

THE
MATHEMATICAL KNOWLEDGE
OF
SECONDARY
PRE-SERVICE TEACHERS

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M.Ed., B.Ed., B.Sc.

A thesis submitted in fulfilment of the
requirement for the degree of
Doctor of Philosophy
University of Tasmania

1999

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ABSTRACT

The problem under investigation in this study is teachers' lack of competence to teach for conceptual understanding of mathematics. It was assumed in this study that pre-service teachers of mathematics go through their teacher education and training with certain insufficiencies in their mathematical understandings and that these insufficiencies will eventually affect the way they teach. The aim of this study, therefore, was to identify what these insufficiencies might be by examining mathematical knowledge that secondary pre-service mathematics teachers bring with them to teacher education programs. The nature of these mathematical knowledge insufficiencies and how these insufficiencies would affect a person's competence to teach were of primary interest.

To explore these knowledge insufficiencies, a multiple-case study design was used. The nineteen cases (secondary mathematics pre-service teachers) from four universities at two Australian states were selected according to their mathematical backgrounds. It was expected that the pre-service teacher participants (university graduates) who majored in mathematics or in other science related areas would show less evidence of knowledge insufficiencies than pre-service teachers who majored in other areas (e.g. economics). Furthermore, it was expected that the participants with mathematics major backgrounds would show more confidence to teach for conceptual understanding of mathematics than participants with mathematics minor backgrounds.

The data collection instrument was a set of three mathematical stimulus items representing trigonometry, logarithm, and statistics. All three items were designed to elicit responses associated with the respondent's knowledge of the mathematics. Written and verbal responses to these items were collected in one-to-one interviews. Skemp's (1978) model of mathematical understanding was the instrument for data analysis.

The results of this qualitative analysis indicated four types of mathematical knowledge deficiencies. In addition, the pre-service teachers' existing mathematical knowledge was highly representative of instrumental understanding of mathematics. These mathematical knowledge insufficiencies were suggested to be the outcomes of learning mathematical content which lacked in essential knowledge aspects, rather than outcomes of rote learning. Furthermore, these insufficiencies tended to reduce the pre-service teachers' confidence and likewise their potential to teach for conceptual understanding of mathematics.

DECLARATION

This thesis may be available for loan and limited copying in accordance with the *Copyright Act 1968*.

I certify that this thesis contains no material which has been accepted for the award of any other degree or diploma in any university and that, to the best of my knowledge and belief, contains no material previously published or written by another persons, except where due reference is made in the text of the thesis.

Lupé M.T. Gates

A handwritten signature in black ink, appearing to read 'LMGates', written in a cursive style.

ACKNOWLEDGMENTS

To those who generously and patiently offered me their support, assistance and helpful criticisms in the preparations of this thesis, I wish to express my sincere thanks and appreciation. I gratefully acknowledge the co-operation of all these people but space permits specific reference to only a few.

It has been a privilege to have Professor John Braithwaite and Associate Professor Lyn English as my key advisers. Their insight, understanding, constructive advice and encouragement have made it a valuable and rewarding learning experience.

The development of the stimulus items for this study was subject to the participation of several mathematics teachers, lecturers and students. I would like to thank sincerely these people for their generous participation and valuable assistance. I would also like to acknowledge the mathematics teachers, pre-service mathematics teachers, university lecturers who generously offered their assistance and participation in the collection of the data.

Finally, I would like to express a special thanks to my husband, Max Gates, for his support and encouragement.

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CHAPTER ONE

INTRODUCTION

This thesis is based on a study about the types of mathematical knowledge prospective secondary teachers bring with them to pre-service mathematics teacher education programs. The documentation of this study includes an outline of relevant research findings and views on mathematical knowledge and the influence of this knowledge on mathematical understanding and competency in the teaching of mathematics. Highlighted in these research findings is a persistent problem concerning teachers' lack of competence in the teaching of mathematics. This lack of teacher competence in mathematics appears to have close connections with teachers' mathematical understanding (or subject-matter knowledge). Although teachers are expected to acquire this understanding from undertaking mathematics teacher education programs, research findings showed that teachers' pre-tertiary learning of mathematics had far more influence on the way teachers teach mathematics (e.g. Ball, 1990). These findings raise several important questions concerning the types of mathematical knowledge that teachers have acquired prior to teacher education. Two of these questions are explored further in the study reported in this thesis: (1) what types of mathematical knowledge secondary pre-service teachers of mathematics bring with them to teacher education, and (2) what influence these types of knowledge might have on their teaching of mathematics.

1.1 Mathematics as a form of knowledge

In addressing a meeting of the National Council of Teachers of Mathematics about the importance of mathematics in modern life, George B. Olds (1928) found it fitting to focus upon the 'power and beauty of mathematics' (p.196). About six decades later the importance of mathematics has had little change, as described in the following quotation.

Mathematics has been called 'the queen of the sciences' for its intrinsic beauty and because it has mothered a host of other sciences. Traditionally, its branches have been arithmetic, algebra, geometry, trigonometry, statistics and logic. It forms the base of many practical sciences such as physics, chemistry, geology and meteorology. It provides the foundation for cultural arts such as music, art and architecture. It is rapidly being adopted as a basic tool by the social sciences and humanities - for studies of population, political trends and economic theories (National Science and Technology Centre, N.S.T.C., 1989, p. ii).

Mathematics has provided the tools and methods which have been largely responsible for the extraordinary advances in the other sciences (Mitchell, 1933). As a form of knowledge, mathematics plays an important part in an individual's formal education because:

- its understanding forms the base for learning other mathematically dependent disciplines (Department of Employment, Education and Training, D.E.E.T., 1989; Australian Education Council, A.E.C., 1991; Leitzel, 1991; McNamara, 1991; De Corte, 1995);
- it contributes to the development and enhancement of skills in the use and appreciation of technology (N.S.T.C., 1989; A.E.C., 1991), and
- it provides the individual with a sense of power and achievement (Skemp, 1986, 1989; Ball, 1990; Leitzel, 1991; Greenwood, 1993).

Mathematical knowledge is also an essential requirement for competency in the workforce, and in everyday life (e.g. Dungan & Thurlow, 1989; A.E.C., 1991; D.E.E.T., 1989, 1992).

1.2 Mathematical understanding in teaching

In most societies it is commonly accepted that knowledge about mathematics is imparted and transferred to individuals through the act of teaching. It is this act of teaching, through organised instruction, that is essential to the recipient of mathematical knowledge (e.g. Olds, 1928; Skemp, 1986; Ball, 1991; Leder, 1992; De Corte, 1995). In order for individuals to be educated in mathematics and to achieve their goals, good teachers of mathematics are of vital importance. Teachers of mathematics also have to have knowledge and understanding about mathematics to provide adequate mathematical teaching. Teachers have the 'power to effect change' (Eisenhart, Borko, Underhill, Brown, Jones & Agard, 1993, p.39) and 'are the key figures in changing the ways in which mathematics is taught and learned in schools' (National Council of Teachers of Mathematics, NCTM, 1991, p.2). Prior to becoming teachers of mathematics, these individuals must be educated sufficiently in mathematics so that they can have the necessary ability to competently transfer knowledge and skills of mathematics to learners (e.g. Kramer, 1933; Shulman, 1987; D.E.E.T., 1992; Berliner, 1994).

Competency in mathematics appears to be an essential attribute required by mathematics teachers. It has been suggested that teacher competence in mathematics involves more than a sound knowledge in mathematics (Skemp, 1986; Leitzel, 1991). It involves two essential components: (i)

knowledge and understanding of mathematics, and (ii) the capacity to communicate or transfer mathematics in a given context (D.E.E.T., 1992). These two components are referred to respectively by several researchers, (e.g. Berliner, Stein, Sabers, Clarridge, Cushing, & Pinnegar, 1988; Ball, 1991; Even, 1993; Berliner, 1994), as teacher subject matter (mathematics) knowledge and pedagogical knowledge. However, it is suggested that for mathematics, pedagogical knowledge is dependent on the teacher's knowledge of mathematics (Ball, 1990; Ball & McDiarmid, 1990; Leinhardt, Putnam, Stein, & Baxter, 1991). The interrelationship between these two components is exemplified in the view of a competent mathematics teacher as one whose performance is underpinned not only by skill but also by knowledge and understanding of mathematics (D.E.E.T., 1992).

It is argued in this thesis that, if pedagogical knowledge is dependent on mathematical knowledge and is important to competent teaching, then teachers of mathematics would need to acquire a substantial level of mathematical knowledge prior to entering the workforce (D.E.E.T., 1992; Eisenhart *et al.*, 1993). Therefore, it is the mathematical knowledge of prospective teachers of mathematics that is the focus of the study reported in this thesis.

1.3 The problem under investigation - teachers' lack of mathematical understanding

The problem under investigation in this study is teachers' lack of mathematical understanding. Studies of both pre-service and inservice mathematics teachers have shown that a large proportion of teachers lack understanding in

mathematics and that this has affected their teaching of mathematics as well as classroom management and organisation (Skemp, 1986, 1989; Shulman, 1987; Leinhardt, 1989; Ball, 1990; Leder, 1991; Leinhardt *et al.*, 1991; McNamara, 1991; Eisenhart *et al.*, 1993; Even, 1993; Wilson, 1994).

Two possible contributing factors to the problem have been identified from the research literature: (1) a current assumption in secondary mathematics teacher education programs that a tertiary mathematics qualification denotes essential pre-requisite knowledge has been acquired, and (2) teachers' prior learning of mathematics is insufficient. These factors are described below.

**(1) A current assumption in secondary mathematics
teacher education programs**

In a recent nationwide Discipline Review of Teacher Education in Mathematics and Science (D.E.E.T., 1987, p.27) it was reported that, although teachers may have acquired 'abstract' or higher level mathematics from their tertiary education, they lack the ability to transfer or communicate such mathematics to their students. This report tends to suggest that gaining a tertiary education, although it provides the individual with higher mathematical learning, is not a guarantee that essential knowledge components have been acquired by the prospective teachers of mathematics. However, the current assumption being promoted in teacher education programs, particularly in secondary programs, is that, prospective teachers, including prospective mathematics teachers, have acquired the essential and necessary subject-matter knowledge and skills from their pre-tertiary and tertiary schooling, and that, when provided with teacher education programs

in their prospective fields, they should become competent teachers (D.E.E.T., 1989, 1992; Leitzel, 1991; McNamara, 1991). This assumption is particularly noticeable in secondary mathematics teacher education programs where the entrant is required to have completed certain mathematics courses or a degree in mathematics or in other science and technology areas. As such the entrant is accepted into the teacher education program as having the essential mathematical pre-requisites. On the other hand, researchers have found that this requirement may actually have negative effects on mathematical competence in that prospective teachers with such mathematical backgrounds find it difficult to adjust their mathematical thinking in order for them to apply their acquired mathematical knowledge to teaching situations (e.g. Ball, 1990; Wilson, 1994; Gates, 1995a).

(2) Teachers' prior learning of mathematics is insufficient

There could be numerous reasons as to why teachers lack competence in mathematics. One likely reason could be that, before undertaking teacher education, teachers did not acquire the appropriate prior knowledge from their pre-tertiary (or college) education. For example, results of a survey conducted in Western Australia showed an increasing number of potential mathematics teachers with inadequate mathematical backgrounds entering teacher education in that state (Western Australian Office of Higher Education, 1992).

In addition, Ball (1990) found that how teachers learned and how they were taught at pre-tertiary level, had far more influence on their mathematical development than the learning achieved from secondary

teacher education programs. Such findings tend to indicate that teacher education programs may not be providing prospective teachers with the necessary mathematical knowledge for competent teaching. In addition, these findings tend to suggest that teachers may go through their training with certain misunderstandings, misconceptions, or gaps in their mathematical knowledge, and that these will eventually affect the way they teach.

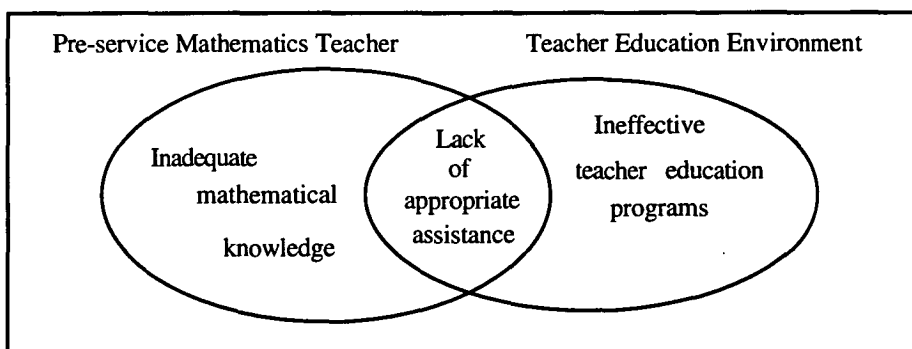
1.4 A contributing factor to a solution - a competent teacher educator of mathematics

It is suggested that those who implement the secondary teacher education pre-service programs play a vital role in preparing prospective mathematics teachers to become competent teachers of mathematics (Ball, 1990; Even, 1993). In addition, Skemp (1986) maintains that a necessary factor in a prospective mathematics teacher's environment is mathematicians or mathematics teacher educators who can competently assist the prospective teachers with the process of mathematical knowledge reconstruction and assimilation. This argument is supported by research findings, for example, Leder's (1991) studies of lecturers acting as mathematics students. Leder (1991) found that 'adapting and responding to students' individual ideas require teachers [mathematics teacher educators] who are confident in their own mathematical knowledge and who themselves have a good grasp of mathematical concepts and ideas' (p.7).

In summarising to this point, two main issues relating to lack of mathematical competence by secondary teachers have emerged. One is

associated with prospective teachers' inadequate prior knowledge of mathematics. The other is related to the ineffectiveness of teacher education programs in providing prospective teachers with environments that can facilitate the reconstruction of mathematical concepts needed for mathematical understanding. These are two separate issues but are linked by a common element, namely, the *lack of appropriate assistance* for prospective teachers (see Figure 1.1). Therefore, appropriate assistance is required as a means to ensure that prospective teachers would gain the essential mathematical knowledge needed to form the basis for competent teaching at the secondary school level.

Figure 1.1: Illustrating the lack of appropriate assistance



1.5 Significance and assumptions of the current study

It appears from the discussion in Section (1.4) that the mathematics teacher educator is partly responsible for providing the assistance needed by pre-service mathematics teachers in reconstructing their existing mathematical knowledge (Skemp, 1986; Leder, 1991). Although the teacher educator may have the mathematical competence and confidence to provide pre-service teachers with such help, a major concern is that

little is known about the inter-relationship between mathematical content knowledge and pedagogical knowledge of secondary mathematics teachers (Even, 1993, p.95). This concern is associated with the shortage of available data on the quality of existing mathematical knowledge of secondary pre-service teachers - or mathematical knowledge pre-service teachers bring with them to teacher education (Ball, 1990; Ball & McDiarmid, 1990). Ball and McDiarmid (1990) indicated that there has been more research on mathematical knowledge and understanding relating to elementary (early-childhood and primary school levels) pre-service teachers than for secondary pre-service teachers. In an endeavour to provide some of this much needed data, this study was designed to explore mathematical knowledge of secondary pre-service teachers of mathematics.

Furthermore, to provide secondary pre-service teachers with appropriate assistance (Figure 1.1), it would be of value for the teacher educator to know more about the types of deficiencies associated with lack of mathematical understanding and the potential of such deficiencies to influence a pre-service teacher's competence in teaching mathematics. This issue is the underlying purpose for this study of pre-service teachers' mathematical knowledge, that is, to identify types of mathematical knowledge deficiencies and their influence on mathematical understanding and competence.

In search for more clarification and elaboration on what has already been documented in the literature regarding teacher competence in mathematics, the study reported here was specifically designed to explore the following assumptions:

- (1) That secondary pre-service teachers of mathematics go through their teacher education and training with certain deficiencies in their mathematical understandings and that these will eventually affect the way they teach.
- (2) That mathematical understanding is dependent on the sufficiencies of procedural and conceptual types of mathematical knowledge. Lack of or a deficiency in either procedural and/or conceptual knowledge could suggest a deficiency in mathematical understanding (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993).
- (3) That pre-service teachers who majored in mathematics or other science related areas (e.g. chemistry, physics, and computer science) would show less evidence of mathematical knowledge deficiencies than pre-service teachers who majored in other areas (e.g. economics and physical education). Furthermore, pre-service teachers with relational understanding (Skemp, 1978) of mathematics would demonstrate more confidence to teach mathematics than pre-service teachers with instrumental understanding (Skemp, 1978).

Having stated the research assumptions, the conceptualisation that provides the theoretical structure of this study is discussed next.

1.6 The conceptual framework of the study

The conceptual framework for this study takes into account both the social constructivist's perspective and the schema theorist's perspective (Derry, 1996; also Reynolds, Sinatra, & Jetton, 1996, for a comprehensive comparison of the views on knowledge). From a social constructivist's view, learners construct knowledge from their experiences and

interactions with others in learning environments such as schools and other social contexts (Reynolds *et al.*, 1996). In this view, schools do make a difference with respect to cognitive development, in the sense that 'the acquisition and growth of the cognitive skills and processes underlying intellectual performances are, to a large degree, the result of learning and teaching in schools' (De Corte, 1995, p.37). From a schema theorist's view, this cognitive development occurs in the mind in the form of schemata or 'packets of knowledge' (Reynolds *et al.*, 1996, p.97; Derry, 1996). The term 'schema' was defined as 'a general term connoting virtually any memory structure' (Derry, 1996, p.167). However, the schema theorists still acknowledge the importance of experiences in providing the raw material from which the mind forms schemata. Knowledge, in schema theory, is fluid and dynamic and it is created out of the interaction between incoming sense perceptions and the inherent capabilities of the mind (Derry, 1996; Reynolds *et al.*, 1996).

In this framework, a mathematics pre-service teacher's existing mathematical knowledge is defined for this thesis as mathematical knowledge the pre-service teacher brings with him or her to the mathematics teacher education program. This existing mathematical knowledge is assumed to have been acquired by the individual from his or her formal schooling, particularly from the mathematics taught at school. This domain-specific knowledge would also include mathematical knowledge gained from university studies in mathematics.

Furthermore, it is assumed that this mathematical knowledge was gained from the individual's own constructions or one's interpretations of experiences from incoming sense perceptions with the help of his or her

prior knowledge (Alexander, Schallert & Hare 1991; Derry, 1996, p.172).

This conceptualisation of mental processes suggests that a pre-service teacher's existing mathematical knowledge is closely inter-related with how the knowledge was acquired (Skemp, 1986; Ball, 1990; De Corte, 1995).

The conceptual framework of this study was developed from the perspective of a pre-service teacher as an active learner or participant in a learning environment. Based on this framework, a pre-service teacher's recorded (verbal or written) interpretations of a mathematical situation would provide data or evidence on the quality of his or her existing mathematical knowledge. In theory, these data should also contain aspects of how mathematical knowledge was acquired (e.g. by rote memorisation) and indicators about how pre-service teachers might teach mathematics.

Since these data are embedded in theoretical structures associated with the selected framework, their interpretation and evaluation require a model of data analysis that encapsulates the theoretical underpinnings of this study. After a substantial review of the literature on how knowledge (or a learned outcome) is assessed (e.g. Biggs & Collis, 1982), particularly for mathematical knowledge, the Skemp (1978) model of mathematical understanding was chosen as the appropriate model for data analysis in this study. Skemp's (1978) model is described in detail in Chapter 3.

How this framework was incorporated into the design of the study is the focus of discussion in the next section, Section (1.7).

1.7 Study design

The conceptual framework outlined above in Section (1.6) indicates that a pre-service teacher's prior or existing mathematical knowledge is the product of cognitive processes which involve both learning experiences and the inherent capabilities of the mind (e.g. Derry, 1996). In order to inquire into this existing mathematical knowledge, a multiple-case study (Yin, 1994) or collective case study (Stake, 1994) design was used. The selection of the cases (or pre-service teachers) follows the 'replication logic' rather than the 'sampling logic' technique (Yin, 1994, p.45). That is, the set of selected cases is not viewed as a sample which is representative of the larger population. Rather, a set of cases is selected because it is believed that understanding these cases would lead to better understanding of a still larger collection of cases (Stake, 1994; Yin, 1994). This design also addresses the issues of validity and reliability through the use of the replication logic for case selection, and by the adoption of certain strategies for data collection and data analysis. These strategies are briefly described next.

In order to collect relevant data from each case, one-to-one interviews were conducted between the participant and the researcher/interviewer. A semi-structured interviewing approach was used in which the participant was required to respond to questions incorporated into mathematical stimulus items. The written responses and verbal responses recorded on audio tapes from these interviews provided the data for analysis.

The data analysis procedures involve identification of patterns of knowledge consistencies and inconsistencies displayed in the data. This identification procedure requires a set of base-line patterns for comparison and evaluation purposes. It is for such purposes that Skemp's (1978) model of mathematical understanding became indispensable to this study, especially in the formation of *predicted response patterns* for a base-line (Yin, 1994, p.106). Detailed descriptions of these patterns are presented in Chapters 3 and 4. However, for the purpose of the discussion in this section a brief description of Skemp's (1978) model is presented here.

Skemp (1978, 1979, 1982, 1986) argued that there are three kinds of mathematical understanding generated in schools (or in formal education institutions): *instrumental*, *relational*, and *symbolic*. These three kinds are in a hierarchy with *symbolic* at the highest level. Although all three kinds are characterised and distinguishable by knowledge types, there are basically only two kinds of learning goals in mathematics: *instrumental and relational*. Skemp (1979, 1982) claimed that the learning goal for symbolic understanding is based on schemata (or knowledge structures) for relational understanding. Furthermore, Skemp (1979) suggested that across these three kinds of mathematical understanding are two modes of cognitive functioning; namely, *intuitive and reflective*. This model is summarised in the matrix presented in Figure 1.2.

Figure 1.2: Predicted response patterns

Three Kinds of Mathematical Understanding			
Modes of Cognitive Functioning	Instrumental (I) Understanding	Relational (R) Understanding	Symbolic (S) Understanding
Intuitive (Int)	(Int, I)	(Int, R)	(Int, S)
Reflective (Ref)	(Ref, I)	(Ref, R)	(Ref, S)

Each cell of the matrix in Figure 1.2 represents a *predicted response pattern*, for example, (Int, I), (Int, R), and (Int, S) are response patterns associated with *intuitive* cognitive functioning. These six predicted response patterns were formed and validated for each of the three stimulus items (detailed in Chapter 4). These predicted patterns form the base-line of patterns for the analysis procedures. That is, each participant’s responses were examined for knowledge characteristics that may match with those of the predicted patterns. This comparison and matching process or ‘pattern-matching logic’ is a recommended method for analysing data associated with multiple-case study designs (Yin, 1994, p.106; Taylor & Bogdan, 1998).

It was expected that the result from the analysis of these response patterns would provide an insight into the pre-service teachers’ existing mathematical knowledge bases. More particularly, this result would provide indicators of the types of mathematical knowledge deficiencies and the effects of such types on knowledge pertaining to teaching or pedagogical knowledge. This expectation is the premise for the research questions described in the next section, Section (1.8).

1.8 Research questions

The two research questions investigated in this study of secondary pre-service teachers' existing mathematical knowledge are:

1. What types of procedural and conceptual mathematical ^{knowledge} exist in pre-service teachers' knowledge bases? _{like}
2. What possible influence could any identified deficiencies in types of mathematical knowledge have on the teaching of mathematics?

In order to form a theoretical base on which to explore these questions, it is important to define what are essential types of mathematical knowledge that constitute mathematical understanding and competence in teaching mathematics. This would include how these essential types of mathematical knowledge might be acquired, and what other types of mathematical knowledge might influence the achievement of mathematical understanding. This information is the focus of the discussion in Chapter 2. The content of the chapters in this document is summarised next.

1.9 Summary of the chapters

To explore further how mathematical knowledge is linked to teacher competence in mathematics and how such has been previously defined, a review of relevant literature is reported in Chapter 2. Also examined in Chapter 2 are the relevant studies associated with what constitutes

knowledge sufficiencies in mathematics. This information provides a comparative framework to assist in examining knowledge deficiencies.

In Chapter 3, a method for analysing response-data on mathematical knowledge is described. The design of the study and the development as well as validation of mathematical stimulus items for data collection are described in Chapter 4. This is followed by the analysis of the study data on secondary pre-service teachers' existing mathematical knowledge in Chapter 5. The discussion of this analysis is presented in Chapter 6. Finally, in Chapter 7 is the conclusion as well as implications arising from the study and recommendations for further research.

CHAPTER TWO

ESSENTIAL MATHEMATICAL KNOWLEDGE IN MATHEMATICAL UNDERSTANDING

Introduction

It was suggested in Chapter 1 that teachers' knowledge of mathematics influenced the way they teach. Despite their formal teacher education and classroom teaching experience, research findings indicate that a high proportion of mathematics teachers lack conceptual understanding of the mathematics they teach to their pupils (e.g. Leinhardt, 1989; Ball, 1990, 1991; Eisenhart *et al.*, 1993).

It was suggested in Chapter 1 (Section 1.3) that teachers' lack of conceptual understanding of mathematics may be the result of inadequate learning of mathematics prior to teacher education. Also in Section (1.3), it was suggested that teachers' pre-tertiary mathematical learning had far more influence on their mathematical development than learning from teacher education programs (Ball, 1990). Such findings tend to indicate that a possible reason for mathematics teachers' lack of understanding may be, that mathematics teachers go through teacher education and training with certain knowledge deficiencies in their mathematical understandings and that these will eventually affect the teacher's competence in teaching mathematics.

It was also suggested in Chapter 1 (Section 1.2) that teacher competence in mathematics involves both content knowledge of mathematics and pedagogical knowledge, and that pedagogical knowledge is dependent on mathematical knowledge. Hence, the focus of this literature review is on essential forms of mathematical knowledge that teachers need in order to gain mathematical understanding for teaching mathematics.

In order to explore what might be considered as mathematical knowledge deficiencies, how they were acquired, and how they might affect a person's competence to teach mathematics, the relevant literature on mathematical knowledge and understanding is examined in this chapter. This examination is discussed with particular emphasis on the following aspects: (i) identifying types of mathematical knowledge that are essential to mathematical understanding for teaching, (ii) defining mathematical competence in relation to the teaching of mathematics, (iii) defining *rote knowledge* as a form of mathematical knowledge, and (iv) identifying an aspect relating to the preparation of pre-service secondary mathematics teachers that needs further research.

2.1 Mathematical knowledge involved in mathematical understanding

To begin the examination of the types of mathematical knowledge involved in mathematical understanding, it is worthwhile to consider the question, what is knowledge? According to Gagné (1962, p.356), knowledge, by definition, 'is that inferred capability ... the individual possesses at any given stage in learning'. Ebel (1972) and Bruner

(1959), for example, would strongly suggest that this 'inferred capability' must be the outcome of thinking and not merely a collection of learned information. Ebel (1972, p.5) argued that 'knowledge is not synonymous with information'. Rather it is constructed out of information by thinking and it is an integrated structure of relationships.

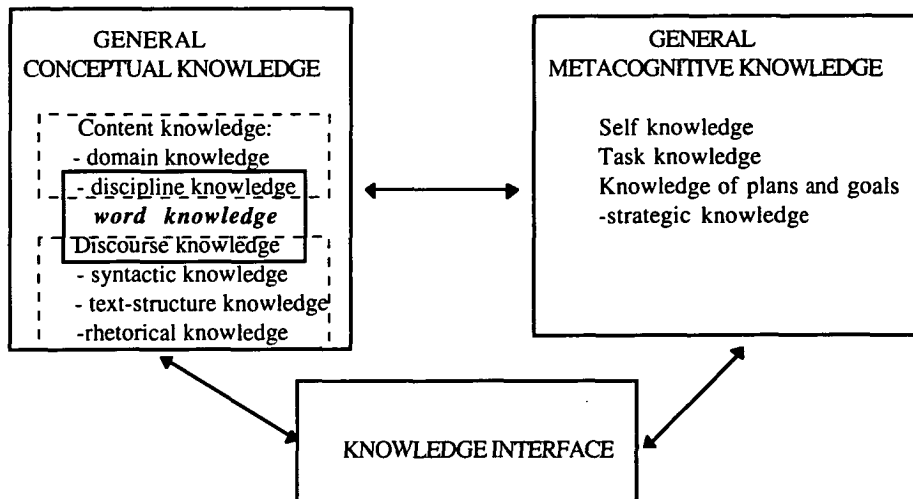
Knowledge, as suggested above, seems to be concerned with mental representations or cognitive processes as well as being a vital element in the formation of understanding. Contents of this mental representation include concepts and procedures which are closely interrelated and often interdependent (Anderson, 1980, 1981; Skemp, 1986; Gick & Holyoak, 1987; Alexander *et al.*, 1991). As such, it must pre-exist as *prior knowledge* or schema in order for it to have a major influence on meaningful reception, integration and retention of new concepts (Anderson, Spiro & Montague 1977; Anderson, 1982; Hiebert & Lefevre, 1986; Skemp, 1986; Kulm, 1994; Derry, 1996). Although this perspective of knowledge represents only one of several approaches for viewing knowledge acquisition (Reynolds, *et al.*, 1996), it does however provide a theoretical framework in which cognition or mental structures can be viewed.

It seems that *prior knowledge* is an important construct to consider in order to gain an understanding of mathematical knowledge. According to Alexander *et al.* (1991, p.324), a person's *prior knowledge* is composed of two general forms of knowledge that interact, namely *conceptual knowledge* and *metacognitive knowledge*. Conceptual knowledge involves *content knowledge*, *discourse knowledge* and their respective subcategories. That is, domain and discipline knowledge are

subcategories of *content knowledge* whilst syntactic, text-structure, rhetorical knowledge are subcategories of *discourse knowledge*. The overlap or link between content and discourse knowledge was suggested as *word knowledge*. Word knowledge according to Alexander *et al.* (1991, p.327) has two parts; the 'label' part which relates to discourse knowledge and the 'concept' part that relates to content knowledge.

Metacognitive knowledge, on the other hand, involves knowledge of self, task, strategy, and knowledge of plans and goals. Furthermore, these authors suggested that in order for an individual to undertake a task within a particular situation, these two forms (conceptual and metacognitive) of knowledge need to be activated and used by way of a *knowledge interface*. This knowledge interface is 'the point of contact between the learner's prior knowledge and other human processes' or a bridge between prior knowledge and external conditions (Alexander *et al.*, 1991, p.330). Derry (1996, p.168) refers to a similar form of knowledge interface as a 'cognitive field' schema. Figure 2.1 is a diagrammatic illustration of the general conceptual framework of prior knowledge suggested by Alexander *et al.* (1991).

Figure 2.1 **A general framework of prior knowledge**
 (Adapted from Alexander *et al.* (1991, p.324))



Legend:

word knowledge - represents the link between Content and Discourse knowledge
 ↔ - represents interactions between knowledge forms

Applying the above general description of *knowledge* to mathematics suggests that in the acquisition of mathematical knowledge (or domain knowledge), there is:

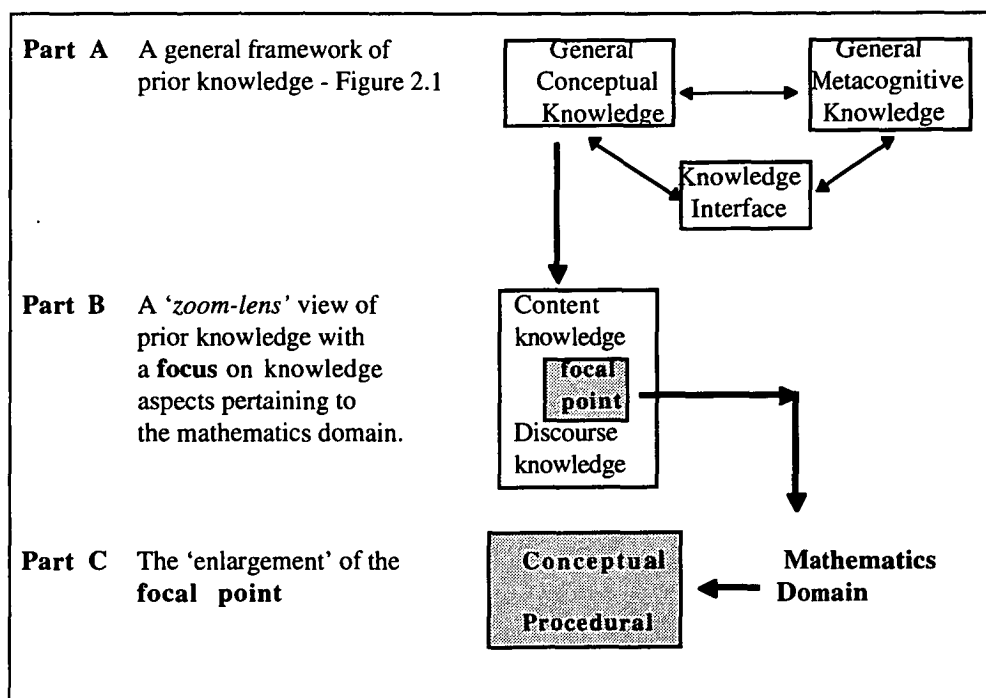
- (1) *an initial stage* - the collection of learned information about mathematics - (e.g. Ebel, 1972; Goodwin & Klausmeier, 1975; Skemp, 1986, 1989; Alexander *et al.*, 1991; Derry, 1996),
- (2) *an intermediate stage* - the formation of knowledge and relationships about mathematics - (e.g. Ebel, 1972; Skemp, 1986; Gick & Holyoak, 1987; Derry, 1996), and
- (3) *an extended stage* - the growth of knowledge in mathematics - (e.g. Ebel, 1972; Skemp, 1979, 1986; Hiebert & Lefevre, 1986; Derry, 1996).

An individual who enters a pre-service teacher education program would have been through these stages in the different levels of formal schooling - primary, secondary and tertiary. However, it appears that it is possible, at any school level, for an individual to acquire conceptual types of mathematical knowledge - the underlying knowledge factors in gaining understanding and achieving competence in mathematics (Skemp, 1986, 1989). For example, Leinhardt (1988) describes a competent primary school student of mathematics as one who can perform 'actions associated with tasks in the area quickly, accurately, flexibly, and inventively under several types of processing constraints, and he or she can explain what was done with reference to broad general principles and demonstrations' (p. 120). This description also seemed to apply to secondary and tertiary level students, for instance, Pask (cited in Entwistle & Ramsden, 1983) argued that such performances are demonstrations of a student's understanding of what has been learned, that is, 'when the student can explain the topic by reconstructing it, and can also demonstrate that understanding by applying the principles learned to an entirely new situation' (p.25).

However, in order to have mathematical understanding, it is said that an individual's existing mathematical knowledge base should consist of *procedural* and *conceptual* types of mathematical knowledge (e.g. Carpenter, 1986; Davis, 1986; Hiebert & Lefevre, 1986; Silver, 1986; Gick & Holyoak, 1987; Eisenhart *et al.*, 1993). The relationship between these two types was suggested to be a necessary element in mathematical understanding (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993; Pirie & Kieren, 1994). Ball (1990) has defined this relationship as substantive understanding of mathematics.

In an endeavour to form a theoretical basis for examining how procedural and conceptual mathematical knowledge may interact and generate mathematical understanding, a *zoom-lens* view of the framework for prior knowledge (Figure 2.1) is used for this purpose. The flowchart presented in Figure 2.2 is an attempt to pinpoint where procedural and conceptual knowledge specific to the mathematics domain might be situated in relation to the general framework of knowledge presented in Figure 2.1.

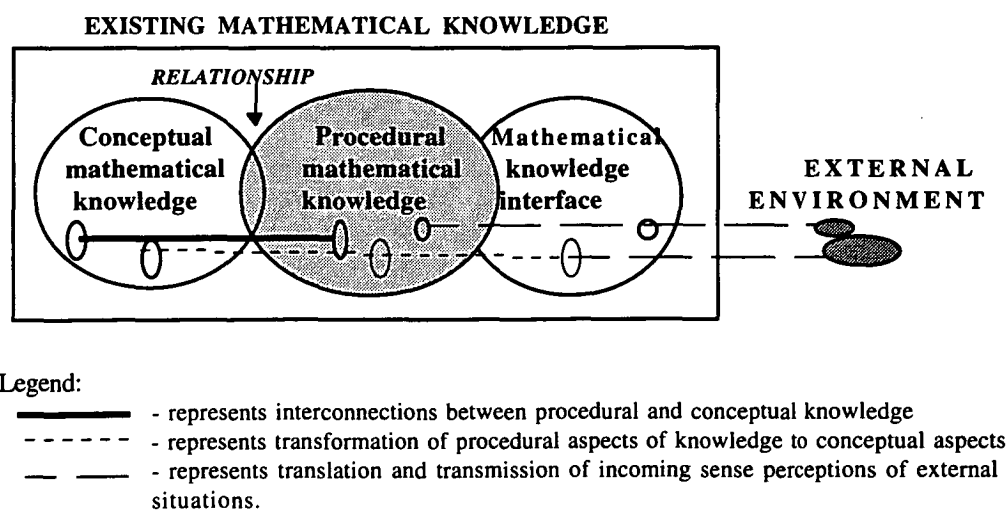
Figure 2.2 Focusing on procedural and conceptual mathematical knowledge



In Figure 2.2 above, it is assumed that both procedural and conceptual mathematical knowledge are components of the general form of *conceptual*

knowledge (Figure 2.1) since both types are required for mathematical understanding (e.g. Hiebert & Lefevre, 1986). This is represented by an arrow from the *conceptual knowledge* frame in part A to another frame in part B of Figure 2.2. To take a closer view of these components, a *zoom-lens* view, for example, of prior knowledge with a focus on procedural and conceptual knowledge aspects associated with the mathematics domain is illustrated in part B of Figure 2.2. It is further assumed that all the functional properties such as interactions between *conceptual knowledge* and *metacognitive knowledge* as well as connections to the *knowledge interface* of the general framework of prior knowledge (Alexander *et al.*, 1991) are retained and inherent in the ‘enlargement’ view of the focal point. This view is further enlarged and illustrated diagrammatically in Figure 2.3 and forms the basis for discussing the relevant literature on mathematical understanding.

Figure 2.3: The relationship between procedural and conceptual mathematical knowledge in mathematical understanding



The two types of mathematical knowledge associated with mathematical understanding illustrated in Figure 2.3 above are discussed further in the following section.

2.2 Conceptual and procedural mathematical knowledge

Conceptual knowledge of mathematics was suggested to be an essential type of knowledge in mathematical understanding. According to Hiebert and Lefevre (1986, p.4), 'growth in conceptual knowledge, the state of knowledge when new mathematical information is connected appropriately to existing knowledge' is mathematical understanding. Skemp (1978, 1986, 1989) defined this growth in mathematical knowledge as *relational understanding*. The type of mathematical understanding which is the product of 'knowing both what to do and why' (Skemp, 1978, p.9).

Conceptual mathematical knowledge or relational understanding (Skemp, 1978, 1979, 1982, 1986, 1989) refers to knowledge of the underlying structures of mathematics. These are the relationships and interconnections (illustrated by a bold line **—** in Figure 2.3) of ideas that explain and give meaning to mathematical procedures (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993; de Jong & Ferguson-Hessler, 1996). It is knowledge which is generally not taught specifically but is associated with the laws, axioms and theory of mathematics, for example, commutative, associative, distributive, and is derived from the students' learning experiences (Skemp, 1986; Leinhardt, 1988; De Corte, 1995; Derry, 1996; de Jong & Ferguson-Hessler, 1996). Conceptual mathematical knowledge is also understanding in the form of

representations, knowledge organisation of sets of propositions, sets of pattern recognition or a mixture of these (Gagné, 1985). Gagné's (1985) characterisation of conceptual knowledge seems to have similarities to the functional aspects of a 'mental model schema' described by Derry (1996, p.168) as a schema which is responsible for the 'process of constructing, testing, and adjusting a mental representation of a complex problem or situation'. Briefly, the goal of mental representations seems to be the construction of an understanding of a phenomenon (Gagné, 1985; Derry, 1996).

Procedural knowledge of mathematics, on the other hand, contains both knowledge of format and syntax of the symbol representation system, and knowledge of rules and algorithms, some of which are symbolic, that can be used to complete mathematical tasks (Eisenhart *et al.*, 1993; de Jong & Ferguson-Hessler, 1996). It is computational knowledge which is primarily numerical or symbolic and is derived from learning sets of procedures (Leinhardt, 1988; de Jong & Ferguson-Hessler, 1996). According to Gagné (1985), procedural knowledge involves computation skills, for example, addition and subtraction. It is knowledge of how to do things. It is more dynamic in that, when activated, the result is not simple recall but a transformation of information (illustrated by a dotted line --- in Figure 2.3). It is accurate and quicker to access than other forms of knowledge. Gagné added that procedural knowledge is used to operate on information to transform it into observable knowledge (illustrated in Figure 2.3 by broken lines — —).

Gagné's (1985) description of procedural knowledge appears to include knowledge types that Alexander *et al.* (1991) have classified as part of a

knowledge interface, particularly the types which are involved in 'the *instantiation* of conceptual knowledge' (Alexander *et al.*, 1991, p.330). The *instantiation* of conceptual knowledge, according to Alexander *et al.* (1991, p.331), occurs 'from the dynamic interaction of existing knowledge structures built on prior experiences with available information from on-going experiences'. In addition, it was suggested that through *instantiation*, individuals particularise the abstract representations, or understandings.

In schema theory, a similar kind of *instantiation* process is referred to as the 'cognitive field schema' (Derry, 1996, p.168). According to Derry (1996), the functions of the cognitive field schema are to 'mediate between experience and learning ... to determine what interpretations and understandings of experience are probable ... [and] also determines which previously existing memory objects and object systems can be modified or updated by an instructional experience' (p.168). It could be suggested from these processings that 'memory object' type schemata which are the 'basic component of stored human knowledge', appropriately describes procedural mathematical knowledge (Derry, 1996, p.167).

The functions of *memory object schemata* appear similar to those associated with procedural type knowledge described above by Gagné (1985) and Alexander *et al.* (1991). That is, memory object schemata allow individuals to recognise and classify patterns in the external world so that they can respond with appropriate mental or physical actions (Derry, 1996). In relation to mathematics, memory objects appear to be complex and structured and incorporate many types of knowledge including visual cue, set relations, mapping and planning procedures, and procedures for constructing numerical expressions (Marshall, 1995; Derry, 1996). Such

memory objects are said to be in a hierarchical structure in which there are lower order schemata or *intuitive schema*, an *integrated schema* and higher order type schemata or the *object family schema* (Derry, 1996, p. 167).

Using this hierarchical conceptualisation of *memory object schemata* as procedural mathematical knowledge, it could be suggested that these are types of knowledge involved in the initial and the intermediate stages of mathematical knowledge acquisition suggested earlier in Section (2.1). For example, the intuitive schema type is said to be relatively unproblematic and 'originates as minimal abstractions of common events' (diSessa, 1993, p105; Derry, 1996). Such minimal abstractions could represent a 'collection of information' required in the initial stage. The integrated schema type is said to be more structured than the intuitive types and appears to function as a *link-mechanism* for the various types of intuitive schemata. This link-mechanism could relate to the processes involved in the 'formation of knowledge' in the second stage.

It is proposed from this schema theory, that part of the functions of procedural knowledge as a *link-mechanism* is to (i) link memory objects in a memory storage (Gagné, 1985; diSessa, 1993; Derry, 1996); (ii) respond to incoming sense perceptions (Alexander *et al.*, 1991; Derry, 1996), and (iii) assist in the formation of new knowledge or memory object schemata (Hiebert & Lefevre, 1986; Derry, 1996). New knowledge could be formed by a process within the link-mechanism which recognises and classifies appropriate intuitive schemata and organises these into a single memory object or by integrating intuitive schemata with other memory objects already in the memory storage (Derry, 1996). This conceptualisation of procedural knowledge as a link-mechanism also seems to apply to higher

order type of memory object schemata or *object-family schema* suggested by Derry (1996). Derry (1996, p.167) characterises this object-family schema as 'loosely organised collections of ideas that tend to work together in certain types of situations'. Derry further suggests that these types 'activate one another and in some ways behave as single memory objects' (Derry, 1996, p.168). Such schema activities or behaviours could appropriately be inter-related and inter-connected by a knowledge link-mechanism proposed here to be mental functionings associated with procedural knowledge.

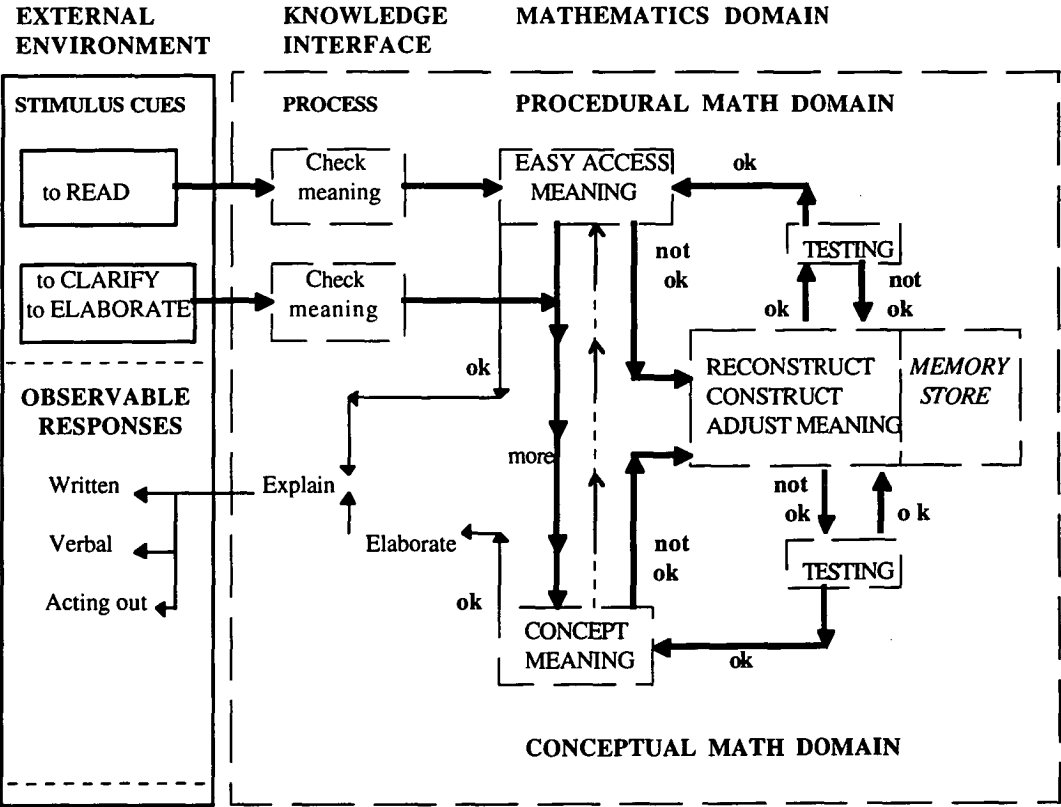
These knowledge interactions seem to appropriately describe those suggested in Figure 2.3 as interconnections between procedural and conceptual types of mathematical knowledge. These interactions include the transformation of procedural knowledge aspects into conceptual types and the translation and transmission of incoming sense perceptions from external situations. Since these knowledge interactions are part of procedural knowledge, hence the perceived 'overlap' of *procedural mathematical knowledge* onto *mathematical knowledge interface* as presented in Figure 2.3. Although Figure 2.3 depicts these mental processes in mathematical understanding in a simplistic way, it has, however, provided a conceptual structure upon which relevant literature can be evaluated and discussed.

In summarising to this point, it seems that procedural type mathematical knowledge is indeed complex and dynamic. Its functions of transformation, translation and transmission of knowledge suggests that it is the *power-house* for the mind for mathematical understanding. Mathematical understanding or the relationship between conceptual and procedural types of mathematical knowledge, seems to depend on the mental functions associated with procedural knowledge types. Although conceptual types of knowledge are

considered important for giving *meaning* to procedural types of knowledge (e.g. Eisenhart *et al.*, 1993; de Jong & Ferguson-Hessler, 1996), it seems that part of this *meaningful* process is also partly generated by procedural types, for example, higher order mental object schemata (Derry, 1996), and in some cases procedural types can provide *meaningful* interactions or ‘particularise the abstract representations, or understandings’ (Alexander *et al.*, 1991, p. 331). Mathematical understanding, therefore, is a dynamic and complex relationship between conceptual types and procedural types of mathematical knowledge.

The purpose of this literature review so far was to identify types of knowledge involved in mathematical understanding. Having examined this, it is important to synthesise these various aspects about mathematical knowledge types and characteristics into a model for the purpose of analysing responses to a mathematical stimulus. This model is presented in Figure 2.4 (next page). Using the same ‘zoom-lens’ idea for a close up view as with *prior knowledge* in Figure 2.2, the model in Figure 2.4 is a ‘close-up’ view of the mental processes involved in mathematical understanding (illustrated in Figure 2.3). Therefore, the tentative modelling of a *response production of mathematical knowledge* (Figure 2.4) is an attempt to illustrate diagrammatically the mental processes, or procedural knowledge, involved in the production of an observable response. These were the processes, illustrated in Figure 2.3, of interconnections, transformation, translation and transmission of incoming sense perceptions from external situations. As in Figure 2.3, the three main components of this model are the *external environment*, *knowledge interface*, and the *mathematics domain* which is composed of a ‘procedural math domain’ and a ‘conceptual math domain’.

Figure 2.4: Modelling a response production of mathematical knowledge



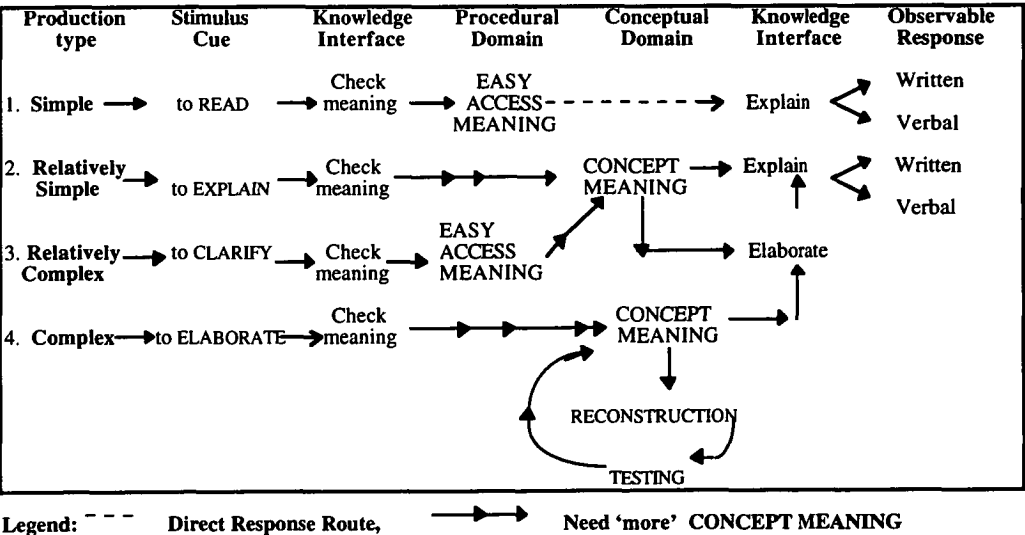
The production of an observable mathematical response, suggested by the modelling of a *response production of mathematical knowledge* (Figure 2.4), involves the inter-connection and inter-relationship of the functions of the three components, namely *external environment*, *knowledge interface*, and the *mathematics domain*. The following are the proposed sequence of events that might occur in a response production:

- (1) The incoming sense perceptions from a *stimulus cue* in the external environment are first processed at the *knowledge interface*. This initial processing, denoted as 'check meaning', would involve the 'instantiation of conceptual knowledge' or a dynamic interaction of existing knowledge structures built on prior experiences with available information from on-going experiences (Alexander *et al.*, 1991, p.331).
- (2) As a result of 'check meaning', the *mathematics domain* is activated at the *procedural math domain*. It is proposed that certain stimulus cues such as 'to read' would activate 'easy access meaning' and other cues such as 'to explain' or 'to clarify' would require 'more' meaning or meaning associated with concepts, denoted as 'concept meaning'.
- (3) At the *mathematics domain*, it is proposed that the *procedural math domain* is the first 'port of call' and the main *link mechanism* for the mental processings involved in producing a response of mathematical knowledge. However, the *conceptual math domain* is also involved in the reconstruction, justification, and testing processes. The processes involving the *conceptual math domain* or 'concept meaning' would be the most desirable for producing 'mathematical understanding'.
- (4) As a result of processings at the *mathematics domain*, it is proposed that an observable response is processed at the *knowledge interface* in either (a) as an outcome from 'easy access meaning' or (b) as

an outcome from ‘concept meaning’. The single outlet point, denoted as ‘explain’, marks the final processings of translation and transformation prior to transfer into observable outcomes (e.g. verbal and written).

For the purpose of describing the production of an observable response, four types of productions are proposed: *simple*, *relatively simple*, *relatively complex*, and *complex*. Each production type is dependent on the stimulus cue and the process routes needed for an observable response. Examples of each type of production is presented in Figure 2.5 below.

Figure 2.5 **Examples of response production pathways**



It is implied by these four examples of response production pathways that knowledge processings and, hence, the observable outcomes are significantly influenced by the stimulus cues. For example, a stimulus cue ‘to read’ may only activate *easily accessible* knowledge from the ‘procedural domain’, hence a *simple* or a more straightforward processing route to an observable

response is produced (e.g. pathway 1). On the other hand, stimulus cues such as 'to explain', 'to clarify', and 'to elaborate' may activate *concept meanings* in the 'conceptual domain' first, via the 'procedural domain' in search for more meaning, prior to an observable response (e.g. pathways 2 & 3). Such cues would also activate a *complex* processing route in which the processes of 'reconstruction', 'justification', and 'testing', as well as the process of 'elaborate' at the *knowledge interface* domain are required prior to an observable response (e.g. pathway 4). Many more production pathways can be generated, particularly if 'reconstruction' and 'testing' process routes are required as illustrated by the production of a *complex* response (pathway 4). However, for observable responses which are demonstrations of mathematical understanding, it is proposed, and highly desirable, that at least a *relatively simple* production (as illustrated by pathway 2) is produced. This modelling of response production of mathematical knowledge forms the conceptual framework for the design of the data collection instrument and the basis for the data collection procedures for this study (Chapter 4).

Nevertheless, in relation to prospective mathematics teachers, it has been suggested that their existing mathematical knowledge bases should be composed of essential aspects of conceptual and procedural types of mathematical knowledge (Hiebert & Lefevre, 1986). For example, a prospective mathematics teacher's mathematical knowledge base should be: broad, sufficient, with in depth conceptualisation of mathematical content (Skemp, 1989; Ball, 1990; Leinhardt *et al.*, 1991; Eisenhart *et al.*, 1993); explicit and involves thinking and making relationships (Ball, 1991; Eisenhart *et al.*, 1993), as well as having understanding based on modern and current conceptions of mathematics (Even, 1993). Having such knowledge

bases would promote the production of *relatively complex* and *complex* types of observable responses (Figure 2.5).

In relation to mathematical competence, Hiebert and Lefevre (1986) suggested that 'students are not fully competent in mathematics if either kind of knowledge [procedural and conceptual] is deficient or if they both have been acquired but remain separate entities' (p.9). It appears that these two types of mathematical knowledge are not only essential to mathematical understanding but are also necessary for the achievement of competence in mathematics. This competence is discussed further in the next section.

2.3 Mathematical competence

It was suggested in Section (2.2) that mathematical competence is closely connected with mathematical understanding and it is therefore not static. Rather it represents growth in relation to a specific domain (mathematics) and comprises evidence of a knowledge base that is increasingly coherent, principled, useful, and goal oriented (Glaser, 1991). This evidence can be described in terms of an individual's ability to demonstrate what he or she understands of mathematics.

Such a demonstration of understanding usually involves a certain quality of knowledge the individual has about mathematics. In addition to this quality of knowledge, competence also appears to relate to how the individual has internalised this knowledge. McAshan (1979), a proponent of this view, also emphasised the importance of understanding to distinguish this notion of competence from behaviours which are based on rote memorisation. He maintained that a competent person is one whose achievements have

‘become part of his or her being to the extent he or she can satisfactorily perform particular cognitive, affective and psychomotor behaviours’ (McAshan, 1979, p. 45).

Having mathematical competency is defined here as the observable outcome or performance by an individual to demonstrate what he or she understands about a topic in mathematics. This competence is based on mathematical understanding and it is characterised by the achievement of specified criteria at a satisfactory standard (McAshan, 1979; Leinhardt, 1988). Another shared characteristic of competence is that the quality of knowledge, understanding and skills are content and context specific. This characteristic of competence has been referred to by Biggs and Collis (1991) as largely accountable for by the acquisition of a well structured knowledge base that bears directly on the task at hand. In teaching, this knowledge base contributes to and ‘plays a significant role in grounding professionalism’ (Donmoyer, 1996, p.98).

In mathematics teaching, Shulman (1986) argued that a teacher’s knowledge pertaining to competence involves: subject-matter (mathematics) content knowledge, pedagogical content knowledge, and curriculum content knowledge. However, pedagogical knowledge and curriculum knowledge in mathematics are dependent on the teacher’s subject-matter (mathematics) knowledge (Ball & McDiarmid, 1990; Leinhardt *et al.*, 1991; Grouws & Schultz, 1996). This dependency is seen by the way in which pedagogical knowledge is defined. For example, pedagogical content knowledge is content knowledge that is useful for teaching (Grouws & Schultz, 1996). As such, Ball and McDiarmid (1990, p.437) have argued that “the myriad tasks

of teaching, such as selecting worthwhile learning activities, ... , all depend on the teachers' understanding" of the subject-matter they teach.

Pedagogical knowledge of mathematics is used here to mean mathematical competence in teaching or the teacher's ability to teach mathematics. That is, the ability: to explain; relate mathematical ideas and procedures; present the mathematics content appropriately and in multiple ways; interpret and appraise students; challenge incorrect notions; flexibly respond to student's questions; extend, as well as formalise intuitive mathematical understanding (e.g. Shulman, 1987; Berliner *et al.*, 1988; Ball, 1990, 1991). Also, the ability to attend simultaneously to the mathematical content and to students' reasoning and understanding of mathematics (Leinhardt *et al.*, 1991), and to teach for conceptual understanding of mathematics (Eisenhart *et al.*, 1993). These abilities are representations of cognitive processes of organisation and integration of knowledge from a well structured mathematical knowledge base (Gagné, 1985; Skemp, 1986; Biggs & Collis, 1991). Hence the importance of focusing on mathematical knowledge in an endeavour to gain a measure of a pre-service teacher's potential to achieve competence in teaching mathematics.

In summarising this section, it is suggested that in order for a prospective teacher to have the competence to teach mathematics at the secondary school level, it is important that he or she must acquire conceptual understanding of the various areas (e.g. measurement, space, function, and statistics) of mathematics and that such understanding is founded on a well structured mathematical knowledge base. In addition, a prospective teacher should have the ability to transform and transfer this knowledge (D.E.E.T., 1992),

and the capacity to internalise this knowledge such that it becomes part of his or her being (Ebel, 1972; McAshan, 1979).

The discussion in this section tends to suggest that, when essential mathematical knowledge aspects associated with the understanding of mathematics are lacking, the learner is unlikely to display competent learned outcomes. It is assumed that such learned outcomes are influenced by an insufficient mathematical knowledge base. It was suggested earlier by Hiebert and Lefevre (1986) that if either procedural or conceptual mathematical knowledge is deficient then this could lead to a lack of mathematical understanding. However, there is another type of mathematical knowledge which appears to have a significant affect on a student's understanding of mathematics. This knowledge is the focus of the following discussion.

2.4 Rote knowledge and mathematical competency

In the previous sections, procedural and conceptual mathematical knowledge were described as two types of knowledge in a mathematical knowledge base. These two types were claimed to be essential in mathematical competency (Hiebert & Lefevre, 1986). One other seemingly important type of knowledge which is commonly associated with mathematics achievement, is knowledge acquired from rote learning. Rote learning concerns the replication or rote memorisation of an external stimulus pattern rather than learning towards an autonomous conceptual understanding (Buxton, 1978; McAshan, 1979; Silver, 1986; Skemp, 1986; von Glasersfeld, 1991). The type of knowledge acquired from such learning is distinguished from procedural and conceptual mathematical knowledge in this thesis by the use of the term

‘rote’, and hereafter is referred to as *rote knowledge*. What is rote knowledge and where does it fit into the two type topology of mathematical competency described in the previous discussion? An attempt to address these two questions is the aim of this section.

Traditionally, knowledge acquired from school learning was measured by the amount of reproduced facts and figures that were learned through memorisation strategies such as recital and continuous practice (Biggs & Moore, 1993; Husén & Postlethwaite, 1994). Although understanding the learned materials was not necessary with this repetitive type of learning, such learning strategies were, and are currently, considered essential and beneficial in developing mathematical knowledge and skills (Brownell, 1956; Tobin, 1986), particularly in the early years of schooling (Reys, Suydam, & Lindquist, 1992). Such methods were also reported to be effective in producing successful outcomes in other areas of learning such as spelling and reading (Moyle, 1972). However, from a cognitivist’s perspective such rote learned outcomes are behavioural or habitual responses rather than knowledge produced from learning, since they argue that learning is self-determined and is actively constructed by the learners themselves and not by their teachers (Ebel, 1972; Skemp, 1986; Biggs & Moore, 1993; De Corte, 1995; Derry, 1996; Reynolds *et al.*, 1996). In more recent studies, learned outcomes from such learning methods are considered as metacognitive knowledge and skills. For example, according to De Corte (1995), ‘metacognitive knowledge includes knowing about the strengths as well as the weakness and limits of one’s cognitive capacities, for example, being aware of the limits of short term memory and knowing that our memory is fallible but that one can use aids (e.g. mnemonics) for retaining certain information’ (p.38). However, Rakow (1992) and Gadanidis (1994) would still argue that mathematical knowledge

acquired by memory alone without understanding or thought, that is by rote, is the direct opposite of meaningful learning as in Ausubel's (1968) theory of learning.

Furthermore, cultural factors may influence the use of rote learning as a strategy. For example, Kember and Gow (1991) suggested from their study of Asian students' reproductive approaches to learning that such may be more a function of teaching practices than an innate tendency. In another study, Asian students gained higher achievements and displayed a more in depth approach to learning tasks than their Australian counterparts (Marton, Dall'Alba & Tse, 1993). This study tends to contradict the notion that rote learning produces limited understanding. According to the Culture Action learning theory, learning occurs 'through shared symbolism (e.g. language) and other practices of a social group' (Walker, 1987, p.12). If such is the case, then what these findings could indicate, is that, Asian teachers may be teaching rote learning strategies as part of a cultural practice.

Two main issues seem to emerge from the above discussion concerning *rote* type knowledge from rote learning. One appears to relate to teaching strategies or external influences of teacher instructional methods (e.g. memorisation and repetitive strategies) on the learner's approach to learning (e.g. Brownell, 1956; Kember & Gow, 1991). The other issue concerns the cognitive processing or self-regulated and self-determined learning by the individual in pursuit of understanding (e.g. De Corte, 1995). This pursuit for understanding may involve metacognitive knowledge and other efforts by the learner. For example, Biggs and Moore (1993, p. 21) argued that efforts by the learner to gain understanding should not be considered as rote learning but 'a means towards acquiring understood and usable knowledge'.

To further clarify the difference between usable rote knowledge (Biggs & Moore, 1993) and habitual learned outcomes in mathematics, consider, for instance, Skemp's (1978, p.9) idea of 'instrumental understanding' or learning mathematical algorithms and 'rules without reasons'. According to Skemp (1978), instrumental understanding describes the kind of knowledge acquired by a learner who was dependent on external guidance (e.g. teachers and textbooks) for learning a new set of given information. An example of instrumental understanding is the rote memorisation of a mathematical formula (e.g. *negative times a negative is equal to a positive*) without understanding why the formula works (Gates, 1995a). Although such form of understanding may just be a collection of learned information rather than knowledge (Ebel, 1972), Brownell (1956) argued that there is a proper place for this habitual learning.

According to Brownell (1956, p.136), the proper time to provide repetitive or habitual learning 'to assure real mastery of skills, real competence in computing accurately, quickly, and confidently', is after the learner has achieved understanding of the learned material. Goodwin and Klausmeier (1975, p.242) also expressed a similar view by suggesting that memorising bits of knowledge might seem pointless but 'without either having the knowledge or being able to obtain it when needed, the individual has nothing to apply or to evaluate'. Could this kind of knowledge be the same as that referred to earlier as a lower order schema or *intuitive schema* (Derry, 1996) or a minimal abstraction of common events (diSessa, 1993)? A possible answer to this question is deferred until Chapter 6.

However, to answer the questions posed earlier about what is *rote knowledge*, it seems that mathematical learned outcomes which are the result of rote learning (or a collection of learned information) are considered as behavioural or habitual responses and may not promote or facilitate the kind of understanding being defined in this thesis as mathematical understanding (Section 2.2). However, there is a possibility that with special attention by the individual learner, this type of mathematical knowledge (rote knowledge) may be transformed into usable knowledge or into an essential type of procedural knowledge (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993), for example, an intuitive schema type (Derry, 1996).

For the purpose of the study reported in this thesis, rote knowledge pertaining to mathematics is referred to as 'usable knowledge' in the same manner that Biggs and Moore (1993, p. 21) defined it, or 'factual knowledge' (Goodwin & Klausmeier, 1975, p.242) to provide stepping-stones for acquiring other knowledge of mathematics. For example, rote memorisation of a formula (e.g. the formula : $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$) could provide a basis for the student to revisit the learning of quadratic functions ($y = ax^2 + bx + c$) and to explore how the formula is related to solving quadratics in other areas (e.g. trigonometry).

Having identified and described the types of knowledge associated with mathematical competence, it is important to consider next how prospective mathematics teachers could be assured of gaining mathematical competence prior to teacher employment.

2.5 Mathematical knowledge versus mathematical environment

The discussions in previous sections provided some insights into procedural, conceptual, and rote mathematical knowledge and the importance of these types of knowledge to mathematical understanding and competence in teaching mathematics. These insights have raised two key questions with respect to prospective teacher's background knowledge of mathematics: (i) At what point are the student-teachers expected to acquire these knowledge types? Is it prior to enrolling in teacher education programs or during teacher education? and (ii) What influence would these knowledge types have on the student-teachers' conceptual understanding of mathematics? The former question is considered first.

At what point are the student-teachers expected to acquire conceptual, procedural and rote mathematical knowledge? To address this question, it is worthwhile to consider the period when the student-teachers enter a pre-service teacher education program. The secondary mathematics pre-service teacher education period in Australia, often only one year duration in a traditional B.Sc Dip. Ed model, is a transition stage for the prospective teachers from being students to becoming classroom teachers (a role reversal). Whilst before, as students, the prospective teachers were learning and studying to meet their own goals, now they have to also consider how to adjust, integrate and reconstruct their prior learning into teachable knowledge in order to include the goals of others - namely the students they intend to teach.

At this transition stage the main source of knowledge about mathematics and teaching is the individual's own knowledge from prior learning experiences. This knowledge source includes learning from the 12 years of pre-university

studies and two or three years of university studies (Ball & McDiarmid, 1990). The prospective teachers' perceptions of what they believe to be important knowledge and ways of teaching mathematics have also been developed and shaped by their earlier learning experiences (McDiarmid & Ball, 1989).

What influence would this prior knowledge have on the student-teachers' conceptual understanding of mathematics? Several researchers found that the quality of the student-teachers' prior mathematical knowledge is fundamental to its reconstruction into teachable (pedagogical content) knowledge (Shulman, 1987; Skemp, 1986, 1989; Ball, 1990; Ball & McDiarmid, 1990; Even, 1993; Wilson, 1994; Gates, 1995a, 1995b). However, researchers (e.g. Ball, 1990; Even, 1993; Gates, 1995a) also found that prospective teachers of secondary level mathematics enter teacher education with inadequate mathematical knowledge, particularly the conceptual type of mathematical knowledge. In order to provide prospective teachers with the necessary knowledge and skills, Even (1993) suggested that they 'need to have environments that foster powerful constructions of mathematical concepts' (p.113).

Powerful constructions, however, require relational understanding (Skemp, 1978, 1986) or relational thinking (Van Hiele, 1986) and substantial knowledge of mathematics (Ball, 1990). In addition to substantial knowledge of mathematics, students' beliefs and approaches to learning can also influence the way they acquire and construct knowledge (Entwistle & Ramsden, 1983; Marton, 1988; Schmeck, 1988; De Corte, 1995). For example, Entwistle and Ramsden's (1983) studies of university students showed that poor background knowledge (especially of concepts in science

and mathematics) was associated with surface (as opposed to deep) approaches of learning. So although appropriate teacher education environments are provided, the prospective teachers may lack even the basic concrete knowledge for them to appreciate the learning they received.

Being unappreciative of the learning one receives is not an uncommon behaviour from students of mathematics (Skemp, 1986; De Corte, 1995). Skemp (1986, p.30) has attributed such behaviours by learners of mathematics to the 'abstractness and generality' of mathematics. He suggested that this behaviour is particularly common among students who are required to learn higher-order concepts (as in the case of prospective mathematics teachers), whilst the necessary lower-order concepts needed for mathematical understanding are lacking. One way for students to gain the required lower-order concepts to enable the achievement of higher-order ones, is by having the assistance of competent mathematics educators (or mathematicians), in conjunction with one's own reflective intelligence (Ebel, 1972; Skemp, 1986; Gates, 1995a, 1995b).

Other researchers (e.g. Ball, 1990; Leder, 1991; Eisenhart *et al.* 1993; Fennema, 1996) have also acknowledged the importance of a mathematics educator in facilitating students' learning of mathematical concepts. This also applies to mathematics teacher educators since they are the key persons in the preparation of mathematics teachers for secondary schools. Teacher educators can provide appropriate environments and opportunities for student-teachers to construct and reconstruct their mathematical knowledge in readiness for teaching.

From the above discussion it appears that in order to provide prospective teachers with appropriate mathematical environments, the mathematics teacher-educator must take into consideration two essential factors: (i) the student-teachers' differences in background or prior knowledge, and (ii) the differences in learning approaches the student-teachers have and will use in acquiring knowledge. In addition, the possibility that learning approaches and knowledge acquisition may be influenced by the person's gender needs consideration. This possibility of gender influence is discussed in the next section.

2.6 Mathematical knowledge and gender

It was suggested in Section (2.5) above that mathematics teacher educators, in providing appropriate environments for student-teachers, would need to consider the differences in prior knowledge and learning approaches student-teachers will bring with them to mathematics teacher education programs. An important question associated with student-teachers' differences is: Does gender have an affect on student-teachers' acquisition of mathematical knowledge pertaining to mathematical teacher competence? In an endeavour to find some answers to this question, the relevant literature on gender associated with mathematics is examined in this section.

A comprehensive summary of previous research findings on gender in mathematics education by Hanna, Kündiger and Larouche (1990, p.86) indicated that gender differences first appear at adolescence and that 'boys are significantly superior to girls in both their mathematical performances and their attitudes towards mathematics'.

To clarify what this 'mathematical performance' might involve, Fennema (1993, video) suggests that the overt participation in learning activities, which teachers, educators and researchers observe, is only part of the mental processing that an individual brings to learning the activity. Other factors such as confidence and interest in the activity all affect the quality of a person's participation. Fennema and Peterson (cited in Fennema & Leder, 1993) argued that in order for individuals to show that they are developing their abilities to work independently in high-cognitive-level activities they must demonstrate autonomous learning behaviours. Autonomous learning behaviours were defined by Fennema and Peterson (Fennema & Leder, 1993) as the active and willing participation by an individual in mathematical tasks that require knowledge and independent thinking. It is claimed that autonomous learning behaviours appropriately model 'the processes through which individuals construct their own knowledge' (Fennema & Leder, 1993, p.7). Using this modelling, a 'mathematical performance' could be viewed as an observable outcome of mental processing of knowledge involving conceptual or relational understanding of mathematics (Skemp, 1978).

Recent research findings by Fennema and Carpenter (1998) suggest that gender differences in mathematical performances may occur as early as grade 3 level, rather than at adolescence as previously thought (Hanna *et al.*, 1990). However, the observed gender differences in these findings were not associated with the children's ability to solve problems, rather with how the children approached the solving of problems. It was reported that 'girls tended to use concrete solution strategies like modeling and counting, and boys tended to use more abstract solution strategies that reflected conceptual

understanding' (Fennema & Carpenter, 1998, p.4). These 'abstract solution strategies' were described as 'invented algorithms [which] usually are generated by children, either individually or [as] interactions with other students' and were contrasted to 'standard algorithms, which generally are learned by automatizing a series of specified procedures' (Fennema & Carpenter, 1998, p.6).

In response to Fennema and Carpenter's (1998) findings, Sowder (1998) showed concern for the lesser degree of conceptual understanding in mathematics that females achieve in earlier years, and suggested that this may lead to learning difficulties and lack of self-confidence in later years. However, Sowder's (1998, p.12) deepest concern was that teachers, in trying to assist females to learn and to gain self-confidence in mathematics, may decide to use 'a more traditional style of teaching, where emphasis is placed on rote learning of rules, [this] may tend to better equalise the advantages of the girls and the boys'. She added that, 'both sexes will then be disadvantaged' (p.12).

Sowder's concerns highlight the importance of having competent mathematics teachers who ensure that children are learning in a collaborative and non-competitive environment in which active cognitive mathematical thinking is promoted (Burton, 1993, video). Further support for minimising gender differences in mathematical performance by students comes from data generated by Fennema's (1996) professional development program - Cognitively Guided Instruction (CGI) -for elementary teachers. Fennema (1996) reported, that when the teachers make instructional decisions based on their mathematical knowledge and understanding of children's thinking, overall, gender differences in mathematical performances by the children

were not found. However, Rhine's (1998) review of teacher professional development programs, including the CGI program, in the United States, particularly the Integrating Mathematics Assessment project, indicated that the same models which help teachers to teach children in the lower grades to achieve understanding of mathematics, may be difficult to implement for the upper grades 'as content in higher grades becomes more complex' (p. 29).

Nevertheless, due to the limited research specifically addressing gender and mathematics in the area of teacher education, particularly from the 'cognitive science' perspective, it is not possible to confirm that gender has an affect on student-teachers' acquisition of mathematical knowledge pertaining to mathematical teacher competence. This shortage of research is acknowledged by Fennema (1996) in the following quotation:

'Cognitive science research ... [provides] insights into teachers' behaviours, knowledge, and beliefs, although little has been done relating to teacher's cognitions about gender. Such studies may lead to deeper understanding of gender differences in mathematics ... and how it influences daily decisions about mathematics. Unfortunately, there are not many studies related to gender that have been done using this perspective [the cognitive science perspective]'. (Fennema, 1996, p.17).

One could surmise from the small quantity of research evidence available, that because knowledge acquisition, particularly at the tertiary level, is so much interwoven with students' career goals, interests, and beliefs, a more accurate measure of any gender differences in student-teachers' existing mathematical knowledge could be achieved by focusing specifically on the types of procedural and conceptual mathematical knowledge they acquired from prior learning.

From the above discussion, there appears to be two distinct types of teaching approaches which could minimise gender differences in mathematics: (1)

the use of traditional style teaching where more emphasis is placed on rote learning rules (Sowder, 1998), and (2) the types of teaching which are based on teachers' knowledge and understanding of students' thinking (Fennema, 1996).

In relation to the preparation and education of pre-service mathematics teachers, it appears that the second teaching approach stated above is applicable to mathematics teacher educators. Particularly, for mathematics teacher educators to have research-based knowledge and understanding of the mathematics that student-teachers bring with them to teacher education (Rhine, 1998). Having information about what student-teachers know of mathematics would assist the mathematics teacher educators in providing appropriate learning environments for pre-service teachers. Fennema (1993), however, would argue that such appropriate environments should foster autonomous learning of higher-order mathematics by the pre-service teachers, particularly the prospective female teachers.

From a cognitive science perspective, the possibility of conflict between a student-teacher who approaches learning instrumentally and a teacher-educator (or lecturer) who teaches relationally or visa versa is high (Skemp, 1978). Such conflicts are inevitable because of the current thinking in teacher education to equip prospective teachers with conceptual understanding of mathematics and to encourage them to teach for relational understanding (e.g. Ball & McDiarmid, 1990; Lietzel, 1991; Eisenhart *et al.*, 1993; Gates, 1995a).

2.7 Extending previous research

The conflict stated above between the mathematics teacher-educator and the student-teachers raises the question: Should prospective teachers be forced to change the way they learn or acquire knowledge? According to Entwistle and Ramsden (1983), 'we [educators] should not try to change a student's learning style, except ... when it is creating serious difficulties for the student' (p.206). However, the authors added that tertiary students (including prospective teachers) would benefit from guided assistance, by lecturers, towards becoming aware of their characteristic style and from showing them how they may most effectively capitalise on their intellectual strengths.

For pre-service teachers of secondary mathematics, Ball (1990, p.465) has highlighted the 'need to know much more than we currently do about how [secondary prospective] teachers can be helped to transform and increase their understanding of mathematics'. One way, she suggested, is by 'working with what they bring [to teacher education] and helping them move toward the kinds of mathematical understanding needed in order to teach mathematics well'. Ball's (1990) suggestion stems from her findings which indicated the low levels of mathematical understanding among the secondary pre-service teachers she observed.

The question then is not who is to change, the student-teacher or the teacher-educator, but how prospective teachers could be helped, by the teacher-educator, to adjust and to integrate new information from teacher education courses so that it becomes part of their own knowledge base.

To facilitate a mathematics teacher-educator in working and helping pre-service teachers toward mathematical competence, a more refined characterisation of the possible mathematical knowledge deficiencies acquired by prospective mathematics teachers would be valuable in determining the best or most suitable method of assistance.

Therefore, the study reported here was organised to find out more about mathematical knowledge deficiencies that secondary pre-service teachers bring with them to teacher education.

2.8 Summary

The purpose of this chapter was to examine the relevant literature on mathematical knowledge in an endeavour to identify the types of knowledge that are essential to mathematical understanding and competence in relation to teaching. This examination was necessary in order to provide a conceptual framework of knowledge in which mathematical knowledge deficiencies could be explored and analysed.

Conceptual and procedural mathematical knowledge were identified as essential knowledge types required for mathematical understanding and that the close relationship or interconnections between these two types is necessary for both mathematical understanding and competence. In short, mathematical competence, or the individual's ability to demonstrate proficiency and accuracy in mathematical understanding, is dependent on one having a well structured mathematical knowledge base. Furthermore, the ability to transform and to transfer mathematical knowledge to

teachable knowledge (or pedagogical knowledge) was defined as mathematical teacher competence.

Mathematical knowledge acquired from the rote memorisation of rules and algorithms was defined as rote mathematical knowledge. As such it was assumed that this type of knowledge is internalised by the learner as isolated bits of information with little or no linkages to conceptual forms of knowledge. It was suggested that without these essential linkages rote knowledge may not facilitate understanding unless the learner can, with special attention, transform it into a usable type of knowledge. This type of knowledge is different from the procedural type of mathematical knowledge - knowledge of rules and algorithms for computations and for completion of mathematical tasks - in that procedural knowledge is closely linked to conceptual types of knowledge.

Notwithstanding the aspects of knowledge required for competence by the prospective teachers, other variables (e.g. learning approaches and gender) have been highlighted as factors having the potential to influence the prospective teacher's learning for mathematical teaching competence. For example, gender could affect teaching practices. Such gender affect, however, could be minimised by having teacher educators who can integrate their knowledge and understanding of student-teachers' thinking into their teaching instructions and strategies.

From the reviewed literature, there was evidence of a shortage of research on the mathematical understanding of pre-service teachers of secondary mathematics. Also there was indication of the need for mathematics teacher educators to learn more about how pre-service teachers of

secondary mathematics can be helped to transform knowledge insufficiencies and increase their mathematical understanding. To extend the previous research on the education of secondary mathematics teachers, two research questions were addressed.

The two research questions investigated in this study of secondary pre-service teachers' existing mathematical knowledge were:

1. What types of procedural and conceptual mathematical knowledge exist in pre-service teachers' knowledge bases?
2. What possible influences could any identified deficiencies in procedural and conceptual types have on the teaching of mathematics?

However, before describing the methods and procedures by which these questions were explored, it is important to consider the theoretical perspective selected as a relevant model for examining and evaluating mathematical knowledge for the study reported in this thesis. In order to show how the selected model incorporates the views on mathematical understanding discussed in this chapter, a brief outline is provided below.

In Sections (2.1) and (2.2) of this chapter, it was argued that an individual's mathematical understanding is dependent on the interrelationship of two types of mathematical knowledge, procedural and conceptual. This interrelationship is illustrated in Figure 2.3.

Mathematical competence was suggested to be the observed outcome of the sufficiencies of both types of mathematical knowledge. In addition, it was suggested in Section (2.4) that rote knowledge, or mathematical knowledge acquired by rote memorisation strategies, may also exist in the

individual's knowledge base but as such it may not facilitate mathematical understanding.

The selected model, Skemp's (1978) model of mathematical understanding, for data analysis in this study is a representation of different kinds of mathematical understanding involving rote, procedural, and conceptual types of mathematical knowledge. These kinds of mathematical understandings are interpreted in terms of how the knowledge was acquired by the individual. For example, mathematical knowledge acquired by rote memorisation (e.g. Brownell, 1956; Goodwin & Klausmeier, 1975) leads to a particular kind of mathematical understanding as compared to mathematical knowledge that was generated by the individual's self pursuit (e.g. von Glasersfeld, 1991; Biggs & Moore, 1993). An examination of this model is the purpose of the next chapter, Chapter 3.

CHAPTER THREE

A MODEL FOR ANALYSING MATHEMATICAL KNOWLEDGE

Introduction

Mathematical knowledge was described in Chapter 2 as the underlying structure for mathematical understanding and the basis for mathematical competence in the teaching of mathematics. It was suggested that there are two types of mathematical knowledge, procedural and conceptual, both of which are essential to mathematical understanding.

Procedural mathematical knowledge is knowledge about the rules, procedures, and the symbolisation associated with mathematical concepts. Also, it is knowledge of how to do computations quickly and accurately. Conceptual mathematical knowledge, on the other hand, is knowledge of the underlying principles of mathematical concepts. It is knowledge that gives meaning to a mathematical procedure and provides the essential linkages from a concept(s) to sets of procedures.

Although both types of mathematical knowledge are essential to mathematical understanding, it was suggested however, in Chapter 2, that conceptual knowledge is fundamental to the growth of mathematical understanding (e.g. Skemp, 1978; Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993; Piere & Kieren, 1994). For prospective teachers of mathematics,

growth in understanding is important to their professional development and competence to teach mathematics.

To gain a measure of this growth in mathematical understanding requires evaluation tools; tools that can appropriately describe and distinguish between knowledge that pertains to mathematical understanding and knowledge pertaining to rote knowledge (described in Chapter 2, Section 2.4). The main purpose of this chapter, Chapter 3, is to describe the model selected as the appropriate 'evaluation tool' for the study data reported in this thesis.

3.1 Skemp's model of mathematical understanding

The theoretical perspective selected for the data analysis task is Skemp's (1978) model of mathematical understanding. Skemp's (1978) suggestion of the two kinds of mathematical understanding, *instrumental* and *relational*, has already been mentioned in Chapter 2. Relational understanding was stated in relation to growth in conceptual mathematical knowledge and instrumental understanding in association with rote knowledge (Chapter 2, Section 2.4). Skemp (1978) argued that both kinds of mathematical understanding are generated by mathematics learning and teaching in schools. This theoretical perspective is important to the study reported here because it provides a basis for examining existing mathematical knowledge that prospective teachers bring with them to mathematics teacher education.

Instrumental mathematical understanding was defined as the product of the rote memorisation of procedures or algorithms, rules, theorems, and their

specific applications (Skemp, 1978; Olive, 1991). In addition, instrumental understanding is externally driven, usually by the teachers, textbooks, calculators, and computers. This definition reasonably describes the type of knowledge that was defined in Section (2.4) as rote knowledge. It was also suggested, in Section (2.4), that this kind of knowledge could be an essential form of procedural knowledge and with special attention by the learner, rote knowledge could be transformed into usable knowledge.

Relational mathematical understanding, on the other hand, was defined by Skemp (1978) as the result of a learner's own constructions of a mathematical situation. The building up of one's own knowledge about a given area of mathematics becomes an intrinsically satisfying goal in itself (Skemp, 1978, p.14). As such, the more complete the learners' schema in an aspect of mathematics, the greater their feelings of confidence in their own ability to find new ways of dealing with mathematical situations without external help (Skemp, 1989; Ball, 1990; Fennema & Leder, 1993; Fennema, 1996). Relational understanding is suggested in this thesis to consist of the two essential mathematical knowledge components described in Chapter 2, Section (2.2); namely procedural and conceptual.

The Skemp model was used successfully, in conjunction with the SOLO (Structure of Observed Learning Outcomes) taxonomy (Biggs & Collis, 1982) and the Van Hiele (1986) levels of thinking, by Olive (1991) to analyse students' mathematical understanding of geometry and their ability to work with Logo programming. Olive (1991) investigated the effects of a computer programming software, Logo, on high school students' learning and understanding of geometry. To achieve this purpose, he needed instruments to analyse three aspects of his study: (i) the complexity of

students' responses to the Logo program (the SOLO taxonomy was used here), (ii) the quality of students' understanding of the mathematics involved (the Skemp model was used here), and (iii) general problem-solving approaches in geometry (the Van Hiele model was used here).

In this study, however, the focus is on the quality of mathematical knowledge assumed to be the result of the learning and teaching an individual received from formal schooling. More specifically, this study is concerned with the types of knowledge which are associated with mathematical understanding and are necessary for competence in teaching mathematics. To achieve this focus, Skemp's model of mathematical understanding was selected as the appropriate method for the analysis in this study because it makes no claims as to when (or time period) the knowledge was acquired or to a developmental stage (or modes in the case of SOLO). Rather, it links the way an individual understands mathematics to how that person has learned or assimilated the learning of mathematics from his or her schooling (Skemp, 1986, 1989).

The use of the Skemp model in the analysis of student-teachers' mathematical knowledge pertaining to mathematical understanding would provide important information relating to competent teaching of mathematics. This model of mathematical understanding provides two essential aspects for the analysis in this study: (i) how to determine the quality of mathematical knowledge in relation to mathematical understanding, and (ii) identifying the types of mathematical understanding that would facilitate and enhance competence in teaching mathematics.

3.2 Illustrating the Skemp model

Richard Skemp has illustrated his two kinds of mathematical understanding, instrumental and relational, in several of his publications (1978, 1979, 1982, 1986, 1989) by using a variety of mathematical examples. In one of Skemp's (1978) earliest publications he used the rule for the area of a triangle as an example to demonstrate instrumental learning versus relational learning. Skemp suggested that it is certainly easier for students to learn that:

$$\text{area of the triangle} = \frac{1}{2} \text{base} \times \text{height}$$

than to learn why this is so. However, this instrumental learning would mean more bits to learn; that is, the students must learn separate rules for rectangles, parallelograms, trapeziums, and circles. Relational understanding consists partly in seeing the areas of all of these shapes in relation to the area of a rectangle. In other words, it is still desirable for the learner to know the separate rules (ie. procedural knowledge); but knowing also how they are inter-related (ie. conceptual knowledge) enables one to remember them as parts of a connected whole.

To further demonstrate how instrumental understanding usually involves a multiplicity of rules rather than fewer principles of more general application, another of Skemp's (1978, p.11) earlier examples is used here as an illustration:

'If the teacher asks a question that does not fit the rule, of course they [students] will get it wrong. ... While teaching area, he [the teacher] became suspicious that the children did not really understand what they were doing. So he asked them:

What is the area of a field 20cms by 15 yards?

The reply was: 300 square centimetres.

He asked: Why not 300 square yards?
Answer: Because area is always in square centimetres.' (p.11)

Skemp (1978) suggested that to prevent the error response, as above, 'the students need another rule (or, of course, relational understanding), that both dimensions must be in the same unit' (p.11).

Relational understanding, on the other hand, seems to represent a network of concepts which are based on conceptual structures referred to by Skemp (1986, p.37) as 'schema'. The notion of a mathematical 'schema' is based on the assumption that higher-order concepts of mathematics are the result of many abstractions which have been derived from earlier or initial abstractions (Skemp, 1986, p.34). This implies that Skemp's (1979) proposed model of mathematical intelligence takes into account the hierarchy of mathematical concepts similar to the schema hierarchy proposed by Derry (1996) in Section (2.2) of Chapter 2.

Derry's (1996) notion of a 'schema' is based on a cognitive schema theory. Similar to Skemp's perspective, she also suggested that lower order or initial schemata are necessary as building blocks for the formation of higher order concepts. These higher order concepts involve *memory object* schemata and *mental model* schemata (Derry, 1996, p.167). A justifiable consideration of the schema theory for this study is the assumption that repeated learning of a situation would result in the formation of a schema. For example, according to Derry (1996):

'If similar kinds of problems and situations are often *repeated* for the student, organisation will gradually emerge because memory objects [or schemata for higher order concepts] that often work together to perform similar classes of tasks tend to acquire stability as a group and activate one another' (p.170, *italics* emphasis added).

However, under similar learning conditions, contrary outcomes may occur whereby schemata representing misconceptions can also be formed into stable patterns by repeated experiences. This assumption appears to take into account Skemp's (1978) idea that a student's mathematical understanding is generated by the mathematics learning and teaching he or she receives in schools, including universities. This schema theory is important to the evaluation of mathematical understanding of prospective mathematics teachers for two reasons: (1) it suggests that mathematical concepts are formed in succession (some sequentially) and contribute to the formation of yet others - therefore each mathematical concept is part of a hierarchy, and (2) that prospective teachers' conceptual understanding of mathematics must at least be relational understanding.

In this model, Skemp suggested a third type of mathematical understanding - formal or logical understanding (Skemp, 1979, p.45). Formal understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. However, this third type of mathematical understanding (formal understanding) was later referred to by Skemp (1982) as *symbolic understanding* to distinguish this from that of other researchers (e.g. Buxton, 1978) who used the same term *formal* to refer to a similar type of understanding. Skemp (1982, p.61) suggested that 'symbolic understanding is a mutual assimilation between a symbol system and a conceptual structure, *dominated by the conceptual structure* (italics in original)'. The function of symbols is for manipulating and communicating mathematical concepts, and these are the true operands in relational mathematics (Skemp, 1979). However, for symbols to be understood

relationally, they must be linked appropriately to their conceptual structures, and be interpreted in terms of the relationships within the structures.

Symbolic understanding of mathematics was suggested as the expected outcome for university students of mathematics, including pre-service teachers of secondary mathematics (Skemp, 1979, p.49).

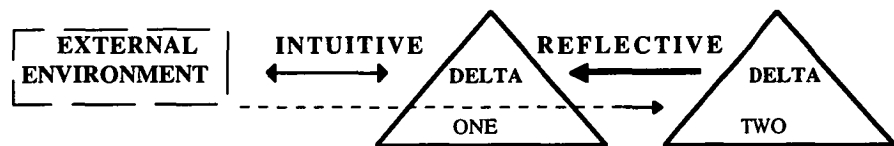
Skemp (1982) defines a symbol system as a set of symbols corresponding to a set of concepts. For example, the symbols 'cos' and '(2x)' taken separately refer to two kinds of mathematical-concepts. When they are written as 'cos(2x)' they specify an operand (the angle size of 2x) and an operation (finding the cosine value of the angle). The expression 'cos(2x)', therefore, represents a mutual assimilation of two schemata: the symbol system, and the structure of mathematical concepts.

Skemp's (1979) three different kinds of mathematical understanding, instrumental, relational, and symbolic are related to different kinds of learning goals. However, these goals were suggested to be based on the schemata for instrumental and relational understanding. That is, there is no separate learning goal for the symbolic type of understanding because it is based on the schema for relational understanding. Skemp (1979, p.47) suggested that logical (symbolic) understanding 'is closely related to the difference between being convinced oneself, for which relational understanding is sufficient, and being able to convince other people'. An example is 'the construction of chains of logical reasoning to produce what we call demonstrations or proofs' (Skemp, 1979, p.47).

In addition to the three types of mathematical understanding, Skemp (1979, p.48) added two modes of 'mental functioning': *intuitive mode* and

reflective mode. In Figure 3.1 below, these two modes are presented in relation to two goal director systems that Skemp (1979, p.44) has referred to as *delta-one* and *delta-two*. According to Skemp (1979), the '*job of delta-one is to direct physical actions, of many kinds. The job of delta-two is goal-directed mental activity, also of many kinds, including learning, but not only learning ... [but also] the construction and testing ... within delta-one*' (p.44).

Figure 3.1: Skemp's two modes of mental activity
(Adapted from Skemp (1979, p.44))



The intuitive mode refers to spontaneous processes within a goal directed system directly operating on the physical environment - this activity is represented in Figure 3.1 by the double-headed arrow \longleftrightarrow . This goal-directed system is referred to as 'delta-one' by Skemp (1979, p.44).

Skemp has likened delta-one to a sensori-motor system. The 'delta-two' is another director system with its operands not in the outside environment, but in delta-one. This activity is represented in Figure 3.1 by the single-headed arrow \longrightarrow . The function of delta-two is to optimise the functioning of delta-one (or the sensori-motor system). Therefore, in the intuitive mode, delta-two takes part either not at all, or not consciously. This activity is represented in Figure 3.1 by the segmented arrow $- - \longrightarrow$. The reflective mode, on the other hand, is referred to as conscious activity by delta-two on delta-one.

A summary of how these modes of mental activity are inter-related with the three kinds of mathematical understanding is provided below in Table 3.1. However, these two mental modes do not relate to different kinds of understanding but, rather, they occur in combination with all three understandings as illustrated by the rows of Table 3.1. Also in Table 3.1, the descriptions of the types of responses for each mode in relation to the three kinds of mathematical understanding are presented in the cells.

Table 3.1: Skemp’s model of mathematical understanding
(Adapted from Skemp (1979, p.48))

MODES OF MENTAL ACTIVITY	Instrumental Understanding	Relational Understanding	Logical (formal) Symbolic Understanding
	An individual uses:	An individual uses:	An individual uses:
Intuitive	rules without reasons	meaning or significance or structure of a problem without explicit reliance on tested techniques of analysis and proof	logical progression of steps for a proof giving evidence of an awareness that something is ‘true’ or ‘false’
Reflective	an assortment of rules to achieve a correct answer	extensive mathematical schemata to relate and verify procedures	‘full mathematical rigour’. Expected evidence of mathematical understanding from college level and university students, including pre-service teachers of mathematics (Skemp, 1979, p.49).

The above modelling of the ‘modes of mental activity’ (*delta-one* and *delta-two*) by Skemp (1979) has close similarity to the model for ‘mathematical understanding’ presented in Chapter 2, Figure 2.3. That is, the functioning of *delta-one* appears to describe what other researchers (e.g. Anderson, 1981; Gagné, 1985) have referred to as the function of procedural

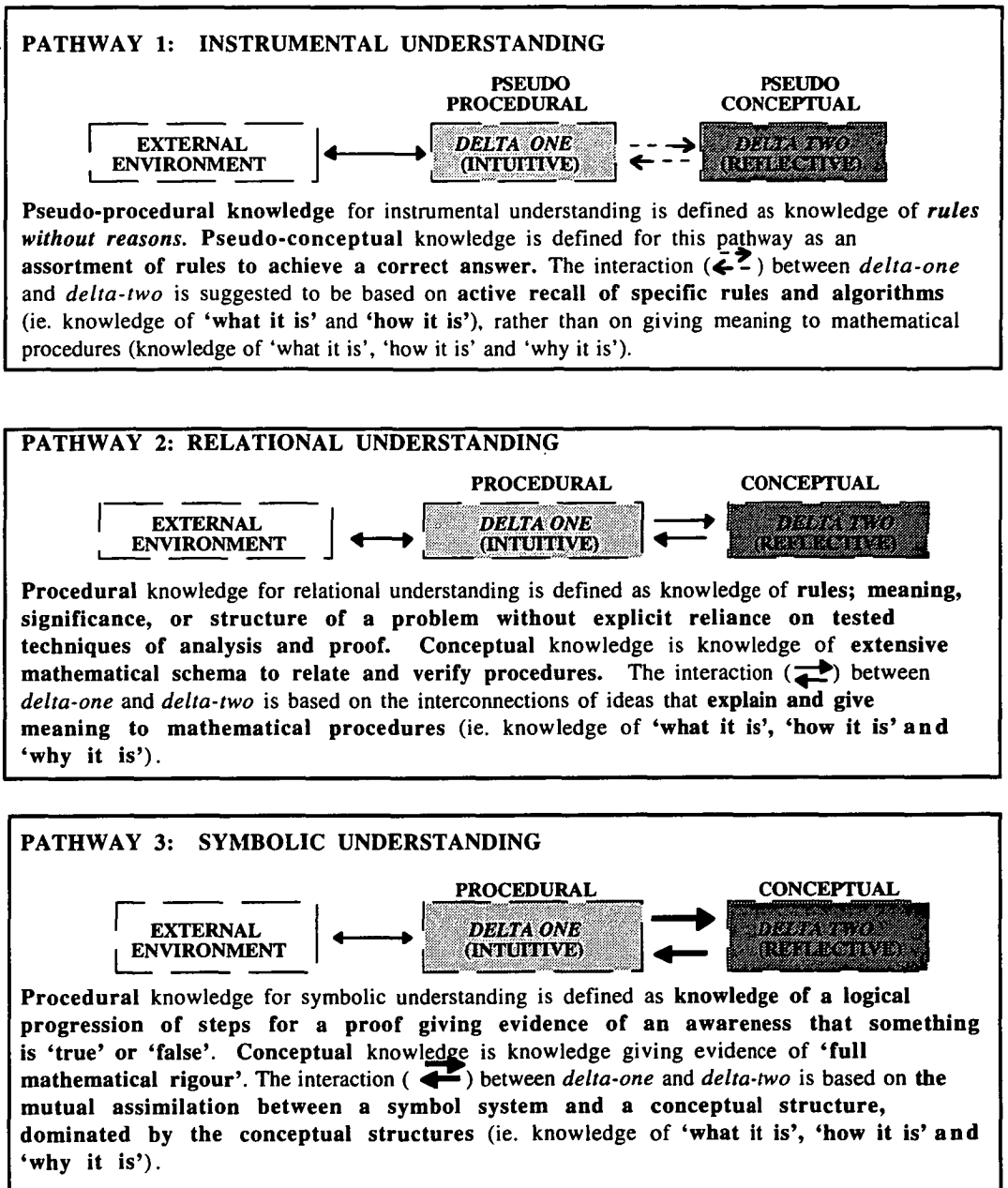
knowledge or knowledge interface (Alexander *et al.*, 1991; Derry, 1996).

Procedural knowledge is knowledge of how to do computations. It is dynamic in that, when activated, the result is not simple recall but a transformation of information (Gagné, 1985; Alexander *et al.*, 1991; Derry, 1996). In addition, procedural knowledge is used to operate on information to transform it into observable knowledge (Gagné, 1985, Alexander *et al.*, 1991; Derry, 1996).

Similarly, *delta-two* seems to appropriately describe the function of conceptual mathematical knowledge in mathematical understanding. That is, conceptual knowledge is knowledge of relationships and interconnections of ideas that explain and give meaning to mathematical procedures (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993; de Jong & Ferguson-Hessler, 1996).

However, what appears to distinguish Skemp's (1979) theory of mathematical understanding from theories by other researchers (e.g. Anderson, 1981; Gagné, 1985; Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993) is the three separate 'pathways' (instrumental, relational, and symbolic mathematical understanding) in achieving a mathematical understanding. It is the criteria for determining procedural and conceptual mathematical knowledge by which these three pathways have been established which are very important to the analysis of mathematical knowledge in this study. An attempt to highlight the criteria by which procedural and conceptual mathematical knowledge are connected to Skemp's (1979) mode of mental activity (intuitive and reflective) is the purpose of Figure 3.2 (next page).

Figure 3.2: The three pathways in mathematical understanding
 (based on Skemp's (1979) model of mathematical understanding)



The three kinds of mathematical understandings in Skemp's (1979) model are presented in Figure 3.2 above by separate pathways. The criteria for 'procedural' and 'conceptual' knowledge are also described in relation to each pathway.

An important point to emphasise here about the *three pathways* as presented in Figure 3.2, especially the pathway for instrumental understanding, is that the type of conceptual knowledge involved in each kind of understanding is dependent on, and influenced by, the type of procedural knowledge acquired by the individual. For example, in the *instrumental understanding* pathway, knowledge is said to be acquired by rote memorisation of rules and procedures. It is therefore assumed, that the types of procedural knowledge aspects which form the basis for *delta-one*, upon which delta-two operates, are rules and algorithms without connectors to knowledge aspects of 'meaning'. These rules and algorithms may (or may not) have connectors to other rules and algorithms. Such unpredictability with the nature of procedural types of knowledge would also affect the quality of the associated conceptual aspects of knowledge. In order to distinguish these kinds of conceptual knowledge aspects from others, the term '*pseudo*' is used for such knowledge aspects, in particularly those associated with instrumental understanding (see Figure 3.2).

Pseudo-conceptual knowledge, the type associated with *instrumental understanding*, is defined for this study as an accumulation of sets of rules and algorithms, most of which are individual bits of memorised information. In other words, a measure of *conceptual* type knowledge in instrumental understanding is determined by the amount (or quantity) of knowledge of

rules, rather than the quality of this knowledge. The term *pseudo-conceptual* is used in this thesis to distinguish the *instrumental understanding* types of conceptual knowledge from the types associated with *relational and symbolic mathematical understanding*, which is defined in this thesis as ‘knowledge of the underlying structures of mathematics that explain and give meaning to mathematical procedures’ (Chapter 2, Section 2.2).

The delineation of Skemp’s (1979) model in this chapter has highlighted important aspects of this model for the analysis of mathematical knowledge. For the purpose of this study, a pre-service teacher of mathematics with the capacity to become competent in teaching mathematics should demonstrate that he or she has acquired mathematical knowledge pertaining to relational and especially symbolic understanding.

3.3 Summary

The Skemp model of mathematical understanding involving instrumental, relational and symbolic understanding, forms the theoretical basis for the analysis of mathematical knowledge addressed in this study. This model is particularly suitable for the examination of student-teachers’ mathematical knowledge because of its basic assumption that mathematical understanding is a product of mathematical learning and teaching from school.

In addition, Skemp’s (1979) recognition of the hierarchical formation of mathematical concepts, leading to a symbol system of mathematical understanding, provides this analysis with a valuable framework for categorising response data associated with prior mathematical learning.

This framework also provides a basis for examining the student-teachers' mathematical competence.

Also for this analysis, knowing how the mathematics was learned or acquired (e.g. rote memorisation or self-determined) would add to the measure of quality and would provide further information on the respondent's potential to teach mathematics for conceptual understanding.

Having described the theoretical perspective selected to form the basis for the 'evaluation tool', the focus of the next chapter, Chapter 4, is on how this evaluation tool is used and incorporated into the design of the study for exploring the research questions stated in Chapter 2.

CHAPTER FOUR

RESEARCH METHODOLOGY

Introduction

The importance of mathematics teachers having a well structured mathematical knowledge base was discussed in Chapter 2. It was suggested that the teachers' mathematical knowledge base should consist of procedural and conceptual mathematical knowledge. Competent mathematical performance was assumed to be dependent on the quality of this knowledge base, particularly conceptual knowledge, and on the individual's ability to transform and transfer this knowledge to competent performance.

Rote knowledge pertaining to mathematics was also described in Chapter 2, Section (2.4) as another type of mathematical knowledge with characteristics of procedural mathematical knowledge. Rote knowledge is distinguished from procedural mathematical knowledge in that rote memorisation is the approach for acquiring the knowledge. Rote memorisation of rules and procedures was classified by Skemp (1978) as instrumental understanding of mathematics. Skemp's model of mathematical understanding was described in Chapter 3 as an appropriate method for the data analysis in this study.

This chapter is organised into two main parts: Part one contains the design of the study, and part two is a detailed account of the procedures involved in developing the data collection instrument and the validation of this instrument.

4.1 PART ONE: STUDY DESIGN

The purpose of this section is to provide a description of the methods and procedures involved in the design of the study. This design is described in eight sections. The research assumptions and the research questions to be addressed in the design are stated in section one. The rationale of the study's design is described in section two. The theoretical framework for selecting case studies is the focus of discussion in section three. This is followed in section four by descriptions of how the cases were to be selected. The procedures for ensuring the validity and reliability of the study data are described in section five. Further discussion on these procedures, particularly on a set of prescribed 'interview cues' for interview data collection, is the purpose of section six. The interview method for data collection is the focus of discussion in section seven. Finally, the description of the procedures for data analysis is the focus of section eight.

4.1.1 Assumptions and research questions

This design addressed two research questions based on the following assumptions:

- (1) Mathematical understanding is dependent on the sufficiencies of procedural and conceptual types of mathematical knowledge. Lack of or a deficiency in either procedural and/or conceptual knowledge types would suggest a deficiency in mathematical understanding (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993).

- (2) Pre-service mathematics teachers go through their teacher education and training with certain deficiencies in their mathematical understandings and that these deficiencies will eventually affect the way they teach.
- (3) Pre-service mathematics teachers who majored in mathematics or other science related areas (e.g. chemistry and computer science) would show less evidence of mathematical knowledge deficiencies than pre-service teachers who majored in other areas (e.g. economics and physical education). Furthermore, pre-service teachers with relational understanding of mathematics would demonstrate more confidence to teach mathematics than pre-service teachers with instrumental understanding.

The two research questions addressed in this study of pre-service teachers' existing mathematical knowledge bases were:

1. What types of procedural and conceptual mathematical knowledge exist in pre-service teachers' knowledge bases?
2. What possible influences could any identified deficiencies in types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?

4.1.2 Rationale of the design

The main focus of this study is on mathematical knowledge. Mathematical knowledge, however, is incorporeal and becomes observable by means of responses (written or verbal) to a stimulus. As such, a study of the quality of a person's knowledge should involve systematic procedures to ensure that collected data provide representative evidence of this knowledge (Yin,

1994; Taylor & Bogdan, 1998). However, because of the incorporeal nature of knowledge and even with the most stringent methods for data collection, the most one could and can expect to obtain is an insight into the quality of knowledge being investigated (Denzin & Lincoln, 1994; Stake, 1994; Yin, 1994; Taylor & Bogdan, 1998). According to Stake (1994) and Yin (1994), an appropriate study method for gaining insights into *knowledge* is a case study approach.

Therefore, the design of this study is based on a multiple case study (Yin, 1994) or collective case study (Stake, 1994) design in which each case (or a pre-service teacher) is selected by following the 'replication logic' rather than the 'sampling logic' procedure (Yin, 1994, p.45). Sampling logic implies that a set of cases is a representative sample of a larger population. Replication logic, on the other hand, is based on the assumption that a selected case or a set of cases would provide relevant and valuable information that would lead to better understanding about a still larger collection of cases (Stake, 1994; Yin, 1994). According to Yin (1994, p.45), replication logic is analogous to procedures used in multiple experiments in which the same results are predicted for each of the cases involved. To apply this replication logic to multiple case studies requires that 'each case must be carefully selected so that it either: (a) predicts similar results (a literal replication) or (b) produces contrasting results but for predictable reasons (a theoretical replication)' (Yin, 1994, p.46). How these criteria (a) and (b) for case selection were addressed in relation to the theoretical assumptions of this study and with respect to mathematical knowledge (or the object of study) is the focus of the discussion in the next section.

4.1.3 Theoretical framework for case selection

For the purpose of the discussion in this section, the conceptual framework discussed in Section (1.6) of Chapter 1 and elaborated on in Sections (2.1) and (2.2) of Chapter 2 provides the theoretical structure for selecting cases for this study. This theoretical framework is illustrated in Figure 2.3 and Figure 2.4 of Chapter 2.

In Chapter 2, Sections (2.1) and (2.2), mathematical knowledge was viewed as a subcategory of *content knowledge*, namely domain knowledge. Content knowledge is illustrated in Figure 2.1 (Chapter 2) as belonging to a broader category of *conceptual knowledge* (Alexander *et al.*, 1991). In addition, mathematical knowledge is said to consist of rote, procedural, and conceptual types of knowledge; and that mathematical understanding is the result of a close relationship between procedural and conceptual knowledge types. This relationship between procedural and conceptual knowledge is illustrated in Figures 2.3 and 2.4. Based on this conceptualisation of knowledge, the target area from which ‘specimens’ (or evidence) of mathematical understanding can be collected would be the *domain (mathematics) knowledge* (Alexander *et al.*, 1991).

A ‘specimen’ is defined here as the evidence or information about mathematical knowledge collected from a case (or a pre-service teacher). A collection of these specimens from a set of cases would together provide a broader perspective or insight into the nature of mathematical knowledge. In relation to replication logic procedures, each case in the

collection is selected such that the ‘specimens’ would either: (a) produce *similarly* predictable results about mathematical knowledge (a literal replication) or (b) produce *contrasting* results about mathematical knowledge but for predictable reasons (a theoretical replication). How these procedures were applied to the selection of cases for this study is the purpose of the next section.

4.1.4 Selection of cases

The theoretical framework described above in Section (4.1.3) provides a structure for the selection of cases for the study. Another important consideration for the selection of the cases concerns the theoretical assumptions of the study, particularly the following assumption:

Pre-service mathematics teachers who majored in mathematics or other science related areas (e.g. chemistry and computer science) would show less evidence of mathematical knowledge deficiencies than pre-service teachers who majored in other areas (e.g. economics and physical education).

To distinguish between these two categories of pre-service teachers, the former category is referred to as ‘maths major’ cases and the latter category as ‘maths minor’ cases.

In order to comply with the replication logic procedures for case selection, each case (or set of cases) was selected to satisfy both the ‘similarity’ criterion (a literal replication) and the ‘contrast’ criterion (a theoretical replication). Accordingly, the first set of cases (‘maths major’ cases) were selected so that each case would contribute to a collection of ‘specimens’ having similar mathematical knowledge types, hence a literal replication.

For a theoretical replication or *contrasting* results to the ‘maths major’ cases, a second set of cases (‘maths minor’ cases) were selected from a different university site to the first set of cases. Each specimen from the ‘maths minor’ cases had been predicted to show evidence of *similar* knowledge types. Thus, both sets of cases (‘maths major’ and ‘maths minor’) provide *similarity* within their respective sets and *contrast* between sets.

These replication procedures for ‘similarity’ and ‘contrasting’ of cases also provided a structure for testing and ensuring the validity and reliability of the study results. The three tests for validity and reliability (external validity, construct validity, and reliability) that were suggested to be applicable to case study designs (Yin, 1994; Taylor & Bogdan, 1998) are discussed in the context of this study in the next section, Section (4.1.5).

4.1.5 Ensuring validity and reliability

The sets of multiple cases, ‘maths major’ and ‘maths minor’ described above in Section (4.1.4) were selected from four universities at two of the Australian states. For the purpose of the discussion in this section, the two Australian states are referred to as State A and State B. In addition, the first set (‘maths major’) and the second set (‘maths minor’) of selected multiple cases from State A are referred to as Set A1 and Set A2 respectively. Similarly, the selected multiple cases from State B are referred to as Set B1 (‘maths major’) and Set B2 (‘maths minor’) (see Table 4.1, next page).

External Validity:

External validity is concerned with establishing a domain in which a study’s findings can be generalised (Grimm & Wozniak, 1990; Yin, 1994). In establishing such domain for this study, the same replication procedures for selecting the multiple cases in State A were applied to cases in another Australian state, State B. These case selections are illustrated in Table 4.1 below.

Table 4.1 Case selection

State A	Set A1 Cases First set of cases in A Background: maths major	Set A2 Cases Second set of cases in A Background: maths minor
State B	Set B1 Cases First set of cases in B Background: maths major	Set B2 Cases Second set of cases in B Background: maths minor

Reliability:

Another important design criterion for the study is the assurance that collected data are reliable *specimens* and ‘errors and biases’ are minimised (Yin, 1994, p.36). Minimising such errors and biases in a study is the goal of *reliability*. Yin (1994, p.36) suggested that for case study designs, reliability is achievable by clearly ‘document[ing] the procedures followed in earlier cases’ so as to allow a repeat of the same study in later cases or by someone else. This criterion was addressed in this study by having a set of prescribed ‘interview cues’ specifically designed for the purpose of collecting data in the interview sessions. These ‘interview cues’ are described in detail in the subsequent section (Section 4.1.6), but it suffices to briefly list the four stages here. These are: (1) silent reading; (2) writing

down responses; (3) explaining written responses, and (4) clarifying and elaborating on written and verbal responses.

Construct validity:

The 'interview cues' listed above also provided an avenue for addressing 'construct validity'. According to Yin (1994, p.34), showing construct validity for case study designs is a challenge for the novice researcher. However, he suggested three 'tactics' that one could employ, one of which is data triangulation or the collection of data from different sources to provide support for 'convergent lines of inquiry' (Yin, 1994, p.34; Taylor & Bogdan, 1998) similar to a test of convergence in quantitative study designs (Grimm & Wozniak, 1990). In theory, convergence implies that the different sources of data (assuming that they are related to the *construct*) would provide the same or similar measures to the *construct* being studied. In relation to this study, the construct of focus is *existing mathematical knowledge* and related sources might be (a) the individual's mathematics results on a written test, (b) data from an interview on the individual's beliefs about the mathematics involved in the test, and (c) data from observations of the individual's attempts to solve a mathematical problem. Accordingly if these different, but assumably related, sources of data show 'convergent lines of inquiry', then there is support for construct validity. However, implicit in such a selection of data sources are the variables or factors concerning the differences in *time* and *place* whereby the formation of a 'convergence' may well be attributable to either one or both of these factors, or by some other factors (Berg, 1989; Taylor & Bogdan, 1998). In other words, triangulation procedures are not a guarantee for construct validity (Taylor & Bogdan, 1998). According to Patton (1987, p.61), 'triangulation is seldom a straightforward process' and

sometimes different data sources provide inconsistent results. In an attempt to avoid such inconsistent results, a more controlled data collection approach was used. The main feature of this controlled approach is the prescribed set of 'interview cues' which were followed for each participant when presenting the stimulus items to them at the interview sessions.

In addition to the set of 'interview cues', the triangulation method was adopted in the selection of stimulus items. Trigonometry, logarithm, and statistics were the three chosen mathematical areas from which the stimulus items were selected. The criteria and procedures for selection and descriptions of these items are detailed in part two of this chapter. The next section, however, is the description of the 'interview cues', Section (4.1.6).

4.1.6 Four interview cues

In Chapter 2, Section (2.2), a conceptual framework for modelling *a response production of mathematical knowledge* is described and illustrated in Figure 2.4. It is suggested from this modelling that specific knowledge productions are provoked by the use of certain stimulus cues. This framework forms the theoretical structure for the set of four prescribed 'interview cues' described here. These 'interview cues' were used in conjunction with the set of stimulus items in the interview sessions. This selected interview method is discussed in the next section. However, the aim of this section is to outline the sequence of procedures and the purpose of these 'interview cues'. These procedures involve four key stages which were followed for each case in a one-to-one interview between the participant and the researcher/interviewer.

The four stages: reading, writing, verbalising, and elaborating, were designed in an attempt to ensure consistency in the collection of 'specimens' and to minimise errors and biases during the interview sessions (Section 4.1.5). Since the target point for the collection of the 'specimens' is the participant's existing mathematical knowledge (Section 4.1.3), emphasis was placed on *stimulating* the participant's attention to focus on, and interact with, the content of the stimulus items. Although the content of the stimulus items would have an influence on the kind of responses exhibited by the participant, it was assumed that with appropriate external cues from the interviewer that the likelihood of activating relevant knowledge would be greatly increased. This assumption was based on a model of a 'response production of mathematical knowledge' illustrated in Figure 2.4 and Figure 2.5 in Chapter 2.

It was proposed from this model that a stimulus cue such as '*to read*' may not necessarily activate a 'complex' response production in which both procedural and conceptual knowledge types are involved (Figure 2.5). Rather, '*to read*' would most likely activate a 'simple' response production involving mainly procedural types of knowledge. On the other hand, stimulus cues such as '*to explain*', '*to clarify*', and '*to elaborate*' would increase the likelihood of activating both procedural and conceptual knowledge types. As a result of this conceptualisation of response production of mathematical knowledge, four stages for eliciting responses were developed. These stages are elaborated on as follow:

(1) **The silent reading stage:**

In this initial stage, the participant was presented with a mathematical stimulus item and was requested to *read* (without writing) the contents

of the stimulus item and begin to formulate mental strategies for giving an answer. It was assumed that having the participant focus on formulating mental strategies would increase the likelihood of a 'complex' response production (Figure 2.5). For example, focused reading may promote the formulation of a plan of action or 'strategic knowledge' in which the sequence of solution responses to a given situation is laid down (e.g. knowing how to organise and interpret given information, and to structure that information in a diagram) (de Jong & Ferguson-Hessler, 1996, p.107). In addition, *reading* may also allow the participant to use strategic knowledge to make 'connections', for example, to comprehend, contextualise, and to assimilate the cues from the stimulus items with his or her existing mathematical knowledge (Hiebert & Lefevre, 1986). There was no set time limit for reading, the participant was simply requested to indicate to the researcher his or her readiness to continue to the next stage.

(2) The **writing** stage:

During this stage the participant was requested to *write* down his or her responses (or answers) to the stimulus questions, assuming that such were the results of knowledge productions from stage one. These written responses constitute a 'specimen' or *actual data* of the study. It was assumed that this specimen contains evidence of the participant's *existing mathematical knowledge*. The next two stages, stages three and four, are important in order to provide an opportunity for the participant *to explain, to clarify and to elaborate* further on the *actual data*. These final two stages are also important for reliability and validity of the study results (Section 4.1.5).

(3) **The verbalising stage:**

In an attempt to validate the written or *actual data* as well as to increase the likelihood of 'complex' response productions, the participant was requested *to explain* to the researcher his or her written attempts (*actual data*). This verbalisation was recorded on audio tapes. During this process the participant was at liberty to modify or add to his or her written records if so desired. This stage was also important in an attempt to overcome the phenomenon of *inert knowledge*. That is, although the knowledge is available and can even be recalled on request, the individual does not spontaneously apply the knowledge in situations where it is relevant for a solution, particularly, new situations (De Corte, 1995; Renkl, Mandl, & Gruber, 1996).

The role of the researcher during this period is to be an *active listener* and to make written notes of any issues (or points) that require further elaboration and clarification. These notes are necessary in the next validation stage, stage 4.

(4) **The elaboration and clarification of responses stage:**

This final stage was essential to assure the validity and reliability of the *actual data* and to provide further opportunity for the participant to produce a 'complex' type response. To begin this exchange, the researcher asks the participant: *Is there anything else (or more) you would like to say about ...* (referring to the stimulus item)? Following any responses to this question, the researcher goes through the list of points or notes recorded from the previous stage and ask the participant to provide an *elaboration*, a *clarification*, or a *confirmation*

of the *actual data*. All exchanges were recorded on audio tape. Finally, in order to provide the opportunity for further addition, deletion or modification to recorded statements, the participant was offered a replay of his or her recorded responses at the end of each recorded exchange.

In summary, the four stages of eliciting responses and collecting 'specimens' on pre-service teachers' existing mathematical knowledge are:

1. Reading - formulating mental strategies.
2. Writing - providing the *actual data* (written data)
3. Explaining - validating the *actual data* (verbal data)
4. Elaborating - further validation of the *actual data* (verbal data).

The interview method for administering the above four stages is described in the next section.

4.1.7 Semi-structured interview method

The interview method for collecting the data is referred to here as a semi-structured method or a 'focused method of interviewing' (Good, 1972, p.244; Berg, 1989; Merton, Fiske, & Kendall, 1990; Oppenheim, 1992). A semi-structured interview method is appropriate for the purposes of this study, particularly for administering the four response stages described above in Section (4.1.6). Open-ended interview methods where the persons are allowed 'to talk about other features of their lives relevant to their feelings' in relation to the topic in question (Phelan, Yu & Davidson, 1994, p.420) were considered inappropriate for the design of this study in

which a person's feelings were regarded as metacognitive knowledge rather than conceptual knowledge (Alexander *et al.*, 1991).

In addition, using an open-ended interview approach as a method for collecting data in case study designs may not be uniformly successful because the respondents can differ in ability, motivation, and in understanding the interview content (Kahn & Cannell, 1978; Berg, 1989; Oppenheim, 1992; Yin, 1994). However, several researchers (e.g. Kahn & Cannell, 1978; Merton, Fiske, & Kendall, 1990; Oppenheim, 1992; Yin, 1994) suggested that the interviewer can provide uniformity by giving assistance to the respondents, for instance, in recall and understanding by providing situations appropriate to the need. For example, the interviewer could provide a diagram or a probing question instead of a verbal explanation. Such methods are important in order for the interviewer and the respondents to have a common frame of reference and a common conceptual language as well as providing adequate opportunity for the respondents to express what they know about the content. In relation to this study, adequate opportunities for the respondents were provided by incorporating the 'four interview cues' (Section 4.1.6) into the interview proceedings.

The semi-structured interviewing method has been used successfully for the purpose of selection and assessment, as for example in interviews conducted with applicants for jobs or with students applying for admission to universities (Kahn & Cannell, 1978; Tutton, 1994). Semi-structured interviews have been used to collect data on communication skills, cognitive style, beliefs and perception, and understanding of mathematical content (Tutton, 1994; Forgasz & Leder, 1996). These examples of applications of

the semi-structured interviewing method lend support to its potential as a suitable means for collecting data for the study reported in this thesis.

There appears to be two essential elements associated with the proper use of semi-structured interviewing methods in research: (i) a set of interview questions based on criteria related to the goals of the research, and (ii) that the interviewer be well conversant with the research aims (Kahn & Cannell, 1978; Berg, 1989; Merton, Fiske, & Kendall, 1990; Tutton, 1994; Yin, 1994). The first of these two elements was satisfied by the use of three stimulus items and the incorporation of the 'interview cues' (Section 4.1.6) into the structure of the interview procedures. The second element was also satisfied by having the researcher as the interviewer.

So far, the discussion in this first part of the chapter has focused on the procedures for the selection of cases and the collection of the data. The procedures by which the data were analysed constitute the final component of the study's design. This then is the focus of the discussion in the next section.

4.1.8 Data analysis procedures

The main procedure for data analysis is pattern-matching (Yin, 1994; Taylor & Bogdan, 1998). Skemp's (1978) model of mathematical understanding (described in Chapter 3) provided the theoretical basis for the formation of predicted response patterns needed for a base-line. Skemp's (1978) model produced six possible response patterns. These response possibilities were illustrated in Figure 1.2, Chapter 1 and were also described in terms of procedural and conceptual mathematical knowledge types in Chapter 3,

(i) What kind of stimulus items would appropriately address the two research questions (Section 4.1.1) of this study?

The two research questions of this study are concerned with the examination of mathematical understanding acquired by pre-service teachers of mathematics. More specifically, the examination of knowledge types pertaining to understanding. For mathematical understanding, these knowledge types were identified in Chapter 2 as procedural and conceptual. The result of a dynamic and complex relationship between these two types is mathematical understanding (Section 2.2, Chapter 2). Therefore, with respect to question (i), the kind of stimulus items needed are those which can appropriately make *interaction* with a person's *knowledge interface* so that an interconnection and interaction of procedural and conceptual knowledge types can occur (Alexander *et al.*, 1991; Derry, 1996). In other words, the kind of stimulus items needed are ones which can act as a probing mechanism into a person's mathematical understanding. Such a mechanism could be in a form of error responses to a mathematical situation. Brownell (1956, p.133) suggested the use of 'error' as an appropriate probing mechanism for determining what type of knowledge has been acquired by a learner.

Gagné (1985) and Derry (1996), for example, also suggested that exposure to irrelevant attributes of stimulus data facilitates cognitive processing of knowledge. This idea is similar to the notion of negative feedback in the information-processing learning theory, that is, negative stimulus causes active cognitive processes to connect the cues to prior knowledge in order to maintain homeostatic equilibrium (Swenson, 1980;

Derry, 1996; Reynolds *et al.*, 1996). A positive stimulus, on the other hand, would produce an amplifying effect similar to positive reinforcements.

In addition, the mathematical content of the stimulus items should reflect higher level mathematics (e.g. college or university) to coincide with the secondary pre-service teachers expected mathematical abilities. That is, the content should be such that it requires *symbolic* type understanding. The type of understanding defined in Chapter 3 as 'a mutual assimilation between a symbol system and a conceptual structure, [but] dominated by the conceptual structure' (Skemp, 1982, p.61).

Based on the above discussions, two important criteria were adopted for selecting stimulus items: (1) that the stimulus item must contain *irrelevant attributes, error responses* or *misconceptions* in mathematics, and (2) the mathematical content should be based on *symbolic* type understanding. Examples of such contents are the 'notions of a function, of limits and infinity, and of the process of mathematical proof' (Tall, 1992, p.495).

The next step in the selection of the items was to determine which areas (e.g. geometry and algebra) of mathematics would provide appropriate triangulation of *symbolic* type knowledge. The approach taken to address this decision is the topic of the next discussion.

(ii) Which mathematical areas should provide appropriate triangulation of items as well as satisfying question (i)?

Traditionally, mathematics is made up of several branches: arithmetic, algebra, geometry, trigonometry, statistics and logic (National Science and Technology Centre, N.S.T.C., 1989). However, the selected items would only be a sample to represent some of these branches. Thus, findings from this study that are based on these items should be viewed as representative in nature rather than definitive.

Based on the notion of data triangulation (Patton, 1987; Yin, 1994; Taylor & Bogdan, 1998), three mathematical items were considered sufficient. In order to gain an understanding of the types of *irrelevant attributes, error responses, or misconceptions* in mathematics that mathematics students might have, the relevant literature concerning students' misconceptions or difficulties in understanding of mathematics, particularly of college or university level mathematics was reviewed. From this review the *function* concept (Grouws, 1992; Tall, 1992; Even, 1993; Coady & Pegg, 1994; Wilson, 1994; Alters, 1996), and statistics and probability (Green, 1983; Grouws, 1992; Shaughnessy, 1993) were areas identified to involve *symbolic understanding* as well as causing understanding difficulties for students.

Knowledge of mathematical functions was suggested to be closely related to knowledge representations of algebraic algorithms, syntax of symbols and format (Even, 1993; Coady & Pegg, 1994; Wilson, 1994; Gates, 1995b). However, it seems that a common misconception of the *function concept* involves misrepresentation of the *concept* as algebraic

formulations (Tall, 1992; Coady & Pegg, 1994) and as graphical representations (Tall, 1992; Alters, 1996). Such misconceptions could be attributed to 'difficulties ... with the variety of different representations [of the function concept] (graph, arrow diagram, formula, table, verbal description, and so on) and the relationships between them' (Tall, 1992, p. 500). Tall (1992) added that in 'emphasising the many representations of the function concept - formula, graph, ... - the central idea of function as a *process* is often overlooked' (p.501). Even's (1993) findings from her study of prospective mathematics teachers' concept of function also indicated difficulties by these prospective teachers in understanding the many representations of the function concept. Alters (1996) study of undergraduate physics students' understanding of logarithmic graphing also indicated the difficulty these students had in distinguishing between the concept of logarithm and the graphical representation. The misconceptions relating to the *function* concept appear to be associated with the way the students were taught to learn it as either (a) a *process* (e.g. a relationship between two variables) or (b) an *object* (e.g. the graph of a relationship) (Tall, 1992, p.501).

Since mathematical *functions* are associated with many representations such as a formula or a graph and variable relationship as well as representations of *topic-matter* as in logarithm, it seems appropriate to choose this as an area (representative of *symbolic understanding*) for the selection of the stimulus items. Therefore, two mathematical topics were chosen from which items would be selected to represent mathematical functions. Logarithm was chosen as one of the topics because it is a required learning for college and university students of mathematics and other science and technology areas (Alters, 1996). Logarithmic functions

involve representations of algebraic formulae and variable relationships between exponentials and linear mathematical expressions (Tall, 1992; Alters, 1996). To supplement these mathematical representations, trigonometry was chosen as the second topic from which items can be selected.

Statistics as a branch of mathematics has at times been viewed in the same way as physics and chemistry or as a practical science (N.S.T.C., 1989). Statistics appeared to require specific mathematical knowledge. For example, learning to calculate the standard deviation for a set of scores is a specific form of mathematical knowledge belonging to statistics. Nevertheless, according to Shaughnessy (1993, p.244), much of students' misconceptions about statistics and probability are embedded in their 'established beliefs about chance' long before they are taught any probability or statistics, and that such 'probabilistic beliefs and conceptions are difficult to change'. However, many of the statistical formulae and algorithms are based on the four arithmetic operations of addition, subtraction, multiplication, and division. For example, addition and division are all that is involved with the arithmetic mean. Also, knowledge associated with graphical representations in statistics are related to representations of the function concept (e.g. the line graph or linear function). It could be suggested that some of the students' difficulties in understanding statistics may relate to the fact that many statistical words represent concepts and not objects (Green, 1983; Miller, 1993). Such words as *mean*, *median*, *standard deviation*, and *variance* describe concepts and have no unique, unambiguous representations in the real world (Miller, 1993; Shaughnessy, 1993). To complement and triangulate

the two stimulus items from the mathematical function topics of logarithm and trigonometry, a third item was chosen from statistics.

It is assumed that items representing logarithm, trigonometry, and statistics would provide a triangulation of data sources on mathematical knowledge. These three items (logarithm, trigonometry, and statistics) were chosen so as to explore the existence of a 'mutual assimilation of two schemata' (Skemp, 1982, p.60): the symbol system associated with algebraic algorithms (these are representative of procedural knowledge), and the conceptual structures related to statistics, trigonometry and logarithm (these are representative of conceptual knowledge).

This 'mutual assimilation of two schemata' is a common feature of trigonometry, logarithm, and statistics. One of the features relates to specific rules (or laws) that govern the algorithms (procedural knowledge) for computation. Another common feature is the requirement of particular conventions for each type, for example, $\sin(2+x)$ is not \sin *multiplied by* $(2+x)$, similarly $\log(4+x)$ is not the same as $\log 4 + \log x$, and $\sum xy$ is not equal to $\sum x \sum y$. Also, the symbols are often used as the embodiment of both the concept and the procedures involved. For example, $y = \sin(2+x)$ is a curve described by a relationship of its domain, the set of values generated by $(2+x)$, and its range (the set of y values). In statistics, the symbol σ or s is often used to represent the standard deviation and its

computation, $\sqrt{\frac{\sum (x - \bar{x})^2}{n}}$.

One of the key assumptions underlying the triangulation of these three stimulus items is that they represent specific types of mathematics and the understanding of each one is dependent on conceptual knowledge of particular mathematical rules (or theorems) and procedures. The close similarity between these specific rules and procedures to the more general forms is of interest for the purpose of this study. For example, the distributive law, $a(b+c) = ab + ac$, is inappropriate for $\log(4+x)$, and yet, a mathematics student with little understanding of logarithm may attempt to use this law to give an expansion: $\log(4+x) = \log 4 + \log x$. How this kind of error (deficiency in knowledge) has been acquired, is the question that is of interest here.

According to Skemp (1978), the learner with a response $\log(4+x) = \log 4 + \log x$, could have acquired this form of knowledge by the rote memorisation of rules and procedures and by a dependency on external guidance (e.g. the teachers, textbooks, and calculators). Mathematical understanding related to this kind of learning approach was defined by Skemp (1978) as *instrumental understanding*. In order for respondents to demonstrate *symbolic understanding* of items representing trigonometry, logarithm and statistics, Skemp (1982) suggested that they would need to show evidence of their abilities to manipulate and appropriately link the symbols to conceptual structures, and interpret these symbols in terms of their relationships within the structures.

In an attempt to distinguish procedural and conceptual types of mathematical knowledge pertaining to *relational* and *symbolic understanding* from those types relating to *instrumental understanding* (Chapter 3), items from trigonometry, logarithm, and statistics were

chosen as stimuli for this purpose. How these items were obtained is the topic of the next discussion, item sampling.

Sampling of the items

(iii) Where would be the appropriate site (e.g. high school, college, or university) for obtaining such items?

It was suggested in the previous discussion (for question (i)) that the mathematical content of the stimulus items should reflect higher level mathematics which requires *symbolic* type understanding (e.g. college or university) to coincide with the secondary pre-service teachers expected mathematical abilities. According to Tall (1992, p.495), 'a major focus in mathematical education at the higher levels is not only to initiate the learner into the complete world of the professional mathematician in terms of the rigour required, but also to provide the experience on which the concepts are founded'. He added that 'traditionally this has been done through a gentle introduction to the mathematical concepts and the process of mathematical proof in school before progressing to present mathematics in a more formally organised and logical framework at college and university' (Tall, 1992, p.495). If this statement is generalised to the Australian education setting, then the appropriate sites for sampling the items that would coincide with the pre-service teachers' abilities would be college (or the last two years of studies prior to university) and university. Furthermore, it was assumed that all the pre-service teachers will have completed college level mathematics prior to university studies. College level mathematics is the link between secondary school mathematics and university mathematics.

(iv) How would these items be obtained?

An important criterion associated with the sampling of items for the data collection instrument is item validity. It was suggested by McAshan (1979) and Griffin and Nix (1991) that for a criterion-referenced evaluation to be valid, the instrument for collecting data must reflect a direct link between what is assessed and the instructional intent or curriculum. This suggests that the stimulus items (or instrument of collecting data) should have a direct link to the mathematics curriculum implemented at college or university settings. This direct link can be achieved appropriately by a classroom teacher or lecturer who has the experience and knowledge of the curriculum and the assessment intention required in order to evaluate the students' learned outcomes (House, 1980; McInerney & McInerney, 1994). With such consideration, each of the three stimulus items was developed and used as a test item, followed by an analysis by either a college teacher or a university lecturer.

In order to satisfy the item selection criteria, as described above, sub-studies for sampling the items involving teachers and their students (study sample) were conducted at two educational settings - college and university. The statistic item was sampled within the college setting and the trigonometric and logarithmic items were sampled within the university setting. In the sampling of the items at each setting, the participating teachers followed three stages: (1) At the completion of teaching the course, an assessment instrument for examining students' mathematical understanding was constructed as in the normal events of assessing students' learning achievements. (2) Students were presented with the assessment instrument under testing or examination conditions. (3) Students' performances were

analysed. As part of the analysis stage, teachers were to identify an item which elicited common misconceptions or understanding difficulties from students. In addition, the teachers were asked to provide a description of the probable causes or reasons for the observed misconceptions by students.

It is acknowledged that time, place, and standard of curricula implementation were variables associated with the selection of items from two different educational settings. Nevertheless, these variables are present with every cohort of students entering secondary teacher education programs. It is accepted that mathematical curricula at these settings are closely linked and many concepts such as trigonometric and logarithmic functions and statistics are common learning areas.

A description of the stimulus items is provided next. These descriptions are organised in three parts: (1) a brief introduction about the participants and the development of the items, (2) the description of the students' misconceptions in relation to the item, and (3) a brief description of the potential items.

The first of these to be described is the statistic item.

Statistical variance

The statistic item was sampled within the college setting. Two secondary mathematics teachers and their 60 students from levels (or year) 11 and 12 agreed to participate in the item sampling. The 60 students had just completed a unit in statistics, and as part of their continuous assessment

they were given a test on statistics. One of the questions in the unit test which was of particular interest for the purpose of this investigation related to the statistical variance. The question was designed to test the students' understanding of the definition of variance as well as their ability to recall the appropriate formulae. The students had been taught the concept of variance as the average of squared deviations from the mean. They had also been taught two methods for calculating the variance. The first method involved setting up tables of computed values corresponding to the

formula: $\frac{\sum (x - \bar{x})^2}{n}$. The second method involved greater use of the

calculator and it required the use of the equivalence formula for the

variance, which is: $\frac{\sum x^2}{n} - \bar{x}^2$.

The following statistics question was identified as a potential stimulus item:

Ten items were measured and four results were produced:

- | | |
|----------------------------------|--|
| 1. $\sum_{i=1}^{10} x_i = 40$ | 2. $\left(\sum_{i=1}^{10} x_i \right)^2 = 1600$ |
| 3. $\sum_{i=1}^{10} x_i^2 = 194$ | 4. $\sum_{i=1}^9 (x_i - \bar{x})^2 = 24$ |

Use these results to find the mean, \bar{x} , and variance, s^2 , for the 10 items.

The evaluation of the 60 college students' responses to the above test item showed them to be highly in favour (86%) of the 4th result,

$$\sum_{i=1}^9 (x_i - \bar{x})^2 = 24.$$

This was indicated by their solutions of 2.4 and 2.7 for the variance. The 4th result was a distraction cue that the teachers used to determine whether the students have learned the second method for the computation of the variance. Although these solutions of 2.4 and 2.7 reflected a common misconception, they also provided a positive indicator, to the teachers, that the respondents did have knowledge of what a variance is. That is, the response of 2.4 was obtained by ignoring the summation to 9 and the result of 24 was divided by 10, the number of items. The response of 2.7 was obtained from dividing the result of 24 by 9. When marking this test item, teachers gave part marks if students produced either of these solutions.

The remaining 14% of the students presented the correct solution of 3.4.

This solution was produced by students who recalled the alternative formula,

$$\frac{\sum x^2}{n} - \bar{x}^2, \text{ and were able to substitute the given information to compute the}$$

correct value, that is $S^2 = 194/10 - 4^2 = 19.4 - 16 = 3.4$.

To find out how teaching is related to these college students' learning, eight students (6 from the respondents with answers of 2.4 and 2.7, and two from those who gave the correct answer of 3.4) who were willing to elaborate on their answers were interviewed in a group setting, as there was limited opportunity to take the students individually away from their normal classes.

Students were asked to respond to three main questions: (1) What is your understanding of the variance? (2) How confident are you in answering a statistical question in the final exam? (3) Why is it important for you to learn statistics? Students did not want their responses to be recorded on tape, but were willing to write their responses on paper first, prior to the group discussion (interview).

The group's response to question (1) reflected the work they did in class, for example, recognising that the summation notation means adding; to find the variance, first find the mean and then the deviations from the mean. The group seemed to have no doubt as to why each deviation from the mean is squared, to avoid a sum of zero. When asked what is a variance? The response was, S^2 and that it is much easier to use the calculator to compute the variance than to compute and set up values in a table form.

The responses to question (2) were mainly in the form: *at this stage, very little, but hopefully I can pick it up again later during study week.* When asked to explain the reason for their lack of confidence, they suggested that *statistics has lots more formulas to remember than the other topics in mathematics* and that *it is best to learn these afresh just before the exam.*

Why is it important for you to learn statistics (question 3)? The following responses indicated why (or if) statistics was important to the group.

Because it's part of the maths course I'm doing. I guess so that I can say I learned statistics. I don't really know, may be because I'll be examined on it later. It's important because it increases my knowledge

of maths. Statistics is different from other maths so learning about it increases my knowledge of maths.

The above responses are by no means a generalisation of the learning for the 60 college students involved. However, the responses do provide an insight into the types of knowledge that could be generated by such an item. Based on the above responses by the eight students, it appears that *rote knowledge* of statistics was the dominant form of knowledge acquired by these students.

From the above evaluation, the following stimulus item was formed:

The Statistical Variance item	
<p>A class of year 11/12 students was asked to find the variance using the information given below:</p> <p>Ten items were measured and four results were provided:</p> <div style="display: flex; justify-content: space-around;"> <div style="text-align: center;"> <p>1. $\sum_{i=1}^{10} x_i = 40$</p> <p>3. $\sum_{i=1}^{10} x_i^2 = 194$</p> </div> <div style="text-align: center;"> <p>2. $\left(\sum_{i=1}^{10} x_i \right)^2 = 1600$</p> <p>4. $\sum_{i=1}^9 (x_i - \bar{x})^2 = 24$</p> </div> </div> <p>The class produced 3 different values for the variance, S^2:</p> <div style="display: flex; justify-content: space-around;"> <p>i) 2.4</p> <p>(ii) 3.4</p> <p>(iii) 2.7</p> </div> <p>Which variance is the correct one? Please explain.</p>	

The items for trigonometry and logarithm are described next.

Trigonometric and logarithmic functions

The following two questions were part of an end of semester examination paper for first year university students who had enrolled in a Science degree program. The mathematics course associated with the examination paper

covered topics in introductory calculus, trigonometry, and logarithm. These two questions were identified as potential stimulus items:

Trig. Evaluate for x , $\cos(2x+1) = 0$ and provide the graph for $y = \cos(2x+1)$.

Log. Simplify and evaluate for x , $\log_{10}(2x+1) = \log_{10}(x-1)$. Justify your answer.

A group of 28 first year university students sat the exam and their responses to the above questions were analysed. Because these responses were given in an examination situation, confidentiality of the students' identification did not allow the opportunity for student interviews.

The group's responses to the trigonometric item showed that 44% made the following type of response when evaluating for x in $\cos(2x+1) = 0$:

$$\begin{aligned}\cos(2x + 1) &= 0 \\ \cos 2x + \cos 1 &= 0 \\ \cos 2x &= -\cos 1 \\ 2x &= -1 \\ \therefore x &= -1/2\end{aligned}$$

There could be several explanations for these university students' responses but two possibilities are described below:

- (1) These students assumed that *cos* was a variable and that the brackets stand for the multiplication operation. Their attempts to graph $\cos 2x$ and $\cos 1$ rather than $\cos(2x+1)$ tend to suggest an inability to disassociate algebraic techniques from conceptual knowledge of special functions such as trigonometric functions.

- (2) The students lacked understanding of the trigonometric rule:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\text{or } [\cos(2x+1) = \cos 2x \cos 1 - \sin 2x \sin 1].$$

For example, it was common for students who recalled the rule correctly to suggest that $\sin 1 = 0$ or $\cos 1 = 1$, leading to $\cos 2x = 0$ and $x = \pi/4$ as their final responses. This type of final outcome appeared to relate to a misconception between the cosine and sine functions as well as degrees and radians, rather than an inability relating to algebraic procedures. In fact, respondents who used the trigonometric rule correctly often failed to complete their computations. However, they were more likely to provide a correct graph for $y = \cos(2x+1)$. The response data by students who recalled the rule correctly seemed to present another type of misconception relating to the stimulus item. This tends to suggest that explanation (1) might be the probable reason for the above learned outcome.

For the logarithmic question, 61% of the respondents presented the following type of logic:

$$\begin{array}{ll} \log(2x+1) = \log(x-1) & \\ \log 2x + \log 1 = \log x - \log 1 & (\log 1=0) \\ \log 2x - \log x = 0, & (\log_{10} x = 0) \\ x = 10^0 \quad \therefore \quad x = 1 & \end{array}$$

Several of this group did not make the connection between logarithms and exponents, as made in the last two lines. These respondents performed the first three rows as algebraic simplification. To 'evaluate for x' they factorised as follows:

$\log 2x - \log x = 0$ $\log(2x - x) = 0$ $x = 0/\log = 0$	$\log 2x - \log x = 0$ $\log x (2-1) = 0$ $\log x = 0$
--	--

In justifying their solutions, all 28 students correctly stated that there was no real solution. However, the 61% group's justification was that their solution for x , $\log x$, or by substitution which produced $\log 1$ or $\log(-1)$, was zero. For example, those who responded with $x=1$, justified their 'no solution' by substituting 1 for x into $\log(2x+1) = \log(x-1)$. This was simplified to $\log 3 = \log 0$ and then to $\log 3 = 0$. The students' use of phrases such as, "you can't have the log of a number equal to zero", "log3 cannot equal zero it's undefined", and "a solution doesn't exist", further indicated the difficulty they had in disassociating algebraic techniques from conceptual understanding of special functions.

From the actual exam questions and the common misconceptions by these 28 first year university students, the following were considered potential stimulus items.

The trigonometric item	The logarithmic item
<p>Evaluate for x, $\cos(2x+1) = 0$</p> <p>A student responded:</p> $\begin{aligned}\cos(2x + 1) &= 0 \\ \cos 2x + \cos 1 &= 0 \\ \cos 2x &= -\cos 1 \\ 2x &= -1 \\ \therefore x &= -1/2\end{aligned}$ <p>Q1. Is the student's response correct?</p> <p>Q2. Please explain why you answered yes/no to Q1.</p>	<p>Simplify and evaluate for x, $\log_{10}(2x+1) = \log_{10}(x-1)$</p> <p>A student responded:</p> $\begin{aligned}\log(2x+1) &= \log(x-1) \\ \log 2x + \log 1 &= \log x - \log 1 \quad (\log 1=0) \\ \log 2x - \log x &= 0, \quad (\log_{10} x = 0) \\ x &= 10^0 \\ \therefore x &= 1\end{aligned}$ <p>Q1. Is the student's response correct?</p> <p>Q2. Please explain why you answered yes/no to Q1.</p>

Having selected the three potential stimulus items the next step was to determine their face validity as an instrument for collecting data on mathematical knowledge of secondary pre-service teachers of mathematics. This step is described in the following sections.

4.2.2 Validating the selected stimulus items

This section is a description of the study conducted for the validation of the selected stimulus items. This study involved a group of 18 experienced secondary mathematics teachers. These teachers were presented with the three stimulus items in a one-to-one interview situation, with interviews recorded on cassette tapes. The teachers were from 6 colleges (2 private and 4 public). The group had experience in teaching secondary mathematics ranging from 5 to 30 years, most with 10 to 25 years experience. Other information collected about the teachers were their mathematics qualifications and the mathematics courses taught - including the units they were currently teaching (see Table 4.2, next page).

The decision to use a sample of experienced mathematics teachers for the validation study was twofold. The main reason was to have the items validated by a group of experts in the field (House, 1980) and to find out whether the items are representative of the types of mathematical knowledge taught in secondary schools. Also, it was important that the item-format be checked by experienced mathematics teachers for appropriateness, particularly the use of 'student error response' as a means for eliciting mathematical knowledge.

Table 4.2: Data on experienced mathematics teachers' background

No. of teaching years	Degree other than Dip.Ed	Maths currently teaching	Maths taught
18	B.Sc, M.Ed	Stage 3, GenM	All college levels.
30	B.Ed	Computing	All high school levels
6	B.Ed (Maths major)	GenM, AppM	Part-time teaching
15	B.Ed (Maths major)	AppM	All high school levels
20	B.Sc	GenM, AppM	All high school levels
13	B.Sc	GenM, AppM	All college levels.
12	B.Sc	GenM, AppM	All college levels + physics
3	M.Sc	High Sch	yr 8, 9, 10 + Science
27	B.Sc	AppM	All college levels.
9	B.Ed	GenM, AppM	Part-time teaching
16	B.Sc	Stage 1,2; Physics	Top college level maths + physics
18	B.Sc	Stage 2,3; GenM	All college levels.
30	B.Sc	AppM	All college levels.
15	M.Ed	AppM	Lower college levels.
11	B.Sc (Maths major)	AppM	All college levels.
19	B.Sc (Biology)	AppM, GenM	Lower college levels.
10	B.Sc	AppM	No data
11	B.Sc	Stage 2,3	All college levels.

Both the **General Mathematics (GenM)** and the **Applied Mathematics (AppM)** are college level mathematics units. General Mathematics covers a wide range of mathematical topics including algebra, trigonometry, finance, and statistics. Applied Mathematics (AppM) covers similar areas in greater depth as well as an introduction to calculus. As its name implies, AppM provides students with knowledge of the application of the mathematics. For example, in the algebraic modelling component, logarithm is treated as a method for linearising exponential functions, and trigonometry as a system of formulae for solving real-life situations involving the computation of angles, distances, and time. The statistics component covers data organisation (eg. distribution tables, graphs), random variables, sampling, measures of central tendency, measures of spread (including standard deviation and variance), and probability.

The **Mathematics Stage 1, 2, and 3** are also college level mathematics units. The Stage 1 mathematics is taken in year 11 as a pre-requisite for Stage 2 and Stage 3 mathematics in year 12. These 3 mathematics units are compulsory pre-requisite courses for entry into a mathematics degree program at most Australian universities. Prospective teachers of secondary school mathematics are assumed to have satisfactorily completed these units of mathematics during their pre-university education.

The second factor in selecting a sample of experienced mathematics teachers was to gather data on mathematical knowledge that are related to experiences of learning and teaching mathematics. It is assumed here that teachers acquire knowledge (or learn) about the mathematics they teach as part of their lesson planning and preparation experience. In other words, a teacher's growth in mathematical knowledge, for a particular mathematics, is influenced by his or her experiences in teaching that particular mathematics. The data from interviewing these teachers were examined in an attempt to collect information on mathematical knowledge associated with teaching. In addition, these interviews provided data for a 'test-run' analysis of data using Skemp's model.

In addition to the validation of the items, the method of collecting the data (described in Section 4.1.6) by interview was put to trial. At the start of the interview, the participants were told that the purpose of the interview was to test items as to their potential in collecting data on knowledge about mathematics from pre-service teachers of mathematics. They were then asked, first of all, to *read* the content of each item prior to giving an explanation, in *written* form, of their responses based on their teaching experiences. After this initial attempt the participants were then interviewed *to elaborate* on their written responses and also to comment on the format of the items and as to whether the items were adequate to provoke thinking and to elicit the kinds of responses the items were designed for.

The interviews with these 18 experienced mathematics teachers provided valuable information on how to improve the stimulus items as a data collection instrument for the main study. For example, it was found that

the one-to-one interviewing situation provided opportunity for the participants to elaborate, clarify, and reflect upon their responses. These responses were more in depth and spontaneous than written responses. Also, the interview data showed that the teaching of the topics, particularly the logarithmic and the statistic items, was closely related to the teachers' own perception of important knowledge for students to learn. For example, several teachers responded to the logarithmic item in a similar manner as the following:

This student doesn't understand logarithm at all and he needs to be shown the basics first. That is, by using the calculator and inputting different values to check out the log laws. E.g. $\log 9 = \log 3 + \log 3$ is $\log ab = \log a + \log b$ and so on. It is really important that students should have good calculator skills, it saves them a lot of valuable time that they can then spend on other important topics. (categorised in Appendix A as 'instrumental')

In addition, it was observed that if teachers have not had the opportunity to teach a particular unit of mathematics, then their responses tended to reflect a lack of knowledge about the mathematics. The statistic item in particular received the majority of these types of responses. Typical responses indicating this lack of knowledge were, for example:

I can't answer this because I have not taught the statistics topic yet, that is coming up next term, (categorised in Appendix A as 'instrumental')
or

This type of statistics was not part of the courses when I was at teachers' college and I have only taught the basic statistics like finding the mean, mode and median, and drawing frequency tables and graphs.

From the outcomes of the 18 experienced mathematics teachers' interviews, two issues were observed to be relevant in examining student-teachers' knowledge of mathematics: (1) the teacher's perception of what is to be important learning for students, and (2) the teacher's teaching experience of the topic.

These findings seem to support the aims of the *second research question* and were therefore incorporated into the initial set of stimulus items with the addition of stimulus questions (2) and (3) to each item. This final set of three stimulus items is presented in the next section.

4.2.2.1 The stimulus items

The set of three stimulus items is presented in Figure 4.1 (next page). In each stimulus item, there are three stimulus questions denoted as SQ1, SQ2, and SQ3. There are two essential components of the stimulus items which correspond to SQ1, SQ2, and SQ3. The first component concerns the stimulus question 1 (SQ1) and it was designed for addressing the first research question. The second component concerns the stimulus questions 2 and 3 (SQ2, SQ3) and it was designed for addressing the second research question.

Figure 4.1 The three mathematical stimulus items.

<p>(1) The trigonometric item:</p> <p>Evaluate for x, $\cos(2x+1) = 0$</p> <p><i>A student responded:</i></p> $\begin{aligned}\cos(2x + 1) &= 0 \\ \cos 2x + \cos 1 &= 0 \\ \cos 2x &= -\cos 1 \\ 2x &= -1 \\ \therefore x &= -1/2\end{aligned}$ <p>If you were the teacher:</p> <p>SQ1. Would you accept the student's response as being correct?</p> <p>SQ2. What do you consider important about the learning of trigonometry that you must teach your students?</p> <p>SQ3. How would you approach the teaching of trigonometry? Please explain and give an example of your teaching method.</p>	<p>(2) The logarithmic item:</p> <p>Simplify and evaluate for x, $\log_{10}(2x+1) = \log_{10}(x-1)$</p> <p><i>A student responded:</i></p> $\begin{aligned}\log(2x+1) &= \log(x-1) \\ \log 2x + \log 1 &= \log x - \log 1, \quad (\log 1=0) \\ \log 2x - \log x &= 0, \quad (\log_{10} x = 0) \\ x &= 10 \\ \therefore x &= 1\end{aligned}$ <p>If you were the teacher:</p> <p>SQ1. Would you accept the student's response as being correct?</p> <p>SQ2. What do you consider important about the learning of logarithm that you must teach your students?</p> <p>SQ3. How would you teach logarithm? Please explain and give an example of your teaching method.</p>				
<p>(3) The statistic item:</p> <p>A class of year 11/12 students was asked to find the variance using the information given below:</p> <p>Ten items were measured and four results were produced:</p> <table><tr><td>1. $\sum_{i=1}^{10} x_i = 40$</td><td>2. $\left(\sum_{i=1}^{10} x_i\right)^2 = 1600$</td></tr><tr><td>3. $\sum_{i=1}^{10} x_i^2 = 194$</td><td>4. $\sum_{i=1}^9 (x_i - \bar{x})^2 = 24$</td></tr></table> <p>The class produced 3 different values for the variance, S^2:</p> <p>(i) 2.4 (ii) 3.4 (iii) 2.7</p> <p>If you were the teacher:</p> <p>SQ1. Which variance would you accept as the correct answer?</p> <p>SQ2. What do you consider important in the learning of statistical variance that your students must learn?</p> <p>SQ3. How would you approach the teaching of variance? Please explain and give an example of your teaching method.</p>		1. $\sum_{i=1}^{10} x_i = 40$	2. $\left(\sum_{i=1}^{10} x_i\right)^2 = 1600$	3. $\sum_{i=1}^{10} x_i^2 = 194$	4. $\sum_{i=1}^9 (x_i - \bar{x})^2 = 24$
1. $\sum_{i=1}^{10} x_i = 40$	2. $\left(\sum_{i=1}^{10} x_i\right)^2 = 1600$				
3. $\sum_{i=1}^{10} x_i^2 = 194$	4. $\sum_{i=1}^9 (x_i - \bar{x})^2 = 24$				

The purpose of the first component of the stimulus items was to elicit responses which are representative of procedural and conceptual types of mathematical knowledge. The three selected mathematical concepts (trigonometry, logarithm, and statistics) for the items were represented by the following mathematical expressions.

1) $\cos(2x+1) = 0$,

2) $\log(2x+1) = \log(x-1)$, and

3) $S^2 = \frac{\sum (x - \bar{x})^2}{n} = \frac{\sum x^2}{n} - \bar{x}^2$

Each mathematical expression is the focal point for the cued-data contained within the stimulus item. Each expression represents 'a mutual assimilation between a symbol system and a conceptual structure' (Skemp 1982, p.61). The function of the symbols is for manipulating and communicating the mathematical concepts. For example, in the first expression, $\cos(2x+1)$, 'cos' specifies the operation (finding the cosine value of the angle) and '2x+1' specifies the operand (the angle size of 2x+1). Similarly in the second expression, $\log(2x+1)$, 'log' specifies the operation (finding the logarithmic value of a non-negative expression) and '2x+1' specifies the operand (the value of the non-negative expression). For the third expression, ' Σ ' specifies the operation (finding the sum of a system of values) and $\frac{(x - \bar{x})^2}{n}$ specifies the operand (a system of values). Each stimulus item, therefore, requires symbolic type understanding, an understanding that is 'dominated by conceptual structures' (Skemp, 1982, p.61).

However, it is quite possible for these three stimulus items to also elicit responses dominated by *procedural structures*. For example, to evaluate for x in $\log(2x+1) = \log(x-1)$, the respondent could proceed as follows:

Cancel 'log' from both sides of the equal sign because it is a common factor. That leaves $(2x+1) = (x-1)$. Solving for x , the result is $x = -2$. Checking that this value of x is appropriate for the equation, substitute $x=-2$ into $\log(2x+1) = \log(x-1)$. The result is, $\log(-3) = \log(-3)$. Since both sides of the 'equal sign' are the same, the value of $x=-2$ is correct. [Response-data from the experienced teachers' interview data]

Although the above response satisfies the request to find the value for x , the respondent's manipulation of the symbols did not communicate his or her knowledge of the mathematical concept in question, namely logarithm. According to Skemp (1982), the symbols must be linked appropriately to their conceptual structures and be interpreted in terms of the relationships within the structures in order for a response-data to be classified as representative of symbolic understanding.

The purpose of the second component (SQ2 and SQ3) of the stimulus items was to elicit responses which are representative of knowledge pertaining to pedagogical content knowledge. It was suggested in Chapter 2, Section (2.3) that teachers' competence to teach mathematics (or pedagogical knowledge) is dependent on their understanding of the mathematics they teach. Pedagogical knowledge, however, also seems to involve informed decision-making by teachers about 'worthwhile learning activities' for students and how to present the mathematics content appropriately to students (Ball & McDiarmid, 1990, p.437).

Similar forms of decision-making by teachers were suggested to reduce gender differences in mathematics (Fennema, 1996; Rhine, 1998). In order to examine how procedural based (or conceptual based) mathematical understanding would influence a pre-service teacher's decisions concerning teaching, SQ2 and SQ3 were incorporated into the design of the three mathematical stimulus items as cues for eliciting response data.

The description of how the data from this validation study were organised and analysed using the Skemp model is the purpose of the next section.

4.2.3 Analysis of data from the validation study using Skemp's model

In Chapter 3, Skemp's model of mathematical understanding was described as the chosen analysis method for the main study. This method involves the formation of predicted response patterns as base-line patterns for comparison and evaluation of responses (Section 4.1.6). Therefore, the main aim in this section is to analyse the interview data from the 18 experienced mathematics teachers and examine these data in terms of Skemp's model in an attempt to form predicted response patterns.

This section is organised in three parts: (1) a description of the initial categorisation procedures and the formation of six predicted response patterns, (2) a detailed description of the response patterns under four categories, and (3) a summary of this analysis.

(1) Initial categorisation of responses

For each of the three stimulus items, the responses were initially categorised according to how a respondent had approached the teaching of the particular mathematics and in relation to Skemp's three kinds of mathematical understanding: *instrumental*, *relational*, and *symbolic*. An example of each categorisation for the *instrumental*, *relational*, and *symbolic* understanding are presented in the following:

A response classified as indicating instrumental type understanding was one where rote memorisation of rules appeared to be the main teaching/learning approach used. For example, the following response was classified as *instrumental*:

The learning of this [referring to the logarithmic item] is usually best done by them [students] being thoroughly acquainted first with the log laws and how to apply them. Otherwise they will have difficulty in doing a question of this kind ... the efficient way of learning these [log laws] and many of the rules in maths is by memorising them, I reckon.

A response indicating an interlinking of knowledge aspects relating to the learning of the particular mathematics was categorised as *relational*. An example of this type is:

My experience in teaching logs seemed to vary according to the students I have. E.g. last year I had a group who needed a lot of specific guidance as to what key information they need to know or bring forward in order to successfully solve a problem. So when I taught them logs, as an algebraic

model, I had to do a lot of back-tracking to graphing skills, the work on indices they've done sometime before ... it was no use assuming that they know the indices work well. My group this year is so different ... when I introduced the idea of logs using graphs and comparing them to exponential graphs, they didn't give me the thumbs down! So its either because I've learned from my last year's experience how to introduce logs or my group now has a better lot of kids.

A response showing evidence of a justification for the given mathematical situation was classified as belonging to *symbolic* understanding. The following is an example:

*I'd start with explaining that $\log(a + b) \neq \log a + \log b$
but $\log a + \log b = \log(ab)$ and $\log a - \log b = \log(a/b)$.
Then $\log(2x+1) = \log(x-1)$
 $\log(2x+1) - \log(x-1) = 0$ -> can use $\log a - \log b = \log(a/b)$ here
 $\log[(2x+1)/(x-1)] = 0$ -> the inverse $\log_{10} x = 0$ -> $x=10^0$ that the student correctly stated is used here.
 $(2x+1)/(x-1) = 10^0$ -> $10^0=1$, the student stated this correctly too.
 $2x+1 = x-1$ -> if $x-1 \neq 0$;
 $x = -2$.
However, $\log x$ is only valid for $x \geq 0$, so $x = -2$ needs checking whether it provides a valid solution. Substituting -2 into $\log(2x+1) = \log(-3)$, this is undefined. Therefore $x = -2$ does not provide a valid solution. That is, $\log(2x+1) = \log(x-1)$ has no real solution. I suspect the outcome of $\log(-3)$ when checking would not alert the student who does not understand the concept of log functions.*

This initial categorisation provided three main sets of response data: (1) instrumental type knowledge, (2) relational type knowledge, and (3) symbolic type knowledge. In order to formulate the predicted response patterns suggested in Section (4.1.8), each set of responses was examined in

terms of *intuitive* and *reflective* modes of thinking (or procedural and conceptual types of knowledge). A classification technique that was found valuable during this examination process was categorising responses according to four response criteria (or categories) based on evidence of what the respondent had indicated that he or she can or cannot do. This technique was based on the model of *a response production of mathematical knowledge* illustrated in Figures 2.4 and 2.5 (Chapter 2) in which four types of mathematical response productions were suggested. These were: *simple*, *relatively simple*, *relatively complex*, and *complex* (Figure 2.5). To avoid confusion between a *suggested* mental process (an unobservable response production) and an actual response (observable response), the suggested four mathematical response productions are referred to as category (1), category (2), category (3), and category (4) respectively. The following are descriptions of knowledge involved in each classification:

- Category (1). Little or no recall of knowledge about the mathematics presented.
- Category (2). Can recognise rules, theorems, symbols, or a system of procedures or methods.
- Category (3). Can carry out computations by applying a rule or a set of procedures.
- Category (4). Can demonstrate as in (2) and (3) as well as providing a justification or reason for a given result.

These four response categories are included in Figure 4.2 (next page) in relation to Skemp's model. These categories, however, do not necessarily correspond to Skemp's (1979) modes of mental functioning (as in Chapter 3, Table 3.1). Particularly, the response-data indicating evidence of 'little or no recall of appropriate knowledge' as in category (1).

However, category (4) type responses seemed to appropriately describe Skemp's third form of mathematical understanding, namely symbolic. These four classification criteria are therefore the author's own interpretation of the types of responses associated with Skemp's categories in Chapter 3, Table 3.1. Figure 4.2 illustrates how category (2), category (3), and category (4) data are linked to Skemp's (1979) three kinds of mathematical understanding. However, category (1) is included in Figure 4.2 as a separate kind of response pattern, at the top, because the response classification does not appear to directly belong to any of Skemp's three types of mathematical understanding. Thus, seven kinds of response patterns are predicted. In addition, the *italics* response data in Figure 4.2 are actual teacher responses (interview data from the 18 experienced teachers).

Figure 4.2: **An adaptation to Skemp's model of mathematical understanding**

Category (1) The individual - Cannot recall appropriate rules or procedures. <i>E.g. I cannot remember because it's a while since I learned this.</i> <i>E.g. I remember doing ... but I didn't really understand it well at the time.</i>		
Skemp's model	Category (2) (Intuitive) Pseudo-Procedural knowledge The individual -	Category (3) (Reflective) Pseudo-Conceptual knowledge The individual -
Instrumental Understanding	Recognised a rule, theorem, symbols, etc. E.g. $\log(axb) = \log a + \log b$ Recognised a given situation by its appearance only. E.g. When given (1) $\cos(2x+1) = 0$ (2) $\cos 2x + \cos 1 = 0$, would ponder over the given situation for some time and then suggest, <i>it appears wrong but I cannot explain why.</i>	Can do computations by applying a rule or a set of procedures. E.g. When asked to evaluate $\log(2x+1)=\log(x-1)$ would immediately remove the 'log' from both sides of the equal sign (=) and proceed to solve for x as follows: $2x+1 = x-1$ $2x-x = -1-1$, therefore $x = -2$.

.... Figure 4.2 continues over to the next page ...

Skemp's model	Category (2) (Intuitive) Relational-Procedural knowledge The individual -	Category (3) (Reflective) Relational-Conceptual knowledge The individual -
Relational Understanding	<p>Recognised a system of related procedures or methods for a given situation.</p> <p>E.g. When given (1) $\cos(2x+1) = 0$ (2) $\cos 2x + \cos 1 = 0$, would immediately suggest that line (2) is not appropriate <i>because</i> $(2x+1)$ represents the angle measure.</p>	<p>Can do computations by applying appropriate theorems and a system of related procedures.</p> <p>E.g. When asked to evaluate $\log(2x+1) = \log(x-1)$ would begin by recalling logarithmic laws: $\log A - \log B = \log A/B$, if $\log B \neq 0$ $\log(2x+1) - \log(x-1) = 0$ $\log[2x+1/(x-1)] = 0$.</p> <p>Recalled another rule or the inverse form: $\log_{10} A = x \rightarrow A = 10^x$ $2x+1/(x-1) = 10^0 \rightarrow$ where $10^0 = 1$ \rightarrow and $x-1 \neq 0$ $2x+1 = x-1$ therefore, $x = -2$.</p>

Skemp's model	Category (4) (Intuitive - Procedural knowledge) The individual -	Category (4) (Reflective - Conceptual knowledge) The individual -
Symbolic Understanding	<p>Provided a justification for a given result using a logical progression of steps giving evidence of an awareness that something is 'true' or 'false'.</p> <p>E.g. Follow on from the Relational Understanding category (2) above: (1) $\cos(2x+1) = 0$ (2) $\cos 2x + \cos 1 = 0$ a justification might be:</p> <p><i>The first mistake is</i> $\cos(2x+1) \neq \cos 2x + \cos 1$. <i>The correct solution for</i> $\cos(2x+1) = 0$ <i>is:</i> (1). Find values of $\cos \theta = 0$, $\theta = \pi/2, 3\pi/2, \dots$ (2). Solve for x, $(2x+1) = \theta = (2n-1)\pi/2$, where $n = \text{integer}$, $2x = (2n-1)\pi/2 - 1$ $\therefore x = (2n-1)\pi/4 - 1/2$</p>	<p>Provided a justification for a given result based on prior learning/teaching experiences. This background knowledge is demonstrated by providing reasons consistent with the situation.</p> <p>E.g. Follow on from Relational Understanding category (3)'s computation of $x = -2$, the next step might be:</p> <p><i>Must check that this value of</i> $x = -2$ <i>is a valid solution,</i> $\log(-3) = \log(-3)$. <i>Although this is a true equation, for logarithm it is invalid because a logarithm of a negative number is undefined in the field of real numbers. Therefore, $x = -2$ is not a valid solution and this equation has no real solution.</i></p>

(2) Description of the response patterns under four categories

For consistency and clarity in this section, all statements in *italics* are actual response data. Also, the categorisation of a response into instrumental and relational understanding are indicated, in brackets, by INST and RELAT respectively. For symbolic understanding, or category (4), the response data are indicated as either belonging to the intuitive or reflective mode of functioning (Skemp, 1979). In addition, the term *cued-data* refers to the content of the stimulus item, particularly the content with respect to the '*student's response*'.

The analysis of the response data from the sample of experienced secondary mathematics teachers is discussed under the four response categories as follow.

Category (1) response-data

Responses showing non-attempts because of lack of knowledge about the mathematics in question were classified as category (1). It was observed in these data (responses from experienced secondary mathematics teachers), that the main reason given for the teachers' lack of knowledge was their lack of teaching opportunity (or experience) in the area. For example:

- a) *I can't do this question [referring to the statistic item] because I don't really have much knowledge in this area. I did not do this type of statistics in high school or at teachers' college and I haven't taught it. [INST]*

- b) *I can't remember, I haven't taught much statistics.* [INST]
- c) *Sorry, I can't answer this question, this is my first year teaching Mathematics Applied and I haven't covered statistics yet. Also it has been a long time since I've done statistics.* [INST]
- d) *I can't remember [referring to the trigonometric item], but I don't think the student [referring to the cued-data] is correct.* [INST]

The categorisation of the above responses as instrumental [INST] understanding was based on the assumption that the notion of 'cannot remember' implies that there was, in the past, knowledge (or information) about statistics or trigonometry but the essential knowledge schema for these mathematics was not well established. Also, the failure to make recall of relevant knowledge, due to lack of teaching opportunity, seemed to suggest a dependency by these teachers to gain knowledge from external means (e.g. other teachers, textbooks, syllabus outline) or incentives (e.g. promotion to senior positions). The dependency on external guidance was suggested by Skemp (1978) to be the main cause of instrumental understanding because it is like '[a] person [teacher] with a set of fixed plans [syllabus outline, textbooks], [who] can find his way from a certain set of starting points to a certain set of goals [in teaching]' (Skemp, 1978, p.14). According to Skemp (1978), the fixed plan tells teachers (or students) what to do at each choice point but this does not give them an awareness of the overall relationship between successive stages, and the final goal.

In addition to the 'lack of knowledge' type responses are those which appear to reflect a category (2) type response (knowledge recognition), but

showed evidence of incomplete (or false and incorrect) attempts. For example:

- a) *The student is incorrect [referring to the given cued-data] because $\log(2x+1) \neq \log 2x + \log 1$. Correction: $\log(2x+1) = \log 2x(\log 1)$ [incorrect recall]. Similarly $\log(x-1) = \log x / \log 1$ [incorrect recall]. The respondent could recognise that certain rules are necessary, however, the attempt was incorrect. This type of response was categorised as instrumental [INST] understanding because it seemed to be dependent on the recall of memorised rules.*
- b) *None of the given values [referring to the statistic item] are correct, because you need $\sum_1^n (x - \bar{x})^2$, but $n=9$ is given, not $n=10$. [INST]*
- c) *The formula that I have been using recently [referring to the statistic item] is not obvious from the data given. So at first glance, I'll say there is not enough information given. [INST]*
- d) *Actually, we only need to find the standard deviation and then square that. But I can't remember the formula in order to answer the question. [INST]*

Response (b) is false because one of the given values in the cued-data was correct, however, the respondent's dependency on a 'fixed plan' or a particular formula appears to be the cause of this partial response.

Response (c) also illustrates a dependency on a 'fixed plan' and like the respondent for response (b), this respondent could not proceed any further.

In response (d), there is evidence to suggest that the respondent has knowledge about the particular mathematics (statistical variance) but in

another form (*standard deviation and then square*), and like (b) and (c), no further action was taken because the specific formula was not available.

In summarising the analysis to this point, the lack of opportunity (or experience) to teach the particular mathematics (e.g. statistics) appeared to contribute to the teachers' lack of knowledge about the mathematics represented by the stimulus items. Another factor that appeared to be associated with category (1) types of mathematical knowledge is teacher dependency on external guidance and knowledge of specific (or fixed) formulae. For example, textbooks could be classified as an external medium to provide teachers with formulae, set procedures and algorithms in computation, and the types of work to give to their students. The type of mathematical knowledge displayed in category (1) seemed to be associated with instrumental understanding of the mathematical concepts represented by the three stimulus items.

Category (2) response-data

This classification was based on the criteria that the response-data would contain evidence of relevant knowledge associated with rules, theorems, symbols, and systems of related procedures for the given cued-data. However, this type of response is limited to the extent that the respondent can recognise only a single relevant knowledge aspect to a given situation.

Consider for example the following response to the cued-data in the logarithmic item:

Given cued-data: $\log(2x+1) = \log(x-1)$

$$\log 2x + \log 1 = \log x - \log 1$$

A category (2) response: $\log(A+B) = \log A + \log B$ is not correct
because this is not $\log(A \times B)$.

This response-data indicated evidence that the respondent recognised a correct rule (a single relevant knowledge) associated with logarithmic computation of the given situation. That is, $\log(A \times B) = \log A + \log B$.

Further examples of category (2) type responses and their categorisation into instrumental [INST] and relational [RELAT] understanding of mathematics are as follow:

- a) *The student used incorrect log theorem because when $x=1$, $\log(3) \neq \log(0)$. By using the calculator the teacher can show the student that $\log(9+1) \neq \log 9 + \log 1$. [INST]*
- b) *The student is incorrect because $\log(2x+1) \neq \log 2x + \log 1$, but I can't explain the rest of what the student has done. I need more time to think about this because the teaching of logs in Maths Applied [mathematics unit currently teaching] takes a different approach ... I mean ... its on application more. [INST]*
- c) *Initially, I would have said 2.7 because $\frac{24}{9} = 2.7$ [referring to the given cued-data in the statistic item where $n=10$ and result (4) was $\sum_{i=1}^9 (x_i - \bar{x})^2 = 24$]. But quickly referring to the formula in the textbook [the interview was conducted in the respondent's office where access to textbooks was possible at a hand's length], it is 3.4. There are too many formulas for students to commit to memory. In statistics where some formulas are rarely required these are best to be obtained*

from other sources, e.g. textbooks, calculators, when called for, like the variance it is available on the calculator. [INST]

- d) *The student is incorrect in treating the problem [referring to the trigonometric item] as a simple algebraic expression because $\cos(2x+1) \neq \cos 2x + \cos 1$, $(2x+1)$ is an angle. The student needs more practice in evaluating simpler forms, e.g. $\sin x = 0$, $\cos x = 1$, first ... [and] knowing how to use the calculator for trig functions is an essential skill to develop before moving on to this form [referring to $\cos(2x+1)=0$]. [RELAT]*

Responses (a), (b) and (c) were classified as instrumental because of the following reasons. In response (a), there was an indication that knowledge of formulae and theorems was important. However, this knowledge was not evident but rather the use of an algebraic procedure to demonstrate that the given cued-data was incorrect. The calculator was also mentioned as a method for teaching logarithm. Response (b) illustrated that the respondent had knowledge of logarithmic laws but was unable to explain the 'student error' due to differences in teaching approach. Response (c) tends to indicate that formulae which are seldom used, such as those in statistics, are perhaps not taught to students as usable knowledge but as external cues (or stimuli) for memory recall. In response (d), there was an indication of relational understanding of trigonometry. The respondent identified the possible cause for the 'student error' in relation to other forms of mathematics (*simple algebraic expression*), and stated the student's error ($\cos(2x+1) \neq \cos 2x + \cos 1$, because $(2x+1)$ is an angle), and also suggested steps for the student to achieve this knowledge more accurately (*... need more practice in evaluating simpler forms*).

Category (3) response-data

This classification was based on the criteria that the response-data would contain evidence of more than one relevant knowledge aspect associated with the rules, theorems, symbols and systems of related procedures for the given cued-data. In addition, there was evidence of computational knowledge.

To illustrate this classification, a response to cued-data in the trigonometric item is given below:

Given cued-data: $\cos(2x+1) = 0$
 $\cos 2x + \cos 1 = 0$

A category (3) response:

- (i) $\cos(A + B) \neq \cos A + \cos B$, because '*cos*' is a function not some algebraic quantity.
- (ii) $\cos 90^\circ = 0$
 $2x + 1 = 90^\circ$
 $2x = 89$
 $x = 44.5^\circ$
- (iii) Or apply the Trig. formula which I've forgotten, something like $\cos(\pi + n) = 0$? (INST)

The respondent's computation in part (ii) was consistent with the statement that '*cos*' is a function (or a symbol denoting a trigonometric function) and not an algebraic quantity (or expression). That is, $(2x+1)$ was recognised as the 'angle measure', this was confirmed by the procedures showing the cosine of 90 degrees has a value of zero. The computation for the value of x concluded with $x = 44.5^\circ$. Although this value was incorrect, this was accepted by the respondent as valid. The closure in part (iii) had no real link to the previous operations and even though the trigonometric formula was referred to, there was indication (*I've forgotten*) of incomplete knowledge.

The above response was categorised as instrumental (INST) understanding since the computational knowledge demonstrated in part (ii) appears to be based on rote memorisation of algorithms for trigonometric situations. Part (iii) tends to confirm the connection of these algorithms to rote learned rules and formulae rather than to knowledge relating to the concept of functions as stated in part (i).

Further examples of response-data classified as belonging to category (3) are described below.

Given cued-data: **Simplify and evaluate for x , $\log_{10}(2x+1) = \log_{10}(x-1)$**

A category (3) response-data is as follows:

*log in base 10 is common on both sides,
so $\log(2x+1) = \log(x-1)$ can be simplified to
 $2x+1=x-1$
 $x = -2$. [INST]*

The above response represents the most common type of response to the logarithmic item from among the group of experienced mathematics teachers. This type of response showed evidence that the respondent had focused on the key 'statement' at the top of the stimulus item (**Simplify and evaluate for x , $\log_{10}(2x+1) = \log_{10}(x-1)$**). The computation that followed was a demonstration (based on computational knowledge) of how to achieve the goal (**Simplify and evaluate for x**). The rest of the cued-data (the student's error response) was ignored and the conclusion was limited to the value of x , ($x = -2$).

A similar approach of focusing on the key 'statement' in the stimulus item followed by correct computations was also commonly used by the

teachers in responding to the trigonometric and the statistic items. This was particularly true for instrumental type responses.

For relational type responses, linkages between the key 'statement' and some sections of the given cued-data were evident. For example, the following is the key 'statement' at the top of the trigonometric stimulus item: **Evaluate for x, $\cos(2x+1) = 0$**

A category (3) response:

The first mistake is $\cos(2x+1) \neq \cos 2x + \cos 1$.

The correct solution for $\cos(2x+1) = 0$ is:

(1). Find values of $\cos \emptyset = 0$, $\emptyset = \pi/2, 3\pi/2, \dots$

(2). Solve for x, $(2x+1) = \emptyset = (2n-1)\pi/2$, where n = integer

$2x = (2n-1)\pi/2 - 1$

$\therefore x = (2n-1)\pi/4 - 1/2$ [RELAT]

A category (3) response to the statistic item is described next. The underlying statement in the statistic stimulus item was:

Explain how to obtain the correct value of the variance for the given data, n=10.

A category (3) response:

I prefer to have students use the formula:

$$s^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n} \right)^2 \quad \text{or} \quad s^2 = \frac{\sum x^2}{n} - \bar{x}^2$$

The mean is 4, so $s^2 = 19.4 - 16 = 3.4$. Incidentally, although I would always mention the idea of variance, our teaching [this teacher was a Head teacher of mathematics] centres entirely on standard deviation. [RELAT]

The categorisation of category (3) type response-data into instrumental understanding was based on evidence that the computational structure was centred around a system of algebraic manipulation without reference to the theorems or laws underlying the particular mathematics. The response to the logarithmic item, described above, is an example of instrumental understanding.

The categorisation into relational understanding, on the other hand, was based on evidence of attempts by the respondent to make use of relevant formulae, algorithms, and procedures closely related to the fundamental theorems or laws of the particular mathematics. The above responses for the trigonometric and the statistic stimulus items are examples of relational understanding.

Category (4) (Symbolic understanding)

The classification into this knowledge category describes a system of knowledge outcomes that give evidence (or justification) to a complex inter-relationship of various mathematical knowledge forms at the cognitive domain. In this evaluation, the classification of a response-data as belonging to category (4) or symbolic understanding was based on several criteria:

- (a) the response-data had evidence that the respondent has conceptual knowledge of the symbolisation structures relating to the underlying principles of the particular mathematics being represented in the stimulus items,
- (b) evidence that most or all of the cued-data (student error) was considered in context,

- (c) evidence of attempts to provide corrective measures for the 'student error' and suggestions of possible learning difficulties related to the acquisition of the particular mathematical knowledge.

An example from each of the logarithmic, trigonometric, and the statistic stimulus items are presented below.

A category (4) response-data for the logarithmic item:

The student has recalled the log law incorrectly and treated $\log(2x+1) = \log(x-1)$ not as belonging to a 'log' function, but has considered the 'log' as a variable. The best way to help the student learn his/her mistake is point out the errors and appraise the correct attempts. So my explanation to the student would be:

I'd start with explaining that $\log(a + b) \neq \log a + \log b$

but $\log a + \log b = \log(ab)$ and $\log a - \log b = \log(a/b)$. Then

$\log(2x+1) = \log(x-1)$

$\log(2x+1) - \log(x-1) = 0$ -> can use $\log a - \log b = \log(a/b)$ here

$\log[(2x+1)/(x-1)] = 0$ -> the inverse $\log_{10} x = 0 \rightarrow x = 10^0$ that the student correctly stated is used here.

$(2x+1)/(x-1) = 10^0$ -> $10^0 = 1$, the student stated this correctly too.

$2x+1 = x-1$ -> if $x-1 \neq 0$

$x = -2$.

However, $\log x$ is only valid for $x \geq 0$, so $x = -2$ needs checking as to whether it provides a valid solution. Substituting -2 into $\log(2x+1) = \log(-3)$, this is undefined. Therefore $x = -2$ does not provide a valid solution. That is, $\log(2x+1) = \log(x-1)$ has no real solution. I suspect the outcome of $\log(-3)$ when checking would not alert the student who does not understand the concept of log functions. [REFLECTIVE MODE]

Next, a category (4) response is presented for the trigonometric item.

The student's response is incorrect and it could be suggested that the reason for the error is related to the following:

1. *If the student was introduced to trig-formulae and was confused or did not understand how to use them then this response could be the result of that.*
I.e. $\cos(2x+1) = \cos 2x + \cos 1$ was the attempt to expand LHS according to $\cos(A+B) = \cos A \cos B - \sin A \sin B$. Although this method could be used, it thus however, requires a sound knowledge of the associated trig-formulae.
2. *The student may not have an understanding of trig functions at all and just treated 'cos' as a variable or an unknown. But this would be an extremely unlikely response for a university student of mathematics.*
3. *The student may have forgotten the basic trig-values, e.g. $\cos 0 = 1$, $\cos \pi/2 = 0$, $\cos \pi = -1$, $\cos 3\pi/2 = 0$, $\cos 2\pi = 1$. The given $\cos(2x+1) = 0$ can be solved by knowing and recalling these values: E.g. $\cos \pi/2 = \cos 3\pi/2 = \cos(2n-1)\pi/2 = 0$, where $n = 1, 2, 3, \dots$, therefore $\cos(2x+1) = \cos(2n-1)\pi/2$.*

The next step of equating the angles, $(2x+1) = (2n-1)\pi/2$, is most probably where students get the belief that 'cos' can be treated as a variable.

Solving for x: $2x = (2n-1)\pi/2 - 1$

$x = (2n-1)\pi/4 - 1/2$, $n = 1, 2, 3, \dots$,

However, weaker students may not use radian form. E.g. they would tend to solve $\cos(2x+1) = 0$ by equating $(2x+1)$ to 90° , $\cos 90^\circ = 0$, $\rightarrow 2x+1 = 90$ (often ignored the degree symbol) concluding that $x = 44.5$. [REFLECTIVE MODE]

A category (4) response-data for the statistic item:

The 3 answers for the given information are all possible and each can be connected to the idea of a variance but with some degree of correctness.

1. *If the student considers the formula which indicates that $s^2 = \frac{\sum (\text{deviations})^2}{n}$, then $\sum_{i=1}^n (x_i - \bar{x})^2 = 24$, is the obvious choice and $24/10 = 2.4$ is the solution. The mistake here is related to carelessness of not checking the data.*
2. *If the student considers the formula in the same way as above and recognises the significance of the summation notation, then $24/9 = 2.7$. The error here is a one degree of freedom and for some cases this is justified.*
3. *The solution for the given information is 3.4 or the (ii) alternative. The focus of this question was on the use of the preferred formula for the variance.*
$$\text{ie. } s^2 = \frac{\sum x^2}{n} - \bar{x}^2 = \frac{194}{10} - 16 = 3.4$$
Students are encouraged to know both forms but this fits the calculator functions and is my preferred method of showing students how to find a variance value. [REFLECTIVE MODE]

In summarising the descriptions for classifying response-data into category (4), it was observed, as illustrated in the above examples, that the response-data which met the specified criteria appear to be representative of what Skemp (1979) described as the *reflective mode* of functioning in mathematics. However, it could be suggested from this analysis that an *intuitive mode* of representation would reflect evidence of knowledge based on procedures and manipulation of symbols to show that something is 'true'

or 'false'. As such, responses displaying *intuitive mode* representation for symbolic understanding were classified as category (3), see Table 4.3 (next page).

An example of this type is illustrated below:

The student seemed to be using a complicated method [implication that the respondent has alternative methods of showing a proof] to compute x , and didn't recognise that \log_{10} can be cancelled because it's a common factor. So $\log_{10}(2x+1) = \log_{10}(x-1)$ is $2x+1=x-1$, therefore $x = -2$. This value is correct in so far as it relates to equations, [implication that the respondent may have conceptual knowledge of logarithm but decided to focus on justifying why the 'student response' was incorrect] e.g. substituting $x = -2$ into $\log(2x+1) = \log(x-1)$ gives $\log(-3)$ on both sides of the equal sign. But the student's answer of $x = 1$ is not correct, ie. $\log 3 \neq \log 0$. [INTUITIVE MODE]

This then completes the analysis of the 18 teachers' interview data. The use of Skemp's model to classify the data into instrumental, relational, and symbolic understanding greatly facilitated this analysis of mathematical knowledge. The model enabled an analysis of responses which were the product of rote memorisation and distinguishes them from those which were the product of the individual's own constructions. Even with response-data based on experienced teachers' knowledge, the use of this model seemed to appropriately describe the quality of mathematical knowledge associated with teacher mathematical competence.

(3) Summary of the analysis of mathematics teachers' interview data

A summary of this analysis and the frequency distribution of the classifications into the four response categories for each of the three stimulus items, abbreviated as LOG, TRIG, and STAT, is presented in Table 4.3 below. Category (1) data are presented at the upper section of Table 4.3 and categories (2), (3) and (4) data are presented at the lower section of Table 4.3 in relation to Skemp's three types of mathematical understanding. In addition to this frequency distribution, the 'category' of each teacher's response-data and the classification of the responses into Skemp's types of mathematical understanding are presented in Table A1 of Appendix A.

Table 4.3: Summary of the analysis of interview data from mathematics teachers

Category	TRIG			LOG			STAT			TOTAL
(1)	2			2			8			12
Skemp's model of mathematical understanding										
Category	Instrumental Understanding			Relational Understanding			Symbolic Understanding			TOTAL
	LOG	TRIG	STAT	LOG	TRIG	STAT	LOG	TRIG	STAT	
(2) Intuitive	5	5	2							12
(3) Reflective	5	4	4	-	2	1	3	1	1	21
(4) Reflective							3	4	2	9
Total	10	9	6		2	1	6	5	3	42

The results of this analysis have provided an insight into the types of mathematical knowledge associated with the teaching of logarithm, trigonometry and statistics at the secondary school level. The frequency distribution of knowledge descriptors presented in Table 4.3 indicated a

strong tendency in favour of instrumental understanding. This result could be the function of several factors (e.g. teacher attitude, teaching method and lack of professional development) which this study was not designed to determine.

However, the high frequency of instrumental type response-data could be associated with the kind of mathematics the teachers were currently teaching at the time. For example, as presented in Table A1 (Appendix A), instrumental type responses were mostly from teachers who were teaching the Applied Mathematics (AppM), Applied Mathematics and General Mathematics (GenM), or High School mathematics. Applied Mathematics and General Mathematics are units for years 11 and 12 (college) students. These two courses are particularly for students who have little intention of pursuing rigorous studies in mathematics but wish to gain a background knowledge of college mathematics.

The teachers' instrumental approach may also be an indicator of the mathematical abilities of the students they taught. That is, if the teachers were teaching a particular group of students who favoured the less rigorous approach to learning mathematics, then this might influence the teachers' responses to the three stimulus items. For example, when one of the teachers was asked about the role of the calculator in teaching logarithm, trigonometry and statistics, the teacher responded: ... *these kids get confused with the nitty-gritty of mathematics ... the calculator provides an avenue for teaching these kids log and trig formulas and particularly the standard deviation ... they need immediate feedback and success to keep them on task, so sound calculator skills is a must in learning these topics.*

Another outcome from this analysis that is worthy of consideration is the 22% (12/57) of the response-data classified as category (1) type knowledge. Although this is a small percentage of all the responses, it is important to note that about a half (44%=8/18) of the teachers had at least one response classified as category (1), particularly for the statistic stimulus item (Table A1 in Appendix A). An example of a typical category (1) response was, *I can't answer this question because I haven't taught this topic yet.* This type of response seemed to suggest a close relation between what the teachers teach and their knowledge of the particular mathematics.

However, the act of teaching alone did not seem to facilitate these teachers in gaining relational or symbolic forms of understanding. For instance, the topics of logarithm, trigonometry and statistics were integral components of the General and Applied Mathematics units (taught by 72% of the respondents), and yet teachers' responses to the set of three stimulus items (representing logarithm, trigonometry and statistics) were mostly of instrumental understanding (68%, category (1) included). This outcome might well be an indication of the teachers' unfamiliarity with the mathematics being represented in the three stimulus items. However, this might only be true of the trigonometric item where the given equation was $\cos(2x+1)=0$ rather than the more familiar form $(\cos 2x = 0)$ being taught in the Applied Mathematics unit.

The teachers' perceptions of their students' mathematical ability could also indirectly affect their understanding of the mathematics they teach. In other words, if the teacher, for example, perceives his or her understanding of the mathematics to be inadequate for teaching the students, then this inadequacy

may influence the teacher to want to explore new ideas and to gain more knowledge about the mathematics to be taught. However, to gain relational or symbolic understanding, the teacher's involvement should be more than studying the syllabus, obtaining appropriate textbooks and other recommended resource materials, and more than studying the text material for procedures and sequencing of the content matter. Rather, the teacher's involvement should show evidence of his or her ability to manipulate and appropriately integrate new conceptual mathematical structures (or symbolic type understanding), and interpret these structures in terms of their relationships with one's prior mathematical knowledge (Skemp, 1982).

In summarising the analysis of the teachers' interview data, three key issues were observed. The first one relates to the 'evaluation tool'. The Skemp model provided an effective means of analysing the quality of mathematical understanding represented by the response-data as well as identifying how the knowledge may have been acquired.

The second observation concerns the strong evidence of instrumental type understanding in the interview data. This outcome might be related to the kinds of mathematics in the courses the teachers were currently teaching at the time. However, this instrumental outcome appears not to be due to the types of mathematics represented by the three stimulus items. Rather, it seemed that the teachers' understanding of the particular mathematics was influenced by what they perceived to be appropriate learning for the students they were teaching at the time.

The third issue relates to the teachers' growth in mathematical (relational) understanding. It could be suggested, from the outcomes of this analysis,

that teachers do not see the value in exploring and studying new ideas about the mathematics they teach, if the courses they teach do not challenge their mathematical understanding.

The above outcomes provide valuable links between teaching and mathematical understanding that could be applied to the education of pre-service mathematics teachers. For example, the skill of presenting mathematics to a class of students does not appear to be a critical factor in teaching, rather it is having an understanding of the mathematics being taught. That is, an understanding which is generated by the individual's pursuit to know substantially more about the mathematics he or she teaches, rather than the understanding based on mathematics textbooks and other curriculum resource materials (e.g. the syllabus). This mathematical understanding is the underlying issue being explored in the study reported in the next chapter, Chapter 5.

CHAPTER FIVE

ANALYSIS OF PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE

Introduction

The purpose of this chapter is to report on the analysis of the data from the study of secondary pre-service teachers' existing mathematical knowledge.

This study addressed three main assumptions:

- (1) Mathematical understanding is dependent on the sufficiencies of procedural and conceptual types of mathematical knowledge. Lack of or a deficiency in either procedural and/or conceptual knowledge types would suggest a deficiency in mathematical understanding.
- (2) Pre-service mathematics teachers go through their teacher education and training with certain deficiencies in their mathematical understandings and that these deficiencies will eventually affect the way they teach.
- (3) Pre-service mathematics teachers who majored in mathematics or other science related areas (e.g. chemistry and computer science) would show less evidence of mathematical knowledge deficiencies than pre-service teachers who majored in other areas (e.g. economics and physical education). Furthermore, pre-service teachers with relational understanding of mathematics would demonstrate more confidence to teach mathematics than pre-service teachers with instrumental understanding.

These assumptions led to the following two research questions:

1. What types of procedural and conceptual mathematical knowledge exist in pre-service teachers' knowledge bases?
2. What possible influences could any identified deficiencies in types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?

In order to address these two research questions, a multiple case study design was employed. The study's design was detailed in the first part of the previous chapter, Chapter 4, in which the procedures for selecting the cases and for data collection were described in Sections (4.1.4 and 4.1.5) and Sections (4.1.6 and 4.1.7) respectively.

In this chapter, the data collection procedures and the analysis of these data are reported in three parts. The first part is the description of the selected multiple cases, the collected data, as well as definitions of the terms used in the analysis. The second part is the reporting of the analysis of the data. The third and final part is a summary of the analysis in view of the two research questions.

5.1 PART ONE: SELECTED CASES, COLLECTED DATA, and DEFINITIONS

5.1.1 Selected cases - pre-service mathematics teachers

The replication procedures applicable for selection of multiple cases in case study designs were used for selection of the pre-service teachers. These procedures were detailed in Section (4.1.4) and presented in Table 4.1 in Chapter 4. For convenience, Table 4.1 is reproduced here as follows:

State A	Set A1 Cases First set of cases in A Background: maths major	Set A2 Cases Second set of cases in A Background: maths minor
State B	Set B1 Cases First set of cases in B Background: maths major	Set B2 Cases Second set of cases in B Background: maths minor

The State of Tasmania was selected as State A. A prospective mathematics teacher may enter the secondary mathematics pre-service teacher education programs at the university of Tasmania in either one of the following pathways:

- (1) Enrolment in a one year Diploma of Education (Dip Ed) program following the completion of a B.Sc degree, or equivalent, with a major in mathematics or in other sciences and technology areas. For example, engineering, physics, chemistry and computer science.
- (2) Enrolment in a four year Bachelor of Education (B.Ed) program and completing the compulsory mathematics units concurrently with the education units.

Pathway (1) was offered to graduate students at the university’s southern campus. The first set of multiple cases were selected from the nine students who were currently enrolled in pathway (1) at the time. However, only five of these nine students accepted the offer to participate in the study. This set of five participants (four males and one female) corresponds to Set A1 in Table 4.1 above.

Pathway (2) was offered to undergraduate students at the university’s northern campus. The target students were those currently enrolled in their third or fourth year of studies. It is important to mention at this point that because it was compulsory for undergraduate students who follow pathway (2) to do the core mathematics units of the B.Sc program, they were referred

to by the university as 'mathematics majors' or credited likewise. As such, it was recommended for students undertaking this pathway to have had completed the appropriate pre-requisite mathematics (mathematics stage 2 and stage 3) courses at the college level. However, this was only a recommendation and not all enrollees follow this advice. Therefore, to meet the 'maths minor' criterion of this study's design, it was those students without the appropriate mathematics background who were invited to participate. Twelve such students were identified and only five (three males and two females) were willing to participate in the study. Three of these participants were third year and two were fourth year students. This second set of cases corresponds to Set A2 in Table 4.1 above.

For external validity of the study results, replication procedures for the selection of cases were repeated at the state of Western Australia, State B. Two (B1 and B2) of the four universities in State B that were involved in this study offered prospective secondary mathematics teachers the following three pathways:

- (1) Enrolment in a one year Diploma of Education (Dip Ed) program following the completion of a B.Sc degree, or equivalent, with a major in mathematics or in other sciences and technology areas. For example, engineering, physics, chemistry and computer science.
- (2) Enrolment in a four year Bachelor of Education (B.Ed) program and completing the compulsory mathematics units concurrently with the education units.

- (3) Enrolment in a one year Dip Ed program following the completion of an undergraduate degree with a minor in mathematics. For example, B. Economics with the completion of at least 2 mathematics units at university level.

University B1 offered both pathways (1) and (2) to prospective secondary mathematics teachers. However, to meet the criterion of ‘maths major’ for this study’s design only those students enrolled in pathway (1) were targeted for selection. Nine such students were identified and only six agreed to be participants (four males and two females) in the study. This set of six participants corresponds to Set B1 in Table 4.1 above.

University B2 offered all three pathways to prospective secondary mathematics teachers. In order to meet the criterion of ‘maths minor’ for this study’s design only those students enrolled in pathways (2) and (3) were targeted for selection. However, due to students’ involvement in school experience during the designated data collection period, only three of the targeted students were available to participate in the study. All three agreed to be participants (three females) and were enrollees of pathway (3). This set of participants corresponds to Set B2 in Table 4.1 above.

The summary of the selected sets of pre-service teachers for the study is presented in Table 5.1 on the next page.

Table 5.1: Distribution of cases - pre-service mathematics teachers

PLACE	Diploma of Education Maths major	Diploma of Education Maths minor	Bachelor of Education Maths minor	TOTAL
TASMANIA	5 (4 males, 1 female)		5 (3 males, 2 females)	10 (7 males, 3 females)
WESTERN AUSTRALIA	6 (4 males, 2 females)	3 (3 females)	-	9 (4 males, 5 females)
TOTAL	11	3	5	19

A description of the procedures for collecting the data from the selected participants is provided in the following section.

5.1.2 Collected data

The collection of the data from the participants in the study was according to the procedures detailed in Sections (4.1.6) and (4.1.7) of Chapter 4. Also detailed in the second part of Chapter 4 were the procedures for the development and validation of the three mathematical stimulus items used in the process of data collection. For the purpose of the discussion in this section, Figure 4.1 containing these three stimulus items is reproduced on the next page.

The collected data were organised according to the three stimulus questions within each of the stimulus items (Figure 4.1). These stimulus questions are defined here as: *stimulus question 1* (or SQ1), *stimulus question 2* (or SQ2), and *stimulus question 3* (or SQ3). The purpose of SQ1, SQ2, and SQ3 was to promote the production of ‘complex’ type responses (Figure 2.5, Chapter

2) at the initial stage of the data collection procedures; the ‘silent reading’ stage (Section 4.1.6, Chapter 4). In addition, SQ1, SQ2, and SQ3 were designed and sequenced in this way for data organisational purposes in preparation for the data analysis procedures. The responses to SQ1 were organised for addressing the first research question: *What types of procedural and conceptual mathematical knowledge exist in pre-service teachers’ knowledge bases?* The response data for SQ2 and SQ3 were organised for addressing the second research question: *What possible influences could any identified deficiencies in the types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?*

Figure 4.1: The three mathematical stimulus items

<p>(1) The trigonometric item:</p> <p>Evaluate for x, $\cos(2x+1) = 0$</p> <p><i>A student responded:</i></p> $\begin{aligned} \cos(2x + 1) &= 0 \\ \cos 2x + \cos 1 &= 0 \\ \cos 2x &= -\cos 1 \\ 2x &= -1 \\ \therefore x &= -1/2 \end{aligned}$ <p>If you were the teacher:</p> <p>SQ1. Would you accept the student's response as being correct?</p> <p>SQ2. What do you consider important about the learning of trigonometry that you must teach your students?</p> <p>SQ3. How would you approach the teaching of trigonometry? Please explain and give an example of your teaching method.</p>	<p>(2) The logarithmic item:</p> <p>Simplify and evaluate for x, $\log_{10}(2x+1) = \log_{10}(x-1)$</p> <p><i>A student responded:</i></p> $\begin{aligned} \log(2x+1) &= \log(x-1) \\ \log 2x + \log 1 &= \log x - \log 1, & (\log 1=0) \\ \log 2x - \log x &= 0, & (\log_{10} x = 0) \\ x &= 10^0 \\ \therefore x &= 1 \end{aligned}$ <p>If you were the teacher:</p> <p>SQ1. Would you accept the student's response as being correct?</p> <p>SQ2. What do you consider important about the learning of logarithm that you must teach your students?</p> <p>SQ3. How would you teach logarithm? Please explain and give an example of your teaching method.</p>
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... Figure 4.1 continues over to the next page ...

(3) The statistic item:

A class of year 11/12 students was asked to find the variance using the information given below:

Ten items were measured and four results were produced:

$$\begin{array}{ll} 1. \sum_{i=1}^{10} x_i = 40 & 2. \left(\sum_{i=1}^{10} x_i \right)^2 = 1600 \\ 3. \sum_{i=1}^{10} x_i^2 = 194 & 4. \sum_{i=1}^9 (x_i - \bar{x})^2 = 24 \end{array}$$

The class produced 3 different values for the *variance*, S^2 :

- (i) 2.4 (ii) 3.4 (iii) 2.7

If you were the teacher:

SQ1. Which variance would you accept as the correct answer?

SQ2. What do you consider important in the learning of statistical variance that your students must learn?

SQ3. How would you approach the teaching of variance? Please explain and give an example of your teaching method.

Another important sequence of organisational procedures in preparation for the analysis of the data are the procedures which were implemented in an attempt to minimise biases during the analysis of the response data. The type of bias that may occur, for instance, when a prior expectation by the researcher could influence his or her perceptions and the analysis of the particular data (Berg, 1989). Hence, in an endeavour to minimise biases, the following procedures were implemented for each of the cases studied.

- a) Shortly (e.g. no more than a day) after the interview session, audio taped responses were transcribed and added to written responses. These responses are referred to as 'raw data'.
- b) The personal information form was labelled with an identification code (ID code) and removed from the 'raw data' to be filed with the other participants' forms in a sealed envelope. Another envelope was

simultaneously labelled with this same ID code and the corresponding raw data were placed inside it ready for the next process.

- c) Each ID code was written onto a small piece of paper and placed in a container. This procedure would allow a random selection of the envelopes during the analysis stage of the data.
- d) At the analysis stage, an ID code is selected from the container and matched to its corresponding envelope which contains the 'raw data' to be analysed. The sequence in which the response data had been randomly selected for analysis is retained and used in the reporting of the analysed data (presented in Figure B1 and Table B2 of Appendix B).

In Chapter 3, Figure 3.2, the 'predicted response patterns' based on Skemp's model were described. The seven response patterns which formed the base-line of patterns for the data analysis of the main study were described in Chapter 4, Section (4.2.2) and presented in Figure 4.2. At this point, it is essential prior to the reporting of the analysis to provide definitions of the various terms involved in determining the particular response patterns. The definitions of these terms are provided in the next section, Section (5.1.3).

5.1.3 Definitions of terms in the analysis

In Chapter 4, Figure 4.2, seven possibilities of response patterns that could be evident in the response data were described. The aim in this section is to define the key terms used in determining and differentiating between these seven patterns. The key terms concerning mathematical understanding are

defined here in relation to Skemp's (1978, 1979, 1982) three kinds of mathematical understanding.

Instrumental understanding is understanding based on active recall of rules and algorithms associated with rote memorisation learning.

- *Pseudo-procedural knowledge* is knowledge of rules without reasons.
- *Pseudo-conceptual knowledge* is knowledge of assorted rules to achieve a correct answer. (Chapter 3, Figure 3.2)

Relational understanding is understanding based on the interconnection of ideas that explain and give meaning to mathematical procedures.

- *Relational-procedural knowledge* is knowledge of rules; meaning, significance, or structure of a problem without explicit reliance on a tested technique of analysis and proof.
- *Relational-conceptual knowledge* is knowledge of extensive mathematical structures (schemata) to relate and verify procedures.
(Chapter 3, Figure 3.2)

Symbolic understanding is understanding based on the assimilation between a symbol system and a conceptual structure, dominated by a conceptual structure.

- *Symbolic-procedural knowledge* is knowledge of a logical progression of steps for a proof giving evidence of an awareness that something is 'true' or 'false'.
- *Symbolic-conceptual knowledge* is knowledge giving evidence of 'full mathematical rigour'. (Chapter 3, Figure 3.2)

The response data were also categorised according to evidence of what the respondent had indicated that he or she can and cannot do. The following are the four response categories:

- Category (1).* Little or no recall of knowledge about the mathematics presented.
- Category (2).* Can recognise rules, theorems, symbols, or a system of procedures or methods.
- Category (3).* Can carry out computations by applying a rule or a set of procedures.
- Category (4).* Can demonstrate as in (2) and (3) as well as providing a justification or reason for a given result. Category (4) is representative of responses belonging to symbolic understanding.

Two other terms used throughout the report and need defining are *cued-data*, and *response-data*.

Cued-data are the stimulus cues within the stimulus items.

Response-data are analysed responses. Analysed responses are actual or raw data which have been through the processes of classification, examination, and analysis.

Having stated the study's assumptions and research questions; described the selected cases as well as the collected data and stimulus items involved, and defined the terms for determining the response patterns in the analysis, the next part of this chapter, part 2, contains the analysis of the data.

5.2 PART TWO: ANALYSIS OF THE DATA

The analysis of the data is reported in three stages corresponding to the three phases of the analysis:

- (1) In the first phase, the responses to SQ1 are analysed for the purpose of addressing the first research question: *What types of procedural and conceptual mathematical knowledge exist in pre-service teachers' knowledge bases?* The responses to SQ2 and SQ3 are also referred to as and when appropriate to provide further support and clarification of meaning to particular response patterns. The reporting of this phase is organised under the *four response categories*. This same organisational approach was used in Section (4.2.3) of Chapter 4 for reporting on the analysis of experienced teachers' interview data.
- (2) The second phase involved the analysis of responses to SQ2 and SQ3 for the purpose of addressing the second research question: *What possible influences could any identified deficiencies in types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?* Reference to the analysis in phase one is made, as and when appropriate, to provide further support and clarification of response data in phase two. The analysis in this phase is reported in terms of the three stimulus items. These items are referred to as TRIG for the trigonometric item, LOG for the logarithmic item, and STAT for the statistic item.
- (3) The third and final phase draws upon the results from phase one and phase two. The aim of this analysis is to address both research questions

in view of any similarities and/or contrasts of knowledge types displayed in the response-data from the 'maths major' (Set A1 and Set B1) groups and the 'maths minor' (Set A2 and Set B2) groups. In this phase, summarised results from phase one and phase two for each case are examined in order to determine 'similarities' within the four sets of multiple cases and 'contrasts' between these four sets (Chapter 4, Section 4.1.4). Reporting is organised under the criteria of similarity and contrast in knowledge types.

The first report is on phase one of the analysis.

5.2.1 Phase one of the analysis

The data to be examined in this phase of the analysis are responses to the three questions in the stimulus items, in particular SQ1. These questions are reviewed below:

Stimulus question 1 (SQ1): If you were the teacher: *Would you accept the student's response as being correct?*

This question was intended to direct the respondent's attention to the main stimulus cued-data, namely the *student's error response*. The aim of this *error response* was to provoke cognitive processing of knowledge (Brownell, 1958; Gagné, 1985; Derry, 1996). It was assumed that responses to this question would provide data (or knowledge representation) about the respondent's knowledge on the particular mathematics in focus. Responses to SQ1 formed the primary source of data for examining the quality of these pre-service teachers' existing mathematical knowledge.

Stimulus question 2 (SQ2): *What do you consider important about the learning of (logarithm, trigonometry, or the statistical variance) that you must teach your students? and*

Stimulus question 3 (SQ3): *How would you teach (logarithm, trigonometry, or the statistical variance)? Please explain and give an example of your teaching method.*

The responses to SQ2 and SQ3 were treated as complementary data to those of SQ1. Responses to SQ2 were important in order to observe any links between what the pre-service teachers know about the mathematics (SQ1) - mathematical knowledge - and what they perceived to be important mathematical learning for students (SQ2) - pedagogical knowledge. Responses to SQ3 were also for exploring the pre-service teachers' pedagogical content knowledge and any links with mathematical knowledge (SQ1). The procedures for analysing the responses to SQ1, SQ2, and SQ3 are described next.

For each of the cases studied, the analysis of response data was according to the following two procedures:

1. The evaluation and classification of responses to SQ1 into three main sets corresponding to the three kinds of mathematical understanding: *instrumental, relational and symbolic* (Skemp, 1978, 1979). This classification process was applied to response data for each of the three stimulus items (TRIG, LOG, and STAT).

2. The response data were examined in terms of procedural and conceptual types of knowledge. This examination also involved the categorisation of responses based on evidence of what the respondent had indicated that he or she can or cannot do:

- Category (1). Little or no recall of knowledge about the mathematics presented.
- Category (2). Can recognise rules, theorems, symbols, or a system of procedures or methods.
- Category (3). Can carry out computations by applying a rule or a set of procedures.
- Category (4). Can demonstrate as in (2) and (3) as well as providing a justification or reason for a given result.

These response-data are presented in Figure B1 and Table B2 of Appendix B. Table B2 summarises the data presented in Figure B1. A frequency distribution of the analysed data is presented in terms of the above response categories and with respect to the three stimulus items in Table 5.2 below.

Table 5.2: Frequency distribution of SQ1 response-data for the three stimulus items

Category	TRIG	LOG	STAT	Total
1	5	6	14	25 (44%)
2	10	8	3	21 (37%)
3	4	5	2	11 (19%)
Total	19	19	19	57 (100%)

The distribution of the analysed data presented in Table 5.2 showed 44% of the response-data were classified as category (1), 37% as category (2) and 19% as category (3), and there was no data for category (4).

The reporting of the first phase of the data analysis is with respect to the four response categories. However, it is appropriate at this stage to provide the reader with more details of how the response-data are reported in this thesis. To distinguish female from male participants, females will be denoted by F1, F2, F3, ... , F8 and males by M1, M2, M3, ... , M11. In an endeavour to illustrate the classification of responses into each category, selected examples of response-data from the respective categories are presented. To provide further clarification about the selected response-data, the author's descriptions and summaries of the types of knowledge involved are inserted as appropriate, inside square-brackets []. The first classification of responses to be described is the 44% of response-data classified as category (1).

5.2.1.1 Category (1) response-data

It was found in the analysis of the experienced teachers' interview data that response-data classified as belonging to category (1) were more likely to represent evidence of instrumental understanding.

A category (1) classification is a measure which indicates that there was sufficient evidence to show that the individual was unable to provide the appropriate mathematical knowledge associated with the stimulus items. This inability or uncertainty appeared to relate to several factors such as lack of memory recall (Lmr), lack of knowledge (Lk) about the topic, and lack of understanding (Lu). In some cases these factors seemed to be interrelated, for example:

- i) *This is hard because I don't even remember [Lmr] what logarithm is about [Lk].*
- ii) *I can't remember [Lmr]. I know that there are log-laws but I can't recall what they are [Lu].*
- iii) *I don't know [Lk], I can't remember [Lmr]. I haven't understood statistics well [Lu] at Uni and I can't remember [Lmr] the variance formula.*

Respondents with category (1) knowledge classifications for SQ1, also provided similar uncertainties in their responses to SQ2 and SQ3, represented respectively as (2) Important learning and (3) Teaching approach. For instance, the responses associated with (i), (ii), and (iii) above are as follow:

- i) (1) *This is hard because I don't even remember [Lmr] what logarithm is about [Lk].*
 (2) Important learning: *Students to learn how to go from $\log(2x+1)$ to $\log 2x + \log 1$.*
 (3) Teaching approach: *Start with log-laws. That's because it's the way I usually work, ie. find a rule and follow that.*

The above responses were from participant F4, a 'maths minor' from State B. The respondent's uncertainty appears to relate to lack of knowledge and misconception about logarithmic functions (*this is hard because I don't even remember what logarithm is about*). This is exemplified by the respondent's inappropriate suggestion of important learning for students: *students to learn how to go from $\log(2x+1)$ to $\log 2x + \log 1$* . Clear evidence of instrumental type understanding was present in the response to SQ3: *Start with log-laws. That's because it's the way I usually work, ie. find a rule and follow that.*

- ii) (1) *I can't remember [Lmr]. I know that there are log-laws but I can't recall what they are [Lu].*
- (2) *Important learning: I don't know. I would have to study up logarithm myself first.*
- (3) *Teaching approach: Log laws and how to apply them. Try to help students understand the difference of logs from algebra.*

The above responses were from participant F7, a 'maths major' from State B. The uncertainty in this case seems to relate to gaps in the respondent's knowledge about logarithmic functions: *I don't know. I would have to study up logarithm myself first.* For the teaching of logarithm, it seems that it may be limited to logarithmic laws and algebraic algorithms for computations: *Try to help students understand the difference of logs from algebra.*

- iii) (1) *I don't know [Lk], I can't remember [Lmr]. I haven't understood statistics well [Lu] at Uni and I can't remember [Lmr] the variance formula.*
- (2) *Important learning: What a variance is. What it tells us about the data. How to calculate the value.*
- (3) *Teaching approach: Find the sample mean and other values from the data. Derive the variance using the formula.*

The above responses were from participant M7, a 'maths major' from State B. The lack of conceptual understanding of what statistics is about appears to be the cause for the uncertainty in this case: *I don't know, I can't remember. I haven't understood statistics well at Uni and I can't remember the variance formula.* However, the respondent seemed to be aware of what is required for a better understanding of the cued-data: *What a variance is. What it tells us about the data. How to calculate the value.* This also seems to summarise how this respondent perceives important learning for understanding a

concept: *What it [the concept] is. What it tells us How to calculate the value.* With respect to teaching, it appears that it is safer when one does not have a sound understanding to stay close to familiar and related areas of the particular mathematics, in this case statistics. For example, *find the sample mean and other values from the data [and then] derive the variance using the formula.*

The above response-data from State B participants (a 'maths minor' and two 'maths major') indicated evidence of uncertainties or lack of confidence in one's own understanding or existing mathematical knowledge. These uncertainties seem to relate to lack of knowledge, gaps in knowledge, misconception, and lack of conceptual understanding of mathematics. Such uncertainties were also evident in responses by participants from State A. The following are examples of both 'maths minor' and 'maths major' cases from State A.

Participant F1 is a 'maths minor' from State A. These are her responses to the LOG item:

- (1) *I've forgotten [lack of memory recall]. I don't know [lack of knowledge] how to do this myself so I can't really say whether [uncertainty] the student is right or wrong. I would need to look up a textbook [dependency on external aid] to remind [insufficient prior learning] myself again.*

Participant M1 is a 'maths minor' from State A. These are his responses to the STAT item:

- (1) *I'm not sure* [uncertainty in relation to the cued-data given for the variance]. *I know it has something to do with statistics, but I'm not sure whether it's the mean something or rather* [uncertainty due to insufficient knowledge].

Participant M8 is a 'maths major' from State A. These are his responses to the TRIG item:

- (1) *The student is treating this as algebra, and there seems to be a misunderstanding between the cos and its angle* [fragments of knowledge about trigonometry]. *But I can't remember how to do this now or explain why this is so* [uncertainty due to insufficient knowledge].

The above response-data from F1, M1, and M8 tend to indicate that lack of confidence in one's own understanding is related to knowledge insufficiencies. These insufficiencies or gaps in knowledge seem to have close association with the respondent's dependency on external aids. For example, F1's uncertainty about her knowledge (*I can't really say whether the student is right or wrong*) was to be compensated by knowledge from a textbook (*I would need to look up a textbook to remind myself again*). Dependency on external aids in order to facilitate one's knowledge of mathematics appears to be a deliberate decision by the individuals and considered by some as an acceptable approach in learning mathematics. F3's responses to the LOG item is an example of this deliberate decision to rely on external aids.

Participant F3 is a 'maths minor' from State B. These are her responses to the LOG item:

- (1) *I have no recall of what logarithm is* [lack of knowledge], *and I'm lost* [uncertainty] *without my calculator* [strong dependency on external aid]. *Because I've always relied on the calculator* [existing knowledge is equated to calculator skills] *for working out logarithm or trigonometry formulas.*

The above response-data from participant F3 tend to suggest that her lack of knowledge and understanding about logarithm is related to a deliberate decision by her to rely on external assistance, namely the calculator, when learning mathematics. This dependency on external assistance was also strongly reflected in participant F3's responses to the STAT item. F3's responses to SQ2 and SQ3 are also included to demonstrate that her understanding of mathematics is rooted in her approaches to learning.

- (1) [An attempt was made to recall isolated aspects relating to statistics, e.g. data collection, graphing, and finding the mean, mode and median. But F3 could not provide an appropriate response for the given cued-data because a *calculator* and the *formula* were not provided.]
- (2) Important learning: *To focus on all statistics rather than just variance.*
- (3) Teaching approach: *I rote-learn a lot of my maths* [evidence of rote memorisation]. *So I'll have to learn on-the-job how to teach this* [lacks confidence in her own knowledge, dependency on external guide] *and adhere closely to the syllabus* [set guidelines].

Participant F3's above response-data indicated lack of essential knowledge and understanding about the mathematics represented by the STAT item. This lack of essential knowledge appeared to be closely linked to the respondent's rote learning of formulae and calculator procedures. Furthermore, it was noted that participant F3 had to repeat the compulsory

mathematics unit required for completion of an Economics undergraduate degree. However, success was achieved a year later by using rote learning strategies. Participant F3 explained rote learning as *getting to know your lecturers, memorise the formulae, and attempt all the tutorial exercises and available past examination papers*. Participant F3 also referred to rote learning as a *trap many students and teachers fall into because it gives immediate results and as such many like myself find it difficult to give up using it*. Participant F3 continues: *Also, the present system of education, particularly at university where the aim is to pass rather than to understand what you were taught, encourages rote learning*.

Rote learning might also result in acquiring specific types of knowledge which are difficult to integrate with the person's existing knowledge.

Consider, for example, F4's responses to the STAT item and M1's responses to the TRIG item.

Participant F4 is a 'maths minor' from State B. These are her responses to the STAT item which tend to indicate that 'standard deviation' was rote learned as a specific type knowledge.

- (1) *I have no idea of [lack of knowledge] what a variance is. I recall doing standard deviation but I can't tell you what it is either [little or no association of the 'standard deviation' to its original source, variance].*

Participant M1 is a 'maths minor' from State A. These are his responses to the TRIG item.

- (1) *I'm not really familiar with these* [cued-data in the TRIG item] *to know how to do them myself. I have to consult a textbook* [dependency on external aid] *to refresh my memory* [lack of memory recall is linked to unfamiliar knowledge].

In a similar way to F4, M1's existing knowledge of trigonometry appears to compose of specific knowledge types which are unrelated to the types represented in the TRIG item. M1's unfamiliarity with the cued-data in the TRIG item, however, also appears to relate to a dependency on external aids as sources of knowledge.

In summarising to this point, it seems that lack of confidence in one's own understanding or existing mathematical knowledge base could be attributed to lack of conceptual understanding or misconceptions, and gaps in knowledge. In order to compensate for their insufficiencies in mathematical understanding, it appears that student-teachers relied upon external aids or textbooks and calculators as sources of mathematical knowledge. However, constructive use of such sources for knowledge should not inhibit the mental processing or formation of knowledge. Rather, these external sources should facilitate mental processings (De Corte, 1995). Rote memorisation of specific mathematical terms and formulae (e.g. standard deviation) also appears to be another contributing factor to the insufficiencies in mathematical understanding. It could be suggested from these participants' response-data that their inability to provide appropriate mathematical knowledge has strong links with the way they had acquired their prior knowledge of mathematics (Skemp, 1978; Anderson, 1981, 1982; Marton, 1988; Biggs & Moore, 1993; Derry, 1996). For example, the acquisition of mathematical knowledge by rote learning methods (Section 2.4, Chapter 2).

Category (1) type response data as described above represented 44% (25/57) of all the response-data for the three stimulus items (Table 5.2). This high frequency tends to indicate that the respondents' uncertainty about the cued-data within the stimulus items may involve several types of knowledge deficiencies. Further analysis of category (1) response-data, in an endeavour to determine what these deficiencies might be, revealed four possible types (all of which are interrelated) of knowledge insufficiencies. These were tentatively identified as: (i) undeveloped, (ii) unproductive, (iii) unrelated or unfamiliar, and (iv) unprocessed.

Undeveloped knowledge is defined as deficiencies of essential knowledge elements needed for understanding. The individual is aware of this inadequacy and acknowledges such by indicating a desire for further learning. The following responses were indicators of undeveloped knowledge:

- *I really don't know because I have not done much learning in this area. I need to do a lot of reading and make sure I understand statistics first.*
- *I don't know. I would have to study up logarithm myself first.*
- *I don't know much about statistics without doing more study on it myself.*

Unproductive knowledge is related to knowledge (or information) from resource materials (e.g. calculators and textbooks). These materials are relied on as the source of mathematical knowledge and without them being available the individual would have difficulty completing the task. The following responses were indicators of unproductive knowledge:

- *I have no recall of what logarithm is, and I'm lost without my calculator. Because I've always relied on the calculator for working out logarithm or trigonometry formulas.*
- *I don't know much about statistics ... I know the basics like the mean, mode, median, bar graphs etc. ... If I need more clarification I will refer to a good statistics textbook.*
- *I've forgotten [referring to the given cued-data in the LOG item]. I don't know how to do this myself ... I would need to look up a textbook to remind myself again.*

Unrelated or unfamiliar knowledge is related to unproductive knowledge in that the individual may find it necessary to consult a textbook or make further enquiries in order to continue with the task. In addition, this type of insufficiency is linked to prior learning of different terminologies and algorithms to what the individual has been presented with at the interview.

The following responses were indicators of unrelated or unfamiliar knowledge:

- *I'm not really familiar with these [referring to the given cued-data in the TRIG item] to know how to do them myself. I have to consult a textbook to refresh my memory.*
- *I have no idea of what a variance is. I recall doing standard deviation but I can't tell you what it is either.*
- *I don't know [referring to the given cued-data in the STAT item] because I need to do more study for me to answer this question. For this kind of knowledge I've always relied on textbooks rather than try and commit it to memory.*

Unprocessed knowledge is associated with knowledge of fixed rules, formulae or algorithms having little or no conceptual bases. The following responses were indicators of unprocessed knowledge:

- *I can't do this [referring to the given cued-data in the STAT item] because a calculator and the formula are not provided. I rote learn a lot of my mathematics.*
- *It is important for students to learn log-rules and ways of manipulating the rules ... because unless the students learn the laws well, they won't know what to do.* [The participant made this comment after providing a response to the LOG item with incorrect recall of the logarithmic law for division].
- *When teaching logarithm, I'd start with the log-laws. That's because it's the way I usually work, ie. find a rule and follow that.*

The four types of mathematical knowledge deficiencies suggested above appear to have instrumental understanding (Skemp, 1978) as their common base. This form of understanding appears to limit an individual's ability to the recall of formulae and algorithms, and places a dependency on the availability of external resources (e.g. textbooks and calculators). Having these knowledge deficiencies, as discussed above, also seem to influence how the individual might teach mathematics. This influence is explored further in the second phase of the analysis. However, it seems appropriate at this stage to provide some examples of category (1) response-data to illustrate the 'crippling' effect that instrumental type understanding could have on the teaching of mathematics. Each of the response-data below gives a glimpse of how these pre-service teachers might teach statistics and trigonometry.

Participant F5 is a 'maths major' from State B. These are her responses to the STAT item. F5's response-data provide an example of *undeveloped knowledge*.

- (1) *I know that 'sigma square' [the symbol is the concept] is the variance and the square-root of it is the standard deviation. But, looking at this [cued-data] I can't recall the formula. I don't really know or have an understanding of what a variance is. I just know it as 'sigma square'. [no understanding, just knowledge of the symbol]*
- (2) *Important learning: Students to learn the actual concept of the formulae. It is important that students learn this first and then the formula and how to use it.*
- (3) *Teaching approach: I don't really know, I guess I will have to follow the syllabus.*

Participant F5's lack of conceptual knowledge about the variance could be considered as an undeveloped knowledge type. That is, knowledge about the variance was acquired, but the essential knowledge elements needed for it to be linked to conceptual knowledge, have not been developed adequately. F5's desire to develop this knowledge is reflected in the response-data for SQ2 (stimulus question 2). However, having a desire to learn is not enough in teaching, rather this should be coupled with guided and expert assistance (Skemp, 1986; Ball, 1990; Leder, 1993). F5's response to SQ3 (stimulus question 3) is a request for assistance and if this is not forthcoming then, *I guess I [she] will have to follow the syllabus.*

Participant F6 is a 'maths minor' from State A. These are her responses to the STAT item. F6's response-data provide an example of *unprocessed knowledge*.

- (1) *I think its $(x - \bar{x})^2$, [recall of symbols connected to the formula] the sum of these divided by n. So the correct variance is 2.4, but it's really only a guess. I didn't really enjoy statistics.*
- (2) Important learning: *An understanding of formulas and what to use them for. E.g. the variance formula and how and where to use it.*
- (3) Teaching approach: *To teach variance I'll introduce standard deviation first. But if the syllabus indicates the variance first then I'll do that (ie. just follow whatever is in the syllabus).*

Participant F6's uncertainties ...*its really only a guess* ... seemed to relate to prior learning experiences, ... *I didn't really enjoy statistics* ... However, F6's knowledge about the variance formulae is rather advanced compared to that

displayed by F5. That is, F6 can recall that the formula, $\frac{\sum (x - \bar{x})^2}{n}$, is

related to the variance but F6 cannot provide a reason why, *it's a guess*. This type of knowledge was classified as unprocessed knowledge because it represents learning of 'fixed rules' without reasons. F6's response-data for SQ2 indicated instrumental type understanding. The response-data for SQ3 demonstrate the negative effect of instrumental understanding on F6's confidence to teach mathematics. Instrumental understanding seems to remove from the individual the initiative or the 'power' to develop his or her understanding (Greenwood, 1993; Gates, 1995a), instead ... *just follow whatever is in the syllabus*.

Participant F2 is a 'maths minor' from State B. These are her responses to the TRIG item. F2's response-data provide an example of *unrelated or unfamiliar knowledge*.

- (1) [A cosine graph, $y=\cos x$, was correctly sketched, but F2 was unable to relate it to find the x-values for $\cos(2x+1)$. Hence, incorrect values were given, but F2 accepted them as being correct.]
- (2) Important learning: *Graphing of trig-functions and how to use the unit-circle.*
- (3) Teaching approach: *Start with graphing the trig-functions and then teach the use of trig-ratios in the calculations of heights and distances.*

Participant F2's responses were classified as an *unrelated knowledge* type. That is, F2 has knowledge of trigonometry but this is not at the same level or standard as the type F2 was presented with. F2 also appears to have learned a fixed strategy (ie. graphing) to deal with trigonometric situations. This learned strategy may well influence the way F2 will teach trigonometry.

This completes the analysis of data classified as category (1). The 37% of response-data (Table 5.2) classified as category (2) are examined next.

5.2.1.2 Category (2) response-data

Category (2) classification is based on evidence of knowledge recognition of rules, algorithms, or systems of procedures. References may be made about computational knowledge but no calculation of values is evident.

In the analysis of experienced teachers' interview data, category (2) type knowledge showed evidence of both instrumental and relational understanding or *transitional types* of knowledge. However, the analysis of the data from the pre-service teachers seems to indicate this transitional knowledge to be mostly of the instrumental kind, in particular, the links with category (1) type knowledge.

Consider, for example, the following responses by participant F6 to the LOG item. Participant F6 is a 'maths minor' from State A. Her responses to the LOG item also provide an example of *pseudo-procedural knowledge* and of links to *unprocessed knowledge* (category (1)).

- (1) $\log(2x+1) \neq \log 2x + \log 1$. $\log[(2x+1)/(x-1)] = 0$ because of the rule: $\log(a/b) = \log a - \log b$ [pseudo-procedural knowledge]. But I don't really know the reason for this [unprocessed knowledge]. And I don't know what to do next. [F6 has knowledge of logarithmic laws but lacks relational-conceptual knowledge which gives meaning to the laws and facilitates computational knowledge.]

F6's response-data for SQ2 and SQ3 are provided here to show further evidence of links to category (1) type knowledge.

- (2) Important learning: *Students to understand log-laws and how to use them. Because that's the problem here, the student didn't understand logarithm.* [Indication that F6 knows the importance of understanding the concept in order for competent performance.]
- (3) Teaching approach: *I don't know really. I need to do more study and see what is required in the syllabus.* [Indication of insufficient existing knowledge and the need for further self-learning. However, there is also an indication that following the set guidelines or syllabus would be a safer option to take if the appropriate assistance is not available.]

Another example of transitional type knowledge is participant F1's response-data for the STAT item. F1 is a 'maths minor' from State A. Her response-data provide an example of *relational-procedural knowledge* with

links to category (1) *unprocessed and undeveloped knowledge types*. F1's response-data for SQ2 and SQ3 are included to provide further evidence of links between knowledge types.

- (1) *The variance is the sum of the deviations from the mean, then divided by (n-1) because it is a sample. So it's 2.4 but the sigma is 1 to 9, but there are 10 items* [recognising from the cued-data a relevant aspect about the variance, ie. $\sum (x - \bar{x})^2 = 24$, and linking this to her existing knowledge - relational-procedural knowledge]. *Now I'm confused and I don't know much about the variance to be able to explain why* [uncertainty due to lack of conceptual knowledge].

F1 appears to have learned a 'fixed' formula and an algorithm (or unprocessed knowledge) for calculating the statistical variance and that this knowledge is still in an undeveloped form ... *I don't know much about the variance*. Although F1 seems to have relational-procedural knowledge, the link of this knowledge to *unprocessed and undeveloped knowledge* (category 1 knowledge) appears to cause interruption of transmission to relational-conceptual knowledge... *I don't know much about the variance to be able to explain why*; hence, her inability to provide a reasonable explanation. The links to unprocessed and undeveloped knowledge rather than to relational-conceptual knowledge are also reflected in F1's response-data for SQ2 and SQ3.

- (2) Important learning: *Understanding what the formula is all about*.
[Consistent with (1)]
- (3) Teaching approach: *I only know what I've been taught* [existing knowledge is linked to prior learning]. *So I will start by emphasising the formula for the variance to derive the standard deviation* [teach the same way as taught]. *Then provide problems using the standard deviation and make sure the students understand* [doing a lot of

exercises or 'problems' is assumed to provide understanding]. *Also emphasise the correct use of calculators.*

Both response-data by F6 and F1 showed evidence of relevant aspects of knowledge relating to the cued-data as well as evidence of lack of appropriate knowledge about the mathematics. The response-data for SQ2 and SQ3 (stimulus questions 2 and 3) provided further data as to how the participants have acquired the knowledge and how this knowledge might be transformed to teachable knowledge.

There was evidence in F6's response-data that indicated the required knowledge could have been acquired through individual pursuit (... *I need to do more study* ...) rather than externally generated by the teacher as in the case of F1 (... *I only know what I've been taught*). F6 also recognised the importance of learning rules for understanding in order to know how to apply them appropriately. F1, on the other hand, seems to emphasise the learning of a formula. In order to achieve this, F1 suggested the doing of practice exercises and the correct use of the calculator.

These response-data also showed elements of knowledge deficiencies (as in category 1) that tended to interfere with further knowledge production. For example, the response-data by F1 could be viewed as undeveloped knowledge (...*I'm confused and I don't know much about the variance to be able to explain why*) and that by F6 as unprocessed mathematical knowledge (... $\log[(2x+1)/(x-1)] = 0$ because of the rule: $\log(a/b) = \log a - \log b$. *But I don't really know the reason for this. And I don't know what to do next*).

In summary, the response-data by F6 and F1 are representative of 81% (46/57) of the data classified as category (1)-(44%) and category (2)-(37%) type knowledge (Table 5.2). The knowledge insufficiencies associated with category (2) type knowledge tended to be connected with deficiencies of conceptual knowledge (as in relational understanding) relating to formulae and algorithms for computations. That is, category (2) type knowledge is mainly pseudo-procedural knowledge and relational-procedural knowledge of mathematics.

Participants who were aware of their insufficiencies tended to demonstrate this by their responses to SQ2 in which considerable emphasis was placed on the importance of understanding mathematical concepts underlying the formulae or rules. The link between the insufficiencies in the respondents' existing knowledge and prior learning experiences was more noticeable in response-data for SQ3. For example, the respondents' uncertainties about their depth of knowledge would either persuade them to: (i) teach according to the way they have been taught (refer to response-data from participant F1) or alternatively, (ii) study the topic further and follow the guidelines in the syllabus (refer to response-data from participant F6).

So far, a large proportion (81%) of the analysed response-data have been accounted for by classifications in category (1) and category (2). The remaining 19% of response-data classified as category (3), see Table 5.2, page 153, are examined in the next section.

5.2.1.3 Category (3) response-data

Response-data classified under category (3) showed evidence of the respondents' ability to recall formulae, algorithms, or systems of procedures with links to computational knowledge. This computational knowledge may indicate relevant relational-conceptual knowledge, relational-procedural knowledge, or both. However, the category (3) classification of these data showed 14% (8/57) as belonging to instrumental understanding and 5% (3/57) as relational understanding. It is important, therefore, to consider some of the response-data from this 14% to demonstrate how or by what evidence they were classified as pseudo-conceptual knowledge, the type ascribed as belonging to instrumental understanding. The first of these response-data to be reported on are those for the LOG item.

There were five response-data for the LOG item classified as category (3). Two of the five response-data were classified as belonging to relational understanding. In the three remaining response-data, there were evidence of the same computational procedures for SQ1 as in the following response-data from participant M7. M7 is a 'maths major' from State B.

- (1) $\log_{10}(2x+1) = \log_{10}(x-1)$, have both expressions on the one side and equal them to zero. [recognised an algorithm]
That is, $\log_{10}(2x+1) - \log_{10}(x-1) = 0$. I could use the log-law for division: $\log A/B = \log A - \log B$, [identified a rule]
but it is easier to cancel the ' \log_{10} ' because this is common to both.
[return to the algorithm recognised earlier]
Therefore, $(2x+1) - (x-1) = 0$ and $x = -2$. [correct computation]
But x is given as equal to 1. I can check that $x = -2$ is the correct value by substituting it into $\log_{10}(2x+1) = \log_{10}(x-1) \Rightarrow \log(-4+1) = \log(-3)$

[no indication of links to the concept of logarithm]. *Both sides of the equation are the same, therefore, $x = -2$ is the correct value. That is, the student's response is unacceptable.*

These response-data were classified as having evidence of pseudo-conceptual knowledge because they are a collection of relevant steps for computation in order to achieve an answer (Figure 3.2, Chapter 3). In addition, there was little evidence of knowledge associated with the *function* concept, namely *logarithm* (Tall, 1992; Alters, 1996).

The response-data for the TRIG item indicated similar computational procedures for SQ1 as the following response-data from M5. Participant M5 is a 'maths major' from State B.

- (1) *$\cos(2x+1) = 0$, that is, cosine of what angle is equal to zero.*
 [recognised the correct symbolisation for the trigonometric concept]
That is, $(2x+1) = 90^\circ$ or $\pi/2$, but $\pi/2$ is correct because $(2x+1)$ is assumed to be radians rather than degrees. [specific knowledge for computation] Therefore, $\cos \pi/2 = 0$, or $(2x+1) = \pi/2$.
Answer: $x = \pi/4 - 1/2$. [a single solution]
- I don't think this student understands trigonometry because $(2x+1)$ is the angle and you cannot expand $\cos(2x+1)$ to $\cos 2x + \cos 1$.*
 [repeat of earlier statement]

These response-data were classified as pseudo-conceptual knowledge because, although references were made about the mathematical concept involved (*$\cos(2x+1) = 0$, that is, cosine of what angle is equal to zero ... $(2x+1)$ is the angle and you cannot expand $\cos(2x+1)$ to $\cos 2x + \cos 1$*), these are knowledge representations of algebraic algorithms, syntax of symbols and format rather than knowledge of the *function* concept (Tall,

1992; Even, 1993). In addition, the single solution (*Answer: $x = \pi/4 - 1/2$*) did not reflect the essential aspect (periodic function) of the concept *trigonometry* (Tall, 1992).

Participant M4's response-data below provide an example of pseudo-conceptual type knowledge for the STAT item.

$$(1) \quad S^2 = \frac{\sum x^2}{n} - \bar{x}^2 = 194/10 - 4^2 = 3.4 \text{ [Recalled the formula}$$

correctly, computed and identified the correct answer of 3.4]. *The variance or S^2 (i.e standard deviation squared) is the sum of the squared-scores divided by n, the number of scores, take away the mean-squared. I know this is right because the other two values, 2.4 and 2.7, are not solutions using this formula. I found with learning statistics that, knowing the correct formula and how to use it is all that's required.*

Although M4 recalled the formula correctly and provided the correct solution, his explanation for the concept *variance* was a description of the formula in terms of the symbols involved and in relation to computation. That is, the term *variance* was perceived as an object (or formula) rather than a representation of a concept (Green, 1983; Miller, 1993; Shaughnessy, 1993). M4's pseudo-conceptual knowledge of statistics appears to explain why he concluded that, *knowing the correct formula and how to use it is all that's required* in learning statistics.

The category (3) classifications described above were based on knowledge about prescribed procedures for computations and the application of appropriate rules or formulae. Although these responses showed evidence of

correct computations with correct solutions, there was little (or no) evidence of 'a mutual assimilation between a symbol system and a conceptual structure' or symbolic understanding (Skemp, 1982, p.61). This lack of evidence, particularly of conceptual structures, might be attributable to a misconception of the 'function concept' (e.g. Tall, 1992; Even, 1993). Tall (1992, p.500) suggested that students' misconceptions of the function concept could be attributed to difficulties with the variety of different representations (e.g. graph, formula, table, and so on) and the relationships between them (Section 4.2.1.2, Chapter 4). From the analysis of these pre-service teachers' mathematical knowledge, it could be suggested that the pre-service teachers' knowledge was a product of learning formulae and other algebraic representations of concepts, rather than a product of symbolic understanding or 'a mutual assimilation between a symbol system and a conceptual structure' (Skemp, 1982, p.61; Green, 1983; Tall, 1992; Even, 1993; Shaughnessy, 1993).



5.2.1.4 Category (4) (Symbolic understanding)

There were no response-data that satisfied the criteria for category (4), symbolic type understanding. Category (4) type response-data would provide evidence of the individual's ability to examine most or all of the given cued-data and integrate these with his or her own knowledge. In addition, the response-data would show evidence that the 'whole' stimulus item has been considered and evaluated in context and that attempts were made to reconcile the inconsistencies or conflicts associated with the cued-data in the context of important learning and teaching strategies. Furthermore, such attempts would be orderly and provide relevant aspects of conceptual knowledge.

This completes the first phase of the analysis of the data from these pre-service teachers of secondary mathematics who agreed to participate in the study. The summary of this analysis is presented in Tables 5.3, 5.4, and 5.5.

Table 5.3: Response-data classified as Category (1)

Knowledge Deficiencies	TRIG	LOG	STAT	Total
Undeveloped		<div>F1</div> <div>F7</div> <div>M8</div>	<div>F2</div> <div>F5</div> <div>M1</div> <div>M8</div> <div>M10</div> <div>M11</div> <div>M6</div>	11 (19%)
Unproductive	<div>F3</div>	<div>F3</div>		2 (4%)
Unrelated	<div>F2</div> <div>F4</div> <div>M1</div>		<div>F4</div>	4 (7%)
Unprocessed	<div>M8</div>	<div>F4</div> <div>M1</div>	<div>F3</div> <div>F5</div> <div>F7</div> <div>M7</div> <div>M2</div>	8 (14%)
Total	5	6	14	25 (25/57=44%)

Legend:  - indicates 'maths minor' from State A
 - indicates 'maths minor' from State B

In Table 5.3, the distribution of response-data classified as category (1) are presented according to four types of knowledge deficiencies. The undeveloped (11/25) and unprocessed (8/25) knowledge were the most common kinds of knowledge deficiencies observed. An undeveloped type of knowledge was defined as knowledge having a deficiency of essential elements that are needed in the formation of relational-conceptual types of knowledge. Another characteristic of this type of deficiency was the respondents' awareness that they need to do further studies in order for them to gain a better understanding of the mathematics. The STAT and the LOG items elicited undeveloped knowledge type responses with a high proportion of such responses for the STAT item.

An unprocessed type of knowledge was defined as knowledge having a deficiency of relational-conceptual type knowledge that gives meaning to knowledge of formulae and algorithms. This form of mathematical knowledge deficiency appears to inhibit further cognitive processing, resulting in an inability by the respondent to provide a reason for the actual recall. The STAT item also seems to attract more of this type of response.

An unproductive type of knowledge was defined as one having a deficiency induced by the dependency of the individual on external resource materials as a source of mathematical knowledge. The lack of a relevant response was often associated with the absence of an external aid to facilitate memory recall. For example, the reliance on a textbook for a formula or the reliance on a calculator for formulae and computation. Hence, the respondent will not continue with the mathematical task if these external aids are not immediately available. This form of knowledge deficiency seems to be closely related to unprocessed type knowledge in that it inhibits further cognitive processing. Only participant F3 explicitly demonstrated this form of deficiency for the TRIG and LOG items.

An unrelated type of knowledge was defined as knowledge having a deficiency of relevant knowledge which is representatively similar to what the respondent has been presented with. Unrelated knowledge could be considered as an undeveloped type of knowledge in that a possible reason for the respondent's unfamiliarity with a given cued-data may be his or her lack of essential knowledge about the mathematics. The TRIG item elicited the most of this type of response.

In summarising the analysis to this point, it is assumed that the participants' lack of essential knowledge elements (undeveloped knowledge) and their dependency on 'fixed' knowledge of formulae and algorithms (unprocessed knowledge) were based on their instrumental understanding of the mathematics represented by the TRIG, LOG, and STAT items.

The distribution of response-data classified as category (2) and category (3) showing instrumental understanding and relational understanding are presented in Table 5.4 and Table 5.5 respectively.

Table 5.4: Response-data classified as Category (2) and Category (3) showing Instrumental Understanding

	INSTRUMENTAL UNDERSTANDING			
	TRIG	LOG	STAT	Total
Category (2) PSEUDO-PROCEDURAL	<div> <div>F1</div> <div>F6</div> <div>M2</div> <div>M3</div> <div>M4</div> </div>	<div> <div>F2</div> <div>F5</div> <div>M2</div> <div>M3</div> <div>M4</div> </div>	<div> <div>F3</div> </div>	11 (19%)
Category (3) PSEUDO-CONCEPTUAL	M5 M6 M7 M10	M5 M6 M7	M4	8 (14%)
TOTAL	9	8	2	19 (33%)



Legend:  - indicates 'maths minor' from State A
 - indicates 'maths minor' from State B

Table 5.5: Response-data classified as Category (2) and Category (3) showing Relational Understanding

	RELATIONAL UNDERSTANDING			
	TRIG	LOG	STAT	Total
Category (2) RELATIONAL PROCEDURAL	F5 F7 F8 M9 M11	F5 F8 M9	F8 M5	10 (18%)
Category (3) RELATIONAL CONCEPTUAL		M10 M11	M9	3 (5%)
TOTAL	5	5	3	13 (23%)

In the category (2) classification representing procedural types of knowledge, 19% (11/57) of the response-data were classified as instrumental understanding (Table 5.4) and 18% (10/57) as relational understanding (Table 5.5). In the category (3) classification representing conceptual types of knowledge, 14% (8/57) of the response-data were classified as instrumental understanding (Table 5.4) and 5% (3/57) as relational understanding (Table 5.5). However, the 14% of pseudo-conceptual knowledge in Table 5.4 is, by definition (Section 5.1.3), procedural type knowledge. The overall result in this study showed 51% (or $19\%+18\%+14\%$) of the response-data had evidence of procedural mathematical knowledge, 5% conceptual mathematical knowledge, and 44% category (1) type knowledge.

Therefore, in addition to category (1) type knowledge, procedural mathematical knowledge was the other most common (51%) type of mathematical knowledge observed in this study (Tables 5.4 and 5.5). In order to gain a further understanding of the nature of this procedural knowledge, it is worthwhile to recall the classification criteria for category (2) and category (3) type data. That is, responses showing evidence of *knowledge recognition of formulae, algorithms, or systems of procedures* were classified as category (2) procedural knowledge. The responses showing evidence of *knowledge of computation involving appropriate rules and formulae, set algorithms, and procedures* were classified as category (3) knowledge pertaining to conceptual knowledge.

Procedural mathematical knowledge on its own, without essential knowledge linkages to conceptual mathematical knowledge (as in relational

understanding), was observed to generate uncertainties or gaps in knowledge. This may explain the high proportion (14%) of category (3) type response-data associated with instrumental understanding as compared with relational understanding (5%).

Continuing the description of the summarised data in Tables 5.3, 5.4 and 5.5, it is observed that gender differences were negligible for category (1) (Table 5.3) and category (2) (Tables 5.4 and 5.5). However, for category (3), all of the response-data (19%) were from male participants, which indicates from these results that there was a gender difference with respect to conceptual types of mathematical knowledge. The dominance of category (3) type responses by males, however, was indicative of instrumental type understanding rather than of the preferred relational mathematical understanding.

In summary, the purpose of the first phase of the data analysis was to address the first research question: *What types of procedural and conceptual mathematical knowledge exist in the pre-service teachers' mathematical knowledge bases?* The results of this analysis of pre-service teachers' mathematical knowledge of trigonometry, logarithm, and statistics, indicated that the predominant types of knowledge were of the procedural types associated with instrumental understanding of mathematics. The 51% of response-data with procedural type mathematical knowledge and the 44% of response-data in category (1) contained evidence of mathematical knowledge deficiencies, particularly the response-data in category (1). These knowledge deficiencies appear to be present regardless of the mathematical backgrounds (e.g. maths major, maths minor) the pre-service teachers had gained from their university undergraduate studies. However, the pre-service teachers with

'maths minor' backgrounds from both State A (F1, F6, M1, M2, and M3) and State B (F2, F3, and F4) seemed more likely to show evidence of knowledge deficiencies for all the three stimulus items as indicated in category (1) (Table 5.3), than participants with 'maths major' backgrounds (Tables 5.4 and 5.5). With respect to the stimulus items, the responses-data for the STAT item showed 56% (14/25) of category (1) type knowledge whilst the response-data for the TRIG and LOG items showed 84% (27/32) of procedural type knowledge. These results are discussed further in Chapter 6.

This completes the analysis of the data addressing the first research question. The second phase of the analysis is the examination of responses to the stimulus questions 2 and 3 (SQ2 and SQ3) in an endeavour to address the second research question: *What possible outcomes could any identified deficiencies in types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?* This analysis is described in the next section, Section (5.2.2).

5.2.2 Phase two of the analysis

The report in this section is on the analyses of responses to SQ2 and SQ3. Some of the response-data for SQ2 and SQ3 were examined in the context of response-data for SQ1 reported in Section (5.2.1) above.

In order to provide a context for examining the response-data associated with SQ2 and SQ3, some of the earlier discussions are briefly revisited here. In Chapter 2, Section (2.3), it was suggested that, in theory, teachers' competence to teach mathematics (or pedagogical knowledge) is dependent on their understanding of the mathematics they teach. However, in practice, teachers' pedagogical knowledge seems to be also influenced by factors such

as knowledge about the students' thinking (e.g. Fennema, 1996; Fennema & Carpenter, 1998; Rhine, 1998) and teachers' own goals and beliefs about teaching (e.g. Ball, 1990; Eisenhart *et al.*, 1993; Alexander, 1995). Another factor is that found in the analysis of experienced teachers' interview data (Section 4.2.2, Chapter 4). It was found that the teachers' perceptions of what is important learning for their students tended to influence the teachers' teaching of mathematics. However, for pre-service teachers of mathematics, this pedagogical knowledge is likely to be rooted in their prior learning experiences, and in their goals and beliefs about mathematics teaching (e.g. Ball, 1990; Alexander, 1995; De Corte, 1995). Therefore, in this study of pre-service teachers' mathematical understanding, it was important to explore the links between mathematical knowledge and knowledge pertaining to pedagogical knowledge. The data for this exploration were the participants' responses to SQ2 and SQ3.

The response-data for SQ2 were the participants' perceptions of what are '*important mathematical learning for students*'. The response-data for SQ3 were the participants' perceptions of appropriate '*teaching approaches*' for teaching the mathematics.

In an endeavour to identify any relevant links between mathematical and pedagogical knowledge from the pre-service teachers' response-data for SQ2 and SQ3, and how these responses are related to response-data for SQ1, (summarised in Tables 5.3 to 5.5), the response-data for SQ2 and SQ3 are examined in terms of the three stimulus item, TRIG, LOG, and STAT. The responses for the TRIG item are examined first.

5.2.2.1 Response-data for the TRIG item

The response-data for SQ2, *important learning in trigonometry for students*, appear to reflect two learning perspectives. These perspectives seem to relate to two forms of representations for the *function concept*, namely formula and graph (Tall, 1992; Even, 1993). Furthermore, these learning perspectives seem to reflect the same kinds of misconceptions about the function concept that were observed in the response-data for SQ1, particularly with category (3) data (Section, 5.2.1). For example, the misconception that *learning trig-ratios ... and their applications to real-life situations*, or *trig-functions and their graphs* are equivalent to understanding the function concept, trigonometry.

In this analysis, one of the perspectives appears to be based on the importance of learning 'abstract rules and formulae' (abstract) and the other on 'visual representations of formulae' (visual). Underlying both these perspectives is the learning of trigonometric formulae and the applications of these formulae. For example, in the 'abstract' perspective, participants F5, M4, M6, M7 and M8 considered the learning by students of *trig-ratios or an understanding of sine, cosine and tan ratios, and their applications to real-life situations* to be important. Similarly, participants M1, M2 and M3 considered the *understanding of the trig-formulae and trig-functions for the calculation of angles* as important learning in trigonometry by students of mathematics.

An interesting result of this analysis is that the 'visual representation of formulae' perspective is favoured more by the female participants while male participants tended to prefer the 'abstract rules and formulae' perspective. For example, the response-data by seven (out of eight) female participants (F1, F2, F3, F4, F6, F7 and F8) and three (out of eleven) male participants (M9, M10 and M11)

indicated the *graphing of the trig-functions and the use of the unit-circle* as important learning in trigonometry. These responses are summarised below:

It is important for students:

- (F1) *To understand trig-functions and their graphical representations.*
- (F2) *To learn graphing of trig-functions and knowing how to use the unit-circle.*
- (F3) *To understand how to graph the trig-functions.*
- (F4) *To understand unit-circles because all trigonometry stems from this.*
- (F6) *To understand trig-functions and their graphs. Because graphs distinguish trigonometry from other functions.*
- (F7) *To have knowledge of trig-graphs and the use of the unit-circle.*
- (F8) *To understand the relationships between what is sine, cosine, and tan. Know how to use the unit-circle and do graphs.*
- (M9) *To be able to draw graphs of the trig-functions.*
- (M10) *To have an understanding of what sine, cosine, and tan functions are in relation to the unit-circle and their graphs.*
- (M11) *To understand the unit-circle diagram and the related graphs.*

These response-data are summarised in Table 5.6 in relation to the categories of response-data for SQ1. This table format is adopted for presenting the summaries to SQ2 for the LOG and STAT items as well.

Table 5.6: Summary of SQ2 response-data for the TRIG item

What is important learning for students?			
PRE-SERVICE TEACHERS' PERSPECTIVE	TRIG		
	Category (1)	Category (2)	Category (3)
ABSTRACT	M1 M8	F5 _R M2 _I M3 _I M4 _I	M5 _I M6 _I M7 _I
VISUAL	F2 F3 F4	F1 _I F6 _I F7 _R F8 _R M9 _R M11 _R	M10 _I

Legend:
 Subscript R indicates response-data classified as Relational understanding.
 Subscript I indicates response-data classified as Instrumental understanding.

In relation to the types of mathematical understanding (Tables 5.4 and 5.5) associated with the pre-service teachers' pedagogical related knowledge (SQ2 response-data), the 47% (9/19 including category (1) response-data) who considered the learning of 'abstract rules and formulae' to be important seemed more likely to have *instrumental* understanding of mathematics than those who considered the 'visual representation of formulae' to be important learning for students (see Table 5.6 above). For those placing importance on 'visual representation of formulae', 32% (6/19) were classified as having *instrumental* understanding (including category (1) responses) and 21% (4/19) as having *relational* understanding (Table 5.6).

The summary of SQ3 response-data for the TRIG item is presented in Table 5.7 below.

Table 5.7: Summary of SQ3 response-data for the TRIG item

Appropriate teaching strategies	
PRE-SERVICE TEACHERS' PERSPECTIVE	TRIG
ABSTRACT	<ol style="list-style-type: none"> 1. Introduce trig-ratios using right-angled triangles. 2. Solve right-angled triangles using trig-ratios. 3. Teach trig-rules and applications. <p>Suggested by participants: F5 M1 M2 M3 M4 M5 M6 M7 and M8.</p>
VISUAL	<ol style="list-style-type: none"> 1. Discuss why the need to learn trig-functions. 2. Teach unit-circle and graphing of trig-functions. 3. Introduce trig-ratios and right-angled triangles. 4. Teach trig-rules in relation to trig-ratios and trig-graphs, and teach how to apply trig-rules to real-life situations. <p>Suggested by participants: F1 F2 F3 F4 F6 F7 F8 M9 M10 and M11.</p>

The response-data for SQ3 or *appropriate teaching approaches* (Table 5.7), appear to indicate strategies which support the response-data for SQ2. For

example, the most common teaching approaches associated with the learning of ‘abstract rules and formulae’ were similar to that suggested by participant M4 (maths major, State B): *Start with the basics of trigonometry (ie. trig-ratios) and then build onto those*, and participant M7 (maths major, State B): *Use triangles to derive trig-ratios, then use trig-ratios to solve triangles, and then the trig-rules and their application*. On the other hand, teaching approaches associated with ‘visual representation of formulae’ tended to be based, for example, initially on the teaching of graphing skills and knowledge of the unit-circle. The following are examples of response-data displaying this ‘visual’ perspective:

Participant F1 (maths minor, State A): *I will start with a discussion on why we need to learn trig-functions, and then move on to graphing the trig-functions, and then introduce trig-ratios in relation to the angles of triangles*.

Participant F3 (maths minor, State B): *Teach calculator skills first and then the unit-circle and graphing, followed by memorising exact trig-values*.

Participant F8 (maths major, State A): *Revise earlier work related to trig. eg. graphing skills, pythagorus rule and right-triangles. Introduce the unit-circle and how it is used in relation to trig-functions before teaching the trig-rules and how to apply them*.

Participant M9 (maths major, State A): *I’ll start with graphing, but the syllabus will give a good guide as to the most appropriate teaching sequence to follow in relation to the students’ ability levels*.

To summarise the analysis of responses for the TRIG item, there appears to be some indication of gender difference in relation to the respondents’ pedagogical content knowledge of trigonometry. That is, the majority of male participants, except participants M9, M10 and M11, seem to consider

the learning of 'abstract rules and formulae' to be important for trigonometry (Table 5.6). On the other hand, the female participants (all except F5) appear to have considered the learning of trigonometry based on 'visual representation of formulae' as being important for students. However, this pedagogical knowledge appears to be based on mathematical knowledge which is predominantly procedural in nature.

An important observation which may assist in the interpretation of the response-data for the LOG and the STAT items, concerns the three male participants M9, M10 and M11. These three participants and the female participants responded to the TRIG item in a similar manner. It is of interest to also note that participants M9, M10 and M11 (all 'maths majors' from State A) were the only ones of the selected cases in the study who majored in computer science. Becker (1990) found from her studies of university graduates in mathematics and computer science that:

'The computer science majors definitely had a different view of mathematics from that of mathematics majors. These students generally did not like the *abstract nature of the subject*, and the emphasis on proving theorems from a certain set of assumptions. What they liked was the *usefulness of the subject*, how it could be applied to solve problems. Problem-solving was definitely the most important feature of computer science to these students.' (*italics added*, p. 123)

Using Becker's (1990) findings, it could be assumed that participants M9, M10 and M11 would tend to base their pedagogical decisions on what they perceive to be the important use of mathematics in solving problems. Adopting this perception, 'usefulness of the mathematics in solving problems', as a descriptor of the kinds of thinking which appear to have been generated by SQ2 and SQ3, the responses by participants M9, M10 and

M11 are used as a ‘gauge’ in an attempt to accurately measure and describe the response-data for the LOG and the STAT items.

Furthermore, it was observed that M9, M10 and M11 were the only participants with responses satisfying the criteria of conceptual types of knowledge associated with relational understanding for the LOG and the STAT items (Table 5.5, page 178). The response-data for the LOG item are examined next.

5.2.2.2 **Response-data for the LOG item**

The summary of SQ2 response-data for the LOG item is presented in Table 5.8 in relation to the ‘abstract’ and ‘visual’ perspectives identified for the TRIG item in Section (5.2.2.1) above.

Table 5.8: Summary of SQ2 response-data for the LOG item

What is important learning for students?			
PRE-SERVICE TEACHERS' PERSPECTIVE	LOG		
	Category (1)	Category (2)	Category (3)
ABSTRACT	M1 M8	F6 _I F5 _I F8 _R M2 _I M3 _I M4 _I M9 _R	M5 _I M6 _I M7 _I M10 _R M11 _R
VISUAL	F1	F2 _I	
DON'T KNOW	F3 F4 F7		

Legend:
Subscript R indicates response-data classified as Relational understanding.
Subscript I indicates response-data classified as Instrumental understanding.

The response-data by participants M9, M10 and M11 for SQ2 of the LOG item are summarised as follows:

Participant M9: *It is important for students to understand the log laws and their relation to indices.*

Participant M10: *It is important for students to gain a good grounding and understanding of the log laws.*

Participant M11: *It is important for students to understand the relationship of logs to exponentials, and to have the knowledge that log functions are for specific applications, in the same way trig-rules are specifically for solving problems related to measurement of lengths and angles.*

Using the above responses as a 'gauge' or a measure of thinking based on the 'usefulness of logarithms in problem solving', it seems that an understanding of the logarithmic laws and the relationship of these laws to exponential functions are useful and essential learning in solving problems related to logarithm.

It was observed that the response-data by participants F5, F6, F8, M1, M2, M3, M4, M5, M6, M7 and M8 also indicated the same or similar content to the response-data by participants M9, M10 and M11 (see Table 5.8). For example, *it is important for students to learn the relationships between indices and logarithmic laws and how to use and apply the laws to solve problems* (participant F8, 'maths major', State A). *It is important for students to memorise the log rules thoroughly to make it easier for them to learn and use these rules in solving problems* (participant M6, 'maths major', State B), and *for students to learn the log laws and to understand that logarithm is for linearising exponential equations* (participant M5, 'maths major', State B). These response-data tend to reflect the importance of learning 'abstract rules and formulae' (Section 5.2.2.1 above).

The 'visual representation of formulae' perspective was evident in the response-data by female participants F1 and F2 (Table 5.8). For example, participant F1 (maths minor, State A) suggested that *graphing is important to learning functions, that is, seeing what a log-graph looks like compared to other graphs like quadratic graphs is important at the initial stage when introducing students to logs*. Similarly, participant F2 (maths minor, State B) suggested that *it is important for students to learn the different graphs as well as the use of graphs to represent the relationship between logarithm and exponential functions*.

However, not all of the participants had sufficient confidence in their own understanding of logarithm for them to suggest what could be important learning for students. Three female participants: F3 (maths minor, State B), F4 (maths minor, State B) and F7 (maths major, State B), responded to SQ2 with '*I don't know*' responses (Table 5.8). These respondents also showed, by their responses to SQ1 (category (1), Table 5.3), that they lack essential knowledge in this area.

In relation to the two perspectives identified for the TRIG item, the response-data for the LOG item tended to reflect the learning of 'abstract rules and formulae' more so than the 'visual representation of formulae' (Table 5.8). The high proportion (14/19=74%) of the respondents (mostly males, 11/14) showing (see Table 5.8, page 188) a preference for the learning of 'abstract rules and formulae' in logarithm may be an indication that logarithm is perceived more as a specific set of algebraic rules or laws which are useful for *specific applications* (participant M11), particularly in relation to *indices*

or exponents (participants F8, M9, M11), for example, *linearising exponential equations* (participant M5).

However, in a similar way to the response-data for the TRIG item, these response-data also showed evidence of misconceptions associated with the function concept, namely logarithm. For example, the equating of representations such as *graphs* to the concept (Tall, 1992). This is illustrated in the response-data, for example, by participant F2: *graphing (representation) is important to learning functions (concept), that is, seeing what a log-graph looks like compared to other graphs like quadratic graphs is important at the initial stage when introducing students to logs*. Another misconception is equating the *concept* to a specific operation (e.g. participant M5's response-data: *students to understand that the logarithm (concept) is for linearising (operation) exponential equations*) or the concept to an *object* (e.g. participant M11's response-data: *students to have the knowledge that log functions (concept) are for specific applications, in the same way trig-rules (objects) are specifically for solving problems related to measurement of lengths and angles*).

The summary of SQ3 response-data for the LOG item is presented in Table 5.9 (next page).

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Table 5.9: Summary of SQ3 response-data for the LOG item

Appropriate teaching strategies	
PRE-SERVICE TEACHERS' PERSPECTIVE	LOG
ABSTRACT	<ol style="list-style-type: none"> 1. Introduce log-laws in relation to indice-laws. 2. Solve algebraic problems using log-laws. 3. Application of log-laws by doing lots of type-examples and written exercises. <p>Suggested by participants: F5 F6 F8 M1 M2 M3 M4 M5 M6 M7 M8 M9 M10 and M11</p>
VISUAL	<ol style="list-style-type: none"> 1. Discuss real-life situations in which logarithms are used. 2. Compare log-graphs to other graphs, e.g. quadratics. 3. Teach log-laws and show relations to indice-laws. 4. Application of log-laws to word problems. 5. Solve complex algebraic problems of the form Y^X using log-laws. <p>Suggested by participants: F1 and F2.</p>
DON'T KNOW	<ol style="list-style-type: none"> 1. Teach calculator skills. 2. Solve algebraic logarithmic problems using log-laws. 3. Do lots of written exercises. <p>Suggested by participants: F3 F4 and F7</p>

The *teaching approaches* associated with the perception of logarithm as a 'useful set of laws' for specific applications, appear to focus on strategies relating to teaching students *how to use and apply the log-laws to solve algebraic problems* (participant M9) and for students to *do lots of type-examples and written exercises* (participant M10). Such approaches tend to promote the rote learning of rules and teacher reliance on external sources (e.g. textbooks and calculators). The following are further examples of teaching approaches associated with the 'abstract' perspective (Table 5.8).

Participant F8 (maths major, State A): *I would start with indices and log-laws and make sure students understand the difference between distributive law and log-laws.*

Participant F5 (maths major, State B): *I'll teach log laws because that is how I learn logs and make sure students do lots of problems, including word problems.*

Participant M8 (maths major, State A): *First, I'll teach the log-laws and make sure students understand their relation to the indice-laws.*

Participant M6 (maths major, State B): *I would emphasise the importance of log rules for students to remember and for students to do lots of examples.*

Participant M1 (maths minor, State A): *I would emphasise the log-laws because unless the students learn the laws well, they won't know what to do.*

These response-data indicate a dominance of teaching logarithm as abstract rules and formulae. Even with a 'visual' perspective (e.g. participant F2: *First, discuss how logarithmic functions are used to solve real-life problems, then compare graphs of $x = \log_b y$ and $y = b^x$*), the teaching of logarithmic laws and their application in solving algebraic problems still seemed to dominate (participant F2 continues: *Teach the log-laws and how to apply them and how to use log-laws to solve complex functions of the form y^x*).

Although the female participants (F3, F4 and F7) with '*I don't know*' response-data for SQ2 lacked essential knowledge in logarithm (category (1), Table 5.3), the *teaching approaches* they suggested were similar to strategies suggested by respondents with an 'abstract' view of learning logarithm (Table 5.9). For example, teaching students *how to use the calculator* (F3), *how to apply the log-laws to solve algebraic problems* (F4, F7), and for students to *do lots of written exercises* (F7). These response-data tend to suggest that the act of teaching could still be performed without sound knowledge of the topic so long as the relevant resource materials, such as calculators and textbooks with formulae and exercises, are available.

It could be suggested from these response-data for SQ3 that there is a tendency for the pre-service teachers to rely on resource materials for *teaching*. Such a reliance on external sources may be more a reflection of how the pre-service teachers were taught logarithm in secondary school, rather than indicators of pedagogical knowledge based only on the individual's mathematical understanding. For example, participant F3's

response-data for the LOG item indicated her lack of understanding of logarithm (category (1) type knowledge), yet she confidently suggested that [she] *I will teach logarithm using the calculator because that is how [she] was taught*. Participant F4, indicated a similar confidence in teaching logarithm based on previous learning experiences: *I'll start [teaching] with log laws, that's because it's the way I usually work (i.e. find a rule and follow that)*. These types of response-data could be attributed to the pre-service teachers' lack of classroom teaching experience.

Nevertheless, it is a concern to observe that pre-service teachers are basing their confidence on 'thin and rule-based understanding' of mathematics, rather than on substantive mathematical understanding (Ball 1990, p.464). Ball (1990) argued that mathematics teachers with confidence based on 'thin and rule-based understanding, can pose a threat to student learning if teachers confidently proclaim wrong ideas or portray mathematics in misleading ways' (p.464).

In relation to the links between the respondents' mathematical understanding and their pedagogical knowledge of logarithm, it seems that procedural knowledge associated with *instrumental* and *relational* understanding is the major source of knowledge contributing to the respondents' decisions about what is important learning and what is appropriate teaching.

In summarising the response-data for the LOG item, it appears that 'abstract logarithmic rules and formulae' were considered by most pre-service teachers to be the more important type of knowledge students should acquire for learning how to solve problems involving logarithm. The participants appear to have been influenced by their lack of teaching

experience and lack of relevant knowledge of logarithm to the extent that they base their suggestions of *appropriate teaching approaches* on how they themselves were taught logarithm.

However, having to reflect on how logarithm was learned and taught in high school tends to suggest that university mathematics did not adequately provide the pre-service teachers with the type of understanding which promotes confidence in teaching. This kind of result is a concern.

According to Ball (1990), the ‘view they [pre-service teachers] do hold is likely to shape not only the way in which they teach mathematics once they begin teaching but also the way in which they approach, learning to teach mathematics’ (p.463).

With respect to gender differences in teaching, there was an indication that female respondents tended more so to consider the ‘visual representation of logarithmic formulae’ as important learning for students, and were more likely to show lack of essential knowledge about logarithm than male participants.

5.2.2.3 Response-data for the STAT item

The summary of SQ2 response-data for the STAT item is presented in Table 5.10 (next page).

Table 5.10: Summary of SQ2 response-data for the STAT item

PRE-SERVICE TEACHERS' PERSPECTIVE	What is important learning for students?		
	STAT		
	Category (1)	Category (2)	Category (3)
ABSTRACT	F3 F5 F6 F7 M3 M7	F1 _I F8 _R M5 _R	M4 _I M9 _R
VISUAL			
DON'T KNOW	F2 F4 M1 M2 M6 M8 M10 M11		

Legend:

Subscript R indicates response-data classified as Relational understanding.

Subscript I indicates response-data classified as Instrumental understanding.

The response-data by participants M9, M10 and M11 for the STAT item were also examined for descriptors on the 'usefulness of statistics in problem solving'. However, it was found that only participant M9's responses could provide some helpful guidance. The following are the responses to SQ2 and SQ3 by participants M9, M10 and M11.

Participant M9: *It is important for students to have an understanding of the different forms of the variance formula and the skills involved in using and applying the formulae* (the response to SQ2). [It is worthwhile to note that M9 was the only participant who provided a response (to SQ1) categorised as *conceptual knowledge* associated with relational understanding for the STAT item]. However, M9's response to SQ3 did not show the same confidence as in the above responses: *Again* [referring to previous responses for the TRIG item], *it's important to get guidance from the syllabus for the appropriate teaching sequence to follow.*

Participant M10: *I don't know, I guess knowledge of what a variance is* (the response to SQ2). Participant M10's lack of knowledge about the variance was also reflected in his response to SQ1 (Table 5.3). This lack of knowledge also seems to have influenced M10's response to SQ3: *Statistics is not my strength, so for teaching, I'll just stick to the syllabus.*

Participant M11: *I'm not really sure* (the response to SQ2). This response was similar to M11's response to SQ1: *I don't really know, I know the mean and the deviation from the mean is squared, but that's about it.* The response to SQ3 by M11 also reflects the same insufficiency of knowledge: *I need to learn and do more studies in this area before teaching it.*

Participant M9's responses, *learning different forms of the ... formula ... and applying the formulae*, seem to reflect the importance of learning 'abstract rules and formulae'. However, the teaching approach for statistics suggested by M9 is relatively the same as that suggested by M10, and there is also a likelihood of the same strategy of *following the syllabus guidelines* by participant M11.

For the STAT item, it appears that either the respondents had knowledge of the statistical variance or they did not know enough about the variance formula for them to provide a suggestion of what is important learning for students and how they intend to teach it (Table 5.10). It is worthwhile to note that a similar result for the STAT item was also observed in the interview data from the 18 experienced mathematics teachers. The analysis of these interview data is reported in Section (4.2.2) of Chapter 4. However, the teachers' lack of knowledge about 'variance' was related to their lack of

opportunities in teaching (e.g. *I can't answer this because I have not taught the statistics topic yet, that is coming up next term*) as well as lack of specific knowledge (e.g. *this type of statistics was not part of the course when I was at teachers' college and I have only taught the basic statistics like finding the mean, mode, and median, and drawing frequency tables and graphs*).

The respondents who appear to have some knowledge of the *statistical variance* tended to suggest that what is important learning for students is: *Understanding what the variance is and it's use in data analysis* (participants F5, F8, M5 and M7). *Understanding the variance formula, and how to use and apply it, or knowing which variance formula to apply in solving problems* (participants F1, F6, M3, M4 and M9). *Applications of the formula to real-life situations. Focus on all statistics including the mean and standard deviation* (participants F3 and F7). The learning of 'abstract rules and formulae' and the appropriate applications of these formulae were considered important by 58% (11/19) of the participants (Table 5.10).

The other 42% (8/19) of the participants (F2, F4, M1, M2, M6, M8, M10 and M11) showed insufficient knowledge of the *statistical variance* concept. This was indicated by the participants' uncertain responses (e.g. *I'm not sure*) about what learning is important for students, or by their lack of knowledge (e.g. *I don't know*) about statistics for them to give an appropriate response (Table 5.10).

It could be suggested from these response-data that the understanding difficulties the pre-service teachers had with statistics, and in particular with the 'variance', are related to the fact that many statistical terms such as the *variance* represent *concepts* and not *objects* (Green, 1983; Miller, 1993;

Shaughnessy, 1993). For example, the response-data by participants F1, F6, M3, M4 and M9 (*understanding the variance, and how to use and apply it, or knowing which variance formula to apply in solving problems*) show evidence of misunderstanding the concept (variance) as an object (variance formula). This misconception is further exemplified by responses such as *‘students to understand applications of the formula (object) to real-life situations. Focus on all statistics (concept) including the mean and standard deviation.*

The summary of SQ3 response-data for the STAT item is presented in Table 5.11 below.

Table 5.11: Summary of SQ3 response-data for the STAT item

Appropriate teaching strategies	
PRE-SERVICE TEACHERS' PERSPECTIVE	STAT
ABSTRACT	<ol style="list-style-type: none"> 1. Introduce statistics using real-life situations. 2. Teach the mean and variance formulae. 3. Application of the variance formulae to solve problems. 4. Interpret results. <p>Suggested by participants: F1 F7 F8 M4 M5 and M7.</p>
VISUAL	
DON'T KNOW	<p>The pre-service teacher:</p> <ol style="list-style-type: none"> 1. Needs to do further studies and to understand statistics first. 2. Find a good textbook. 3. Learn on-the-job. 4. Follow the syllabus. <p>Suggested by participants: F2 F3 F4 F5 F6 M1 M2 M3 M6 M8 M9 M10 and M11.</p>

An important observation with respect to teaching (SQ3 response-data) is participant M9’s response-data for the STAT item. It was observed that although M9’s response-data for SQ1 and SQ2 indicated evidence of conceptual understanding of the statistical variance, his response-data for

SQ3 showed evidence (*follow the syllabus*) suggesting a lack of confidence in teaching this topic (Table 5.11). This observation tends to contradict the assumption that an individual with relational mathematical understanding would demonstrate more confidence to teach mathematics than an individual with instrumental understanding (Section 4.1.1).

This lack of confidence to suggest an appropriate *teaching approach* for the variance was also expressed in the response-data for SQ3 by participants (F2, F4, M1, M2, M6, M8, M10 and M11) who claimed that they are '*not sure*' or '*don't know*' much about the variance formula. Evidence of this lack of confidence to teach statistics was reflected in responses such as: *I need to do further studies and to understand statistics first; find a good textbook on statistics; learn on-the-job, and follow the syllabus* (Table 5.11).

It could be suggested from these response-data that the lack of confidence to teach statistics may be related to the pre-service teachers' lack of teaching opportunities and lack of relevant knowledge specifically associated with the STAT item (a similar result was shown for experienced teachers in Section 4.2.2, Chapter 4). In addition, it seems that in teaching, although having relational understanding (e.g. participant M9) would enable the individuals to decide what is important learning for students, having little or no teaching experience tends to encourage the individuals to draw on their past experiences of how they were taught mathematics in high school or teachers' college (Section 4.2.2, Chapter 4). However, Ball (1990, p.464) would argue that this lack of confidence to teach would not necessarily be increased by having teaching experience, rather the '*increases in their [pre-service teachers] substantive understanding [and] changes in their ideas or feelings*

about mathematics' would ensure the kinds of confidence expected of competent mathematics teachers.

Nevertheless, the *teaching approaches* suggested by participants F1, F7, F8, M4, M5 and M7 seem to follow a sequential pattern: (a) introduce statistics using *real-life situations such as collecting and organising information about population and economic growths*, (b) *teach the mean and then the variance formula and how it is used*, and (c) *provide knowledge of how to interpret results* (Table 5.11).

Furthermore, the response-data for SQ3 of the STAT item tended to indicate that any observed differences in mathematical understanding between a pre-service teacher (e.g. F2, F4, M1, M2, M6, M8, M10 and M11) who has knowledge deficiencies (as in category (1), Table 5.3) and a pre-service teacher with relational understanding (M9) of mathematics, (*relational understanding* as in category (3), Table 5.5), are likely to be equalised once the pre-service teachers begin their teaching career. That is, regardless of background, they will *need to do further studies and to understand statistics first; find a good textbook on statistics; learn on-the-job*, and *follow the syllabus*.

To summarise the analysis of the response-data for the STAT item, it could be suggested that knowledge of the formula (object) is not sufficient for competent learning of the *statistical variance* (concept). Rather, important learning aspects, for example, a conceptual *understanding of what a variance* (concept) *is and how it is used in data analysis* (or its function) appear to be needed. The pre-service teachers' pedagogical knowledge appears to be mainly 'external knowledge' or knowledge from a *good textbook* or from

the syllabus, and by *learning on-the-job* (Table 5.11). This external knowledge could contribute to the high proportion (13/19=68%) of respondents who were not confident to suggest a teaching strategy. However, the pre-service teachers' lack of confidence could be attributed to their lack of teaching experience. Nevertheless, the concern is that this lack of confidence in teaching may also be attributed to the pre-service teachers' abstract and rule-based perspective of learning mathematics, which may not have 'afford[ed] them substantial advantage in articulating and connecting underlying concepts, principles, and meanings' (Ball 1990, p.463).

Differences due to gender were not evident in the response-data for the STAT item.

Summary of the second phase of data analyses

As an overall summary of the analyses of response data for SQ2 and SQ3, the tables of summaries (Tables 5.6, 5.8, 5.10) for SQ2 corresponding to the TRIG, LOG, and STAT items are combined in Figure 5.1. Similarly, the tables of summaries (Tables 5.7, 5.9, 5.11) for SQ3 corresponding to the three stimulus items are combined in Figure 5.2. These two Figures, 5.1 and 5.2, are presented in the following pages. These analyses indicated that there are differences in the respondents' knowledge pertaining to teaching (or pedagogical knowledge) not only within, but also across the three stimulus items (TRIG, LOG and STAT). In the subsequent discussions these differences are summarised in relation to knowledge insufficiencies identified in this phase and in the first phase of the data analysis (Section 5.2.1).

Figure 5.1: Summary of SQ2 response-data for the TRIG, LOG, and STAT items

What is important learning for students?			
PRE-SERVICE TEACHERS' PERSPECTIVE	TRIG		
	Category (1)	Category (2)	Category (3)
ABSTRACT	M1 M8	F5 _R M2 _I M3 _I M4 _I	M5 _I M6 _I M7 _I
VISUAL	F2 F3 F4	F1 _I F6 _I F7 _R F8 _R M9 _R M11 _R	M10 _I

PRE-SERVICE TEACHERS' PERSPECTIVE	LOG		
	Category (1)	Category (2)	Category (3)
ABSTRACT	M1 M8	F6 _I F5 _I F8 _R M2 _I M3 _I M4 _I M9 _R	M5 _I M6 _I M7 _I M10 _R M11 _R
VISUAL	F1	F2 _I	
DON'T KNOW	F3 F4 F7		

PRE-SERVICE TEACHERS' PERSPECTIVE	STAT		
	Category (1)	Category (2)	Category (3)
ABSTRACT	F3 F5 F6 F7 M3 M7	F1 _I F8 _R M5 _R	M4 _I M9 _R
VISUAL			
DON'T KNOW	F2 F4 M1 M2 M6 M8 M10 M11		

Legend:
 Subscript R indicates response-data classified as Relational understanding.
 Subscript I indicates response-data classified as Instrumental understanding.

Figure 5.2: Summary of SQ3 response-data for the TRIG, LOG, and STAT items

Appropriate teaching strategies	
PRE-SERVICE TEACHERS' PERSPECTIVE	TRIG
ABSTRACT	<ol style="list-style-type: none"> 1. Introduce trig-ratios using right-angled triangles. 2. Solve right-angled triangles using trig-ratios. 3. Teach trig-rules and applications. <p>Suggested by participants: F5 M1 M2 M3 M4 M5 M6 M7 and M8.</p>
VISUAL	<ol style="list-style-type: none"> 1. Discuss why the need to learn trig-functions. 2. Teach unit-circle and graphing of trig-functions. 3. Introduce trig-ratios and right-angled triangles. 4. Teach trig-rules in relation to trig-ratios and trig-graphs, and teach how to apply trig-rules to real-life situations. <p>Suggested by participants: F1 F2 F3 F4 F6 F7 F8 M9 M10 and M11.</p>

PRE-SERVICE TEACHERS' PERSPECTIVE	LOG
ABSTRACT	<ol style="list-style-type: none"> 1. Introduce log-laws in relation to indice-laws. 2. Solve algebraic problems using log-laws. 3. Application of log-laws by doing lots of type-examples and written exercises. <p>Suggested by participants: F5 F6 F8 M1 M2 M3 M4 M5 M6 M7 M8 M9 M10 and M11</p>
VISUAL	<ol style="list-style-type: none"> 1. Discuss real-life situations in which logarithms are used. 2. Compare log-graphs to other graphs, e.g. quadratics. 3. Teach log-laws and show relations to indice-laws. 4. Application of log-laws to word problems. 5. Solve complex algebraic problems of the form Y^x using log-laws. <p>Suggested by participants: F1 and F2.</p>
DON'T KNOW	<ol style="list-style-type: none"> 1. Teach calculator skills. 2. Solve algebraic logarithmic problems using log-laws. 3. Do lots of written exercises. <p>Suggested by participants: F3 F4 and F7</p>

PRE-SERVICE TEACHERS' PERSPECTIVE	STAT
ABSTRACT	<ol style="list-style-type: none"> 1. Introduce statistics using real-life situations. 2. Teach the mean and variance formulae. 3. Application of the variance formulae to solve problems. 4. Interpret results. <p>Suggested by participants: F1 F7 F8 M4 M5 and M7.</p>
VISUAL	
DON'T KNOW	<p>The pre-service teacher:</p> <ol style="list-style-type: none"> 1. Needs to do further studies and to understand statistics first. 2. Find a good textbook. 3. Learn on-the-job. 4. Follow the syllabus. <p>Suggested by participants: F2 F3 F4 F5 F6 M1 M2 M3 M6 M8 M9 M10 and M11.</p>

The observed differences *within items* are considered first:

For the TRIG item (Figure 5.1), the observed difference concerns the predominance of females showing a preference for the 'visual' perspective and the predominance of males indicating a preference for the 'abstract' perspective. The frequency distribution of the response-data according to knowledge categories (1, 2, and 3) provided little indication to suggest that this gender difference in teaching perspectives was related to an influence of procedural and conceptual types of knowledge (Tables 5.2, 5.3, 5.4).

However, it could be suggested from the results in Figure 5.2 that the males would be more likely to teach trigonometry using algebraic representations than the females. Nevertheless, underlying these perspectives ('abstract' and 'visual'), is an understanding of trigonometry based on knowledge of formulae and graphical representations (objects) for specific applications rather than conceptual knowledge of trigonometry.

For the LOG item (Figure 5.1), the observed difference is the high proportion ($14/19=74\%$) of participants indicating their preference for the 'abstract' perspective. This difference appears to be an outcome that is related to procedural mathematical knowledge. For example, all of the participants, except F2, with response-data classified as categories (2) and category (3) indicated a preference for the 'abstract' perspective. Gender differences in teaching perspectives were observed. That is, males tended to prefer the 'abstract' perspective whilst the females tended to prefer the 'visual' perspective and were more likely to show lack of confidence in suggesting what is important learning for students. However, the understanding of *logarithm* as a concept was not evident from the response-data. Rather, as with trigonometry, logarithm was perceived as an object (e.g. *log-laws*) for

solving specific mathematical problems (e.g. *linearising exponential equations*).

Another observed difference with the response-data for the LOG item concerns the teaching strategies, particularly the strategies corresponding to the 'don't know' perspective (Figure 5.2). These strategies were suggested by participants F3, F4 and F7 who indicated (Figure 5.1) their insufficient knowledge of logarithm to the extent that they were unable to suggest what could be important learning for students. However, insufficiency of knowledge did not appear to inhibit these participants to suggest what they perceived as appropriate teaching strategies. However, such strategies tended to reflect 'thin and rule-based understanding' (Ball, 1990, p.465) that these participants had demonstrated in their response-data for SQ1 (Tables 5.3, 5.4, 5.5).

For the STAT item (Figure 5.1), the observed difference is the contrast in the participants' perspective about the statistical variance. That is, the participants would either say 'don't know' or suggest that the 'abstract learning of formulae' is important in learning statistics. This difference appears to relate to three factors: (1) The lack of specific knowledge about statistics. This is indicated by the majority of 'don't know' responses classified as category (1) response-data (Figure 5.1). (2) A misconception of 'variance' as an object (*variance formula*) rather than a concept. This misconception would result in 'abstract' response-data and category (1) type data. (3) A perception that knowledge of *variance* is 'external knowledge' contained in, for example, textbooks and calculators rather than a mental knowledge or schema (Skemp, 1979, 1982; Derry, 1996).

Another observed difference with the response-data for the STAT item concerns the teaching strategies associated with the 'don't know' perspective (Figure 5.3). That is, it appears that one (e.g. participant M9) could have relational understanding of the variance formulae and could suggest what is important learning for students, but may not have the confidence to suggest how the topic could be taught, except to *follow the syllabus* (or external knowledge). On the other hand, one (e.g. participant M2) may have insufficient knowledge about the variance and be *unsure* of what is important learning for students, while at the same time, he or she may show confidence in teaching by *relying on a good textbook because this sort of topic is too much to commit to memory, ... and of course I would need to follow the syllabus* (participant M2). These observations tend to imply that for these pre-service teachers, good teaching of mathematics is afforded by having information on *how to follow the syllabus*, and *how and where to find relevant resource materials*.

Observed differences *across the three items* are considered next.

The main difference observed across the three items is the progressive shift from the 'visual' perspective with the TRIG item to the 'abstract' perspective with the LOG and the STAT items. There could be several factors which influenced the participants' decisions to respond to the items differently. According to Ball (1990), based on her studies of prospective secondary mathematics teachers, "teacher candidates' knowledge, ways of thinking, beliefs, and feelings, jointly affected their responses ... [also] their approaches to figuring out problems were shaped by their self-confidence, [and] their repertoire of strategies" (p.461). The following discussion,

therefore, is an attempt to describe what might be the factors contributing to this difference.

It is suggested that the mathematics represented by each item may be one contributing factor. Another factor might be related to the participants' beliefs of what is important mathematics in learning and teaching.

The mathematics represented by the TRIG item involves the periodic nature of trigonometric functions. Although this periodic nature was presented to the participants in an algebraic form, it was possible for the individuals to use their understanding involving knowledge of trigonometric ratios and rules in relation to the unit-circle and trigonometric graphs. In addition, the TRIG item represented aspects of measurement and spatial knowledge of the sine, cosine, and tangent ratios in relation to the unit-circle and the sides of a right-angled triangle. The responses by the pre-service teachers to SQ2 and SQ3 indicated many of these knowledge aspects; the most common aspect was the trigonometric ratios. It could be suggested from the responses to SQ2 and SQ3 that females prefer the 'visual' aspect (or representations) of mathematics and that males prefer the 'algebraic or abstract' nature of mathematics.

The mathematics represented by the LOG item involves knowledge of specific formulae or laws, algorithms, and computational skills. Although these laws could be represented graphically, it seems that not all the participants, except F1 and F2, may have been familiar with or learned to illustrate logarithmic functions diagrammatically. From the response-data for SQ2 and SQ3, it could be suggested that most of the participants, regardless of gender, were only familiar with the algebraic formulation and

computations involving the logarithmic laws. This algebraic knowledge of logarithm appears to have contributed to the shift from ‘visual’ to the ‘abstract’ perspective for the LOG item. It could also be suggested from these results that these pre-service teachers will most likely teach logarithm using algebraic formulation (see Figure 5.2, page 204).

The mathematics represented by the STAT item also involves knowledge of a particular concept, *statistical variance*. However, it appears that the pre-service teachers had little knowledge of what the *variance* is, and that they perceived the *statistical variance* formula to constitute a system of symbols and procedures that are not essential to the general understanding of mathematics. For example, *this sort of topic is too much to commit to memory, so I’ve always relied on textbooks rather than try and commit it to memory* (participant M2).

In addition, it seems that a large proportion of the participants have had little study in statistics, including the *variance*. For example, *I really don’t know because I have not done much learning in this area* (participant F2). An important observation from the analysis of the response-data for the SQ1 of the STAT item (category (1) in Table 5.3) which seems to relate to these data is that, *undeveloped* and *unprocessed* knowledge deficiencies were observed to occur most frequently with the STAT item (Section 5.2.1). That is, there is a deficiency of essential knowledge elements due to lack of learning (*undeveloped knowledge*) and a deficiency of conceptual knowledge which gives meaning to formulae (*unprocessed knowledge*) that may have contributed to the outcomes summarised in Figures 5.1 and 5.2.

Furthermore, it is proposed that the 'abstract' perspective the participants preferred for the STAT item, may be a reflection of their 'beliefs' on what is important learning in statistics, rather than an indication of their mathematical understanding about statistics. For example, participant F5's responses to SQ1 indicated that she knows the symbols, (*I know that 'sigma squared' is the variance and the square-root of it is the standard deviation. But, looking at this [cued-data] I can't recall the formula*), but lacks the necessary understanding of these symbols for her to decide what is appropriate learning to teach students (*I don't really know or have an understanding of what a variance is. I just know it as 'sigma squared'*). However, when asked what is important learning to teach students, participant F5 replied: *The actual concept of the formulae students have to learn*. When asked about a teaching strategy, F5 replied: *I don't really know, I guess I will have to follow the syllabus*. It could be suggested from the responses to SQ2 and SQ3 for the STAT item that the 'abstract' classification is based on the pre-service teachers' *beliefs* about what might be important for students when learning the statistics formulae.

In summary, the aim of this second phase of the data analysis was to address the second research question: *What possible influences could any identified deficiencies in types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?* From the results of these analyses, it appears that deficiencies in conceptual knowledge of mathematics can influence: (i) student-teachers' dependency on mathematical knowledge obtained from external sources (e.g. textbooks, calculators, and syllabi), and (ii) student-teachers to make pedagogical decisions based on what they *believed* to be important, rather than on their symbolic understanding (Skemp, 1982) or substantive

understanding (Ball, 1990) of mathematics. In addition, a gender difference in teaching perspectives appears to be associated with a deficiency in conceptual knowledge. For example, females tended to prefer teaching using the 'visual' perspective whilst males tended to prefer teaching using the 'abstract' perspective.

This completes the second phase of the analysis. The following section is the report on the third phase of the analysis.

5.2.3 Phase three of the analysis

This phase of the analysis draws on the results from phase one and phase two. The aim of this phase of the analysis is to address both research questions in view of any 'similarities' and/or 'contrasts' of knowledge types displayed in the response-data from 'maths major' participants (Set A1 and Set B1) and 'maths minor' participants (Set A2 and Set B2). To achieve this aim, the results for the individual cases from the analysis in the first phase are tabulated into their respective groups in Figure 5.3 (next page). For consistency with the description of the cases in Chapter 4 (Table 4.1), the summarised response-data from each of the cases are presented in Figure 5.3 in their respective groups, namely Set A1, Set B1, Set A2, and Set B2.

Figure 5.3: Summary of the data analysis for each case studied

SET A1: 'MATHS MAJOR' - STATE A

STATE A UNI: A1 PARTICIPANTS	DEGREE/ MAJOR	TRIG Category:1,2,3,4 Skemp's Model	LOG Category:1,2,3,4 Skemp's Model	STAT Category:1,2,3,4 Skemp's Model
Female 8 A1: major	B.Sc (Maths-Hons) M. Chem Engin Dip.Ed	Category 2 Rel-Procedural	Category 2 Rel-Procedural	Category 2 Rel-Procedural
Male 8 A1: major	B.Sc (Engineer) Dip.Ed	Category 1	Category 1	Category 1
Male 9 A1: major	B.Sc (Computer Sc) Dip.Ed	Category 2 Rel-Procedural	Category 2 Rel-Procedural	Category 2 Rel-Conceptual
Male 10 A1: major	B.Sc (Computer Sc) Dip.Ed	Category 3 Pseudo-Conceptual	Category 3 Rel-Conceptual	Category 1
Male 11 A1: major	B.Sc (Computer Sc) Dip.Ed	Category 2 Rel-Procedural	Category 3 Rel-Conceptual	Category 1

SET B1: 'MATHS MAJOR' - STATE B

STATE B UNI: B1 PARTICIPANTS	DEGREE/ MAJOR	TRIG Category:1,2,3,4 Skemp's Model	LOG Category:1,2,3,4 Skemp's Model	STAT Category:1,2,3,4 Skemp's Model
Female 5 B1: major	B.Sc (Chem) Dip.Ed	Category 2 Rel-Procedural	Category 2 Rel-Procedural	Category 1
Female 7 B1: major	B.Sc (Maths) Dip.Ed	Category 2 Rel-Procedural	Category 1	Category 1
Male 4 B1: major	B.Sc (Chem) Dip.Ed	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 3 Pseudo-Conceptual
Male 5 B1: major	B.Sc (Maths) Dip.Ed	Category 3 Pseudo-Conceptual	Category 3 Pseudo-Conceptual	Category 2 Pseudo-Procedural
Male 6 B1: major	B. Sc (Engineer) Dip.Ed	Category 3 Pseudo-Conceptual	Category 3 Pseudo-Conceptual	Category 1
Male 7 B1: major	B.Sc (Maths/Physic) Dip.Ed	Category 3 Pseudo-Conceptual	Category 3 Pseudo-Conceptual	Category 1

SET A2: 'MATHS MINOR' - STATE A

STATE A UNI: A2 PARTICIPANTS	DEGREE/ MAJOR	TRIG Category:1,2,3,4 Skemp's Model	LOG Category:1,2,3,4 Skemp's Model	STAT Category:1,2,3,4 Skemp's Model
Female 1 A2: minor	B.Ed (Maths) (final year)	Category 2 Pseudo-Procedural	Category 1	Category 2 Pseudo-Procedural
Female 6 A2: minor	B.Ed (Maths) (3 rd year)	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 1
Male 1 A2: minor	B.Ed (Maths) (3 rd year)	Category 1	Category 1	Category 1
Male 2 A2: minor	B.Ed (Maths) (3 rd year)	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 1
Male 3 A2: minor	B.Ed (Maths) (final year)	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 1

SET B2: 'MATHS MINOR' - STATE B

STATE B UNI: B2 PARTICIPANTS	DEGREE/ MAJOR	TRIG Category:1,2,3,4 Skemp's Model	LOG Category:1,2,3,4 Skemp's Model	STAT Category:1,2,3,4 Skemp's Model
Female 2 B2: minor	B.A Dip.Ed (Maths Minor)	Category 1	Category 2 Pseudo-Procedural	Category 1
Female 3 B2: minor	B. EconDip.Ed (Maths Minor)	Category 1	Category 1	Category 1
Female 4 B2: minor	B.A Dip.Ed (Maths Minor)	Category 1	Category 1	Category 1

Legend:

Rel-Procedural - Relational Procedural Knowledge

Rel-Conceptual - Relational Conceptual Knowledge

Pseudo-Procedural - Instrumental Procedural Knowledge

Pseudo-Conceptual - Instrumental Conceptual Knowledge

5.2.3.1 Similarities of knowledge types

In Figure 5.3 above, all the participants in Set A1 (maths majors from State A), except M8, responded to the TRIG and LOG items with mainly procedural type knowledge. Participant M8's response-data for all three stimulus items were classified as category (1) type knowledge. Although M8 was able to recognise the 'errors' represented in the stimulus items, his knowledge seemed rather fragmented. For example, M8's responses to the TRIG item indicated that he did have knowledge of trigonometry: *The student is treating this as algebra, and there seems to be a misunderstanding between the cos and its angle* (interpreted the symbols correctly). However, his knowledge of trigonometry was limited (*But I can't remember how to do this now or to explain why this is so*). For the LOG item, although M8 was able to identify an 'error' (*The student fails to check the final answer to see that $\log(3) \neq \log(0)$*) and attempted to give a correction ($\log_{10}100 = 2$, ie. $10^2 = 100$), he was unable to proceed further (*Sorry, I can't remember*). M8's responses to the STAT item were simply, *Sorry, but I can't remember any of this*. The other cases in Set A1, except F8 and M9, also indicated category (1) type knowledge for the STAT item.

The response-data for the participants in Set B1 (maths majors from State B) indicated mostly pseudo-conceptual knowledge which is, by definition (Section 5.1.3), procedural type knowledge for the TRIG, LOG, and STAT items (Figure 5.3). Although F7 was able to provide a category (2) type response-data for the TRIG item, she was not able to do the same for the LOG item (e.g. *I can't remember, I know that there are log-laws but I can't*

remember what they are). All the participants, except M4 and M5, indicated category (1) type knowledge for the STAT item.

The response-data for the participants in Set A2 (maths minors from State A) indicated little differences between these cases (Figure 5.3). That is, the pseudo-procedural type knowledge was dominant for all the cases, except M1 with category (1) type response-data for all the three items. However, category (1) type knowledge was more evident with response-data for the STAT item. Both pseudo-procedural knowledge and category (1) type knowledge are rule-based knowledge associated with instrumental understanding, these are the types of knowledge referred to in Section (2.4) of Chapter 2 as *rote knowledge*.

The response-data for the three participants in Set B2 (maths minors from State B) indicated category (1) type knowledge as the dominant type, particularly for the STAT item. Although F2's response-data had evidence of category (2) type knowledge for the LOG item, these were pseudo-procedural type knowledge (or rote knowledge). However, F2's response-data for the TRIG and STAT items were classified as category (1), similar to the other two participants.

In summarising the results in this section in relation to the first research question, it appears that the dominant types of knowledge that exists in these pre-service teachers' mathematical knowledge bases were procedural knowledge and category (1) type knowledge (or rote knowledge). These results are consistent with those observed in the first phase of the analysis (Section 5.2.1). However, procedural knowledge pertaining to relational understanding was observed more in the response-data from Set A1 (maths

major from State A) than Set B1 (maths major from State B). Group Set B1's response-data, on the other hand, appear to show more evidence of procedural knowledge (the 'pseudo' type) classified as belonging to instrumental understanding. Nevertheless, both groups' procedural type mathematical knowledge is not indicative of these participants' mathematical backgrounds (majors in mathematics, physics, chemistry, engineering, and computer science).

Pseudo-procedural knowledge was also the dominant type of knowledge observed in the response-data for the Set A2 group (maths minors from State A). Category (1) type knowledge, on the other hand, was observed as the dominant type in the response-data by the participants in Set B2 (maths minors from State B). However, the underlying knowledge structures (or schemata) for both these knowledge types, category (1) and pseudo-procedural knowledge, are those based on rote knowledge (Section 2.4, Chapter 2) or instrumental mathematical understanding (Section 5.2.1).

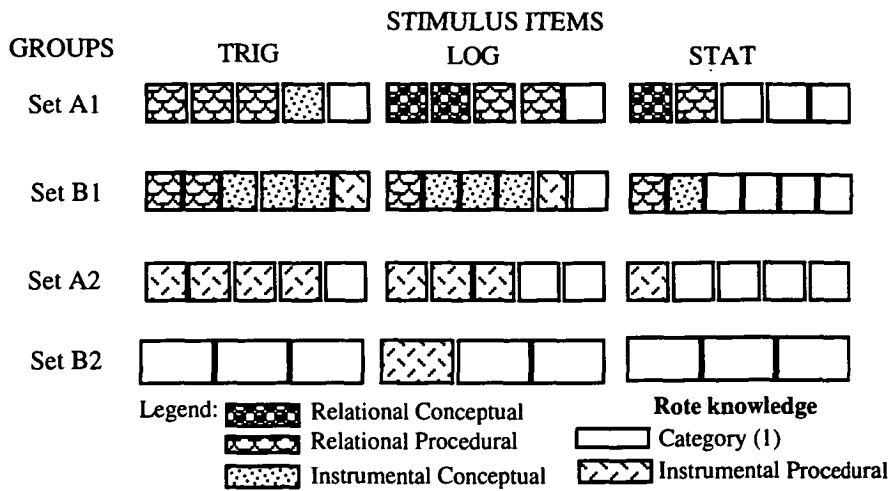
With respect to the second research question in relation to teaching knowledge (pedagogical knowledge), both Set A1 and Set B1 groups' response-data for SQ2 and SQ3 (Section 5.2.2) indicated similar results (Figure 5.1 and 5.2). That is, both groups indicated the importance for mathematics students to learn 'abstract rules and formulae'. Although there were some observed differences with the groups' response-data for the TRIG and STAT items (e.g. for the TRIG item, group Set A1 tended to prefer the 'visual' perspective and group Set B1 the 'abstract' perspective), the fact that both the 'abstract' and 'visual' perspectives were based on knowledge of formulae (or objects) rather than knowledge of concepts suggests that there were minimal conceptual structures on which

pedagogical practices could be built (Section 5.2.2). For the 'maths minor' (Set A2 and Set B2) groups' teaching knowledge (pedagogical knowledge), both groups' response-data for SQ2 and SQ3 indicated very similar results based on rote knowledge or 'thin and rule-based knowledge' (Ball, 1990).

Another similarity was in relation to the STAT item in which the main type of knowledge appears to be category (1). Although M4 and M9 (both 'maths majors') indicated category (3) type knowledge for the STAT item (Table 5.5), their response-data for SQ2 and SQ3 showed a similar lack of confidence in their understanding to those with 'maths minor' backgrounds.

Figure 5.4 (next page) summarises the data presented in Figure 5.3. In Figure 5.4, the knowledge types displayed in the response-data are illustrated in association with each stimulus item (TRIG, LOG, and STAT). In addition, the response-data from each participant is represented by a 'rectangular shape'. For example, in relation to the TRIG item, each of the five rectangles associated with Set A1 group represents a participant's response-data in this group. Likewise, each of the three rectangles associated with Set B2 group represents a participant's response-data in this group.

Figure 5.4 Similarities and contrasts in knowledge types of the four groups



5.2.3.2 Contrasts of knowledge types

From Figure 5.4 above, there appears to be more similarities than there are contrasts of knowledge types. For example, the similarities of knowledge types in the response-data with respect to the participants' mathematical backgrounds; referred to as 'maths major' and 'maths minor'. The 'maths major' participants seemed to have similar knowledge types, namely procedural mathematical knowledge (Figure 5.4). On the other hand, the 'maths minor' participants tended to show more evidence of rote type knowledge, particularly Set B2 (Figure 5.4).

However, the more obvious contrasting results are associated with the three stimulus items. For example, there is a noticeable difference of results for the TRIG and LOG items with respect to Set A1 (maths major) and Set B2 (maths minor) groups (Figure 5.4). In addition, the STAT item has elicited more category (1) type knowledge than the TRIG and LOG items. Both the

TRIG and the LOG items, on the other hand, tended to elicit procedural type knowledge. These results tend to suggest that there might be different knowledge structures (or schemata) for trigonometry, logarithm, and statistics. Also, that these schemata may not interlink but exist as separate 'memory objects' (Derry, 1996). It could be suggested that 'contrasts' between these schemata types are based on the 'demands' placed upon them during knowledge production (Figures 2.4, 2.5, Chapter 2) or the amount of repeated usage in meeting the individual's goal (Alexander, 1995; Derry, 1996). Based on such an approach, it could be suggested that because the participants had less need for statistics, minimal demand was placed on the 'statistics schema'; hence the majority of category (1) type knowledge for the STAT item.

In summarising this section, it appears that this notion of 'separate schema' for each mathematical area is feasible in theory and it could explain the contrast of knowledge types displayed in the response-data by groups Set A1 and Set B2 and the contrast of knowledge types observed between the function items (TRIG and LOG) and the statistic item (STAT). However, for pre-service teachers of mathematics, it appears essential that one of their goals should be to identify areas of weakness and to increase the 'demands' on these mathematical schemata in their pursuit to become competent teachers of mathematics.

This completes the analysis of the data of this study of secondary pre-service mathematics teachers' existing mathematical knowledge pertaining to trigonometry, logarithm, and statistics. A summary of the analyses in terms of the two research questions is presented in the following section, Section (5.3).

5.3 PART THREE: SUMMARY OF THE DATA ANALYSIS

The analysis of the data, particularly in the first phase, was for the examination of responses by nineteen secondary pre-service mathematics teachers to three mathematical stimulus items (TRIG, LOG, and STAT). These responses were assumed to be representative of the pre-service teachers' mathematical thought processes based on their existing mathematical knowledge bases. Therefore, it was assumed that the outcome of this analysis would appropriately address the first research question: *What types of procedural and conceptual mathematical knowledge exist in pre-service teachers' knowledge bases?* These outcomes are outlined below.

1. The 44% of the response-data for the three mathematical stimulus items showed evidence of mathematical knowledge deficiencies. These knowledge deficiencies were identified as:
 - a) *Undeveloped knowledge* or a deficiency of knowledge elements that are essential in the formation of conceptual types of knowledge.
 - b) *Unprocessed knowledge* or a deficiency of conceptual type knowledge that gives meaning to knowledge of formulae and algorithm.
 - c) *Unproductive knowledge* or a deficiency induced by the dependency of the individual on external resource materials (external knowledge) as a source of mathematical knowledge.
 - d) *Unrelated knowledge* or a deficiency of relevant knowledge which is similar to what the respondent was presented with.

It was observed that *undeveloped knowledge* and *unprocessed knowledge* were the most common types of knowledge deficiencies.

2. The other 56% of the response-data were representative of the *instrumental* and *relational* types of mathematical understanding. Of this 56%, 51% was representative of procedural type knowledge and 5% of conceptual type knowledge.
3. It was observed that procedural type knowledge on its own, without essential links to conceptual knowledge, generated uncertainties and gaps in knowledge, particularly procedural knowledge associated with instrumental understanding. Therefore, the 51% of procedural type knowledge also contained elements of knowledge deficiencies, particularly the 33% that was classified as characteristic of instrumental understanding. In addition, although 23% of the response-data were characteristic of relational mathematical understanding, there was little evidence of these, except for the 5% in category (3), displaying relevant links to conceptual knowledge.
4. The lack of knowledge pertaining to symbolic understanding (or 'a mutual assimilation between a symbol system and a conceptual structure, [but] dominated by the conceptual structure' (Skemp, 1982, p.61)) was observed to relate to misconceptions of the *function concept*. The concepts *trigonometry* and *logarithm* were treated and understood as objects (or formulae, rules, ratios, laws, and graphs) rather than concepts. Similarly, the statistics or *variance* concept was also understood as an object or a system of symbols such as 'sigma squared'. The lack of symbolic understanding also seems to be associated with a dependency on 'external knowledge' or knowledge from textbooks and calculators, particularly in the case of statistics.

In addressing the first research question, the outcomes of the first phase of the analysis tended to indicate that instrumental understanding was the dominant form of understanding by the pre-service teachers. In addition, the pre-service teachers' existing mathematical knowledge associated with the mathematics represented by the TRIG, LOG, and STAT items was largely based on procedural knowledge and possibly rote knowledge of formulae and algorithms.

The second phase of the analysis was particularly for the examination of responses by the pre-service teachers to two questions (SQ2 and SQ3) which focused on the thought processes involved in making pedagogical decisions. The two stimulus questions were concerned with *what is important mathematics that students must learn* (SQ2), and *which strategies would be appropriate for teaching the mathematics* (SQ3). It was assumed that these pedagogical decisions were dependent on, and influenced by, the pre-service teachers' existing mathematical knowledge and understanding (outcomes relating to SQ1). As such the outcomes from these decisions will provide data that would appropriately address the second research question: *What possible influences could any identified deficiencies in types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?* These outcomes are summarised below.

1. There was evidence of a gender difference in teaching perspectives with respect to trigonometry. The female participants tended to favour the teaching of trigonometry to students using visualisation strategies (e.g. graphs of trigonometric functions, unit-circle, right-angled triangles and trigonometric ratios). The male participants, on the other hand, tended to favour the teaching of trigonometry to students using

'abstract' rules and formulae. A similar gender difference in teaching perspectives was evident with responses concerning the teaching of logarithm. However, with respect to statistics where there was a large proportion (14/19) of the response-data showing evidence of category (1) type knowledge deficiencies, there was no observable gender difference in teaching perspectives.

2. The affects of knowledge deficiencies identified as *undeveloped*, *unprocessed*, *unproductive*, and *unrelated* on pedagogical decisions appear to be more pronounced with statistics than with trigonometry and logarithm. The *undeveloped* and *unprocessed* types of knowledge deficiencies were observed more frequently with the STAT item and appeared to relate to the participants' lack of prior learning in statistics (Green, 1983; Shaughnessy, 1993). Having knowledge deficiencies also seemed to influence the pre-service teachers to make pedagogical decisions which are based on their *beliefs* of how important and relevant the mathematics is to future achievements. It was observed that, as a consequence of having knowledge deficiencies, the pre-service teachers' confidence to teach for conceptual understanding is reduced while, at the same time, their dependency on rule-based teaching strategies, on 'external knowledge' (e.g. textbook knowledge) and on prescribed teaching strategies (e.g. the syllabus, learn on-the-job) increases.
3. The dominance of procedural knowledge and instrumental type understanding seemed to influence and promote pedagogical decisions which embraced mathematical learning of algorithms and algebraic formulae. Although the conceptual understanding of these abstract

formulations were considered important by the pre-service teachers, their suggested teaching strategies indicated they lacked confidence in their own understandings to suggest methods which promote conceptual mathematical learning by students. Such lack of confidence appeared to relate to the pre-service teachers' past experiences, particularly their high school experiences, of how they were taught mathematics as meaningless sets of algebraic procedures, and sets of symbols and formulae.

- 4, Conceptual knowledge pertaining to relational understanding of mathematics was observed in only a few of the cases. However, the pedagogical decisions associated with these cases showed little or no difference to decisions suggested by pre-service teachers with deficiencies or gaps in their knowledge of mathematics.

In addressing the second research question, it could be suggested from these outcomes that deficiencies of essential procedural and conceptual types of mathematical knowledge reduce confidence and likewise the potential of a secondary pre-service mathematics teacher to teach students for conceptual understanding of mathematics. In addition, having knowledge deficiencies would tend to influence pre-service teachers to depend on their past experiences of rule-based teaching strategies and on 'external' resource materials (e.g. textbooks, calculators, syllabus) as appropriate 'models' of teaching. Furthermore, it could be suggested from these results that mathematics teachers with teacher-confidence based on misconceptions or insufficiencies of mathematical knowledge and on the availability of resource materials are more likely to generate similar learning outcomes (knowledge deficiencies) from their students of mathematics.

The purpose of the third and final phase of the data analysis was to address both research questions in relation to the individuals' mathematical backgrounds ('maths major' and 'maths minor'). As such, the results for each of the nineteen cases were examined within and between groups. The following were similarities and contrasts in knowledge types observed within and between the four (A1, B1, A2, B2) groups.

1. Similarities of knowledge types observed within groups:
 - (i) Participants in Set A1 tended to display relational-procedural knowledge, particularly for the TRIG and LOG items.
 - (ii) Participants in Set B1 tended to display more pseudo-conceptual (or instrumental) knowledge than relational-procedural knowledge, particularly for the TRIG and LOG items.
 - (iii) Participants in Set A2 tended to display more pseudo-procedural than category (1) knowledge for the TRIG and LOG items.
 - (iv) Participants in Set B2 tended to display category (1) knowledge for all the three stimulus items.
2. Similarities of knowledge types observed between groups:
 - (i) Participants with 'maths major' backgrounds (A1 and B1) tended to display more procedural mathematical knowledge than category (1) type knowledge for the TRIG and LOG items.
 - (ii) Participants with 'maths minor' backgrounds (A2 and B2) tended to display rote knowledge for all the three stimulus items, particularly the participants in Set B2.
 - (iii) Participants in Set A2 displayed similar procedural ('pseudo') type knowledge to participants in Set B1, particularly for the TRIG and LOG items.

3. Similarities of knowledge types in relation to the three stimulus items.
 - (i) The STAT item appeared to elicit category (1) type knowledge from the participants in all the four groups.
 - (ii) The TRIG and LOG items appeared to elicit procedural type knowledge from the participants, particularly those in Set A1, Set B1, and Set A2.
4. Contrasts of knowledge types were observed only for, 'between group' results and not for, 'within group' results.
 - (i) Group Set A1 (maths majors) tended to display more procedural type knowledge than Set B2 (maths minor), particularly in relation to the TRIG and LOG items.
5. Contrasts of knowledge types observed between the three items.
 - (i) The results for the STAT item indicated more category (1) type knowledge than the results for the TRIG and LOG items.

In relation to teaching knowledge (pedagogical knowledge) associated with trigonometry, logarithm, and statistics, the results indicated little difference between the four groups. That is, all participants tended to display pedagogical knowledge based on 'external knowledge' from textbooks, calculators, and syllabi.

The similarities and contrasts in the types of mathematical knowledge evident in this final phase of the analysis of the response-data, give further support to the outcomes in phase one and phase two. In relation to the first research question, the pre-service teachers with 'maths major' (Set A1 and Set B1)

backgrounds tended to show more evidence of procedural mathematical knowledge than those with 'maths minor' (Set A2 and Set B2) backgrounds. However, the response-data for group Set A2 (maths minor), third and fourth year pre-service teachers in the B.Ed program, indicated relatively similar knowledge types (i.e. instrumental types) to those of Set B1 (maths major). The category (1) type knowledge was observed more with response-data for statistics (14/19) than with logarithm (6/19) or trigonometry (5/19). It was suggested in Section (5.2.3) that the high level of category (1) and procedural types of knowledge observed in the data could be the result of a greater reliance on 'external knowledge' by the pre-service teachers. As a consequence of this 'external knowledge', less demand was placed on the mathematical schemata associated with trigonometry, logarithm, and in particular statistics.

In relation to the second research question, it could be suggested from these results that, unless the individual makes it his or her 'goal' to place 'rigorous demands' on the production of mathematical knowledge structures (or schemata), then the likelihood of a dependency on 'external knowledge' would increase. For the pre-service teachers of mathematics, it is highly recommended that their goals in teaching are that of identifying their areas of weakness and increasing the 'rigorous demand' on these mathematical schemata, particularly if they are to develop a sound foundation of conceptual structures upon which pedagogical knowledge could be established.

These results and findings from the analysis of the data are discussed further in relation to the research questions and assumptions of this study in the next chapter, Chapter 6.

CHAPTER SIX

DISCUSSION

Introduction

In chapter 5, the mathematical knowledge that the nineteen secondary pre-service mathematics teachers brought with them to mathematics teacher education programs was examined in order to provide some answers to two research questions. The main focus of this chapter, therefore, is to address these two questions in relation to the outcomes of the previous analyses.

6.1 Addressing the first research question

1. What types of procedural and conceptual mathematical knowledge exist in pre-service teachers' knowledge bases?

In order to address appropriately the above question, it is worthwhile to restate the assumptions which formed the basis of this question. These were:

- (1) Pre-service mathematics teachers who majored in mathematics or other science related areas have acquired the necessary mathematical knowledge pre-requisites from their pre-tertiary and university studies in mathematics.
- (2) Mathematical understanding is dependent on the sufficiencies of procedural and conceptual types of mathematical knowledge. Lack of or a deficiency in either procedural or conceptual mathematical knowledge

types would suggest a deficiency in mathematical understanding (Hiebert & Lefevre, 1986; Eisenhart *et al.*, 1993). Therefore, it was expected in this study, that pre-service teachers with qualifications in mathematics or in other science related areas would provide stronger indicators of conceptual types of mathematical knowledge than those with qualifications in other areas (e.g. economics).

However, although the majority (11/19) of the study participants were university graduates with study majors in mathematics and other science related areas, the quality of the observed mathematical knowledge was not indicative of the mathematics rigour that is associated with university level mathematics. Instead, as a reflection of the quality of the mathematical knowledge examined in this study, 44% of the response-data displayed deficiencies in both procedural and conceptual types of mathematical knowledge. In addition, the quality of knowledge was mostly procedural in form (51%) and highly representative of 'instrumental mathematical understanding' (Skemp, 1978, 1979, 1982, 1986).

Hence, the quality of observed mathematical knowledge in this study does not support the expectation that pre-service teachers who majored in mathematics would provide stronger indicators of conceptual types of mathematical knowledge.

It could be suggested from the above results that most of the participants did not have sufficient conceptual knowledge of the mathematics related to the TRIG, LOG, and STAT items. Although 56% of the response-data tended to fit the evaluation criteria of Skemp's (1979) model of mathematical understanding, a high proportion (44%) that was classified as category (1)

response-data did not fit this criteria, and needed further clarification.

Although other studies have found that lack of conceptual mathematical understanding contributes to learning difficulties and incompetent teaching of mathematics (e.g. Skemp, 1986; Hiebert & Lefevre, 1986; Ball, 1990; Tall, 1992; Eisenhart *et al.*, 1993; Even, 1993; Greenwood, 1993; Wilson, 1994; Gates, 1995a, 1995b), these studies have not explicitly identified the nature of these mathematical knowledge insufficiencies and the effects on pre-service teachers' potential to teach for conceptual understanding of mathematics. Therefore, the following discussion, for the remainder of this section, is an attempt to provide a clarification for the category (1) data.

In the analysis of category (1) response-data in Chapter 5 (Section 5.2.1), four types of mathematical knowledge deficiencies were tentatively identified. These knowledge deficiencies were described as *undeveloped knowledge*, *unproductive knowledge*, *unrelated/unfamiliar knowledge*, and *unprocessed knowledge*. These descriptions of knowledge deficiencies evolved from an attempt to rationalise why a high percentage (44%) of the response-data to seemingly very familiar mathematics (represented by the TRIG, LOG, STAT items) showed evidence of lack of knowledge - *This is hard because I don't even remember what logarithm is about; I don't know, I can't remember, I haven't understood statistics well at Uni ...; I have no recall of what logarithm is...; I've forgotten, I don't know how to do this myself so I can't really say whether the student is right or wrong, I would need to look up a textbook*. One could argue that these responses are outcomes related to the recency of when the learning of the mathematics represented in the stimulus items took place or to how meaningful such learning was to these individuals.

Such an argument is feasible in the light of the fact that individuals have certain learning goals, and a choice in the type of information and how they learn or commit this information to memory (Ball, 1990; Alexander, 1995; De Corte, 1995). However, it was expected that university graduates in mathematics, assumed to be above average achievers in mathematics, would retain evidence of the mathematical schemata relating to the mathematics represented in the TRIG, LOG, and STAT items, unless such schemata were not developed at all (Skemp, 1982; Alexander *et al.*, 1991; diSessa, 1993; Derry, 1996).

Given that the response-data in category (1) were representative of participants from both Australian states and also from four universities, tends to suggest that it is not a localised affect, rather, there could be other underlying factors than recency and meaningfulness of learning. Three possible factors are considered below.

1. Deficiencies in lower-order mathematical concepts

One possible factor associated with *undeveloped* type knowledge (Section 5.2.1), relates to deficiencies of essential '*primary abstraction*' or lower-order concepts which, according to Skemp (1986, p.24), diSessa (1993) and Derry (1996), are required for achieving the understanding of higher-order mathematical concepts. In other words, the participants may have been required, as part of their prior learning of mathematics, to learn higher-order concepts (connected with the TRIG, LOG, and STAT items) without having an understanding of the primary abstractions of these concepts. For example, the summing of a group of scores to find their mean (or average) is an earlier abstraction of the concept variance - average of squared deviations

from the mean. In the light of the schema theory (Skemp, 1986; diSessa, 1993; Derry, 1996; Reynolds *et al.*, 1996), a deficiency of essential earlier abstractions or lower-order concepts in statistics might be the reason why 74% of the participants could not even make a recall of what a variance is, but openly acknowledged their lack of knowledge and understanding in this area (Section 5.2.1). However, some of the participants have referred to the way they were taught to rote learn the formulae in statistics as a reason for their lack of knowledge about the variance.

Failing to achieve lower-order concepts prior to learning higher-order concepts may contribute to lower retention of prior learning and could explain some of the deficiencies observed in the response-data (Section 5.2.1). However, the rote learning of formulae also emerges as a contributing factor to lower retention and deficiencies in prior knowledge.

2. Rote learning rules without reasons

Rote learning of *rules without reasons* is another factor known to influence deficiencies in mathematical understanding (Skemp, 1986, p.111; Greenwood, 1993; De Corte, 1995; Gates, 1995a). According to Skemp (1986), to teach students to learn mathematics using *rules without reasons*, 'would be equivalent to destruction of these schemata - the mental equivalent of bodily injury' (p.111). This suggests that, the insufficiency of knowledge indicated in category (1) response-data might not be a lack of learning ability in trigonometry, logarithm, and statistics by these individuals. Rather, it is likely that the prior knowledge relating to trigonometry, logarithm and statistics was the result of learning a series of meaningless rules which, when assimilated into one's schema, retards knowledge growth (Skemp, 1986;

Rakow, 1992; Biggs & Moore, 1993; Greenwood, 1993; Gadanidis, 1994; Gates, 1995a).

3. Rote learning is unavoidable

In addition, this study tends to show that learning *rules without reasons* (or rote learning) is an unavoidable learning approach in mathematics, particularly at the university level. The following category (1) response-data from participant F3 on rote learning is given here as an example: *Rote learning is getting to know your lecturers, memorise the formulae, and attempt all the tutorial exercises and available past examination papers. Rote learning is a trap many students and teachers fall into because it gives immediate results and as such many like myself find it difficult to give up using it.* Participant F3 continues: *Also, the present system of education, particularly at university where the aim is to pass rather than to understand what you were taught, encourages rote learning.* This kind of response tends to suggest that there were other factors (e.g. the goal to pass written examinations) which may have deterred the pre-service teachers from learning mathematics for conceptual or relational understanding.

Taking into consideration the above three factors: (1) deficiencies in lower-order concepts, (2) rote learning rules without reasons, and (3) unavoidable rote learning of mathematics, it could be suggested that the knowledge deficiencies observed (in relation to the TRIG, LOG, and STAT items) in category (1) response-data were the outcomes of rote learning, hence *rote knowledge* (Section 2.4, Chapter 2). This perspective also tends to provide an explanation for the high proportion of response-data classified as *instrumental understanding* of mathematics, this type of mathematical

understanding was suggested to be based on knowledge from rote learning (Skemp, 1978). This also suggests that the proportion (33%) of procedural and conceptual knowledge associated with *instrumental* mathematical understanding could well be the result of rote learning. These results tend to imply that *rote knowledge* is the dominant type of mathematical knowledge that the pre-service teachers brought with them to mathematics teacher education programs.

Should this outcome, dominance of *rote knowledge*, be the result of 'unavoidable' rote learning by the pre-service teachers because of the need to pass written examinations, then an important next step in this investigation of mathematical knowledge is to find out more about the role of *rote knowledge* in mathematical competence and its relationship to procedural and conceptual mathematical knowledge. The following discussion, therefore, is an attempt to re-examine category (1) and category (2) response-data in order to provide an alternative explanation to the views expressed above and to determine the value, if any, of *rote knowledge* to mathematical understanding.

6.2 An alternative view of rote knowledge in mathematics

The alternative view argued here, is that the *act of rote memorisation* (or rote learning) may not be the cause of deficiencies in mathematical knowledge. Rather, that the *mathematical content* being rote memorised by the individual was deficient of essential knowledge elements. This view is analogous to 'food consumption' in that it is not the 'act of eating' which could cause a deficiency in one's dietary requirements. Rather, it is the

'content' of the diet one indulges and consumes that can cause the deficiency. This perspective could be considered as an extension of Skemp's (1986, p.111) argument against having students to learn mathematical *rules without reasons*. That is, it is argued here that if students were presented with *mathematical contents* of abstract rules and formulae (or representations of concepts) and they perceived little or no real reason or purpose for learning these rules apart from that of being examinable material and for passing the course, then the rote memorisation of mathematical rules becomes 'unavoidable' (participant F3).

Under such systems of learning mathematics, it is not surprising that the pre-service teachers will bring with them to teacher education mostly, if not all, rote knowledge of mathematics. Other researchers in the field (e.g. Ball, 1990; Eisenhart *et al.*, 1993; Even, 1993; Wilson, 1994; Gates, 1995a) also indicated similar findings from their studies. It is important therefore, from the point of view of educating mathematics teachers, to consider closely how the pre-service teachers could be assisted with the reconstruction and transformation of *rote knowledge* they acquired into usable or teachable mathematical knowledge (Section 2.4, Chapter 2). An attempt to explore further this issue by considering the relationship of *rote knowledge* to procedural and conceptual knowledge is the focus of the following argument.

1. Rote knowledge as a source of supply for procedural knowledge

It was suggested in Chapter 2, Section (2.1) that mathematical knowledge is acquired in three stages, such that growth in conceptual and procedural knowledge (stage 3) is dependent upon the collection and formation of prior knowledge (stages 1 and 2). This growth is mathematical understanding,

which is important to mathematical teacher competence. Also in Chapter 2, Section (2.4), it was argued that rote knowledge (or factual knowledge) is a usable form of mathematical knowledge in providing links to other knowledges.

It is argued here that the observed insufficiency of conceptual knowledge, in this study, results from deficiencies in essential *knowledge links* which are required to connect new mathematical information to existing mathematical knowledge. When this 'connection' takes place, mathematical understanding occurs (Hiebert & Lefevre, 1986). The appropriate connection of new knowledge to existing knowledge was suggested in Chapter 2, Section (2.2) to be a function of procedural knowledge. Procedural knowledge is also required in the transformation of information to observable knowledge (Gagné, 1985; Alexander *et al.*, 1991; Derry, 1996). This implies that procedural knowledge is essential in knowledge production (Chapter 2, Figure 2.4) and a vital medium in the acquisition of conceptual knowledge (Chapter 2, Section 2.1). These functions are important to the acquisition of mathematical understanding.

However, it is proposed, based on the results of this study, that procedural knowledge alone is not sufficient in order to effectively activate the mental processes involved in knowledge acquisition (Section 5.2.1). These are the mental processes of: (1) transformation of information into knowledge, (2) connection of new knowledge appropriately to existing knowledge, and (3) translation and transfer of knowledge aspects to observable outcomes (Gagné, 1985; Alexander *et al.*, 1991; Derry 1996). It is argued here, that to effectively activate these processes, procedural knowledge requires a storage

mechanism or a *source of supply* to draw on. This *source of supply* can be appropriately produced by *rote knowledge*.

2. Rote knowledge is necessary at the initial stage of knowledge acquisition

At the initial stage of mathematical knowledge acquisition, rote knowledge is suggested to be present in the form of *information* (Ebel, 1972) or as factual knowledge (Goodwin & Klausmeier, 1975) or an intuitive schema (diSessa, 1993; Derry, 1996). This stage is important in developing a repertory of *information* as a *source of supply* for procedural knowledge. Rote memorisation or repetitive learning is one of the learning strategies associated with the acquisition of rote knowledge (Section 2.4, Chapter 2). Although these types of learning strategies and their outcomes (e.g. rote knowledge) were found by some researchers to inhibit conceptual understanding (e.g. Skemp, 1978; Rakow, 1992; Greenwood, 1993; Gadanidis, 1994; Gates, 1995a), others have found value in rote memorisation strategies (Brownell, 1956; Goodwin & Klausmeier, 1975; De Corte, 1995; Derry, 1996), while others observed learners with high achievements and understanding of mathematics using rote memorisation as a method for acquiring knowledge (Marton, Dall'Alba & Tse, 1993). Nevertheless, it is argued here, that without a supply of information or factual knowledge at the initial stage, procedural knowledge cannot be activated effectively to transform information into new knowledge. As Goodwin and Klausmeier (1975, p.242) maintained, 'without either having the knowledge [acquired by rote] or being able to obtain it [from a source of supply] when needed, the individual has nothing to apply or to evaluate'.

3. Rote knowledge as a source of knowledge links

Information (or intuitive schema) acquired by rote learning methods in the initial stage of knowledge acquisition, is a necessary source for the second stage of knowledge acquisition. This second stage is the formation of knowledge and relationships to other knowledge types (Section 2.1). It is proposed that when a respondent considers a mathematical cued-data (e.g. logarithm in LOG item) and provides the following response: *I can't remember. I know that there are log-laws but I can't recall what they are* (from category (1) response-data), it implies that the *information* about *log-laws* was acquired or rote learned (*I know that there are log-laws*), and it is in storage or memory. However, the 'knowledge links' required to activate procedural knowledge to transform the information (*log-laws*) into new knowledge and for the transfer into observable knowledge were not present (*I can't remember*). It is suggested from this example, that it is not just the lack of (or absence of) information (or intuitive schema) that restricts the functioning of procedural knowledge, but also that essential links (or knowledge links) to existing knowledge are deficient.

A '*knowledge link*' is defined here as *information* which contains an element(s) of knowledge; such knowledge, however, is already in existence in the individual's knowledge base.

More specifically, a deficiency in essential 'knowledge links' would suggest that the types of *mathematical information* (or content) being rote memorised by the individual contained little or no correlation with knowledge previously acquired or with prior knowledge.

It is argued that, the importance of *rote knowledge* as a source of supply in support of procedural knowledge is not in terms of quantity (the amount of facts) or how it was acquired (rote learned), but in terms of its quality (the type of facts). This quality is defined as the types of 'knowledge links' which are representative of knowledge already in existence in the individual's mathematical knowledge base. For example, the symbol, $\sqrt{\quad}$, is often learned by students as the 'square-root sign', and also, that $\sqrt{4} = 2$. If they were then given $4^{\frac{1}{2}}$ and asked to evaluate and simplify, the likely answer, if they have not learned that $\sqrt{4} = 4^{\frac{1}{2}}$, will be, *I don't know*. The *I don't know* response is the indicator that this information ($4^{\frac{1}{2}}$) is not representative of the knowledge already in memory. The point here is that, when students learned about 'square-roots', an essential 'knowledge link' ($x^{\frac{1}{2}}$, where x stands for any number) was lacking. In other words, the information (\sqrt{x}) that was rote memorised is deficient or the essential knowledge link ($x^{\frac{1}{2}}$) required for procedural knowledge to operate on $4^{\frac{1}{2}}$ was not part of the initial rote memorisation of the 'square-root sign'.

It is suggested that to activate the mental functionings of procedural knowledge, essential 'knowledge links' are required for the efficient processes of transformation and connection of newly acquired *information* to existing knowledge. Using this perspective, a function of rote knowledge is to maximise the functioning of procedural knowledge by providing a source of supply of 'knowledge links'. This function is required,

particularly, at the stage of transformation of acquired *information* into knowledge aspects, namely stage 2 (Section 2.1).

Based on this view of *rote knowledge*, it is suggested that the use of rote memorisation strategies by an individual may not cause a lack of conceptual knowledge as was suggested in Chapter 2 (Section 2.4). Rather, the *mathematical content* rote memorised was deficient of essential 'knowledge links' to existing procedural type knowledge. That is, rote memorisation of mathematical content containing 'knowledge links', or essential knowledge aspects, would enhance existing knowledge and effectively maximise processing of procedural knowledge, which in turn, facilitates the acquisition of conceptual knowledge (Derry, 1996). This theory of *rote knowledge* may explain why the Asian students, observed by Marton *et al.* (1993) to have used rote learning strategies to learn mathematics, gained higher achievements and more in-depth understanding of mathematics than their Australian counterparts.

As an example to demonstrate how deficiencies of 'knowledge links' in *rote knowledge* could retard the functioning of procedural knowledge, consider the response-data by participant F5:

Participant F5 (maths major from State B): Response-data for the STAT item. [Example of category (1) - undeveloped knowledge]

- (1) *I know that 'sigma squared' [the symbol is the concept] is the variance and the square-root of it is the standard deviation. But, looking at this [cued-data] I can't recall the formula. I don't really know or have an understanding of what a variance is. I just know it as 'sigma squared'. [no understanding, just knowledge of the symbol]*

It is suggested that participant F5 had acquired *information* about the concept, variance, by rote memorising the symbols for the variance and the standard deviation formulae (*I know that 'sigma squared' is the variance and the square-root of it is the standard deviation*). However, the essential 'knowledge links' to existing knowledge were absent when this *information* was acquired (*I just know it as 'sigma squared'*). As such, '*sigma squared*' remains a deficient type of *rote knowledge* because there were no appropriate links to activate procedural knowledge processing. In other words, '*sigma squared*' (rote knowledge) could not be transformed into new knowledge (*I don't really know or have an understanding of what a variance is*) or to be translated to observable outcomes (*But, looking at this [cued-data] I can't recall the formula*).

Having suggested the importance of rote knowledge to the acquisition of mathematical knowledge, and as an 'alternative' explanation to category (1) response-data, it is worthwhile at this stage to reconsider the four types of knowledge deficiencies, tentatively identified from category (1) response-data, in relation to *rote knowledge*.

Undeveloped knowledge was described in Chapter 5 (Section 5.2.1) as knowledge having a deficiency of knowledge elements that are required in the formation of conceptual knowledge. This type of deficiency was identified more with responses to the STAT item. The statistics involved, namely the value of the variance, is a single numerical figure. However, the understanding of this 'value' lies in the computations or in the manipulation of symbols associated with the formula (or the algebraic representation of the concept variance).

The symbolisation of the variance formula involves several symbols (e.g. σ^2 , Σ , $\sqrt{\quad}$), each of which may signify both an operation and the order of that operation. It is suggested that *undeveloped knowledge* was the result of rote memorisation of the variance formula without appropriate 'knowledge links' for the symbols and their computational aspects to existing procedural knowledge. For instance, $\Sigma(x - \bar{x})^2$ signifies that each 'deviation from the mean', $(x - \bar{x})$, is 'squared' followed by the computation of the 'sum'. Without the appropriate 'knowledge links', the end result is knowledge deficiency as illustrated by participant F5's knowledge of the variance as '*sigma squared*'. That is, the *information* (rote knowledge) cannot be developed into new or usable knowledge or be used for activating procedural knowledge.

Unproductive knowledge, on the other hand, was described in Section (5.2.1) as internalised knowledge cues of external stimuli which the individual can use to facilitate memory recall. Calculators and textbooks are examples of external stimuli to assist with accessing, what is believed to be, appropriate knowledge for a given situation. *Unproductive information* was demonstrated in responses such as *a calculator contains many formulae in its memory and it can do computations much quicker than I can, so why try to commit these formulae to memory, or, there are many excellent textbooks containing information on mathematics now available to students and teachers, so if I need to know any formula I just look for it in a textbook.* This type of information does not require 'knowledge links' or elements in the existing knowledge base because its operant does not appear to be on internal mental functioning but on the external environment. Therefore,

this type of information will contribute little to the production of mathematical knowledge (Figure 2.4, Chapter 2). However, some of the symbolisation of formulae and algorithms associated with *unproductive information* may be retained in memory by the individual as *specific forms* or 'unrelated/unfamiliar' *information*.

Unrelated/unfamiliar knowledge is similar to *unproductive* information in that, it also relies on external stimuli. However, the external stimuli are internalised as specific knowledge elements of symbols, rules, algorithms and mathematical terms. As such, accessibility of internalised *unrelated/unfamiliar information* (rote knowledge) is highly dependent on an 'exact match' with the external cues in order to activate procedural knowledge processing to transform this type of *information* into new or usable knowledge.

The specific knowledge elements associated with *unrelated/unfamiliar* knowledge deficiencies could be similar to what Bastick (1993) has defined as *context cues*. Context cues are implicit and not directly transferable into new learning contexts, and as such, students 'cannot independently initiate them for themselves or use them outside the classroom' (Bastick, 1993, p.87).

Unprocessed knowledge was described in Section (5.2.1) as knowledge elements internalised by the individual as 'fixed' symbols, rules, and algorithms without links to knowledge pertaining to understanding. However, *unprocessed knowledge* of rules and algorithms appears to be a deficiency associated with procedural knowledge rather than that of rote knowledge. It is proposed that *unprocessed knowledge* is a type of procedural knowledge which is deficient of appropriate linkages to existing

knowledge in the domain of conceptual knowledge. This deficiency, which was identified as the respondent's inability to *explain the why* of rules and formulae, is suggested to be a 'breakdown' in procedural knowledge interaction with conceptual knowledge. This proposal is based on the model of a *response production of mathematical knowledge* suggested in Chapter 2, Section (2.2) and illustrated in Figure 2.4. The interaction between procedural and conceptual knowledge was illustrated in Figure 2.4 by a 'two-way flow' between EASY ACCESS MEANING (procedural domain) and CONCEPT MEANING (conceptual domain). As an example to illustrate this 'breakdown', consider the response-data from category (2), representative of procedural knowledge, by participant F6 below:

Participant F6 (maths major from State A): Response-data for the LOG item.

- (1) $\log(2x+1) \neq \log 2x + \log 1$. $\log[(2x+1)/(x-1)] = 0$ because of the rule: $\log(a/b) = \log a - \log b$ [procedural knowledge]. But I don't really know the reason for this [unprocessed knowledge]. And I don't know what to do next. [F6 has procedural knowledge of logarithmic laws but lacks conceptual knowledge which gives meaning to the laws and facilitates computational knowledge.]

Participant F6 is assumed to have learned a rule $(\log[(2x+1)/(x-1)] = 0$ because of the rule: $\log(a/b) = \log a - \log b$), and when this rule was learned it contained elements of procedural knowledge that were already part of F6's prior or existing knowledge. These elements are suggested to be knowledge that participant F6 might have in memory from prior learning involving 'indices' (e.g. 'division links with subtraction' and 'multiplication links with addition'). Having these procedural elements present in the existing knowledge base made it possible for the connection of the 'given cued-data'

to existing rules and the immediate transfer to the observed outcomes ($\log(2x+1) \neq \log 2x + \log 1$). However, in order for this procedural knowledge (or unprocessed knowledge: *the rule: $\log(a/b) = \log a - \log b$*) to be connected appropriately to conceptual knowledge, it seems that a vital 'knowledge link' has not yet formed (*But I don't really know the reason for this*). This knowledge link (the 'two-way' flow in Figure 2.4) or *knowledge of why and how a rule works* is fundamental to the achievement of mathematical understanding. It is suggested that without the *knowledge of why and how a rule works*, procedural knowledge cannot be activated to translate and transfer conceptual knowledge to observable outcomes, resulting in outcomes such as, *And I don't know what to do next*.

In summary, the knowledge deficiencies tentatively identified in category (1) response-data appear to reflect deficiencies associated with rote knowledge, with the exception of 'unprocessed knowledge'. Unprocessed knowledge represents a deficiency of 'knowledge links' assumed to be necessary for the interconnection and interaction between procedural and conceptual types of mathematical knowledge (Figure 2.4, Chapter 2).

Although this alternative view of *rote knowledge* needs further exploration, it does, however, provide an insight into how *rote knowledge* could be transformed into a source of usable knowledge for relational understanding of mathematics. The success of this transformation (a function associated with procedural knowledge) is dependent on both the kinds of mathematics rote learned and the existence of appropriate prior knowledge in the individual's mathematical knowledge base. The main function of *rote knowledge* in the process of mathematical knowledge acquisition, is to maximise the mental processing of procedural knowledge which in turn

interacts with conceptual knowledge, resulting in mathematical understanding. In this way, *rote knowledge* is an essential factor in achieving relational understanding of mathematics.

It could be suggested from this alternative view of *rote knowledge* that the insufficiencies in conceptual understanding of the mathematics observed in this study, is more a reflection of the low quality of mathematics that was taught to the pre-service teachers, rather than of the pre-service teachers having used rote memorisation strategies.

This study of pre-service teachers' existing mathematical knowledge also highlighted the fact that many other factors, including the individual's goal for acquiring knowledge, are involved in knowledge acquisition (Alexander, 1995; De Corte, 1995; Derry, 1996). Although this factor (learning goal) was not specifically examined in this study, it was, however, an important contributing element to consider when making inferences from the study outcomes. The individuals' choices to select which mathematics to learn and how to learn in order to accommodate their learning goals may have a compounding affect on the instrumental understanding of mathematics observed in this study. For example, an 'unproductive' form of knowledge deficiency appears to be associated with a deliberate decision by the individual to keep certain information or knowledge as 'external source items' (e.g. calculators and textbooks) rather than committing this information to memory. Also, it is suggested that the deficiency related to 'unprocessed knowledge' may have been induced by the individual's learning goal. That is, if the goal for mathematical knowledge acquisition was for passing an examination, then it is likely that conceptual knowledge may be suppressed or considered by the individual as unnecessary.

Another factor that seemed to relate to the individual's goal and could have an influence on the individual's existing mathematical knowledge, is the initial career pathway chosen by the individual. This study did not have a sufficiently large sample to provide the data to explore this issue, but there were indicators suggesting that the pre-service teachers' initial goal for studying mathematics was not for a career in teaching, but rather for a career in other areas such as Engineering, Commerce and Computer programming. The affects of this change in career pathways on the pre-service teachers' potential to study mathematics for teaching needs further investigation.

Further discussion of these knowledge outcomes in relation to teaching is the topic of the next section.

6.3 Addressing the second research question

2. What possible influences could the identified deficiencies in the types of procedural and conceptual mathematical knowledge have on the teaching of mathematics?

The results of the analysis of the pre-service teachers' mathematical knowledge bases indicated more existing rote and procedural knowledge than conceptual knowledge. The potential effects of these knowledge types (or lack of conceptual knowledge) on teaching is the question that is addressed in this section. The research assumptions which formed the basis for this question were:

- (1) Pre-service mathematics teachers with relational understanding of mathematics would demonstrate more confidence to teach mathematics than pre-service teachers with instrumental understanding.
- (2) Pre-service mathematics teachers go through their teacher education and training with certain deficiencies in their mathematical understandings and that these deficiencies will eventually affect the way they teach.

With respect to the first assumption, the response-data (responses to stimulus questions 2 and 3 (SQ2, SQ3)) examined in relation to the teaching of mathematics showed evidence of the pre-service teachers' lack of confidence and uncertainties in teaching the mathematics (represented in the TRIG, LOG, and STAT items) regardless of their mathematical background (Section 5.2.2; Section 5.2.3). Although 5% of the response-data displayed relational type knowledge (Section 5.2.1), the analysis in Section (5.2.2) and Section (5.2.3) did not show any evidence in support of assumption (1). Rather, it appears that the deficiencies in the respondents' conceptual knowledge, (i) would create for them uncertainty and lack of confidence in their own understanding. This lack of confidence, (ii) encourages a dependency on "something" which may provide misleading information when seeking more understanding in mathematics, and (iii) promotes replicative teaching or teaching the same way as being taught. The following response-data are examples showing evidence of these three factors as consequences of having rote knowledge.

Rote knowledge of mathematics:

(i) creates uncertainty and lack of confidence in the respondents' understanding. For example, *I don't really know* [referring to the STAT item], *I guess I will have to follow the syllabus*; *To teach variance I'll introduce standard deviation first. But if the syllabus indicates the variance first then I'll do that (ie. just follow whatever is in the syllabus)*; *Hopefully I don't have to teach a year 11 and 12 class* [referring to the STAT item]; *I will teach whatever and however it's required in the syllabus*; *Probably start with graphs* [referring to the LOG item]. *But I'll just follow the syllabus. Usually there are textbooks available for teachers and students to use.*

(ii) encourages a dependency on 'something' (*syllabus, textbooks or resource materials*) rather than on 'someone' (*mathematician or mathematics educator*) when essential knowledge for teaching was lacking. For example, *I don't know. I need to do more research and see what is required in the syllabus*; *I don't really know, I guess I will have to follow the syllabus*; *If I need more clarification I will refer to a good statistic textbook*; *I will teach whatever and however it's required in the syllabus.*

(iii) promotes replicative teaching or teaching the same way as one was taught. For example, *Start [teaching logarithm] with log-laws. That's because it's the way I usually work. That is, find a rule and follow that; I'd teach [logarithm] using calculators because that is how I was taught; I only know what I've been taught. So I will start by emphasising the formula for the variance to derive the standard deviation. Then provide problems [practice work exercises] ... to make sure students understand. Also emphasise the correct use of the calculator.*

Procedural knowledge on its own without links to conceptual knowledge was suggested in Chapter 2, Section (2.3) as a deficiency associated with mathematical competence. The predominance of this type of knowledge in category (2) and category (3) suggests that it may be connected with the pre-service teachers' 'visual' and 'abstract' perceptions of concepts. A gender difference in pedagogical knowledge associated with the 'visual' and 'abstract' perspectives was observed. For example, the respondents with category (3) type responses were all males. The main criterion for the classification of responses into category (3) was based on evidence showing the individual's ability to do computations. It could be suggested that the males' abilities to do computations may have influenced them to decide on using strategies involving algebraic procedures and formulae (or abstract rules and formulae) as appropriate methods for teaching students trigonometry and logarithms.

On the other hand, the classification into category (2) was based on evidence showing the individual's ability to recognise appropriate mathematical aspects or 'distinguishable features' relating to the stimulus item. The ability to recognise or to identify and differentiate one mathematical aspect from another by using 'distinguishable features' such as graphical representations was observed more in response-data from the females, particularly with the TRIG item, than in response-data from males. For example, participant F6's responses on student learning of trigonometry, *I believe it is important for students to understand trig-functions and their graphs. Because graphs distinguish trigonometry from other functions.*

The gender differences in teaching approaches observed in this study have some similarities to the findings of a study on primary school children's abilities to solve mathematical problems (Fennema & Carpenter, 1998). Fennema and Carpenter (1998) found that girls tended to approach the solving of mathematical problems using concrete or visual modeling while the boys tended to use abstract methods.

However, it appears that with mathematical situations such as statistics in which rote knowledge is the dominant source and basis for making pedagogical decisions by both females and males, gender difference in teaching is negligible. This observation tends to reaffirm Sowder's (1998, p.13) fear of mathematics teachers using 'traditional style of teaching, where emphasis is placed on rote learning of rules', in that, she believes this traditional teaching style would 'tend to better equalise the [differing] advantages of the girls and the boys'.

Although the study involving the 18 experienced mathematics teachers was mainly for trialing and testing the data-collection instrument (Section 4.2.3, Chapter 4), two important observations that relate to the teaching of mathematics at the secondary school level are worthy of mentioning here. First, it was observed that the mathematics course, teachers currently and frequently teach, seemed to have a considerable influence on the teachers' understanding of the mathematics they teach. Second, it was suggested, as an outcome of this item validation study, that the sampled teachers tend to perceive *the act of teaching* as an important factor for their growth in mathematical understanding, rather than to gain this growth from understanding the mathematics they teach.

The second assumption is now addressed: *Pre-service mathematics teachers go through their teacher education and training with certain deficiencies in their mathematical understandings and that these deficiencies will eventually affect the way they teach.* It could be suggested from the results and findings of this study that the deficiencies in mathematical understanding, namely rote knowledge and procedural knowledge, would reduce the potential of the pre-service teachers to gain conceptual understanding of mathematics from studies in teacher education programs. Therefore, if the pre-service teachers' mathematical understanding is deficient, then the conceptual knowledge structures upon which their pedagogical knowledge is based, would also be deficient. Hence, the pre-service teachers' rote knowledge and procedural knowledge of mathematics would eventually reduce their confidence to teach students for conceptual understanding of mathematics.

An important question now is: *what can the pre-service teachers with such insufficiencies in mathematical knowledge do in order to gain relational or conceptual understanding of mathematics?*

It has been suggested by Skemp (1986) that acquiring an understanding of higher-order mathematical concepts (relational understanding) is a difficult process because,

Mathematics cannot be learnt directly from the everyday environment, but only indirectly from other *mathematicians*, in conjunction with one's own reflective intelligence. (Skemp, 1986, p.30, italics added)

Therefore, it is argued here, in the light of the present findings, that pre-service teachers of mathematics need mathematics teacher educators that have the

mathematical competence to guide them through their reconstruction of 'instrumental' mathematical understanding. Competent teacher educators can help pre-service teachers in: (i) identifying their areas of weaknesses or 'injuries' (or deficiencies of knowledge links) caused by past teaching, and (ii) repairing and perhaps developing 'primary abstractions' of some of the concepts (Skemp, 1986, p.111).

Such assistance should provide confidence and encourage the pre-service teachers to seek help from "someone" (*mathematician or mathematics educator*), when they know they have a deficiency of essential mathematical knowledge, rather than placing a reliance on "something" (*syllabus, textbooks or resource materials*).

In summary, the outcomes of the study of secondary pre-service mathematics teachers' existing mathematical knowledge indicated that these pre-service teachers brought with them to teacher education mainly procedural and rote knowledge of mathematics. The successful completion of studies in mathematics during high school and university years seemed to have provided these pre-service teachers with more instrumental understanding of mathematics than relational understanding.

It was suggested that instrumental understanding of mathematics may be an unavoidable result of studying mathematics, particularly at the university level, because certain study requirements (e.g. completion of study within a set time and to pass examinations) do not always appear to facilitate the achievement of relational understanding of mathematics. Assuming that this might be the case (unavoidable rote learning of mathematics) for most of the pre-service teachers who participated in this study, an alternative

interpretation of the outcomes of the study was proposed. This alternative view focused on rote knowledge and how this type of mathematical knowledge could be transformed and reconstructed into 'usable' and teachable knowledge. It was suggested that learning mathematical content containing essential knowledge linkages to knowledge already in existence in the individual's knowledge base is an appropriate method for transforming rote knowledge into usable mathematical knowledge.

In addition, the deficiencies in conceptual mathematical knowledge appeared to reduce teacher-confidence and likewise the potential of the pre-service teachers to teach for conceptual understanding of mathematics. In order to assist the pre-service teachers in their reconstruction of rote knowledge and to gain confidence in their own mathematical understanding, it was suggested that competent mathematics teacher educators are important to ensure that pre-service teachers receive the appropriate mathematical environment that promotes reconstruction of knowledge.

The purpose of this chapter, Chapter 6, was to discuss the outcomes of the study in terms of the research assumptions and the research questions. Having achieved this, it is important to review the significant points of this study of secondary mathematics pre-service teachers to draw out conclusions following from the data analyses and discussions and to suggest areas for further research. This then is the purpose of the next chapter, Chapter 7.

CHAPTER SEVEN

CONCLUSION

7.1 Review

This study arose from the concern that while mathematics teacher education is providing essential skills and knowledge for the development of teacher mathematical competence, there was evidence of teachers lacking competence in teaching mathematics. This lack of teacher mathematical competence could be related to certain insufficiencies in mathematical knowledge that teachers had prior to teacher employment.

Two types of mathematical knowledge were identified in the reviewed literature as contributing to mathematical competence, namely procedural and conceptual knowledge. Both of these knowledge types were suggested as necessary and essential components of mathematical competence, with the latter being especially desirable for teachers' mathematical competence.

Rote knowledge, the outcome of rote learning, was described as another type of mathematical knowledge. This type of knowledge may not promote mathematical understanding. However, it was suggested that it might be possible, with special attention by the learner, to transform rote knowledge (or factual knowledge) into usable knowledge.

Competency to teach mathematics was suggested to involve the interaction between mathematical knowledge and pedagogical knowledge. Although

this interaction has not been fully investigated and understood, recent studies indicated that pedagogy is directly influenced and shaped by mathematical knowledge.

This study focused on the types of mathematical knowledge elements associated with knowledge insufficiencies in an endeavour to determine how these insufficiencies may influence mathematical competence with respect to teaching. In order to explore what these knowledge insufficiencies might be, 19 secondary pre-service mathematics teachers' existing mathematical knowledge was investigated. A multiple-case study design was used in which the selection of the 19 cases was by replication procedures. The 19 participants were selected from four universities in two Australian states. These cases were selected according to the participants' mathematical background and the teacher education program they enrolled in (Diploma of Education 'maths major', Diploma of Education 'maths minor', and Bachelor of Education in secondary mathematics 'maths minor').

In addition to 19 pre-service mathematics teachers, a sample of 18 experienced secondary mathematics teachers participated in the study for validating the stimulus items. These experienced teachers were from a number of colleges, both state and private, in one of the states where some of the pre-service teacher participants were selected. The item validation study was conducted to explore the validity of the data collection instrument and to examine a model for analysing data on mathematical knowledge. The outcomes of the analysis of data from the item validation study provided useful information for the improvement of the data collection instrument and data organisation procedures.

A set of three mathematical stimulus items constituted the data collection instrument. The sampling of the stimulus items was conducted at two educational settings, namely a college and a university. The original source of these stimulus items was from assessment instruments constructed by mathematics teachers for the evaluation of their students' learning in trigonometry, logarithm, or statistics. Common misconceptions and misunderstandings associated with learning the mathematical concepts represented in the 'original' assessment items were incorporated as cued-data in the stimulus items.

Responses to the stimulus items were collected by semi-structured interviews. It is acknowledged that the mathematics represented by these items represents a very small area of mathematical knowledge a pre-service teacher of mathematics would have acquired from schooling and tertiary education. As such, the responses generated by these stimulus items and the study results should be viewed as representative rather than definitive in nature.

The model of mathematical understanding by Skemp (1979) was used to analyse the response-data for both the item validation study and the main study. The response-data were examined according to Skemp's (1979) three types of mathematical understanding, namely instrumental, relational, and symbolic. To assist classification of the response-data into these three types of mathematical understanding, four response categories were used. Category (1) represented responses showing evidence of knowledge deficiencies. Category (2) and category (3) represented responses with evidence of knowledge types belonging to instrumental and relational

understanding, whilst the category (4) type response-data were representative of symbolic understanding.

The overall result from the analysis of the study data indicated a high proportion (44%) of responses in category (1). Although 56% of responses were classified as category (2) and category (3) type data and were representative of instrumental and relational understanding, these responses were mostly of procedural types of mathematical knowledge (51%). There were no responses which were representative of symbolic understanding.

The majority of the knowledge outcomes, therefore, represent deficiencies in conceptual mathematical knowledge. Conceptual knowledge is essential in relational or substantive understanding of mathematics (Skemp, 1978; Ball, 1990). Furthermore, it was observed that mathematical knowledge deficiencies can reduce teacher confidence and the potential of a pre-service teacher to teach for conceptual understanding of mathematics.

A gender difference in mathematical knowledge was observed in relation to category (3) type data with more males indicating computational knowledge of trigonometry, logarithm and statistics than females. In addition, males were more likely to use algebraic procedures and formulation in teaching students trigonometry and logarithm than females. The females, on the other hand, tended to use graphical representation at the initial stage of teaching trigonometry and logarithm or prior to the teaching of algebraic procedures and formulae. However, with respect to statistics, the gender differences in mathematical knowledge and teaching approaches were 'equalised', particularly in relation to teaching strategies.

7.2 Conclusion

In conclusion, the mathematical knowledge pertaining to trigonometry, logarithm, and statistics, that the secondary pre-service mathematics teachers brought with them to teacher education programs, was limited to mathematical knowledge based on rote knowledge and procedural knowledge of mathematics. It was argued that these types of knowledge were representative of mathematical knowledge deficiencies. It was also argued that the deficiencies of lower-order mathematical concepts is one likely cause for the deficiencies of higher-order mathematical concepts.

Another likely cause of these deficiencies in mathematical knowledge, concerns the deficits in essential elements of procedural knowledge. It was argued in Chapter 6 (Section 6.2) that rote learning mathematics (or the use of rote memorisation strategies) may not be the cause of conceptual knowledge insufficiencies, but rather, it is caused by the deficiencies of essential knowledge aspects in the content of the mathematics rote learned. In addition, it was suggested that rote knowledge is an essential factor in the acquisition of mathematical understanding. However, the individual's goal in learning and one's freedom to be selective of information to commit to memory, appear to influence the potential of rote knowledge as a usable type of knowledge.

Furthermore, the existence of knowledge deficiencies in the pre-service teachers' existing mathematical knowledge bases was observed to generate lack of confidence and reduce the pre-service teachers' potential to teach for conceptual understanding of mathematics. It was suggested that, one of the ways for increasing this potential is to increase the development of higher-

order mathematical schemata. It was also suggested that the formation, or lack thereof, of higher-order schemata is dependent on the individual's goal in learning. That is, if one's goal is to acquire in-depth understanding of the mathematics (e.g. statistics), then this places greater demands upon the mental processes involved with the 'statistics schema'. Such demands would facilitate the processes of integration and assimilation of the lower-order schemata to higher-order schemata. If this is the case, then it is essential for pre-service teachers to endeavour to identify areas of mathematical weakness and increase demand on these mathematical schemata in their pursuit to become competent teachers of mathematics.

The gender differences in teaching approaches observed in this study were interlinked with the pre-service teachers prior learning experiences, particularly with experiences of how they were taught the mathematics. Although females were more likely to teach using 'visual' representations and males to teach using 'abstract formulations', these differences were not present when essential conceptual knowledge was lacking. Furthermore, it was observed that having conceptual knowledge which is associated with computational knowledge (e.g. category (3) relational knowledge) was not sufficient to engender confidence to teach for conceptual understanding of mathematics. In addition, any differences in how females and males acquired mathematical knowledge and differences in how they would teach this knowledge were 'equalised' if pre-service teachers did not have the confidence in their own understanding of mathematics. The result of this lack of confidence, regardless of gender, is the teacher's dependency on 'unproductive' external knowledge or knowledge 'kept' in textbooks and calculators or computer memories.

In addition, the outcomes of this study did not support the assumption that the pre-service teachers of secondary mathematics had gained the necessary pre-requisites (conceptual mathematical knowledge) by completing a university degree in mathematics. Likewise, there was little evidence of relational understanding to support the assumption that pre-service teachers with relational understanding would demonstrate more confidence to teach mathematics than those with instrumental understanding. Rather, there was evidence to support the assumption that pre-service teachers go through teacher education with insufficiencies in their understanding of mathematics. These insufficiencies were exemplified by the pre-service teachers' lack of confidence in their own understanding, misconceptions, and gaps in their knowledge. Such evidence would suggest that if the pre-service teachers do not receive appropriate help, then the insufficiencies they had would eventually affect their competence in teaching mathematics.

This study has provided empirical evidence showing that pre-service teachers of mathematics should be provided, at the mathematics teacher education level, with opportunities to (i) examine and evaluate their mathematical knowledge, and (ii) reconstruct and transform the insufficiencies of their existing knowledge to teachable mathematical knowledge, prior to employment as classroom teachers. This process of self-examination and self-evaluation by the pre-service teachers should involve the assistance of a competent mathematics teacher educator in order to ensure that essential mathematical knowledge has been acquired.

7.3 Implications of the study

The results discussed in Chapter 6 highlighted some of the influences of mathematical knowledge deficiencies on a pre-service teacher's mathematical competence. It was suggested that the observed knowledge insufficiencies associated with rote knowledge of mathematics may relate more to the quality of the mathematical content acquired by the pre-service teachers, rather than how the content was learned, for example, by rote memorisation. That is, the pre-service teachers' lack of essential conceptual knowledge is more akin to the kinds of mathematics they acquired, than to their mathematical abilities and methods they used to acquire the mathematics.

An important implication of these results to mathematics teacher educators, the author being one of them, is not to assume that pre-service teachers of secondary mathematics have acquired from their prior learning the necessary mathematical knowledge pre-requisites needed for transformation or reconstruction into teachable mathematical knowledge. Instead, mathematics teacher educators should acknowledge that pre-service mathematics teachers will bring to teacher education mostly rote knowledge of mathematics. As such, it is essential that pre-service mathematics teachers should be provided with learning environments that would allow them to examine their knowledge and for them to identify deficiencies or gaps in their prior knowledge. Furthermore, the pre-service teachers should be presented with mathematical contents that promote the enhancement, readaptation and reconstruction of existing knowledge, which includes rote knowledge, into usable and teachable knowledge prior to teacher employment. It is vitally important for the pre-service teachers to achieve this knowledge prior to employment because, as indicated from the outcomes of the study with the

18 experienced teachers, there is little assurance that the experience gained from teaching mathematics would provide the necessary conceptual understanding and confidence needed to teach secondary level mathematics.

It was also observed in this study that, the pre-service teachers' decisions to consult a *textbook* or the *syllabus*, when they *don't know how to teach a topic*, were mostly associated with rote knowledge and instrumental understanding. This kind of decision has several important implications concerning a pre-service teacher's potential to become an effective and a competent classroom teacher:

- (1) It implies that pre-service teachers perceive the teaching of mathematics as more to do with the availability of appropriate resource materials than to understand the content of these materials.
- (2) It implies that the pre-service teachers had not reached the stage of 'readiness' to move into a goal (or role) reversal situation, for example, from a mathematics student for physics, chemistry, economics, or engineering to a secondary mathematics student-teacher.
- (3) It implies that the mathematics teacher educators, when instructing pre-service teachers, may have over-emphasised the benefits of teacher resource materials (or a reliance on 'something' rather than on 'someone') and not placed enough emphasis on how to gain a conceptual understanding of the mathematics to be taught.
- (4) It implies a 'vicious circle' in the understanding and teaching of mathematics. The teaching of mathematics to students by a teacher whose understanding is based on instrumental mathematical understanding would result in the students also acquiring instrumental understanding of mathematics.

The four aspects stated above further suggest the importance that, mathematics teacher educators provide the secondary pre-service teachers with appropriate assistance prior to teacher employment. Once employed, the new teachers should be encouraged to continue examining and re-assessing their respective situations with the help of other competent mathematics colleagues.

Furthermore, although the focus of this study was on secondary pre-service teachers' mathematical knowledge, the results also have important implications for the education of primary pre-service teachers. This study showed that these pre-service mathematics teachers with 'maths major' and 'maths minor' backgrounds had insufficiencies in their conceptual understanding of mathematics. It is implied by this finding that prospective teachers of primary school mathematics, with little or no tertiary level mathematics, would inevitably go through pre-service teacher education with even less conceptual mathematical understanding. As a consequence, the teaching of important lower-order mathematical concepts or intuitive schemata at primary school levels would most likely be based on rote knowledge and instrumental understanding of mathematics. The result of this teaching would be the continuation of the 'vicious circle' of learning and teaching for instrumental understanding of mathematics. This 'cycle' must be broken. From the results of this study, it could be suggested that the appropriate 'break' point is at the teacher education level. This implies that competent mathematics teacher-educators are needed to ensure that prospective teachers, for all school levels of mathematics, gain effective reconstruction of mathematical knowledge.

In addition, it was demonstrated in Chapter 6, Section (6.2) that the four types (or descriptors) of mathematical knowledge deficiencies, identified from the analysis of the present data, provided a meaningful evaluation of the quality of *rote knowledge*. An important implication for research involving mathematical understanding is the potential use of these 'descriptors' for qualitative analysis of mathematical knowledge.

7.4 Limitations and recommendations for further research

One of the inherent limitations of the study is related to the mathematical stimulus items chosen as the data collection instrument. These items were based on logarithm, trigonometry, and statistics, but other mathematical areas might be considered for investigation in follow-up research.

Another limitation of the study is the adoption of a single method for qualitative analysis, the model of mathematical understanding (Skemp, 1978). Although this model was valuable in classifying mathematical knowledge on teaching and according to the individual's mathematical understanding, there may well be other methods that might be considered in follow-up studies, for example, a triangulation of methods of analysis.

This study was an extension of previous research (e.g. Ball, 1990; Eisenhart *et al.*, 1993; Even, 1993) on how prospective teachers could be helped to transform their knowledge insufficiencies and to increase their understanding of mathematics. This extension focussed on a more refined characterisation of mathematical knowledge deficiencies prospective teachers brought with them to pre-service teacher education programs (Chapter 2, Section 2.7). Such characterisation of knowledge deficiencies is of value to teacher-

educators for planning programs which would help prospective teachers in reconstructing their knowledge insufficiencies (Chapter 1, Section 1.4). This study has highlighted four characteristics of knowledge deficiencies in mathematics. These deficiencies were associated with *rote knowledge* (outcome of rote learning) and appear to have the potential to undermine the basic structure of mathematical understanding upon which mathematical teacher competence (pedagogical knowledge) is established. However, these findings (knowledge deficiencies) are limited to knowledge structures (or schemata) associated with trigonometry, logarithm, and statistics. As a follow-up study, it is recommended that there be further investigation of secondary pre-service teachers' mathematical knowledge insufficiencies taking into consideration other mathematical areas.

In addition, it was suggested, as an approach for transforming deficiencies in mathematical knowledge to 'usable knowledge', that prospective teachers, with assistance from teacher-educators, need to identify their areas of weakness and endeavour to gain conceptual understanding of these areas by learning mathematical contents which contain essential 'knowledge links' (Chapter 6, Section 6.2). An additional follow-up study would be to explore approaches, other than the one suggested here, for transforming knowledge such as *rote knowledge* into usable and teachable mathematical knowledge.

Furthermore, several of the participants in this research did not study university mathematics with the goal to enter a mathematics pre-service teacher education program. It was suggested that not having the initial goal to study mathematics for the purpose of becoming a teacher may have a bearing on the pre-service teachers' existing mathematical knowledge.

Information on the affect of a change of career goal in favour of teaching would also be valuable to teacher-educators in their planning and developing of mathematics curriculum for pre-service teacher education programs. As such, another recommended follow-up study would be to investigate how a change of career pathway in favour of teaching mathematics would influence a pre-service teacher's ability to acquire the competency to teach mathematics.

Information from such studies would greatly assist our understanding of how teacher mathematical competence promotes quality in the teaching of mathematics.

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APPENDIX A

Table A1: Summary of experienced mathematics teachers' interview data

No. of teaching years	Degree other than Dip.Ed	Maths currently teaching	LOG Category: 1,2,3,4/ Skemp	TRIG Category: 1,2,3,4/ Skemp	STAT Category: 1,2,3,4/ Skemp	Maths taught
18	B.Sc, M.Ed	Stage 3 GenM	4 Symbolic (Reflective)	4 Symbolic (Reflective)	4 Symbolic (Reflective)	All college levels.
30	B.Ed	Computing	4 Symbolic (Reflective)	3 Relational	3 Relational	All high school levels
6	B.Ed (Maths major)	GenM AppM	4 Symbolic (Intuitive)	4 Symbolic (Intuitive)	1 Instrumental	Part-time teaching
15	B.Ed (Maths major)	AppM	3 Instrumental	2 Instrumental	1 Instrumental	All high school levels
20	B.Sc	GenM AppM	3 Instrumental	3 Relational	3 Instrumental	All high school levels
13	B.Sc	GenM AppM	2 Instrumental	4 Symbolic (Intuitive)	3 Instrumental	All college levels.
12	B.Sc	GenM AppM	2 Instrumental	4 Symbolic (Intuitive)	3 Instrumental	All college levels+physics
3	M.Sc	High Sch	3 Instrumental	3 Instrumental	1 Instrumental	yr 8, 9, 10 + Science
27	B.Sc	AppM	2 Instrumental	3 Instrumental	4 Symbolic (Intuitive)	All college levels.
9	B.Ed	GenM AppM	2 Instrumental	2 Instrumental	1 Instrumental	Part-time teaching
16	B.Sc	Stage 1,2 Physics	4 Symbolic (Reflective)	4 Symbolic (Intuitive)	2 Instrumental	Top college level maths + physics
18	B.Sc	Stage 2,3 GenM	4 Symbolic (Intuitive)	2 Instrumental	1 Instrumental	All college levels.
30	B.Sc	AppM	2 Instrumental	2 Instrumental	2 Instrumental	All college levels.
15	M.Ed	AppM	3 Instrumental	1 Instrumental	1 Instrumental	Lower college levels.
11	B.Sc (Maths major)	AppM	3 Instrumental	3 Instrumental	4 Symbolic (Intuitive)	All college levels.
19	B.Sc (Biology)	AppM GenM	1 Instrumental	1 Instrumental	1 Instrumental	Lower college levels.
10	B.Sc	AppM	1 Instrumental	2 Instrumental	1 Instrumental	No data
11	B.Sc	Stage 2,3	4 Symbolic (Intuitive)	3 Instrumental	3 Instrumental	All college levels.

Legend:

GenM: General Mathematics
AppM: Applied Mathematics

Stage 1,2,3: Mathematics Stage 1,2,3
High Sch: High School Mathematics

APPENDIX B

Figure B1: Summarised interview response-data from the nineteen pre-service mathematics teachers

FEMALE 1: A2	TRIG-Category (2) Instrument	LOG - Category (1)	STAT-Category (2) Instrument
B.Ed maths/sci (Final Year)	Identified student error: <i>Students have confused trig with algebra by considering 'cos' as a variable and used the distributive law. ie. $\cos(2x+1) \neq \cos 2x + \cos 1$</i>	<i>I've forgotten. I don't know how to do this myself so I can't really say whether the student is right or wrong. I would need to look up a textbook to remind myself again.</i>	<i>The variance is the sum of the deviations from the mean, then divided by (n-1) because it is a sample. So it's 2.4 but the sigma is 1 to 9, but there are 10 items. Now I'm confused and I don't know much about the variance to be able to explain why.</i>
Important learning	<i>To understand the concept of trig-functions and their graphical representations.</i>	<i>To understand graphing, ie. seeing what a log-graph looks like.</i>	<i>Understanding what the formula is all about.</i>
Teaching approach	<i>Teach students to understand: 1. Why they need to learn trig-functions. 2. Graphs of trig-functions. 3. Trig-ratios in relation to right-angled triangles.</i>	<i>Probably start with graphs. But I'll just follow the syllabus. Usually there are textbooks available for teachers and students to use.</i>	<i>I only know what I've been taught, so I will start: 1. I will emphasise the formula for the variance to derive the standard deviation. 2. Give problems using the standard deviation & make sure the students understand. Also emphasise the correct use of calculators.</i>
FEMALE 2: B2	TRIG - Category (1)	LOG-Category (2) Instrument	STAT - Category (1)
BA Italian major & maths minor	<i>Sketched a cosine graph ($y=\cos x$) correctly but was unable to make corrections (ie. the errors were accepted).</i>	<i>Identified student error and gave a correction using the log-law: $\log(a/b) = \log a - \log b$. Solving for x, $x=-2$. However, she accepted $\log(-3)$ as a valid solution when it is not valid.</i>	<i>Responses were mostly irrelevant Eg. 'Variance is to do with measuring whatever, like length & volume.'</i>
Important learning	<i>Graphing of trig-functions and how to use the unit-circle.</i>	<i>To learn the different graphs as well as the use of graphs to represent the relationship between $\log x$ and y^x functions.</i>	<i>I really don't know because I have not done much learning in this area.</i>
Teaching approach	<i>1. Graphing. 2. Use trig-ratios to calculate measure of heights and distances.</i>	<i>1. Discuss how log functions are used to solve real-life problems, compare graphs of $\log_b Y = X$ and $y=b^x$. 2. Log laws and how to apply them. 3. The use of $\log x$ to solve complex functions of the form y^x.</i>	<i>I need to a lot of reading and make sure I understand statistics first.</i>
FEMALE 3: B2	TRIG - Category (1)	LOG - Category (1)	STAT - Category (1)
B.Econ maths minor	<i>Irrelevant responses based on learning experiences in geography, ie. after rain, the silt in the drains forms a sine-curve.</i>	<i>I have no recall of what logarithm is, and I'm lost without my calculator. Because I've always relied on the calculator for working out logarithm or trigonometry formulas.</i>	<i>Can recall isolated relevant aspects relating to statistics as a whole. But could not provide a conclusion because a calculator and the formula were not provided.</i>
Important learning	<i>Must understand how to graph the trig-functions.</i>	<i>I don't know, but logs are used a lot in Engineering.</i>	<i>To focus on all statistics rather than just variance.</i>
Teaching approach	<i>1. Teach calculator skills. 2. Unit-circles and graphing. 3. Memorising exact trig-values.</i>	<i>I'd teach using the calculator because that is how I was taught.</i>	<i>I rote-learn a lot of my maths. So I'll have to learn on-the-job how to teach and stick closely to the syllabus.</i>

FEMALE 4: B2	TRIG - Category (1)	LOG - Category (1)	STAT - Category (1)
BA Japanese language maths minor	<i>I can't remember and I can't explain to you why.</i>	<i>This is hard because I don't even remember what logarithm is about.</i>	<i>I have no idea of what a variance is. I recall doing std. deviation but I can't tell you what it is either.</i>
Important learning	<i>Unit-circle because every trig stems from this.</i>	<i>Learn how to go from $\log(2x+1)$ to $\log 2x + \log 1$. [Response is illogical]</i>	<i>I don't know because I haven't covered much statistics including variance.</i>
Teaching approach	<i>1. Unit-circle. 2. Trig-ratios</i>	<i>Start with log laws. That's because it's the way I usually work. ie. Find a rule and follow that.</i>	<i>Hopefully I don't have to teach a year 11 and 12 class.</i>
FEMALE 5: B1	TRIG-Category (2) Relation	LOG-Category (2) Relation	STAT - Category (1)
B.Sc Chem major	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$ because $\cos(A+B) \neq \cos A + \cos B$. [No solution was provided]	Identified student error & provided a correction by using the log-law: $\log(a/b) = \log a - \log b$. [Cancelled 'log' from both sides of $=$, then solved for x , resulting in $x=-2$. And accepted $\log(-3)$ as a valid solution when it is not.]	<i>I know that 'sigma squared' is the variance and the square-root of it is the standard deviation. But looking at this, I can't recall the formula. I don't really know or have an understanding of what a variance is. I just know it as 'sigma squared'.</i>
Important learning	<i>Understanding trig-ratios.</i>	<i>Need to understand the laws and how to apply them.</i>	<i>To learn the actual concept of the formulae, students need to learn the concept first and then the formula and how to use it.</i>
Teaching approach	<i>I was taught trig-rules and that using the rules was [brought] understanding. But that is not so and I would never teach kids that way.</i>	<i>1. Log laws (that's how I learn logs) 2. Problems, including word problems.</i>	<i>I don't really know, I guess I will have to follow the syllabus.</i>
FEMALE 6: A2	TRIG-Category (2) Instrument	LOG-Category (2) Instrument	STAT - Category (1)
B.Ed maths/sci (3rd Year)	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$, because $(2x+1)$ is an angle. There is a trig-rule to solve this but I can't recall what it is now. [Did not provide a solution.]	Identified student error: $\log(2x+1) \neq \log 2x + \log 1$. Provided a correction by using $\log(2x+1)/(x-1) = 0$ because of the rule: $\log(a/b) = \log a - \log b$. But I don't really know the reason for this. And I don't know what to do next.	<i>I think its $(x - \bar{x})^2$ the sum of these divided by n. So the correct variance is 2.4, but it's really only a guess. I didn't really enjoy statistics.</i>
Important learning	<i>Students to understand trig-functions and their graphs. Because graphs distinguish trigonometry from other functions.</i>	<i>Students to understand the log-laws & how to use them. Because that's the problem here, the student didn't understand logs.</i>	<i>An understanding of formulas and what to use them for. Eg. the variance formula and how and where to use it.</i>
Teaching approach	<i>1. Trig-graphs. 2. Trig-ratios and their relation to right-angled triangles. 3. Trig-rules & formulas.</i>	<i>I don't know really. I need to do more study and see what is required in the syllabus.</i>	<i>To teach variance I'll introduce standard deviation first. But if the syllabus indicates the variance first then I'd do that. That is, just follow whatever is in the syllabus.</i>
FEMALE 7: B1	TRIG - Category (2) Relation	LOG - Category (1)	STAT - Category (1)
B.Sc maths major	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$, because $(2x+1)$ is a whole number. [Did not provide a solution.]	<i>I can't remember. I know that there are log-laws but I can't remember what they are.</i>	[Attempted to consider formula and concept but had difficulty in expressing how the formula relates to her understanding.]
Important learning	<i>Knowledge of trig-graphs and the use of the unit-circle.</i>	<i>I don't know. I would have to study up logarithm myself first.</i>	<i>Deviations from the mean and how variance is applied to real-life.</i>
Teaching approach	<i>1. Graphing the trig-functions. 2. Trig-ratios and angles of triangles. 3. Use of the calculator.</i>	<i>1. Log laws and how to apply them. 2. Try to help students understand the difference of logs from algebra.</i>	<i>1. Application using a practical example. 2. The use of the formula and introduce the bell-shape [normal curve] idea.</i>
FEMALE 8: A1	TRIG - Category (2) Relation	LOG - Category (2) Relation	STAT - Category (2) Relation
Master's Chem Eng maths major	Identified the student error and provided a correction: $\cos(A+B) = 0$ implies $A+B=\pi/2$ but you can't do $\cos(A+B) = \cos A + \cos B = 0$	Identified the student error as: Student had a memory lapse of the simple application of log laws and indices. And treated it as a typical expansion $4(x+1) = 4x + 4$. It is wrong because $x = 1$, $\log(3) \neq \log(0)$.	<i>The variance formula is the average of the deviations from the mean. The standard deviation is the square of the variance. By inspection, I'd say 2.4 is the correct answer.</i>
Important Learning	<i>Knowledge of trig-graphs, trig-ratios and how to apply this knowledge in solving problems.</i>	<i>The relationship between indices and logarithm: $a^x a^y = a^{x+y}$ $\log(axb) = \log a + \log b$</i>	<i>To understand how to use the variance formula in solving problems, particularly real life application.</i>

Teaching approach	1. Revise earlier work related to trig. Eg. graphing skills, pythagorus rule and right-triangles. 2. Introduce the unit-circle and how it is used in relation to trig-functions. 3. Trig-rules.	Start with indices and log-laws and make sure students understand the difference between distributive law and log-laws.	Introduce the idea of variance by using real-life situations that students can relate to. Teach the mean and deviations from the mean and how these are connected to the formula. Then apply the formula.
MALE 1: A2	TRIG - Category (1)	LOG - Category (1)	STAT - Category (1)
B.Ed maths/sci (3rd Year)	I'm not really familiar with these to know how to do them myself. I have to consult a textbook to refresh my memory.	Accepted an incorrect form of the log law: $\log a / \log b = \log(a-b)$ and I know this is right. Eg. $\log(x-1) = \log x / \log 1$, (ie. $\log 1 = 0$), $\log x / 0$. And anything divided by zero is infinity. Therefore, the student's answer of $x=1$ is definitely incorrect. [illogical & erroneous response]	I'm not sure. I know it has something to do with statistics, but I'm not sure whether it's the mean something or rather.
Important learning	Students to learn trig-rules	Students to learn log-rules and ways of manipulating the rules ie. algorithms.	I don't know much about statistics without doing more research on it myself. I know the basics like the mean, mode, median, bar graphs etc. Beyond that I'm not confident.
Teaching approach	Emphasise the use of trig-rules in order for students to know how to calculate trig-values.	Emphasise the log-laws because unless the students learn the laws well, they won't know what to do.	I will teach whatever and however it's required in the syllabus. If I need more clarification I will refer to a good statistics textbook.
MALE 2: A2	TRIG-Category (2) Instrument	LOG-Category (2) Instrument	STAT - Category (1)
B.Ed maths/sci (3rd Year)	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$ because $(2x+1)$ is an angle. [No solution was provided]	Irrelevant cues were attended to initially and then by trial-and-error and by using $\log(2x+1)/(x-1) = 0$ and then $2x+1=x-1$, $x=-2$. By substitution noted that $x=-2$ is the correct value. But accepted $\log(-3)$ as a valid solution when it is not.	[Can recall isolated relevant aspects relating to the statistical variance, but could not provide a conclusion because the formula was not provided.]
Important learning	The understanding that trig-functions are for calculations of angles. That is what the student did wrong, was to treat the angle $(2x+1)$ as an algebraic term.	Students need to learn the log laws & how to apply the laws to different problems. Because a knowledge of log is required for university calculus.	I don't know because I think I need to do more studies for me to answer this question. For this kind of knowledge I've always relied on textbooks rather than try and commit it to memory.
Teaching approach	1. Right-angled triangle and trig-ratios. 2. Trig-rules and formulas.	1. Develop a good understanding of the log laws in relation to indices. 2. Make sure the log-laws are applied correctly, ie. do lots of practice exercises.	This sort of topic is too much to commit to memory, so I would rely on a good textbook and of course I would need to follow the syllabus.
MALE 3: A2	TRIG-Category (2) Instrument	LOG-Category (3) Instrument	STAT - Category (1)
B.Ed maths/sci (Final Year)	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$ because $(2x+1)$ is an angle. [No solution was provided]	By trial-&-error and using algebraic method: $\log(2x+1)/(x-1) = 0$ and then $2x+1=x-1$, $x=-2$. Concluding that the student lacks understanding of logs and got mixed up with other algebraic methods along the way.	Attempted to recall the formula but had difficulty in expressing how the given formula relates to his own understanding.
Important learning	Understanding the trig-formulas and functions	Understanding the concept of what logarithm is about. ie what a power of a number has to be in order to make it into another number.	Students to understand why they need to use certain statistics formulas such as variance.
Teaching approach	1. Trig-ratios and right-angled triangles. 2. Students to learn how to differentiate trig-functions from algebraic expressions.	1. Start with log-laws 2. Do lots of examples, eg. $\log_{10} 100=2$ 3. I need to do further study to extend from here.	I would need to do further study in order to have a clear understanding of what to teach. Then just follow the syllabus.
MALE 4: B1	TRIG-Category (2) Instrument	LOG-Category (2) Instrument	STAT-Category (3) Instrument
B.Sc Chem major	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$. [Did not provide a solution.]	Identified student error: $\log(2x+1) \neq \log 2x + \log 1$ By trial-&-error and using algebraic method: $2x+1=x-1$, $x=-2$. Using substitution concluded that $\log(-3) = \log(-3)$.	Recalled the formula correctly and identified the correct answer (3.4). However, why 2.4 & 2.7 were incorrect was discounted as being unnecessary. Concluding that 'knowing the correct formula & how to use it are what's required'.
Important learning	Trig-ratios are crucial to learning of trigonometry.	What logarithm means.	How to manipulate the many formulas.
Teaching approach	1. Start from the basics or the roots of trigonometry (ie. trig-ratios) and then build onto those.	1. Introduce log-laws and then inverse relationship of $\log x$ and y^x . 2. Students to learn the rules.	1. Activities based on real-life, 2. Calculate Variance using the formula.

MALE 5: B1	TRIG-Category (3) Instrument	LOG-Category (3) Instrument	STAT - Category (2) Relation
B.Sc maths major	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$ Explained the reason for the error. <i>Because $(2x+1)$ is an angle.</i> Showed correction: $(2x+1) = \pi/2$ and provided a solution, $x = \pi/4 - 1/2$.	Identified student error: $\log(2x+1) \neq \log 2x + \log 1$ Provided a solution: $\log(2x+1) = \log(x-1)$ using algebraic method: $2x+1=x-1$, $x=-2$. Accepted $\log(-3)$ as a valid solution when it is not.	Recalled the <i>std.deviation as the linearising of the data. The variance is the 'average of deviations'.</i> But I can't remember the proper formula. [No solution was provided.]
Important learning	<i>Trig-ratios and how to use trigonometry in real-life situations.</i>	<i>Log laws and understanding that logarithm is for linearising exponent equations.</i>	<i>The meaning of what a variance is. How significant it is in analysis.</i>
Teaching approach	<i>1. Activities based on real-life situations. 2. Learn formulae to calculate angles.</i>	<i>1. How to apply the log laws. 2. How to simplify non-linear relations. 3. Conclusions about the use of logs.</i>	<i>1. Discuss how data varies from a centre. 2. Show how to use the Variance formula. 3. Interpret results.</i>
MALE 6: B1	TRIG-Category (3) Instrument	LOG-Category (3) Instrument	STAT - Category (1)
B.Sc Engineer	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$ Provided a solution: $\cos(2x+1) = 0$ by letting $(2x+1) = \pi/2$ then $x = \pi/4 - 1/2$.	Identified student error: $\log(2x+1) \neq \log 2x + \log 1$ Provided a solution: $\log(2x+1) = \log(x-1)$ using algebraic method: $2x+1=x-1$, $x=-2$. [Accepted $\log(-3)$ as a valid solution when it is not.]	<i>I studied statistics 17yrs ago and I can only recall doing 'std.deviation'. As a guess, 2.4 is the variance.</i>
Important learning	<i>Clear understanding of trig-rules.</i>	<i>Rules should be learned thoroughly to make it easier to memorise them.</i>	<i>I'm not sure because I haven't covered this topic recently.</i>
Teaching approach	<i>Emphasise the rules and how to use the formulae correctly.</i>	<i>Emphasise the log rules for students to remember and do lots of examples.</i>	<i>I don't know because I last did statistics 17yrs ago.</i>
MALE 7: B1	TRIG-Category (3) Instrument	LOG-Category (3) Instrument	STAT - Category (1)
B.Sc maths & physics major	Identified student error: $\cos(2x+1) \neq \cos 2x + \cos 1$ Provided a solution: $\cos(2x+1) = 0$ by letting $(2x+1) = \pi/2$ then $x = \pi/4 - 1/2$.	Identified student error: $\log(2x+1) \neq \log 2x + \log 1$ Showed knowledge of 'log laws': $\log A/B = \log A - \log B$. Gave a solution by removing 'log' from both sides and solved for x , $x=-2$. Accepted $\log(-3)$ as a valid solution when it is not.	<i>I don't know. I can't remember, I haven't understood statistics well at Uni and I can't remember the formula.</i>
Important learning	<i>An understanding of sine, cosine and tan as ratios. And how these functions represent lengths.</i>	<i>How logarithm is related to exponents. How to use the log laws.</i>	<i>What variance is. What it tells us about the data. How to calculate the value.</i>
Teaching approach	<i>1. Use triangles to derive trig-ratios. 2. Use trig-ratios to solve triangles. 3. Trig-rules and how to apply them.</i>	<i>1. Relationship of logarithm to exponentials. 2. Understand the log laws and how to use them in solving problems.</i>	<i>1. Find the sample mean and other values from the data. 2. Derive the variance using the formula.</i>
MALE 8: A1	TRIG - Category (1)	LOG - Category (1)	STAT - Category (1)
B.Sc Electronic Engineer	<i>The student is treating this as algebra, and there seems to be a misunderstanding between the cos and its angle. But I can't remember how to do this now or to explain why this is so.</i>	Identified student error: <i>Student fails to check final answer to see that $\log(3) \neq \log(0)$.</i> Attempted to give a correction: $\log_{10} 100 = 2$, ie. $10^2 = 100$ but couldn't proceed any further.	<i>Sorry, but I can't remember any of this.</i>
Important learning	<i>Knowledge of trig-ratios and their relation to right triangles.</i>	<i>Knowledge about the concept of logarithm.</i>	<i>All about statistics not just the variance.</i>
Teaching approach	<i>I would follow the syllabus to make sure that I teach the right topics at the appropriate levels.</i>	<i>Teach the log-laws and make sure students understand their relation to the indice-laws.</i>	<i>Similar to trigonometry, I would follow the syllabus to make sure I'm on the right track.</i>

MALE 9: A1	TRIG - Category (2)- Relation	LOG - Category (2)-Relation	STAT - Category (3) Relation
B.Sc Computer Science	Identified student error and provided an explanation: $(2x+1)$ is the angle so $\cos(2x+1) \neq \cos 2x + \cos 1$. This could be demonstrated by the use of graphs to show that $y=\cos(2x+1)$ is not the same as $y = \cos 2x + \cos 1$.	Identified student error: Student didn't understand log laws and how to apply them. Provide a solution: Log is of base 10, $\log_{10}(2x+1)$. Also by applying logs one should realise that $2x+1 = x-1$. Then solve for x , ie. $x = -2$.	If $\Sigma(x - \text{mean})^2$ then $24/10=2.4$. But the Σ is to 9 not to 10. If Σ is to 9 then $24/9=2.7$. So it can't be 2.4 or 2.7. There is another formula for the variance: $S^2 = \Sigma x^2/n - (\text{mean})^2$ and by substitution $194/10 - 4^2 = 3.4$ So $S^2 = 3.4$ is the correct value.
Important learning	Being able (the ability) to graph the trig-functions.	The understanding of log laws and their relation to indices.	Students to have an understanding of the different forms of the variance formula and the skills involved in using and applying the formulae.
Teaching approach	The syllabus will be a good guide as to what is appropriate to teach to the students.	Introduce the log-laws and show how to apply them to solve algebraic problems.	Again, it's important to get guidance from the syllabus for the appropriate teaching sequence to follow.
MALE 10: A1	TRIG-Category (3) Instrument	LOG - Category (3) Relation	STAT - Category (1)
B.Sc Computer Science	Identified student error: Student has ignored the function (cos) and treated 'cos' as a pronumeral and then used the distributive law. Provided a correction: The question is to find the value of x in relation to cos(of an angle). That is, $\cos(2x+1)=0$ [Draw a unit circle and label the angles] So $2x+1 = \pi/2, 3\pi/2, \dots$, etc. Solve for x .	Identified student error: Incorrect use of log laws. Eg. $\log(2x+1) \neq \log 2x + \log 1$ because $\log AB = \log A + \log B$. Using the fact that \log_{10} is on both sides of the equal sign (=), [cancelled log] then $2x+1=x-1$ where $(x > 0)$. But $x = -2$. Since $x < 0$, there is no solution.	I'd say 2.4 because $24/10 = 2.4$, but that can't be because the 'sum' is to 9 not 10. If that is the case then $S^2 = 24/9 = 2.7$.
Important learning	An understanding of what sine, cosine and tan functions are in relation to the unit-circle and their graphs.	Students to gain a good grounding and understanding of the log-laws.	I don't know, I guess knowledge of what a variance is.
Teaching approach	Introduce trig-functions using unit-circle and make sure students understand the use of trig functions.	Introduce log-laws and give lots of practice exercises.	Statistics is not my strength, so for teaching, I'll just stick to the syllabus.
MALE 11: A1	TRIG - Category (2) Relation	LOG - Category (3) Relation	STAT - Category (1)
B.Sc Computer Science	Identified student error: By substituting -0.5 into $\cos(2x+1) = \cos(0)$ and $\cos(0)=1$ not 0. Provided a correction: $\cos(2x+1) = 0$ implies the angle $(2x+1)$ is either $\pi/2$ or $3\pi/2$ [draws the unit circle]. This confirms that $x \neq -0.5$.	Identified student error: Student has considered 'log' as a variable or recalled rules incorrectly. Provided a correction: 'log' is a function relating to some 'operation'. Eg. $\log AB = \log A + \log B$ and $\log_{10} X = a$ implies $10^a = X$. However, finding x in $\log(2x+1)=\log(x-1)$ is easier by using algebra, ie. $2x+1 = x-1$. Solve for x and $x = -2$. I was taught that for a valid solution of logs $x > 0$. [He couldn't explain why x has to be > 0 , but just remembers it that way]	I really don't know. I know the mean is 4, and the deviation from the mean is squared, but that's about it.
Important learning	To understand the unit-circle diagram and the related trig-graphs.	To understand the relationship of logs to exponentials, and to have the knowledge that log functions are for specific applications, in the same way trig-rules are specifically for solving problems related to measurement of lengths and angles.	I'm not really sure.
Teaching approach	Start with the unit-circle and trig-graphs and then teach the trig-ratios.	Start by making sure students understand the relation between indices and log-laws. Then move on to show how to apply these laws to real-life situations.	I need to learn and do more studies in this area before teaching it.

Table B2: Summary of the analysed data presented in Figure B1

PARTICIPANT GROUP	DEGREE/ MAJOR	TRIG Category:1,2,3,4 Skemp's Model	LOG Category:1,2,3,4 Skemp's Model	STAT Category:1,2,3,4 Skemp's Model
Female 1 A2	B.Ed (Maths) (final year)	Category 2 Pseudo-Procedural	Category 1	Category 2 Pseudo-Procedural
Female 2 B2	B.A Dip.Ed (Maths Minor)	Category 1	Category 2 Pseudo-Procedural	Category 1
Female 3 B2	B. EconDip.Ed (Maths Minor)	Category 1	Category 1	Category 1
Female 4 B2	B.A Dip.Ed (Maths Minor)	Category 1	Category 1	Category 1
Female 5 B1	B.Sc (Chem) Dip.Ed	Category 2 Rel-Procedural	Category 2 Rel-Procedural	Category 1
Female 6 A2	B.Ed (Maths) (3 rd year)	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 1
Female 7 B1	B.Sc (Maths) Dip.Ed	Category 2 Rel-Procedural	Category 1	Category 1
Female 8 A1	B.Sc Dip.Ed (Maths-Hons) M. Chem Engi	Category 2 Rel-Procedural	Category 2 Rel-Procedural	Category 2 Rel-Procedural
Male 1 A2	B.Ed (Maths) (3 rd year)	Category 1	Category 1	Category 1
Male 2 A2	B.Ed (Maths) (3 rd year)	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 1
Male 3 A2	B.Ed (Maths) (final year)	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 1
Male 4 B1	B.Sc (Chem) Dip.Ed	Category 2 Pseudo-Procedural	Category 2 Pseudo-Procedural	Category 3 Pseudo-Conceptual
Male 5 B1	B.Sc (Maths) Dip.Ed	Category 3 Pseudo-Conceptual	Category 3 Pseudo-Conceptual	Category 2 Rel-Procedural
Male 6 B1	B. Sc Dip.Ed (Engineer)	Category 3 Pseudo-Conceptual	Category 3 Pseudo-Conceptual	Category 1
Male 7 B1	B.Sc Dip.Ed (Maths/Physic)	Category 3 Pseudo-Conceptual	Category 3 Pseudo-Conceptual	Category 1
Male 8 A1	B.Sc Dip.Ed (Engineer)	Category 1	Category 1	Category 1
Male 9 A1	B.Sc Dip.Ed (Computer Sc)	Category 2 Rel-Procedural	Category 2 Rel-Procedural	Category 3 Relational
Male 10 A1	B.Sc Dip.Ed (Computer Sc)	Category 3 Pseudo-Conceptual	Category 3 Rel-Conceptual	Category 1
Male 11 A1	B.Sc Dip.Ed (Computer Sc)	Category 2 Rel-Procedural	Category 3 Rel-Conceptual	Category 1

Legend:

Rel-Procedural - Relational Procedural Knowledge,

Rel-Conceptual - Relational Conceptual Knowledge

Pseudo-Procedural - Instrumental Procedural Knowledge, Pseudo-Conceptual - Instrumental Conceptual Knowledge