delta_{ij} = 1 \quad i = j$$
$$= 0 \quad i \neq j$$

Once the coefficients have been found, it is a simple matter to calculate the electric field.

Substituting eq. (4) in eq. (1) one finds that at the earth's surface

$$\bar{E} = \bar{E}^{i} + (k_3^2 - k_2^2) \int_V \bar{\bar{G}}_s(R|R') \cdot \sum_{n=1}^{\infty} \sum_{m=0}^{n} (A\underset{o}{e}mn\overline{SM}\underset{o}{e}mn(k_3) + B\underset{o}{e}mn\overline{SN}\underset{o}{e}mn(k_3))\, dv'. \tag{19}$$

$\bar{\bar{G}}_s(R|R')$ is given in the Appendix.

When the host medium is a halfpsace $\bar{\bar{G}}_s(R|R')$ is $\bar{\bar{G}}_{00}(R|R')$ of result (10) in the Appendix.

When the host medium is layered ground then $\bar{\bar{G}}_s(R|R')$ is $\bar{\bar{G}}_{20}(R|R')$ of eq. (11) in the Appendix.

The integrations in eq. (19) can be performed by using results (9 (i, ii)) and (6 (i, ii, iii)) in the Appendix.

One finds that

$$\bar{E} = \bar{E}^{i} + \frac{i}{4\pi} \int_0^{\infty} d\lambda \sum_{r=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} \left[g\bar{M}\underset{o}{e}r\lambda(h_0) \left(\sum_{s=r}^{\infty} (\mp k_2 r) \right.\right.$$

$$\times \left[i^{s-1}(2s+1) P_s^r(\cos\psi) \frac{(s-r)!/((s+r)!)}{s(s+1)} \{CN\underset{e}{o}rs\} \right]$$

$$+ k_2 \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{(2s+1)} P_s^r(\cos\psi)(s-r)!/(s+r)! \left\{ \frac{(s-r+1)}{s+1} CM\underset{o}{e}r(s+1) \right.$$

$$\left.\left. + \frac{r+s}{s} CM\underset{o}{e}r(s-1) \right\} \right)$$

$$+ h\bar{N}\underset{o}{e}\lambda(h_0) \left(\mp k_2 r \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{s(s+1)} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} \cdot (CM\underset{e}{o}rs) \right.$$

$$+ k_2 \sum_{s=r}^{\infty} i^{s-1} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} \cdot \left(\left[\frac{(s-r+1)}{s+1} CN_{\mathfrak{g}}r(s+1) \right.\right.$$

$$\left.\left. + \frac{s+r}{s} CN_{\mathfrak{g}}r(s+1) \right] \right) \Bigg) \Bigg] \tag{20}$$

Here the coefficients g and h are determined by the number of layers in the ground. See the Appendix.

From eq. (20) it is a very easy matter to calculate the magnetic fields.

From Maxwell's equations one has

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{21}$$

However,

$$\nabla \times \bar{N}(k) = k\bar{M}(k)$$

and

$$\nabla \times \bar{M}(k) = k\bar{N}(k).$$

Here N and M represent the respective N and M wave functions in the Appendix. Thus an expression for $i\omega\bar{B}$ may be obtained from eq. (20) by simply replacing $N(k)$ by kM and $M(k)$ by kN.

When the fields are required beneath the layer they are easily found to be:

$$\bar{E} = \bar{E}^i + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\overline{SM}^{(1)}{}_{\mathfrak{g}}mn(k_2) CM_{\mathfrak{g}}mn + \overline{SN}^{(1)}{}_{\mathfrak{g}}mn(k_2) CN_{\mathfrak{g}}mn \right)$$

$$\times \left(\frac{ik_2}{4\pi} \frac{(2-\delta_0)(2n+1)}{n(n+1)} \frac{(n-m)!}{(n+m)!} \right)$$

$$+ \frac{i}{4\pi} \int_0^{\infty} d\lambda \sum_{r=0}^{\infty} \frac{(2-\delta_{0m})}{\lambda h_2} g\bar{M}_{\mathfrak{g}}r\lambda(h_2) \left(\sum_{s=r}^{\infty} (\mp k_2 r) \right.$$

$$\times \left[i^{s-1}(2s+1) P_s^r \frac{(\cos\psi)(s-r)!}{s(s+1)} / ((s+r)!) \left\{ CN_{\mathfrak{g}}rs \right\} \right]$$

$$+ k_2 \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{(2s+1)} P_s^r(\cos\psi)(s-r)!/((s+r)!) \left\{ \frac{(s-r+1)}{s+1} CM_{\mathfrak{g}}r(s+1) \right.$$

$$\left.\left. + \frac{r+s}{s} CM_{\mathfrak{g}}r(s-1) \right\} \right)$$

$$+ h\bar{N}_{\mathfrak{g}}r\lambda(h_2) \left(\mp k_2 r \sum_{s=r}^{\infty} \frac{i^{s-1}(2s+1)}{s(s+1)} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} (CM_{\mathfrak{g}}rs) \right.$$

$$+ k_2 \sum_{s=r}^{\infty} i^{s-1} P_s^r(\cos\psi) \frac{(s-r)!}{(s+r)!} \left[\frac{(s-r+1)}{s+1} CN_o^g r(s+1) \right.$$

$$\left. \left. + \frac{s+r}{s} CN_o^g r(s+1) \right] \right) \tag{22}$$

Here the functions g and h are selected from the Appendix in accordance with the number of layers.

The magnetic fields can be found in a manner analogous to that described above.

3. *Transient response*

The transient response may be obtained by taking the inverse Fourier transform of the electric field, that is,

$$\overline{\varepsilon(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \bar{E}(\omega)\, d\omega.$$

A problem of practical importance is the transient response of a sphere in a layered medium. In this case the Fourier transform can be evaluated by contour integration once a suitable path is selected in the lower ω plane.

Since the frequency response is damped in the lower half plane it is only necessary to consider the behavior of the fields for small $k_2 b$. The precise meaning of this statement will emerge as the analysis proceeds. Thus the first step in the evaluation of eq. (23) is to derive an expression for $\bar{E}(\omega)$ that is valid when $k_2 b$ is small.

An inspection of the equations for the coefficients A_o^emn and B_o^emn shows that these quantities depend upon DM_o^emn, DN_o^emn, WM_o^emn and WN_o^emn. For small $(k_2 b)$, then, these quantities are easily estimated. There are two important features to notice about these quantities. First, the $WM_o^emn(j_n, h_n^1)$ and the $WN_o^emn(j_n, h_n^1)$ are inversely proportional to some power of $k_2 b$. Secondly the division of $WM_o^emn(j_n, h_n^1)$ or $WN_o^emn(j_n, h_n^1)$ into the left hand side of the eqs (15), (16), (17) and (18) yields terms in powers of $k_2 b$. Consequently, for small $k_2 b$ only the first term in the equation is relevant.

Suppose that a large loop of radius 'a' is moved across, but not in contact with, the ground, and that a current of $-I_0/i\omega$ flows in the loop (see Fig. 3).

For this case one finds that the only relevant coefficients are

$$Ae_{01} = 3\mu_0 a T_0 I_0/(4k_2^2 WMe_{01}(j_1(k_3 b), h_1^1(k_2 b))$$
$$Ae_{11} = -3\mu_0 a T_{1a} I_0/(4k_2^2 WMe_{01}(j_1(k_3 b), h_1^1(k_2 b))$$

and

$$B_{011} = 3\mu_0\, a T_{1b}\, I_0/(4k_2\; WN_{011}^1(j_1(k_3\, b), h_1^1(k_2\, b))$$

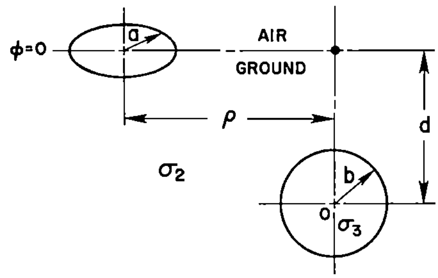


Figure 3
Loop traversing a buried sphere.

Where

$$T_0 = + \int_0^\infty \frac{e^{ih_1 d}\, 2\lambda^2 J_0(\lambda\rho) J_1(\lambda a)}{(h_1 + i\lambda)i}\, d\lambda$$

$$T_{1a} = \int_0^\infty \frac{k_2^2 2 P_2^1(\cos\psi)\, e^{ih_1 d} J_1(\lambda a) J_1(\lambda\rho)}{3(i\lambda + h_1)i}\, d\lambda$$

$$T_{1b} = - \int_0^\infty \frac{2\lambda\, e^{ih_1 d}}{i\lambda + h_1} J_1(\lambda a) J_1(\lambda\rho)\, d\lambda \tag{25}$$

for a half space. For a layered ground

$$T_0 = \int_0^\infty -\lambda g\, e^{ih_0 d} J_0(\lambda\rho) J_0(\lambda a)\, d\lambda$$

$$T_{1a} = k_2^2 \int_0^\infty - \frac{P_2^1(\cos\psi)\, e^{ih_0 d} g J_1(\lambda a) J_1(\lambda\rho)}{3\lambda}\, d\lambda$$

$$T_{1b} = + \int_0^\infty ig\, e^{ih_0 d} J_1(\lambda\rho) J_1(\lambda a)\, d\lambda \tag{26}$$

See the Appendix (eq. (11 (i))).

In eqs (25) and (26) ρ is the distance of the centre of the loop from the vertical axis of the sphere.

Unless the ground is a half space the integrals in eqs (25) and (26) must be done numerically. For the special case of a half space it is possible to evaluate the integrals analytically. Indeed, if we follow the procedure of WAIT and CAMPBELL (1953) we find that for a half space

$$T_0 = \frac{1}{\pi k_2^2} \int_0^{2\pi} \frac{\partial}{\partial a} U\, d\phi$$

$$T_{1a} = - \frac{i}{\pi k_2^2} \int_0^{2\pi} \frac{\partial}{\partial d} U \cos\phi\, d\phi$$

$$T_{1b} = + \frac{i}{\pi k_2^2} \int_0^{2\pi} U \cos\phi\, d\phi \tag{27}$$

Here

$$U=\left[\left(\frac{3d^2}{R^5}-\frac{1}{R^3}+\left(\frac{3d^2}{R^4}-\frac{1}{R^2}\right)\sqrt{-k_2^2}-\frac{k_2^2d^2}{R^3}\right)e^{-R\sqrt{-k_2^2}}+\frac{1}{2R^3}\right.$$
$$\times\left\{-\alpha^2\cos\theta(1-3\cos^2\theta)I_0K_0-\alpha((\alpha^2\sin^2\theta\cos\theta)\right.$$
$$+(1-3\cos^2\theta)(1+\cos\theta))I_0K_1+\alpha(\alpha^2\sin^2\theta\cos\theta$$
$$\left.\left.-(1-3\cos^2\theta)(1-\cos\theta))I_1K_0+3\alpha^2\sin^2\theta\cos\theta I_1K_1\right\}\right]$$

and

$$\theta=\tan^{-1}(r/d),\qquad \alpha=R\sqrt{-k_2^2}$$
$$R=\sqrt{d^2+r^2},\qquad r=\sqrt{a^2+\rho^2-2\rho a\cos\phi} \tag{28}$$

The argument of I_1 and I_0 is $((R-d)\sqrt{-k_2^2})/2$ while that of K_1 and K_0 is $((R+d)\sqrt{-k_2^2})/2$.

The important feature to notice about this equation is that T_0, T_{1a} and T_{1b} are analytic in the lower half plane except for a branch cut along the negative imaginary ω axis. We return to this point later.

Since it is instructive to consider the analytic form of the solution for the transient response of a sphere in a half space the remainder of the analysis concentrates on that case. This will clarify the meaning of the statement for small $k_2 b$.

4. *Transient response of a sphere in a half space*

For the case where there is no conductive overburden one finds from eq. (20) that the voltage $V(t)$ induced in a receiving loop at the earth's surface is

$$V(t)=\int_0^\infty \bar{\varepsilon}\cdot d\ell$$

$$V(t)=V_p(t)+V_s(t) \tag{29}$$

$$V_p=\frac{a\mu_0I_0\sqrt{\pi}}{t}\sqrt{\frac{a^2\sigma_2\mu_0}{4t}}\sum_{j=0}^{\infty}\frac{2(-1)^j(2j+2)!}{(2j+5)j!(j+1)!(j+2)!}\left(\frac{a^2\sigma_2\mu_0}{4t}\right)^{j+1}$$

$$V_s=\int_{-\infty}^{\infty}e^{-i\omega t}\left\{iaT_{1b}WN_{011}(j_1(k_3b),j_1(k_2b))B_{011}\right. \tag{30}$$

$$\left.+WMe_{11}(j_1(k_3b),j_1(k_2b))\times[iT_0Ae_{01}-iT_{1a}Ae_{11}]\frac{a}{k_2}\right\}d\omega \tag{31}$$

In eq. (29) l specifies the contour of the receiving loop. V_p is the voltage that would be observed if there were no sphere present. See LEE and LEWIS (1974).

The contribution from V_s may be obtained by contour integration. As was noted above one must integrate about the branch cut, that is, along the negative imaginary ω

axis. This same sort of singularity arises in the expressions $WM\underset{0}{g}mn$ and $WN\underset{0}{g}mn$ because of the term $k_2 b$.

In evaluating the integral one has a choice as to which expression for T_0, T_{1a} and T_{1b} to use, that given in eqs (27) and (28) or that given in eq. (25). Since numerical results have to be obtained it is simpler to use the expressions given in eq. (25).

An important feature to notice about eq. (25) is that h_1 is defined to have a positive imaginary part. The consequence of this is that one must place a cut in the complex η plane. Here

$$\eta = k_2^2 - \lambda^2. \tag{32}$$

With this cut $\sqrt{\eta}$ is uniquely defined. When $|k_2| > \lambda$ and k_2^2 is to the left of $P-$ but very close to the negative imaginary ω axis

$$\eta = -\sqrt{k_2^2 - \lambda^2} \tag{33}$$

Similarly, when k_2 is to the right of $P+$ but very close to the negative imaginary ω axis

$$\eta = \sqrt{k_2^2 - \lambda^2} \tag{34}$$

when $|k_2| \leq \lambda$

$$\eta = i\sqrt{\lambda^2 - k_2^2}\cdot \tag{35}$$

As well as the branch cut contribution there are also possible contributions from poles which are the zeros of $WM^1\underset{0}{g}mn$ and $WN^1\underset{0}{g}mn$. The zeros of $WM^1\underset{0}{g}mn$ are given by

$$j_n(\cancel{\alpha} z)h_n^{1\prime}(\alpha z) - \alpha j_n'(\cancel{\alpha} z)h_n^1(\alpha z) = 0$$

where

$$\alpha = \sqrt{\frac{\sigma_2}{\sigma_3}} \tag{36}$$

A restriction on the zero is that the imaginary part of z must be greater than zero.

Alternatively the zeros are given by:

$$j_n(\bar{z})h_n^{2\prime}(\alpha\bar{z}) - \alpha j_n'(\bar{z})h_n^2(\alpha\bar{z}) = 0. \tag{37}$$

In eq. (37) $\bar{z}$ is the complex conjugate of z. The restriction now is that the imaginary part of $\bar{z}$ is less than zero. SINGH (1973) has made a careful study of the zeros of eq. (37) and he showed that there were no zeros for which the imaginary part of $\bar{z}$ was less than zero. One concludes, then, that there are no zeros of eq. (36) such that the imaginary part of z is greater than zero.

The zeros of $WN^1\underset{0}{g}$ are given by

$$(n+1)WN'mn\underset{0}{g} + nWM'\underset{0}{g}mn + 1 = 0 \tag{38}$$

Equation (38) may be rearranged as

$$zj_n(z)\frac{d}{dz}zh_n^1(\alpha z)-\alpha^2 zh_n^1(\alpha z)\frac{d}{dz}zj_n(\alpha z)=0. \qquad (39)$$

Equation (39) implies that if z is a zero then

$$\bar{z}j_n(\alpha\bar{z})\frac{d}{d\bar{z}}\bar{z}h_n^2(\alpha\bar{z})-\alpha^2\bar{z}h_n^2(\alpha\bar{z})\frac{d}{d\bar{z}}\bar{z}j_n(\alpha\bar{z})=0. \qquad (40)$$

Here $\bar{z}$ is the complex conjugate of z. Once again SINGH (1973) has shown that there are no zeros of eq. (40) for $\bar{z}$ with an imaginary part greater than zero. Therefore there are no zeros of eq. (38) for an imaginary part of z less than zero.

If I_1 represents the contributions from the residue series, then, with the approximations given above, I_1 is found to be given by

$$I_1=0. \qquad (41)$$

Further, a suitable path for the contour integration is shown in Fig. 4.

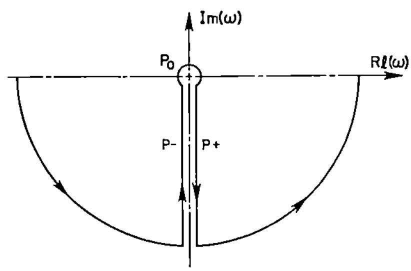


Figure 4
Path for the contour integral.

Let I_2 represent the contribution from the integration around the branch point. This contribution is the sum of three terms which involve integrations along the paths $P-$, P_0 and $P+$. The contributions from the arcs vanish because of Jordon's theorem. Notice that the contributions from P_0 are zero because the spectrum of $V_s(t)$ is zero at the origin. The contributions from the paths $P+$ and $P-$ may be found by writing

$$\omega=(\sigma_2\mu_0 b^2)^{-1}s\,e^{-i\pi/2} \qquad \text{on } P+$$

and

$$\omega=(\sigma_2\mu_0 b^2)^{-1}s\,e^{i\pi+\pi/2i} \qquad \text{on } P- \qquad (42)$$

Since all the square roots have been defined by eqs (33), (34), (35) and (36) one easily finds that

$$
\begin{aligned}
I_2 = \frac{3}{4}\frac{I_0 a^2 b}{\sigma_2} \int_0^\infty e^{-st/(\sigma_2 \mu b^2)} \Bigg\{ & \frac{WM_{e11}(j_1(\sqrt{s/\alpha^2}), j_1(\sqrt{s}))}{s} \\
& \times \Big[T_0^2(+s)/(\sqrt{s}\, WMe_{01}(j_1(\sqrt{s/\alpha^2}), h_1^1(\sqrt{s})) \\
& + T_0^2(-s)/(\sqrt{s}\, WMe_{01}(j_1(\sqrt{s/\alpha^2}), h_1^2(\sqrt{s})) \\
& - T_{1a}^2(+s)/(\sqrt{s}\, WMe_{11}(j_1(\sqrt{s/\alpha^2}), h_1^1(\sqrt{s})) \\
& - T_{1a}^2(-s)/(\sqrt{s}\, WMe_{11}(j_1(\sqrt{s/\alpha^2}), h_1^2(\sqrt{s})) \Big] + \frac{WN_{011}(j_1(\sqrt{s/\alpha^2}), j_1(\sqrt{s}))}{b^2} \\
& \times \Big[T_{1b}^2(+s)/(\sqrt{s}\, WN_{011}(j_1(\sqrt{s/\alpha^2}), h_1^1(\sqrt{s}))) \\
& + T_{1b}^2(-s)/(\sqrt{s}\, WN_{011}(j_1(\sqrt{s/\alpha^2}), h_1^2(\sqrt{s}))) \Big] \Bigg\} ds
\end{aligned}
\tag{43}
$$

In eq. (43)

$$T_0(\pm s) = -\int_0^\infty 2\lambda^2 f_0(\lambda) J_0(\lambda\rho) J_1(\lambda a)\, d\lambda$$

where

$$
\begin{aligned}
f_0(\lambda) &= \frac{e^{\pm id\sqrt{s/b^2-\lambda^2}}}{\lambda \mp i\sqrt{s/b^2-\lambda^2}} && s/b^2 \geq \lambda^2 \\
&= \frac{e^{-d\sqrt{-s/b^2+\lambda^2}}}{\lambda + \sqrt{-s/b^2+\lambda^2}} && s/b^2 < \lambda^2
\end{aligned}
\tag{44}
$$

and

$$T_{1a}(\pm s) = -\int_0^\infty 2\lambda f_{1a}(\lambda) J_1(\lambda a) J_1(\lambda\rho)\, d\lambda$$

where

$$
\begin{aligned}
f_{1a}(\lambda) &= \frac{\pm i\sqrt{s/b^2-\lambda^2}\, e^{\pm id\sqrt{s/b^2-\lambda^2}}}{\lambda \mp i\sqrt{s/b^2-\lambda^2}}, && s/b^2 \geq \lambda^2 \\
&= \frac{-\sqrt{\lambda^2 - s/b^2}\, e^{-d\sqrt{\lambda^2-s/b^2}}}{\lambda + \sqrt{\lambda^2 - s/b^2}}, && s/b^2 < \lambda^2
\end{aligned}
\tag{45}
$$

and

$$T_{1b}(\pm s) = \int_0^\infty 2i\lambda J_1(\lambda a) J_1(\lambda\rho) f_{1b}(\lambda)\, d\lambda$$

where

$$f_{1b}(s) = \frac{e^{\pm id\sqrt{s/b^2-\lambda^2}}}{\lambda \mp i\sqrt{s/b^2-\lambda^2}}, \qquad s/b^2 \geq \lambda^2$$
$$= \frac{e^{-d\sqrt{\lambda^2-s/b^2}}}{\lambda + \sqrt{\lambda^2 - s/b^2}}, \qquad s/b^2 < \lambda^2 \tag{46}$$

In practice $t/(\sigma_2\mu_0 b^2)$ is at least unity for $t = 0.1$ ms. Consequently, for most of the time range used in prospecting, the major part of the integrand arises when s is less than unity. Since e^{-5} is about 0.006 one would intuitively expect an accuracy of a few percent provided one could limit the upper range of integration to $5s_l$.

$$s_l \geq 5\sigma_2\mu_o b^2/t \tag{47}$$

i.e.

$$|k_2 b| \geq \sqrt{5\sigma_2\mu_0 b^2/t} \tag{48}$$

We now choose t such that

$$\sqrt{5\sigma_2\mu_0 b^2/t} < 0.5. \tag{49}$$

If we estimate $|k_2 b|$ by taking the equality in eq. (48) then with the condition specified by eq. (49) the spherical Bessel functions may be estimated by the first term in their power series. Thus the approximations made in arriving at an expression for the transient voltage are valid.

One finds, then, that the voltage induced in the receiving loop can now be found from eqs (29), (43) and (41) to be

$$V(t) = V_p + I_1 + I_2 \tag{50}$$

Equation (50) is valid provided $5\sigma_2\mu_0 b^2/t$ is about 0.25. In practice eq. (50) is valid for times greater than about 1 ms.

In writing down the final estimates we have ignored the higher order modes. The reason for this is that they have a much more rapid decay than the early ones. Despite our efforts to obtain good estimates they are still rather vague. Because of this we provide an additional check on our results by comparing our calculated results with some model results. This check will show that eq. (49) is more restrictive than is needed.

5. *Calculations*

PALMER (1974) has carried out a careful scale model experiment for studying the transient response of a sphere in a conductive environment. For this experiment he used a copper sphere in a lead block. The model was designed to represent a 50 m loop transversing a sphere which was at a depth of 100 m. The radius of the sphere was 75 m and the conductivities of the ground and the sphere were ~~0.56~~ 0·049 s/m and ~~0.049~~ 0·56 s/m respectively. Some of his results are shown in Fig. 5.

Equation (49) shows that for this geometry good agreement should be reached for times greater than 6.9 ms. Figure 5 shows, however, that this estimate is quite conservative; the results are very good at 2 ms and even at 1 ms they are reasonable.

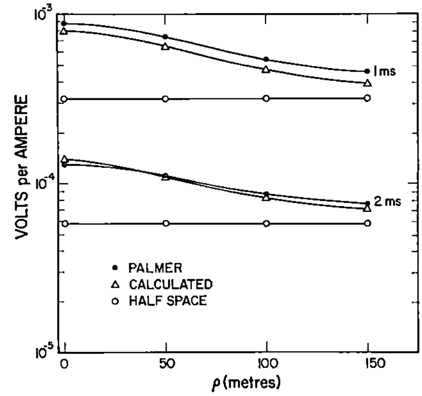


Figure 5
Comparison of some calculated results with some scale model results of PALMER (1974).

6. *Discussion*

The theory given above lays the foundation for an extensive numerical investigation of the transient electromagnetic response of a sphere in a layered medium.

We have not specified the source so as to be able to cater for any of the prospecting methods that are currently used. In particular it should be noted that the equations given above allow for the fields to be computed beneath the ground. This is important for drill hole measurements. In subsequent papers we will explore some of the more interesting situations that can be modelled by the theory given above.

One result should be immediately noted and that is there are no poles in the continued spectrum of the electromagnetic response of a sphere in a layered medium. This is quite different from the transient electromagnetic response of layered spheres in free space. These spheres have pole type responses. Consequently such spheres are only poor models for the effect of host rock on the electromagnetic transient.

Acknowledgments

The author would like to acknowledge Dr James R. Wait for suggesting the project. Dr Chen-To Tai generously made available an unpublished report. Thanks are also due to Roz Oscarson for typing what can only be regarded as a difficult manuscript.

References

D'Yakonov, B. P. (1959), *The Diffraction of Electromagnetic Waves by a Sphere Located in a Halfspace.* Bull. Izv. Acad. Science, USSR, Geophysics Series No. 11.

Hohmann, G. W. (1971), *Electromagnetic Scattering by Conductors in the Earth Near a Line Source of Current*, Geophysics *36*, 101–131.

Howard, A. Q. (1972), *Electromagnetic Fields of a Subterranean Cylindrical Inhomogeneity Excited by a Line Source*, Geophysics *37*, 975–984.

Kristensson, S. G. (1979), *Scattering of Stationary Waves from Buried Three-Dimensional Inhomogeneities*, Akademisk Avhandling For Avläggande Av Filosofie Doktorsexamen I Teoretisk Fysik Avhandlingen kommer enligt beslut av fysiska avdelningen vid Göteborgs universitet och Chalmers tekniska högskola att offentligt försvaras torsdagen den 22 November 1979 kl 10.15 i sal 6306, Forskarhuset, Institutionen för teoretisk fysik, GV/CTH, Fysikgränd 3^{VI}, Goteborg. Avhandlingen försvaras på engelska.

Lee, T. J. (1974), *Transient Electromagnetic Response of a Sphere in a Layered Medium*, unpublished Ph.D. thesis, Macquarie University, Australia, p. 107.

Lee, T. J. (1975), *An Integral Equation and its Solution for some Two and Three Dimensional Problems in Resistivity and Induced Polarization, J. R. Astr. Soc. 42*, 81–95.

Palmer, T. M. (1974), *TEM Modelling – Sphere in a Half Space*, unpublished report for L. A. Richardson and Associates, Pty Ltd, Chatswood, NSW, Australia.

Singh, S. K. (1973), *Electromagnetic Response of a Conducting Sphere Embedded in a Conductive Medium*, Geophysics *38*, 864–893.

Tai, Chen-To (1971), *Dyadic Green's Functions in Electromagnetic Theory*, Intext, San Fransisco, pp. 246.

Tai, Chen-To (1973), *Eigen-Function Expansion of Dyadic Green's Function, Mathematics Notes, No. 28.* A.F.W.L. Kirtland Airforce base, Albuquerque, New Mexico, USA.

Tai, Chen-To (1980), *Singular Terms in the Eigen-Function Expansion of Dyadic Green's Function of the Electric Type, Mathematics Notes No. 65.* A.F.W.L. Kirtland Airforce base, Alburquerque, New Mexico, USA.

Wait, J. R., and Campbell, L. L. (1953), *The Fields of an Electric Dipole in a Semi-infinite Conducting Medium*, J. Geophy. Res. *58*, 21–28.

Yaghjian, A. (1980), *Electric Dyadic Green's Function in the Source Region*, Proceedings of the I.E.E.E. *68*, 248–263.

(Received 21 August 1980)

Appendix

Some intermediate results

(1) Let

$$\bar{\psi}_{\substack{e\\o}n\lambda} = J_n(\lambda r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\phi\, e^{ihz}$$

and

$$h^2 = k^2 - \lambda^2$$

(i)
$$\nabla^2 \bar{\psi}_{\substack{e\\o}n\lambda} + (\lambda^2 + h^2)\bar{\psi}_{\substack{e\\o}n\lambda} = 0$$

(ii)
$$\bar{M}_{\substack{e\\o}n\lambda} = \nabla \times \bar{\psi}_{\substack{e\\o}n\lambda}\hat{\mathbf{z}}$$

$$= \left[\mp \frac{nJ_n(\lambda r)}{r} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} n\phi\hat{\mathbf{r}} - \frac{\partial}{\partial r} J_n(\lambda r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\phi\hat{\phi}\right] e^{ihz}$$

Tai (1971, p. 84)

(iii) $$\bar{N}_{\stackrel{e}{o}}n\lambda = \frac{1}{k}\nabla\times\nabla\times\bar{\psi}_{\stackrel{e}{o}}n\lambda\hat{\mathbf{z}}$$

$$= \frac{e^{ihz}}{k}\left[ih\frac{\partial}{\partial r}J_n(\lambda r)\begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\hat{\mathbf{r}} \mp \frac{ihn}{r}J_n(\lambda r)\begin{matrix}\sin\\ \cos\end{matrix}\, n\phi\hat{\boldsymbol{\phi}} + \lambda^2 J_n(\lambda r)\begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\hat{\mathbf{z}}\right]$$

TAI (1971, p. 84)

(iv) $$\nabla\times\nabla\times\begin{matrix}\bar{N}_{\stackrel{e}{o}}n\lambda(h)\\ \bar{M}_{\stackrel{e}{o}}n\lambda(h)\end{matrix} - k^2\begin{matrix}\bar{N}_{\stackrel{e}{o}}n\lambda(h)\\ \bar{M}_{\stackrel{e}{o}}n\lambda(h)\end{matrix} = 0$$

TAI (1970, p. 70)

(2) $$\int_0^\infty\int_0^{2\pi}\bar{M}_{\stackrel{e}{o}}n\lambda(h)\cdot\bar{M}_{\stackrel{e}{o}}n'\lambda'(-h')r'\,d\theta'\,dr'$$

$$= 0 \qquad n \neq n'$$

$$= (1+\delta_0)\pi\lambda\,\delta(\lambda-\lambda')\,e^{iz(h-h')} \qquad n = n'$$

$$\delta_0 = 1 \qquad n = 0$$

$$\delta_0 = 0 \qquad n \geq 1$$

TAI (1971, p. 92)

(3) $$\int_0^\infty\int_0^{2\pi}\bar{N}_{\stackrel{e}{o}}n\lambda(h)\cdot\bar{N}_{\stackrel{e}{o}}n'\lambda'(-h')r'\,d\theta'\,dr'$$

$$= (1+\delta_0)\pi\lambda\,\delta(\lambda-\lambda')e^{iz(h-h')} \qquad n = 0$$

$$= 0 \qquad n \neq n'$$

TAI (1971, p. 72)

(4) $$\bar{\bar{G}}_0(\bar{R}\,|\,\bar{R}') = \int_0^\infty d\lambda\int_{-\infty}^\infty dh\sum_{n=0}^\infty\frac{(2-\delta_0)}{(4\pi^2\lambda)(h^2+\lambda^2-k^2)}\cdot[\bar{M}_{\stackrel{e}{o}}n\lambda(h)\bar{M}'_{\stackrel{e}{o}}n\lambda(-h)$$

$$+ \bar{N}_{\stackrel{e}{o}}n\lambda\bar{N}'_{\stackrel{e}{o}}n\lambda(-h)]$$

Where $\bar{\bar{G}}_0(R\,|\,\bar{R}')$ is a solution of

$$\nabla\times\nabla\times\bar{G}_0(R\,|\,\bar{R}') - k^2\bar{G}_0(\bar{R}\,|\,\bar{R}') = \bar{\bar{I}}\,\delta(R-R').$$

TAI (1971, p. 94, 95) or from eqs (2) and (3) above.

(5) Let

$$\psi_{\stackrel{e}{o}}mn(k) = j_n(kR)P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi.$$

Then

(i)
$$\nabla^2 \psi_{\substack{e\\o}}mn(k) + k^2 \psi_{\substack{e\\o}}mn(k) = 0$$

$$\overline{SM}_{\substack{e\\o}}mn(k) = \nabla \times (\psi_{\substack{e\\o}}mn(k)\mathbf{R})$$
$$= j_n(kR)\left[\mp \frac{m}{\sin\theta} P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix}\, m\phi\hat{\boldsymbol{\theta}} - \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\hat{\boldsymbol{\phi}}\right]$$

TAI (1971, p. 170)

(ii)
$$\overline{SN}_{\substack{e\\o}}mn(k) = \frac{1}{k}\nabla \times \nabla \times \psi_{\substack{e\\o}}mn(k)\mathbf{R}$$
$$= \frac{n(n+1)}{kR} j_n(kR) P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\hat{\mathbf{R}} + \frac{1}{kR}\frac{\partial}{\partial R}$$
$$\times [Rj_n(kR)] \left[\frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\hat{\boldsymbol{\theta}}\right.$$
$$\left. \mp \frac{m}{\sin\theta}\cdot P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix}\, m\phi\hat{\boldsymbol{\phi}}\right]$$

TAI (1971, p. 170)

(iii)
$$\nabla \times \nabla \times \begin{matrix}\overline{SM}_{\substack{e\\o}}mn(k)\\ \overline{SN}_{\substack{e\\o}}mn(k)\end{matrix} - k^2 \begin{matrix}\overline{SM}_{\substack{e\\o}}mn(k)\\ \overline{SN}_{\substack{e\\o}}mn(k)\end{matrix} = 0$$

(6) (i)
$$\int_{\text{sphere}} \overline{SM}_{\substack{e\\o}}mn(k)\cdot \overline{SN}_{\substack{e\\o}}m'n'(k')\,dv = 0$$
$$m' \neq m, \qquad n \neq n'$$

(ii)
$$\int_{\text{sphere}} \overline{SN}_{\substack{e\\o}}mn(k)\cdot \overline{SN}_{\substack{e\\o}}m'n'(k')\,dv = \frac{\delta mm'\,\delta nn'\,2(1+\delta_{0m})\pi n(n+1)}{kk'(2n+1)}$$
$$\times \frac{(n+m)!}{(n-m)!}\cdot kk' \int_0^b \frac{R^2}{2n+1}$$
$$\times ((n+1)j_{n-1}(kR)j_{n-1}(k'R)$$
$$+ nj_{n+1}(kR)j_{n+1}(k'R))\,dR$$

Where

$$\begin{aligned} &\delta_{0m} = 1 && m = 0 \\ &\delta_{0m} = 0 && m \neq 0 \qquad n \neq 0 \\ &\delta_{ij} = 1 && i = j \\ &\quad\;\; = 0 && i \neq j \end{aligned}$$

TAI (1971, p. 172) Actually Tai has, by mistake, omitted from the second factor the term kk'.

$$\text{(iii)}\quad \int_{\text{sphere}} \overline{SM}_{\mathcal{E}}mn(k)\cdot \overline{SM}_{\mathcal{E}}m'n'(k')\,dv = \pi(1+\delta_{0m})\frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!} \times \int_0^b R^2 j_n(kR) j_n(k'R)\,dR\cdot \delta mm'\,\delta nn'$$

$$\text{(7) (i)}\quad \frac{1}{k}\nabla\times[\bar{\psi}_{\mathcal{E}}mn(k)\hat{\mathbf{z}}] = \overline{SM}^z_{\mathcal{E}}(k) \mp \frac{m}{n(n+1)}\overline{SN}_{\mathcal{E}}mn + \frac{1}{2n+1} \times\left[\frac{n-m+1}{n+1}\overline{SM}_{\mathcal{E}}m(n+1) + \frac{n+m}{n}\overline{SN}_{\mathcal{E}}m(n-1)\right]$$

TAI (1971, p. 228)

$$\text{(ii)}\quad \frac{1}{k^2}\nabla\times\nabla\times\psi_{\mathcal{E}}mn(k)\hat{\mathbf{z}} = \overline{SN}^z_{\mathcal{E}}mn(k) = \mp\frac{m}{n(n+1)}\overline{SM}_{\mathcal{E}}mn + \frac{1}{(2n+1)} \times\left[\frac{n-m+1}{n+1}\overline{SN}_{\mathcal{E}}m(n+1) + \frac{n+m}{n}\overline{SN}_{\mathcal{E}}m(n-1)\right]$$

TAI (1971, p. 228)

$$\text{(8)}\qquad e^{ihz}J_m(\lambda r) = \sum_{n=m}^{\infty}(2n+1)i^{n-1}j_n(kR)\cdot P_n^m(\cos\psi)P_n^m(\cos\theta)\frac{(n-m)!}{(n+m)!}$$

$$h=\sqrt{k^2-\lambda^2}$$

$$R=\sqrt{z^2+r^2},\qquad k\sin\psi=\lambda$$

LEE (1974, p. 33)

(9) Expansion of the cylindrical wave functions in spherical wave functions.
(i) First the M functions:

$$\bar{M}_{\mathcal{E}}m\lambda = \nabla\times\bar{\psi}_{\mathcal{E}}m\lambda(h)\hat{\mathbf{z}}$$

$$= \nabla\times J_m(\lambda r)\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\, e^{ihz}\hat{\mathbf{z}}$$

$$= \sum_{n=m}^{\infty} i^{n-1}(2n+1)P_n^m(\cos\psi)(n-m)!/(n+m)!\cdot\nabla\times\left[\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\, j_n(kR)P_n^m(\cos\theta)\hat{\mathbf{z}}\right]$$

[result (8) above]

$$= \sum_{n=m}^{\infty} ki^{n-1}(2n+1)P_n^m(\cos\psi)(n-m)!/(n+m)!\cdot\left\{\mp\frac{m}{n(n+1)}\overline{SN}_{\mathcal{E}}mn + \frac{1}{2n+1}\left[\frac{n-m+1}{n+1}\overline{SM}_{\mathcal{E}}m(n+1) + \frac{n+m}{n}\overline{SM}_{\mathcal{E}}m(n-1)\right]\right\}$$

[result (7) above]

$$
\begin{aligned}
&= \mp km \sum_{n=m}^{\infty} \frac{i^{n-1}(2n+1)P_n^m(\cos\psi)(n-m)!/(n+m)!}{n(n+1)} \cdot \{\overline{SN}_{\substack{e\\o}}mn\} \\
&\quad + k \sum_{n=m}^{\infty} i^{n-1} \frac{(2n+1)}{2n+1} P_n^m(\cos\psi)(n-m)!/(n+m)! \cdot \\
&\quad \times \left[\left(\frac{n-m+1}{n+1}\right) \overline{SM}_{\substack{e\\o}}m(n+1) + \frac{n+m}{n} \overline{SM}_{\substack{e\\o}}m(n-1)\right]
\end{aligned}
$$

For $m = 0$ the first term is $k\overline{SM}_{\substack{e\\o}01}/i$

(ii) Expansion of the N functions

Now

$$\bar{N}_{\substack{e\\o}}m\lambda = \frac{1}{k}\nabla \times \bar{M}_{\substack{e\\o}}m\lambda$$

$$\nabla \times \bar{N}_{\substack{e\\o}}m\lambda = k\bar{M}_{\substack{e\\o}}m\lambda$$

$$\overline{SN}_{\substack{e\\o}}m\lambda = \frac{1}{k}\nabla \times \overline{SM}_{\substack{e\\o}}m\lambda$$

$$\nabla \times \overline{SN}_{\substack{e\\o}}m\lambda = \frac{1}{k}\nabla \times \overline{SM}_{\substack{e\\o}}m\lambda$$

TAI (1971)

Therefore

$$
\begin{aligned}
\bar{N}_{\substack{e\\o}}m\lambda &= \sum_{n=m}^{\infty} i^{n-1}(2n+1)P_n^m(\cos\psi)(n-m)!/(n+m)! \\
&\quad \left\{\mp \frac{km}{n(n+1)} \overline{SM}_{\substack{o\\e}}mn + \frac{1}{2n+1}\left[\frac{(n-m+1)}{n+1}\right.\right. \\
&\quad \left.\left. \times k\overline{SN}_{\substack{e\\o}}m(n+1) + \frac{(n+m)}{n} kSN_{\substack{e\\o}}m(n-1)\right]\right\} \\
&= \mp km \sum_{n=m}^{\infty} \frac{i^{n-1}(2n+1)}{n(n+1)} P_n^m(\cos\psi) \frac{(n-m)!}{(n+m)!} \overline{SM}_{\substack{o\\e}}mn \\
&\quad + k \sum_{n=m}^{\infty} i^{n-1} P_n^m(\cos\psi)(n-m)!/(n+m)! \cdot \left[\left(\frac{n-m+1}{n+1}\right) \overline{SN}_{\substack{e\\o}}m(n+1)\right. \\
&\quad \left. + \frac{n+m}{n} \overline{SN}_{\substack{e\\o}}m(n-1)\right]
\end{aligned}
$$

For $m = n = 0$ the first term is $\frac{k}{i} SN_{\substack{e\\o}01}$.

(10) The Green's dyadics for the halfspace.

The geometry for the Green's dyadics is as shown in Figs. 2a and 2b.

(i) The Green's dyadics for the source.

For the case where $\sigma_1 = \sigma_2$ and $\varepsilon_1 = \varepsilon_2$ the layered structure reduces to a halfspace and the relevant Green's dyadic has been given by Tai (1970).

$$G^l = G^l_{00}(R|R') + G^l_{s0}(R|R') + g \qquad z > d$$
$$G^l = G^l_1(R|R') \qquad z < d$$

Where

$$G^l_{00}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0} \begin{cases} \bar{M}_{\mathcal{g}}n\lambda(h_0)\bar{M}'_{\mathcal{g}}n\lambda(-h_0) + \bar{N}_{\mathcal{g}}n\lambda(h_0)\bar{N}'_{\mathcal{g}}(-h_0), & z \geq z' \\ \bar{M}_{\mathcal{g}}n\lambda(-h_0)\bar{M}'_{\mathcal{g}}n\lambda(h_0) + \bar{N}_{\mathcal{g}}n\lambda(-h_0)\bar{N}'_{\mathcal{g}}(h_0), & z < z' \end{cases}$$

$$G^l_{s0}(R|R') = \frac{i}{4\pi}\int_o^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}[a_1\bar{M}_{\mathcal{g}}n\lambda(h_0)M'_{\mathcal{g}}n\lambda(+h_0) + b_1\bar{N}_{\mathcal{g}}n\lambda(h_0)\bar{N}'_{\mathcal{g}}n\lambda(h_0)]$$

$$G^l_1(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}\cdot[c_1\bar{M}_{\mathcal{g}}n\lambda(-h_1)\bar{M}'_{\mathcal{g}}n\lambda(h_0)$$
$$+ d_1\bar{N}_{\mathcal{g}}n\lambda(-h_1)\bar{N}'_{\mathcal{g}}n\lambda(h_0)]$$

$$g = -\hat{z}\hat{z}\,\delta(R-R')/k_0^2$$

Where

$$a_1 = \left[\frac{h_0 - h_1}{h_1 + h_0}\right] e^{-2h_0 dl}$$

$$c_1 = \frac{2h_0\, e^{l(h_1-h_0)d}}{h_1 + h_0}$$

$$b_1 = \frac{k_1^2 h_0 - k_0^2 h_1}{k_1^2 h_0 + k_0^2 h_1} e^{-2lh_0 d}$$

$$d_1 = \frac{2k_1 h_0 k_0}{k_1^2 h_0 + k_0^2 h_1} e^{ld(h_1-h_0)}$$

Note. In the notation of Fig. 1, $h_1 = h_2$ and $k_1 = k_2$.

(ii) The Green's dyadics for the scatterer.

Once again the dyadics can be easily constructed by following the method given by Tai (1970).

$$\bar{\bar{G}} = \bar{\bar{G}}_0(R|R') + \bar{\bar{G}}_{s1}(R|R') - \hat{z}\hat{z}\delta(R - R')/k_2^2, \qquad z < d$$
$$\bar{\bar{G}} = \bar{\bar{G}}_{00}(R|R'), \qquad z > d$$

Where

$$\bar{\bar{G}}_0(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_1} \begin{cases} \bar{M}_{\substack{e\\o}n\lambda}(h_1)\bar{M}'_{\substack{e\\o}n\lambda}(-h_1) + \bar{N}_{\substack{e\\o}n\lambda}(h_1)\bar{N}_{\substack{e\\o}n\lambda}(-h_1), & z \geq z' \\ \bar{M}_{\substack{e\\o}n\lambda}(-h_1)\bar{M}'_{\substack{e\\o}}(h_1) + \bar{N}_{\substack{e\\o}n\lambda}(-h_1)\bar{N}'_{\substack{e\\o}n\lambda}(h_1), & z < z' \end{cases}$$

$$\bar{\bar{G}}_{s1}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_1} \cdot [A\bar{M}_{\substack{e\\o}n\lambda}(-h_1)\bar{M}'_{\substack{e\\o}n\lambda}(h_1) + B\bar{N}_{\substack{e\\o}n\lambda}(-h_1)\bar{N}'_{\substack{e\\o}n\lambda}(h_1)]$$

$$\bar{\bar{G}}_{00}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_1} \cdot [C\bar{M}_{\substack{e\\o}n\lambda}(h_0)\bar{M}'_{\substack{e\\o}n\lambda}(h_1) + D\bar{N}_{\substack{e\\o}n\lambda}(h_0)\bar{N}'_{\substack{e\\o}n\lambda}(h_1)]$$

Where

$$A = \left[\frac{h_1 - h_0}{h_0 + h_1}\right] e^{2h_1 di}$$

$$C = \frac{2h_1}{h_1 + h_0} e^{id(h_1 - h_0)}$$

$$B = + \frac{e^{2ih_1 d}(h_1 k_0^2 - h_0 k_1^2)}{k_1^2 h_0 + k_0^2 h_1}$$

$$D = \frac{2k_0 h_1 k_1 e^{i(h_1 - h_0)d}}{h_0 k_1^2 + h_1 k_0^2}$$

Note. In the notation of Fig. 1, $h_1 = h_2$, $k_1 = k_2$.

(11) Green's Dyadics for the layered medium.

The expressions for the dyadic Green's function for a layered medium involve the same wave functions as for the non-layered case and can be found by the method discussed by TAI (1970).

(i) The Green's dyadics for the source

$$\bar{\bar{G}}^l = \bar{\bar{G}}^l_{00}(R|R') + \bar{\bar{G}}^l_s(R|R') + g \qquad z > d$$
$$\bar{\bar{G}}^l = \bar{\bar{G}}^l_1(R|R') \qquad d - t \leq z \leq d$$
$$\bar{\bar{G}}^l = \bar{\bar{G}}^l_2(R|R') \qquad z < d - t$$

Where

$$g = -\hat{z}\hat{z}\,\delta(R - R')/k_0^2$$

$$\bar{\bar{G}}_{00}^{l}(R|R') = \frac{i}{4\pi}\int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0} \begin{cases} \bar{M}_{\substack{e\\o}}n\lambda(+h_0)\bar{M}'_{\substack{e\\o}}n\lambda(-h_0) + \bar{N}_{\substack{e\\o}}n\lambda(h_0)\bar{N}'_{\substack{e\\o}}n\lambda(-h_0), & z \geq z' \\ \bar{M}_{\substack{e\\o}}n\lambda(-h_0)\bar{M}'_{\substack{e\\o}}n\lambda(h_0) + \bar{N}_{\substack{e\\o}}n\lambda(-h_0)\bar{N}_{\substack{e\\o}}'(h_0), & z < z' \end{cases}$$

$$\bar{\bar{G}}_{s}^{l}(R|R') = \frac{i}{4\pi}\int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}[A\bar{M}_{\substack{e\\o}}n\lambda(h_0)\bar{M}_{\substack{e\\o}}'n\lambda(h_0) + B\bar{N}_{\substack{e\\o}}n\lambda(h_0)\bar{N}'_{\substack{e\\o}}n\lambda(h_0)]$$

$$\bar{\bar{G}}_{1}^{l}(R'|R') = \frac{i}{4\pi}\int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}[C\bar{M}_{\substack{e\\o}}n\lambda(h_1)\bar{M}'_{\substack{e\\o}}n\lambda(h_0) + D\bar{M}_{\substack{e\\o}}n\lambda(-h_1)\bar{M}'_{\substack{e\\o}}n\lambda(h_0)$$
$$+ E\bar{N}_{\substack{e\\o}}n\lambda(h_1)\bar{N}'_{\substack{e\\o}}n\lambda(h_0) + F\bar{N}_{\substack{e\\o}}n\lambda(-h_1)\bar{N}'_{\substack{e\\o}}n\lambda(h_0)]$$

$$\bar{\bar{G}}_{2}^{l} = \frac{i}{4\pi}\int_0^{\infty} d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_0}[G\bar{M}_{\substack{e\\o}}n\lambda(-h_2)\bar{M}'_{\substack{e\\o}}n\lambda(h_0) + H\bar{N}_{\substack{e\\o}}n\lambda(-h_2)\bar{N}'_{\substack{e\\o}}n\lambda(h_0)]$$

The functions A to H are found by requiring that $\hat{\mathbf{z}} \times \bar{\bar{G}}^{l}$ and $\hat{\mathbf{z}} \times \nabla \times \bar{\bar{G}}^{l}$ be continuous at $z = d - t$ and $z = d$. With $z_1 = d$ and $z_2 = d - t$ one finds that:

$$A = -e^{-2h_0z_1} + \frac{2h_0(h_2+h_1)e^{-ih_1z_1 - ih_0z_1} - 2h_0(h_0-h_1)e^{ih_1z_1 - 2ih_1z_2}}{(h_0+h_1)(h_2+h_1)e^{-ih_1z_1} - (h_0-h_1)(h_2-h_1)e^{ih_1z_1-2ih_1z_2}}$$

$$B = e^{-2ih_0z_1} + \frac{k_0}{h_0}e^{-ih_0z_1}\left[\frac{h_1}{k_1}E\,e^{ih_1z_1} - \frac{Fh_1\,e^{-ih_1z_1}}{k_1}\right]$$

Where E and F are given below.

$$C = \frac{-2h_0(h_0-h_1)\,e^{-2ih_1z_2}}{(h_0+h_1)(h_2+h_1)\,e^{-ih_1z_1} - (h_0-h_1)(h_2-h_1)\,e^{ih_1z_1-2ih_1z_2}}$$

$$D = \frac{2h_0(h_2+h_1)\,e^{-ih_0z_1}}{(h_0+h_1)(h_2+h_1)\,e^{-ih_1z_1} - (h_0-h_1)(h_2-h_1)\,e^{+ih_1z_1-2ih_1z_2}}$$

$$E = \frac{-2h_0k_1k_0\,e^{-ih_0z_1}(h_1k_2^2 + k_1^2h_2)}{e^{ih_1z_1}(h_1k_0{}^2 - k_1^2h_0)(h_1k_2^2 + k_1^2h_2) - e^{-ih_1z_1-ih_1z_2}(h_1k_2^2 - k_1^2h_2)(h_1k_0^2 + k_1^2h_0)}$$

$$F = \frac{-2h_0k_1k_0\,e^{-ih_0z_1}(h_1k_2^2 - k_1^2h_2)}{e^{ih_1z_1}(h_1k_0^2 - k_1^2h_0)(h_1k_2^2 + k_1^2h_2) - e^{-ih_1z_1-ih_1z_2}(h_1k_2^2 - k_1^2h_2)(h_1k_0^2 + k_1^2h_0)}$$

$$G = e^{ih_2z_2}[C\,e^{ih_1z_2} + D\,e^{-ih_1z_2}]$$

$$\mathrm{H} = \frac{e^{ih_2z_2}}{k_2}(k_1E\,e^{ih_1z_2} + k_1\,e^{-ih_1z_2}F)$$

(ii) The Green's dyadics for the scatterer.

The general form for the dyadics for the medium is:

$$\bar{\bar{G}} = \bar{\bar{G}}_{22}(R|R') + \bar{\bar{G}}_{00}(R|R') - \hat{\mathbf{z}}\hat{\mathbf{z}}\,\delta(R-R')/k_2^2 \qquad z < d-t$$
$$\bar{\bar{G}} = \bar{\bar{G}}_{21}(R|R') \qquad d-t < z < d$$
$$\bar{\bar{G}} = \bar{\bar{G}}_{20}(R|R') \qquad z > d$$

Where

$$\bar{\bar{G}}_{00}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} \begin{cases} \bar{M}_{\substack{e\\o}} n\lambda(h_2)\bar{M}'_{\substack{e\\o}} n\lambda(-h_2) + \bar{N}_{\substack{e\\o}} n\lambda(h_2)\bar{N}'_{\substack{e\\o}} n\lambda(-h_2) & z \geq z' \\ \bar{M}_{\substack{e\\o}} n\lambda(-h_2)\bar{M}'_{\substack{e\\o}} n\lambda(h_2) + \bar{N}_{\substack{e\\o}} n\lambda(-h_2)\bar{N}'_{\substack{e\\o}} n\lambda(h_2) & z < z' \end{cases}$$

$$\bar{\bar{G}}_{22}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} [a_2 \bar{M}_{\substack{e\\o}} n\lambda(-h_2)\bar{M}'_{\substack{e\\o}} n\lambda(h_2) + b_2 \bar{N}_{\substack{e\\o}} n\lambda(-h_2)\bar{N}'_{\substack{e\\o}} n\lambda(h_2)]$$

$$\bar{\bar{G}}_{21}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} [c_2 \bar{M}_{\substack{e\\o}} n\lambda(-h_1)\bar{M}'_{\substack{e\\o}} n\lambda(h_2) + d_2 \bar{M}_{\substack{e\\o}} n\lambda(h_1)\bar{M}'_{\substack{e\\o}} n\lambda(h_2) + e_2 \bar{N}_{\substack{e\\o}} n\lambda(-h_1)\bar{N}'_{\substack{e\\o}} n\lambda(h_2) + f_2 \bar{N}_{\substack{e\\o}} n\lambda(h_1)\bar{N}'_{\substack{e\\o}} n\lambda(h_2)]$$

$$\bar{\bar{G}}_{20}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_{n=0}^{\infty} \frac{2-\delta_{0m}}{\lambda h_2} [g_2 \bar{M}_{\substack{e\\o}} n\lambda(h_0)\bar{M}'_{\substack{e\\o}} n\lambda(h_2) + h\bar{N}_{\substack{e\\o}} n\lambda(h_0)\bar{N}'_{\substack{e\\o}} n\lambda(h_2)]$$

The functions a_2 to h are found by requiring that $\hat{\mathbf{z}} \times \bar{\bar{G}}$ and $\hat{\mathbf{z}} \times \nabla \times \bar{\bar{G}}$ be continuous at $z = d - t$ and $z = d$.

With $z_1 = d$ and $z_2 = d - t$ one finds that:

$$a_2 = -e^{2ih_2z_2} + e^{ih_2z_2}(c_2 e^{-ih_1z_2} + d_2 e^{ih_1z_2})$$

c_2 and d_2 defined below.

$$b_2 = -e^{-2ih_2z_2} + \frac{e^{-ih_2z_2}}{k_2}[k_1 e_2\, e^{-ih_1z_2} + k_1 f_2\, e^{ih_1z_2}]$$

e_2 and f_2 defined below.

$$c_2 = \frac{2h_2\, e^{ih_2z_2}(h_0 - h_1)}{(h_2-h_1)(h_0-h_1)\, e^{-ih_1z_2} - (h_0+h_1)(h_1+h_2)\, e^{ih_1z_2 - 2ih_1z_1}}$$

$$d_2 = -\frac{2h_2\, e^{ih_2z_2}(h_0 + h_1)}{(h_2-h_1)(h_0-h_1)\, e^{-ih_1z_2} - (h_0+h_1)(h_1+h_2)(e^{ih_1z_2 - 2ih_1z_1})}$$

$$e_2 = \frac{2h_2 k_1 k_2 \, e^{ih_2 z_2}(h_1 k_0^2 - k_1^2 k_0)}{(k_1^2 h_2 - k_2^2 h_1)(h_1 k_0^2 - k_1^2 h_0) + (k_1^2 h_2 + k_2^2 h_1)(k_1^2 h_0 + k_0^2 h_1) \, e^{-2ih_1 z_1}}$$

$$f_2 = \frac{2h_2 k_1 k_2 \, e^{ih_2 z_2}(k_1^2 h_0 + k_2^2 h_1) \, e^{-2ih_1 z_1 + h_2 z_2 i}}{(k_1^2 h_2 - k_2^2 h_1)(h_1 k_0^2 - k_1^2 h_0) + (k_1^2 h_2 + k_2^2 h_1)(k_1^2 h_0 + k_0^2 h_1) \, e^{-2ih_1 z_1}}$$

$$g_2 = e^{-ih_0 z_1}(e^{-ih_1 z_1} c_2 + d_2 \, e^{ih_1 z_1})$$

$$h = \frac{e^{-ih_0 z_1}}{k_0} [k_1 e_2 \, e^{-ih_1 z_1} + f_2 k_1 \, e^{ih_1 z_1}]$$

(12) Primary Green's function in spherical coordinates

$$\bar{\bar{G}}_0(\bar{R} \mid \bar{R}') = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (2 - \delta_{0m}) \frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}$$

$$\cdot \begin{cases} \overline{SM}^{(1)} \underset{o}{e} mn(k) \overline{SM}' \underset{o}{e} mn(k) + \overline{SN}^{(1)} \underset{o}{e} mn(k) \overline{SN} \underset{o}{e}' mn(k), & R > R' \\ \overline{SM} \underset{o}{e} mn(k) \overline{SM}'^{(1)} \underset{o}{e} mn(k) + \overline{SN} \underset{o}{e}(k) \overline{SN}'^{(1)} \underset{o}{e} mn(k), & R < R' \end{cases}$$

$$- [\hat{R}\hat{R} \, \delta(\bar{R} - \bar{R}')]/k^2$$

Where $\overline{SM}^{(1)}(k)$ means that the vector wave function is defined with respect to the spherical Hankel function of the first kind.
i.e.

$$h_n^1(kR) \quad \text{replaces} \quad j_n(kR).$$

$\overline{SN}^{(1)}(k)$ is similarly defined.

$$\delta_{0m} = 1 \qquad m = 0$$
$$= 0 \qquad m \neq 0$$

TAI (1971, p. 174)
See also the errata. TAI (1980, p. 5)

Transient Electromagnetic Waves Applied to Prospecting

Terry Lee

Reprinted from *IEEE Transactions on Proceedings,* Vol. 67, No. 7, July 1979
0018-9219/79/0700-1016$00.75

Transient Electromagnetic Waves Applied to Prospecting

TERRY LEE

Abstract—An argument based on a high-frequency filter can be used to show that the later stages of a transient are strongly controlled by the low-frequency component of its spectrum. In particular for a sphere of radius 50 m and conductivity 2 mho/m, frequencies greater than 3100 Hz are unimportant for times greater than 3 ms. For the later stages of the transient, where only low frequencies are important, it is possible to provide solutions for the transient response of spheres, cylinders, and dykes. These solutions have the attractive feature that the singular part of the Green's function is the most important term in this function for these calculations. The early stages of the transient are useful for depth estimations and for these times, different methods of calculation must be used. The various approaches to the problem of calculating transient electromagnetic responses utilize various singularities of the transfer function of the ground. This has shown that the contributions from the integration about poles as well as branch cuts of the transfer function must be considered. Alternatively, if the singularities are seen as specifying the structure, then deriving transient responses for frequency-domain data is likely to be a very ill-conditioned problem. A further conclusion is that care must be given to the choice of pulse shape and measuring time.

I. Introduction

There is a strong tendency to equate the transient electromagnetic method of prospecting with some sort of multifrequency method of prospecting. This is unfortunate because from a mathematical point of view there is a choice as to how to represent the transient. One way is to express it as a sum of steady-state responses. Another way is to represent it in terms of a sum of integrals each of which is taken about a particular singularity of the transfer function of the ground. The purpose of this paper is to show that not only has the former method led to gross simplification of the physical process, but also to argue that the latter method offers considerable insight into the observed transients.

The results of a transient electromagnetic survey, have been viewed from either a frequency-domain or time-domain point of view. Because of this, two ways of calculating analytically the transient electromagnetic response of geological structures have been advocated, and in neither case have the advantages been thoroughly explored. The first of these is to calculate the response entirely in the time-domain [1], [2]. The second way is to convert the frequency-domain response to the time-domain by the use of the inverse Laplace transform. Unfortunately, this last operation must, in general, be done numerically because of the extreme difficulty of finding the inverse transform analytically. Despite the difficulty, a number of analytic solutions is known for dipoles in or on stratified media. [3]-[12]. A third approach has been to solve Maxwell's equations numerically by using a time-stepping procedure [13]. The results presented from this last study were those that modeled the magnetotelluric method of prospecting. In that study the size of the body was of the order of a skin depth. Of all the methods of calculating transient responses, then, this method has received the least attention.

The former analytic method of calculating the transient response usually involves the assumption that the time-domain response can be represented as a sum of exponentials. The decay constants are then found by solving a transcendental equation which is derived from the boundary conditions. The difficulty here is that a series of exponentials corresponds to a series of poles in the frequency-domain, and so the latter method must be used for those geometries that are associated with transfer functions which require the introduction of branch cuts to ensure that they are single valued in the complex plane. These branch cuts produce a response like $1/t^{v}$, where t is time and v is an arbitrary index. For early stages then, these terms can dominate the response and so make depth estimation difficult [14], [15]. An example of models where these terms are important is a slab on a uniform half-space [15].

It has not been possible to write down suitable time-domain expressions and so almost all calculations of the transient electromagnetic response of a geological structure have involved the numerical inversion of a Laplace transform. At this stage the the problem is often simplified by concentrating on the low-frequency part of the transfer function of the ground. There is a danger here that instrument design will be influenced by the results of this theory. Should this happen, it is possible that the transient electromagnetic methods of prospecting will not provide any more information than a less sophisticated steady-state system.

There is a real need then, to be able to estimate the frequency content of the later stages of an electrical transient. In Section II, we develop a method that is capable of estimating the highest possible set of frequencies that are significant at the later stages. The conclusions from this section can be used to place approximate methods of solutions of the type given in Section III on a more rigorous foundation. Alternatively, they can be used to argue (if it is known that a broad frequency range is needed to resolve geological structures) that much more emphasis should be placed on the earlier stages of this transient. Sections IV and V are concerned with showing that a considerable amount of insight can be gained by viewing the transients to be composed of a sum of integrals about the singularities of the transfer function of the ground.

II. Equating Frequency-Domain and Time-Domain Methods

As mentioned above the time-domain response and frequency-domain response of a geological structure are related to each other by the Laplace transform and this is true regardless of the configuration used in prospecting. Unfortunately this relationship makes qualitative conclusions with regard to

Manuscript received June 29, 1979; revised November 7, 1978.
The author is with L. A. Richardson and Associates, Pty., Ltd., Gordon, N.S.W. 2072, Australia.

0018-9219/79/0700-1016$00.75 © 1979 IEEE

frequency-domain data and time-domain data difficult. The problem here then, is to determine the stage in the transient time-domain method of prospecting that provides a broader spectrum than a competing multifrequency-domain method of prospecting.

In particular, how much effect does the higher frequency part of the spectrum have on the later stages of the transient? It is shown below that much of the higher frequency content needs to be removed for an appreciable relative effect on the transient at later stages. Thus we are led to compare the later stages of a filtered transient against an unfiltered transient.

Let the spectrum of the transient be multiplied by exp $((as)^2)$ where s is the parameter of the Laplace transform and "a" is a positive real constant. Notice, that if the substitution $s = i\omega$ is made where ω is the angular frequency, then the inverse Laplace transform is converted to an inverse Fourier transform and the attenuation of the filter is revealed.

If $G(t)$ is the time-domain response of the unfiltered function and the convolution theorem is used, then the difference between the filtered response and the unfiltered response is:

$$G(t) - \int_{-\infty}^{\infty} \frac{G(t-\mu)\cdot \exp(-\mu^2/4a^2)}{(4\pi a^2)^{1/2}}\, d\mu. \qquad (1)$$

Therefore, the relative change E is

$$E = \left[G(t) - \int_{-\infty}^{\infty} \frac{G(t-\mu)\cdot \exp(-\mu^2/4a^2)}{(4\pi a^2)^{1/2}}\, d\mu\right] \Big/ G(t). \qquad (2)$$

For a step-function excitation, $G(t)$ is zero for negative time t, and if "a" is also small then

$$E \simeq \frac{1}{2}\operatorname{erfc}\left(\frac{t}{2a}\right) - \exp\left(\frac{-t^2}{4a^2}\right)\cdot \frac{a}{(\pi)^{1/2}}\frac{\partial}{\partial t}\ln(G(t)). \qquad (3)$$

Frequently, the later stages of a transient process may be described by means of a single exponential such as $Ae^{-\gamma t}$ [15], [16].

Hence, an estimate for relative change is:

$$E < \frac{1}{2}\operatorname{erfc}\left(\frac{t}{2a}\right) + \frac{a\gamma}{(\pi)^{1/2}}\exp\left(\frac{-t^2}{4a^2}\right). \qquad (4)$$

As a rough rule, the result will be true if

$$t/a > 4 \quad \text{and} \quad a\gamma < 1. \qquad (5)$$

Or in terms of the decay constant

$$t \geqslant 4/\gamma \text{ and the filter is } \exp(-\omega^2/\gamma^2). \qquad (6)$$

For example, if a sphere has radius of 50 m, a conductivity of 2S/m and the permeability of free space, then

$$\gamma = 500\,\pi s^{-1}. \qquad (7)$$

Hence for $t > 8/\pi$ ms (i.e., frequencies less than 3100 Hz) the observed transient will show a relative change of 2 percent or less. Incidentally if the conductivity is increased by a factor of ten, then the results are still true for times greater than 30 ms.

As a consequence of the above result, it can be seen that if the decay constant or the behavior of the later stages of the transient response can be determined, then the result can b used to construct completely the late stages of the transien response from a limited amount of low-frequency data. Th same sort of analysis can be repeated for other forms c decay such as $1/t^v$.

If it is decided from the outset that only the low-frequenc response are required then there are approximate methoc which can be used to compute the later stage sequence. Thes conclusions are particularly relevant for those systems that ar designed to measure beyond 100 ms [17]. The point to b made here then, is that this type of analysis shows just ho much the low-frequency component of the spectrum dom nates the later stages of the transient. Hence, if the argumen concerning the desirability of measuring high frequencies a valid, then much more attention should be paid to the earl stages [18].

III. Low-Frequency Response Calculations

A. Approximations

Most analytic calculations that are attempted for the numer cal modeling of the transient electromagnetic response c geological structures involve some sort of low-frequency ap proximation. An almost universal approximation is to neglec displacement currents, that is, the dielectric constant is se equal to zero. From the frequency-domain point of view suc an approximation is reasonable for earth materials if the fre quencies are in the kilohertz range or less [19]. In the time domain, however, this approximation is less obvious and it usual to refer to the results of Wait [4], or Fuller and Wa [20].

It is possible that other effects besides those of strictly di placement currents are relevant. The problem is that there ar too few reliable measurements of conductivity spectra of rock over the frequency ranges that might be used. One of th reasons this is so is that many of the measurements that ar made are only over a very limited frequency range often onl up to 10 Hz [21]. This is disturbing because if there are rock with an appreciable phase difference for the conductivity fun tion then this is sufficient to explain the anomalous negativ transients that have been observed [22], [23].

Recently *in situ* conductivity spectra of rocks have bee measured over a broad-frequency range [24]. This last stud reported, for some cases, significant phase changes at highe frequencies. These results would therefore seem to reinforc the results of the previous time-domain studies.

In practice the problems that are treated are further sin plified by approximating the frequency-domain response a low frequencies by a function that readily admits a Laplac transform. The validity for these approximations lies in th initial and final value theorem of the Laplace transform [25] [27]. The usefulness of such an approach has been demon strated by Luke [28]–[30]. Such an approach for exampl has been used by several authors to study the transient respons of a conducting layer [31], [32]. Thus the viewing of th transient from a frequency-domain point of view naturall leads to gross simplification of the process. The simplificatio concentrates on the later stages, where it now seems possib to obtain results for a variety of structures [53], [54].

Recently it has been proved possible to calculate the steady state response of a conductor in a half-space excited by a lir or loop source, by approximating the integral operators ir volved. That such an approach might prove useful for solvir scattering problems was suggested by Noble [33].

For the case of a cylinder in a half-space excited by a line source there is a perturbation solution as well as a mode-matched one [34], [35]. The mode-matched solution is particularly interesting because the resulting scattering matrix was dominated by the contribution from the primary, or singluar term of the Green's function. This work shows that for a line-source excitation, cylindrical structures, and moderate frequencies, the resulting integral equations can be approximated by neglecting the nonsingular term. Later this same conclusion was reached for the case of exciting a buried spherical conductor by a coaxial loop source [36]. For time-domain calculations it is essential that a cheap way be found for calculating the steady-state responses for a number of different frequencies. Since the approach to approximating the integral equations involved appears to provide a solution to this problem, it is worthwhile looking at how this approch can be extended. One such extension is demonstrated below for the case of a dipping dyke. It is realised that this approach does not specify the time after which the results are valid. However, if some scale model results are available or if the decay constants can be estimated from simple models, then this restriction can be overcome. The way to overcome it is to use this approach outlined in Section II.

B. Extension of the Methods for Low-Frequency Calculations—An Example

Consider the geometry shown in Fig. 1. A line source of current carrying a current of $Ie^{i\omega t}$ is situated at x_1, dyke of thickness γ, conductivity σ_2, depth to the top α, and dip β is situated x_0 meters. The conductivity of the half-space is σ_1, and all permeabilities have been set equal to μ_0 the permeability of free space. The geometry is described by the coordinate system (x, y, z) and as is shown in the diagram. Notice that the y-axis is parallel to the strike of the dyke and to the line source.

For this geometry it has been shown that for moderate frequencies the secondary term in the Green's function may be neglected if an accuracy of only a few percent is required in the calculation of the electric and horizontal magnetic field [37]. This is not a severe restriction because it is clear from Maxwell's equations that the induced voltage in a receiving loop is just the electric field integrated around that loop.

For this geometry the electric field across the dyke may be described by the integral equation [38]

$$E = E^i - \frac{\mu_0(\sigma_2 - \sigma_1)\, i\omega}{2\pi} \int_A GE\, dx^1\, dz^1$$

here

$$G = K_0(k_1 R), \qquad k_1 = (i\omega\mu_0\sigma_1)^{1/2}$$
$$R = ((x - x^1)^2 + (z - z^1)^2)^{1/2} \tag{8}$$

and E^i is the incident electric field.

For a uniform half-space E^i is given by:

$$E^i = \frac{-i\omega\mu_0 I}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-n_1 z + i\lambda(x - x_1))}{(|\lambda| + n_1)}\, d\lambda \tag{9}$$

where

$$n_1 = (\lambda^2 + k_1^2)^{1/2}.$$

In writing down this equation the secondary term in the Green's function has been neglected for the reasons already given. This equation can now be reduced to one originally studied by Fok [39] first by assuming that the dyke is thin enough such that the integration can be approximated across the dyke, and second, by Fourier transforming out the coordinate of the line source in (8). The approximation is accurate to a few percent when

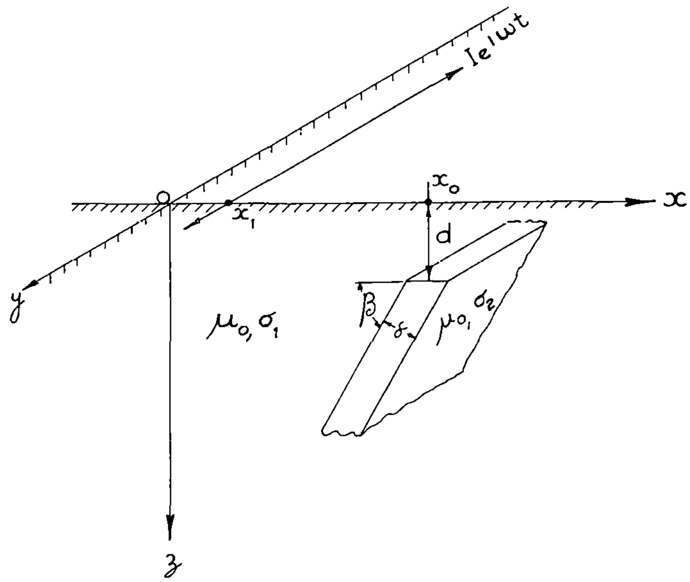


Fig. 1

$$\gamma((\sigma_2\mu_0\omega)/2)^{1/2} < \eta. \tag{10}$$

Here η may be 0.6 or even as large as 0.9 depending upon the accuracy required [38].

Denoting the Fourier transform by a bar yields:

$$\bar{E} = \bar{E}^i + \frac{\Delta}{\pi} \int_0^{\infty} K_0(k_1 |l - l^1|)\, \bar{E}\, dl^1 \tag{11}$$

where

$$\Delta = -(\sigma_2 - \sigma_1)\, \mu_0 \gamma i\omega/2$$
$$\bar{E}^i = B \exp(-k_1\, l \cos\psi)$$
$$B = \frac{-i\omega\mu_0 I \exp(-n_1 d + i\lambda(x_0))}{(\pi(|\lambda| + n_1))}$$
$$\psi = \beta + \phi + \pi$$
$$z - d = l \sin\beta, \quad x_0 - x = l\cos\beta$$
$$\lambda = ik_1 \cos\phi.$$

Also l and l^1 are lengths, measured from the top of the dyke, along its length.

Although Fok only considered the case where k_1 is unity, his method still applies for the case described above. Fok introduces two functions $G_+(\lambda)$ and $G_+(-\lambda)$ which are analytic in the upper and lower complex plane. These functions satisfy the relations:

$$\ln(G_+(\lambda)) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln\left(1 - \frac{\Delta}{(u^2 + k_1^2)^{1/2}}\right) \frac{du}{u - \lambda} \tag{12}$$

$$G_+(\lambda)\, G_+(-\lambda) = \frac{(\lambda^2 + k_1^2)^{1/2}}{(\lambda^2 + k_1^2)^{1/2} - \Delta}. \tag{13}$$

One then finds that the solution to (11) can be written down as:

$$\overline{E}(l) = B\left\{\frac{(\xi^2 + k_1^2)^{1/2} \exp(i\xi l)}{(\xi + k_1^2)^{1/2} - \Delta} + \frac{a^2\, G_+(\xi) \exp(i\alpha l)}{\alpha\, G_+(\alpha)\,(\xi - \alpha)} + \frac{\Delta\, G_+(\xi)\, k_1^2\, i}{\pi} \cdot \int_1^\infty \frac{\exp(-k_1 s l)\,(s^2 - 1)^{1/2}\, ds}{G_+(ik_1 s)\,(\Delta^2 + k_1^2 s^2 - k_1^2)\,(\xi - ik_1 s)}\right\}. \tag{14}$$

Here $\xi = ik_1 \cos\psi$ and $\alpha = (\Delta^2 - k_1^2)^{1/2}$. For small k_1, then

$$E = \int_{-\infty}^{\infty} \overline{E} \exp(-i\lambda x_1)\, d\lambda. \tag{15}$$

Equation (15) then, is a solution for the electric field within the dyke. By using it and (8) the electric field can be calculated anywhere.

The approach to approximating the integral operators would therefore seem to be a fruitful field for future research. It should be realized however, that the real problem is to determine not whether a solution can be obtained but whether it is useful. In particular it is questionable whether these types of solutions are sufficient to interpret the geology. The sort of additional information that is required can be found by looking at the very early stages of the transient.

IV. High-Frequency Response Calculations

The low-frequency analysis very often gives only a single decay constant that is related to the shape and size of a conducting body. The attempts which have been made to determine the depth to a conductor have used mainly the early stages of the transient, i.e., the high-frequency part. One of the reasons this approach is attractive is that the success of the low-frequency calculations depend upon the quantity $|k_1 a|$ being small. Here k_1 is the propagation constant of the medium and "a" is some length associated with the body. In the low-frequency calculations then, the relative dimensions of the body become less important. The perturbation solution for the buried cylinder is a case that illustrates this [34].

The first approach to the problem of determining the depths of a conductor made use of the theory of matched filters [40]. In that study that transient wave in the ground was shown to consist of a series of waves, that were reflected back and forth between the conductor and a wave that was only associated with the air–earth interface. The conclusion reached was that the transient electromagnetic data were difficult to process because of the interference of the ground wave [14]. A later study showed how these various waves could be transformed into a convenient solution for late time calculations [15]. What was being exploited here was the transfer function of the ground which could be expressed in different ways. One way necessitated the integration about a branch cut and the other about a series of poles. The first expression then, yields a series of waves while the second produces a series of modes.

The following example makes this quite clear. A common model for a conductive overburden has been a conducting slab of thickness h, conductivity σ, and permeability μ_0. The frequency-domain response for the electric field E, of the structure to a step-function magnetic field of height H_0 is

$$E = \frac{2H_0}{(i\omega\mu_0\sigma)^{1/2}}\left[1 + 2\sum_{n=0}^{\infty} \exp(-2\,nh(\sigma i\omega\mu_0)^{1/2})\right]. \tag{16}$$

Here $i\omega$ is the parameter of the Laplace transform. The equation can be checked easily by taking the inverse transform and comparing the result to the known time-domain formula [15].

Now there are two ways in which the expression can be transformed back into the time domain. The first is simply to take the inverse Laplace transform of each term in the expression. This will involve an integration about the branch cut along the negative $i\omega$ axis. The result of this is that the resulting transient consists of a series of waves of the type already described. This formula is useful for calculations at the early stages. The resulting ground wave is useful for determining the conductivity of the ground and this information can be used if the first wave term can be estimated to yield the depth to the conductor [14]. The second way is to sum the series first; hence

$$E = \frac{2H_0}{(i\omega\sigma\mu_0)^{1/2}}\left[\frac{\cosh(h(\sigma i\omega\mu_0)^{1/2})}{\sinh(h(\sigma i\omega\mu_0)^{1/2})}\right]. \tag{17}$$

The transient response can be obtained by summing a series of residues. This method produces a series of modes that decay exponentially. This formula is useful for calculations at later stages where the field decays like $\exp(-\pi^2 t/(\sigma\mu_0 h^2))$ and because of this depth, estimation is difficult [15].

This example shows that the frequency-domain approach to the analysis of transients is simply the concentration on specific singularities of the transfer function. At the same time other singularities are simply ignored. The suggested approach to the analysis of transients then, is to follow Baum [41] and to view the transients in terms of all of the singularities of the transfer function of the ground. The problem now is to be able to interpret the transient in terms of these singularities and to this end the choice of pulse shape is important.

V. Pulse Shape

Since the Laplace transform theorem relates frequency-domain data to time-domain data, it is clear from the frequency-domain point of view that the choice of pulse shape is important because it modifies the response function. Moreover as shown above, if early stages are neglected then the only part of the spectrum of the pulse that is being used is the low-frequency part. In view of this it is worthwhile looking at the results of research into multifrequency methods of prospecting.

Recently, it has been argued that a good way to choose the set of frequencies for multifrequency-electromagnetic survey is to make use of the theory of the generalized inverse [18]. The procedure is to require that the frequencies chosen be the ones that are capable of providing a stable inverse to a given set of data. The conclusion from these studies was that a very broad, frequency range should be used. This approach has been extended to obtain estimates for the parameter error bounds that are related to the parameter solution and the measurement of noise [42].

In contrast to this, the time-domain problem has been attacked in a much less sophisticated manner [43]. For the time-domain study, however, only forward solutions were computed for different types of pulse shape. The pulse shapes chosen were square and half-sinusoidal. The criteria for detectability was maximizing the difference between half-space response and the layered-ground response. The conclusions reached were that the detectability increases with increasing transmitter–receiver separation and that the square pulse excitation was the best. Another study of the transient electromagnetic response of a conducting sphere for step, sawtooth, ramp, rectangular, and impulse pulses, showed that for sawtooth pulses, a fast rise time should be used so as to obtain a greater secondary signal [44]. One way of stating the results of all authors is to say that they found that the best pulse to use was the one that had the widest frequency range. The difficulty with these latter studies then, is that they only approach the problem indirectly.

A more direct approach to this problem is to ask how stable is the process of inversion of the Laplace transforms, or since the inverse Laplace transform is determined by the singularities of the kernel, how much these can vary and still yield a transient response that is approximately the same as the observed response in the considered time range. Viewed in this light then, the proper choice of pulse shape and time interval over which to measure the transient response is simply the problem of choosing that pulse and time interval which will be sensitive to the position of singularities of the transfer function in the complex plane. The singularity expansion method of viewing transients then, allows for an alternative approach to answering the questions concerning the stability of the geophysical inverse problem.

For example, if the position of the nearest pole to the origin is to be determined, then the later stages of the transient are important because a small error in the decay constant will only become apparent when the product of error and time is appreciable, and this will only occur at later stages. Interestingly, this is simply saying that low frequencies are important in this case because the positions of these frequencies lie closest to the pole. However, it should be borne in mind that the position of this pole is only one of the singularities in the kernel that fixes the geology. Another one is a branch cut from the origin. Now a step-function pulse contains a significant amount of energy in the low-frequency part and so this pulse will tend to accentuate the ground wave. As noted above this can be a hindrance.

One approach is to select a pulse that has a significant amount of energy in the frequencies that were indicated from the generalized inverse approach. Leaving aside for the moment, any practical difficulties in measuring the resulting transient, there are still problems. First, by limiting oneself to the $i\omega$ axis in the complex plane, one is immediately faced with a difficult numerical inverse Laplace transform in order to find the behavior of the fields in the time domain. The reason for this is that the numerical computation of the inverse Laplace transform is an ill-conditioned problem [45]. Specifically, this is likely to be the case when there is a branch point at infinity [46]–[49]. As already seen this is frequently the case in the geophysical problem. The transform can be inverted, but high accuracy is needed and this is generally not available from geophysical modeling programs. Thus from an integration point of view it is necessary to choose points off the axis in order to perform the inverse Laplace transform [50]. Hence it is important to look at how the spectrum of the pulse continues off the $i\omega$ axis. Thus from a numerical point of view, the singularity expansion approach to the inverse problem would seem to be a very desirable one. At this stage, the frequencies ω, should be thought of as complex frequencies.

From a geophysics point of view, if the earth structure gives rise to a series of singularities in the continued spectrum, and the position of the singularities can vary a great deal without significantly changing the observed spectrum, then the earth structure will be hard to resolve.

Thus the approach to choosing the correct pulse shape is simply one of viewing the spectrum of the pulse as coupling with the singularities in the transform function of the ground. Further, such an approach allows for the suppression of unwanted earth characteristics, if they can be identified as arising from specific singularities in the complex plane. Alternatively, if the time window used for sampling the decay curve is too wide, then there is the possibility of not being able to separate out specific poles and hence, specific geological structures.

A serious difficulty, however, is that the electromagnetic waves undergo dispersion in the ground [51]. The consequence of this is that it may be necessary to modify the pulse shape when changing prospecting areas. One way to do this is to have a whole series of rectangular pulses. By measuring the transients following each of these pulses, it would be possible to use the results later to construct the response from a fairly arbitrary pulse. Alternatively the sequency rate of the pulse could be changed.

Whatever pulse shape is used, it will be necessary to make measurements at very early times as well as the later ones. The reasons for this, besides the ones given above, are that some studies have reported the possibility of detecting induced polarization effects by using electromagnetic transients and these effects were observed at the very early stages [22], [23]. At the present time the only instruments operating in the time domain that have considered some of these problems are UTEM of Toronto [23], and Newmont EMP System [52].

VI. Conclusion

The concept of applying transient electromagnetic waves to prospecting is attractive because it allows for the measurement of the secondary fields in the absence of the primary field. This combined with averaging makes for a good signal-to-noise ratio. There are difficulties however, and these are mainly concerned with the strong emphasis that is placed on the later time portion of the transient. A reason for this emphasis is the comparative ease with which it has been possible to obtain numerical results. In view of the work that has been done on the inversion of multifrequency-electromagnetic data much more emphasis should be devoted to the early stages.

The approach to viewing transients as a simple collection of steady-state responses, then, has led to gross simplifications of the physical processes involved in the transients. A more natural way to view the transients is in terms of the singularities of the transfer function of the ground. As shown above, this approach provides a way of overcoming the limitation generated by the simple multifrequency approach to transients. Moreover, this last appoach would appear to be very successful in relating specific singularities with specific geological features. The method also provides a new insight into experimental design.

References

[1] S. Hjelt, "The transient electromagnetic field of a two-layer sphere," *Geoexploration*, vol. 9, pp. 213–229, 1971.

[2] M. N. Nabighain, "Quasistatic transient response of a conducting permeable two-layer sphere in a dipolar field," *Geophysics*, vol. 32, pp. 303-309, 1971.
[3] J. R. Wait, "A basis of electrical prospecting employing time-varying electromagnetic fields," Ph.D. dissertation, Univ. Toronto, Toronto, Canada, 1951.
[4] ——, "Propagation of electromagnetic pulses in a homogeneous conducting earth," *Appl. Sci. Res.*, Part B, vol. 9, pp. 213-253, 1960.
[5] J. R. Wait, and K. P. Spies, "Transient fields for an electric dipole in a dissipative medium," *Can. J. Phys.* vol. 48, pp. 1858-1862, 1970.
[6] B. K. Bhattacharyya, "propagation of transient electromagnetic waves in a medium of finite conductivity," *Geophysics*, vol. 22, pp. 75-84, 1957.
[7] ——, "Propagation of an electric pulse through a homogeneous and isotropic media," *Geophysics*, vol. 22, pp. 905-921, 1957.
[8] ——, "Electromagnetic fields of a transient electromagnetic dipole on the earth's surface," *Geophysics*, vol. 24, pp. 89-108, 1959.
[9] J. R. Wait, and D. A. Hill, "Transient signals from a buried magnetic dipole," *J. Appl. Phys.*, vol. 42, pp. 3866-3869, 1971.
[10] ——, "Electromagnetic surface fields produced by a pulse-excited loop buried in the earth," *J. Appl. Phys.*, vol. 43, pp. 3988-3991, 1972.
[11] ——, "Transient magnetic fields produced by a step-function-excited loop buried in the earth," *Electron. Lett.*, vol. 8, No. 11, pp. 294-295, 1972.
[12] J. R. Wait, "Transient electromagnetic propagation in a conducting medium," *Geophysics*, vol. 16, pp. 213-221, 1951.
[13] R. D. Watts, "Magnetotelluric fields over round structures," Ph.D. dissertation, Univ. Toronto, Toronto, Canada, p. 138, 1972.
[14] Clay *et al*, "Matched filter detection of electromagnetic transient reflections," *Geophysics*, vol. 39, pp. 683-691, 1974.
[15] T. Lee, "Estimation of depth to conductors by the use of electromagnetic transients," *Geophys. Prospect.*, vol. 25, pp. 61-75, 1977.
[16] ——, "Transient electromagnetic response of a sphere in a layered medium," Ph.D. dissertation, Macquarie Univ., NSW, Australia, pp. 107, 1974.
[17] G. Buselli, and B. O'Neill, "SIROTEM: A new portable instrument for multi-channel transient electromagnetic measurements," *ASEG., Bull.*, vol. 8, pp. 82-87, 1977.
[18] S. H. Ward, W. J. Peeples, and J. Rhu, "Analysis of geoelectromagnetic data," *Methods in Computational Physics*, vol. 13, *Geophysics*, B. A. Bolt, Ed. New York: Academic Press, pp. 163-238.
[19] F. S. Grant and G. F. West, *Interpretation Theory in Applied Geophysics.* New York: McGraw-Hill, p. 469, 1965.
[20] J. A., Fuller, J. R. Wait, "A pulsed dipole in the earth," in *Transient Electromagnetic Waves*, vol. 10, *Topics in Applied Physics*, L. B. Felsen, Ed. New York: Springer Verlag, pp. 238-270, 1976.
[21] G. D. Van Voorhis, "Complex resistivity spectra of porphyry copper mineralisation," *Geophysics*, vol. 38, pp. 49-60, 1973.
[22] T. Lee, "Sign reversals in the transient method of electrical prospecting (one loop version)" *Geophys. Prospect.*, vol. 23, pp. 653-662, 1975 b.
[23] Yves. Lamontagne, "Applications of wide band, time domain, E. M. measurements in mineral exploration," Ph.D. dissertation, Univ. of Toronto, Toronto, Canada, p. 329, 1975, unpublished.
[24] W. H. Pelton *et al.*, "Mineral discrimination and removal of inductive coupling with multifrequency IP., *Geophysics*, vol. 43, pp. 588-609, 1978.
[25] G. A. Korn, and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (2nd enlarged ed). New York: McGraw-Hill, p. 1130, 1961.
[26] S. Goldstien, "Some two-dimensional diffusion problems with circular symmetry," *Proc. London Math. Soc.*, vol. 34, pp. 51-88., 1932.
[27] N. W. McLachlan, "Complex variables and operational calculus with technical applications *Cambridge Univ. Press*, p. 91, 1939.
[28] Y. L. Luke, "On economic representations of transcendental functions," *J. Math. Phys.*, vol. 38, pp. 279-294, 1960.
[29] ——, "On the approximate inversion of some Laplace transforms," in *Proc. 4th U.S. Nat. Congress Appl. Math.*, pp. 269-276, 1962.
[30] ——, "Approximate inversion of class of Laplace transforms applicable to supersonic flow problems," *Quart. J. Mech. Appl. Math.*, vol. 17, pp. 91-103, 1964.
[31] F. M. Kamenetski, "Transient processes using combined loops for a two-layer section with a non-conducting base," *Izv. Vuzov, Sec. Geology Prospecting*, No. 6, pp. 108-113, 1968.
[32] J. R. Wait, "The magnetic dipole over the horizontally stratified earth," *Can. J. Phys.*, vol. 29, pp. 577-592, 1951.
[33] "Integral equation perturbation methods in low-frequency diffraction," in *Proc. Symp. conducted by the Mathematics Research Center, U.S. Army, (Univ. Wisconsin, Madison, April 10-12, 1961)*, R. E. Langer, Ed. Madison, WI: Univ. Press, pp. 323-360, 1961.
[34] J. R. Wait, "The effect of a buried conductor on the subsurface fields for line source excitation," *Radio Sci.*, vol. 7, pp. 587-591, 1972.
[35] A. Q. Howard, "The electromagnetic fields of a subterranean cylindrical inhomogeneity excited by a line source," *Geophysics*, vol. 37, pp. 975-984, 1972.
[36] T. Lee, "Transient electromagnetic response of a sphere in a layered medium," *Geophys. Prospecting*, vol. 23, pp. 492-511, 1975 a.
[37] R. Lewis, private communication, 1977.
[38] G. W. Hahmann, "Electromagnetic scattering by conductors in the earth near a line source of current," *Geophysics*, vol. 36, pp. 101-131, 1971.
[39] V. A. Fok, "On the solution of an integral equation of mathematphysics, *Mathematical Sbornik*, vol. 14, pp. 3-50, 1944.
[40] G. Turin, "An introduction to matched filters," *IRE Trans. Inform. Theory*. vol. IT-6, pp. 311-329, 1960.
[41] C. E. Baum, "The singularity expansion method" *Transient Electromagnetic Waves*, vol. 10. *Topics in Applied Physics*, L. B. Felsen, Ed. New York: Springer Verlag, pp. 130-176, 1976.
[42] K. Vozoff, and D. L. B. Jupp, "Effective search for a buried layer: an experimental design in geophysics," *ASEG Bull.*, vol. 8, pp. 6-15, 1977.
[43] K. Mallick, and R. K. Verma, "Detectability of intermediate conductive & resistive layers by time-domain electromagnetic sounding," presented at the 47th Annual Convention of the S.E.G., Calgery, Alberta, Canada, Sept. 18-22, 1977.
[44] S. K. Verma, "Transient electromagnetic response of a conducting sphere excited by different types of input pulses," *Geophys. Prospect.* vol. 20, pp. 752-770, 1972.
[45] R. Bellman, R. E. Kalaba, and J. Lockett, Numerical Inversion of the Laplace Transform. New York: American Elsevier Publishing Co., p. 249, 1966.
[46] R. Pressens, "On a numerical method for the calculation of transient responses," *J. Franklin Inst.*, vol. 292, pp. 57-64, 1971.
[47] ——, "Gaussian quadrature formulas for the numerical integration of Bromwich's integral and the inversion of the Laplace transform," *J. Eng. Math.*, 5, pp. 2-9, 1971.
[48] I. M. Longman, "On the numerical inversion of the Laplace transform of a discontinuous original," *J. Inst. Math. Its Appl.*, vol. 4, pp. 320-328, 1968.
[49] I. M. Longman, M. Sharir, "Laplace transform inversion of rational functions," *Geophysics, J. Roy. Astr. Soc.*, vol. 25, pp. 299-205, 1971.
[50] R. Pressens, "A new numerical method for the inversion of the Laplace transform," *J. Inst. Math. Its Appl.*, vol. 10, pp. 185-192, 1972.
[51] H. F. Morrison, R. J. Phillips, and D. P. O'Brien, "Quantitative interpretation of transient electromagnetic fields over a layered half-space," *Geophys. Prospect.*, vol. 17, pp. 82-101, 1969.
[52] M. Nabighian, "The Newmont EMP System," presented at a Workshop on Mining Geophysics, Salt Lake City, UT, Dec. 5-8, 1976.
[53] R. G. Geyer, "Transient electromagnetic response near a fault zone," *Geophys. Prospect.*, vol. 20, pp. 828-846, 1972.
[54] ——, "Effect of a dipping contact on the behaviour on the electromagnetic field," *Geophys.*, vol. 37, pp. 337-350, 1972.

Paper

The Transient Electric Fields about a Loop on a Halfspace

R. Lewis* and T. Lee**

*Geology Department, University of Tasmania, G.P.O. Box 252C, Hobart, Tasmania 7001

**Geopeko, 27 Merriwa Street, Gordon, N.S.W. 2072

ı transient electric field excited by a step current loop of radius a on the surface of a ground of ·ductivity σ and permeability μ_0 are ›centrated in a toroidal zone. At early times zone moves slightly inward from the loop and ·n but at later time the field maximum moves ıg a cone dipping at 30° away from the loop. later times the maximum occurs beneath a ıt on the surface at radius R where $R^2 \approx$ 6 $t/\sigma\mu_0$. At this maximum the time rate of nge of the radial magnetic induction is zero.

·roduction

·e domain electromagnetic prospecting systems that a single large loop are becoming increasingly important ·he search for sulphides (Vozoff 1978). Despite this e have been few detailed calculations of the fields in ground. This is surprising because the significance of steady state case has been pointed out by Wait and ɜs (1971) and Wait and Hill (1972) have pointed out ' the distortion of an electromagnetic pulse travelling n a subsurface source to the surface is critically ·endent on the relative positions of the source and ɜiver.

·re recently Nabighian (1978) has indicated approximate .ribution of transient fields in the ground for various ırces. Also, Lee (1977) has argued that the propagation ·transients have important directional characteristics.

·e we first formulate a time domain expression for the ısmitted field in the ground and apply the result to dying the distribution of current in the ground as a ·ction of time and position.

ıeory

e electric field E^i at a depth z and radius r due to a rizontal loop of radius a carrying a current I $e^{i\omega t}$ on a ·form ground of conductivity σ and permeability μ_0 is en by:

$$= - i\omega\mu_0 Ia \int_0^\infty \frac{\lambda J_1(\lambda a)\, J_1(\lambda r)}{n_1 + \lambda} e^{-n_1 z} d\lambda \qquad (1)$$

$$\text{where } n = \sqrt{\lambda^2 + i\omega\mu_0\sigma} \qquad \text{(Lee 1974)} \qquad (2)$$

Here we have assumed the quasistatic case i.e. displacement currents are ignored.

The transient electric field ϵ in the ground due to a current step I in the loop then is:

$$\epsilon = \frac{aI}{2\sigma\sqrt{\pi}}\left(\frac{\sigma\mu_0}{t}\right)^{\frac{3}{2}} e^{-(z^2 + a^2 + r^2)\left(\frac{\sigma\mu_0}{4t}\right)}\, I_1\left(\frac{ar\sigma\mu_0}{2t}\right)$$

$$- \frac{aI}{\sigma} \int_0^\infty \lambda^2 J_1(\lambda a)\, J_1(\lambda r)\, e^{\lambda z} \operatorname{erfc}\left(\tfrac{1}{2} z\sqrt{\frac{\sigma\mu_0}{t}} + \lambda\sqrt{\frac{t}{\sigma\mu_0}}\right) d\lambda \qquad (3)$$

In deriving the expression we have made use of results of Oberhettinger and Badii (1973 No. 5.95) and Gray and Mathews (1966 p.69 No. 18). We also notice in passing that this result is equivalent to that previously given by Lee and Lewis (1974) for z = 0.

A convenient dimensionless form of (3) can be obtained by making the substitutions:

$$\tau = \frac{\sigma\mu_0 a^2}{4t}$$

$$R = \frac{r}{a}$$

$$Z = \frac{z}{a}$$

$$\epsilon^* = \frac{2\sigma\epsilon a^2}{I} \qquad (4)$$

Then

$$\epsilon^* = \frac{4}{\sqrt{\pi}} \tau^{\frac{3}{2}} e^{-(Z^2 + 1 + R^2)\tau}\, I_1(2R\tau)$$

$$- \int_0^\infty \lambda^2 J_1(\lambda)\, J_1(\lambda R)\, e^{Z\lambda} \operatorname{erfc}\left(Z\sqrt{\tau} + \tfrac{1}{2}\sqrt{\frac{\lambda^2}{\tau}}\right) d\lambda\ . \qquad (5)$$

The integral in (5) may be conveniently evaluated by numerical integration between successive zeros of the term $J_1(\lambda)\,J_1(\lambda R)$.

Calculation

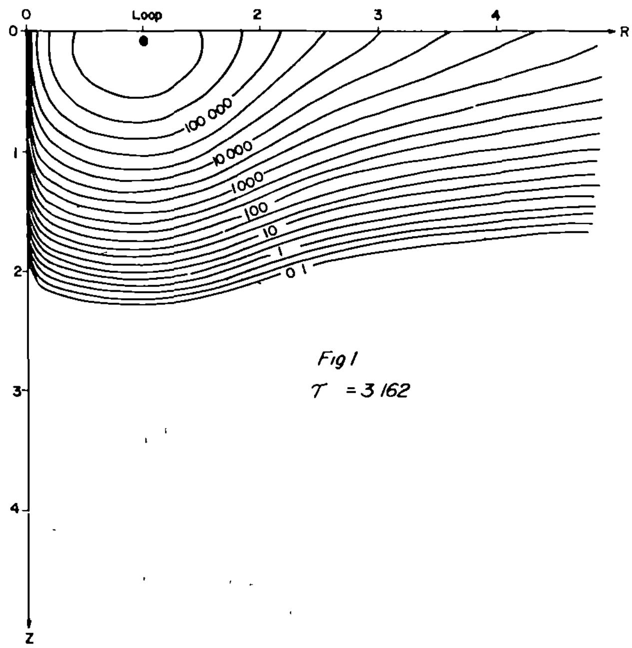
We have calculated ϵ^* to an accuracy of better than 1% relative error for values of τ spaced at a factor of $\sqrt{10}$ in the interval ($\sqrt{10}$, 10^{-5}) on various grids of R and Z. Here we present a subset of these calculations. For large τ a section 5 loop radii square was used but for small τ or late times a grid 50 loop radii square was used. These results have been contoured with a logarithmic scale at 3 contours per decade and are shown in figures 1-7. We have extracted the maximum values from the grids and used them to construct a locus of maximum ϵ^* shown in figure 8.

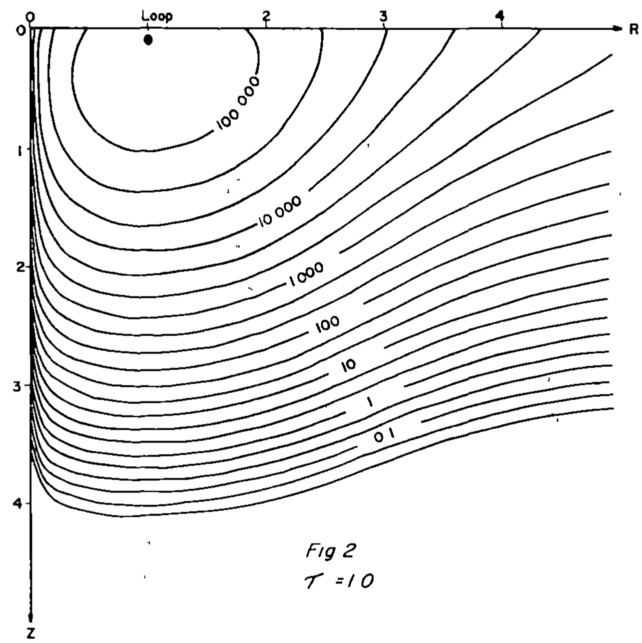
We have also calculated ϵ^* as a factor of τ at selected points for a range of τ extending up to 10 and plotted them as waveforms in figures 9-13.

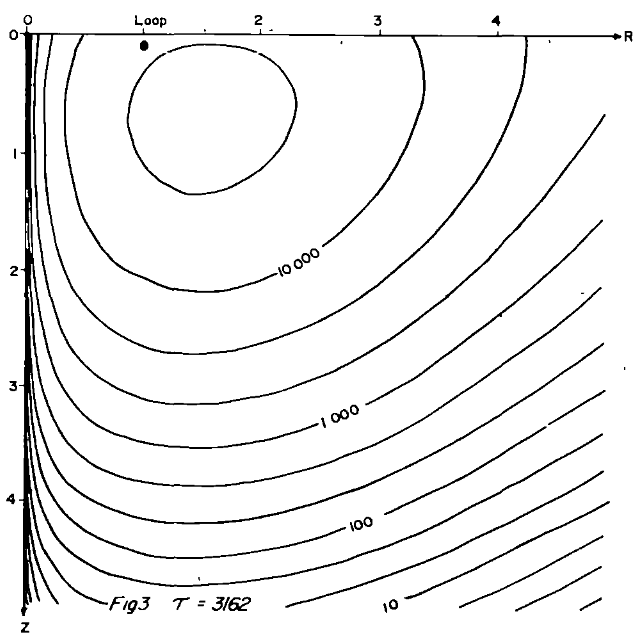
Discussion

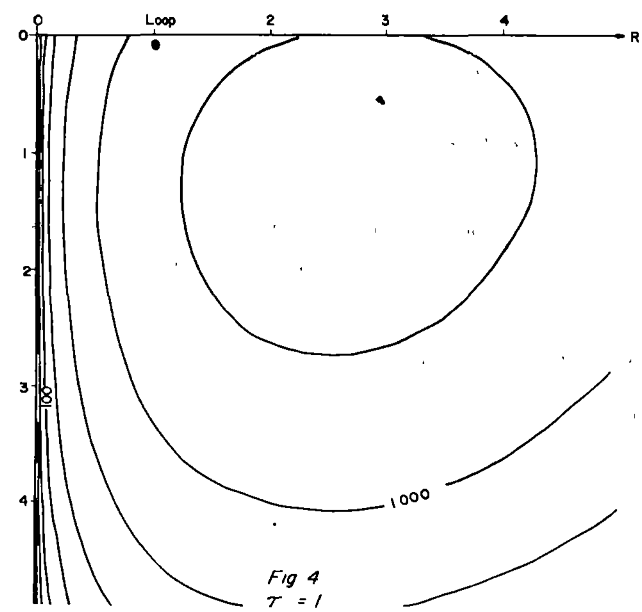
The calculations show that at a large τ (short time) the locus of maximum ϵ^* briefly moves slightly towards the centre of the loop and downwards. At later time the maximum moves along a cone dipping at about 30°

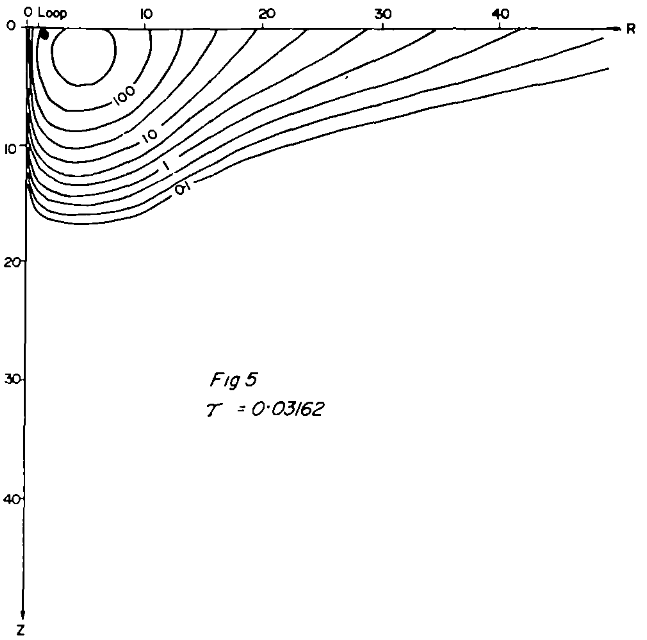
FIGURES 1-4
The Distribution of $\epsilon^* \times 10^6$. Contours are logarithmic with 3 contours per decade.

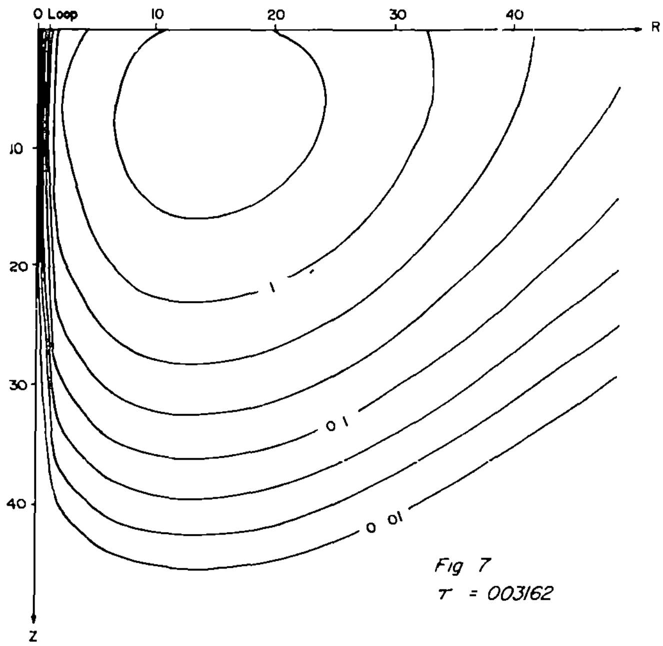


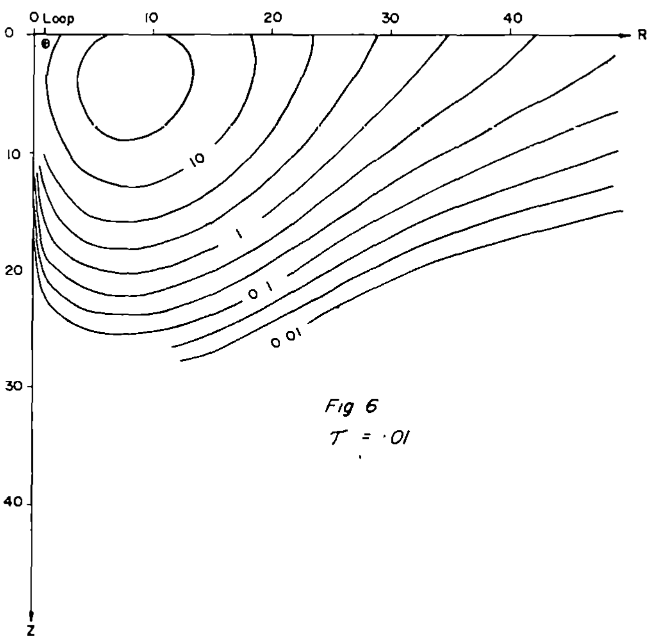






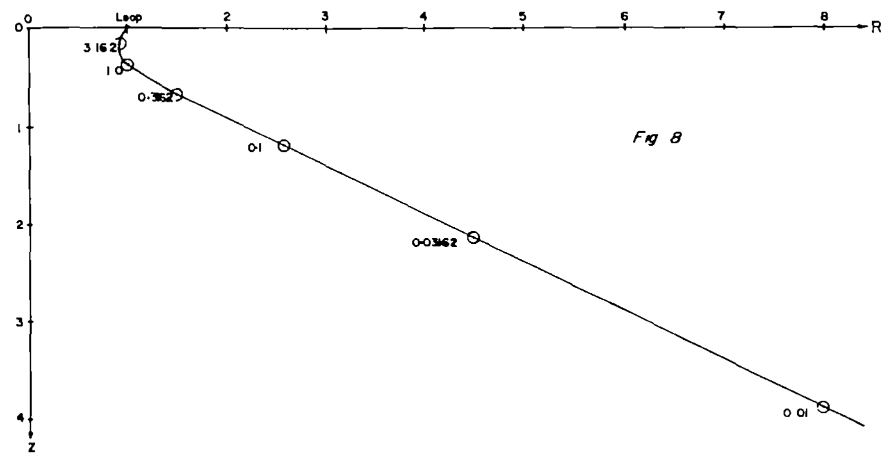




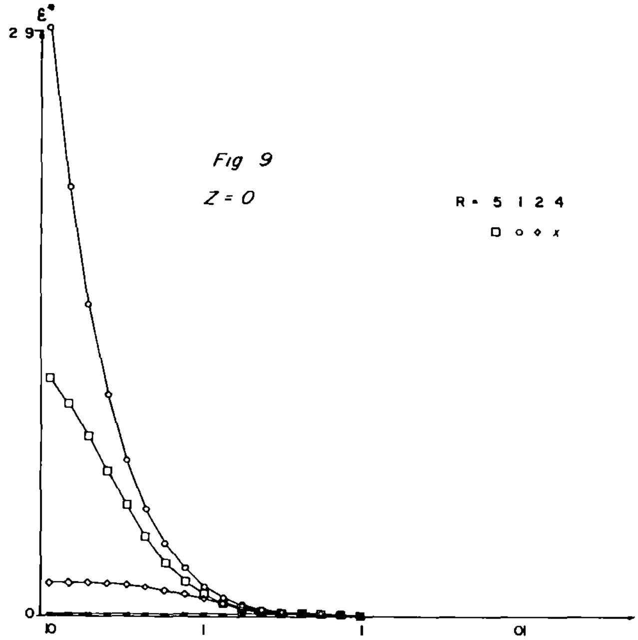


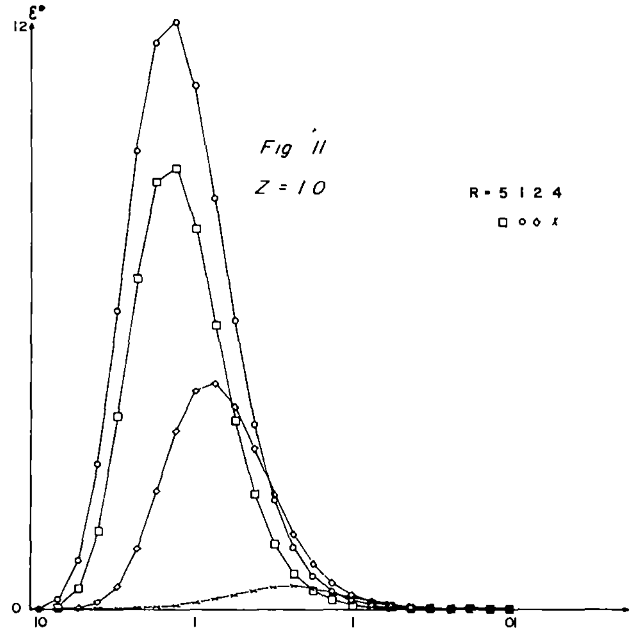
FIGURES 5-7
The distribution of $\epsilon^* \times 10^6$. Note that the scales on figures 5-7 are 10 times those in figures 1-4. Contours are logarithmic with 3 contours per decade.

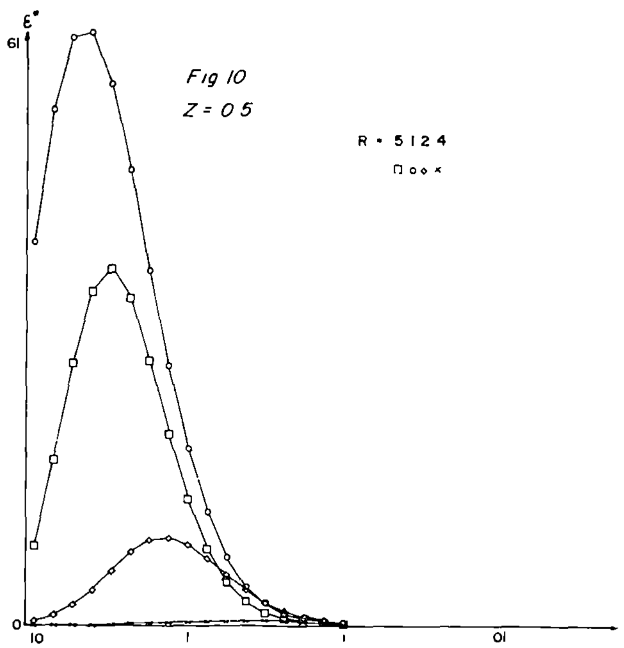
FIGURE 8
The locus of maximum ϵ^*. The numbers along the curve are the values of τ corresponding to the observed peak value.

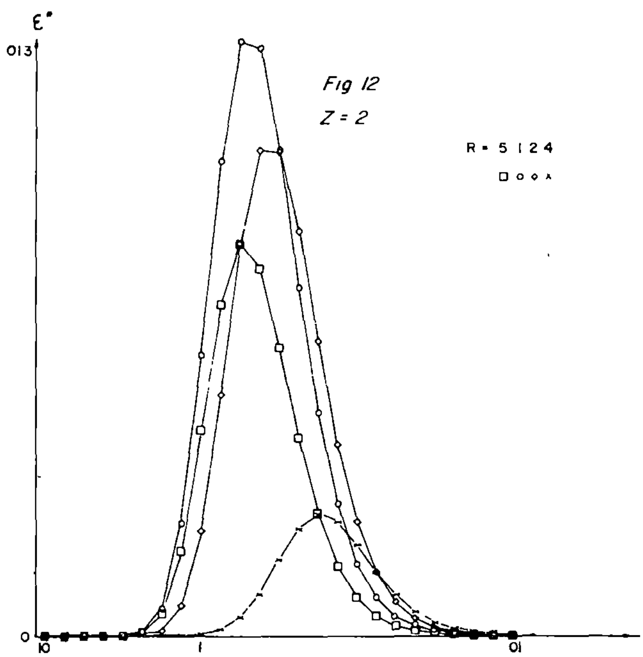


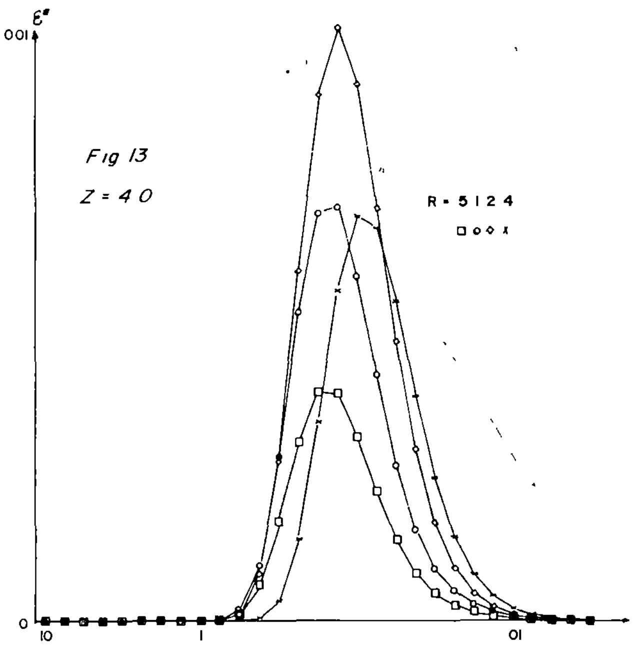
FIGURES 9-13
Waveforms of ϵ^* at selected points in and on the ground. Each figure shows waveforms at corresponding points at a different Z value. Within each figure the different curves represent different values of R. The vertical axes are on a linear scale.











and propagates at a decreasing velocity away from the loop. Since the section has radial symmetry the results indicate approximately a torus of current expanding outwards with the maximum current on a cone at late time and on a more complex surface at early time. An empirical relationship can be derived from figure 8 but it should be remembered that its accuracy is slightly limited because the data was calculated on a grid.

At late time the maximum ϵ^* occurs at radius R_m given by:

$$R_m^2 \ \tau \approx 0.64 \qquad (6)$$

An interesting feature shown by figures 1-7 is that not only is there a maximum in current but that the gradients in the current system decrease with increasing time as evidenced by the increase in contour spacing and drop in actual values. Now at the maximum

$$\frac{\partial \epsilon^*}{\partial z} = \frac{\partial \epsilon^*}{\partial_r} = 0 \qquad (7)$$

and by application of Maxwells equation

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad (8)$$

It is apparent that

$$\frac{\partial \underline{Br}}{\partial t} = 0 \qquad (9)$$

where Br is the radial component of the magnetic inducation. The term in 9 is associated with the excitation of a current flowing in a vertical tangential plane in an anomalous body.

Cases are known in which current modes in a scattering body decouple (Lee 1978, Thio 1977) i.e. the body current modes are merely those of the incident field. The above indicates that certain modes may be very weakly excited in parts of a scattering body.

It is also instructive to consider the radiation pattern in terms of the transmitted waveforms shown in figures 9-13. The waveforms show a general attenuation and time delay with distance from the loop but, as expected from the contour plots the relationship is not simple. However, an interesting feature is that in the R, Z ranges 0-5 loop radii the width of the pulse where significant amplitudes occur does not exceed about 3 decades in τ.

The results have important applications to field measurements. They allow the removal of the incident field from the loop at any point rendering feasible measurements using separate detectors on the surface or down boreholes. Also, with the removal of the primary field the scattered field due to inhomogeneities can be isolated and may be more amenable to interpretation than the combined fields particularly at early times.

Acknowledgements

We are indebted to Geopeko for continued support, in particular for travel expenses making the co-operative effort possible.

References

GRAY, A., & MATHEWS, G.B., 1966. *Treatise on Bessel Functions and Their Applications to Physics,* 2nd Edition, Dover N.Y.

LEE, T., & LEWIS, R., 1974. Transient EM Response of a Large Loop on a Layered Ground, *Geophysical Prospecting 22,* 430-444.

LEE, T., 1974. *Transient Electromagnetic Response of a Sphere in a Layered Medium.* Unpublished Ph.D. Thesis, Macquarie University.

LEE, T., 1977. Estimation of Depth to Conductors by the Use of Electromagnetic Transients. *Geophysical Prospecting* 25, 61-75.

LEE, T., 1978. *Transient Electromagnetic Waves Applied to Prospecting,* Invited Paper Presented at the 19th General Assembly of U.R.S.I., Helsinki, 1-8 August, 1978.

NABIGHAN, M., 1978. As reported in B. Brabham *et al,* 1978, Lecture Notes from the U.S. Australia Electromagnetic Workshop. *Bull. Aust. Soc. Explor. Geophys. Vol. 9,* 2-33.

OBERHETTINGER, F., & BADII, L., 1973. *Tables of Laplace Transforms,* Springer-Verlag, Berlin.

THIO, Y.T., 1977. *Riemann Boundary Value Problems in Electromagnetism,* Unpublished Ph D. Thesis, Monash University.

VOZOFF, K., 1978. *Electromagnetic Methods in Applied Geophysics,* Paper given at the Fourth Workshop on Electromagnetic induction in the Earth and Moon, Murnau, G.F.R., 12 September, 1978.

WAIT, J.R., & SPIES, K.P., 1971. Subsurface Electromagnetic Fields of a Line Source on a Conducting Half-space, *Radio Science, Vol. 6,* 781-786.

WAIT, J.R., HILL, D.A., 1972 Transient Magnetic Fields Produced by a Step Function — Excited Loop buried in the Earth, *Electronic Letters, Vol. 8,* No. 11, 213-221.

Paper

The Effect of Loop Height in Transient Electromagnetic Modelling or Prospecting

T. J. Lee

L. A. Richardson & Assoc. P/L.,
P.O. Box 217,
Gordon, N.S.W. 2072

The voltage induced in a horizontal loop over a uniform ground at height, h, has been calculated for the case where the loop is excited by a step current. If the loop has a radius b and the ground has a permeability of μ and conductivity then at time t :

$$\left(\frac{\partial V}{\partial h}\ \frac{100}{V}\right)\Big|_{h=o} \simeq 125\ \sqrt{\frac{\delta\mu\pi}{t}}$$

For $\sigma\mu b^2/(4t)$ in the range 10^{-4} to 1 the maximum error in using this formula is less than 44%. If is less than 0.1 the error is less than 5%.

It is suggested, then, that for small h,

$$V(h) \simeq V(o) + \frac{\partial V}{\partial h}\Big|_{h=o} h$$

and that the above formulae will give first order corrections for the voltage induced in the loop for changes in elevation of the loop.

Introduction

At the present time all the type curves for the interpretation of transient electromagnetic data assume that the traverses are carried out on flat terrain. Further, in the case of model studies carried out by using metallic models all the dimensions are very small. A consequence of this is that the actual field situation being modelled is for a loop several metres above the ground. There is a need, therefore, to be able to allow for the elevation changes of these loops.

The Calculation of Elevation Changes

If V is the voltage observed in a loop of radius b over a uniform half-space of conductivity δ and h is the height of the loop above the ground, then:

$$V(h,\sigma,b) = V(o,\sigma,b) + \sum_{n=1}^{\infty} \frac{\partial^n}{\partial h^n} V(h,\sigma,b)\Big|_{h=o} \frac{h^n}{n!} \qquad (1)$$

Therefore for small elevations the percentage change per metre of height is approximately

$$\frac{(V(h,\sigma,b) - V(o,\sigma,b)) \times 100}{h\,V(o,\sigma,b)} = \frac{\dfrac{\partial V(h,\sigma,b) \times 100}{\partial h}}{V(o,\sigma,b)} \qquad (2)$$

From Lee and Lewis (1974) (equation 2 and 13), the secondary voltage E induced in the loop carrying a e^{iwt} current is:

$$\bar{E} = i\pi b^2 \omega\mu I \int_0^{\infty} e^{-2\lambda h}\left[J_1(\lambda b)\right]^2\left(\frac{n_1-\lambda}{n_1+\lambda}\right) d\lambda$$

$$\text{where} \qquad n_1 = \sqrt{\lambda^2 + i\omega\mu\sigma} \qquad (3)$$

Here μ is the permeability of the ground and is assumed to be equal to the permeability of free space, μo.

The voltage, $\overline{V}$, due to a step function wave form is then:

$$\bar{V} = \pi b^2 \mu I \int_0^{\infty} \bar{e}^{2\lambda h}\left[J_1(\lambda b)\right]^2\left(\frac{n_1-\lambda}{n_1+\lambda}\right) d\lambda \qquad (4)$$

Therefore

$$\frac{\partial \bar{V}}{\partial h}\Big|_{h=o} = -2\pi b^2 \mu I \int_0^{\infty} \lambda \left[J_1(\lambda b)\right]^2\left(\frac{n_1-\lambda}{n_1+\lambda}\right) d\lambda \qquad (5)$$

Writing $x = \lambda b$ and denoting the inverse Laplace transform by $\mathcal{L}^{-1}$ yields for $t > 0$.

$$\frac{\partial V}{\partial h}\Big|_{h=o} = \frac{4\pi b \mu I}{b}\int_0^{\infty} x^2 \left[J_1(x)\right]^2 \mathcal{L}^{-1}\left(\frac{1}{m+x}\right) dx \qquad (6)$$

where $\quad m_1 = \sqrt{x^2 + i\omega\mu\sigma b^2} \quad$ and $\quad \mathcal{L}^{-1}(\bar{V}) = V$

Therefore

$$\frac{\partial V}{\partial h}\Big|_{h=o} = \frac{4\pi\mu I}{b^2\sigma\mu\sqrt{\pi\tau}}\int_0^{\infty} x^2 \left[J_1(x)\right]^2 \bar{e}^{\tau x^2}\left(1 - x\sqrt{\pi\tau}\, e^{\tau x^2}\,\mathrm{erfc}(x\sqrt{\tau})\right) dx \qquad (7)$$

Here $\tau = t/\sigma\mu b^2$

The above expression can be evaluated by expanding the Bessel functions as a power series and then evaluating the resulting integrals by using the definition of the gamma function and the result given by Gradshteyn and Ryzhik (1965, p648, Number 6.281):

$$\int_0^{\infty} x^{m-1}\,\mathrm{erfc}(x)\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{m\sqrt{\pi}} \tag{8}$$

Therefore

$$\left.\frac{\partial V}{\partial h}\right|_{h=0} = \frac{4\pi\mu I}{\sigma\mu b^2}\sum_{m=0}^{\infty}\left\{\frac{(-1)^m\,(2m+2)!\,(2m+3)!!}{2^m m!\,(m+1)!\,(m+1)!\,(m+2)!\,(m+3)}\left(\frac{\sigma\mu b^2}{4t}\right)^{m+3}\right\} \tag{9}$$

From Lee and Lewis (1974, equation 11)

$$V = -\frac{2b\mu I\sqrt{\pi}}{t}\left(\frac{\sigma\mu b^2}{4t}\right)^{3/2}\sum_{m=0}^{\infty}\left\{\frac{(-1)^m(2m+2)!\left((\sigma\mu b^2)/4t\right)^m}{(m+1)!\,(m+2)!\,(2m+5)}\right\} \tag{10}$$

Therefore

$$\left.\left(\frac{\partial V}{\partial h}\,\frac{100}{V}\right)\right|_{h=0} = 25\sqrt{\frac{\sigma\mu\pi}{t}}\left\{\frac{\sum_{m=0}^{\infty}\frac{(-1)^m(2m+2)!(2m+3)!!\left((\sigma\mu b^2)/(4t)\right)^m}{2^m(m)!(m+1)!(m+1)!(m+2)!(m+3)}}{\sum_{m=0}^{\infty}\frac{(-1)^m(2m+2)!\left((\sigma\mu b^2)/(4t)\right)^m}{m!(m+1)!(m+2)!(2m+5)}}\right\} \tag{11}$$

If $\frac{\sigma\mu b^2}{4t}$ is small, then:

$$\left.\left(\frac{\partial V}{\partial h}\,\frac{100}{V}\right)\right|_{h=0} \simeq 125\sqrt{\frac{\sigma\mu\pi}{t}} \tag{12}$$

Discussion

Figure 1 shows the results of comparing the exact formula with the approximate formula for $\frac{\sigma\mu b^2}{4t}$ in the range 10^{-4} to 1. As the figure shows the percentage error is very small. In the practical field case the mean height can be of the order of one metre due to bushes, topography etc., however, this figure shows that the resultant error is always less than 1%. So for this case it would not be necessary to apply the correction.

However, in scale model experiments when the modelling materials are metals then the correction is important because the scaled coils are at a much greater distance from the surface.

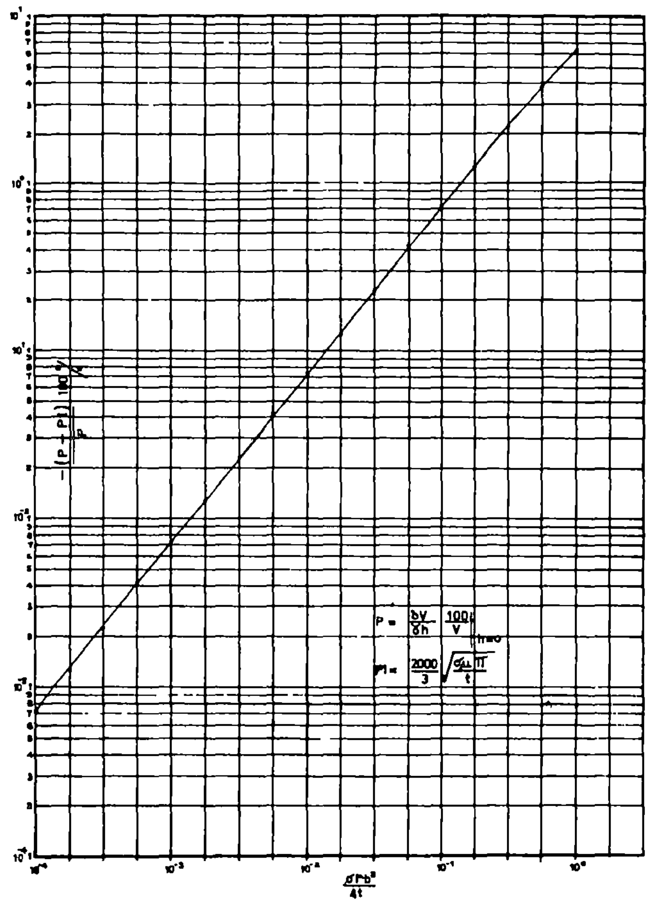


FIGURE 1
First Order Corrections for elevation changes for single loop TEM equipment.

Acknowledgements

The author wishes to acknowledge L. A. Richardson & Assoc. Pty. Ltd., for the support of this work.

References

GRADSHTEYN, I.S., & RYZSHIK, I.M., 1965: Tables of Integrals Series and Products, 4th Edition, Tran. Academic Press, New York, 1086p.

LEE, T., LEWIS, R., 1974: Transient E.M. Response of a Large Loop on a Layered Ground. *Geophysicaly Prospecting* **22**, 430-444

Paper

Depth Estimation with PEM —A Cautionary Note

T.J. Lee

L.A. Richardson & Assoc.
P.O. Box 217,
Gordon, N.S.W. 2072

The method advocated by Crone (1977) for determining the position of a "current axis" for a conducting ore body will give a positive result even if there is no ore body.

The depth, ℓ, to this axis in a uniform ground of conductivity σ and permeability μ_o is given by:

$$\ell = \frac{8}{5} \cdot \frac{1}{\sqrt{\pi}} \sqrt{\frac{t}{\sigma \mu_o}}$$

In this expression t is time and consequently the result can be used to determine the conductivity profile of a nearly uniform ground.

1. Introduction

There are few methods of estimating the depth and position of a conductive body that are available to the field geophysicist. It is pleasing, then, to learn of the method advocated by Crone (1977) to do this task. In what follows it will be argued that this method should be treated with caution as it will tive a positive result over a uniform ground.

The method under discussion has been described by Crone (1977) as follows: "In this method a vertical detailed transmitting loop is placed over the weak anomaly for the purpose of selectively energising the conductor at depth. A short traverse with six to eight readings spaced twenty to twenty-five metres apart is carried out over the suspected coneuctor, perpendicular to its strike. Readings are taken at both horizontal and vertical receiving loop positions and the dip angles of the secondary field at all eight samples are plotted. If perpendicular lines are drawn from the dip angle then they should converge at the eddy current axis of that sample".

Figure 1 shows the construction used in the interpretation.

2. Theory

As the loops used in this field survey are quite small they can be approximated by a horizontal magnetic dipole. The electromagnetic fields about such a vertical loop resting on a uniform half space of conductivity σ and magnetic permeability μ_o have been given by Morrison *et al* (1969), and Ward *et al* (1973).

Whence,

$$H_x = \frac{-Q}{x 4\pi} \int_0^\infty (1 + R(\lambda))\lambda\left[-J_1(\lambda x) + \lambda x J_0(\lambda x)\right] d\lambda \tag{1}$$

$$H_y = 0 \tag{2}$$

$$H_z = \frac{+Q}{4\pi} \int_0^\infty (R(\lambda) - 1)\lambda^2 J_1(\lambda x)\, d\lambda \tag{3}$$

$$R(\lambda) = (\sqrt{\lambda^2 + i\omega\sigma\mu_o} - \lambda)/(\sqrt{\lambda^2 + i\omega\sigma\mu_o} + \lambda) \tag{4}$$

In these equations Q is the dipole moment, i.e. $N_1 A_1 I$ where N_1 is the number of turns in the loop and A_1, the area of the loop and I the current flowing in the loop. Here displacement currents have been neglected and an e^{iwt} time dependence has been assumed.

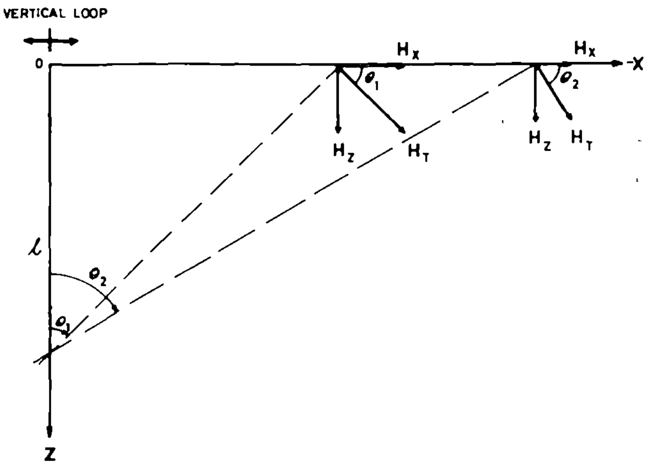
H_x, H_y and H_z are the three components of the magnetic field. The axes chosen have their origin at the centre of the loop with the x axis directed out along the axis of the loop. This axis defines the traverse line. The z axis is directed vertical and the y axis is into the page. See Figure 1.

The spectra of the voltages induced in a small receiving loop ($\bar{V}_x$, $\bar{V}_z$) orientated vertically or horizontally for a step-like current flowing in the primary loop are given by:

$$\bar{V}_x = -A_2 N_2 H_x \tag{5}$$

$$\bar{V}_z = -A_2 N_2 H_z \tag{6}$$

Here we notice that the spectrum of a step function is $1/\omega$ and this cancels with the $i\omega$ arising from the time derivative of the magnetic field. Also A_2 is the area of



the small receiving loop of N_2 turns.

The time domain voltages ($V_x(t)$, $V_z(t)$) therefore are:

$$V_x(t) = -\frac{Q}{x}\frac{N_2A_2}{4\pi}\int_0^\infty \frac{2\lambda^2}{\sigma\mu_0}F(\lambda,t)\left[-J_1(\lambda x)+\lambda x J_0(\lambda x)\right]d\lambda \tag{7}$$

$$V_z(t) = +\frac{Q\,N_2A_2}{4\pi}\int_0^\infty \frac{2\lambda^3}{\sigma\mu_0}F(\lambda,t)J_1(\lambda x)\,d\lambda \tag{8}$$

If t is time then for $t>0$.

$$F(\lambda,t) = \frac{a}{\sqrt{\pi}}e^{-\lambda^2/a^2} - \lambda\,\mathrm{erfc}\left(\frac{\lambda}{a}\right) \tag{9}$$

where $a = \sqrt{\sigma\mu_0/t}$

(Lee and Lewis (1974)).

These integrals are readily evaluated by firstly using the power series expansion for the Bessel functions, Abromowitz and Stegun (1965, F.9.1.10) and secondly the results of Gradshteyn and Rhzshik (1965, No. 6.281).

Expanding the Bessel function as a power series and integrating term by term the resulting integrands of equations 7 and 8 yields:

$$V_x = \frac{-QA_2N_2}{x\sigma\mu_0}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{2n+1}\frac{(-1)^n\,a^{2n+5}\,(2n+1)}{n!\,4\pi\,\sqrt{\pi}\,(2n+5)} \tag{10}$$

$$V_z = \frac{+QA_2N_2}{\sigma\mu_0}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{2n+1}\frac{(-1)^n\,a^{2n+6}\,\Gamma(n+5/2)}{n!\,(n+1)!\sqrt{\pi}\,(2n+6)\,4\pi} \tag{11}$$

The angle that fixes the "current axis" is seen from Figure 1 to be defined by

$$\tan\theta = -V_z/V_x \tag{12}$$

This expression simplifies for $ax<<1$ to be:

$$\tan\theta = \frac{5\sqrt{\pi}}{8}\,ax \tag{13}$$

This result shows that even for a uniform half space there is a "current axis" at a depth ℓ, where

$$\ell = \frac{8}{5}\cdot\frac{1}{\sqrt{\pi}}\cdot\sqrt{\frac{t}{\sigma\mu_0}} \tag{14}$$

This equation shows that the apparent depth is independent of station position.

3. Discussion

The above result does not show that the method will not yield the "current axis" of a steeply dipping conductor beneath the transmitter. It does show, however, that a positive result will be obtained even when there is no conductor present. For this reason the method of depth estimation should be treated with care.

Alternatively equation 14 can be used to estimate the conductivity of a reasonably uniform ground once ℓ is known. As noted above this quantity can be found from Figure 1.

4. Acknowledgements

I am grateful to Dr. Peter Gunn of Aquataine for suggesting this problem and to L.A. Richardson & Assoc. for their continued support.

5. References

ABRAMOWITZ, M., STEGUN, I., 1965. Handbook of Mathematical Functions, Dover, New York, pp.1046.

CRONE, D. 1977. Ground Pulse EM — Examples of Survey Results in the Search for Massive Sulphides and New Equipment Developments, *Bull. Aust. Soc. Explor. Geophys.* **8**, pp.38-42.

GRADSHTEYN, I.S., RYZSHIK, I.M , 1965. Tables of Integrals Series and Products, 4th Edition, Academic Press, New York, pp.1086.

LEE, T., LEWIS, R.J.G., 1974. Transient EM Response of a Large Loop on a Layered Ground, *Geophysical Prosp.* **22,** pp 430-444

MORRISON, H.F., PHILLIPS, R.J., OBRIEN, D.P., 1969. Quantitative Interpretation of Transient Electromagnetic Fields over a Layered Half-space, *Geophysical Prospecting,* **17**, pp.82-101.

WARD, S H., PEEPLES, W.J , RHU, J , 1973. Analysis of Geomagnetic Data Methods in Computational Physics, Vol 13, *Geophysics,* Edited by B.A. Bolt, Academic Press, New York, pp.163-238

ESTIMATION OF DEPTH TO CONDUCTORS BY THE USE OF ELECTROMAGNETIC TRANSIENTS

BY

T. LEE

Reprinted from:

GEOPHYSICAL PROSPECTING, Volume XXV, No. 1, 1977

ESTIMATION OF DEPTH TO CONDUCTORS BY THE USE OF ELECTROMAGNETIC TRANSIENTS *

BY

T. LEE **

ABSTRACT

LEE, T., 1977, Estimation of Depth to Conductors by the Use of Electromagnetic Transients, Geophysical Prospecting 25, 61-75

The transient response of a layered structure to plane wave excitation can be considered to be composed of a series of waves and a ground wave. For the case of a half-space of conductivity σ and permeability μ the maximum in the electric field is found at a depth z and time t when $t = z^2\sigma\mu/2$.

This formula can be used to estimate the depth to a buried horizontal conductor with an accuracy that depends upon the resistive contrast at the conductor's surface.

The above ray type of solution can be converted to a solution composed of a number of modes by the use of a Poisson transform and the transformed solutions yield decay constants that are consistent with the previously reported results.

In the case of a finite source, the maximum in the electric field is strongly directed. The direction depends upon the geometry of the source and the air-earth interface. Although the maximum varies with direction it can be shown that in some directions similar laws to that above are valid.

The depth to a conductor can be estimated from the early part of the transients when the ground wave is removed. The removal of the ground wave from the transient is facilitated by the use of an apparent conductivity formula.

Although these results were obtained under restrictive conditions they do provide some insight into the electrical transients that are encountered by studying more complex models.

1. INTRODUCTION

The most common instrument for the measurement of electromagnetic transients that are generated from a square wave in a large loop is the Russian-made MPPO-1 equipment. This equipment measures the decaying voltage for various times between 0.5 ms and 15 ms (Velekin, Bulgakov, Gigoryeu, and Polikorpov 1971). A recent paper by Buscelli (1974) suggests that it may soon be possible to measure the voltage over the time range 0.25 to 100 ms.

* Received August 1975

** L. A. Richardson & Assoc. Pty. Ltd., 33 Bertram Street, Chatswood, N.S.W. 2067, Australia

Although the Russian made equipment has been widely used there does not appear to be available any means by which the depth to a conductor can be estimated from the recorded signals.

Lee (1975) has shown that the effect of depth of a buried spherical conductor could be seen in the early part of the transient signal. At later stages the fields decayed purely exponentially. Indeed the usual method used to calculate type curves for simple shaped conductors is to obtain the solution as a series of terms of the form $A\ \mathrm{e}^{-at}$. Unfortunately this series converges slowly for small values of the time parameter t. Recently, however, Kunetz (1974) has presented a series of decay curves for the transient response of layered structures. Kunetz (1974) obtained his solutions as a sum of terms of the form $B\ \mathrm{e}^{-b/t}$. These series are very rapidly convergent for the early stages of the transient.

In this paper we exhibit the relationship between the two types of solution and examine the possibility of estimating the depths to a conductor from the arrival time of the maximum in the secondary transients.

In what follows displacement currents have been neglected. This means that our solutions will be valid provided that $t \gg \varepsilon/\sigma$. Here ε is the dielectric constant and σ is a conductivity (Wait 1960). In addition it will be assumed that the magnetic permeability μ is equal to the permeability of free space everywhere

2. Propagation of a Plane Transient Electromagnetic Wave in a Half-Space

Since the depth of a conductor is to be estimated from the secondary signal observed at the ground surface it is expedient to examine first the propagation of a transient electromagnetic wave in a half-space.

Kunetz (1972) has shown that if the electric field is generated by a magnetic field whose wave function is:

$$H = H_0 \mathrm{H}(t), \tag{1}$$

where $\mathrm{H}(t)$ is the Heaviside step function and H_0 is the amplitude of the step function, then the associated electric field E is:

$$E = \frac{2H_0\mu_0\, \mathrm{e}^{-z^2\sigma\mu/(4t)}}{\sqrt{\pi\sigma\mu_0 t}}. \tag{2}$$

Here μ is the magnetic permeability, σ the conductivity of the half-space, and z the depth beneath the ground surface

Therefore, a maximum in the electric field intensity at a depth z occurs when

$$t = z^2\sigma\mu/2. \tag{3}$$

The question arises, then, as to whether or not the depth to a layer can be estimated by using this formula with $z = 2h$, where h is the depth to the second layer and t the time of arrival of the secondary pulse at the earth's surface. In order to see if this is possible we shall briefly examine the transient response of a two-layered structure.

3. Transient Electromagnetic Response of a Two Layered Ground

Consider a ground of two layers the upper one of which has a thickness h and a conductivity σ_1. The basal layer has a conductivity σ_2. Both layers are assumed to have a magnetic permeability μ_0. The electric field produced at the earth's surface due to a step-like change in the inducing magnetic field is:

$$E = \frac{2H_0\mu_0}{\sqrt{\pi\sigma_1 t\mu_0}}\left[1 + 2\sum_{n=1}^{\infty} r^n\, e^{-n^2 h^2 \sigma_1 \mu/t}\right], \tag{4}$$

where

$$r = (\sqrt{\sigma_1} - \sqrt{\sigma_2})/(\sqrt{\sigma_1} + \sqrt{\sigma_2})$$

(Kunetz 1972).

This equation has been interpreted by Clay, Greischar, and Kan (1974) to be a ground wave and a series of rays which are being reflected back and forth between the two surfaces of the top layer. The approach used by Clay et al. (1974) was to design a matched filter so that the arrival of the first reflection could be accurately estimated. These authors found that for best results the ground wave had to be first removed from the measured electric field and in many instances more than the first reflection had to be considered.

The previous section of this paper suggests that the maximum in the secondary wave occurs when:

$$\frac{h^2\sigma\mu}{4t} = 0.5 = y(0.5)$$

An inspection of equation (4) shows that this is likely to be an excellent estimate when r is small or moderate. Figure 1 shows a graph of y against the reflection coefficient r.

These results show that for typical host rock conductivities there is useful information needed to estimate the depth to a flat lying conductor in the very early portion of the transient signal.

An interesting feature of equation (4) is that for early times

$$E \approx \frac{2H_0\mu_0}{\sqrt{\pi\sigma\,\mu_0 t}}\left[1 + 2r\, e^{-h^2\sigma\mu/t}\right]. \tag{5}$$

This equation shows that the ground wave is first decreased if the basal layer is a conductor while it is increased if the basal layer is a resistor. In view of this some care should be exercised when interpreting the early part of a transient decay curve. The effects have also been observed in previous calculations in involving more complex sources by Morrison, Phillips, and O'Brien (1969) and Lee and Lewis (1974).

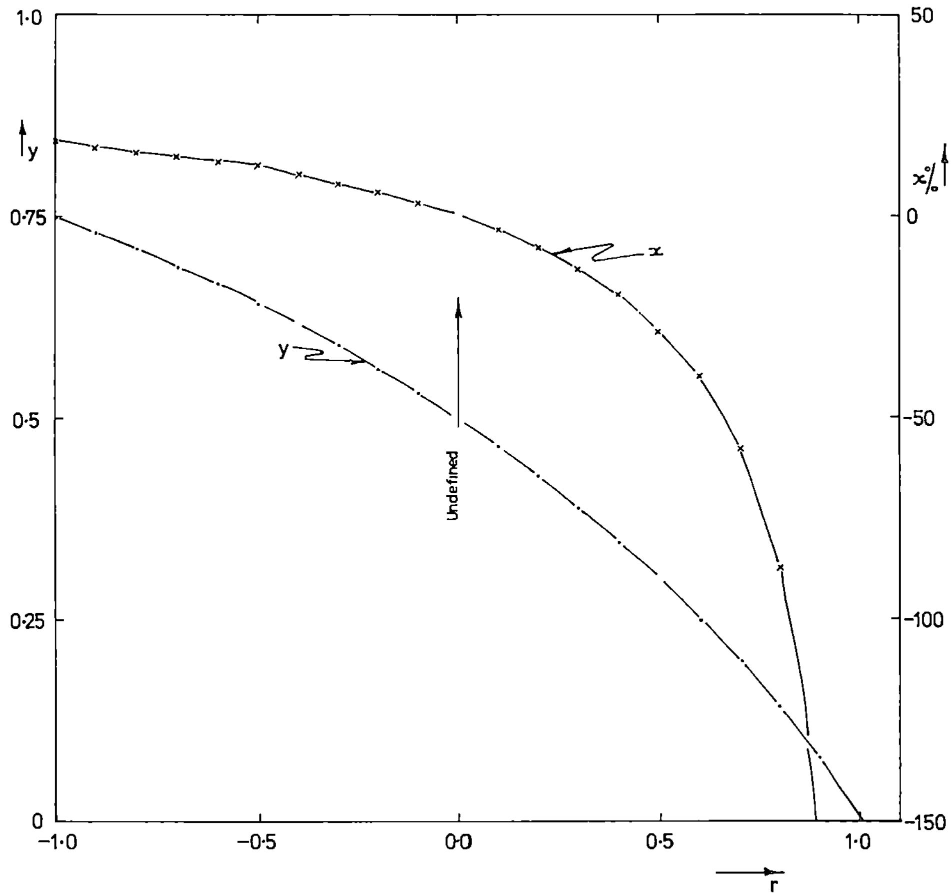


Fig 1 A graph of the reflection factor $r = (\sqrt{\sigma_1} - \sqrt{\sigma_2})/(\sqrt{\sigma_1} + \sqrt{\sigma_2})$ against the value of $y(h^2\sigma_1\mu_0/(4t))$ at which the maximum in the secondary transient occurs Also shown is the error incurred from estimating the depth with $y = 0.5$

Equation (5) might also form the basis for estimating the depth by subtracting the ground wave and plotting t times the secondary response as a function of $1/t$ on semilog paper. For early times the graph will be asymptotic to a straight line whose gradient is proportional to h^2.

The expression on the right hand side of (4) can be considered to be composed of a series of terms that are the various reflected rays in which the time parameter is found as a reciprocal. The consequence of this is that it is difficult to see the behaviour of the electric field at later stages. Since it is the response during these later stages that is indicative of conductors at depth (Lee and Lewis 1974) it is useful to be able to transform the ray solution to a mode theory solution in which the time parameter occurs as t and not $1/t$ This is also handy if computations are to be performed, as a ray theory solution will give a solution in a few terms for early times while a mode theory solution will yield an efficient solution at later stages.

4. Transformation of Ray Theory Solutions to Mode Theory Solutions

The transformation from one type of solution to another can be readily effected by the use of the Poisson transform

The Poisson transform can be formally written as:

$$\beta^{1/2} \sum_{n=-\infty}^{\infty} g(\beta n) = \sum_{n=-\infty}^{\infty} \frac{\alpha^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{-i\alpha n y}\, dy, \tag{6}$$

where $\alpha\beta = 2\pi$ (Courant and Hilbert 1965, p. 77).

When $g(y)$ is an even function this equation becomes:

$$\beta^{1/2}[g(0) + 2 \sum_{n=1}^{\infty} g(\beta n)] = \sqrt{\alpha}[f(0) + 2 \sum_{n=1}^{\infty} f(n\alpha)] \tag{7}$$

where

$$f(n\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos\,(n\alpha y)\, g(y)\, dy.$$

By inspection, then, when this result is applied to equation (4) and the more complex formulae such as those of Kunetz (1972) we obtain a series of terms that have a time parameter t and not $1/t$

For example, consider the transient response of a conductive plate above a resistive half-space. Applying the result given in equation (7) with $\beta = 2\pi\sqrt{t/(h^2\sigma_1\mu_1)}$ yields:

$$E = \frac{2H_0\,\mu_0}{h_0\,\sigma\mu_0}\,[\exp\,(-\log\,(r^{-2})/(4\alpha^2)) \cdot \operatorname{erfc}\,(\log\,(1/r)/(2\alpha)) + 2 \sum_{n=1}^{\infty} \exp\,(-n^2\pi^2 t/(\sigma\mu h^2))]. \tag{8}$$

In evaluating the integrals—except the first—it has been assumed that $\log\,(1/r)/\alpha + 2in\pi/\alpha \approx 2in\pi/\alpha$

Therefore

$$E \approx \frac{2H_0\,\mu_0}{\sqrt{(\sigma_0\mu_0 h)}} + \frac{4H_0\,\mu_0}{h\sigma_1\mu_0} \sum_{n=1}^{\infty} e^{-n^2\pi^2 t/(\sigma\mu_0 h^2)}. \tag{9}$$

Hence the contribution E_t to the transient electric field from the upper layer can be approximated by:

$$\frac{4H_0\,\mu_0}{h\sigma\mu_0}\,e^{-\pi^2 t/(\sigma\mu_0 h^2)}.$$

when t is large,
i.e.

$$E_t \approx \frac{4H_0\,\mu_0}{h\sigma\mu_0}\,e^{-\pi^2 t/(\sigma\mu_0 h^2)}. \tag{10}$$

A similar result has been given by Velekin and Bulgakov (1967) for a thin plate excited by a large loop of wire in which a primary step-like current flows. These authors found that the electromagnetic fields at later times decay exponentially with a decay constant of $1/(\sigma\mu_0 hl)$.

Here l was a length parameter that was related to the size of the plate along the dip and the location of the plate with respect to the loop.

As a second example, consider the transient response of a slab of thickness h above a conductive half-space such that the parameter r in equation (4) is approximately -1. Applying the Poisson transform to equation (4) with $r = -1$ yields:

$$E = \frac{2H_0\,\mu_0}{2h\mu_0\sigma}\,\left(\left(2 + 2\sum_{n=1}^{\infty} e^{-\beta^2 n^2/4}\cdot 2\cos h(\beta^2 n/4)\right) e^{-\beta^2/16}\right) \tag{11}$$

where

$$\beta = 2\pi\sqrt{t/\sigma\mu h^2}.$$

Hence for later times

$$E_t \approx \frac{4H_0\,\mu_0}{h\sigma\mu_0}\,e^{-\pi^2 t/(\sigma\mu_0 4h^2)}. \tag{12}$$

There is no ground wave term in this case because, as can be seen from equation (1), as the conductivity of the second layer increases the ground wave decreases for all $t > 0$. Therefore, this structure is equivalent to a slab of thickness $2h$. As stated previously, the difference between the two structures is readily distinguished from the early part of the decay curve.

These mode type solutions, then, are complementary to the wave type solution presented previously. That is to say, they are useful for large values of the time parameter.

The above results show, then, that much information has been ignored in the past by people using the transient electromagnetic method for geophysical prospecting by concentrating on the later parts of the transients.

As pointed out by Clay et al. (1974), who studied the transient response of thin plates, depth estimates could be made with the electrical transients if the ground wave was first subtracted. In order to subtract the ground wave it is necessary to know the conductivity of the uppermost layer. It will be shown in the next section how this parameter can be estimated from the transients by an apparent conductivity function.

5. Apparent Conductivity Functions

The previous theory has shown that for a resistor over a conductor the depth to the conductor can be approximately estimated from the maximum in the secondary transient wave.

From equation (2) the conductivity of a uniform ground can be found to be:

$$\sigma = \left[\frac{2H_0\mu_0}{E\sqrt{\pi t\mu_0}}\right]^2 \tag{13}$$

In this expression the electric field is measured at the ground's surface.

In general, equation (13) can be used to define an apparent conductivity σ_0. From equation (1) it is clear that for early times the electric field is rapidly attenuated with depth. Therefore it is to be expected that any layered structure will, at best, behave as a two-layered structure for early times.

Hence for layered structures and early times

$$\sigma_a \approx \frac{\sigma}{[1 + 2r\,e^{-h^2\sigma\mu_0/t}]^2} \approx \sigma[1 - 4r\,e^{-h^2\sigma\mu_0/t}]. \tag{14}$$

Equation (14) shows that if the conductivity of the top layer is to be estimated from the transient electric fields then the fields must be measured at very short times or else the secondary structures must be very deep.

6. Extension to Other Sources

Although the previous discussion has been devoted exclusively to the transient response of a layered structure by a plane wave, it will now be shown that the above conclusions are consistent with features derived from transient curves derived from other sources. This conclusion has been noted previously by Lee (1975) who studied the transient response of a sphere in a layered medium which was excited by a large co-axial loop. Lee (1975) found, for example, that for later times the fields decreased at the same rate of decay as would be expected from the previous studies made using plane wave sources.

6.1 *Line Sources*

Wait (1971) studied the transient response of a uniform ground to a line source of current situated at the earth's surface. He showed that for the line source that carried an impulsive current, the electric field $e(t)$ at a depth h below the surface is described by:

$$e(t) = \frac{8}{\pi\sigma^2\mu_0 h^4} \cdot F\left(\frac{4t}{\sigma\mu_0 h^2}\right),$$

where

$$F(T) = \frac{1}{T^2}\left[\frac{1}{2} - \frac{1}{T} + \left(\frac{1}{\pi T}\right)^{1/2}\right] \exp\,(-1/T)$$

and

$$T = 4t/(\sigma\mu h^2). \tag{15}$$

Therefore, the electric field $E(t)$ to be expected from a step current flowing in the wire is:

$$E(t) = \int_0^t e(t)\,dt. \tag{16}$$

The electric field has a maximum when

1.043

$$h^2\sigma\mu/(4t) = \cancel{1.4205}. \tag{17}$$

The transient electric field E at the earth's surface at a distance x from the line source is:

$$E = \frac{-\mu_0 I}{\pi}\int_0^\infty L^{-1}\left(\frac{\cos\lambda x}{n+\lambda}\right)d\lambda \tag{18}$$

where L^{-1} denotes the inverse Laplace transform and $n = \sqrt{(\lambda^2 + p\sigma\mu_0}$ p is the parameter of the Laplace transform (Wait 1971, eq. (1))

Therefore,

$$E = \frac{-I\mu_0}{4t}\left[\frac{1}{x^2}\,(1 - e^{-x^2})\right],$$

where

$$x^2 = \frac{\sigma\mu x^2}{4t}. \tag{19}$$

This equation shows that the maximum in the electric field at any time occurs when $x = 0$. The effect of half-space and a two dimensional source is to direct the maximum in the electric field.

6.2 *Ring Sources*

The study of the transients that are produced by axially symmetric sources is, in general, difficult because of the need to evaluate a series of very difficult integrals.

Wait (1960) found that if an electric dipole in an infinite medium is excited by a step current source, the two components e_r and e_θ of the electric field are:

$$e_r = \frac{c}{R^3} A\left(\sqrt{\frac{\sigma\mu_0 R^2}{4t}}\right) \cos\theta \tag{20}$$

and

$$e_\theta = \frac{c}{2R^3} B\left(\left\{\sqrt{\frac{\sigma\mu_0 R^2}{4t}}\right\}\right) \sin\theta, \tag{21}$$

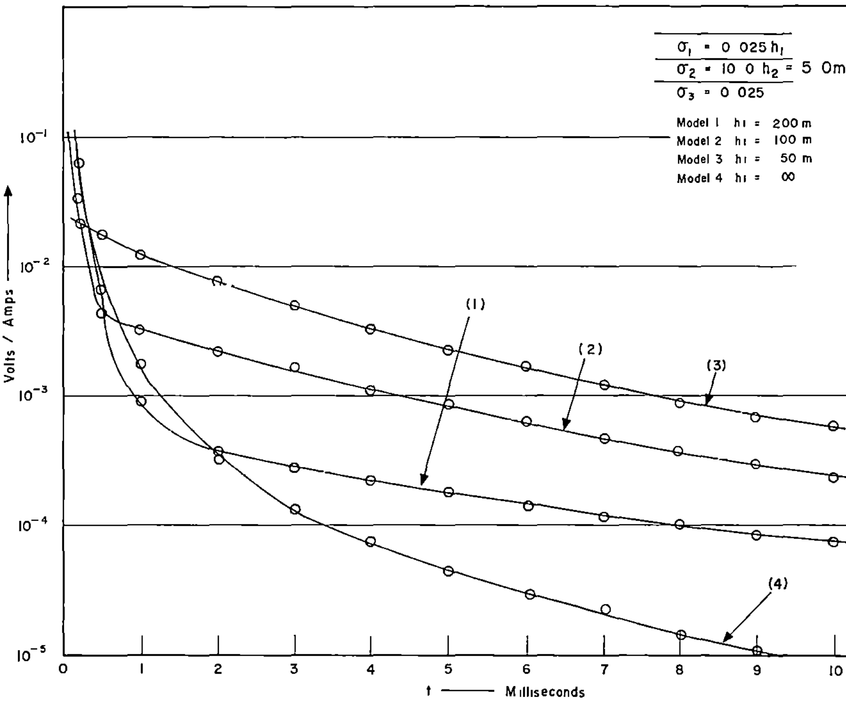


Fig 2 A set of three layer transient decay curves which show the effect of a single conductive layer at various depths in otherwise uniform ground

where

$$A(x) = \operatorname{erfc}(x) + 2x\, e^{-x^2}/\sqrt{\pi}$$

and

$$B(x) = \operatorname{erfc}(x) + (x + 2x^3)\, e^{-x^2} \cdot \frac{2}{\sqrt{\pi}}.$$

In these equations R and θ are spherical polar coordinates and c is a constant. Therefore at a point R, θ the maximum arrives when $\sigma\mu R^2/(4t) = 1$ for the radial component e_r and $\sigma\mu R^2/(4t) = (1 + \sqrt{3})/2$ for the angular component e_θ. The presence of the angular functions shows that the absolute maximum is directed along two orthogonal directions.

A large circular loop of wire in which a step current flows has been used by Lee and Lewis (1974) and by Lee (1975) as an approximation to the rectangular loops used in the field when carrying out single loop transient electromagnetic measurements.

That the fields from such loops may exhibit many of the features found in the transient signals discussed above can be seen from inspection of figure 2. This figure shows the normalised transient voltage E that would be observed in a 100 meter radius loop. The loop lies on the surface of the structure shown in the diagram. At the early stages of the transient the signal approaches that to be expected from the ground wave. A little later the fields decay more rapidly than that of the ground wave. Equation (5) suggests that the nearest structure is a conductor.

To show that this is in fact the case it is sufficient to consider the transient response of a two layered ground at early times.

Lee and Lewis (1974, equation (16)) showed that if E_1 is the normalised primary voltage and E_2 is the normalised anomalous voltage, then:

$$E = E_1 + E_2,$$

where

$$E_2 = -\mu_0 b^2 \pi \int_0^\infty J_1(\lambda b)^2 L^{-1} \left(R_2 \frac{(R_1^2 - 1)\, e^{-2n_1 d}}{1 + R_1 R_2\, e^{-2n d}} \right) d\lambda, \tag{22}$$

$$R_1 = (n_1 - \lambda)/(n_1 + \lambda),\ R_2 = (n_2 - n_1)/(n_2 + n_1),$$

and

$$n_j = \sqrt{(\lambda^2 + p\sigma_j \mu_0)}.$$

Here p is the parameter of the inverse Laplace transform and σ_1 and σ_2 are the conductivities of the top and basal layers respectively. d is the depth to the layer, and the magnetic permeability is assumed to be μ_0.

Utilization of the technique developed by Goldstein (1932) to obtain the approximate value of the inverse Laplace transform at early times yields:

$$E_2 = -cr \int_0^\infty \frac{J_1(\lambda b)^2 \lambda e^{-\lambda^2 t/(\sigma_1\mu) - d^2\sigma_1\mu/t}\, d\lambda}{\sqrt{\pi\sigma\mu_0 t}} \tag{23}$$

where $c = \mu_0 b^2 \pi$ and $r = (\sqrt{\sigma_2} - \sqrt{\sigma_1})/(\sqrt{\sigma_1} + \sqrt{\sigma_2})$.

Here we have used the inverse Laplace transform pair No. 823 of Campbell and Foster (1948).

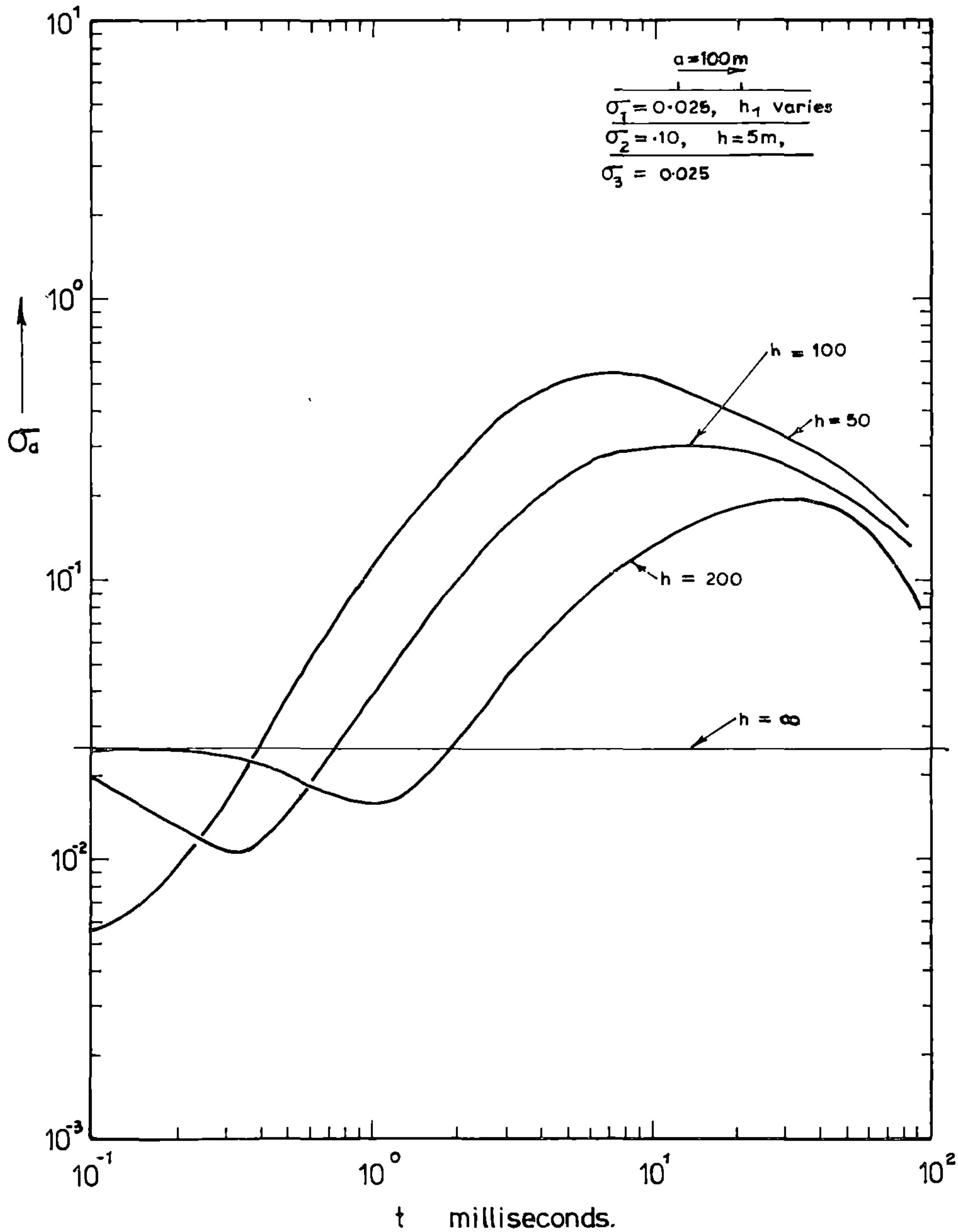


Fig 3. The apparent conductivity function for the decay curves shown in figure 2

Equation (23) can be simplified by using results given by Gay and Mathews (pp. 69 No. 18) and the asymptotic expansion of the modified Bessel functions.

Therefore,

$$E_2 = cr/(t\pi b) \cdot e^{-d^2\sigma_1\mu_0/t} \tag{24}$$

Hence the total normalized voltage E from Lee and Lewis (1974, equation (14)) and equation (24) is found to be·

$$E = \frac{c}{tb}\left[\frac{1 - re^{-\sigma_1\mu_0 d^2/t}}{\pi}\right]. \tag{25}$$

Equation (25) shows that the conclusions drawn from equation (5) are also applicable here.

The computer print-out showed that the maximum of the secondary electric field arrived at approximately 0.45 ms, 0.88 ms, and 1.7 ms for depths to the conductor of 50 m, 100 m, and 200 m respectively These results suggest that for the case of a large loop the time for the maximum response to arrive is proportional to the depth.

Figure 2 also shows that for later times the fields decay exponentially. For the case of the deepest conductor a decay constant of $0.29 \cdot 10^{+3}s^{-1}$ was found. This result compares favourably with the theoretical value of $\pi \cdot 10^{+4}s^{-1}$ that is predicted from equation (10). For the shallowest conductor the decay constant was only $0.24 \cdot 10^{+3}s^{-1}$. The error is due to the fact that for shallow conductors the maximum in the secondary field arrives when the primary field is still appreciable.

Figure 3 shows a graph of apparent conductivity σ_a as a function of time. This function is defined analogously to the functions described in section 5 of this paper. A procedure for the rapid computation of this function is described in appendix 1. As is expected from equation (25), σ_a only approaches the true value of the conductivity of the top layer provided that $\sigma_1\mu_0 d^2/t$ is large. It is expected that this function can be used to estimate the conductivity of the top layer and to provide some idea of the range of conductivities in the geological section An interesting feature of these curves that has not yet been explained is that the stationary points for early stages of this example approximately satisfy the relation $\sigma_1\mu d^2/t = 1$.

7 Conclusion

The theory developed above shows that theoretical studies of the transient responses of geological structure to plane wave sources do provide some insight into the processes occurring in the transients that are generated from more complex sources. These results indicate that it may be possible to develop useful rules to estimate the depth to conductors from the arrival time of the maximum secondary response. It seems, however, that the relative maxima arrive very early and consequently the existing instruments will need to be modified if this information is to be used in the interpretation of field results. Alternatively, if the existing instruments cannot be modified, then resistivity surveys should be carried out in conjunction with transient electromagnetic surveys, so as to provide information on the near surface conductivities of the area.

From a computational point of view there are two ways of obtaining formulae for the calculation of a transient response The first of these, which is suitable for the early stages, is a ray theory approach, and the second, suitable for later stages, is a mode theory approach. The two sets of results may be transformed from one to the other by use of the Poisson transform.

The above results are also in agreement with Ward et al. (1974) who found that if electromagnetic data were to be successfully inverted measurements over a very wide frequency range are required.

Appendix 1

Lee and Lewis (1974) studied the transient response of a uniform half-space of conductivity σ and magnetic permeability μ_0. The eddy currents were induced into the half-space by instantaneously switching off a steady current of strength I in a large loop of radius a. These authors found that the voltage induced in the loop could be described by the expression:

$$V = -\frac{2a\mu\sqrt{\pi}\cdot y}{t} \tag{A1}$$

where

$$y = 2\sqrt{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^m(2m+2)!\,\alpha^{m+1}}{m!\,(m+1)!\,(m+2)!\,2(2m+5)}$$

and

$$\alpha = \sigma\mu a^2/(4t).$$

Therefore,

$$y^{\frac{1}{3}} = \sqrt{\alpha}(1/5)^{1/3}\left[1-\frac{10\alpha}{7}+\frac{25\alpha^2}{18}-\frac{35\alpha^3}{33}\cdots\cdots\right]^{1/3} \tag{A2}$$

$$\approx x[0.584803-0.27848x^2+0.13813x^4-0.05414x^6\ \ldots\ ] \tag{A3}$$

where $x = \sqrt{\alpha}$.

This series can be inverted by the result of Abramowitz and Stegun (1965, P. 16, No. 3. 6.25) to yield:

$$x \approx (((-34.5864y^{2/3} + 6.513181)y^{2/3} + 2.38667)^{2/3} + 1.70998)y^{1/3} \dots \quad \text{(A4)}$$

Table A1 shows the results of a series of calculations using this formula. It is obvious that the error incurred in using this formula is less than 1.2% for $x \leqslant 0.5$ For values of $y \geqslant 178.272 \cdot 10^{-1}$. x can be found by the use of table A2 and a suitable interpolation formula.

TABLE A1

Range of x for which formula (A4) is useful

$y \times 10^4$	x_c (calculated)	x	error $\left(\frac{x - x_c}{x}\right)$
0 24911	0 05	0 05	0 000%
0 49600	0 06295	0 06295	0 000
0 98639	0 07925	0 07924	—0 013
1 9578	0 09976	0 09976	0 000
3 8743	0 12560	0 12559	—0 008
7 6300	0 15812	0.15811	—0 006
14 9144	0 19905	0 19904	—0.005
28 8138	0 25054	0 25059	0.020
54 66	0 31519	0 31548	0 092
100 87	0 39586	0.39716	0 323
178 272	0 49432	0.5000	1 136
298 95	0 61127	0 62946	2 890
462 612	0 73663	0.79245	7 044
653 52	0 85450	0 99763	14 347
840 014	0 94773	1 25594	32 522

TABLE A2

Relation of x and y for y > 0.01

x	y
0 39716	0 01005
0 500	0 017897
0 62946	0 029896
0 79245	0.046262
0 99763	0 06535
1 25594	0.084001
1 58114	0 099678
1 99054	0 111711
2 5059	0 120543
3 15478	0.126866
3 9717	0 131320
5 0	0 134419

Acknowledgements

We wish to acknowledge L. A. Richardson and Associates for partial support of the research summarized here.

References

ABRAMOWITZ, M, and STEGUN, I A, 1965, Handbook of mathematical functions with graphs, and mathematical tables, New York, Dover publications, pp 1046

BUSELLI, G, 1974, Multi channel transient electromagnetic measurements near cloncurry, Bull Aust Soc Exploration Geophysics 5, 31-47

CAMPBELL, G, and FOSTER, R M, 1948, Fourier Integrals for practical applications, Van Nostrand Co, New York, pp 177

CLAY, C S, GREISCHAR, L L., and KAN, T K, 1974, Matched filter detection of electromagnetic transient reflections, Geophysics 39, 683-691

COURANT, R, and HILBERT, D, 1965, Methods of mathematical physics Vol 1, Interscience Publishers Inc, New York, pp 561

GOLDSTEIN, S, 1932, Some two dimensional diffusion problems with circular symmetry, Proc London. Math Soc. (2), 34, 51-88

GRAY, A, and MATHEWS, G B, 1966, A treatise on Bessel functions and their applications to physics. Second edition prepared by Gray, A, and Macrabert, T M, Dover, New York

KUNETZ, G, 1972, Processing and interpretation of magnetotelluric soundings, Geophysics 37, 1005-1021.

LEE, T, and LEWIS, R, 1974, Transient EM response of a large loop on a layered ground, Geophysical Prospecting 22, 430-444

LEE, T, 1975, Transient electromagnetic response of a sphere in a layered medium, Geophysical Prospecting 23, 492-512

MORRISON, H F, PHILLIPS, R J, and O'BRIEN, D P, 1969, Quantitative Interpretation of transient electromagnetic fields over a layered half space, Geophysical Prospecting 17, 82-101

VELEKIN, A B, and BULGAKOV, JU I, 1967, Transient method of electrical prospecting (one loop version) Presented to the UNO international seminar on Geophysical methods of prospecting for ore minerals Ministry of Geology of the U S S R, Moscow

VELEKIN, A B, BULGAKOV, JU I, GIGORYEU, G O, and POLIKORPOV, A M, 1971, Equipment for the method of transient processes, the MPPO-1 Apparatus Paper No 9 in a collection of papers on the Transient process method in the search for sulphide ore bodies,' "Nedra" Leningrad, pp 243.

WARD, S H, RYU, J, GLENN, W E, HOHMANN, G W, SMITH, B D, and DEY, A, 1974, Electromagnetic methods in conductive terrains, Geoexploration, 12, 121-184

WAIT, J R, 1960, Propagation of electromagnetic pulses in a homogeneous conducting earth App Sc Res B 8, 213-253

WAIT, J R, 1971, Transient excitation of the earth by a line source of current, Proc IEEE (Letters) 59, 1287-1288