A CROSS RATIO IN CONTINUOUS GEOMETRY

by

B.D.JONES, B.Sc.

submitted in fulfilment of the requirements for the degree of

MASTER OF SCIENCE

UNIVERSITY OF TASMANIA

HOBART

(25-2-61)

INTRODUCTION

A. Fuhrmann [8] generalizes Sperner's definition of the cross ratio of four collinear points with coordinates in a division ring to apply to four linear varieties over a division ring. The form of his cross ratio is still very classical. In Part 2 of this a cross ratio is defined for a configuration of subspaces of a continuous or discrete geometry. Although this cross ratio could'nt be further removed in appearance from the classical form we give a simple proof (Section 4) to show that, for the case of four points on a line , it does in fact agree with the usual cross ratio. Moreover, it will follow from the results of Sections 5,6 that for the case of finite dimensional (i.e discrete) geometries, Fuhrmann's cross ratio is essentially the same as the one introduced here. The cross ratio has the desired property of invariance under collineations (Theorem 3).

Results, similar to classical ones are obtained for the properties of the cross ratio under permutations of order in the configuration.(Theorems 4-9).

In Section 5 a representation, in terms of three "fixed" subspaces and a fourth subspace depending only on the cross ratio, is obtained for an arbitrary conf-

-iguration (satisfying a minimum of position conditions)

Section 6 is devoted to studying the invariance properties of this representation. Several complete sets of invariants for the representation , under collineatio -ns , are obtained. These involve a cross ratio and some position conditions (Lemmas 4,5). Theorem 12 goes half way towards providing a complete set of invariants in terms of cross ratios alone. However, the converse of this theorem is out of reach at present. Lemma 3 reduces the question of invariance to one of a special kind of similarity involving the cross ratio.

Part 1 contains a general discussion of continuous geometries. The exposition follows the original papers [1]-[5] of von Neumann, who invented the subject. Also, in Section 6 of Part 1 we collect some other results needed for Part 2. The references given for these are mainly to [7], since the original work [6] is unavailable.

The author wishes to acknowledge that the results contained in Part 2, Sections 1-5 were obtained in collaboration with Dr. R.J.Smith; the remainder of Part 2 is due to the author alone.

PART 1

CONTINUOUS GEOMETRIES AND REGULAR RINGS

The axioms and basic properties of the systems called continuous geometries were developed by J. von Neumann in the sequence of papers [1] - [5] . The following brief summary of these results forms a background to Part 2.

1. AXIOMS: Consider a class L of at least two different elements a, b, c ,......, and in it a relation "$\leq$", with the following properties:

A. $a \leq b$ is a partial ordering of L, relative to which L is a complemented, modular lattice with unit and zero elements, i.e

(i) For each pair a, b there are(necessarily) unique elements $a \cup b$(join), $a \cap b$(meet) satisfying

$$a \cup b \leq c \iff a \leq c \text{ and } b \leq c$$

$$c \leq a \cap b \iff c \leq a \text{ and } c \leq b \text{ .}$$

(ii) The zero(unit) element O(1) is also unique and satisfies $0 \leq a$($a \leq 1$) for all a in L.

(iii) The modular law holds in L,i.e

$$a \cap (b \cup c) = b \cup (a \cap c) \text{ whenever } b \leq a.$$

(iv) For every $\mathfrak{a}$ in L there exists at least one $\mathfrak{b}$ (called a complement of $\mathfrak{a}$) in L such that

$$\mathfrak{a} \cup \mathfrak{b} = 1 \quad \text{and} \quad \mathfrak{a} \cap \mathfrak{b} = 0 .$$

B. L is complete, relative the partial order $\leqslant$, i.e for any subset N of L there exists an element $\cup N$ ($\cap N$) of L such that : $\cup N \leqslant \mathfrak{a} \iff \mathfrak{b} \leqslant \mathfrak{a}$ for all $\mathfrak{b}$ in N ($\mathfrak{a} \leqslant \cap N \iff \mathfrak{a} \leqslant \mathfrak{b}$ for all $\mathfrak{b}$ in N).

C. The lattice L is irreducible i.e $0 = \cap L$, $1 = \cup L$ are the only elements of L with unique complements.

D. A limit notion can be introduced in L, as follows: Let Ω denote an infinite aleph and consider a sequence S of $\mathfrak{a}_\alpha$ from L, where α runs over all ordinals $< \Omega$. Define

(i) If $\alpha < \beta < \Omega$ implies $\mathfrak{a}_\alpha \leqslant \mathfrak{a}_\beta$ then $\lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha) = \cup S$

(ii) If $\alpha < \beta < \Omega$ implies $\mathfrak{a}_\beta \leqslant \mathfrak{a}_\alpha$ then $\lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha) = \cap S$

Otherwise $\lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha)$ is undefined.

E. The lattice operations $\cup$, $\cap$ are continuous in L i.e , $\alpha < \beta < \Omega \Rightarrow \mathfrak{a}_\alpha \leqslant \mathfrak{a}_\beta$ then $\lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha \cap \mathfrak{b}) = \lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha) \cap \mathfrak{b}$

and if $\alpha < \beta < \Omega \Rightarrow \mathfrak{a}_\beta \leqslant \mathfrak{a}_\alpha$ then $\lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha \cup \mathfrak{b}) = \lim^*_{\alpha \to \Omega}(\mathfrak{a}_\alpha) \cup \mathfrak{b}$

The axioms (A)-(E) are invariant under the dualization obtained by reversing the partial order, along with the resulting interchanges of $\cup$, $\cap$ and 0,1 . Hence the dual of any theorem derived from these axioms is also derivable from them.

The axioms can clearly be satisfied when L is the lattice of subspaces of a finite-dimensional projective geometry; in particular, the notion of $\lim^*$ is void.

<u>2. THE DIMENSION FUNCTION</u>: It is possible to define, in exactly one way, a function $D = D(a)$ on the elements of L such that the following conditions are satisfied:

(i) D takes real values $\geqslant 0$, $\leqslant 1$.

(ii) $D(a) = 0 \iff a = 0$, $D(1) = 1$.

(iii) $D(a \cup b) + D(a \cap b) = D(a) + D(b)$.

(iv) The range of D is one of the sets

$$\underline{D}_n = \{ 0, 1/n, 2/n, \ldots, 1 \}, \text{ some integer}$$

n or, $$\underline{D}_\infty = \{ \text{all real numbers } \geqslant 0, \leqslant 1 \}$$

If D has range $\underline{D}_n$ then $L = L_n$ is the lattice of subspaces of an irreducible projective geometry with ordinary dimension n+1. In this case L is called a discrete geometry. On the other hand, if D has range $\underline{D}_\infty$, then $L = L_\infty$ is the lattice of "subspaces" of an irreducible continuous geometry. All geometries mentioned will be irreducible.

3. IDEALS IN RINGS: We consider only rings $\mathcal{R}$ with unit 1. The results stated below for right ideals have obvious counterparts for left ideals.

A subset $\mathfrak{a}$ of $\mathcal{R}$ is a right ideal if (i) x,y in $\mathfrak{a}$ => x+y in $\mathfrak{a}$ and (ii) x in $\mathfrak{a}$ => xy in $\mathfrak{a}$, all y in $\mathcal{R}$. The principal right ideal generated by an a in $\mathcal{R}$ is the set $(a)_r = \{ ax;\ x \varepsilon \mathcal{R} \}$.

In any ring with unit the right ideals form a lattic L under the relation of set inclusion; lattice meet is set intersection, whilst lattice join is defined by $\mathfrak{a} \cup \mathfrak{b} = \{ x+y;\ x \text{ in } \mathfrak{a},\ y \text{ in } \mathfrak{b} \}$. L has as zero elemen the empty ideal $\{0\}$ and as unit element the ideal $\mathcal{R}$.

An element e in $\mathcal{R}$ is called idempotent if $e^2 = e$. If e is idempotent so also is 1-e. For idempotent e we have : x belongs to $(e)_r$ <=> ex = x.

Two principal right ideals $\mathfrak{a}$, $\mathfrak{b}$ are inverse if and only if they are complements in the lattice of right idea-ls. This being, there exist a unique idempotent e such that $\mathfrak{a} = (e)_r$, $\mathfrak{b} = (1-e)_r$.

4. REGULAR RINGS: $\mathcal{R}$ is said to be regular if it satisfies one of the following equivalent conditions: for every a in $\mathcal{R}$ there exists

(i) x in $\mathcal{R}$ such that axa = a

(ii) an idempotent e in $\mathcal{R}$ such that $(e)_r = (a)_r$

(iii) an idempotent f such that $(f)_l = (a)_l$.

(iv) a right ideal inverse to $(a)_r$.

(v) a left ideal inverse to $(a)_l$.

Every division ring is regular($x = a^{-1}$ in (i)).

Let $\mathcal{R}$ be a regular ring. If $\mathfrak{a}$ is a right ideal in $\mathcal{R}$ then the left annihilator of $\mathfrak{a}$ is defined as

$$\mathfrak{a}^l = \{ x \text{ in } \mathcal{R} ;\ xy = 0 \text{ for all } y \text{ in } \mathfrak{a} \}$$

The set $\mathfrak{a}^l$ is a left ideal, principal if $\mathfrak{a}$ is principal. In particular, if $\mathfrak{a} = (e)_r$ for idempotent e, then $\mathfrak{a}^l = (e)_r^l = (1-e)_l$.

Denote by $R_{\mathcal{R}}$ ($L_{\mathcal{R}}$) the set of all principal right (left) ideals in a regular ring $\mathcal{R}$. Then

(i) $R_{\mathcal{R}}$,$L_{\mathcal{R}}$ are complemented modular lattices.

(ii) $R_{\mathcal{R}}$ is anti-isomorphic to $L_{\mathcal{R}}$ under the one-one inverse mappings

$\mathfrak{a}$ in $R_{\mathcal{R}} \to \mathfrak{a}^l$, $\mathfrak{b}$ in $L_{\mathcal{R}} \to \mathfrak{b}^r$.

(cf. Lemma 1)

The centre $\mathfrak{Z}$ of a regular ring is a commutative regular ring. An ideal in a regular ring which is both left and right is called a two-sided ideal. A principal ideal $\mathfrak{a}$ is a two-sided ideal if and only if it is generated by an idempotent in the centre.

A reduction of a ring $\mathcal{R}$ is a decomposition of $\mathcal{R}$ into two two-sided ideal direct summands. If $\mathcal{R}$ is regular its only reductions are of the form $(e)_* \cup (1-e)_*$ where e is a central idempotent and $()_*$ denotes two-sided ideal. Since the only idempotents in a division ring are 0,1 it follows that $\mathcal{R}$ is an irreducible regular ring if and only if its centre is a field.

5. COORDINATIZATION: It is well known that any discrete (projective) geometry L_n satisfying Desargue's theorem (in particular with n>3) can be coordinatized, using homogeneous coordinates for the points of the geometry. v.Neumann expressed this classical result in the followi -ng way: Let Ψ denote a division ring and Ψ_n the ring of all n×n matrices over Ψ ; then for each (n-1)-dimension al projective geometry L_n(satisfying Desargue's theorem) there exists a suitable division ring Ψ such that the class of all linear subspaces of L_n, partially ordered by inclusion, can be put in lattice isomorphism with the class of all right ideals of Ψ_n. Generalizing this result von Neumann showed that for any complemented modular lattice L possessing a homogeneous basis of order n(cf. 6 for the definition of this) a regular ring $\mathcal{R}$ can b -e found such that L is isomorphic to the lattice of

principal right ideals of $\mathcal{R}$. Moreover, for $n \geq 3$ this ring $\mathcal{R}$ is uniquely determined by L, to within ring isomorphism.

Thus, the lattice of subspaces of every discrete or continuous geometry is isomorphic to the $R_{\mathcal{R}}$ of an irreducible regular ring $\mathcal{R}$, unique to within isomorphism(except for the exceptions mentioned).

If L is a discrete geometry L_n then $\mathcal{R}$ is the complet matrix ring Φ_n over a division ring Φ , whilst if L is a continuous geometry L_∞ , then the corresponding irreducible regular ring is called a continuous ring.

An algebraic characterization of the continuous ring may be obtained as follows: For any geometry L the numbers

$$D((a)_r) \; ; \; D'((a)_l) \; , \; 1-D((a)_r^l) \; , \; 1-D'((a)_l^r)$$

are equal, for any a in the ring of the geometry, where D(D') is the dimension function defined on $R_{\mathcal{R}}$ ($L_{\mathcal{R}}$). Their common value is called the rank, R(a) , of a. The function R(a) has the following properties

(i) $0 \leq R(a) \leq 1$ for all a in $\mathcal{R}$.

(ii) $R(a) = 0 \iff a = 0$.

(iii) $R(a) = 1 \iff a^{-1}$ exists in $\mathcal{R}$.

(iv) $R(a) = R(b) \iff a = ubv$, where u^{-1}, v^{-1} exist.

(v) $R(ab) \leqslant \min\{ R(a), R(b) \}$

(vi) $R(a+b) \leqslant R(a) + R(b)$.

(vii) For $e^2 = e$, $f^2 = f$, $ef = fe = 0$ we have

$$R(e+f) = R(e) + R(f) .$$

The function $R(a-b)$ serves as a metric in $\mathcal{R}$ (the rank distance). If $\mathcal{R}$ is the ring of a discrete or continuous geometry , it is complete in the topology of the rank distance i.e, if $a_1, a_2, a_3, \ldots$ are in $\mathcal{R}$ then the existence of a in $\mathcal{R}$ with

$$\lim_{\pi \to \infty} R(a_\pi - a) = 0$$

is equivalent to

$$\lim_{\pi, \omega \to \infty} R(a_\pi - a_\omega) = 0$$

If, conversely, a rank function with the properties (i)-(vii) is defined on the elements of an irreducible regular ring $\mathcal{R}$, then $\mathcal{R}$ will be the ring of a discrete or continuous geometry if and only if it is complete in the topology of the rank distance. It will be a discrete ring(i.e matrix ring) if the range of the rank function is the set $\{0, 1/n, 2/n, \ldots, 1\}$ for some integer n, and a continuous ring if the range of the rank function is the set of all real numbers $\geqslant 0, \leqslant 1$. In either case $\mathcal{R}$

is called a complete rank ring. In this way a one-one correspondence between continuous geometries and continuous rings is obtained.

6. It is convenient to list below some other definitions , theorems and lemmas which will be drawn upon in Part 2. We give these results in the setting of a system L satisfying axioms (A)-(E), and the associated complete rank ring $\mathcal{R}$ (i.e L is $R_{\mathcal{R}}$), although many of them remain valid in a more general setting.

The operations $^{\mathfrak{l},r}$ introduced in 4 have the additional properties given by

<u>LEMMA 1:</u> For each $\mathfrak{a}, \mathfrak{b}$ in L

(i) $\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a}^{\mathfrak{l}} \geq \mathfrak{b}^{\mathfrak{l}}$

(ii) $\mathfrak{a} \leq \mathfrak{a}^{\mathfrak{l}r}$

(iii) $\mathfrak{a}^{\mathfrak{l}} = \mathfrak{a}^{\mathfrak{l}r\mathfrak{l}}$

(iv) $(\mathfrak{a} \cup \mathfrak{b})^{\mathfrak{l}} = \mathfrak{a}^{\mathfrak{l}} \cap \mathfrak{b}^{\mathfrak{l}}$.

A proof is given in [7; Hilfssatz 1.3, ch. VI]

<u>LEMMA 2:</u> If e,f are idempotent elements of $\mathcal{R}$,then

(i) $(e)_r \cup (f)_r = (e+g)_r$, where g is any idempotent such that $(g)_r = ((1-e)f)_r$

(ii) $(e)_r \cap (f)_r = (f-fg)_r$, where g is any idempotent such that $(g)_{\mathfrak{l}} = (f-ef)_{\mathfrak{l}}$.

[7; Hilfssatz 3.2,3.3, ch VI]

<u>LEMMA 3</u>: If e,f are idempotents elements of $\mathcal{R}$,then

(i) $(e)_r = (f)_r \iff e = f+fy(1-f)$, some y in $\mathcal{R}$

(ii) $(e)_\ell = (f)_\ell \iff e = f+(1-f)zf$, some z in $\mathcal{R}$

[7; Hilfssatz 1.5, ch VI]

Call an element a of $\mathcal{R}$ non-singular if a^{-1} exists in $\mathcal{R}$. Then

<u>LEMMA 4</u>: If e,f are idempotents of the same rank in $\mathcal{R}$ then $e = sfs^{-1}$, for some non-singular s in $\mathcal{R}$.

[9; Lemma 9, p 400]

<u>LEMMA 5</u>: $(a)_r = (b)_r \iff a = bv$ for some non-singular v in $\mathcal{R}$.

[7; Hilfssatz 1.6, ch VII]

<u>LEMMA 6:</u> (i) If e is a non-zero idempotent in $\mathcal{R}$, then $e\mathcal{R}e$ ($= \{exe; x \text{ in } \mathcal{R}\}$) is the complete regular rank ring corresponding to $L((e)_r)$ (= the sublattice of L of all $\mathfrak{a}$ such that $\mathfrak{a} \leq (e)_r$) .

(ii) If $\mathfrak{Z}$ is the centre of $\mathcal{R}$ then $e\mathfrak{Z}e$ is the centre of $e\mathcal{R}e$.

[9; Lemma 10, p 400]

<u>RELATIVE</u> <u>COMPLEMENTS:</u> If $\mathfrak{a} \leqslant \mathfrak{b}$ in L then a relative complement of $\mathfrak{a}$ in $\mathfrak{b}$ is an element $\mathfrak{c}$ (not necessarily unique) of L such that $\mathfrak{a} \cap \mathfrak{c} = 0$ and $\mathfrak{a} \cup \mathfrak{c} = \mathfrak{b}$. Since L is complemented and modular it is also relatively complemented, i.e whenever $\mathfrak{a} \leqslant \mathfrak{b}$ there exists at least one relative complement of $\mathfrak{a}$ in $\mathfrak{b}$ (for, if $\mathfrak{c}$ is a complement of $\mathfrak{a}$ then $\mathfrak{b} \cap \mathfrak{c}$ is a relative complement of $\mathfrak{a}$ in $\mathfrak{b}$).

<u>LEMMA 7:</u> If a,e in $\mathfrak{R}$ are such that $e^2=e$ and $(e)_r \leqslant (a)_r$, then $(a-ea)_r$ is a relative complement of $(e)_r$ in $(a)_r$.

[11; 3.2(iii)]

<u>INDEPENDENCE</u>: A finite set $\{\mathfrak{a}_i ; i = 1,\ldots,k\}$ of elements of L is called independent if

(1) $(\mathfrak{a}_1 \cup \ldots\ldots \cup \mathfrak{a}_i) \cap \mathfrak{a}_{i+1} = 0$, all $i = 1,\ldots,k-1$.

Equation (1) holds if and only if for every two disjoint subsets I_1, I_2 of the index set $\{1,\ldots,k\}$ we have

(2) $$\Big(\bigcup_{i\varepsilon I_1} \mathfrak{a}_i\Big) \cap \Big(\bigcup_{i\varepsilon I_2} \mathfrak{a}_i\Big) = 0$$

[7; Definition 1.15, Satz 1.8, ch I]

For independent $\mathfrak{a}_i (i=1,\ldots,k)$ write $\mathfrak{a}_1 \oplus \ldots\ldots \oplus \mathfrak{a}_k$ in place of $\mathfrak{a}_1 \cup \ldots\ldots \cup \mathfrak{a}_k$.

A set $\{e_i;\ i = 1,\ldots,k\}$ of idempotents is called independent if

$$e_i e_j = \begin{cases} e_i & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad i,j=1,\ldots,k$$

<u>THEOREM 1:</u> If $e_1,\ldots,e_k$ are independent idempotents from $\mathcal{R}$ then the $(e_i)_r$ are independent subspaces of L and

$$(e_{i_1})_r \oplus \ldots\ldots \oplus (e_{i_\pi})_r = (e_{i_1} + \ldots\ldots + e_{i_\pi})_r$$

for every set of distinct integers $\{i_1,\ldots,i_\pi\}$ on the range $\{1,\ldots,k\}$. (the $e_{i_1}+\ldots+e_{i_\pi}$ are idempotent)

<u>THEOREM 2:</u> $\mathfrak{a}_1,\ldots,\mathfrak{a}_k$ are independent if and only if there exist independent idempotents $e_1,\ldots,e_k$ such that

$$\mathfrak{a}_i = (e_i)_r\ ,\ i = 1,\ldots,k.$$

[7; Satz 1.3,Satz1.4, ch VI]

<u>HOMOGENEOUS</u> <u>BASIS</u>: If two elements $\mathfrak{a}, \mathfrak{b}$ of L possess a common complement $\mathfrak{c}$ (i.e $\mathfrak{a} \oplus \mathfrak{c} = \mathfrak{b} \oplus \mathfrak{c}$) then we say $\mathfrak{a}$ is perspective to $\mathfrak{b}$ (written: $\mathfrak{a} \sim \mathfrak{b}$) with axis $\mathfrak{c}$. Notice that the conditions (i) $\mathfrak{a} \oplus \mathfrak{c} = \mathfrak{b} \oplus \mathfrak{c}$ some $\mathfrak{c}$ and (ii) $\mathfrak{a} \oplus \mathfrak{c} = \mathfrak{b} \oplus \mathfrak{c} = (1)_r$ some $\mathfrak{c}$, are equivalent.

[1; par 6.2]

n independent subspaces $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ are said to form a basis for L if $\mathfrak{a}_1 \oplus \ldots. \oplus \mathfrak{a}_n = 1$. If also

$$\mathfrak{a}_i \sim \mathfrak{a}_j \text{ , each } i,j = 1,..,n$$

then the basis is called homogeneous. n is called the order of the basis.

We also require

<u>LEMMA</u> 8: $\mathfrak{a} \sim \mathfrak{b} \Leftrightarrow D(\mathfrak{a}) = D(\mathfrak{b})$

(in fact this is how v Neumann defines D; see [1])

<u>EXISTENCE</u> <u>OF</u> <u>MATRIX</u> <u>UNITS</u>: If $\mathfrak{a} \cap \mathfrak{b} = 0$ for some $\mathfrak{a}$, $\mathfrak{b}$ in L then we denote by $L_{\mathfrak{b}\mathfrak{a}}$ the set of all relative complements of $\mathfrak{a}$ in $\mathfrak{a} \oplus \mathfrak{b}$,i.e

$$\tau \in L_{\mathfrak{b}\mathfrak{a}} \Leftrightarrow \mathfrak{a} \oplus \tau = \mathfrak{a} \oplus \mathfrak{b}$$

<u>THEOREM</u> 3: Let e_1, e_2 be independent idempotents such that $\mathfrak{a} = (e_1)_r$, $\mathfrak{b} = (e_2)_r$. Then τ is in $L_{\mathfrak{b}\mathfrak{a}}$ if and only if there exists a unique e_{12} in $\mathfrak{R}$ such that

$$e_1 e_{12} e_2 = e_{12} \quad \text{and} \quad \tau = (e_2 - e_{12})_r$$

<u>THEOREM</u> 4: With the same notation as in Theorem 3, τ is in both $L_{\mathfrak{b}\mathfrak{a}}$ and $L_{\mathfrak{a}\mathfrak{b}}$ if and only if there exist unique elements e_{21}, e_{12} in $\mathfrak{R}$ such that

$$e_1 e_{12} e_2 = e_{12},\ e_2 e_{21} e_1 = e_{21},\ e_{21} e_{12} = e_2,\ e_{12} e_{21} = e_1,$$
$$\text{and } \tau = (e_1 - e_{21})_r = (e_2 - e_{12})_r$$

[7; Hilfssatz 4.1, Satz 4.2, ch VI]

The left multiplications in $\mathfrak{R}$ carry right ideals into right ideals and so induce mappings in L. Denote by γ_t the mapping $(x)_r \to (tx)_r$, all x and some fixed t in $\mathfrak{R}$. Then

<u>LEMMA</u> 9: (i) γ_t is an order-preserving endomorphism of L, i.e

$$\mathfrak{a} \leqslant \mathfrak{b} \Rightarrow \gamma_t \mathfrak{a} \leqslant \gamma_t \mathfrak{b}$$

(ii) If t is non-singular then $(\gamma_t)^{-1}$ exists and equals $\gamma_{t^{-1}}$ and γ_t is a lattice automorphism, i.e $\quad \mathfrak{a} \leqslant \mathfrak{b} \Leftrightarrow \gamma_t \mathfrak{a} \leqslant \gamma_t \mathfrak{b}$

[9; Lemma 3, p 399]

It follows easily that the γ_t with non-singular t have the additional properties

$$\gamma_t(\mathfrak{a} \cup \mathfrak{b}) = \gamma_t \mathfrak{a} \cup \gamma_t \mathfrak{b}$$
$$\gamma_t(\mathfrak{a} \cap \mathfrak{b}) = \gamma_t \mathfrak{a} \cap \gamma_t \mathfrak{b}$$
$$D(\gamma_t \mathfrak{a}) = D(\mathfrak{a})$$

A system of n^2 elements $e_{ij}(i,j = 1,\dots,n)$ of $\mathcal{R}$ is called a system of matrix units if

$$\text{(i)} \quad e_{ij}e_{kh} = \begin{cases} 0 \text{ if } j \neq k \\ e_{ih} \text{ if } j = k \end{cases}$$

$$\text{(ii)} \quad \sum_i e_{ii} = 1$$

If the e_{ij} are a system of matrix units then the e_{ii} are independent idempotents.

<u>LEMMA 10</u>: $\{\mathfrak{a}_i \,;\, i = 1,\dots,n\}$ is a homogeneous basis for L if and only if there exist n^2 matrix units e_{ij} in $\mathcal{R}$ such that $\mathfrak{a}_i = (e_{ii})_r$.
[7; Satz 1.1,1.2,ch IX]

<u>LEMMA 11</u>: Γ_n ,the n×n matrix ring over Γ , is a regular ring if and only if Γ is regular.
[7; Satz 2.1, ch IX]

<u>THEOREM 5</u>: L has order n if and only if there exists a regular ring Γ_n such that $\mathcal{R}$ is isomorphic to Γ_n

Lemma 10 provides a system of matrix units e_{ij}; taking $\Gamma = e_1\mathcal{R}e_1$, $e_1 = e_{11}$, we have an isomorphism as follows: $x\varepsilon\ \mathcal{R}$ and $(x^{ij})\varepsilon\ \Gamma_n$ correspond if and only if

$$x = \sum_{i,j} e_{i1}x^{ij}e_{1j}$$

and

$$x^{ij} = e_{1i} x e_{j1} \quad , \quad i,j = 1,\ldots,n$$

[7; Hilfssatz 2.2,ch IX]

<u>THE CORRESPONDENCE BETWEEN MODULES AND IDEALS:</u> Let Γ be a regular ring and denote by V_Γ^n the n-dimensional vector space over Γ . A subset M of V_Γ^n is called a Γ right submodule if (i) $v_1, v_2 \varepsilon M \Rightarrow v_1+v_2 \varepsilon M$ and (ii) $v \varepsilon M \Rightarrow v\gamma \varepsilon M$ all $\gamma \varepsilon \Gamma$. For every right submodule M set

$$\mathfrak{r}(M) = \{(\xi_{ij}); \quad (\xi_{1j},\ldots,\xi_{nj}) \varepsilon M, \ j = 1,\ldots,n\}.$$

Then $\mathfrak{r}(M)$ is a right ideal in Γ_n . Conversely, given a right ideal $\mathfrak{r}$ in Γ_n let

$$M(\mathfrak{r}) = \{ (\xi_1,\ldots,\xi_n); \begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix} = A \begin{pmatrix}1\\ 0\\ \vdots \\ 0\end{pmatrix}, \text{some } A \ \varepsilon \ \mathfrak{r} \}$$

Then $M(\mathfrak{r})$ is a right submodule of V_Γ^n and

$$\mathfrak{r}(M(\mathfrak{r})) = \mathfrak{r} \ , \quad M(\mathfrak{r}(M)) = M.$$

THEOREM 6 : (i) The correspondence $M \leftrightarrow \mathfrak{r}(M)$ is a lattice isomorphism between the lattice of all Γ- right submodules of V_Γ^n and the lattice of right ideals of Γ_n .

(ii) Under this isomorphism the right submodules which correspond to principal ideals are exactly those which are finitely generated.

[7; pp 186–188]

PART 2

A CROSS RATIO IN CONTINUOUS GEOMETRY

In all of what follows, $\mathcal{R}$ will always denote, without explicit mention to the contrary, a complete rank ring and L its lattice of principal right ideals.

1. Firstly, we prove two theorems which will be useful later on.

THEOREM 1: For any a,b in $\mathcal{R}$, with $(h)_l = (a)_r^l$,

$$\text{(i)} \quad D((a)_r \cup (b)_r) = D((hb)_r) + D((a)_r) \quad ,$$

and

$$\text{(ii)} \quad D((a)_r \cap (b)_r) = D((b)_r) - D((hb)_r) \quad .$$

PROOF: Let $\mathfrak{m}_1 = (a)_r$, $\mathfrak{m}_2 = (b)_r$, $\mathfrak{r} = \mathfrak{m}_1 \cap \mathfrak{m}_2$ and let $\mathfrak{n}$ be a relative complement of $\mathfrak{r}$ in $\mathfrak{m}_2$, i.e $\mathfrak{r} \oplus \mathfrak{n} = \mathfrak{m}_2$. Then

$$\mathfrak{m}_1 \cup \mathfrak{m}_2 = \mathfrak{m}_1 \cup \mathfrak{r} \cup \mathfrak{n}$$

$$= \mathfrak{m}_1 \cup \mathfrak{n} \qquad \text{, as } \mathfrak{r} \leq \mathfrak{m}_1$$

Also, $\mathfrak{m}_1 \cap \mathfrak{n} = (0)_r$, as

$$\begin{aligned} D(\mathfrak{m}_1 \cap \mathfrak{n}) &= D(\mathfrak{m}_1) + D(\mathfrak{n}) - D(\mathfrak{m}_1 \cup \mathfrak{n}) \\ &= -D(\mathfrak{m}_1 \cup \mathfrak{m}_2) + D(\mathfrak{m}_1) - D(\mathfrak{m}_1 \cap \mathfrak{m}_2) + D(\mathfrak{m}_2) \\ &= 0 \end{aligned}$$

so that

$$(1)_{\mathfrak{l}} = (0)_{r}^{\mathfrak{l}} = (\mathfrak{m}_1 \cap \mathfrak{n})^{\mathfrak{l}}$$
$$= (\mathfrak{m}_1^{\mathfrak{l}r} \cap \mathfrak{n}^{\mathfrak{l}r})^{\mathfrak{l}}$$
$$= (\mathfrak{m}_1^{\mathfrak{l}} \cup \mathfrak{n}^{\mathfrak{l}})^{r\mathfrak{l}}$$
$$= \mathfrak{m}_1^{\mathfrak{l}} \cup \mathfrak{n}^{\mathfrak{l}} \quad ,$$

[Part 1 ; Lemma 1 and its dual]

In other words,

$$(h)_{\mathfrak{l}} \cup \mathfrak{n}^{\mathfrak{l}} = (1)_{\mathfrak{l}}$$

and for some y in $\mathfrak{R}$, p in $\mathfrak{n}^{\mathfrak{l}}$ we have

$$yh + p = 1 \quad .$$

From $\mathfrak{m}_2 = \mathfrak{r} \oplus \mathfrak{n}$ it follows, by [Part 1 ; Theorem 2] , that there exist independent idempotents e_1, e_2 such that

$$\mathfrak{r} = (e_1)_r \,, \quad \mathfrak{n} = (e_2)_r \,, \quad \mathfrak{m}_2 = (e_1 + e_2)_r$$

and hence that

$$b = (e_1 + e_2)b = e_1 b + e_2 b \;.$$

Then, since $e_1 b \in \mathfrak{r} \leqslant \mathfrak{m}_1$,

$$hb = he_2 b = hn$$

where $n = e_2 b$ and clearly $R(n) = D(\mathfrak{n})$.

Thus

$$R(hb) = R(hn) \geqslant R(yhn)$$
$$= R((yh + p)n)$$
$$= R(n)$$

And, as obviously $R(n) \geqslant R(hn)$, equality must hold, i.e

$$R(hb) = R(n) = D(\mathfrak{n}).$$

In addition we have

$$D(\mathfrak{n}) = D(\mathfrak{m}_2) - D(\mathfrak{m}_1 \cap \mathfrak{m}_2)$$
$$= D(\mathfrak{m}_2 \cup \mathfrak{m}_1) - D(\mathfrak{m}_1)$$

and therefore

$$R(hb) = D((hb)_r) = D(\mathfrak{m}_1 \cup \mathfrak{m}_2) - D(\mathfrak{m}_1).$$

This proves (i); (ii) follows immediately.

Now let $a = cf$, $b = c(1-f)$ for some c,f in $\mathfrak{R}$.

Then $\mathfrak{m}_1 \cup \mathfrak{m}_2 = (cf)_r \cup (c(1-f))_r$ contains $cf + c(1-f) = c$.

But cf, $c(1-f)$ belong to $(c)_r$, which therefore contains $\mathfrak{m}_1 \cup \mathfrak{m}_2$, and so equality holds i.e in this case we have

$$D(\mathfrak{m}_1 \cup \mathfrak{m}_2) = D((c)_r)$$

and we have proved

<u>COROLLARY 1:</u> $R(c) = R(hc(1-f)) + R(cf)$, for any c,f in $\mathfrak{R}$ with $(h)_{\mathfrak{r}} = (cf)_r^{\mathfrak{r}}$.

Furthermore, let $c = df + e(1-f)$; then $cf = df$, $C(1-f) = d(1-f)$ and Corollary 1 gives

<u>COROLLARY 2:</u> For any d,e,f in $\mathcal{R}$ with $(h)_{\mathfrak{r}} = (df)_r^{\mathfrak{r}}$,

$$R(df + e(1-f)) = R(df) + R(he(1-f)) .$$

<u>THEOREM 2:</u> Let a,f in $\mathcal{R}$ be such that $f^2 = f$, $(h)_{\mathfrak{r}} = (a(1-f))_r^{\mathfrak{r}}$. Then

$$R(haf) = R(h) = R(f)$$

if and only if

$$(a(1-f))_r \oplus (af)_r = (1)_r .$$

<u>PROOF:</u> Necessity: $R(haf) = R(h) = R(f) \Rightarrow$ a is non-singular by [10] . Then $(a(1-f))_r \cup (af)_r$ contains $a(1-f) + af = a$, i.e

$$(a(1-f))_r \cup (af)_r \geq (a)_r = (1)_r$$

and equality must hold. Also

$$(a(1-f))_r \cap (af)_r = (0)_r$$

for, if x belongs to the left-hand side, then for some y,z in $\mathcal{R}$, $x = a(1-f)y = afz$; thus $a^{-1}x = (1-f)y = = fz$, and left multiplying by f gives $fz = a^{-1}x = 0$.

Sufficiency: Suppose $(a(1-f))_r \oplus (af)_r = (1)_r$. Then a is non-singular, since for some y,z in $\mathcal{R}$

$$a\{(1-f)y+fz\} = 1 \ .$$

Also,

$$h = ha(1-f)y + hafz = hafz$$

so that

$$R(h) \leqslant R(haf) \leqslant R(h)$$

i.e

$$R(h) = R(haf) \leqslant R(f) \ .$$

But,

$$R(h) = 1 - R(a(1-f)) \geqslant 1 - R(1-f) = R(f) \ .$$

This completes the proof.

<u>COROLLARY 3:</u> For any b,c,f in $\mathcal{R}$ with $f^2 = f$,

$$(h)_\ell = (b(1-f))_r^\ell \ ,$$

$$R(hcf) = R(h) = R(f)$$

if and only if

$$(b(1-f))_r \oplus (cf)_r = (1)_r$$

<u>PROOF</u> : Let $a = b(1-f)+cf$ in Theorem 2 .

2. THE CROSS RATIO

By a configuration we mean, as always, an ordered quadruple. Let $P_n(K)$ be an n-dimensional projective space over a field K. Suppose that two hyperplanes ω_1, ω_2 are defined by the $(n+1)^{th}$-order row vectors r_1, r_2, and the two points π_1, π_2 by means of the $(n+1)^{th}$-order column vectors r_3, r_4. Then the cross ratio of the configuration $\{\omega_1, \omega_2, \pi_1, \pi_2\}$ can be defined uniquely as

$$\mathbb{R}(\omega_1, \omega_2; \pi_1, \pi_2) = \frac{r_1 \cdot r_3 \; r_2 \cdot r_3}{r_1 \cdot r_4 \; r_2 \cdot r_4}$$

For K a division ring, Sperner defines the cross ratio as

$$\mathbb{R} = (r_1 \cdot r_4)^{-1}(r_1 \cdot r_3)(r_2 \cdot r_4)^{-1}(r_2 \cdot r_3)$$

and shows that it is uniquely determined up to inner automorphisms of K.

If $\{V_1, V_2, V_3, V_4\}$ is any configuration of subspaces of $P_n(K)$ (K a division ring), where V_i is spanned by the columns of a matrix M_i and the null-space of V_i by the rows of a matrix $^{\bullet}M_i$, Fuhrmann [8] defines a formal matrix

$$A = (^{\bullet}M_1 . M_4)^{-1}(^{\bullet}M_1 . M_3)(^{\bullet}M_2 . M_4)^{-1}(^{\bullet}M_2 . M_3)$$

and shows that an N.S.C for A to exist is

$$V_1 \oplus V_4 = V_2 \oplus V_3 = V^{(n)}$$

the space spanned by the columns of the $(n+1)^{th}$-order unit matrix. This being so , the class of all matrices similar to A is invariant under non-singular collineations, and is by definition the cross ratio $\text{℞}(V_1,V_2;V_3,V_4)$ of the configuration.

For a configuration $\{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}\}$ from L the conditions

$$(1) \qquad \mathfrak{a} \oplus \mathfrak{b} = (1)_r = \mathfrak{c} \oplus \mathfrak{d}$$

hold if and only if there are unique idempotents e,f with

$$(2) \quad \mathfrak{a} = (e)_r \, , \quad \mathfrak{b} = (1-e)_r \, , \quad \mathfrak{c} = (f)_r \, , \quad \mathfrak{d} = (1-f)_r$$

For any a in $\mathcal{R}$, <a> will denote the set of all sas^{-1} , for non-singular s in $\mathcal{R}$. We make the

<u>DEFINITION 1:</u> The cross ratio of a configuration $\{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}\}$ satisfying (1) is

$$\text{℞}(\mathfrak{a},\mathfrak{b};\mathfrak{c},\mathfrak{d}) = \langle efe \rangle$$

where e,f are given by (2).

<u>THEOREM 3:</u> The cross ratio is invariant under all non-singular collineations γ_t . [Part 1 ; p 15]

<u>PROOF :</u> Since γ_t is a lattice automorphism , it follows from (1) that

$$\gamma_t \mathscr{u} \oplus \gamma_t \mathscr{f} = (1)_r = \gamma_t \tau \oplus \gamma_t \mathscr{v}$$

and since $\gamma_t \mathscr{u} = (te)_r = (tet^{-1})_r$,...., it follows that tet^{-1}, tft^{-1} are the unique idempotents in the sense of (2) . Hence $\mathrm{R}_x(\gamma_t \mathscr{u}, \gamma_t \mathscr{f} ; \gamma_t \tau , \gamma_t \mathscr{v})$ is defined and equals

$$\langle tet^{-1} tft^{-1} tet^{-1} \rangle = \langle tefet^{-1} \rangle = \langle efe \rangle .$$

3. PERMUTATION PROPERTIES OF THE CROSS RATIO

All told there are 24 possible cross ratios, corresponding to permutations of order in the configuration . Of course, different conditions (1) are required to define them ,and therefore it is reasonable to expect some cross ratios to be equal.

<u>THEOREM 4:</u> $\mathrm{R}_x(\mathscr{u}, \mathscr{f} ; \mathscr{v} , \tau) = \langle e-efe \rangle$

<u>PROOF</u>: Obvious.

<u>THEOREM 5:</u> $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{c} \oplus \mathfrak{d} = \mathfrak{a} \oplus \mathfrak{d} = (1)_r$ imply

$$\mathbf{R}(\mathfrak{c}, \mathfrak{d}; \mathfrak{a}, \mathfrak{b}) = \mathbf{R}(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d})$$

<u>PROOF</u>: There exists a unique idempotent g such that $\mathfrak{a} = (e)_r = (g)_r$ and $\mathfrak{d} = (1-f)_r = (1-g)_r$ (i.e, $(f)_\ell = (g)_\ell$) Hence, [Part 1 ; Lemma 3], for some x, y in $\mathcal{R}$

$$g = e + ex(1-e) \ , \ f = g + (1-g)yg$$

and as $fg = f$, $eg = g$ we have

$$fef = fefg = fefeg = (g + (1-g)yg)efeg$$

Let $s_1 = (1-g)yg$, $s_2 = ex(1-e)$. Then $s_i^2 = 0$, $(1+s_i)^{-1} = 1-s_i$ $(i = 1,2)$, $gs_1 = s_2e = 0$, $s_1g = s_1$, $es_2 = s_2$ and

$$
\begin{aligned}
(1+s_2)&(1-s_1)fef(1+s_1)(1-s_2) \\
&= (1+s_2)(1-s_1)(g + (1-g)yg)efeg(1+s_1)(1-s_2) \\
&= (1+s_2)efeg(1-s_2) \ , \text{ as } ge = e \\
&= (1+s_2)efe(1+s_2)(1-s_2) \\
&= efe
\end{aligned}
$$

i.e, we have shown $\langle fef \rangle = \langle efe \rangle$, which is the desired result.

THEOREM 6: $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{c} \oplus \mathfrak{d} = \mathfrak{b} \oplus \mathfrak{c} = (1)_r$ imply

$$\mathbf{R}(\mathfrak{c}, \mathfrak{d}; \mathfrak{a}, \mathfrak{b}) = \mathbf{R}(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d})$$

PROOF: Interchange $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{c}, \mathfrak{d}$ respectively, in Theorem 5 .

THEOREM 7: $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{c} \oplus \mathfrak{d} = \mathfrak{b} \oplus \mathfrak{c} = \mathfrak{a} \oplus \mathfrak{d} = (1)_r$ imply $R_x(\mathfrak{a}, \mathfrak{d}; \mathfrak{c}, \mathfrak{b}) = \langle \overline{efe} \rangle$. (where $\overline{efe}$ denotes the inverse of efe in $e\mathcal{R}e$, the subring with unit e.)

PROOF: There exist unique idempotents g,h such that $\mathfrak{a} = (e)_r = (g)_r$, $\mathfrak{b} = (1-e)_r = (1-h)_r$, $\mathfrak{c} = (f)_r = (h)_r$, $\mathfrak{d} = (1-f)_r = (1-g)_r$.

Since $R_x(\mathfrak{a}, \mathfrak{d}; \mathfrak{c}, \mathfrak{b}) = \langle ghg \rangle$ we are required to show that $\langle ghg \rangle = \langle \overline{efe} \rangle$. As $ge = e$, $he = h$, $hf = f$, $gf = g$ we have

$$eghgefe = ghefe = ghfe = gfe = ge = e ,$$

so that $\overline{efe}$ exists and equals $eghge$. Then, as $eg = g$ and $g = e + ex(1-e)$ for some x in $\mathcal{R}$,

$$ghg = eghgeg = eghge(e + ex(1-e)) .$$

Let $s = ex(1-e)$. Then $se = 0$, $es = s$ so that

$$(1+s)ghg(1-s) = (1+s)eghge(e + ex(1-e))(1 - ex(1-e))$$
$$= eghge .$$

Hence $\langle ghg \rangle = \langle eghge \rangle = \langle \overline{efe} \rangle$.

THEOREM 8: $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{c} \oplus \mathfrak{d} = \mathfrak{a} \oplus \mathfrak{c} = (1)_r$ imply

(i) $R_x(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d}) = R_x(\mathfrak{d}, \mathfrak{c}; \mathfrak{b}, \mathfrak{a})$ and

(ii) $R_x(\mathfrak{c}, \mathfrak{d}; \mathfrak{a}, \mathfrak{b}) = R_x(\mathfrak{b}, \mathfrak{a}; \mathfrak{d}, \mathfrak{c})$

PROOF: There exists a unique idempotent g such that $\mathfrak{m} = (e)_r = (g)_r$, $\mathfrak{c} = (f)_r = (1-g)_r$(or, $(1-f)_{\mathfrak{l}} = (g)_{\mathfrak{l}}$)

ad (i): For some x,y in $\mathcal{R}$ we have

$$g = e + ex(1-e) \ , \ 1-f = g + (1-g)yg \ .$$

Then, since $(1-f)g = 1-f$, $eg = g$,

$$(1-f)e(1-f) = (1-f)e(1-f)g = (1-f)e(1-f)eg \ .$$

Let $s_1 = (1-g)yg$, $s_2 = ex(1-e)$ .Then, since $ge = e$,

$(1+s_2)(1-s_1)(1-f)e(1-f)eg(1+s_1)(1-s_2)$

$= (1+s_2)(1-s_1)(g+(1-g)yg)e(1-f)eg(1+s_1)(1-s_2)$

$= (1+s_2)ge(1-f)eg(1-s_2)$

$= (1+ex(1-e))e(1-f)e(e+ex(1-e))(1-ex(1-e))$

$= e(1-f)e$,

so that $\langle(1-f)e(1-f)\rangle = \langle e(1-f)e\rangle$. Also,

$(1+s_2)(1-s_1)(1-f)(1+s_1)(1-s_2)$

$= (1+s_2)(1-(1-g)yg)(g+(1-g)yg)(1+(1-g)yg)(1-s_2)$

$= (1+s_2)g(1-s_2)$

$=(1+s_2)e(1+s_2)(1-s_2)$

$= e$,

so that $\langle 1-f\rangle = \langle e\rangle$. Finally, since

$(1-f)(1-e)(1-f) = 1-f-(1-f)e(1-f)$, $efe = e-e(1-f)e$,

we must have $\langle(1-f)(1-e)(1-f)\rangle = \langle efe\rangle$, which is (i) .

ad (ii): $(f)_r = (1-g)_r$ and $(1-e)_{\mathfrak{l}} = (1-g)_{\mathfrak{l}}$ mean ,for some y,z in $\mathcal{R}$,

$1-g = f + fy(1-f)$ and $1-e = 1-g + gz(1-g)$. Let $s_1 = gz(1-g)$, $s_2 = fy(1-f)$. Then, as $(1-e)(1-g) = 1-e$, $f(1-g) = 1-g$,

$$(1+s_2)(1-s_1)(1-e)f(1-e)(1+s_1)(1-s_2)$$
$$= (1+s_2)(1-s_1)(1-e)f(1-e)f(1-g)(1+s_1)(1-s_2)$$
$$= (1+s_2)(1-g)f(1-e)f(1-g)(1-s_2)$$
$$= f(1-e)f,$$

so that $\langle(1-e)f(1-e)\rangle = \langle f(1-e)f\rangle$; similarly, using the same non-singular $t = (1+s_2)(1-s_1)$ we find $t(1-e)t^{-1} = f$, i.e, $\langle 1-e\rangle = \langle f\rangle$. Finally we have

$$\langle(1-e)(1-f)(1-e)\rangle = \langle fef\rangle,$$

which is (ii).

<u>THEOREM 9:</u> $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{c} \oplus \mathfrak{d} = \mathfrak{b} \oplus \mathfrak{d} = (1)_r$ imply

(i) $R_x(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d}) = R_x(\mathfrak{d}, \mathfrak{c}; \mathfrak{b}, \mathfrak{a})$ and

(ii) $R_x(\mathfrak{c}, \mathfrak{d}; \mathfrak{a}, \mathfrak{b}) = R_x(\mathfrak{b}, \mathfrak{a}; \mathfrak{d}, \mathfrak{c})$.

<u>PROOF:</u> Interchange $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{c}, \mathfrak{d}$ respectively in Theorem 8.

From Theorems 4-9 we can deduce the situation illustrated in Table 1, where we have made the identifications : $\mathscr{a} = 1, \mathscr{b} = 2, \mathscr{c} = 3, \mathscr{d} = 4$. Relations in the margin are necessary and sufficient for the existence of the cross ratios in the corresponding rows. Either condition at the end of a double arrow is sufficient for the equality of the cross ratios connected. In general the cross-ratios in any one column are distinct, whilst equality in one row indicates the corresponding equality in the other rows. When $\mathscr{a} \oplus \mathscr{b} = \mathscr{c} \oplus \mathscr{a} = \mathscr{a} \oplus \mathscr{d} = \mathscr{c} \oplus \mathscr{d} = (1)_r$, it is clear that, in general , there are exactly six distinct cross ratios(viz., those in the same column).

TABLE 1

$$1+2,\ 3+4 \begin{cases} (12;34) = \langle efe\rangle \quad (34;12)=\langle fef\rangle \quad (43;21)=\langle(1-f)(1-e)(1-f)\rangle \quad (21;43)=\langle(1-e)(1-f)(1-e)\rangle \\ \qquad 1+4 \longleftrightarrow 2+3 \qquad\qquad 1+4 \longleftrightarrow 2+3 \\ (12;43) = \langle e(1-f)e\rangle \quad (34;21)=\langle f(1-e)f\rangle \quad (43;12)=\langle(1-f)e(1-f)\rangle \quad (21;34)=\langle(1-e)f(1-e)\rangle \end{cases}$$

$$1+3 \longleftrightarrow 2+4$$

$$1+3 \longleftrightarrow 2+4$$

$$2+3,\ 1+4 \begin{cases} (14;32) = \langle\overline{efe}\rangle \quad (32;14)=\langle\overline{fef}\rangle \quad (41;23)=\langle\overline{(1-f)(1-e)(1-f)}\rangle \quad (23;41)=\langle\overline{(1-e)(1-f)(1-e)}\rangle \\ \qquad 1+2 \longleftrightarrow 3+4 \qquad\qquad 1+2 \longleftrightarrow 3+4 \\ (14;23) = \langle e-\overline{efe}\rangle \quad (32;41)=\langle f-\overline{fef}\rangle \quad (41;32)=\langle(1-f)-\overline{(1-f)(1-e)(1-f)}\rangle \quad (23;14)=\langle(1-e)-\overline{(1-e)(1-f)(1-e)}\rangle \end{cases}$$

$$1+3 \longleftrightarrow 2+4$$

$$1+3 \longleftrightarrow 2+4$$

$$2+4,\ 1+3 \begin{cases} (13;42) = \langle\overline{e(1-f)e}\rangle \quad (31;24)=\langle\overline{f(1-e)f}\rangle \quad (42;13)=\langle\overline{(1-f)e(1-f)}\rangle \quad (24;31)=\langle\overline{(1-e)f(1-e)}\rangle \\ \qquad 1+2 \longleftrightarrow 3+4 \qquad\qquad 1+2 \longleftrightarrow 3+4 \\ (13;24) = \overline{\langle e-\overline{efe}\rangle} \quad (31;42)=\langle\overline{f-\overline{fef}}\rangle \quad (42;31)=\langle\overline{(1-f)-(\overline{1-f)(1-e)(1-f})}\rangle \quad (24;13)=\langle\overline{(1-e)(1-\overline{e)(1-f)(1-e)}}\rangle \end{cases}$$

$$1+4 \longleftrightarrow 2+3$$

$$1+4 \longleftrightarrow 2+3$$

4. FOUR POINTS ON A LINE

Let $\mathcal{R}$ be the 2×2 matrix ring Γ_2 over a division ring Γ . We will establish the connection of our cross ratio with the one usually ascribed in this case.

Consider two Γ-submodules(points) P_i of V_Γ^2 ,

$$P_i = \{ \begin{pmatrix} \lambda_i \\ 1 \end{pmatrix} \rho \ ; \text{ fixed } \lambda_i, \text{ arbitrary } \rho \text{ in } \Gamma \}, \ (i = 1,2),$$

and $\lambda_1 \neq \lambda_2$. From $\lambda_1 \neq \lambda_2$ it follows that P_1 and P_2 are complementary. Now, under the one-one correspondence between submodules of V_Γ^2 and principal right ideals in Γ_2 described in [Part 1; Theorem 6], P_i corresponds to the principal right ideal

$$\mathfrak{r}_i = \left(\begin{vmatrix} 1 & 0 \\ \lambda_i^{-1} & 0 \end{vmatrix} \right)_r$$

where $\begin{vmatrix} 1 & 0 \\ \lambda_i^{-1} & 0 \end{vmatrix} = e_i$, say, is idempotent. Moreover, the lattice isomorphic nature of the correspondence shows

$$\mathfrak{r}_1 \oplus \mathfrak{r}_2 = \Gamma_2$$

and hence there exists a unique idempotent e in Γ_2 such that $(e_1)_r = (e)_r$ and $(e_2)_r = (1-e)_r$. Therefore, for some x_1, x_2 in Γ_2 , we must have

$$e = e_1 + e_1x_1(1-e_1) = 1 - (e_2 + e_2x_2(1-e_2)) .$$

Now,

(1) $e_1+e_1x_1(1-e_1) = \begin{vmatrix} 1-b\lambda_1^{-1} & b \\ \lambda_1^{-1}-\lambda_1^{-1}b\cdot\lambda_1^{-1} & \lambda_1^{-1}b \end{vmatrix}, x_1 = \begin{vmatrix} * & b \\ * & * \end{vmatrix}$

and

(2) $1-e_2-e_2x_2(1-e_2) = \begin{vmatrix} c\lambda_2^{-1} & -c \\ \lambda_2^{-1}c\,\lambda_2^{-1}-\lambda_2^{-1} & 1-\lambda_2^{-1}c \end{vmatrix}, x_2 = \begin{vmatrix} * & c \\ * & * \end{vmatrix}$

Equating (1) and (2) we find

$b=-c,\ c\lambda_2^{-1} = 1-b\lambda_1^{-1},\ -\lambda_2^{-1}+\lambda_2^{-1}c\lambda_2^{-1}=\lambda_1^{-1}-\lambda_1^{-1}b\lambda_1^{-1},$

and $1-\lambda_2^{-1}c = \lambda_1^{-1}b$, i.e

(3) $-b\lambda_2^{-1} = 1-b\lambda_1^{-1}$

(4) $-\lambda_2^{-1}-\lambda_2^{-1}b\lambda_2^{-1} = \lambda_1^{-1}-\lambda_1^{-1}b\lambda_1^{-1}$

(5) $1+\lambda_2^{-1}b = \lambda_1^{-1}b$

(3) and (5) give the same solution, viz

(6) $b = (\lambda_1^{-1}-\lambda_2^{-1})^{-1} = \lambda_2(\lambda_2-\lambda_1)^{-1}\lambda_1 = \lambda_1(\lambda_2-\lambda_1)^{-1}\lambda_2$

whilst (4) holds whenever (3) does, for (4) is,using (3)

$-\lambda_2^{-1}(1+b\lambda_2^{-1}) = -\lambda_2^{-1}b\,\lambda_1^{-1} = (1-\lambda_1^{-1}b)\lambda_1^{-1}.$

so that, finally, we have

$$e = \begin{vmatrix} 1-\lambda_2(\lambda_2-\lambda_1)^{-1} & \lambda_2(\lambda_2-\lambda_1)^{-1}\lambda_1 \\ \lambda_1^{-1}-\lambda_1^{-1}\lambda_2(\lambda_2-\lambda_1)^{-1} & \lambda_1^{-1}\lambda_2(\lambda_2-\lambda_1)^{-1}\lambda_1 \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda_1(\lambda_2-\lambda_1)^{-1} & \lambda_1(\lambda_2-\lambda_1)^{-1}\lambda_2 \\ -(\lambda_2-\lambda_1)^{-1} & (\lambda_2-\lambda_1)^{-1}\lambda_2 \end{vmatrix}$$

Similarly, if P_3, P_4 are Γ-submodules with different parameters λ_3, λ_4, we find for the unique idempotent f such that $\mathfrak{n}_3 \oplus \mathfrak{n}_4 = \Gamma_2$,

$$f = \begin{vmatrix} -\lambda_3(\lambda_4-\lambda_3)^{-1} & \lambda_3(\lambda_4-\lambda_3)^{-1}\lambda_4 \\ -(\lambda_4-\lambda_3)^{-1} & (\lambda_4-\lambda_3)^{-1}\lambda_4 \end{vmatrix}$$

Using (6) we find

$$\begin{vmatrix} 1 & 0 \\ -\lambda_1^{-1} & 1 \end{vmatrix} e \begin{vmatrix} 1 & \lambda_2 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} -\lambda_1(\lambda_2-\lambda_1)^{-1} & 0 \\ 0 & 0 \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & 0 \\ -\lambda_3^{-1} & 1 \end{vmatrix} f \begin{vmatrix} 1 & \lambda_4 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} -\lambda_3(\lambda_4-\lambda_3)^{-1} & 0 \\ 0 & 0 \end{vmatrix}.$$

Then

$$efe = - \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} (\lambda_2-\lambda_1)^{-1} & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -\lambda_2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \lambda_3 & 0 \\ 1 & \lambda_3 \end{vmatrix}$$

$$\begin{vmatrix} (\lambda_4-\lambda_3)^{-1} & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -\lambda_4 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} (\lambda_2-\lambda_1)^{-1} & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -\lambda_2 \\ 0 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} (\lambda_2-\lambda_1)^{-1}(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)^{-1} & 0 \\ 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} (\lambda_1-\lambda_4)(\lambda_2-\lambda_1)^{-1} & 0 \\ (\lambda_2-\lambda_1)^{-1} & 0 \end{vmatrix} \begin{vmatrix} 1 & -\lambda_2 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} \psi & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} (\lambda_2-\lambda_1)^{-1} & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} -1 & \lambda_2 \\ 0 & -1 \end{vmatrix}$$

where

$$\psi = (\lambda_2 - \lambda_1)^{-1}(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)^{-1}(\lambda_1 - \lambda_4)$$

and just as well

$$efe = \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} 1 & -(\lambda_2 - \lambda_1)^{-1}\lambda_2\lambda_1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \psi & 0 \\ 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} (\lambda_2 - \lambda_1)^{-1} & 0 \\ \lambda_1^{-2} & \lambda_1^{-2}(\lambda_2-\lambda_1) \end{vmatrix} \begin{vmatrix} -1 & \lambda_2 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix}^{-1}$$

$$= \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} 1 & -\eta \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \psi & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & \eta \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix}^{-1}$$

where $\eta = (\lambda_2 - \lambda_1)^{-1}\lambda_2\lambda_1$

$$= b \begin{vmatrix} \psi & 0 \\ 0 & 0 \end{vmatrix} b^{-1} \quad , \quad b = \begin{vmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{vmatrix} \begin{vmatrix} 1 & -\eta \\ 0 & 1 \end{vmatrix}$$

and finally, we conclude that, due to the isomorphism between $\begin{vmatrix} \Gamma & 0 \\ 0 & 0 \end{vmatrix}$ and Γ , there is a one-one correspondence

$$\langle efe \rangle \leftrightarrow \langle \psi \rangle .$$

When $\mathcal{R} = \Gamma_n$ Theorems 11 and 12 are the same as [8; Theorems 21,22]; hence, an application of Lemma 2 will show that our cross ratio coincides with that of [8] in this case.

5. THE NORMAL FORM OF A CONFIGURATION

The following theorem will ensure the existence of matrix units.

THEOREM 10: Let e_1, e_2 be idempotents in $\mathcal{R}$ with the following properties: (i) $e_1 e_2 = e_2 e_1 = 0$ and (ii) $R(e_1) = R(e_2)$. Then there exist unique elements e_{12}, e_{21} in $\mathcal{R}$ such that

$$e_1 e_{12} e_2 = e_{12},\ e_2 e_{21} e_1 = e_{21},\ e_{21} e_{12} = e_2 \text{ and } e_{12} e_{21} = e_1$$

PROOF: It suffices to show the existence of a τ in L such that : $(e_1)_r \oplus (e_2)_r = (e_1)_r \oplus \tau = (e_2)_r \oplus \tau$. Then the existence and uniqueness of e_{12}, e_{21} is guaranteed by [Part 1; Theorem 4]. Let $\mathfrak{r} = (e_1)_r$, $\mathfrak{t} = (e_2)_r$. Then $D(\mathfrak{r}) = D(\mathfrak{t})$ is equivalent to $\mathfrak{r} \sim \mathfrak{t}$. Hence there is a $\mathfrak{v}$ in L such that $\mathfrak{r} \oplus \mathfrak{v} = \mathfrak{t} \oplus \mathfrak{v}$. Then

$$\begin{aligned}
(\mathfrak{r} \cup \mathfrak{v}) \cap (\mathfrak{r} \cup \mathfrak{t}) &= \mathfrak{r} \cup ((\mathfrak{r} \cup \mathfrak{v}) \cap \mathfrak{t}) \\
&\qquad \text{as } \mathfrak{r} \leqslant \mathfrak{r} \cup \mathfrak{v} \\
&= \mathfrak{r} \cup ((\mathfrak{t} \cup (\mathfrak{t} \cap \mathfrak{v})) \\
&\qquad \text{as } \mathfrak{r} \cup \mathfrak{v} = \mathfrak{t} \cup \mathfrak{v} \\
&= \mathfrak{r} \cup \mathfrak{t}
\end{aligned}$$

But this is equivalent to $\mathfrak{r} \cup \mathfrak{v} \geqslant \mathfrak{r} \cup \mathfrak{t}$. Let

$$\tau = \mathfrak{v} \cap (\mathfrak{r} \cup \mathfrak{t})$$

and we have

$$\mathfrak{r} \cap \mathfrak{c} = \mathfrak{r} \cap \mathfrak{v} \cap (\mathfrak{r} \cup \mathfrak{t})$$

$$= (0)_r \text{ , as } \mathfrak{r} \cap \mathfrak{v} = (0)_r$$

and

$$\mathfrak{r} \cup \mathfrak{c} = \mathfrak{r} \cup (\mathfrak{v} \cap (\mathfrak{r} \cup \mathfrak{t}))$$

$$= (\mathfrak{r} \cup \mathfrak{v}) \cap (\mathfrak{r} \cup \mathfrak{t}) \text{ , as } \mathfrak{r} \leqslant \mathfrak{r} \cup \mathfrak{t}$$

$$= \mathfrak{r} \cup \mathfrak{t} \text{ , i.e}$$

$$\mathfrak{r} \oplus \mathfrak{c} = \mathfrak{r} \oplus \mathfrak{t} \text{ .}$$

Similarly, $\mathfrak{r} \oplus \mathfrak{t} = \mathfrak{t} \oplus \mathfrak{c}$. This completes the proof.

The configuration $\{\mathfrak{r}, \mathfrak{t}, \mathfrak{c}, \mathfrak{v}\}$ will be denoted by C . If C, $\hat{C}$ are two configurations such that for some non-singular a in $\mathfrak{K}$,

$$\gamma_a \mathfrak{r} = \hat{\mathfrak{r}}, \quad \gamma_a \mathfrak{t} = \hat{\mathfrak{t}}, \quad \gamma_a \mathfrak{c} = \hat{\mathfrak{c}}, \quad \gamma_a \mathfrak{v} = \hat{\mathfrak{v}}$$

then we say that there exists a non-singular collineation of C onto $\hat{C}$ and write $\gamma_a : C \to \hat{C}$.

We will now give a generalization of the well-known theorem which states that any four distinct collinear points can be projected onto the points $0, 1, \infty$, r, where r is their cross ratio.

Suppose(from now on) that the members of C satisfy

(1) $\mathfrak{r} \cup \mathfrak{t} = \mathfrak{r} \oplus \mathfrak{b} = \mathfrak{t} \oplus \mathfrak{v} = \mathfrak{r} \oplus \mathfrak{v} = (1)_r$,

and let $\mathfrak{m}_1$, $\mathfrak{m}_2$ be relative complements of $\mathfrak{r} \cap \mathfrak{t}$ in $\mathfrak{r}$, $\mathfrak{t}$ respectively, i.e

$$(\mathfrak{r} \cap \mathfrak{t}) \oplus \mathfrak{m}_1 = \mathfrak{r} \, , \quad (\mathfrak{r} \cap \mathfrak{t}) \oplus \mathfrak{m}_2 = \mathfrak{t} \, .$$

Then

$$\begin{aligned}(\mathfrak{r} \cap \mathfrak{t}) \cup \mathfrak{m}_1 \cup \mathfrak{m}_2 &= ((\mathfrak{r} \cap \mathfrak{t}) \cup \mathfrak{m}_1) \cup ((\mathfrak{r} \cap \mathfrak{t}) \cup \mathfrak{m}_2) \\ &= \mathfrak{r} \cup \mathfrak{t} \\ &= (1)_r .\end{aligned}$$

Also

$$\begin{aligned}((\mathfrak{r} \cap \mathfrak{t}) \cup \mathfrak{m}_1) \cap \mathfrak{m}_2 &= \mathfrak{r} \cap \mathfrak{m}_2 \\ &= \mathfrak{r} \cap \mathfrak{m}_2 \cap \mathfrak{t}, \text{ as } \mathfrak{m}_2 \leq \mathfrak{t} \\ &= (\mathfrak{r} \cap \mathfrak{t}) \cap \mathfrak{m}_2 \\ &= (0)_r .\end{aligned}$$

and since

$$(\mathfrak{r} \cap \mathfrak{t}) \cap \mathfrak{m}_1 = (0)_r$$

we conclude that $\mathfrak{r} \cap \mathfrak{t}, \mathfrak{m}_1, \mathfrak{m}_2$ are independent and [Part 1;Theorem 2] there exist unique , independent idempotents e_1, e_2, e_3 such that $(e_1)_r = \mathfrak{r} \cap \mathfrak{t}$, $(e_2)_r = \mathfrak{m}_1$, $(e_3)_r = \mathfrak{m}_2$ and $e_1 + e_2 + e_3 = 1$.

Since e_1, e_2, e_3 are independent,

$\mathfrak{a} = (e_1)_r \oplus (e_2)_r = (e_1 + e_2)_r$, $\mathfrak{c} = (e_1 + e_3)_r$.

Taking, as in Section 2, e and f to be the unique idempotents defined by the conditions

$$\mathfrak{a} \oplus \mathfrak{b} = (1)_r = \mathfrak{c} \oplus \mathfrak{d}$$

(i.e, $\mathfrak{a} = (e)_r$, $\mathfrak{b} = (1-e)_r$, $\mathfrak{c} = (f)_r$, $\mathfrak{d} = (1-f)_r$) we have : $(e_1+e_2)e = e$, $e(e_1+e_2) = e_1+e_2$, $(e_1+e_3)f = f$, $f(e_1+e_3) = e_1+e_3$, and since $(e_1)_r \leq (e)_r$, also $ee_i = e_i$, $(i = 1,2)$; similarly $fe_i = e_i (i = 1,3)$.

Moreover, from $(e_1+e_2)_r = (e)_r$ and $1-(e_1+e_2) = e_3$ follows: $(e_3)_\ell = (e)_r^\ell = (1-e)_\ell$; similarly, $(e_2)_\ell = (1-f)_\ell$.

From the relations (1) we deduce

(2) $D(\mathfrak{a}) = D(\mathfrak{c})$, $D(\mathfrak{b}) = D(\mathfrak{d})$, $D(\mathfrak{d}) + D(\mathfrak{c}) = 1$.

Hence, also $D(\mathfrak{m}_1) = D(\mathfrak{m}_2)$,i.e e_2, e_3 are independent idempotents of the same rank, and by Theorem 10 there exist unique matrix units

$$e_{22} = e_2,\ e_{23}, e_{32},\ e_{33} = e_3$$

such that

$$e_{ij}e_{pq} = \begin{cases} e_{iq}, \text{if } j = p \\ 0, \quad \text{if } j \neq p \end{cases} \quad (i,j,p,q = 2,3).$$

Since $e_{32} = e_{32}e_2$, $e_2 = e_{23}e_{32}$, it follows that $(e_{32})_\ell = (e_2)_\ell$; similarly, $(e_{23})_\ell = (e_3)_\ell$, so that

$$R(e_{32}) = R(e_{23}) = R(e_2) = R(e_3) .$$

Since, by assumption

$$\mathfrak{r} \oplus \mathfrak{v} = (1)_r ,$$

we have $(e_1+e_2)_r \oplus (1-f)_r = (1)_r$. Hence there is an idempotent h such that $(e_1+e_2)_r = (h)_r$, $(1-f)_r = (1-h)_r$. The latter means, for some z in $\mathcal{R}$, $1-h = 1-f+(1-f)zf$. Also, $(h)_r^{\mathfrak{r}} = (e_3)_{\mathfrak{r}}$ and by Theorem 2,

$(h)_r \oplus (1-h)_r = (1)_r$ implies $R(e_3(1-h)) = R(e_3) = R(1-h)$

But, $R(e_3(1-h)) = R(e_3(1-f)(1+(1-f)zf)) = R(e_3(1-f))$, since $1+(1-f)zf$ is non-singular; so that, using $(1-f)e_2 = (1-f)$, we have

$$R(e_3(1-f)e_2) = R(e_3) = R(e_2)$$

But clearly $(e_3(1-f)e_2)_{\mathfrak{r}} \leqslant (e_2)_{\mathfrak{r}} = (e_{32})_{\mathfrak{r}}$; hence equality must hold,i.e

$$(e_3(1-f)e_2)_{\mathfrak{r}} = (e_{32})_{\mathfrak{r}}$$

and , for some x in $\mathcal{R}$, we have

$$e_{32} = xe_3(1-f)e_2 = e_3xe_3(1-f)e_2 .$$

Clearly, this means

$$R(e_{32}) \leqslant R(e_3xe_3) \leqslant R(e_3) = R(e_{32})$$

so that equality holds,i.e

$$R(e_3xe_3) = R(e_3)$$

and e_3xe_3 has an inverse, $\overline{e_3xe_3}$, in the subring $e_3\mathcal{R}e_3$.

Consider

$$a = e_1 - e_1(1-f)e_2 + e_2 + e_3xe_3.$$

a is non-singular; for,

$$a(1 + e_1(1-f)e_2)(e_1 + e_2 + \overline{e_3xe_3}) = e_1 + e_2 + e_3 = 1 .$$

Let us examine the effect of the non-singular collineatio-n γ_a on C.

$$\text{(i)}\ \gamma_a \mathfrak{r} = \gamma_a(e_1+e_2)_r$$

$$= (e_1 - e_1(1-f)e_2 + e_2)_r$$

$$= (e_1+e_2)_r = \mathfrak{r} \quad ,$$

since: $(e_1-e_1(1-f)e_2+e_2)(1+e_1(1-f)e_2) = e_1+e_2$ implies $\gamma_a\mathfrak{r} \geqslant (e_1+e_2)_r$. The reverse inclusion is clear since $\mathfrak{r}$ contains e_1, e_2 and so must contain $e_1-e_1(1-f)e_2+e_2$.

$$\text{(ii)}\ \gamma_a \tau = \gamma_a(e_1+e_3)_r$$

$$= (e_1+e_3xe_3)_r$$

$$= (e_1+e_3)_r = \tau \quad ,$$

since: $(e_1+e_3xe_3)(e_1+\overline{e_3xe_3}) = e_1+e_3$ implies $\gamma_a\tau \geqslant$ $(e_1+e_3)_r$;on the other hand, since $(e_1+e_3)(e_1+e_3xe_3) =$ $e_1+e_3xe_3$, it follows that $\gamma_a\tau \leqslant \tau$.

(iii) Since $(1-f)e_2 = 1-f$, $e_2(1-f) = e_2$, we have $1-f = (e_1+e_2+e_3)(1-f)e_2 = e_1(1-f)e_2+e_2+e_3(1-f)e_2$, so that

$$\begin{aligned}
\Upsilon_a \mathscr{v} &= \Upsilon_a (1-f)_r \\
&= \Upsilon_a (e_1(1-f)e_2+e_2+e_3(1-f)e_2)_r \\
&= (e_1(1-f)e_2-e_1(1-f)e_2+e_2+e_{32})_r \\
&= (e_2+e_{32})_r .
\end{aligned}$$

(iv) Since $(1-e)e_3 = 1-e$, $e_3(1-e) = 1-e$ we have

$$\begin{aligned}
1-e &= (e_1+e_2+e_3)(1-e)e_3 \\
&= e_1(1-e)e_3+e_2(1-e)e_3+e_3
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_a \mathscr{t} &= \Upsilon_a (1-e)_r \\
&= \Upsilon_a (e_1(1-e)e_3+e_2(1-e)e_3+e_3)_r \\
&= (e_1(1-e)e_3-e_1(1-f)e_2(1-e)e_3+e_2(1-e)e_3+e_3xe_3)_r \\
&\geqslant ((e_1(1-e)e_3-e_1(1-f)e_2(1-e)e_3+e_2(1-e)e_3+e_3xe_3)(e_3(1-f)e_2))_r \\
&= (b+e_2(1-e)e_3(1-f)e_2+e_3xe_3(1-f)e_2)_r \\
&= (b+e_2(1-e)e_3(1-f)e_2+e_{32})_r ,
\end{aligned}$$

where $b = (e_1(1-e)e_3-e_1(1-f)e_2(1-e)e_3)e_3(1-f)e_2$.

Equality must hold as

$$D(\Upsilon_a \mathscr{t}) = D(\mathscr{t}) = 1-D(\mathscr{v}) = 1-R(e_1+e_2) = R(e_3)$$

whilst

$$\begin{aligned}
R(b+e_2(1-e)e_3(1-f)e_2+e_{32}) &\geqslant R(e_3(b+e_2(1-e)e_3(1-f)e_2+e_{32})) \\
&= R(e_3e_{32}) \\
&= R(e_{32}) = R(e_3) .
\end{aligned}$$

i.e,

$$\Upsilon_a \mathscr{t} = (b+e_2(1-e)e_3(1-f)e_2+e_{32})_r$$

But, since $(1-f)_l = (e_2)_l$, $(1-e)_l = (e_3)_l$, $ee_2 = e_2$,

$$\begin{aligned} e_2(1-e)e_3(1-f)e_2 &= e_2(1-e)(1-f)e_2 \\ &= e_2-e_2ee_2-e_2fe_2+e_2efe_2 \\ &= e_2efe_2 = e_2efee_2, \end{aligned}$$

and

$$\begin{aligned} b &= \{e_1(1-e)e_3-e_1(1-f)e_2(1-e)e_3\}e_3(1-f)e_2 \\ &= e_1\{1-e-(1-f)(1-e)\}(1-f)e_2 \\ &= e_1f(1-e)(1-f)e_2 \\ &= e_1be_2 \end{aligned}$$

so that, finally

$$\gamma_a\mathfrak{t} = (e_1be_2+e_2efee_2+e_{32})_r$$

Writing $efe = \mathfrak{R}$ we have the

<u>LEMMA 1:</u> (i) $\langle e_2\mathfrak{R}e_2\rangle = \mathfrak{R}(\mathfrak{v}, \mathfrak{r}; \mathfrak{t}, \mathfrak{n})$

(ii) $\langle e_1+e_1be_2+e_2\mathfrak{R}e_2\rangle = \mathfrak{R}(\mathfrak{n}, \mathfrak{t}; \mathfrak{r}, \mathfrak{v})$

<u>PROOF</u>: Since $(e_2)_l = (1-f)_l$ we have, for some y in $\mathcal{R}$, $e_2 = 1-f+fy(1-f)$. Then

$$\begin{aligned} e_2\mathfrak{R}e_2 &= e_2(1-e)(1-f)e_2 \\ &= \{1-f+fy(1-f)\}(1-e)(1-f) \end{aligned}$$

Let $s = fy(1-f)$; then $(1-f)(1+s) = 1-f$ and

$$\begin{aligned} (1-s)e_2\mathfrak{R}e_2(1+s) &= \{1-fy(1-f)\}\{1+fy(1-f)\}(1-f)(1-e)(1-f) \\ &= (1-f)(1-e)(1-f) \end{aligned}$$

i.e, $\langle e_2 \mathcal{R} e_2 \rangle = \langle (1-f)(1-e)(1-f) \rangle$

which proves (i).

Again, since $(e)_r = (1-e_3)_r$ and $(f)_r = (1-e_2)_r$, we have for some x, z in $\mathcal{R}$

$$e = 1-e_3+(1-e_3)xe_3 \ , \ f = 1-e_2+(1-e_2)ze_2$$

and so

$$efe = \{1-e_3+(1-e_3)xe_3\}\{1-e_2+(1-e_2)ze_2\}\{1-e_3+(1-e_3)xe_3\}$$
$$= (1-e_3)(1+ze_2+xe_3+xe_3ze_2+xe_3ze_2xe_3)$$

Let $s_1 = e_1xe_3$, $s_2 = e_1ze_2xe_3$, $s_3 = e_1ze_2$, $s_4 = e_1xe_3ze_2$ and $s_5 = e_2xe_3$. Then $s_i^2 = 0$ $(i = 1, \ldots, 5)$ and, after a simple calculation,

$$(1+s_5)\ldots(1+s_1)efe(1-s_1)\ldots(1-s_5)$$
$$= e_1+e_1xe_3ze_2+e_1ze_2xe_3ze_2+e_2xe_3ze_2 \ .$$

On the other hand,

$$e_1+e_1be_2+e_2\mathcal{R}e_2 = e_1+e_1f(1-e)(1-f)+e_2(1-f)(1-e)(1-f)$$
$$= e_1+e_1(1-e)(1-f)+(e_2-e_1)(1-f)(1-e)(1-f)$$

and easy calculations show, successively

$(1-e)(1-f) = -e_3ze_2+(1-e_3)xe_3ze_2$, $e_1(1-e)(1-f) = e_1xe_3ze_2$; $(1-f)(1-e)(1-f) = e_2xe_3ze_2-(1-e_2)ze_2xe_3ze_2$, $(e_2-e_1)(1-f)(1-e)(1-f) = e_2xe_3ze_2+e_1ze_2xe_3ze_2$.

so that

$$e_1+e_1be_2+e_2\mathcal{R}e_2 = tefet^{-1}, \ t = (1+s_5)\ldots(1+s_1).$$

This completes the proof of (ii).

Denoting by N (normalform) the configuration $\{(e_1+e_2)_r, (e_1be_2+e_2\mathfrak{R}e_2+e_{32})_r, (e_1+e_3)_r, (e_2+e_{32})_r\}$ we have proved the

<u>THEOREM 11:</u> For any configuration C satisfying (1) there is a non-singular collineation

$$\gamma_a : C \to N$$

such that

$$\mathfrak{R}(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d}) = \langle e_1+e_1be_2+e_2\mathfrak{R}e_2\rangle$$

and

$$\mathfrak{R}(\mathfrak{d}, \mathfrak{c}; \mathfrak{b}, \mathfrak{a}) = \langle e_2\mathfrak{R}e_2\rangle \ .$$

When, in addition to (1) the members of C satisfy $\mathfrak{a} \cap \mathfrak{c} = (0)_r$, N reduces to

$$\{(e_2)_r, (e_2\mathfrak{R}e_2 + e_{32})_r, (e_3)_r, (e_{32}+e_2)_r\} \ .$$

Corollary 4 will show that, in this case, N actually characterizes the cross ratio, i.e. we could use N as a means of defining $\mathfrak{R}$. This is actually what Baer [12; p 72] does for four points on a line, over a field.

Conversely, given a non-singular collineation

$$\gamma_a : C \Rightarrow N$$

then the members of C must satisfy the relations (1).

For, by [Part 1; Lemma 2],

(i) $(e_1+e_2)_r \cup (e_1+e_3)_r = (e_1+e_2+g)_r$,

where $(g)_r = ((1-e_1-e_2)(e_1+e_3))_r = (e_3)_r$, i.e $g = e_3$,

and so $\gamma_a \mathfrak{n} \cup \gamma_a \tau = (1)_r$.

(ii) $(e_1+e_2)_r \oplus (e_2+e_{32})_r = (e_1+e_3)_r \oplus (e_2+e_{32})_r$

$$= (1)_r$$

The "$\cup$" part follows as in (i) whilst $x \varepsilon (e_1+e_2)_r \cap (e_2+e_{32})_r \Rightarrow x = (e_1+e_2)x = (e_2+e_{32})x \Rightarrow e_3x = 0 = e_{32}x$, i.e $e_2x = 0$; but then $x = 0$. Similarly for the other case

(iii) $(e_1+e_2)_r \oplus (e_1be_2+e_2Re_2+e_{32})_r = (1)_r$ .For,

$x = (e_1+e_2)x = (e_1be_2+e_2Re_2+e_{32})z \Rightarrow e_3x = 0 = e_{32}z$

$\Rightarrow e_2z = 0$, i.e $x = 0$; whilst

$$R(e_1be_2+e_2Re_2+e_{32}) \geqslant R(e_3(e_1be_2+e_2Re_2+e_{32})) = R(e_{32})$$
$$= R(e_3)$$

$\Rightarrow D((e_1+e_2)_r) \oplus D((e_1be_2+e_2Re_2+e_{32})_r) \geqslant 1$, so that equality must hold.

Finally, as

$$\gamma_a (\mathfrak{n} \cup \mathfrak{t}) = (1)_r \{ \gamma_a(\mathfrak{n} \cap \mathfrak{t}) = (0)_r \}$$

if and only if

$$\mathfrak{n} \cup \mathfrak{t} = (1)_r \{ \quad \mathfrak{n} \cap \mathfrak{t} = (0)_r \}$$

we see that C does satisfy (1) .

6. SIMILARITY INVARIANTS FOR THE NORMALFORM

The object of this section is to determine the extent to which the cross ratios provide similarity invariants for the normalform N. We will see that being given a non-singular collineation between two configurations $C, \hat{C}$ implies a special kind of similarity of their cross ratios [Theorem 12]; on the other hand, if $C, \hat{C}$ have a cross ratio in common it is , in general, necessary to assume that their members satisfy some further conditions(i.e in addition to 5.1) before there will be a non-singular collineation between them [Lemmas 4,5].

Notice that the existence of a non-singular collineation $\gamma_n : C \to \hat{C}$ implies the existence of one between the normalforms,viz $\gamma_{\hat{a}na^{-1}} \{ = \gamma_{\hat{a}}(\gamma_n(\gamma_a^{-1} \,..)) \} : N \to \hat{N}$, where $a, \hat{a}$ are the non-singular elements for $C, \hat{C}$ respectively in the sense of Theorem 11. Conversely, $\gamma_n : N \to \hat{N}$ $\Rightarrow \gamma_{\hat{a}^{-1}na} : C \to \hat{C}$.

Let the normalform $\hat{N}$, provided by Theorem 11, of $\hat{C}$ be

$$\{(f_1+f_2)_r, (f_1\hat{b}f_2+f_2\hat{R}f_2+f_{32})_r, (f_1+f_3)_r, (f_2+f_{32})_r\}$$

where the matrix units f have similar properties to their counterparts e in N.

THE GENERAL COLLINEATION: Assume that there exists a non-singular collineation

$$\gamma_n : N \to \hat{N} .$$

Clearly, corresponding members have the same dimension, and therefore corresponding matrix units have the same ra-nk. This information allows us to replace the f's by the e's, for

LEMMA 2: If $\{e_{ij}\},\{f_{ij}\}$ are two systems of n^2 matrix units(cf.[Part 1; p 16]) such that

$$R(e_{ii}) = R(f_{ii}) , \ i = 1,\dots,n$$

then there exists a non-singular m in $\mathcal{R}$ such that

$$mf_{ij}m^{-1} = e_{ij}, \ i,j = 1,\dots,n.$$

PROOF: By [Part 1; Lemma 4] we have non-singular $s_i (i = 1,\dots,n)$ in such that

$$s_i f_i s_i^{-1} = e_i \ (e_i = e_{ii}, \ f_i = f_{ii}) .$$

Hence

$$e_i s_i f_i s_i^{-1} e_i = e_i .$$

Let

$$m = \sum_{i=1}^{n} e_i s_i f_i \ , \ p = \sum_{i=1}^{n} f_i s_i^{-1} e_i$$

and then, since the f_i are independent idempotents,

$$\begin{aligned} mp &= \sum_i e_i s_i f_i \sum_j f_j s_j^{-1} e_j \\ &= \sum_i e_i s_i f_i s_i^{-1} e_i \\ &= \sum_i e_i = 1, \end{aligned}$$

by definition of the e_{ij}. Thus m is non-singular; furthermore

$$mf_im^{-1} = e_is_if_is_i^{-1}e_i = e_i \ (i = 1,..,n).$$

Let

$$mf_{ij}m^{-1} = g_{ij} \ (i,j = 1,..,n).$$

Then $g_{ii} = e_i$ and $\{g_{ij}\}$ is a system of n^2 matrix units, for $\sum_i g_{ii} = 1$ and $g_{ij}g_{kh} = mf_{ij}f_{kh}m^{-1}$

$$= \begin{cases} 0 \text{ if } j \neq k \\ mf_{ih}m^{-1} = g_{ih} \text{ if } j=k \end{cases}$$

for all $i,j,k,h = 1,..,n$. Now, [Part 1; Theorem 4],

$$(f_i)_r \oplus (f_i - f_{ji})_r = (f_i)_r \oplus (f_j)_r$$

and since m is non-singular (i.e since γ_m is a lattice automorphism) it follows that

$$\gamma_m(f_i)_r \oplus \gamma_m(f_i - f_{ji})_r = \gamma_m(f_i)_r \oplus \gamma_m(f_j)_r$$

i.e

$$(e_i)_r \oplus (e_i - g_{ji})_r = (e_i)_r \oplus (e_j)_r$$

and, by the uniqueness part of the last theorem referred to, we can only have $g_{ij} = e_{ij}$, all i,j .

Now, even though the matrix units of $N,\hat{N}$ do not form a complete 3^2 system (the $e_{1j}, f_{1j}, e_{j1}, f_{j1}$ with $j = 2,3$ are absent), it is still clear that the assertion of Lemma 2 holds for them, with exactly the same proof. Therefore, there exists a non-singular m in $\mathcal{R}$ such that

$$e_1 = mf_1m^{-1}, \ e_{ij} = mf_{ij}m^{-1} \ (i,j = 2,3).$$

Then $\gamma_m: \hat{N} \to \hat{N}^*$, where $\hat{N}^*$ is
$\{(e_1+e_2)_r, (e_1\hat{b}^*e_2+e_2\hat{R}^*e_2+e_{32})_r, (e_1+e_3)_r, (e_2+e_{32})_r\}$
and $\hat{b}^* = m\hat{b}m^{-1}$, $\hat{R}^* = m\hat{R}\, m^{-1}$. Consequently, we may as well assume, from the outset, that $\hat{N}$ is the configuration obtained by replacing $\hat{b}^*$ and $\hat{R}^*$ in $\hat{N}^*$ with $\hat{b}$ and $\hat{R}$ respectively. This assumption will apply for the remainder of the section.

The relations $(n(e_1+e_2))_r = (e_1+e_2)_r$, $(n(e_1+e_3))_r = (e_1+e_3)_r$ and $(n(e_2+e_{32}))_r = (e_2+e_{32})_r$ are equivalent to

(i) $(e_1+e_2)n(e_1+e_2) = n(e_1+e_2)$

(ii) $(e_1+e_3)n(e_1+e_3) = n(e_1+e_3)$

(iii) $(e_2+e_{32})n(e_2+e_{32}) = n(e_2+e_{32})$.

Right multiplying (i) by e_1, e_2((ii) by e_1, e_3) and then left multiplying the two resulting equations by $e_3(e_2)$ gives

(1) $$e_3ne_1 = e_3ne_2 = e_2ne_1 = e_2ne_3 = 0.$$

Since $e_1+e_2+e_3 = 1$, we can write (iii) as

(iv) $(e_1+e_3-e_{32})n(e_2+e_{32}) = 0$

and left multiplying (iv) by e_1 we find $e_1ne_2 = -e_1ne_{32}$, i.e

(2) $$e_1ne_3 = -e_1ne_{23}$$

Finally, taking (1),(2) into account, (iv) becomes

$$e_3ne_{32} = e_{32}ne_2 + e_{32}e_2ne_3e_{32} = e_{32}ne_2 ,$$

that is

(3) $$e_3ne_3 = e_{32}ne_{23} .$$

Then, as $e_1+e_2+e_3 = 1$, $n = (e_1+e_2+e_3)n(e_1+e_2+e_3)$ and (1),(2),(3) combine to give

(4) $$n = \begin{vmatrix} e_1ne_1 & e_1ne_2 & -e_1ne_{23} \\ 0 & e_2ne_2 & 0 \\ 0 & 0 & e_{32}ne_{23} \end{vmatrix}$$

where the matrix denotes the sum of its entries. Since also $\gamma_{n^{-1}}: \hat{N} \to N$,

(5) $$n^{-1} = \begin{vmatrix} e_1n^{-1}e_1 & e_1n^{-1}e_2 & -e_1n^{-1}e_{23} \\ 0 & e_2n^{-1}e_2 & 0 \\ 0 & 0 & e_{32}n^{-1}e_{23} \end{vmatrix}$$

and it is clear that n, n^{-1} have the most general form possible under the conditions. We note also, for future reference, that

(6) $$\begin{cases} e_ine_i = e_in \ , \ e_in^{-1}e_i = e_in^{-1} \ (i = 2,3) \\ e_1ne_1 = ne_1 \ , \ e_1n^{-1}e_1 = n^{-1}e_1 . \end{cases}$$

$\gamma_n: N \to \hat{N}$ also implies

$$(n(e_1be_2+e_2Re_2+e_{32}))_r = (e_1\hat{b}e_2+e_2\hat{R}e_2+e_{32})_r$$

that is, for some y in $\mathcal{R}$,

$$n(e_1be_2+e_2\mathfrak{R}e_2+e_{32}) = (e_1\hat{b}e_2+e_2\hat{\mathfrak{R}}e_2+e_{32})y .$$

Then

$$e_{23}n(e_1be_2+e_2\mathfrak{R}e_2+e_{32}) = e_{23}e_{32}y = e_2y ,$$

and as $e_3n = e_3ne_3 = e_{32}ne_{23}$ we have

$$e_2y = e_{23}ne_{32} = e_{23}e_{32}ne_{23}e_{32} = e_2ne_2 .$$

Hence

(7) $$n(e_1be_2+e_2\mathfrak{R}e_2+e_{32}) = (e_1\hat{b}e_2+e_2\hat{\mathfrak{R}}e_2+e_{32})\ ne_2 .$$

But, by (3), $e_1ne_1 = ne_1$, $e_2ne_2 = e_2n$ and therefore

$$e_1ne_1be_2+e_1ne_2\mathfrak{R}e_2+e_1ne_{32} = e_1\hat{b}e_2ne_2 = e_1\hat{b}e_2n ;$$

and as $e_1ne_2 = -e_1ne_{32}$ this yields

(8) $$e_1\hat{b}e_2 = e_1ne_1be_2n^{-1} - e_1ne_2(e_2-e_2\mathfrak{R}e_2)n^{-1}.$$

Also from (7),

$$e_2ne_2\mathfrak{R}e_2 = e_2\hat{\mathfrak{R}}e_2ne_2 = e_2\hat{\mathfrak{R}}e_2n , \text{ i.e}$$

(9) $$e_2\hat{\mathfrak{R}}e_2 = e_2ne_2\mathfrak{R}e_2n^{-1} .$$

But, from (1),(2),(3),

$$n^{-1}e_2n = e_2n^{-1}e_2ne_2 = e_2n^{-1}ne_2 = e_2 ,$$

so that

$$e_2\hat{\mathfrak{R}}e_2 = ne_2\mathfrak{R}e_2n^{-1}, \text{ i.e}$$

(10) $$\langle e_2\hat{\mathfrak{R}}e_2\rangle = \langle e_2\mathfrak{R}e_2\rangle .$$

Writing

$$n_{12} = \begin{vmatrix} e_1ne_1 & e_1ne_2 \\ 0 & e_2ne_2 \end{vmatrix}, \quad \bar{n}_{12} = \begin{vmatrix} e_1n^{-1}e_1 & e_1n^{-1}e_2 \\ 0 & e_2n^{-1}e_2 \end{vmatrix}$$

and $g = \begin{vmatrix} e_1 & e_1be_2 \\ 0 & e_2\mathfrak{R}e_2 \end{vmatrix}$, we have the

<u>LEMMA 3</u>: The following condition is necessary and sufficient for the existence of a non-singular collineation of N onto $\hat{N}$: for some non-singular n in $\mathcal{R}$ we have

$$(11) \left\{ \begin{array}{l} n_{12}g\bar{n}_{12} = \hat{g}, \\ \text{where } n_{12}\bar{n}_{12} = \bar{n}_{12}n_{12} = e_1+e_2 . \end{array} \right.$$

<u>PROOF:</u> Assume γ_n is a non-singular collineation of N onto $\hat{N}$. Then

$n_{12}g\bar{n}_{12} =$

$$\begin{vmatrix} e_1ne_1n^{-1}e_1 & e_1ne_1n^{-1}e_2+e_1ne_1be_2n^{-1}e_2+e_1ne_2\mathfrak{R}e_2n^{-1}e_2 \\ 0 & e_2ne_2\mathfrak{R}e_2n^{-1}e_2 \end{vmatrix}$$

and as $e_1n^{-1}e_1 = n^{-1}e_1$ we have $e_1ne_1n^{-1}e_1 = e_1$. Also, by (9), $e_2ne_2\mathfrak{R}e_2 = e_2\hat{\mathfrak{R}}e_2n$, that is

$$e_2ne_2\mathfrak{R}e_2n^{-1}e_2 = e_2\hat{\mathfrak{R}}e_2 .$$

Finally, by (8),

$e_1ne_1n^{-1}e_2+e_1ne_1be_2n^{-1}e_2+e_1ne_2\mathfrak{R}e_2n^{-1}e_2$

$=$

$$= e_1\hat{b}e_2+e_1ne_2n^{-1}e_2+e_1ne_1n^{-1}e_2$$

$$= e_1\hat{b}e_2 \ ,$$

for, since $e_1+e_2+e_3 = 1$ and $e_3n^{-1}e_2 = 0$,

$$0 = e_1e_2 = e_1nn^{-1}e_2 = e_1n(e_1+e_2+e_3)n^{-1}e_2$$

$$= e_1ne_1n^{-1}e_2+e_1ne_2n^{-1}e_2 \ .$$

We have shown that

$$n_{12}g\bar{n}_{12} = \hat{g} \ .$$

Also, as $e_2ne_2 = e_2n$, it follows that

$$e_2ne_2n^{-1}e_2 = e_2$$

and hence that

$$n_{12}\bar{n}_{12} = \left|\begin{array}{cc} e_1ne_1n^{-1}e_1 & e_1ne_1n^{-1}e_2+e_1ne_2n^{-1}e_2 \\ 0 & e_2ne_2n^{-1}e_2 \end{array}\right|$$

$$= e_1+e_2 \ .$$

Conversely, assume that there is a non-singular n such that (11) holds. Then letting

$$m = \left|\begin{array}{cc:c} \multicolumn{2}{c:}{n_{12}} & -e_1ne_{23} \\ \cdot & \cdot & 0 \\ 0 & 0 & e_{32}ne_{23} \end{array}\right| , \quad p = \left|\begin{array}{cc:c} \multicolumn{2}{c:}{\bar{n}_{12}} & -e_1n^{-1}e_{23} \\ \cdot & \cdot & 0 \\ 0 & 0 & e_{32}n^{-1}e_{23} \end{array}\right|$$

we have

$$mp =$$

$$= \left| \begin{array}{ccc} n_{12}\bar{n}_{12} & \vdots & -e_1ne_1n^{-1}e_{23}-e_1ne_{23}e_{32}n^{-1}e_{23} \\ \cdots & \vdots & 0 \\ 0 \quad 0 & & e_{32}ne_{23}e_{32}n^{-1}e_{23} \end{array} \right|$$

$= \ e_1+e_2+e_3 = 1,$

as , by (11), $e_1ne_1n^{-1}e_2+e_1ne_2n^{-1}e_2 = 0$, that is $e_1ne_1n^{-1}e_{23}+e_1ne_2n^{-1}e_{23} = 0$; also by (11)

$$e_{32}ne_{23}e_{32}n^{-1}e_{23} = e_{32}e_2ne_2n^{-1}e_2e_{23} = e_{32}e_{23} = e_3.$$

Thus m is non-singular. In order that $\Upsilon_m: N \to \hat{N}$ it suffices that $\Upsilon_m(e_1be_2+e_2\mathrm{R}e_2+e_{32})_r = (e_1\hat{b}e_2+e_2\hat{\mathrm{R}}e_2+e_{32})_r$. But this is so, since (11) implies (8) and (9) and hence

$$\left| \begin{array}{ccc} n_{12} & \vdots & -e_1ne_{23} \\ \cdots & \vdots & 0 \\ 0 \quad 0 & & e_{32}ne_{23} \end{array} \right| \left| \begin{array}{ccc} 0 & e_1be_2 & 0 \\ 0 & e_2\mathrm{R}e_2 & 0 \\ 0 & e_{32} & 0 \end{array} \right| \left| \begin{array}{ccc} \bar{n}_{12} & \vdots & -e_1n^{-1}e_{23} \\ \cdots & \vdots & 0 \\ 0 \quad 0 & & e_{32}n^{-1}e_{23} \end{array} \right|$$

$$= \left| \begin{array}{ccc} 0 & e_1\hat{b}e_2 & 0 \\ 0 & e_2\hat{\mathrm{R}}e_2 & 0 \\ 0 & e_{32} & 0 \end{array} \right| .$$

Combining Lemmas 1 and 3 with (10) we have

THEOREM 12: The existence of a non-singular collineation of N onto $\hat{N}$ implies

$$\mathrm{R}_x(\mathfrak{u},\mathfrak{t};\mathfrak{r},\mathfrak{v}) = \mathrm{R}_x(\widehat{\mathfrak{u},\mathfrak{t};\mathfrak{r},\mathfrak{v}}) \quad \text{and}$$

$$\mathrm{R}_x(\mathfrak{v},\mathfrak{r};\mathfrak{t},\mathfrak{u}) = \mathrm{R}_x(\widehat{\mathfrak{v},\mathfrak{r};\mathfrak{t},\mathfrak{u}}).$$

COROLLARY 4: If $e_1 = 0$ and C has the form

$$\{(e_2)_r, (e_2re_2+e_{32})_r, (e_3)_r, (e_2+e_{32})_r\}$$

for some r in $\mathcal{R}$, then

$$\mathrm{R}_x(\mathfrak{u},\mathfrak{t};\mathfrak{r},\mathfrak{v}) = \langle e_2re_2\rangle$$

PROOF: $x=e_2x = (e_2re_2+e_{32})z \Rightarrow e_{32}z = 0$, i.e $e_2z = 0$; hence $x = 0$. Since $e_2+e_3 = 1$ and $R(e_2re_2+e_{32}) \geqslant R(e_{32}) = R(e_3)$ we must have $(e_2)_r \oplus (e_2re_2+e_{32})_r = (1)_r$ so that the cross ratio is defined. Then Theorems 11,12, and 6 combine to give the result.

<u>RANK</u> <u>RELATIONS</u>: Recall that $\gamma_a(\mathfrak{m}\cap\mathfrak{c}) = \mathfrak{m}\cap\mathfrak{c} = (e_1)_r$, $\gamma_a\mathfrak{v} = (e_2+e_{32})_r$ and $\gamma_a\mathfrak{t} = (e_1be_2+e_2Re_2+e_{32})_r$.

Since $(e_2+e_{32})_r = (e_3+e_{23})_r$ and both e_2+e_{32} and e_3+e_{23} are idempotent, we have

$$(1-e_3-e_{23})_\ell = (e_2+e_{32})_r^\ell .$$

An easy calculation shows that

$$e_2-e_2Re_2-e_1be_2 = -(1-e_3-e_{23})(e_1be_2+e_2Re_2+e_{32}) .$$

Hence, by Theorem 1(ii),

$$\begin{aligned} R(e_2-e_2Re_2-e_1be_2) &= R(1-e_3-e_{23})(e_1be_2+e_2Re_2+e_{32})) \\ &= D(\gamma_a\mathfrak{t}) - D(\gamma_a\mathfrak{t}\cap\gamma_a\mathfrak{v}) \\ &= D(\gamma_a\mathfrak{t}) - D(\gamma_a(\mathfrak{t}\cap\mathfrak{v})) \\ &= D(\mathfrak{t}) - D(\mathfrak{t}\cap\mathfrak{v}). \end{aligned} \tag{12}$$

By [Part 1; Lemma 1] we have

$$\begin{aligned} ((\mathfrak{m}\cap\mathfrak{c})\cup\gamma_a\mathfrak{v})^\ell &= (\mathfrak{m}\cap\mathfrak{c})^\ell\cap(\gamma_a\mathfrak{v})^\ell \\ &= (1-e_1)_\ell\cap(1-e_3-e_{23})_\ell ; \end{aligned}$$

and, as $e_1+e_2+e_3 = 1$, we see that

$$1-e_3-e_{23} = e_1+(e_2-e_{23}) ;$$

since e_1, e_2-e_{23} are independent idempotents , this means

$$(1-e_3-e_{23})_\ell = (e_1)_\ell \oplus (e_2-e_{23})_\ell ;$$

hence,

$$((\mathfrak{m}\cap\mathfrak{c})\cup\gamma_a\mathfrak{v})^\ell = ((e_1)_\ell\oplus(e_2-e_{23})_\ell)\cap(1-e_1)_\ell$$

$= (e_2-e_{23})_{\mathfrak{r}}$,

by modularity, since $(e_2-e_{23})(1-e_1) = e_2-e_{23}$ implies $(e_2-e_{23})_{\mathfrak{r}} \leq (1-e_1)_{\mathfrak{r}}$.

Then, as

$$e_2-e_2\mathfrak{R}e_2 = (e_{23}-e_2)(e_1be_2+e_2\mathfrak{R}e_2+e_{32}) ,$$

we have, by Theorem 1(i),

$$\begin{aligned}
R(e_2-e_2\mathfrak{R}e_2) &= R((e_{23}-e_2)(e_1be_2+e_2\mathfrak{R}e_2+e_{32}))\\
&= D(((\mathfrak{n}\cap\mathfrak{c})\cup\gamma_a\mathfrak{v})\cup\gamma_a\mathfrak{t})-D((\mathfrak{n}\cap\mathfrak{c})\cup\gamma_a\mathfrak{v})\\
&= D(\gamma_a((\mathfrak{n}\cap\mathfrak{c})\cup\mathfrak{v}\cup\mathfrak{t}))-D(\gamma_a((\mathfrak{n}\cap\mathfrak{c})\cup\mathfrak{v}))\\
&= D((\mathfrak{n}\cap\mathfrak{c})\cup\mathfrak{t}\cup\mathfrak{v}) - D((\mathfrak{n}\cap\mathfrak{c})\cup\mathfrak{v}) \qquad (13)\\
&= D(\mathfrak{n}\cap\mathfrak{c})+D(\mathfrak{v}\cup\mathfrak{t})-D((\mathfrak{n}\cap\mathfrak{c})\cap(\mathfrak{v}\cup\mathfrak{t}))\\
&\qquad -D((\mathfrak{n}\cap\mathfrak{c})\cup\mathfrak{v})\\
&= D(\mathfrak{n}\cap\mathfrak{c}) + D(\mathfrak{v}) + D(\mathfrak{t}) - D(\mathfrak{v}\cap\mathfrak{t})\\
&\qquad - D((\mathfrak{n}\cap\mathfrak{c})\cap(\mathfrak{v}\cup\mathfrak{t})) - D((\mathfrak{n}\cap\mathfrak{c})\cup\mathfrak{v}),
\end{aligned}$$

since $(\mathfrak{n}\cap\mathfrak{c})\cap\mathfrak{v} = (\mathfrak{n}\cap\mathfrak{v})\cap(\mathfrak{c}\cap\mathfrak{v}) = (0)_r$. Thus,

(14) $\quad R(e_2-e_2\mathfrak{R}e_2) = D(\mathfrak{t}) - D(\mathfrak{v}\cap\mathfrak{t}) - D((\mathfrak{n}\cap\mathfrak{c})\cap(\mathfrak{v}\cup\mathfrak{t}))$

and taking (12) into account

(15) $R(e_2 - e_2\mathcal{R}e_2) = R(e_2 - e_2\mathcal{R}e_2 - e_1 b e_2) - D((\mathfrak{m}\cap\mathfrak{c})\cap(\mathfrak{v}\cup\mathfrak{b})$

<u>LEMMA 3:</u> If e_1, e_2 are independent idempotents and u, v arbitrary elements in $\mathcal{R}$, then for some w in $\mathcal{R}$

$$(e_1 u + e_2 v)_{\mathfrak{r}} = (e_2 v)_{\mathfrak{r}} \oplus (e_1 u - e_1 w e_2 v)_{\mathfrak{r}} .$$

<u>PROOF:</u> $(e_2 v)_{\mathfrak{r}} \leq (e_1 u + e_2 v)_{\mathfrak{r}}$, since $e_2 v = e_2(e_1 u + e_2 v)$. If h is any idempotent such that $(h)_{\mathfrak{r}} = (e_2 v)_{\mathfrak{r}}$, then, [Part 1; Lemma 7], a relative complement of $(h)_{\mathfrak{r}}$ in $(e_1 u + e_2 v)_{\mathfrak{r}}$ is given by

$$((e_1 u + e_2 v)(1-h))_{\mathfrak{r}} = (e_1 u - e_1 u h)_{\mathfrak{r}} ,$$

since $e_2 v h = e_2 v$. Now, $h = z e_2 v$ for some z in $\mathcal{R}$ and hence $e_1 u h = e_1 u z e_2 v$. Let $w = e_1 u z$ and we have

$$(e_1 u + e_2 v)_{\mathfrak{r}} = (e_2 v)_{\mathfrak{r}} \oplus (e_1 u - e_1 w e_2 v)_{\mathfrak{r}}.$$

Clearly, we also have

$$R(e_1 u + e_2 v) = R(e_2 v) + R(e_1 u - e_1 w e_2 v) .$$

Lemma 3 shows that for some w in $\mathcal{R}$,

(16) $R(e_2 - e_2\mathcal{R}e_2 - e_1 b e_2) = R(e_2 - e_2\mathcal{R}e_2) + R(e_1 b' e_2)$,

where

$$(17)\qquad e_1b'e_2 = e_1be_2 - e_1we_2(e_2-e_2\mathfrak{K}e_2) .$$

Moreover, (17) shows that

$$\begin{vmatrix} e_1 & e_1we_2 \\ 0 & e_2 \end{vmatrix}\begin{vmatrix} e_1 & e_1be_2 \\ 0 & e_2\mathfrak{K}e_2 \end{vmatrix}\begin{vmatrix} e_1 & -e_1we_2 \\ 0 & e_2 \end{vmatrix}$$

$$= \begin{vmatrix} e_1 & e_1b'e_2 \\ 0 & e_2\mathfrak{K}e_2 \end{vmatrix}$$

i.e Lemma 3 applies and gives a non-singular collineation carrying N into

$$\{(e_1+e_2)_r,(e_1b'e_2+e_2\mathfrak{K}e_2+e_{32})_r,(e_1+e_3)_r,(e_2+e_{32})_r\} .$$

For this reason we may assume (16) to hold with b' replaced by b.

COMPLEMENTS TO LEMMA 3: Notice that the similarity of (10),(11) is actually in the following stronger sense: if $a,b \in e\mathcal{R}e$ (some idempotent e) then write $\langle a\rangle = \langle b\rangle$ if $xay = b$ where $x,y \in e\mathcal{R}e$ and $xy = yx = e$. From now on $\langle\ \rangle$ will have this meaning. Similarity in the new sense implies similarity in the old sense; for, $x+1-e$ is non-singular with inverse $y+1-e$ and $(x+1-e)a(y+1-e) = b$. Let $n = x+1-e$; then $ene = x$ and $en^{-1}e = y$.

In general, in order to obtain a non-singular collineation of N onto $\hat{N}$, it is not enough to be given one cross ratio. However, by assuming in addition that the members of $N,\hat{N}$ satisfy one other condition , we can arrive at this position.

LEMMA 4: If (i) $\langle g\rangle = \langle \hat{g}\rangle$ and (ii) both $C,\hat{C}$ satisfy $\mathfrak{n} \cap \mathfrak{l} = (0)_r$, then there exists a non-singular collineation of N onto $\hat{N}$.

PROOF: By (i) , we have for some non-singular r,

$$(e_1+e_2)r^{-1}(e_1+e_2)g(e_1+e_2)r(e_1+e_2) = \hat{g}$$

where $(e_1+e_2)r(e_1+e_2)r^{-1}(e_1+e_2) = (e_1+e_2)r^{-1}(e_1+e_2)r(e_1+e_2)$
$$= e_1+e_2.$$

Hence, just as well

$$g(e_1+e_2)r(e_1+e_2) = (e_1+e_2)r(e_1+e_2)\hat{g} ,$$

i.e, subtracting each side from $(e_1+e_2)r(e_1+e_2)$,

$$(e_2-e_2\mathfrak{R}e_2-e_1be_2)((e_1+e_2)r(e_1+e_2))$$
$$= ((e_1+e_2)r(e_1+e_2))(e_2-e_2\hat{\mathfrak{R}}e_2-e_1\hat{b}e_2)$$

and hence

(18) $$(e_2-e_2\mathfrak{R}e_2-e_1be_2)e_2re_1 = 0 \quad .$$

But, from (ii), we have using (12)

$$R(e_2-e_2\mathfrak{R}e_2-e_1be_2) = R(1-e) = R(e_2),$$

and since, obviously , $(e_2-e_2\mathfrak{R}e_2-e_1be_2)_{\mathfrak{r}} \leq (e_2)_{\mathfrak{r}}$ we must have equality. Hence also

$$(e_2-e_2\mathfrak{R}e_2-e_1be_2)^{r}_{\mathfrak{r}} = (e_2)^{r}_{\mathfrak{r}}$$

i.e, (18) holds if and only if $e_2re_1 = e_2e_2re_1 = 0$. Similarly $e_2r^{-1}e_1 = 0$. We have shown that (11) holds, and therefore there is a non-singular collineation of N onto $\hat{N}$.

<u>LEMMA 5</u>: If (i) $\langle e_2\mathfrak{R}e_2\rangle = \langle e_2\hat{\mathfrak{R}}e_2\rangle$ and (ii) both $C,\hat{C}$ satisfy $(\mathfrak{n} \cap \mathfrak{c}) \cap (\mathfrak{v} \cup \mathfrak{t}) = (0)_r$, then there is a non-singular collineation of N onto $\hat{N}$.

PROOF: (ii) implies, by (15),

$$R(e_2 - e_2 \mathfrak{R} e_2) = R(e_2 - e_2 \mathfrak{R} e_2 - e_1 b e_2)$$

i.e, by (16),

$$R(e_1 b e_2) = 0$$

so that $e_1 b e_2 = 0$, and similarly $e_1 \hat{b} e_2 = 0$.

(i) implies, for some non-singular r,

$$e_2 \mathfrak{R} e_2 = e_2 r e_2 \hat{\mathfrak{R}} e_2 r^{-1} e_2 \ ,$$

where $e_2 r^{-1} e_2$ is the inverse of $e_2 r e_2$ in $e_2 \mathcal{R} e_2$. Then an easy calculation shows that

$$(e_1 + e_2 r^{-1} e_2) g (e_1 + e_2 r e_2) = \hat{g}$$

i.e (11) holds.

When $\mathfrak{m} \cap \mathcal{C} = (0)_r$ (i.e $e_1 = 0$) Lemma 5 is the converse of Theorem 12. For then $\langle g \rangle = \langle e_2 \mathfrak{R} e_2 \rangle$ and Lemma 5 shows that $\langle g \rangle = \langle \hat{g} \rangle \Rightarrow$ the existence of a non-singular collineation of N onto $\hat{N}$. In the general case however, we have been unable to establish this converse .

7.

When $(\mathfrak{u} \cap \mathfrak{r}) \cap (\mathfrak{v} \cup \mathfrak{t}) = (0)_r$ we have seen (Lemma 5) that $e_1 b e_2 = 0$. In this case N can be decomposed into

$$\{(e_1)_r, (0)_r, (e_1)_r, (0)_r\} \oplus \{(e_2)_r, (e_2 \mathfrak{R} e_2 + e_{32})_r, (e_3)_r, (e_2 + e_{32})_r\}$$

where the direct sum means component-wise direct sum. The left hand summand is obviously uninteresting and from now on we neglect it. We may as well assume that $e_1 = 0$, i.e $\mathfrak{u} \cap \mathfrak{r} = (0)_r$. Then

$$(1') \quad \mathfrak{u} \oplus \mathfrak{r} = \mathfrak{u} \oplus \mathfrak{t} = \mathfrak{r} \oplus \mathfrak{v} = \mathfrak{u} \oplus \mathfrak{w} = (1)_r.$$

From (1') we deduce that $\mathfrak{u}, \mathfrak{t}, \mathfrak{r}, \mathfrak{w}$ all have dimension 1/2. Moreover, since $e_2 + e_3 = 1$ and $R(e_2) = R(e_3)$ it follows that $\mathfrak{u}, \mathfrak{r}$ form a homogeneous basis of order two for L [Part 1; Lemma 10].

We can extend the range of (1') as follows: Let $\mathfrak{Z}$ denote the centre of $\mathfrak{R}$ ($\mathfrak{Z}$ is a field). Then $e_2 \mathfrak{Z} e_2$ is the centre of $e_2 \mathfrak{R} e_2$ [Part 1; Lemma 6], and is isomorphic to $\mathfrak{Z}$ under the correspondence : $\mathfrak{Z} \ni x \leftrightarrow e_2 x e_2$. Let

$$\mathfrak{u}(a,b) = (e_2 a e_2 + e_{32} b e_2)_r$$

where $a, b \in \mathfrak{Z}$, and define

$$M = \{\mathfrak{u}(a,b);\ a,b \in \mathfrak{Z}\}.$$

Clearly, $b \neq 0 \iff \mathfrak{v}(a,b) \neq \mathfrak{v}$. For, since $\overline{e_2ae_2}$ exists in $e_2\mathfrak{Z}e_2$, $\mathfrak{v}(a,0) = (e_2ae_2)_r = (e_2)_r$. Conversely if $\mathfrak{v} = \mathfrak{v}(a,b)$ then $e_2ae_2+e_{32}be_2 = e_2z$, some z in $\mathfrak{R}$, $\Rightarrow e_{32}be_2 = 0 \Rightarrow e_2be_2 = 0 \Rightarrow b = 0$. Since $\mathfrak{v}(a,b) = (e_2ab^{-1}e_2+e_{32})_r$ if $b \neq 0$, if we agree to write $\mathfrak{v} = \mathfrak{v}_\infty$ then we can (symbolically) parametrize M as

$$M = \{ \mathfrak{v}_a = (e_2ae_2+e_{32})_r;\ a \in \mathfrak{Z} \} \cup \{ \mathfrak{v}_\infty \} .$$

M includes : $\mathfrak{v} = \mathfrak{v}_\infty$, $\mathfrak{r} = \mathfrak{v}_0$, $\mathfrak{u} = \mathfrak{v}_1$. Moreover, when ever $a \neq 0$, we have

(2) $\mathfrak{v}_a \oplus \mathfrak{v}_0 = \mathfrak{v}_a \oplus \mathfrak{v}_\infty = \mathfrak{v}_0 \oplus \mathfrak{v}_\infty = (1)_r$.

To see this, notice that $\mathfrak{m}$ is a subspace satisfying

(3) $\mathfrak{v} \oplus \mathfrak{m} = \mathfrak{r} \oplus \mathfrak{m} = \mathfrak{v} \oplus \mathfrak{r}$

if and only if there exist [Part 1; Theorem 4] unique matrix units u,v such that

$uv = e_3$, $vu = e_2$, $e_3ue_2 = u$, $e_2ve_3 = v$ and

$$\mathfrak{m} = (e_2-u)_r = (e_3-v)_r.$$

Then, since by [7; Satz 3.3] we have a one-one correspondence between the $\mathfrak{m}$ satisfying (3) and the invertible elements of $e_2\mathfrak{R}e_2$, given by

$$\mathfrak{m} \leftrightarrow e_2ae_2 = -e_{23}u, \text{ where } \overline{e_2ae_2} = -ve_{32},$$

it follows that (2) holds.

Also, if $a \in \mathfrak{Z}$, then $e_2-e_2ae_2 \in e_2\mathfrak{Z}e_2$ and therefore $\overline{e_2-e_2ae_2}$ exists in $e_2\mathfrak{Z}e_2$. Then

$x = (e_2+e_{32})x = (e_2ae_2+e_{32})z \Rightarrow e_{32}x = e_{32}z \Rightarrow e_2x = e_2z$. Hence $(e_2-e_2ae_2)x = 0$, i.e $e_2x = 0$ and we can only have $x = 0$. This shows that $\mathfrak{a}_1 \cap \mathfrak{a}_a = (0)_r$, for $a \neq 1$. Since $R(e_2ae_2+e_{32}) \geq R(e_{32}) = R(e_3)$ we have

$$D(\mathfrak{a}_a \oplus \mathfrak{a}_1) = D(\mathfrak{a}_a) + D(\mathfrak{a}_1) \geq R(e_3)+R(e_2) = 1,$$

so that equality must hold, i.e

(3) $\qquad \mathfrak{a}_a \oplus \mathfrak{a}_1 = (1)_r$, whenever $a \neq 1$.

(2) and (3) show that cross ratios are defined for all permutations of $\{\mathfrak{a}_\infty, \mathfrak{a}_a, \mathfrak{a}_0, \mathfrak{a}_1\}$; moreover, cross ratios in the rows of Table 1(with $\mathfrak{t}$ replaced by $\mathfrak{a}_a$) are equal.

Define

$$\mathfrak{M} = \mathfrak{M}(\infty,0,1) = \{x \in \mathfrak{R} ; x \in \mathfrak{a}_a \text{ some } \mathfrak{a}_a \in M\}.$$

$\mathfrak{M}$ is closed for multiplication, but not for addition.

For any $a \in \mathfrak{Z}$ let $\mathfrak{t}(a) = (e_2(\mathfrak{R}-a)e_2)^r_{\mathfrak{l}}$, i.e $x \in \mathfrak{t}(a) \Leftrightarrow e_2\mathfrak{R}e_2x = e_2ae_2x$. Now, if $y \in \mathfrak{t} \cap \mathfrak{M}$, then $y = (e_2\mathfrak{R}e_2+e_{32})z_1 = (e_2ae_2+e_{32})z_2$, for some $a \in \mathfrak{Z}$ and $z_1, z_2 \in \mathfrak{R}$, so that $e_3y = e_{32}z_1 = e_{32}z_2$, i.e $e_2z_1 = e_2z_2 = z$, say. Hence $y = (e_2\mathfrak{R}e_2+e_{32})z = (e_2ae_2+e_{32})z$ and therefore $z \in \mathfrak{t}(a)$. Conversely, assume that $y \in \gamma_t\, \mathfrak{t}(a)$, where $t = e_2\mathfrak{R}e_2+e_{32}$. Then $y = (e_2\mathfrak{R}e_2+e_{32})x$ where

$x \in \mathcal{L}(a)$. Hence also $y = (e_2ae_2+e_{32})$, i.e $y \in \mathcal{L} \cap \mathcal{M}$. Finally, if $R(e_2\mathfrak{R}e_2-e_2ae_2) = R(e_2)$ then clearly $\mathcal{L}(a) = (e_3)_r$ and hence $\gamma_t\mathcal{L}(a) = (0)_r$. Defining e_2ae_2 to be an eigenvalue of $e_2\mathfrak{R}e_2$ in $e_2\mathfrak{Z}e_2$ if $R(e_2(\mathfrak{R}-a)e_2) < R(e_2)$, we have shown

(4) $$\mathcal{L} \cap \mathcal{M} = \{ x;\ x\varepsilon\ \gamma_t\mathcal{L}(a),\ a\ \varepsilon\ \Lambda \}$$

where $e_2\Lambda e_2$ denotes the set of eigenvalues of $e_2\mathfrak{R}e_2$ in $e_2\mathfrak{Z}e_2$.

By Corollary 4, $R(\alpha_\infty, \alpha_a; \alpha_0, \alpha_1) = \langle e_2ae_2\rangle$. Hence a permutation of $\mathcal{U}, \mathcal{T}, \mathcal{W}$ amongst themselves (obviously leaves $\mathcal{M}$ unchanged) induces the corresponding permutation of the parameters. Moreover, assume that the inverses referred to in Table 1 exist, and we have

$$(e_2-e_2\mathfrak{R}e_2)-(e_2-e_2ae_2) = -(e_2\mathfrak{R}e_2-e_2ae_2) \text{ and}$$
$$\overline{e_2\mathfrak{R}e_2} - \overline{e_2ae_2} = -\overline{e_2ae_2\mathfrak{R}e_2}(e_2\mathfrak{R}e_2-e_2ae_2) .$$

Hence the eigenvalues transform under permutations down the columns of Table 1 in exactly the same way as the cross ratios. This enables to determine the relative position (4) for configurations defined in terms of another cross ratio, provided the latter is obtained by a permutation of the above type. Hence it suffices to consider this relative position in relation to the permutations of the Klein four group.

8. SOME COLLINEATIONS INDUCING PERMUTATIONS

Case (i): $\overline{e_2\mathfrak{R}e_2}$ exists in $e_2\mathfrak{R}\,e_2$. Then $n = e_{32}+e_2\mathfrak{R}e_{23}$ is non-singular with $n^{-1} = e_{23}+e_{32}\overline{e_2\mathfrak{R}e_2}$ and

$$\gamma_n\{\mathfrak{u},\mathfrak{t},\tau,\mathfrak{v}\}$$

$$= \{(e_{32})_r,(e_{32}\mathfrak{R}e_2+e_2\mathfrak{R}e_2)_r,(e_2\mathfrak{R}e_{23})_r,(e_{32}+e_2\mathfrak{R}e_2)_r\}$$

$$= \{(e_3)_r,(e_2+e_{32})_r,(e_2)_r,(e_2\mathfrak{R}e_2+e_{32})_r\}$$

$$= \{\tau,\mathfrak{v},\mathfrak{u},\mathfrak{t}\}.$$

Case (ii): $\overline{e_2-e_2\mathfrak{R}e_2}$ exists. Then $n= e_2+e_{32}-e_2\mathfrak{R}e_{23}-e_3$ is non-singular with $n^{-1} = (e_2+e_{32})(\overline{e_2-e_2\mathfrak{R}e_2})-(e_2\mathfrak{R}e_2 + e_{32})(\overline{e_2-e_2\mathfrak{R}e_2})e_{23}$,

and $\gamma_n\{\mathfrak{u},\mathfrak{t},\tau,\mathfrak{v}\} =$

$$\{(e_2+e_{32})_r,(e_{32}\mathfrak{R}e_2-e_{32})_r,(e_2\mathfrak{R}e_{23}+e_3)_r,(e_2-e_2\mathfrak{R}e_2)_r\}$$

$$= \{(e_2+e_{32})_r,(e_3)_r,(e_2\mathfrak{R}e_2+e_{32})_r,(e_2)_r\}$$

$$= \{\mathfrak{v},\tau,\mathfrak{t},\mathfrak{u}\}.$$

Case (iii): Both $\overline{e_2\mathfrak{R}e_2}$ and $\overline{e_2-e_2\mathfrak{R}e_2}$ exist. Then

$$n = e_{23}+e_{33}-e_2-e_{32}\overline{e_2\mathfrak{R}e_2}$$

is non-singular with

$$n^{-1}= \overline{(e_2-\overline{e_2\mathfrak{R}e_2})}(e_2-e_{23})+e_{32}(\overline{e_2-e_2\mathfrak{R}e_2})+e_{32}\overline{(e_2-\overline{e_2\mathfrak{R}e_2})}e_{23}$$

and then

$$\gamma_n\{\mathfrak{r}, \mathfrak{t}, \tau, \mathfrak{v}\}$$

$$= \{(e_2+e_{32}\overline{e_2 R_x e_2})_r, (e_{23})_r, (e_2+e_{32})_r, (e_{32})_r\}$$

$$= \{(e_2 R_x e_2+e_{32})_r, (e_2)_r, (e_2+e_{32})_r, (e_3)_r\}$$

$$= \{\mathfrak{t}, \mathfrak{r}, \mathfrak{v}, \tau\}.$$

Hence, corresponding to (i),(ii),(iii) , the positio-n of $\mathfrak{t}$ relative to $\mathfrak{m}(\infty, 0, 1)$ corresponds to the position of

(i') $\mathfrak{v}$ relative to $\mathfrak{m}(0, \infty, R_x)$

(ii') τ relative to $\mathfrak{m}(1, R_x, \infty)$

(iii') $\mathfrak{r}$ relative to $\mathfrak{m}(R_x, 1, 0)$

respectively.

REFERENCES

J.von Neumann, Proc. Nat. Acad. Sci. U.S.A ,

(1) On continuous geometries, 22(36)92-100.

(2) Examples of continuous geometries, 22(36)101-8.

(3) On Regular Rings, 22(36)707-713.

(4) Algebraic theory of continuous geometries, 23(37)16-22.

(5) Continuous Ri ngs and their arithmetics, 23(37)341-347.

(6) Princeton Lectures, vols I-II-III,1936-7.

(7) F.Maeda, Kontinuierliche Geometrien, Springer 1958.

(8) A.Fuhrmann, Klassen Ahnlicher Matrizen als verallgemeinerte Doppelverhaltnisse, Math.Zeit. 62(55)211-240.

(9) G.Ehrlich, A continuous geometry within the unit group, Trans. Amer. Math. Soc. 83(56)397-416.

(10) R.J.Smith, A generalization of Ore's determinant, Proc. S. Dak. Acad. Sci. 37(59)196.

(11) I.Halperin and K.Fryer, The von Neumann coordinatization theorem, Acta. Sci. Math.(Szeged) 17(56)203-249.

(12) R.Baer , Linear Algebra and Projective Geometry, Academic Press, 1952.