SINGULAR INTEGRAL EQUATIONS

by

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Submitted in fulfilment of the requirements

for the degree of Doctor of Philosophy

UNIVERSITY OF TASMANIA

HOBART

(January, 1977)

Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any university, and that, to the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by another person, except when due reference is made in the text.

Murray Dow

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ACKNOWLEDGEMENTS

I wish to thank my supervisor, Professor D. Elliott for his constant interest and advice.

I also thank Miss G. Gafar for her diligent typing, the Australian Government for a Commonwealth post-graduate research award, the staffs of the Commonwealth department of education and University of Tasmania, and finally the Australian people, who provided the funds for this research.

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SUMMARY

The classical analytic solution of the dominant singular integral equation

$$a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = f(t) \quad , \quad -1<t<1 \quad ,$$

is found by transforming the equation into a Riemann boundary problem. (The above integral is interpreted as a Cauchy principal value.)

This analytic solution is not very useful for numerical work, since it requires the evaluation of a Cauchy principal value integral of a function Z, which has, in general, algebraic-logarithmic singularities at ±1.

Three alternative solutions are found which avoid this problem. These solutions are used as the basis of an algorithm for the numerical solution of the above singular integral equation. Convergence of this algorithm is proved, and four examples are given, including the H equation of Chandrasekhar.

An algorithm and a computer program is also given for the solution of the complete singular integral equation

$$a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau + \int_{-1}^{1} K(t, \tau)\phi(\tau)d\tau = f(t) \quad , \quad -1<t<1 \ .$$

Two examples are given, including the numerical solution of a singular integral equation arising in neutron transport theory.

CHAPTER I

SUMMARY OF RELEVANT THEORY

§1. Introduction

We begin with a definition.

Definition 1.1 *Singular integral equations of the form*

$$(1.1)\quad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau - t}\,d\tau + \int_{-1}^{1} K(t,\tau)\phi(\tau)d\tau = f(t), \quad -1<t<1,$$

will be called "complete" singular integral equations, and equations of the form

$$(1.2)\qquad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau - t}\,d\tau = f(t)\ , \quad -1<t<1,$$

will be called "dominant" singular integral equations.

The singular integrals in (1.1) and (1.2) are interpreted as Cauchy principal value integrals, which are defined by

$$(1.3)\qquad \int_{-1}^{1}\frac{\phi(\tau)}{\tau - t}\,d\tau = \lim_{\epsilon\to 0}\left\{\int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1}\right\}\frac{\phi(\tau)}{\tau - t}\,d\tau\ , \quad \epsilon>0\ .$$

A sufficient condition for the existence of this integral is given in Lemma 2.3.

The dominant equation (1.2) was first solved by Carleman [4] in 1922, in the case b = constant. In 1941, Muskhelishvili [32] and Gakhov [16] generalized this solution, allowing b to be a function of t. (For a historical summary, see Gakhov [16]).

Using these methods, (1.1) can be reduced to a Fredholm integral equation, which has a rather complicated kernel, see [32, §109 - 111]. While useful for studying the properties of (1.1), this reduction is not very useful for numerical work.

The accurate numerical solution of (1.1) or (1.2) is very difficult, as it involves the evaluation of a Cauchy principal value integral of a function Z (see §4 - 5), which has algebraic-logarithmic singularities at the points ±1. In fact, assuming that a and b are real, MacCamy [31] showed that if

$\xi = \frac{1}{2\pi i} \log \frac{a(-1) + i\, b(-1)}{a(-1) - i\, b(-1)}$, $-1<\xi\leq 0$, then

$$(1.4) \qquad \phi(t) \sim (1+t)^{\xi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}(1+t)^{m+n} \log^{n}(1+t) ,$$

where the symbol ~ means "asymptotically equal to as $t\to -1$". (A similar result holds at t = 1). Hence, if an algorithm for the solution of (1.1) is to give good results, it must be able to cope with singularities of this type.

Singular integral equations on an open contour (e.g., [-1, 1] or [0, ∞)) present a more difficult problem than those on a closed contour (e.g., the unit circle in the complex plane, or (-∞, ∞)). For instance, consider the equation

$$(1.5) \qquad a(t)\phi(t) + \frac{1}{\pi i} \int_{c} \frac{H(t,\tau)\phi(\tau)}{\tau - t}\, d\tau = f(t) \ , \ t \in c ,$$

where c is a closed contour. Assuming that a, f, and H are analytic in the domain enclosed by c (H analytic in both variables), Case [5] showed, under certain conditions, that the solution of (1.5) is simply $\phi(t) = \frac{f(t)}{a(t) + H(t,t)}$, $t \in c$.

We note that (1.4) implies that, using only elementary functions, it is impossible to transform (1.1) into an equation on a closed contour without introducing singularities. Thus, the mappings given in Achieser and Glasmann [2, §50], and Ivanov [20, §11.1] introduce square root singularities, and while these are suitable for equations of the first kind, they only complicate equations of the second kind.

We shall now summarise general numerical methods for the approximate solution of equation (1.1), which have been proposed previously.

Singular integral equations of the first kind

If $a \equiv 0$, then (1.1) is said to be an equation of the first kind. The vast majority of singular integral equations discussed in the literature are of this type; efficient algorithms for their approximate solution can be constructed using the integrals

$$(1.6) \qquad \frac{1}{\pi}\int_{-1}^{1} \frac{T_n(\tau)}{\sqrt{1-\tau^2}} \frac{d\tau}{\tau - t} = U_{n-1}(t) \ , \ n=0,\ 1,\ldots; \quad -1<t<1 \ ,$$

where T_n , U_n are Chebyshev polynomials of the first and second kinds ($U_{-1} \equiv 0$). For example, Kalandiya [23] has used the trigonometric form of (1.6). It should be noted that the determinant of his system of equations (10) is zero, so that his method needs modifying.

Singular integral equations of the second kind with constant coefficients

If a and b are constant, then we shall call (1.1) an equation with constant coefficients. In his book on elasticity, Muskhelishvili [33, §110] proposed a method for solving equations of this form. While his approach seems quite viable (and is similar to the methods used in this thesis), a better method is to use the following identity in Jacobi polynomials, given by Karpenko [24] (see also Tricomi [43]):

$$(1.7)\quad \frac{1}{\pi}\int_{-1}^{1}\frac{(1-\tau)^{\alpha}(1+\tau)^{\beta}P_n^{(\alpha,\beta)}(\tau)}{\tau - t}\,d\tau = \cot(\pi\alpha)(1-t)^{\alpha}(1+t)^{\beta}P_n^{(\alpha,\beta)}(t) - $$

$$- \frac{2^{\alpha+\beta}}{\sin(\pi\alpha)}P_{n+\alpha+\beta}^{(-\alpha,-\beta)}(t),$$

$$\alpha,\beta>-1\ , \quad \alpha+\beta = -1,0,1,\ldots \quad ; \quad -1<t<1\ .$$

Using this, Karpenko [24] in 1966 gave an algorithm for the numerical solution of equations with constant coefficients, and estimated the error incurred.

Using similar methods, Erdogan and Gupta [12], Erdogan, Gupta and Cook [13] and Krenk [27], [28] have also considered such equations.

Singular integral equations with variable coefficients

For equations in which a and b are variable, only Ivanov [19] in 1956 and in his book [20] (1968) has published a general method of solution. (The method proposed by MacCamy [31] does not appear to be practicable).

In his first method [20, §13], Ivanov defines a polynomial R of degree $2r + 1$ such that

$$(1.8)\qquad \frac{d^{\nu}}{dt^{\nu}}\Big\{R(t) - \log[a(t) - i\ b(t)]\Big\}_{t=\pm1} = 0\quad ,$$
$$\nu = 0,1,\ldots,r.$$

Now, if a or b is not differentiable at ±1 (a common occurence), then it is necessary to choose $r = 0$, and consequently the convergence of his algorithm will be very poor. Ivanov then transforms the contour of the singular integral equation into the unit circle, the coefficients of the resulting equation having discontinuities in their $r + 1^{th}$ derivatives.

This approach is rather tedious, and completely obscures useful properties of the original equation. The same criticisms apply to the method given in [19].

In Ivanov's third method [20, §11.3], poor convergence will again result if a or b are not differentiable. Also required is the summation of an infinite series (eqn 11.45), the convergence of which is unknown, and whose coefficients are not easy to evaluate.

Ivanov does not give any worked examples of these methods.

In 1963, Pken [37] obtained a simple expression for the solution of

$$(1.9) \quad P_n(t)\phi(t) + \sqrt{1-t^2}\, Q_m(t)\, \frac{1}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t) \ , \ -1<t<1 \ ,$$

by assuming that P_n and Q_m are polynomials. Using this result, he proposed a general numerical method for equations of the form $a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t)$, $-1<t<1$, approximating a by P_n and b by $\sqrt{1-t^2}\, Q_m$. This approach, while quite good, is clearly limited by poor convergence if b does not have square root singularities at ± 1, and if a is not a sufficiently smooth function.

In §2, we give results from the theory of Cauchy integrals and Cauchy principal value integrals which we need in this thesis.

As we have noted, the solution of (1.2) is given, for example, by Muskhelishvili [32] and Gakhov [16]. These authors derive the solution of (1.2) by transforming it into a Riemann boundary problem (see §3), solving the Riemann boundary problem and then deriving the solution of (1.2). This is the approach used in this thesis; however, since we use a simpler form of the solution of (1.2), we will explain the differences between the form of this solution derived by Muskhelishvili and in this thesis.

An important quantity in the theory of (1.2) is the index κ; for its definition, see §4. The index takes only integral values, and, if a and b are continuous on [-1,1], depends only on the zeros of a and b on [-1,1]. If the index κ is positive, then the general solution of (1.2) may be written as

$$(1.10) \qquad \phi = \phi_0 + \sum_{i=1}^{\kappa} c_i \phi_i ,$$

where ϕ_0 satisfies (1.2), the ϕ_i are a linearly independent set which satisfy the homogenous equation

$a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = 0$, $-1<t<1$, and the c_i are arbitrary constants. If the index is zero, then (1.2) has a unique solution. If the index κ is negative, then a solution of (1.2) exists if and only if $-\kappa$ additional conditions are satisfied, such a solution being unique.

In this thesis, we simply derive the general solution to (1.2), in the form (1.10), whereas by varying the index, Muskhelishvili is able to derive particular solutions which are bounded at ±1. Thus, if the general solution is of index κ, then solutions bounded at +1 or -1 are of index κ-1, and solutions bounded at both ±1 are of index κ-2. (The only exception to this is if b is zero at +1 or -1, in which case the solution ϕ is automatically bounded at that end-point). Allowing the index to vary in this way leads to the necessity of classifying the solutions according to their behaviour at ± 1. The inclusion of these particular solutions complicates the work and obscures the nature of the index. Because of these complications, and because these particular solutions (if they exist) may be derived from the general solution by suitable choice of the arbitrary constants c_i in (1.10), we will derive the general solution only of (1.2) in this thesis.

Thus, in §3, we define and solve the Riemann boundary problem on the contour $[-1,1]$, and in §4 solve the dominant singular integral equation (1.2) by converting it into a Riemann boundary problem. Thus §1 - 4 consist mostly of known methods and results, and comprise chapter one.

Chapter two consists of §5 - 7, §5 being an introduction. In §6, we give a method for the analytic evaluation of the type of Cauchy principal value integrals encountered in (1.1), and in §7 apply these results to give an alternative analytic solution of (1.2), and two exact solutions which assume that b or f are polynomials. These three alternative solutions are used later in this thesis.

Chapter three is concerned with the numerical solution of the dominant equation (1.2). The algorithm is given in §8, including a proof of convergence; §9 - 10 give details of the algorithm, and in §11 we give several examples.

In chapter four we consider the complete equation, §12 giving the algorithm, §13 - 14 details of the algorithm, and in §15 give two examples, one of these being a complete singular integral equation arising in neutron transport theory. In §16 we give a computer program for the numerical solution of the complete equation.

The methods of this thesis may be generalised to singular integral equations on arbitrary arcs in the complex plane; alternatively, if the arc is sufficiently smooth, it may be mapped on to $[-1, 1]$.

§2. Cauchy integrals and Cauchy principal value integrals

In this section, we summarise various results from the theory of the Cauchy integral and Cauchy principal value integral, most of which is taken from Muskhelishvili [32]. First, we define the classes H and H*.

Definition 2.1 *A function* ϕ *is said to be Hölder continuous on an interval* I *if there exists positive constants* A, α, *with* $0 < \alpha \leq 1$, *such that* $|\phi(s) - \phi(t)| \leq A|s - t|^{\alpha}$ *for any* $s, t \in I$. *We denote this by writing* $\phi \in H(I)$.

If the interval is omitted (i.e., $\phi \in H$), then it will be assumed to be $[-1, 1]$. In §8, it will be necessary to refer explicitly to the exponent α of the class H to which ϕ belongs, so we will write $\phi \in H_{\alpha}$.

Definition 2.2 *If* ϕ *is Hölder continuous on any closed subinterval of* $(-1, 1)$ *which does not include* ± 1, *and if*

$$\phi(t) = \begin{cases} \phi_1{}^*(t)(1+t)^{-\gamma} & \text{for } t \text{ near } -1 \\ \phi_2{}^*(t)(1-t)^{-\gamma} & \text{for } t \text{ near } +1 \end{cases}$$

where $0 < \text{Re}\,\gamma < 1$ *and* $\phi_1^*, \phi_2^* \in H$, *then* ϕ *is said to belong to the class* H*.

We now consider the Cauchy integral, defined by

$$\text{(2.1)} \qquad \Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - z}\, d\tau \quad , \quad z \notin [-1, 1].$$

Lemma 2.1 *If* $\phi \in H^*$, *then* Φ *(defined by* (2.1)) *is analytic in the complex plane excluding* $[-1, 1]$, *and is zero at infinity.*

Proof That Φ is analytic in the complex plane excluding $[-1, 1]$ follows because its derivative, $\Phi'(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\phi(\tau)}{(\tau-z)^2}\, d\tau$, exists for all $z \notin [-1, 1]$.

To show that $\lim_{z\to\infty} \Phi(z) = 0$, we substitute the series $\frac{1}{\tau - z} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\tau}{z}\right)^k$ in (2.1), and since this series is uniformly convergent in a neighbourhood of the point at infinity, we may interchange the order of integration and summation in (2.1), thus proving the lemma. #

We now consider the behaviour of Φ near the points ± 1. In the following lemma, we assume that the functions $\log (z\pm 1)$ are single-valued in the complex plane near ∓ 1, cut from ∓ 1 along $[-1, 1]$ and then to infinity.

Lemma 2.2 *Let* $\phi \in H$, *and* Φ *be defined by* (2.1). *Then, for* z *in a neighbourhood of* -1, *but* $z \notin [-1, 1]$, *we have*

$$\Phi(z) = -\frac{\phi(-1)}{2\pi i}\log (z+1) + \Phi_0(z) ,$$

where Φ_0 *is a bounded function near* -1, *and tends to a definite limit as* $z\to -1$ *along any path.*

A similar result holds for z *near* 1, $z \notin [-1, 1]$, *viz:*

$$\Phi(z) = \frac{\phi(1)}{2\pi i}\log (z-1) + \Phi_1(z) ,$$

where Φ_1 *is bounded near the point* 1.

Proof See Muskhelishvili [32, §29]. #

We now consider the limiting value of $\Phi(z)$ as z tends to a point $t \in [-1, 1]$, from either side of $[-1, 1]$. Thus we define the functions $\Phi^{\pm}$ on $[-1, 1]$ by

(2.2)
$$\Phi^{+}(t) = \lim_{z \to t} \Phi(z) \ , \qquad \operatorname{Im} z > 0 \ , \quad t \in [-1, 1]$$
$$\Phi^{-}(t) = \lim_{z \to t} \Phi(z) \ , \qquad \operatorname{Im} z < 0 \ , \quad t \in [-1, 1],$$

provided that these limits exist. The following theorem, due to Sokhotski and Plemelj, gives an expression for these limits.

<u>Theorem 2.1</u> *If* $\phi \in H^*$, *then the limits* (2.2) *exist everywhere on* $(-1, 1)$, *and are given by the formulae*

(2.3)
$$\Phi^{\pm}(t) = \pm\tfrac{1}{2}\,\phi(t) + \frac{1}{2\pi i}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau \ , \quad t \in [-1, 1].$$

However, $\Phi^{\pm}$ *may not exist at either* +1 *or* -1 *if* ϕ *is non zero there.*

<u>Proof</u> See Muskhelishvili [32, §17] and Gakhov [16, §4]. #

The integrals in equations (2.3) are interpreted in the sense of the Cauchy principal value, defined in §1.

A sufficient condition for the existence of a Cauchy principal value integral is given by the following lemma.

<u>Lemma 2.3</u> *If a function* ϕ, *defined on* $[-1, 1]$, *is Hölder continuous in a neighbourhood of a point* $t_0 \in (-1, 1)$, *and integrable elsewhere, then* ψ , *which is defined by the singular integral*

$$\psi(t) = \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau \ ,$$ *is Hölder continuous in a neighbourhood of* t_0.

<u>Proof</u> See Muskhelishvili [32, §19] and Gakhov [16, §5]. #

The next two lemmas concern the change of order of integration of Cauchy principal value integrals.

Lemma 2.4 (Poincaré - Bertrand) *Let* $\phi \in H^*$, $\psi \in H^*$ *and the product* $\phi\psi \in H^*$. *Then*

$$\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t} \int_{-1}^{1} \frac{\psi(s)}{s - \tau}\, ds d\tau = - \int_{-1}^{1} \frac{\psi(s)}{s - t} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - s}\, d\tau ds +$$

$$+ \int_{-1}^{1} \frac{\psi(\tau)}{\tau - t}\, d\tau \int_{-1}^{1} \frac{\phi(s)}{s - t}\, ds - \pi^2 \phi(t)\psi(t) \ , \quad -1<t<1 \ .$$

Proof See, for example, Levinson [29] or Tricomi [43]. #

Lemma 2.5 (Parseval) *Let* $\phi \in H^*$, $\psi \in H^*$ *and* $\phi\psi \in H^*$. *Then*

$$\int_{-1}^{1} \phi(\tau) \int_{-1}^{1} \frac{\psi(s)}{s - \tau}\, ds d\tau = \int_{-1}^{1} \psi(s) \int_{-1}^{1} \frac{\phi(\tau)}{s - \tau}\, d\tau ds \ .$$

Proof See Tricomi [43], or replace $\phi(\tau)$ in Lemma 2.4 with $\phi(\tau)(\tau-t)$. #

In the following lemma, we give a representation for functions which are analytic in the complex plane excluding [-1, 1].

Lemma 2.6 *Let* χ *be an arbitrary function which is analytic in the complex plane excluding* [-1, 1], *and which is either analytic at infinity or has a pole of order* n *at infinity. Let* h *be a polynomial of degree* n *such that*

$$\lim_{z\to\infty} \left\{ h(z) - \chi(z) \right\} = 0 \ . \tag{2.4}$$

We assume that $\chi^{\pm} \in H^*$.

Then the following representation for χ *is valid:*

$$\chi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\chi^+(\tau) - \chi^-(\tau)}{\tau - z}\, d\tau + h(z) \ , \quad z \notin [-1, 1] \ . \tag{2.5}$$

Proof See Muskhelishvili [32, §78]. #

The following lemma provides a powerful technique for the evaluation of singular integrals.

Lemma 2.7 *Let* χ *and* h *be as in the preceding lemma. Then*

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{\chi^{+}(\tau) - \chi^{-}(\tau)}{\tau - t} \, d\tau = \chi^{+}(t) + \chi^{-}(t) - 2\, h(t) \;, \quad -1<t<1 \;.$$

Proof Follows immediately on applying Theorem 2.1 to the Cauchy integral in (2.5). #

§3. The Riemann boundary problem on an open contour

In this section, we shall solve a Riemann boundary problem, that is to say we seek a function Φ which is analytic in the complex plane excluding [-1, 1], zero at infinity, subject to the following condition:

$$(3.1) \qquad \Phi^+(t) = G(t)\,\Phi^-(t) + g(t) \ , \quad -1<t<1 \ ,$$

where G and g are known functions.

The material in this section is drawn from Muskhelishvili [32, Chap. 10] and Levinson [29].

<u>Definition 3.1</u> *Given a non-vanishing function* $G \in H$, *then the "canonical function"* X *of the Riemann boundary problem* (3.1) *is defined to be any function having the following properties:*

(i) X *is analytic in the finite complex plane excluding* [-1, 1], *and has finite degree at infinity, i.e.* $X(z) = O(z^{-\kappa})$ *as* $z\to\infty$, *where* κ *is some integer (positive, negative or zero).*

(ii) $$X^+(t) = G(t)\, X^-(t) \ , \quad -1<t<1 \ .$$

(iii) X *has no zeros in the complex plane.*

(iv) $X^{\pm} \in H^*$, *and* X *has the bounds*

$$0 < A_1 \le |X(z)| \le \frac{B_1}{|z+1|^{\beta_1}} \quad \text{for } z \text{ near } -1 \ , \text{ and}$$

$$0 < A_2 \le |X(z)| \le \frac{B_2}{|z-1|^{\beta_2}} \quad \text{for } z \text{ near } +1 \ ,$$

where $0 < \beta_1, \beta_2 < 1$ *and* A_1, A_2, B_1, B_2 *are constants.*

Equivalently, we may say that the functions $X^{\pm}$ *are integrable and nonzero at the points* ± 1.

Lemma 3.1 *For* G *non-vanishing and Hölder continuous on* $[-1, 1]$, *the canonical function* X *is given by*

$$(3.2) \qquad X(z) = (1 - z)^{\sigma} (1 + z)^{-\kappa-\sigma} \exp\left(\frac{1}{2\pi i} \int_{-1}^{1} \frac{\log G(\tau)}{\tau - z} d\tau \right) ,$$

$$z \notin [-1, 1] ,$$

where σ *and* κ *are given by* (3.4), *and the branch of the logarithm function is chosen such that* $\log G$ *is continuous and single valued on* $[-1, 1]$.

Proof Define $\Gamma(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\log G(\tau)}{\tau - z} d\tau$, $z \notin [-1, 1]$. Since G does not vanish on $[-1, 1]$, we can always choose a branch of the logarithm function such that log G is continuous and single valued on $[-1, 1]$. Since $G \in H$, then by Lemma 2.1, Γ is analytic in the complex plane excluding $[-1, 1]$, and $\Gamma(\infty) = 0$. Hence $X(z) = O(z^{-\kappa})$ as $z \to \infty$, and so X satisfies (i).

Using Theorem 2.1, it is not difficult to show that (3.2) satisfies (ii). Clearly, condition (iii) is also satisfied.

To show that (iv) is satisfied, we consider the behaviour of X at $z = \pm 1$. By Lemma 2.2, we can write

$\Gamma(z) = \Gamma_0(z) - \frac{1}{2\pi i} \log G(-1) \cdot \log(z+1)$, where Γ_0 is bounded in a neighbourhood of the point -1. Putting

$$(3.3) \qquad \gamma_1 + i\,\delta_1 = -\frac{1}{2\pi i} \log G(-1) \, , \; \gamma_2 + i\,\delta_2 = \frac{1}{2\pi i} \log G(1) ,$$

we obtain

$$X(z) = (1-z)^{\sigma}\,(1+z)^{-\kappa-\sigma+\gamma_1+i\delta_1} \exp \Gamma_0(z) \quad \text{near } z = -1 \;,$$

and similarly

$$X(z) = (-1)^{\gamma_2+i\delta_2}\,(1-z)^{\sigma+\gamma_2+i\delta_2}\,(1+z)^{-\kappa-\sigma} \exp \Gamma_1(z) \quad \text{near } z = 1 \;,$$

where Γ_1 is bounded in a neighbourhood of the point 1. Using these two expressions, we now choose σ and κ so that (iv) is satisfied. Thus, to ensure that $X^{\pm}$ are nonzero and integrable at the points ± 1, we need $-1<\sigma+\gamma_2 \leq 0$ and $-1<-\kappa-\sigma+\gamma_1 \leq 0$, which together imply that

$$\sigma = \lfloor -\gamma_2 \rfloor$$

(3.4) $$\kappa = \begin{cases} \lfloor \gamma_1 \rfloor - \sigma + 1 & \text{if } \gamma_1 \text{ is non integral} \\ \gamma_1 - \sigma & \text{if } \gamma_1 \text{ is an integer,} \end{cases}$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Thus we have shown that (3.2) satisfies the conditions of Definition 3.1, and the proof is complete. #

It is possible to prove that under the conditions of Definition 3.1, the canonical function is unique up to an arbitrary multiplicative constant; however, it is not necessary that we do this.

In Muskhelishvili [32, §79] , the canonical function is not unique, because it is allowed to be zero at ± 1. This leads to the necessity of defining the class of the canonical function according to its behaviour at the endpoints ± 1. While this approach is useful if particular solutions (bounded at -1 or +1) of the Riemann boundary problem (3.1) are required, our assumption that the canonical function is nonzero at ± 1 does not cause loss of solutions to the Riemann boundary problem, as we shall prove in Theorem 3.1, below.

We also note that (3.2) differs from Muskhelishvili's definition of the canonical function [32, §79] by the factor $(-1)^{\sigma}$, and from Gakhov's [16, §43] by the factor $(-1)^{\kappa}$.

Definition 3.2 *The integer* κ, *defined by* (3.4), *will be called the "index" of the canonical function* X *and of the associated Riemann boundary problem* (3.1).

As we shall see, the index determines the number of linearly independent solutions of the Riemann boundary problem.

Definition 3.3 *If* γ_1 (γ_2) *is integral, then the point* -1 $(+1)$ *is called a "special end".*

From the definitions (3.3) of γ_1 and γ_2, we see that an end is special if G is real and positive at that end, and it also follows that the canonical function is bounded at special ends, and unbounded otherwise.

Using the canonical function, we can now give the solution of the Riemann boundary problem (3.1).

Let g, G $\in$ H , with G non-vanishing on $[-1, 1]$. Then the canonical function X exists, and is given by (3.2), and the index κ is given by (3.4).

Theorem 3.1 *Let* Φ *be analytic in the complex plane excluding* $[-1, 1]$, *zero at infinity, with* $\Phi^{\pm} \in H^{*}$. *Suppose that* Φ *satisfies the Riemann boundary problem*

(3.5) $$\Phi^{+}(t) = G(t)\, \Phi^{-}(t) + g(t) \quad , -1<t<1 .$$

Then

(3.6) $$\Phi(z) = X(z)\left(\frac{1}{2\pi i}\int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)}\,\frac{d\tau}{\tau - z} + P_{\kappa-1}(z) \right),$$

$$z \notin [-1, 1],$$

where $P_{\kappa-1}$ *is an arbitrary polynomial of degree* $\kappa-1$ *if* $\kappa>0$ *, and zero otherwise. If* $\kappa<0$ *, then* (3.5) *is solvable if and only if*

(3.7) $$\int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)}\,\tau^{k-1}d\tau = 0 , \quad k = 1, 2,\ldots, -\kappa,$$

the solution Φ *being given by* (3.6).

We shall refer to (3.7) as the "consistency condition".

<u>Proof</u> In the first part of this proof, we assume that (3.5) is solvable, and show that the solution Φ has the form (3.6).

We define the function ψ by

(3.8) $$\psi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)}\,\frac{d\tau}{\tau - z} , \quad z \notin [-1, 1] .$$

Then from Theorem 2.1, we have

$$\psi^{+}(t) - \psi^{-}(t) = g(t)/X^{+}(t) , \quad -1<t<1 ,$$

and recalling that $G = X^{+}/X^{-}$, we can write (3.5) as

(3.9) $$\frac{\Phi^{+}(t)}{X^{+}(t)} - \psi^{+}(t) = \frac{\Phi^{-}(t)}{X^{-}(t)} - \psi^{-}(t) , \quad -1<t<1 .$$

Defining

(3.10) $$Y(z) = \frac{\Phi(z)}{X(z)} - \psi(z) , \quad z \notin [-1, 1] ,$$

we see that Y is analytic in the complex plane; in particular, by (3.9),

Y is continuous across the arc $(-1, 1)$, and its only possible singularities are at the points ± 1 and ∞. Let us examine the behaviour of Y at these points.

Now, $X^{\pm}$ is nonzero at ± 1, and $\Phi^{\pm} \in H^*$, hence Φ/X cannot have poles at ± 1. Also, by Lemma 2.2, ψ does not have poles at ± 1. Thus, Y has no poles at ± 1, and since Y is continuous across $(-1, 1)$, the singularities at ± 1 must be isolated and removable. Hence, Y is analytic in the complex plane.

We now determine the behaviour of Y at infinity. Since $\Phi(\infty) = 0$, and Φ is analytic in a neighbourhood of infinity, we have (at most) $\Phi(z) = O(z^{-1})$ as $z \to \infty$. By Lemma 2.1, $\psi(z) = O(z^{-1})$ and from Definition 3.1, $X(z) = O(z^{-\kappa})$ as $z \to \infty$. Then from (3.10),

$$Y(z) = O(z^{\kappa-1}) + O(z^{-1}) \quad \text{as } z \to \infty . \tag{3.11}$$

We consider two cases.

(i) Index $\kappa \leq 0$

If the index κ is nonpositive, then by (3.11), $Y(\infty) = 0$. Further, since Y is analytic in the complex plane, it follows from Liouville's Theorem that $Y \equiv 0$. Hence by (3.10), $\Phi = X\psi$, which is (3.6) with $P_{\kappa-1} \equiv 0$.

(ii) Index $\kappa > 0$

If the index is positive, then by (3.11), $Y(z) = O(z^{\kappa-1})$ as $z \to \infty$, and since Y is analytic, it must be a polynomial of degree κ-1. Identifying Y with $P_{\kappa-1}$, we have from (3.10) that

$\Phi = X \left(\psi + P_{\kappa-1} \right)$, which again is (3.6).

Thus we have shown that if (3.5) has a solution, then it is given by (3.6).

In the second part of the proof, we show that (3.6) satisfies the conditions of the Theorem. Using Theorem 2.1, it is easy to show that Φ , defined by (3.6), satisfies (3.5). From Lemma 2.1 we can show that Φ is analytic in the complex plane excluding $[-1, 1]$, and from Lemma 2.2 we can show that $\Phi^{\pm} \in H^*$. If the index $\kappa \geq 0$, then, since $X(z) = O(z^{-\kappa})$, $\psi(z) = O(z^{-1})$, $P_{\kappa-1}(z) = O(z^{\kappa-1})$ as $z \to \infty$, it is evident that $\Phi(\infty) = 0$. If the index $\kappa < 0$, then to determine the behaviour of Φ at infinity, we expand ψ in powers of z^{-1}:

$$\psi(z) = \frac{1}{2\pi i} \sum_{k=1}^{\infty} z^{-k} \int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)} \tau^{k-1} \, d\tau \ , \quad |z|>1 \ .$$

Then it follows that $\Phi = X\psi$ will be zero at infinity if and only if

$$\int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)} \tau^{k-1} \, d\tau = 0 \ , \quad k=1, 2, \ldots, -\kappa \ .$$

This completes the proof of the theorem. #

§4. The analytic solution of the dominant equation

In this section, we will solve the dominant singular integral equation

$$(4.1) \qquad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t) \;,\; -1<t<1 \;,$$

by converting it into a Riemann problem of the type considered in the last section. We will derive the general solution of (4.1), as we said in §1.

We assume that a, b and f are real, that

$$(4.2) \qquad a,\, b,\, f \in H \quad , \quad \phi \in H^*$$

and that

$$(4.3) \qquad a^2(t) + b^2(t) \neq 0 \quad \text{for} \quad -1 \le t \le 1 \;.$$

Assuming that a solution ϕ of (4.1) exists, we define the function Φ by

$$(4.4) \qquad \Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - z}\, d\tau \,, \qquad z \notin [-1, 1] \;.$$

Then by Lemma 2.1, Φ is analytic in the complex plane excluding $[-1, 1]$, and is zero at infinity. Applying the Sokhotski-Plemelj formulae (2.3) to (4.4), we obtain from (4.1) the following Riemann boundary problem for Φ:

$$(4.5) \qquad \Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad , \quad -1<t<1 \;,$$

where

$$(4.6) \qquad G(t) = \frac{a(t) - i\, b(t)}{a(t) + i\, b(t)} \quad , \quad g(t) = \frac{f(t)}{a(t) + i\, b(t)} \quad , \quad -1 \le t \le 1 \;.$$

Using the results of §3, we can write down the solution of (4.5).

First, we make the following definitions.

Let Arctan denote the multivalued function, and arctan the principal value; thus $-\pi/2 < \arctan x < \pi/2$ for x real, and $\text{Arctan } x = \arctan x + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

<u>Definition 4.1</u> *Let θ be a real continuous function, defined on* $[-1, 1]$, *such that for every* $t \in [-1, 1]$, θ *equals one of the values of the multivalued function* $\frac{1}{\pi} \text{Arctan} \frac{b(t)}{a(t)}$, *and such that*

$$(4.7) \qquad -1 < \theta(-1) \leq 0 \ .$$

Thus $\theta(t) = \frac{1}{\pi} \arctan \frac{b(t)}{a(t)} + N(t)$, where N is a possibly discontinuous function of t which takes only integral values. Using the identity

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \pi/2 & , \ x > 0 \\ -\pi/2 & , \ x < 0 \ , \end{cases}$$

we can also write

$$\theta(t) = -\frac{1}{\pi} \arctan \frac{a(t)}{b(t)} + N(t) + \begin{cases} \tfrac{1}{2} & \text{if } \frac{b(t)}{a(t)} > 0 \\ -\tfrac{1}{2} & \text{if } \frac{b(t)}{a(t)} < 0 \ , \end{cases}$$

which is useful if t is near a zero of a. We note that since $a, b \in H$, then $\theta \in H$.

If we define

$$(4.8) \qquad r(t) = [a^2(t) + b^2(t)]^{\frac{1}{2}} \ , \ -1 \leq t \leq 1 \ ,$$

then we can write

$$(4.9) \qquad a(t) + i\, b(t) = r(t) \exp [\pi\, i\, \theta(t)] \ , \ -1 \leq t \leq 1 \ ,$$

on choosing the positive sign of the square root in (4.2). Hence, (4.7), (4.9) imply that

$$(4.10) \qquad b(-1) \le 0 \; .$$

It is important that this condition is satisfied; thus, it may be necessary to multiply (4.1) by -1. It would be possible to remove this restriction by changing (4.7), but this will complicate later equations, for example (4.13).

From (4.9) we obtain $a(t) - i\,b(t) = r(t) \exp[-\pi i\,\theta(t)]$, and so (4.6) can be written

$$(4.11) \qquad G(t) = \exp[-2\pi i\,\theta(t)] \; , \; -1 \le t \le 1 \; .$$

It was specified in Lemma 3.1 that log G be continuous on [-1, 1]; thus we choose that branch of the logarithm so that $\log G(t) = -2\pi i\,\theta(t)$, $-1 \le t \le 1$. From (3.3) and (3.4), it follows that κ and σ must satisfy

$$(4.12) \qquad \kappa + \sigma = 0$$

and

$$(4.13) \qquad \kappa = -[\theta(1)] \; ,$$

where $[x]$ denotes the greatest integer not exceeding x.

From (3.2), the canonical function is given by

$$(4.14) \qquad X(z) = (1-z)^{-\kappa} \exp\left\{ - \int_{-1}^{1} \frac{\theta(\tau)}{\tau - z}\, d\tau \right\} , \quad z \notin [-1, 1] \; .$$

We note that it follows from Lemma 2.1 that

$$(4.15) \qquad X(z) = (-z)^{-\kappa} + O(z^{-\kappa-1}) \quad \text{as} \quad z \to \infty \; .$$

With these definitions, the solution of the Riemann boundary problem (4.5) is, by (3.6):

$$(4.16) \qquad \Phi(z) = X(z)\left\{ \frac{1}{2\pi i} \int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)} \frac{d\tau}{\tau - z} + P_{\kappa-1}(z)\right\}, z \notin [-1, 1] ,$$

where $P_{\kappa-1}$ is an arbitrary polynomial of degree κ-1 if κ>0, and identically zero otherwise. If κ<0, then (4.16) is a solution of the Riemann problem if and only if

$$(4.17) \qquad \int_{-1}^{1} \frac{g(\tau)}{X^{+}(\tau)} \tau^{k-1} \, d\tau = 0 , \quad k = 1, 2, \ldots, -\kappa .$$

We define

$$(4.18) \qquad Z(t) = (1-t)^{-\kappa} \exp\left\{ - \int_{-1}^{1} \frac{\theta(\tau)}{\tau - t} d\tau \right\}, \quad -1<t<1 ,$$

and on applying the Sokhotski-Plemelj formulae (2.3) to (4.14), we have the useful expressions

$$(4.19) \qquad X^{+}(t) = \frac{a(t) - i\, b(t)}{r(t)} Z(t) , \; X^{-}(t) = \frac{a(t) + i\, b(t)}{r(t)} Z(t) ,$$

$$Z(t) = [X^{+}(t)\, X^{-}(t)]^{\frac{1}{2}} , \quad -1<t<1 .$$

Using (2.3) again, we have from (4.4) that $\phi(t) = \Phi^{+}(t) - \Phi^{-}(t)$, $-1<t<1$, and after some algebra we obtain the solution of (4.1), which we present as a theorem.

Theorem 4.1 *The solution in the class* H* *of the singular integral equation*

$$(4.1) \qquad a(t)\phi(t) + \frac{b(t)}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t} d\tau = f(t) , \quad -1<t<1 ,$$

where a, b, f $\in$ H *and* $a^2(t) + b^2(t) \neq 0$ *for* $-1\leq t\leq 1$, *is*

$$(4.20) \qquad \phi(t) = \frac{a(t)\, f(t)}{r^2(t)} - \frac{b(t)\, Z(t)}{\pi\, r(t)} \int_{-1}^{1} \frac{f(\tau)}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - t} +$$

$$+ \frac{b(t)\, Z(t)}{r(t)} P_{\kappa-1}(t), \quad -1<t<1 .$$

If the index κ *is negative, then from* (4.6), (4.17) *and* (4.19) *we have that* (4.1) *is solvable if and only if*

$$(4.21) \qquad \int_{-1}^{1} \frac{f(\tau)}{r(\tau)\, Z(\tau)}\, \tau^{k-1}\, d\tau = 0 \ , \quad k = 1, 2, \ldots, -\kappa \ ,$$

the solution being given by (4.20).

Since in deriving (4.20), we assumed that (4.1) had a solution, we will show that (4.20) satisfies (4.1). We note that Muskhelishvili [32] and Gakhov [16] did not prove this, and Tricomi [44, §4.4] proved this in the case b = constant only (i.e., index = 1).

First, we need the following lemma.

<u>Lemma 4.1</u> *Let* χ_1 *be the polynomial of degree* $-\kappa$ *which satisfies*

$$(4.22) \qquad \lim_{z\to\infty} \{ \chi_1(z) - X(z) \} = 0 \ . \quad \textit{If } \kappa > 0, \textit{ we choose } \chi_1 \equiv 0 \ .$$

Then

$$(4.23) \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{b(\tau)\, Z(\tau)}{r(\tau)} \, \frac{d\tau}{\tau - t} = - \frac{a(t)\, Z(t)}{r(t)} + \chi_1(t) \quad , \quad -1<t<1 \ .$$

Also, let $P_{\kappa-1}$ *be an arbitrary polynomial of degree* $\kappa-1$ *(as above); then for* $\kappa > 0$,

$$(4.24) \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{b(\tau)\, Z(\tau)}{r(\tau)} \, \frac{P_{\kappa-1}(\tau)}{\tau - t}\, d\tau = - \frac{a(t)\, Z(t)\, P_{\kappa-1}(t)}{r(t)} \quad , \quad -1<t<1 \ .$$

<u>Proof</u> Define $Q = X - \chi_1$, with χ_1 given by (4.22), and X as in (4.14). Then by (4.19), $Q^{\pm}(t) = \dfrac{a(t) \mp i\, b(t)}{r(t)}\, Z(t) - \chi_1(t)$, $-1<t<1$, and since Q is zero at infinity, on applying Lemma 2.7, we obtain (4.23). To prove (4.24), proceed similarly with $Q = X\, P_{\kappa-1}$. #

The polynomial χ_1 may be constructed using the methods of §10, below.

We now prove that (4.20) satisfies (4.1).

Proof that (4.20) satisfies (4.1) First, it follows immediately from (4.24) that, for $\kappa > 0$, $\frac{bZ\,P_{\kappa-1}}{r}$ satisfies the homogenous equation $a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = 0$. Secondly, substituting (4.20) in the left side of (4.1), we obtain

$$a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = \frac{a^2(t)\,f(t)}{r^2(t)} -$$

$$- \frac{a(t)b(t)Z(t)}{\pi\, r\,(t)}\int_{-1}^{1} \frac{f(\tau)}{r(\tau)Z(\tau)}\,\frac{d\tau}{\tau - t} + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{a(\tau)f(\tau)}{r^2(\tau)}\,\frac{d\tau}{\tau - t} -$$

$$- \frac{b(t)}{\pi}\int_{-1}^{1} \frac{b(\tau)Z(\tau)}{r(\tau)(\tau-t)}\,\frac{1}{\pi}\int_{-1}^{1} \frac{f(s)}{r(s)Z(s)}\,\frac{ds}{s - \tau}\,d\tau \;.$$

Reversing the order of the double singular integral using Lemma 2.4, and then using (4.23), we obtain

$$(4.25)\quad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = f(t) +$$

$$+ \frac{b(t)}{\pi}\int_{-1}^{1} \frac{f(\tau)}{r(\tau)Z(\tau)}\,\frac{\chi_1(\tau) - \chi_1(t)}{\tau - t}\,d\tau\,, \quad -1<t<1\;.$$

Now, if the index $\kappa \geq 0$, then $\frac{\chi_1(\tau) - \chi_1(t)}{\tau - t} \equiv 0$, (since $\chi_1 \equiv 0$ if $\kappa > 0$, and χ_1 is constant if $\kappa = 0$), and so (4.20) clearly satisfies (4.1). If $\kappa < 0$, then $\frac{\chi_1(\tau) - \chi_1(t)}{\tau - t}$ is a polynomial in τ of degree $-\kappa-1$; if the consistency condition (4.21) is satisfied, then the integral on the right side of (4.25) is zero, and so (4.20) again satisfies (4.1). #

We observe that if the consistency condition (4.21) is not satisfied, then (4.20) satisfies the equation (4.25), which is (4.1) with a modified right side.

In Muskhelishvili's notation [32, §79], (4.20) is the solution of the class h_0 , and is the general solution of (4.1).

We note that Khvedelidze [25], [26] has shown that Theorem 4.1 holds (almost everywhere) if a, b and f are continuous, but not necessarily Hölder continuous.

We now consider the function θ and the calculation of the index in more detail.

If a and b are differentiable, then we can derive a useful expression for θ as follows. From Definition 4.1, $\theta(t) = \frac{1}{\pi} \operatorname{Arctan} \frac{b(t)}{a(t)}$; differentiating and integrating, we obtain

$$\theta(t) = \theta(t_0) + \frac{1}{\pi} \int_{t_0}^{t} \frac{a(\tau)b'(\tau) - a'(\tau)b(\tau)}{a^2(\tau) + b^2(\tau)} d\tau \ , \quad -1 \le t,\ t_0 \le 1 \ . \tag{4.26}$$

It is clear that this expression holds, in particular, near those points at which the function a changes sign; thus, despite the change of branch of Arctan, θ is continuous and differentiable, provided that a and b are differentiable.

The following lemma provides a simple method of calculating the index without constructing the function θ.

Let b have μ zeros on [-1, 1], and λ zeros $\beta_1, \beta_2, \ldots, \beta_\lambda$ in the open interval (-1, 1), i.e. $\beta_i \neq \pm 1$, $i = 1, 2, \ldots, \lambda$. Define ϵ_i, $i = 1, 2, \ldots, \lambda$, γ^+ and γ^- by

$$\epsilon_i = \begin{cases} 1 & \text{if } b/a \text{ is increasing at } \beta_i \text{ , and b changes sign there} \\ -1 & \text{if } b/a \text{ is decreasing at } \beta_i \text{ , and b changes sign there} \\ 0 & \text{if b does not change sign at } \beta_i \end{cases}$$

$$\gamma^{\pm} = \begin{cases} 1 \text{ if b has a zero at } \pm 1 \text{ and b/a is increasing there} \\ 0 \text{ otherwise} \end{cases}$$

Lemma 4.2 *The index* κ *of the dominant equation* (4.1) *and of its associated Riemann problem* (4.5) *is*

$$\kappa = 1 - \gamma^{+} - \gamma^{-} - \sum_{i=1}^{\lambda} \epsilon_i$$

Proof Using (4.9), it is easy to show that if b/a is increasing at β_i, then $[\theta(\beta_i+)] = [\theta(\beta_i-)] + 1$, and if b/a is decreasing at β_i, then $[\theta(\beta_i+)] = [\theta(\beta_i-)] - 1$, if b changes sign. If b does not change sign at β_i, then $[\theta(\beta_i+)] = [\theta(\beta_i-)]$. Now, if b has a zero at the point +1, then

$$[\theta(1)] = [\theta(1-)] + \begin{cases} 1 \text{ if b/a is increasing at 1-} \\ 0 \text{ if b/a is decreasing at 1-} \end{cases}$$

Then
$$\begin{aligned}[\theta(1)] &= [\theta(1-)] + \gamma^{+} \\ &= [\theta(-1+)] + \gamma^{+} + \sum_{i=1}^{\lambda} \epsilon_i \\ &= -1 + \gamma^{+} + \gamma^{-} + \sum_{i=1}^{\lambda} \epsilon_i \quad \text{using (4.7),}\end{aligned}$$

and the lemma follows on applying (4.13). #

Example Choose $a(t) = -t$, $b(t) = \frac{1}{4} - t^2$. At the point -1, we have $\theta(-1) = (1/\pi)$ Arctan (-3/4), and so by (4.7) we must choose

$$\theta(-1) = -(1/\pi) \arctan (3/4) = -0.204833 .$$

Thus

(4.27) $\quad \theta(t) = (1/\pi) \arctan [(t^2 - \frac{1}{4})/t] \quad$ for $\quad -1 \le t < 0$.

At $t = 0$, a has a zero, and so we pass to another branch:

(4.28) $\quad \theta(t) = 1 + (1/\pi) \arctan [(t^2 - \frac{1}{4})/t] \quad$ for $\quad 0 < t \le 1$.

We can also derive

(4.29) $\theta(t) = \frac{1}{2} - (1/\pi)\arctan[t/(t^2 - \frac{1}{4})]$, $-\frac{1}{2}<t<\frac{1}{2}$.

Alternatively, using (4.26) and choosing $t_0 = -1$, we obtain

$$\theta(t) = \theta(-1) + (2/\pi)\arctan 2 + (2/\pi)\arctan(2t) .$$

As above, we have $\theta(-1) = -(1/\pi)\arctan(3/4)$, which gives

(4.30) $\theta(t) = \frac{1}{2} + (2/\pi)\arctan(2t)$, $-1\le t\le 1$.

Of course, (4.27) - (4.30) coincide over common intervals of definition.

It is useful to draw graphs of $b(t)/a(t)$ against t, and θ against b/a, to ensure that θ is constructed correctly. If b/a is an increasing function, then θ will be too, and vice versa.

CHAPTER II

THE ANALYTIC EVALUATION OF CAUCHY PRINCIPAL VALUE INTEGRALS

§5. Introduction

While (4.20) gives the solution of the dominant singular integral equation (4.1), it is not very useful for numerical work, due to the difficulty of evaluating the Cauchy principal value integral of f/rZ. This numerical difficulty is due to the singularities of Z at ± 1. Let us illustrate this by an example.

Example 5.1 Let $a(t) = \cos\pi(\alpha+\beta t)$, $b(t) = -\sin\pi(\alpha+\beta t)$, and assume for simplicity that $-1<\beta-\alpha\leq 0$, $-1<\beta+\alpha\leq 0$. Then from Definition 4.1, $\theta(t) = -\alpha-\beta t$, and by (4.13) the index is zero. From (4.18) we obtain (in this case)

$$Z(t) = e^{2\beta}\left(\frac{1-t}{1+t}\right)^{\alpha+\beta t} \quad , \quad -1<t<1 .$$

However, the mathematical techniques required for the analytic evaluation of singular integrals involving the function Z are available in Muskhelishvili [32] and Tricomi [43]. For example, we have already used these methods in Lemma 4.1.

In §6 we present two theorems which provide methods for the analytic evaluation of Cauchy principal value integrals involving Z.

In §7 we use these theorems to evaluate the singular integral of f/rZ in various ways, and thus give alternative analytic solutions of the dominant equation (4.1).

§6. Cauchy principal value integrals of Z

In this section, we give two theorems which provide a method for the analytic evaluation of Cauchy principal value integrals which involve the function Z. We also give some simple examples.

First, we recall the definitions of Z, see (4.18), X, (4.14) and r (4.8).

Let Q be meromorphic, that is a function whose only singularities in the finite plane are poles. Q may only have poles on [-1, 1] if

$$bQZ/r \in H^* . \tag{6.1}$$

Thus the poles of Q on [-1, 1] must coincide with the zeros of b, and the order of these poles of Q must be no greater than the order of the corresponding zeros of b.

Let R be a meromorphic function, with its poles lying at the poles of Q, such that QX - R is analytic in the finite plane excluding [-1, 1], and such that (on [-1, 1])

$$aQZ/r - R \in H^* . \tag{6.2}$$

Let Ω be an entire function, that is a function whose only singularity is at infinity, such that QX - R - Ω is zero (and analytic) at infinity.

The following two theorems are applications of Lemma 2.7, but because of their importance, we present them as theorems.

Theorem 6.1 *With* Q, R *and* Ω *as above, then*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)Q(\tau)Z(\tau)}{r(\tau)(\tau-t)}\, d\tau = -\frac{a(t)Q(t)Z(t)}{r(t)} + R(t) + \Omega(t) \quad , \ -1<t<1. \tag{6.3}$$

Proof Consider the function F, defined by

$$F(z) = Q(z)X(z) - R(z) - \Omega(z) \quad , \quad z \notin [-1, 1] .$$

By (4.19), its limiting values on $[-1, 1]$ are

$$F^{\pm}(t) = Q(t)Z(t)\,\frac{a(t) \mp ib(t)}{r(t)} - R(t) - \Omega(t) \ , \ -1<t<1 .$$

Because of the way Q, R and Ω are chosen, F is analytic in the complex plane excluding $[-1, 1]$, and is zero at infinity. By (6.1) and (6.2), its limiting values $F^{\pm} \in H^*$. Thus we can apply Lemma 2.7, and obtain (6.3). #

Let Q be as above, except that we replace (6.1) by

$$\frac{bQ}{rZ} \in H^* , \tag{6.4}$$

Let S be a meromorphic function, with its poles lying at the poles of Q, such that Q/X - S is analytic in the complex plane excluding $[-1, 1]$, and

$$\frac{aQ}{rZ} - S \in H^* . \tag{6.5}$$

Let χ be an entire function such that $Q/X - S - \chi$ is zero at infinity.

Theorem 6.2 *With* Q, S *and* χ *as above, then*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)Q(\tau)}{r(\tau)Z(\tau)}\,\frac{d\tau}{\tau - t} = \frac{a(t)Q(t)}{r(t)Z(t)} - S(t) - \chi(t) \ , \ -1<t<1 .$$

Proof Define the function G by

$$G(z) = Q(z)/X(z) - S(z) - \chi(z) \ , \quad z \notin [-1, 1] \ ,$$

and proceed as in the previous proof. #

In practice, the function Q is specified, and then R and Ω are determined (or S and χ).

We need to consider the existence of the functions R, Ω, S and χ. It is sufficient to consider only R and Ω.

If Q has an infinity of poles, then the only possible limit point of these poles is at infinity. Then Q will have an essential singularity at infinity, and Ω can be chosen to be the principal part in the Laurent expansion of $QX - R$ at infinity. R can be constructed using Weierstrass's factor theorem - see, e.g. Copson [10, §7.2].

In practice, Q will have a finite number of poles and perhaps a pole at infinity. Then Q will be a rational function, R will also be rational, and Ω will be a polynomial.

Further generalizations of the above two theorems are possible; for example, we could allow Q to have a cut on $[-1, 1]$, e.g. $Q(z) = (z^2 - 1)^{\frac{1}{2}}$.

Example 6.1 With a, b as in Example 5.1, we choose $Q \equiv 1$, and thus $R \equiv 0$. Since the index is zero, it follows from (4.15) that $\lim_{z\to\infty} X(z) = 1$, and hence we choose $\Omega \equiv 1$. Then Theorem 6.1 gives the singular integral

$$\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-\tau}{1+\tau}\right)^{\alpha+\beta\tau}\frac{\sin\pi(\alpha+\beta\tau)}{\tau-t}\,d\tau = \cos\pi(\alpha+\beta t)\cdot\left(\frac{1-t}{1+t}\right)^{\alpha+\beta t} -$$

$$- e^{-2\beta} \quad , \quad -1<t<1 \ .$$

It can be shown that this result holds provided that $[(1-t)/(1+t)]^{\alpha+\beta t}\sin\pi(\alpha+\beta t) \in H^*$

As further applications of these Theorems, we present the following two lemmas, which do not appear to have been published previously.

In these lemmas, we assume that the singular integral equation

$$a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau-t}\,d\tau = f(t) \ , \quad -1<t<1 \ , \tag{6.6}$$

is soluble; then under the conditions of Theorem 4.1, its solution is

$$\phi(t) = \frac{a(t)f(t)}{r^2(t)} - \frac{b(t)Z(t)}{\pi r(t)}\int_{-1}^{1}\frac{f(\tau)}{r(\tau)Z(\tau)}\frac{d\tau}{\tau-t} + \frac{b(t)Z(t)}{r(t)}P_{\kappa-1}(t) \ , \quad -1<t<1 \ . \tag{6.7}$$

<u>Lemma 6.1</u> *Let* Δ_1 *be a polynomial of degree* $n-\kappa$ *which satisfies*

$$\lim_{z\to\infty}\left\{\Delta_1(z) - z^n X(z)\right\} = 0 \ , \tag{6.8}$$

where n *is an integer. If* $n-\kappa<0$, *put* $\Delta_1 \equiv 0$.

Then

$$\int_{-1}^{1}\phi(\tau)\tau^n\,d\tau = \int_{-1}^{1}\left(\frac{f(\tau)\Delta_1(\tau)}{Z(\tau)} + \tau^n\, b(\tau)Z(\tau)P_{\kappa-1}(\tau)\right)\frac{d\tau}{r(\tau)} \ . \tag{6.9}$$

Proof Choose $Q(z) = z^n$, and $\Omega = \Delta_1$. Since Q has no poles, put $R \equiv 0$. By (6.8), $QX - \Delta_1$ is zero at infinity, and so by Theorem 6.1 we obtain

$$(6.10)\quad \frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)Z(\tau)\ \tau^n}{r(\tau)}\ \frac{d\tau}{\tau - t} = \frac{a(t)Z(t)\ t^n}{r(t)} + \Delta_1(t)\ ,\ -1<t<1\ .$$

Substituting (6.7) in the left side of (6.9), we have

$$\int_{-1}^{1} \phi(\tau)\tau^n\, d\tau = \int_{-1}^{1}\left(\frac{a(\tau)f(\tau)}{r(\tau)} - \frac{b(\tau)Z(\tau)}{\pi}\int_{-1}^{1} \frac{f(s)}{r(s)Z(s)}\ \frac{ds}{s - \tau} + \right.$$
$$\left. + b(\tau)Z(\tau)\ P_{\kappa-1}(\tau)\right) \frac{\tau^n}{r(\tau)}\, d\tau\ .$$

Inverting the double integral using Lemma 2.5, and using (6.10) to evaluate the resulting singular integral, we obtain (6.9), thus proving the lemma. #

Lemma 6.2 *Let* Δ_2 *be a polynomial of degree* $n + \kappa$ *such that*

$$(6.11)\qquad \lim_{z\to\infty} \left\{ \Delta_2(z) - z^n X^{-1}(z)\right\} = 0\ .$$

If $n + \kappa < 0$, *we choose* $\Delta_2 \equiv 0$.

Then

$$(6.12)\qquad \int_{-1}^{1} \frac{f(\tau)\ \tau^n}{r(\tau)Z(\tau)}\, d\tau = \int_{-1}^{1} \phi(\tau)\ \Delta_2(\tau)d\tau\ .$$

Proof Choose $Q(z) = z^n$, $\chi = \Delta_2$, $S \equiv 0$. From Theorem 6.2 we have

$$(6.13)\qquad \frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)\ \tau^n}{r(\tau)Z(\tau)}\ \frac{d\tau}{\tau - t} = \frac{a(t)\ t^n}{r(t)Z(t)} - \Delta_2(t)\ ,\ -1<t<1\ .$$

Substituting (6.6) in the left side of (6.12), using Lemma 2.5, and (6.13), we obtain (6.12), thus proving the lemma. #

For example, if the index κ is negative, by putting successively $n = 0, 1, \ldots, -\kappa-1$ in Lemma 6.2 we have in each case $\Delta_2 \equiv 0$, and thus obtain

$$\int_{-1}^{1} \frac{f(\tau)\ \tau^n}{r(\tau)Z(\tau)}\, d\tau = 0 \quad , \ n = 0,\ 1, \ldots,\ -\kappa-1 \ , \tag{6.14}$$

which is the consistency condition (4.21). Thus, by assuming that (6.6) had the solution (6.7) and that the index was negative, we have shown that (6.14) must be satisfied.

§7. Alternative forms of the solution of the dominant equation

In this section, we show how Theorem 6.2 may be used to write the solution of the dominant singular integral equation (4.1) in more useful forms, which do not involve Cauchy principal value integrals of the function Z.

Theorem 7.1 *Let* χ_2 *be the polynomial of degree* κ *which satisfies*

$$(7.1) \qquad \lim_{z\to\infty}\left\{ \chi_2(z) - X^{-1}(z)\right\} = 0 \ . \ (\mathit{If}\ \kappa<0\ ,\ \chi_2 \equiv 0).$$

Then under the conditions of Theorem 4.1, *the solution of the dominant singular integral equation*

$$(7.2) \qquad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = f(t) \quad , \quad -1<t<1$$

can be written as

$$(7.3) \quad \phi(t) = \frac{Z(t)}{r(t)}\left\{f(t)\chi_2(t) - \frac{1}{\pi}\int_{-1}^{1} \frac{f(\tau)b(t) - f(t)b(\tau)}{\tau - t}\ \frac{d\tau}{r(\tau)Z(\tau)} + \right.$$
$$\left. + \ b(t)\ P_{\kappa-1}(t)\right\} , \quad -1<t<1 \ .$$

Proof Adding and subtracting a term, we can rewrite the solution (4.20) of (7.2) as

$$(7.4) \quad \phi(t) = \frac{a(t)f(t)}{r^2(t)} - \frac{Z(t)}{\pi r(t)}\left\{ f(t)\int_{-1}^{1} \frac{b(\tau)}{r(\tau)Z(\tau)}\ \frac{d\tau}{\tau - t} + \right.$$
$$\left. + \int_{-1}^{1} \frac{f(\tau)b(t) - b(\tau)f(t)}{\tau - t}\ \frac{d\tau}{r(\tau)Z(\tau)}\right\} +$$
$$+ \ \frac{b(t)Z(t)}{r(t)}\ P_{\kappa-1}(t) \ .$$

To evaluate the singular integral, we put $Q \equiv 1$, $\chi = \chi_2$, $S \equiv 0$ in Theorem 6.2 to obtain

$$\frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - t} = \frac{a(t)}{r(t)Z(t)} - \chi_2(t) \ , \ -1<t<1 \ .$$

where χ_2 satisfies (7.1). Substituting this in (7.4) and cancelling, we obtain (7.3), thus proving the theorem. #

We note that (7.3) can also be written as

$$(7.5) \quad \phi(t) = \frac{Z(t)}{r(t)}\left\{ f(t)\chi_2(t) + \frac{f(t)}{\pi}\int_{-1}^{1} \frac{b(\tau) - b(t)}{\tau - t} \frac{d\tau}{r(\tau)Z(\tau)} - \right.$$

$$\left. - \frac{b(t)}{\pi}\int_{-1}^{1} \frac{f(\tau) - f(t)}{\tau - t} \frac{d\tau}{r(\tau)Z(\tau)} + b(t)\, P_{\kappa-1}(t) \right\} , \quad -1<t<1 \ ,$$

which we will use in §8.

In the next two theorems, we assume that b_m is a polynomial of degree m, with μ zeros, which are at the points β_i, and are of multiplicity α_i , $i = 1, 2,\ldots, \mu$. Thus we have

$$(7.6) \qquad b_m(z) = \gamma \prod_{i=1}^{\mu} (z-\beta_i)^{\alpha_i} \ , \quad m = \sum_{i=1}^{\mu} \alpha_i \ , \quad \gamma = \text{constant} \ .$$

Theorem 7.2 *Let* b_m *be given by* (7.6) . *Let* Ω_2 *be a polynomial of degree* κ-m *such that*

$$(7.7) \qquad \lim_{z\to\infty} \left\{ \Omega_2(z) - \frac{1}{b_m(z)X(z)} \right\} = 0 \ . \quad \textit{If } \kappa\text{-}m<0,\ \Omega_2 \equiv 0 \ .$$

Let R_2 *be a polynomial of degree* m-1 *such that*

$$(7.8) \quad \begin{aligned} &\frac{d^j}{dz^j}\left\{R_2(z) - \frac{1}{X(z)}\right\}_{z=\beta_i} = 0 \quad \textit{if} \quad \beta_i \notin [-1, 1] \\ &\frac{d^j}{dt^j}\left\{R_2(t) - \frac{a(t)}{r(t)Z(t)}\right\}_{t=\beta_i} = 0 \quad \textit{if} \quad \beta_i \in [-1, 1] \ , \end{aligned}$$

$j = 0, 1,\ldots, \alpha_i-1, \qquad i = 1, 2,\ldots, \mu.$ *If* m=0, *put* $R_2 \equiv 0$.

To ensure that R_2 ***exists**, we assume that the first* $\alpha_i - 1$ *derivativesof* a *are Hölder continuous in a neighbourhood of the points* β_i, *if* $\beta_i \in [-1, 1]$, $i = 1, 2, \ldots \mu$.

Then the solution of

$$(7.9) \qquad a(t)\phi(t) + \frac{b_m(t)}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t) \;, \; -1<t<1 \;,$$

is given by

$$(7.10) \quad \phi(t) = \frac{Z(t)}{r(t)} \Big\{ f(t)[R_2(t) + b_m(t)\Omega_2(t)] - $$
$$- \frac{b_m(t)}{\pi} \int_{-1}^{1} \frac{f(\tau) - f(t)}{\tau - t} \frac{d\tau}{r(\tau)Z(\tau)} + b_m(t)\, P_{\kappa-1}(t) \Big\} \,.$$

Proof The solution of (7.9) is given by (4.20); by adding and subtracting a term, it can be written as

$$(7.11) \quad \phi(t) = \frac{a(t)f(t)}{r^2(t)} - \frac{b_m(t)Z(t)}{r(t)} \Big\{ \frac{f(t)}{\pi} \int_{-1}^{1} \frac{1}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - t} + $$
$$+ \frac{1}{\pi} \int_{-1}^{1} \frac{f(\tau) - f(t)}{\tau - t} \frac{d\tau}{r(\tau)Z(\tau)} + P_{\kappa-1}(t) \Big\} \;, \quad -1<t<1 \,.$$

In Theorem 6.2 we choose $Q = 1/b_m$, $S = R_2/b_m$ and $\chi = \Omega_2$. Since b_m is a polynomial, and the first $\alpha_i - 1$ derivatives of a are Hölder continuous, it can be shown that the first $\alpha_i - 1$ derivatives of Z exist at β_i , if $\beta_i \in [-1, 1]$, $i = 1, 2, \ldots, \mu$. Also, since X is analytic in the complex plane excluding $[-1, 1]$, and has no zeros, then it can be shown that R_2 exists and satisfies (7.8). It is evident that the choices of Q and χ satisfy (6.4) and (6.5), and so from Theorem 6.2 we have

$$(7.12) \quad \frac{1}{\pi} \int_{-1}^{1} \frac{1}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - t} = \frac{1}{b_m(t)} \Big\{ \frac{a(t)}{r(t)Z(t)} - R_2(t) \Big\} - \Omega_2(t) \;,$$
$$-1<t<1 \,.$$

Substituting this in (7.11), we obtain (7.10), thus proving the theorem. #

To construct R_2, it may be necessary to differentiate the function Z, which involves a Cauchy principal value integral. These may be differentiated using the identity

$$(7.13)\quad \frac{d^n}{dt^n}\left\{\int_{-1}^{1} \frac{\theta(\tau)}{\tau - t}\, d\tau\right\} = \int_{-1}^{1} \frac{\theta^{(n)}(\tau)}{\tau - t}\, d\tau - \sum_{j=0}^{n-1}(-1)^{n-j}(n-j-1)!\ \times$$

$$\times \left\{\theta^{(j)}(1)(t-1)^{j-n} - \theta^{(j)}(-1)(t+1)^{j-n}\right\}, \quad -1<t<1 .$$

If derivatives of Z are required at or near +1, the following expression (which can be derived from (4.18)) is useful:

$$Z(t) = (1+t)^{-\kappa} \exp\left\{-\int_{-1}^{1} \frac{\theta(\tau)+\kappa}{\tau - t}\, d\tau\right\}, \quad -1<t<1 .$$

We now assume that f_n is a polynomial of degree n.

<u>Theorem 7.3</u> *Let* f_n *and* b_m *be polynomials of degree* n *and* m *respectively, with* b_m *as in* (7.6) . *Let* Ω_3 *be the polynomial of degree* $n+\kappa-m$ *such that*

$$(7.14)\qquad \lim_{z\to\infty}\left\{\Omega_3(z) - \frac{f_n(z)}{b_m(z)X(z)}\right\} = 0 . \quad \textit{If } n+\kappa-m<0\ ,\ \Omega_3 \equiv 0 .$$

Let R_3 *be a polynomial of degree* m-1 *such that*

$$\frac{d^j}{dz^j}\left\{R_3(z) - \frac{f_n(z)}{X(z)}\right\}_{z=\beta_i} = 0 \quad , \quad \beta_i \notin [-1, 1]$$

(7.15)

$$\frac{d^j}{dt^j}\left\{R_3(t) - \frac{a(t)f_n(t)}{r(t)Z(t)}\right\}_{t=\beta_i} = 0 \quad , \quad \beta_i \in [-1, 1] ,$$

$j = 0, 1, \ldots, \alpha_i-1$, $i = 1, 2, \ldots, \mu$. *If* m=0, $R_3 \equiv 0$.

Again, we assume that the first $\alpha_i - 1$ derivatives of a *are Hölder continuous at β_i, if $\beta_i \in [-1, 1]$.*

Then the solution of the singular integral equation

$$(7.16) \qquad a(t)\phi(t) + \frac{b_m(t)}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f_n(t) \;, \quad -1<t<1 \;,$$

is given by

$$(7.17) \qquad \phi(t) = \frac{Z(t)}{r(t)} \left\{ R_3(t) + b_m(t)\Omega_3(t) + b_m(t)\, P_{\kappa-1}(t) \right\} \;, \quad -1<t<1 \;.$$

<u>Proof</u> The solution of (7.16) is given by (4.20); to evaluate the singular integral $\frac{1}{\pi}\int_{-1}^{1} \frac{f_n(\tau)}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - t}$ we choose $Q = f_n/b_m$, $\chi = \Omega_3$ and $S = R_3/b_m$, and obtain from Theorem 6.2

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f_n(\tau)}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - t} = \frac{1}{b_m(t)} \left\{ \frac{a(t)f_n(t)}{r(t)Z(t)} - R_3(t) \right\} - \Omega_3(t) \;, \quad -1<t<1 \;.$$

Substituting this in (4.20) we obtain (7.17). #

The construction of R_3 and Ω_3 is discussed in §10; χ_2, Ω_2 and R_2 can be found similarly.

We note that singular integral equations of the form

$$(7.18) \qquad a(t)\phi(t) + \frac{Q_1(t)}{Q_2(t)} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = \frac{Q_3(t)}{Q_4(t)} \;, \quad -1<t<1 \;,$$

where Q_1, Q_2, Q_3 and Q_4 are polynomials (Q_2 and Q_4 have no zeros on $[-1, 1]$) can be solved using Theorem 7.3 by first multiplying (7.18) by Q_2Q_4. Alternatively (and equivalently), the singular integral $\int_{-1}^{1} \frac{Q_3(\tau)}{Q_4(\tau)r(\tau)Z(\tau)} \frac{d\tau}{\tau - t}$ can be evaluated using Theorem 6.2, the solution of (7.18) then being obtained using (4.20).

CHAPTER III

AN ALGORITHM FOR THE NUMERICAL SOLUTION OF THE DOMINANT EQUATION

§8. General description of the algorithm and proof of convergence

We will now give a numerical method for the approximate solution of the singular integral equation

$$(8.1) \qquad a(t)\varphi(t) + \frac{b_m(t)}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t) \;, \quad -1<t<1 \;.$$

We assume that b_m is a polynomial of degree m:

$$(8.2) \qquad b_m(t) = \gamma \prod_{i=1}^{\mu} (t - \beta_i)^{\alpha_i} \;, \quad \text{with} \quad m = \sum_{i=1}^{\mu} \alpha_i \;,$$

as in (7.6). If, in a given equation

$$(8.3) \qquad a^*(t)\phi(t) + \frac{b^*(t)}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f^*(t) \;, \quad -1<t<1 \;,$$

b^* is not a polynomial, or is a polynomial which has some or all its zeros not on [-1, 1] , then provided that the zeros of b^* on [-1, 1] are either simple or multiple (see examples below), we may multiply (8.3) by a function h, defined on [-1, 1] , such that $b_m = b^*h$ is a polynomial of degree m. If a^* and b^* are bounded on [-1, 1] , then h must be nonvanishing on [-1, 1]; otherwise b_m and $a = a^*h$ will vanish simultaneously, thus violating (4.3).

We emphasise that we are not approximating b^* by a polynomial. In the method below, we will be approximating f by a polynomial f_n , and thus it may be necessary to choose h carefully to ensure that

$f = f^*h$ can be approximated readily by f_n.

Example 8.1 Consider (8.3) with $b^*(t) = \sin(\pi t)(\exp(t)-1)$, $-1<t<1$. Then b^* has a double zero at $t = 0$, and simple zeros at ± 1. We choose $b_m(t) = t^2(1-t^2)$, and so $h = b_m/b^*$ will be non-vanishing on $[-1, 1]$. Multiplying (8.3) by h, we obtain

$$a^*(t)h(t)\phi(t) + \frac{t^2(1-t^2)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f^*(t)h(t) \ , \ -1<t<1 \ ,$$

which is now in the form (8.1).

Example 8.2 Suppose $b^*(t) = \exp(-1/|t|)$, $-1<t<1$. Since b^* has a zero of infinite order at $t = 0$, there does not exist a polynomial b_m such that $h = b_m/b^*$ is a bounded non-vanishing function on $[-1, 1]$. Consequently, in this case, we cannot transform (8.3) into the form (8.1), and the following algorithm is not applicable. A similar statement holds if $b^*(t) = (1+t)^{\frac{1}{2}}$, for example.

We return to the description of the algorithm.

We approximate to f by a polynomial f_n of degree n, and using Theorem 7.3, we solve

$$(8.4) \qquad a(t)\phi_n(t) + \frac{b_m(t)}{\pi}\int_{-1}^{1} \frac{\phi_n(\tau)}{\tau - t}\, d\tau = f_n(t) \ , \ -1<t<1$$

exactly, giving

$$(8.5) \qquad \phi_n(t) = \frac{Z(t)}{r(t)}\,[R_3(t) + b_m(t)\Omega_3(t) + b_m(t)\, P_{\kappa-1}(t)], \ -1<t<1 \ .$$

If the index is negative, the integrals (4.21) may be evaluated using the quadrature formula in §13. R_3 and Ω_3 can be found using the method of §9 and §10.

We note that if $\kappa>0$, and $n\leq m-1$ or if $\kappa\leq 0$ and $n\leq m-\kappa$, then it is not necessary to construct Ω_3, since it is of no higher degree than the arbitrary polynomial $P_{\kappa-1}$ (see (8.5)), to which it is added.

Examples of this method will be given in §11.

We now examine the convergence of ϕ_n to the exact solution ϕ of (8.1). First, we need the following results.

Definition 8.1 *Let* F_n *denote the space of all polynomials of degree not greater than* n. *Then, assuming that* $f \in H_\alpha$ *, we define*

$$E_n(f) = \min_{g_n \in F_n} \max_{-1 \le x \le 1} |f(x) - g_n(x)| .$$

It is well known that there exists a unique polynomial p_n say, for which $\max\limits_{-1\le x\le 1} |f(x) - p_n(x)| = E_n(f)$. The polynomial p_n is called the polynomial of best approximation to f.

The constants $A_1, A_2, \ldots, A_7$ used below are all positive and independent of n.

Lemma 8.1 *If* $f \in H_\alpha$ *, then* $E_n(f) \le A_3\, n^{-\alpha}$.

Proof Jackson's theorem; see [21].

Definition 8.2 *Let* $-1 \le x_0 < x_1 < \ldots < x_n \le 1$ *be a set of* n+1 *distinct points. We define the Lagrangian interpolation polynomial of degree* n *to be* $L_n(f; x) = \sum_{i=0}^{n} \ell_i(x) f(x_i)$, *where* ℓ_i *are polynomials of degree* n *such that* $\ell_i(x_j) = \delta_{ij}$, $i, j = 0, 1, \ldots, n$.

Lemma 8.2 *If we choose*

(8.6) $$x_i = \cos\left(\frac{\pi}{2} \cdot \frac{2i+1}{n+1}\right) \quad , \quad i = 0, 1, \ldots, n$$

then

$$\max_{-1\le x\le 1} \sum_{i=0}^{n} |\ell_i(x)| \le A_1 + A_2 \log n .$$

Proof See Rivlin [38, Thm 4.5]. #

Lemma 8.3 *With* x_i *as in* (8.6), *we have*

$$\max_{-1\le x\le 1} |f(x) - L_n(f;\, x)| \le E_n(f)\ (A_7 + A_2 \log n) ,$$

where $A_7 = A_1 + 1$.

Proof See Rivlin [38, Thm 4.1]. #

Lemma 8.4 *Let* $f \in H_\alpha$. *Let there be a polynomial* g_n *of degree* n *such that for any* n

$$\max_{-1\le x\le 1} |f(x) - g_n(x)| \le A_4\, n^{-\alpha} .$$

Then

$$\sup_{-1\le x,s\le 1} \frac{|\delta_n(x) - \delta_n(s)|}{|x - s|^{\beta}} \le \frac{A_5}{n^{\alpha-2\beta}} , \qquad x \ne s ,$$

where $\delta_n(x) = f(x) - g_n(x)$, $-1\le x\le 1$, *and* $0<2\beta<\alpha$.

Proof See Kalandiya [22] . #

Definition 8.3 *With* r *and* Z *as above, and* $f \in H$, *define the operator* I *by*

$$I(f;\, t) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(\tau) - f(t)}{\tau - t} \frac{d\tau}{r(\tau)Z(\tau)} , \quad -1\le t\le 1 .$$

Lemma 8.5 *If* $f \in H_\alpha$, *then* I *is bounded on* [-1, 1] .

Proof We have

$$|I(f;\ t)| \le \sup_{-1\le x,s\le 1} \frac{|f(x) - f(s)|}{|x - s|^{\alpha}} \max_{-1\le t\le 1} \frac{1}{\pi}\int_{-1}^{1} \frac{|\tau - t|^{\alpha-1}}{r(\tau)Z(\tau)}\, d\tau \ ,$$

$$-1\le t\le 1 \ .$$

Since $f \in H_\alpha$ the first quotient is bounded. Also, since rZ is nonzero in $[-1, 1]$, it follows that the integral exists for $\alpha>0$, and thus I is bounded. #

We can now prove that the solution ϕ_n of (8.4) converges to the solution of (8.1). Since (8.5) is an exact solution of (8.4), we can use (7.5) and Definition 8.3 to write the solution of (8.4) in the form

$$(8.7)\quad \phi_n(t) = \frac{Z(t)}{r(t)}\Big\{f_n(t)[\chi_2(t) + I(b_m;t)] - b_m(t)I(f_n;t) + $$

$$+ b_m(t)\ P_{\kappa-1}(t)\Big\} \ , \quad -1<t<1.$$

Similarly, the exact solution of (8.1) is

$$(8.8)\quad \phi(t) = \frac{Z(t)}{r(t)}\Big\{f(t)[\chi_2(t) + I(b_m;t)] - b_m(t)I(f;\ t) + $$

$$+ b_m(t)\ P_{\kappa-1}(t)\Big\} \ , \quad -1<t<1 \ .$$

Subtracting these and defining

$$(8.9)\quad \delta_n = f - f_n \ , \quad \psi = r\phi/Z \ , \quad \psi_n = r\phi_n/Z \ ,$$

we have

$$(8.10)\quad \psi(t) - \psi_n(t) = \delta_n(t)[\chi_2(t) + I(b_m;t)] - b_m(t)I(\delta_n;t) \ ,$$

$$-1<t<1 \ ,$$

where we have assumed that the arbitrary polynomial $P_{\kappa-1}$ is the same in (8.7) and (8.8).

<u>Theorem 8.1</u> *If* $f \in H_\zeta$ *and* f_n *is chosen so that* $f_n(x_i) = f(x_i)$,

where $x_i = \cos\left(\frac{\pi}{2}\frac{2i+1}{n+1}\right)$, $i = 0, 1, \ldots, n$, *then the solution* ϕ_n *of* (8.4) *converges to the solution* ϕ *of* (8.1) *in the sense that*

$$\lim_{n\to\infty} \max_{-1\le x\le 1} |\psi(x) - \psi_n(x)| = 0 .$$

<u>Proof</u> Since $f \in H_\zeta$, then by Lemma 8.3 and Lemma 8.1 we have

$$\max_{-1\le x\le 1} |\delta_n(x)| \le \frac{A_3}{n^\zeta}(A_7 + A_2 \log n) \le \frac{A_6}{n^\alpha} , \quad \text{where} \quad 0<\alpha<\zeta .$$

Then, since f is also in the class H_α , Lemma 8.4 is applicable.

From (8.10) we have

$$\max_{-1\le t\le 1} |\psi(t) - \psi_n(t)| \le \max_{-1\le t\le 1} |\delta_n(t)| \cdot \max_{-1\le t\le 1} |\chi_2(t) + I(b_m;t)| +$$

$$+ \max_{-1\le t\le 1} |b_m(t)| \cdot \max_{-1\le t\le 1} |I(\delta_n;t)| .$$

Now

$$\max_{-1\le t\le 1} |I(\delta_n;t)| \le \sup_{-1\le x,s\le 1} \frac{|\delta_n(x) - \delta_n(s)|}{|x - s|^\beta} \max_{-1\le t\le 1} \frac{1}{\pi}\int_{-1}^{1} \frac{|\tau - t|^{\beta-1}}{r(\tau)Z(\tau)} d\tau ,$$

$$x \neq s ,$$

where $0<2\beta<\alpha$. We showed in Lemma 8.5 that the above integral is bounded. Applying Lemma 8.4, we have

$$\max_{-1\le t\le 1} |I(\delta_n;t)| \le \frac{A_5}{n^{\alpha-2\beta}} .$$

Hence

$$\max_{-1\le t\le 1} |\psi(t) - \psi_n(t)| \le \frac{A_6}{n^{\zeta}} \max_{-1\le t\le 1} |\chi_2(t) + I(b_m;t)| + $$
$$+ \frac{A_5}{n^{\alpha-2\beta}} \max_{-1\le t\le 1} |b_m(t)| \ .$$

Since I, χ_2 and b_m are bounded and independent of n, and $0<2\beta<\alpha$, on taking the limit, the theorem follows. #

§9. Expansions of X and X^{-1}

In §8, and later in this thesis, we require expansions of X and X^{-1} in descending powers of z.

From (4.14) we have

$$(9.1)\qquad \begin{aligned} X(z) &= (1 - z)^{-\kappa} \exp\left(-\int_{-1}^{1} \frac{\theta(\tau)}{\tau - z}\, d\tau\right) \\ X^{-1}(z) &= (1 - z)^{\kappa} \exp\left(\int_{-1}^{1} \frac{\theta(\tau)}{\tau - z}\, d\tau\right) . \end{aligned} \qquad z \notin [-1, 1]$$

We will consider the expansion for X in detail; the expansion for X^{-1} will then follow analogously.

Using the expansion

$$(9.2)\qquad \frac{1}{\tau - z} = -\frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{\tau}{z}\right)^{k} , \quad \left|\frac{\tau}{z}\right| < 1 ,$$

and defining the moments of θ to be

$$(9.3)\qquad C_k = \int_{-1}^{1} \theta(\tau)\tau^{k}\, d\tau , \quad k = 0, 1, \ldots ,$$

we can write (9.1) as

$$(9.4)\qquad X(z) = (1 - z)^{-\kappa} \exp\left(\sum_{k=0}^{\infty} C_k\, z^{-k-1}\right) , \quad |z| > 1.$$

The moments C_k , $k = 0, 1, \ldots, N$ say are found as accurately as possible. We then define the coefficients $bb_k^{(m)}$ (bb denoting a single symbol) by

$$(9.5)\qquad \sum_{k=0}^{\infty} bb_k^{(m)}\, z^{-k} = \exp\left(\sum_{k=0}^{m} C_k\, z^{-k-1}\right) , \quad m = 0, 1, \ldots, N,$$

Then it follows that

(9.6) $$bb_k^{(0)} = C_0{}^k/k! \quad , \quad k = 0, 1, \ldots \quad ,$$

and for $m > 0$,

$$\sum_{k=0}^{\infty} bb_k^{(m)} z^{-k} = \exp\left(\sum_{k=0}^{m-1} C_k z^{-k-1}\right) . \exp\left(C_m z^{-m-1}\right)$$

$$= \sum_{k=0}^{\infty} bb_k^{(m-1)} z^{-k} . \sum_{j=0}^{\infty} \frac{(C_m)^j}{j!} z^{-(m+1)j}$$

$$= \sum_{k=0}^{\infty} z^{-k} \sum_{j=0}^{\left[\frac{k}{m+1}\right]} bb_{k-(m+1)j}^{(m-1)} (C_m)^j/j!$$

Hence we can write $bb_k^{(m)}$ in terms of $bb_k^{(m-1)}$:

(9.7) $$bb_k^{(m)} = \sum_{j=0}^{[k/(m+1)]} bb_{k-(m+1)j}^{(m-1)} (C_m)^j/j! \ , \ k = 0, 1, \ldots; \quad m = 1, 2, \ldots, N .$$

Thus the $bb_k^{(m)}$ can be found using (9.6) and (9.7).

If we define the e_k^* by

(9.8) $$\sum_{k=0}^{\infty} e_k^* z^{-k} = \exp\left(\sum_{k=0}^{\infty} C_k z^{-k-1}\right) ,$$

then from (9.7) we see that $e_k^* = bb_k^{(N)}$, $k = 0, 1, \ldots, N+1$, and so in practice we only need to calculate $bb_k^{(j)}$, $k = 0, 1, \ldots, N+1$, $j = 0, 1, \ldots, N$ to obtain e_k^* , $k = 0, 1, \ldots, N+1$, for any given N. Substituting (9.8) in (9.4), we obtain the required expansion:

(9.9) $$X(z) = (1 - z)^{-\kappa} \sum_{k=0}^{\infty} e_k^* z^{-k} \quad , \quad |z| > 1 .$$

We shall give the first few e_k^* :

$$(9.10)\qquad e_0^* = 1$$

$$e_1^* = C_0$$

$$e_2^* = C_0^{\,2}/2! + C_1$$

$$e_3^* = C_0^{\,3}/3! + C_0C_1 + C_2$$

$$e_4^* = C_0^{\,4}/4! + C_0^{\,2}C_1/2! + C_1^{\,2}/2! + C_0C_2 + C_3$$

$$e_5^* = C_0^{\,5}/5! + C_0^{\,3}C_1/3! + C_1^{\,2}C_0/2! + C_2C_0^{\,2}/2! + C_1C_2 + C_0C_3 + C_4$$

Similarly for the expansion of X^{-1} we define

$$(9.11)\qquad \sum_{k=0}^{\infty} aa_k^{(m)}\, z^{-k} = \exp\left(-\sum_{k=0}^{m} C_k\, z^{-k-1}\right) \;,\quad m = 0, 1, \ldots, N \;,$$

and obtain

$$(9.12)\qquad aa_k^{(0)} = (-C_0)^k/k! \quad,\quad k = 0, 1, \ldots$$

$$aa_k^{(m)} = \sum_{j=0}^{[k/(m+1)]} aa_{k-(m+1)j}^{(m-1)}\,(-C_m)^j/j! \;,\; k = 0, 1, \ldots; \quad m = 1, 2, \ldots, N \;,$$

giving the expansion for X^{-1}

$$(9.13)\qquad X^{-1}(z) = (1-z)^{\kappa} \sum_{k=0}^{\infty} e_k\, z^{-k} \quad,\quad |z| > 1 \;,$$

where $\qquad e_k = aa_k^{(N)} \quad,\quad k = 0, 1, \ldots, N+1 \;.$

The first e_k are

$$e_0 = 1$$

$$e_1 = -C_0$$

$$e_2 = C_0^{\,2}/2! - C_1$$

$$(9.14)\quad e_3 = -C_0{}^3/3! + C_0C_1 - C_2$$

$$e_4 = C_0{}^4/4! - C_0{}^2C_1/2! + C_1{}^2/2! + C_0C_2 - C_3$$

$$e_5 = -C_0{}^5/5! + C_0{}^3C_1/3! - C_1{}^2C_0/2! - C_2C_0{}^2/2! + C_1C_2 + C_0C_3 - C_4 .$$

§10. The polynomials R_3 and Ω_3

The solution (8.5) of the dominant equation required the polynomial Ω_3 (of degree $n+\kappa-m$) defined by (see (7.14))

$$(10.1) \qquad \lim_{z\to\infty}\left(\Omega_3(z) - \frac{f_n(z)}{b_m(z)X(z)}\right) = 0 ,$$

and the polynomial R_3 (of degree m-1), whose definition can be written (see (7.15)):

$$(10.2) \qquad \frac{d^j}{dz^j}[R_3(z) - F(z)]_{z=\beta_i} = 0 , \; j = 0, 1,\ldots, \alpha_i-1; \quad i = 1, 2,\ldots, \mu ,$$

where

$$F(z) = \begin{cases} f_n(z)/X(z), & z \notin [-1, 1] \\ f_n(z)a(z)/(r(z)Z(z)) & z \in [-1, 1] . \end{cases}$$

We consider Ω_3 first. Now (10.1) implies that at most

$$\Omega_3(z) - \frac{f_n(z)}{b_m(z)X(z)} = O(z^{-1}) \quad \text{as} \quad z \to \infty .$$

Multiplying by b_m and using (9.13), this becomes

$$(10.3) \quad b_m(z)\Omega_3(z) - f_n(z)(1-z)^\kappa\left(\sum_{k=0}^{N+1} e_k z^{-k} + O(z^{-N-2})\right) = O(z^{m-1}) .$$

If $\kappa\geq 0$, we find Ω_3 by equating the powers $z^{n+\kappa}$, $z^{n+\kappa-1}$, ... , z^m, giving $n+\kappa+1-m$ equations for the coefficients of Ω_3 . Clearly, we need to choose $N\geq n+\kappa-m-1$ to ensure that sufficient e_k are known. If $\kappa<0$, multiply (10.3) by $(1-z)^{-\kappa}$ and equate the powers z^n , z^{n-1} , ... , $z^{m-\kappa}$ to find Ω_3 .

Of course if $n+\kappa-m<0$, then $\Omega_3 \equiv 0$.

The calculation of R_3 is straightforward if all the zeros of b are simple, i.e. if $\alpha_i = 1$ for $i = 1, 2,\ldots, \mu$.

Then $m = \mu$, and the Lagrangian interpolation formula gives

$$R_3(z) = \sum_{k=1}^{m} F(\beta_k) \prod_{\substack{j=1 \\ j \neq k}}^{m} \frac{z - \beta_j}{\beta_k - \beta_j}$$

However, if b_m has multiple zeros, then the construction of R_3 may be more difficult. An explicit representation for R_3 can be obtained using the method of Spitzbart [41] or Goncharov [17] . See also Ivanov [20, eqn. 11.17] .

§11. Examples - Dominant Equation

In this section we give four examples which illustrate the preceding theory. In Examples 11.2 and 11.4 the solution of the dominant integral equation is reduced to the evaluation of the function Z by quadrature. In the other two examples, Z can be obtained analytically.

Example 11.1 Constant coefficients

Consider the singular integral equation

$$(11.1)\qquad a\phi(t) + \frac{b}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = \frac{\alpha + t}{\beta + t} \quad , \quad -1<t<1 \ ,$$

where α, β, a and b are real constants, such that $a^2 + b^2 = 1$, $|\beta| > 1$, and to satisfy (4.10), $b \le 0$. Karpenko [24] has considered equations of this form; however, he assumed that the right side of (11.1) was a polynomial.

From Definition 4.1, we have

$$(11.2)\qquad \theta = (1/\pi)\arctan(b/a) + N = \text{constant} \ ,$$

where $$N = \begin{cases} 0 & \text{if} \quad b/a < 0 \\ -1 & \text{if} \ b/a > 0 \ . \end{cases}$$

Then from (4.13), the index $\kappa=1$, and from (4.18)

$$(11.3)\qquad Z(t) = (1 - t)^{-1-\theta}\,(1 + t)^{\theta} \quad , \quad -1<t<1 \ .$$

From Theorem 4.1 the solution of (11.1) is

$$(11.4)\qquad \phi(t) = a\,\frac{\alpha + t}{\beta + t} - \frac{bZ(t)}{\pi}\int_{-1}^{1} \frac{\alpha + \tau}{\beta + \tau}\, \frac{1}{Z(\tau)}\, \frac{d\tau}{\tau - t} + cZ(t) \quad , \quad -1<t<1 \ ,$$

where c is an arbitrary constant. To evaluate the singular integral, we use Theorem 6.2, and choose $Q(z) = (\alpha+z)/(b(\beta+z))$, $S(z) = d/(\beta+z)$ where d is to be found, and χ to be a polynomial of degree one, such that $\lim_{z\to\infty}\left(\frac{\alpha+z}{\beta+z}\,\frac{1}{bX(z)} - \chi(z)\right) = 0$. To determine χ, we have from (9.13) and (9.14) that $X^{-1}(z) = -z + (1+C_0) + O(z^{-1})$, where $C_0 = \int_{-1}^{1} \theta(t)dt = 2\theta$.

Hence, after a little algebra, we obtain

$$\chi(z) = (-z + 1 + 2\theta - \alpha + \beta)/b \; . \tag{11.5}$$

The constant d is chosen so that $QX^{-1} - S$ is analytic at the point $-\beta$; hence we obtain

$$d = \frac{\alpha - \beta}{bX(-\beta)} \; , \tag{11.6}$$

where, by (4.14),

$$X(-\beta) = \frac{1}{1+\beta}\left|\frac{\beta+1}{\beta-1}\right|^{-\theta} .$$

Then applying Theorem 6.2, we have the singular integral

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\alpha+\tau}{\beta+\tau}\,\frac{1}{Z(\tau)}\,\frac{d\tau}{\tau - t} = \frac{a}{b}\,\frac{\alpha+t}{\beta+t}\,\frac{1}{Z(t)} - \frac{d}{\beta+t} - \chi(t) \;, \quad -1<t<1 \;,$$

and substituting this in (11.4) we obtain

$$\phi(t) = (1-t)^{-1-\theta}(1+t)^{\theta}\left((1+\beta)\frac{\alpha-\beta}{\beta+t}\left|\frac{\beta+1}{\beta-1}\right|^{\theta} + A - t\right) , \quad -1<t<1 \;, \tag{11.7}$$

where $A = 1 + C_0 - \alpha + \beta + c$, which is arbitrary, since it includes c.

This completes the solution of (11.1), but we will discuss the construction of some special solutions.

Since Z is unbounded at ± 1, the solution ϕ will also be unbounded at ± 1, except for particular choices of the arbitrary constant A. If we choose

$$(11.8) \qquad A = -(1+\beta)\frac{\alpha-\beta}{\beta-1}\left|\frac{\beta+1}{\beta-1}\right|^{\theta} - 1$$

then $\phi(-1) = 0$ and $\phi(1)$ is unbounded, and for

$$(11.9) \qquad A = -(\alpha-\beta)\left|\frac{\beta+1}{\beta-1}\right|^{\theta} + 1 \ ,$$

$\phi(1)$ will be zero and $\phi(-1)$ unbounded.

These two choices of A (or c) give solutions which, in Muskhelishvili's [32, §107] method, are of index zero. The two values of A will in general be distinct, but if

$$(11.10) \qquad -(1+\beta)\frac{\alpha-\beta}{\beta-1}\left|\frac{\beta+1}{\beta-1}\right|^{\theta} - 1 = -(\alpha-\beta)\left|\frac{\beta+1}{\beta-1}\right|^{\theta} + 1$$

then the values of A coincide, and so the corresponding particular solution ϕ will be bounded at both ± 1. In Muskhelishvili's method, this is a solution with an index of minus one, and (11.10) is the condition that such a solution exist.

The general solution (11.7) has, of course, an index of one in this thesis and in Muskhelishvili's work.

<u>Example 11.2</u> <u>b having a pair of complex conjugate zeros</u>

Consider

$$(11.11) \qquad -\phi(t) - \frac{(t+\alpha)^2+\beta^2}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau-t}\,d\tau = -1 \quad , \quad -1<t<1 \ .$$

We assume for simplicity that $\beta \neq 0$.

From Definition 4.1, we have

$$\theta(t) = \frac{1}{\pi} \arctan [(t + \alpha)^2 + \beta^2] - 1 \ , \ -1 \le t \le 1 \ ,$$

and by Lemma 4.2, the index $\kappa = 1$. Z is given by (4.18), and r by (4.8). The degree of b is m = 2, the degree of f is n = 0. From Theorem 7.3, the solution of (11.11) is

$$\phi(t) = \frac{Z(t)}{r(t)} \left(R_3(t) - [(t+\alpha)^2 + \beta^2] [\Omega_3 + c] \right) \ , \quad -1<t<1 \ , \tag{11.12}$$

where R_3 is of degree one, $\Omega_3 \equiv 0$ and c is an arbitrary constant. From (7.15), R_3 is defined by

$$R_3(z) = -X^{-1}(z) \quad \text{at} \quad z = \alpha \pm i\beta \ . \tag{11.13}$$

Putting $R_3(z) = A + Bz$, then since X (and X^{-1}) has the property $X(\bar{z}) = \overline{X(z)}$, we can put $1/X(\alpha \pm i\beta) = p \pm iq$, where p and q are found from the definition of X.

From (11.13) we obtain $A = -p + \alpha q/\beta$, $B = -q/\beta$, and the solution of (11.11) is

$$\phi(t) = \frac{Z(t)}{r(t)} \left(A + Bt - c[(t+\alpha)^2 + \beta^2] \right) \quad , \ -1<t<1 \ .$$

Thus we have reduced the problem to the evaluation of $X(\alpha \pm i\beta)$ and Z, which would presumably be found by quadratures.

<u>Example 11.3</u> <u>$a(t) = -\sqrt{1 - t^2}$</u>

Consider

$$-\sqrt{1 - t^2}\, \phi(t) + \frac{t - \beta}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = 1 + t \ , \quad -1<t<1 \ . \tag{11.14}$$

We assume that $-1<\beta<1$. The form of $a(t)$ makes it possible to find the function Z without evaluating the singular integral of θ.

We first note that (11.14) satisfies (4.10), i.e. $b(-1)\leq 0$. From Lemma 4.2, since b/a is decreasing at the zero of b, the index $\kappa = 2$.

Z can be found numerically, but in this case it is possible to find Z analytically, as follows. We define

$$b(z) = z - \beta \tag{11.15}$$

and

$$J(z) = -b(z) - \sqrt{z^2 - 1} \quad , \quad z \in [-1, 1] \; , \tag{11.16}$$

choosing that branch of $\sqrt{z^2 - 1}$ which is $O(z)$ at infinity. We also define

$$\omega(t) = \sqrt{1 - t^2} \quad , \quad -1\leq t\leq 1 \; . \tag{11.17}$$

Then the limiting values of J on $[-1, 1]$ are

$$\begin{aligned} J^{\pm}(t) &= -b(t) \mp i\omega(t) \; , \\ J^{+}(t)J^{-}(t) &= r^2(t) = 1 + \beta^2 - 2\beta t \; , \; -1\leq t\leq 1 \; . \end{aligned} \tag{11.18}$$

From Definition 3.1, and (4.6), the canonical function X satisfies

$$\frac{X^{+}(t)}{X^{-}(t)} = - \frac{J^{-}(t)}{J^{+}(t)} \quad , \quad -1<t<1 \; . \tag{11.19}$$

We also define

$$M(z) = (z^2 - 1)^{-\frac{1}{2}} \quad , \quad z \notin [-1, 1] \quad , \tag{11.20}$$

with the branch chosen so that M is zero at infinity.

Then

$$M^{\pm}(t) = \pm \frac{1}{i\omega(t)} \quad , \quad -1<t<1 \quad , \tag{11.21}$$

and so $M^+(t)/M^-(t) = -1$. Thus we can write (11.19) as

$$(11.22) \qquad \frac{X^+(t)J^+(t)}{M^+(t)} = \frac{X^-(t)J^-(t)}{M^-(t)} \quad , \quad -1<t<1 \ .$$

Hence the function

$$(11.23) \qquad \Phi = XJ/M$$

is analytic in the complex plane, and by (11.22) is continuous across $(-1, 1)$, the singularities at ± 1 being removable.

To determine the behaviour of Φ at infinity, we proceed as follows.

Since the index of (11.14) is two, then by (4.15),

$$(11.24) \qquad X(z) = z^{-2} + 0(z^{-3}) \quad \text{as} \quad z \to \infty \ .$$

Also, from (11.16) and (11.20) we have $M(z) = z^{-1} + 0(z^{-2})$, $J(z) = -2z + 0(1)$ as $z \to \infty$ and so $\Phi(z) = -2 + 0(z^{-1})$ as $z \to \infty$. Hence, by Liouville's Theorem, $\Phi(z) \equiv -2$, and so by (11.23) ,

$$(11.25) \qquad X(z) = -\frac{2}{\sqrt{z^2 - 1}} \frac{1}{J(z)} \quad , \quad z \notin [-1, 1] \ .$$

It is easy to show that (11.25) satisfies the other conditions of Definition 3.1. Finally, from (4.19), (11.25) and (11.18) we have, since Z is non-negative,

$$(11.26) \qquad Z(t) = \frac{2}{r(t)\omega(t)} = \frac{2}{[(1-t^2)(1+\beta^2-2\beta t)]^{\frac{1}{2}}} \quad , \quad -1<t<1 \ .$$

We will solve (11.14) using Theorem 7.3. Ω_3 is of degree two, and is given by (7.14):

$$(11.27) \qquad \lim_{z\to\infty} \left(\Omega_3(z) - \frac{z+1}{(z-\beta)X(z)} \right) = 0 \ .$$

Since Ω_3 will be added to the arbitrary polynomial $P_{\kappa-1}$ (which is of degree one), it is only necessary to find the coefficient of z^2 in Ω_3. Thus, from (11.24) we obtain

$$\Omega_3(z) = z^2 + O(z) . \tag{11.28}$$

R_3 is a constant, and is, by (7.15) and (11.26):

$$R_3 = -\frac{1+\beta}{Z(\beta)} = -\tfrac{1}{2}(1+\beta)(1-\beta^2) . \tag{11.29}$$

Hence the solution of (11.14) is, using (7.17):

$$\phi(t) = \frac{2}{\sqrt{1-t^2}\,(1+\beta^2-2\beta t)} \left(-\tfrac{1}{2}(1+\beta)(1-\beta^2) + (t-\beta)(t^2+At+B) \right) , \quad -1<t<1 , \tag{11.30}$$

where A and B are arbitrary constants, obtained by adding Ω_3 and $P_{\kappa-1}$.

<u>Example 11.4</u> <u>The H functions of Chandrasekhar</u>

The H functions of Chandrasekhar satisfy the equation

$$1 + \tfrac{1}{2}\lambda\mu H(\mu) \int_0^1 \frac{H(x)}{x+\mu}\,dx = H(\mu) . \tag{11.31}$$

This equation is valid for any μ in the complex plane excluding $[-1, 0)$, and for $0 \le \mu \le 1$, it becomes a nonlinear integral equation for H.

Busbridge [3, §11] has shown that for $\lambda = 1$, (11.31) has a unique solution continuous in $[0, 1]$, and for $0<\lambda<1$ has two solutions; we shall consider only the solution which satisfies

$$\tfrac{1}{2}\lambda \int_0^1 H(\mu)\,d\mu = 1 - \sqrt{1-\lambda} . \tag{11.32}$$

Stibbs & Weir [42] have shown that the solution of (11.31) which satisfies (11.32) is given by

$$(11.33)\qquad H(\mu) = \exp\left(-\frac{\mu}{\pi}\int_0^{\pi/2} \frac{\log(1-\lambda\zeta\cot\zeta)}{\cos^2\zeta + \mu^2\sin^2\zeta}\,d\zeta\right),\quad 0\le\mu\le1,\ 0\le\lambda\le1\ ,$$

and have used this expression to tabulate values of H.

Sobolev [40, p 106] , and Fox [15] in 1961 (by a different method) have transformed (11.31) into the linear singular integral equation

$$(11.34)\qquad H(\nu)\left(1+\tfrac{1}{2}\lambda\nu\log\frac{1-\nu}{1+\nu}\right) - \tfrac{1}{2}\lambda\nu\int_0^1 \frac{H(\zeta)}{\zeta-\nu}\,d\zeta = 1\ ,\quad 0\le\nu\le1\ .$$

We shall solve (11.34) using the methods of this thesis, and then, since (11.34) does not have a unique solution, use (11.32) to locate the particular solution of (11.34) which also satisfies (11.31).

We do not suggest that solving (11.34) provides the best method of solving (11.31), because the logarithmic singularity at $\nu = 1$ makes it difficult to evaluate the Cauchy principal value integral of θ, which is needed in Z, below. The expression (11.33) appears to be more practical. The equation (11.34) was chosen as an example because it was one of the few singular integral equations I could find in the literature which had been solved accurately by other methods.

To transform (11.34) into the type of equation considered in this thesis, put $t = 2\nu-1$, $\tau = 2\zeta-1$, $\phi(t) = H(\nu)$, which gives

$$(11.35)\qquad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^1 \frac{\phi(\tau)}{\tau - t}\,d\tau = f(t)\ ,\ -1<t<1\ ,$$

where

$$(11.36)\qquad a(t) = 1 + \tfrac{1}{4}\lambda(1+t)\log\frac{1-t}{3+t}\ ,\quad b(t) = -\tfrac{1}{4}\pi\lambda(1+t)\ ,\quad f(t) = 1\ ,\quad -1<t<1\ .$$

Sobolev [40, p 109] gave an exact solution of (11.35) subject to (11.32). In terms of ϕ, (11.32) is

$$\tag{11.37} \tfrac{1}{4}\lambda \int_{-1}^{1} \phi(\tau)d\tau = 1 - \sqrt{1-\lambda} \quad , \quad 0<\lambda\leq 1 \ .$$

Sobolev's method was to substitute

$$\tag{11.38} \phi(t) = [A - B\ b(t)]\ Z(t)/r(t) \quad , \quad -1<t<1$$

(where Z is given by (4.18) and r by (4.8)) in (11.35), and by noting that bZ/r satisfies the homogenous equation $a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\,d\tau = 0$, $-1<t<1$, (see Theorem 4.1), he obtained the following condition on A:

$$\tag{11.39} \tfrac{1}{4}\ A\lambda \int_{-1}^{1} \frac{Z(\tau)}{r(\tau)}\,d\tau = 1 \ .$$

With A given by this, since the index of (11.35) is one (see below), (11.38) gives the general solution of (11.35), with B an arbitrary parameter. Substituting (11.38) in (11.37), and using (11.39) Sobolev then obtained

$$\tag{11.40} \tfrac{1}{4}\ B\lambda \int_{-1}^{1} \frac{b(\tau)Z(\tau)}{r(\tau)}\,d\tau = \sqrt{1-\lambda} \quad ,$$

which can be used to determine B.

Thus Sobolev has given the solution of (11.35) by (11.38), with A and B determined by integrals which involve Z. Since Z is not easy to evaluate, and has logarithmic singularities at ± 1, we shall show how the methods of this thesis can be used to evaluate A and B analytically.

We now solve (11.35) using Theorem 7.3.

We observe that in (11.36), a is infinite at $t = 1$. By dividing (11.35) by $\log((1-t)/4)$, the corresponding a, b, and f become bounded but not Hölder continuous. However, as Khvedelidze [25] has shown, Theorem 4.1 still holds, and the method below is still applicable, though we shall not prove this.

In (11.36), a has a zero on $[-1, 1]$, which we denote by t_0. For $\lambda = 1$, $t_0 = 0.667113$, and as $\lambda \to 0$, $t_0 \to 1$. From Definition 4.1 we obtain

$$\theta(t) = \begin{cases} \frac{1}{\pi}\arctan\,(b(t)/a(t)) & , \quad -1 \le t < t_0 \\ -\frac{1}{2} - \frac{1}{\pi}\arctan\,(a(t)/b(t)) & , \quad -1 < t \le 1 \\ -1 + \frac{1}{\pi}\arctan\,(b(t)/a(t)) & , \quad t_0 < t \le 1 \;. \end{cases}$$

Of course, these expressions coincide over common intervals of definition.

Thus $\theta(1) = -1$, and by (4.13), $\kappa = 1$. The endpoint -1 is a special end, and so Z will be bounded there.

In the notation of Theorem 7.3, we have $n = 0$ and $m = 1$, and so Ω_3 and R_3 are constants, given by $\Omega_3 = \lim\limits_{z\to\infty} \frac{1}{b(z)X(z)}$, $R_3 = 1/Z(-1)$. From (4.15) and (11.36) we have $\Omega_3 = 4/(\pi\lambda)$, and so from (7.17) the general solution of (11.35) is

$$\text{(11.41)} \qquad \phi(t) = \frac{Z(t)}{r(t)}\left(\frac{1}{Z(-1)} + b(t)\left(\frac{4}{\pi\lambda} + c\right)\right) \quad , \quad -1 \le t < 1 \quad ,$$

where c $(= P_{\kappa-1})$ is an arbitrary constant.

To determine the value of c so that (11.41) corresponds to a solution of (11.31), proceed as follows.

In Lemma 6.1, we choose $n = 0$, and since $\kappa = 1$, then $\Delta_1 \equiv 0$ and (6.9) becomes

$$\int_{-1}^{1} \phi(\tau)d\tau = c \int_{-1}^{1} \frac{b(\tau)Z(\tau)}{r(\tau)}\, d\tau \quad , \tag{11.42}$$

since $c = P_{\kappa-1}$ = constant. To evaluate the integral on the right, we choose in Theorem 6.1, $Q(t, z) = z - t$ (where t is a parameter) . Since Q has no poles, $R\equiv 0$, and since by (4.15) $X(z) = -z^{-1} + O(z^{-2})$, $Q(t, z) = z + O(1)$ as $z \to \infty$, then ψ is a constant, given by $\psi = \lim_{z\to\infty} Q(t, z)X(z) = -1$. Hence, since $Q(t, t) = 0$, Theorem 6.1 gives

$$\frac{1}{\pi} \int_{-1}^{1} \frac{b(\tau)Z(\tau)}{r(\tau)}\, d\tau = -1 \quad , \tag{11.43}$$

(We note that this integral can be obtained more directly using §14, or by integrating X round a contour enclosing [-1, 1].)

Using this result, (11.42) and (11.37), we obtain

$$c = -\frac{4}{\pi\lambda}\,(1 - \sqrt{1-\lambda}) \tag{11.44}$$

and so (11.41) becomes, with this choice of c:

$$\phi(t) = \frac{Z(t)}{r(t)} \left(\frac{1}{Z(-1)} + b(t)\, \frac{4}{\pi\lambda}\, \sqrt{1-\lambda} \right), \quad -1<t<1 \quad , \tag{11.45}$$

which is the particular solution of (11.35), subject to (11.37), and corresponds to (11.33) using $\phi(t) = H(\frac{1+t}{2})$. Using (11.37) (11.43) and (11.45) we also obtain

$$\int_{-1}^{1} \frac{Z(\tau)}{r(\tau)}\, d\tau = \frac{4}{\lambda}\, Z(-1) \quad . \tag{11.46}$$

Thus we see that (11.43) and (11.46) give simple expressions for the integrals (11.40), (11.39).

If $\lambda = 1$, then the solution of (11.35) becomes, from (11.45)

$$\phi(t) = \frac{Z(t)}{r(t)} \cdot \frac{1}{Z(-1)} \quad , \quad -1<t<1 \ . \tag{11.47}$$

But from Busbridge [3, §12] we have that

$$\int_{-1}^{1} (1+\tau)\phi(\tau)d\tau = 8/\sqrt{3} \quad ,$$

and so using (11.43), (11.36) and (11.46) we obtain , if $\lambda = 1$, that $Z(-1) = \sqrt{3}/2$.

CHAPTER IV

AN ALGORITHM FOR THE NUMERICAL SOLUTION OF THE COMPLETE EQUATION

§12. General description of the algorithm

The complete equation

$$(12.1)\quad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau - t}\,d\tau + \int_{-1}^{1} K(t,\tau)\phi(\tau)d\tau = f(t)\ , \quad -1<t<1$$

may be reduced to a Fredholm integral equation by defining

$$(12.2)\qquad g(t) = f(t) - \int_{-1}^{1} K(t,\tau)\phi(\tau)d\tau\ , \quad -1<t<1$$

and then using any of the solutions of the dominant equation. For example, this was done by Muskhelishvili [32, § 109] and Gakhov [16, § 48], using the solution of §4. However, the resulting Fredhom integral equation is, in general, not useful for numerical work, as it involves Cauchy principal value integrals of the function Z.

As we have seen in §7, there are many different ways of writing the solution of the dominant equation and consequently many possible algorithms for the solution of the complete equation. The algorithm which we describe in this chapter has been chosen in an attempt to minimize the amount of computation required.

We assume that b is a polynomial of degree m, of the form

$$(12.3)\qquad b(z) = \gamma \prod_{i=1}^{\mu} (z - \beta_i)^{\alpha_i}\ , \text{ where } m = \sum_{i=1}^{\mu} \alpha_i\ .$$

For this to be so, it may be necessary to multiply the original equation by some function h, as described in §8.

We approximate g (see 12.2) by a polynomial g_n of degree n, and solve

$$(12.4)\qquad a(t)\phi_n(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi_n(\tau)}{\tau - t}\,d\tau = g_n(t)\quad,\ -1<t<1\ ,$$

exactly using Theorem (7.2), which gives

$$(12.5)\qquad \phi_n(t) = \frac{Z(t)}{r(t)}\Bigg(g_n(t)[R_2(t) + b(t)\Omega_2(t)] - \frac{b(t)}{\pi}\int_{-1}^{1}\frac{g_n(\tau)-g_n(t)}{\tau - t}\,\frac{d\tau}{r(\tau)Z(\tau)} + b(t)\,P_{\kappa-1}(t)\Bigg)\ ,\quad -1<t<1\ .$$

where Ω_2 and R_2 are defined by (7.7) and (7.8), and Z, r and $P_{\kappa-1}$ as in §4. Defining g^* by

$$(12.6)\qquad g^*(t) = f(t) - \int_{-1}^{1} K(t,\tau)\phi_n(\tau)d\tau\quad,\ -1<t<1\ ,$$

we arbitrarily choose g_n so that it interpolates g^* at the points

$$(12.7)\qquad x_j = \cos\left(\frac{\pi}{2}\,\frac{2j+1}{n+1}\right)\quad,\ j = 0, 1,\dots, n\ .$$

Then g_n can be expressed in terms of the Chebyshev polynomials T_i as follows (see, for example, Hildebrand [18, §9.7]):

$$(12.8)\qquad g_n(t) = {\sum_{i=0}^{n}}' G_i\,T_i(t)\quad,\ -1<t<1\ ,$$

where the dash indicates a sum whose first term is halved. The G_i are given by

$$(12.9)\qquad G_i = \frac{2}{n+1}\sum_{j=0}^{n} g^*(x_j)T_i(x_j)\quad,\ i = 0, 1,\dots, n\ ,$$

the T_i being defined by $T_i(x) = \cos(i \arccos x)$, $i = 0, 1, \dots$, $-1\le x\le 1$. We choose these Chebyshev polynomials and the points x_j because of the simplicity of the relations (12.8), (12.9) .

Of course, any set of polynomials could be used.

We note that the Chebyshev polynomials satisfy

$$(12.10)\qquad \begin{gathered} T_0 \equiv 1 \;, \quad T_1(t) = t \;, \\ T_{i+1}(t) = 2t\,T_i(t) - T_{i-1}(t) \quad , \quad i = 1, 2, \ldots \quad , \\ \int_{-1}^{1} \frac{T_n(t)T_m(t)}{(1-t^2)^{\frac{1}{2}}}\, dt = \begin{cases} \pi & m = n = 0 \\ \pi/2 & m = n \neq 0 \\ 0 & m \neq n \end{cases} \end{gathered}$$

We define the polynomials W_i, of degree i, by

$$(12.11)\qquad W_i(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_{i+1}(\tau) - T_{i+1}(t)}{\tau - t}\, \frac{d\tau}{r(\tau)Z(\tau)} \quad , \quad i = -1, 0, 1, \ldots$$

Then (12.10) gives

$$(12.12)\qquad W_{-1} \equiv 0 \quad , \quad W_0 \equiv \frac{1}{\pi}\int_{-1}^{1} \frac{d\tau}{r(\tau)Z(\tau)} \quad ,$$

and the recurrence relation

$$(12.13)\qquad W_i(t) = 2tW_{i-1}(t) - W_{i-2}(t) + \frac{2}{\pi}\int_{-1}^{1} \frac{T_i(\tau)}{r(\tau)Z(\tau)}\, d\tau \quad , \quad i = 1, 2, \ldots$$

In practice, the W_i will be evaluated using (12.12) and (12.13), which require the evaluation of the *modified moments*

$$d_i = \frac{1}{\pi}\int_{-1}^{1} \frac{T_i(\tau)}{r(\tau)Z(\tau)}\, d\tau \quad , \quad i = 0, 1, \ldots, n-1 \; .$$

Then using (12.5), (12.8) and (12.11) we have

$$(12.14)\qquad \phi_n(t) = \frac{Z(t)}{r(t)}\left(\sum_{j=0}^{n} G_j\,[T_j(t)(R_2(t)+b(t)\Omega_2(t)) - b(t)W_{j-1}(t)] + \right. \\ \left. + \; b(t)\,P_{\kappa-1}(t) \right) .$$

Using (12.9) and by defining

$$(12.15)\qquad \Gamma_i(t) = \frac{2}{n+1}\sum_{j=0}^{n} T_j(x_i)\Big(T_j(t)[R_2(t)+b(t)\Omega_2(t)] - b(t)W_{j-1}(t)\Big),$$

$$i = 0, 1,\dots, n, \quad -1<t<1,$$

(12.14) can be written

$$(12.16)\qquad \phi_n(t) = \frac{Z(t)}{r(t)}\Big(\sum_{i=0}^{n} \Gamma_i(t)g^*(x_i) + b(t)P_{\kappa-1}(t)\Big), \quad -1<t<1 .$$

We define a new dependent variable ψ_n by

$$(12.17)\qquad \psi_n = r\,\phi_n/Z ,$$

and eliminating g^* using (12.6), (12.16) becomes

$$(12.18)\qquad \psi_n(t) + \sum_{j=0}^{n} \Gamma_j(t)\int_{-1}^{1} K(x_j, \tau)\,\frac{Z(\tau)}{r(\tau)}\,\psi_n(\tau)d\tau =$$

$$= \sum_{j=0}^{n} \Gamma_j(t)f(x_j) + b(t)P_{\kappa-1}(t), \quad -1<t<1 .$$

This is a Fredholm integral equation with a separable kernel, and can be solved in the usual way. Defining

$$(12.19)\qquad \xi_j = \int_{-1}^{1} K(x_j, \tau)\,\frac{Z(\tau)}{r(\tau)}\,\psi_n(\tau)d\tau \;; \quad j = 0, 1,\dots, n ,$$

multiplying (12.18) by $K(x_i, t)Z(t)/r(t)$ and integrating, we obtain

$$(12.20)\qquad \xi_i + \sum_{j=0}^{n} \xi_j\int_{-1}^{1} K(x_i, t)\,\frac{Z(t)}{r(t)}\,\Gamma_j(t)dt =$$

$$= \sum_{j=0}^{n} f(x_j)\int_{-1}^{1} K(x_i, t)\,\frac{Z(t)}{r(t)}\,\Gamma_j(t)dt +$$

$$+ \int_{-1}^{1} K(x_i, t)\,\frac{Z(t)}{r(t)}\,b(t)P_{\kappa-1}(t)dt , \quad i = 0, 1,\dots, n .$$

Defining

$$F_i = \sum_{j=0}^{n} f(x_j) \int_{-1}^{1} K(x_i, t) \frac{Z(t)}{r(t)} \Gamma_j(t)dt ,$$

$$A_{ij} = \delta_{ij} + \int_{-1}^{1} K(x_i, t) \frac{Z(t)}{r(t)} \Gamma_j(t)dt \quad , \quad i, j = 0, 1,\ldots, n ,$$

$$Q_k(t) = \int_{-1}^{1} K(t, \tau) \frac{Z(\tau)}{r(\tau)} b(\tau)L_{k-1}(\tau)d\tau \quad , \quad k = 1, 2,\ldots, \kappa ,$$

then (12.20) can be written

$$\text{(12.21)} \qquad \sum_{j=0}^{n} A_{ij}\, \xi_j = F_i + \sum_{k=1}^{\kappa} \rho_k\, Q_k(x_i) \ , \ i = 0, 1,\ldots, n ,$$

where the ρ_k are arbitrary constants and the L_k can be chosen to be polynomials of degree k, with $P_{\kappa-1} = \sum_{k=1}^{\kappa} \rho_k L_{k-1}$. If the index κ is zero or negative, then the term $\sum_{k=1}^{\kappa} \rho_k\, Q_k(x_i)$ is omitted.

Thus we have reduced the complete singular integral equation (12.1) to the Fredholm integral equation (12.18), which was then reduced to the system of linear algebraic equations (12.21). Since (12.21) may still be soluble if the matrix A is singular, we need to consider two cases.

Matrix A non-singular

If the matrix A is non-singular, then its inverse A^{-1} exists. Thus (12.21) can be solved, giving formally

$$\xi_i = \sum_{j=0}^{n} A^{-1}_{ij} F_j + \sum_{k=1}^{\kappa} \rho_k \sum_{j=0}^{n} A^{-1}_{ij} Q_k(x_j) \quad , \quad i = 0, 1,\ldots, n ;$$

where A^{-1}_{ij} denotes the (i,j) element of A^{-1}.

Then, to find ψ_n, we use (12.18) and (12.19):

$$\text{(12.22)} \qquad \psi_n(t) = v(t) + \sum_{k=1}^{\kappa} \rho_k\, u_k(t) \ , \quad -1<t<1 ,$$

where

$$v(t) = \sum_{j=0}^{n} \Gamma_j(t) \left(f(x_j) - \sum_{k=0}^{n} A_{jk}^{-1} F_k \right) \quad ,$$

(12.23)

$$u_k(t) = b(t)L_{k-1}(t) - \sum_{j=0}^{n} \Gamma_j(t) \sum_{\ell=0}^{n} A_{j\ell}^{-1} Q_k(x_\ell) \ .$$

We then use (12.17) to obtain ϕ_n, our approximate solution of (12.1).

The first term on the right side of (12.22) corresponds to an approximate particular solution of (12.1), and the other term corresponds to the κ approximate solutions of the homogenous equation

$$a(t)\phi(t) + \frac{b(t)}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t} d\tau + \int_{-1}^{1} K(t, \tau)\phi(\tau)d\tau = 0 \ , \ -1<t<1 \ .$$

Matrix A singular

If the matrix A in (12.21) is singular, then its inverse does not exist; however it may still be possible to solve (12.21). This corresponds to (12.1) being an eigenvalue problem.

From Noble [34, Thm 10.22], we have that (12.21) is soluble if and only if

$$(12.24) \qquad \sum_{i=0}^{n} \left(F_i + \sum_{k=1}^{\kappa} \rho_k Q_k(x_i) \right) y_i = 0 \ ,$$

where y_i are any solutions of

$$(12.25) \qquad \sum_{j=0}^{n} A_{ji} \, y_j = 0 \quad , \quad i = 0, 1, \ldots, n \ .$$

If A is of rank s, then (12.25) has n+1-s linearly independent solutions. If (12.24) is soluble, then the solution of (12.21) is of the form

$$\xi_i = z_i^{(0)} + \sum_{j=1}^{n+1-s} B_j\, z_i^{(j)} , \quad i = 0, 1, \ldots, n ,$$

where the $z_i^{(0)}$ are a particular solution of (12.21), with the ρ_k satisfying (12.24), the $z_i^{(j)}$ are the $n+1-s$ linearly independent solutions of $\sum_{j=0}^{n} A_{ij}\, z_j = 0$, $i = 0, n$, and the B_j are arbitrary constants.

Negative index If the index of (12.4) is negative, we need to check that the consistency condition (4.21) is satisfied; i.e., ϕ is a solution of (12.1) if and only if

$$\int_{-1}^{1} [f(t) - \int_{-1}^{1} K(t, \tau)\phi(\tau)d\tau] \frac{t^k}{r(t)Z(t)}\, dt = 0 , \quad k = 0, 1, \ldots, -\kappa-1 . \tag{12.26}$$

We approximate ϕ by ϕ_n, and consider the numbers δ_k

$$\delta_k = \int_{-1}^{1} [f(t) - \int_{-1}^{1} K(t, \tau)\phi_n(\tau)d\tau] \frac{t^k}{r(t)Z(t)}\, dt , \quad k = 0, 1, \ldots, -\kappa-1. \tag{12.27}$$

If (12.1) is soluble, then (12.26) will be satisfied, and so the numbers δ_k, $k = 0, 1, \ldots, -\kappa-1$ should be zero to within the order of approximation of ϕ by ϕ_n.

The numerical evaluation of Cauchy principal value integrals

To find ϕ_n from ψ_n using (12.17), and if b has zeros on $[-1, 1]$, then to find R_2 (and R_1 below), it is necessary to evaluate the function Z accurately. In some cases Z can be found exactly by analytic means (sse examples in §11); however it may often be necessary to use numerical methods to evaluate the Cauchy principal value integral involved in Z (see (4.18)). For such methods see Paget [35] , Paget & Elliott [36] and Davis & Rabinowitz [11, §2.12.8].

This completes the general description of the method of numerical solution of the complete equation. This algorithm will be complete if we can evaluate the modified moments

$$d_i = \frac{1}{\pi}\int_{-1}^{1} \frac{T_i(\tau)}{r(\tau)Z(\tau)}\, d\tau \quad , \quad i = 0, 1, \ldots, n-1 \quad ,$$

and provide quadrature formulae for the integrals in (12.20) and (12.27).

We have not had time to prove that this algorithm (for the solution of the complete equation) converges. However, it appears that Karpenko's [24] method for error estimates is applicable.

§13. Quadrature formulae

To evaluate the integrals in (12.20) and (12.27), we need quadrature formulae suitable for integrals of the form

$$(13.1) \qquad I = \int_{-1}^{1} \frac{h(\tau)}{r(\tau)Z(\tau)}\, d\tau \ , \qquad I^* = \int_{-1}^{1} \frac{Z(\tau)}{r(\tau)}\, h(\tau) d\tau \ ,$$

where h is an arbitrary, known function.

A possible method of deriving suitable quadrature formulae is to approximate h by a polynomial h_ν of degree ν, using (12.8) and (12.9):

$$(13.2) \qquad h_\nu(t) = \sum_{j=0}^{\nu} h\,(x_j)\frac{2}{\nu+1} \sum_{i=0}^{\nu}{}' \; T_i(t)T_i(x_j) \quad ,$$

where h_ν interpolates h at the points $x_j = \cos\left(\frac{\pi}{2}\ \frac{2j+1}{\nu+1}\right)$, $j = 0, 1, \ldots, \nu$. We then define I_ν , I^*_ν , which are approximate values for I, I* by replacing h by h_ν in (13.1). Thus defining

$$(13.3) \qquad \begin{aligned} \omega_j &= \frac{2}{\nu+1} \sum_{i=0}^{\nu}{}' \; T_i(x_j) \int_{-1}^{1} \frac{T_i(\tau)}{r(\tau)Z(\tau)}\, d\tau \ , \\ \omega^*_j &= \frac{2}{\nu+1} \sum_{i=0}^{\nu}{}' \; T_i(x_j) \int_{-1}^{1} \frac{Z(\tau)}{r(\tau)}\, T_i(\tau) d\tau \ , \ j = 0, 1, \ldots, \nu \ , \end{aligned}$$

we have

$$(13.4) \qquad I_\nu = \sum_{j=0}^{\nu} \omega_j h(x_j) \ , \quad I^*_\nu = \sum_{j=0}^{\nu} \omega^*_j\, h(x_j) \ ,$$

which will be our quadrature formulae for I, I* .

However, to calculate ω_j and ω^*_j , $j = 0, 1, \ldots, \nu$, we will need the modified moments

$$(13.5)\qquad d_i = \frac{1}{\pi}\int_{-1}^{1} \frac{T_i(\tau)}{r(\tau)Z(\tau)}\,d\tau\ ,\quad d_i^* = \frac{1}{\pi}\int_{-1}^{1}\frac{Z(\tau)}{r(\tau)}\,T_i(\tau)d\tau\ ,\qquad i = 0, 1,\ldots, \nu;$$

a method for the evaluation of these integrals is given in the next section.

In practice, it is easiest to choose $\nu = n$.

We will now use these formulae for the approximate evaluation of the integrals (12.27). Applying the first formula of (13.4), with $\nu = n$, (12.27) becomes

$$\delta_k = \sum_{j=0}^{n} \omega_j \left(f(x_j) - \int_{-1}^{1} K(x_j, \tau)\phi_n(\tau)d\tau \right) x_j^{\,k} + R\ ,$$

where R is the remainder which depends on n and k. From (12.19) and (12.17) this becomes

$$(13.6)\qquad \delta_k = \sum_{j=0}^{n} \omega_j\,[f(x_j) - \xi_j]\,x_j^{\,k} + R\ ,\quad k = 0, 1,\ldots, -\kappa-1\ .$$

Thus the δ_k can be readily evaluated to within the accuracy of the quadrature formula.

Another method of using the modified moments to provide quadrature formulae is given by Sack & Donovan [39]. This approach also generates the recurrence relations for a set of polynomials orthogonal to $1/rZ$, and a set orthogonal to Z/r , and thus could possibly be used as the basis for another algorithm for the numerical solution of (12.1). However, we shall not pursue this here.

§14. The modified moments

We complete the description of the algorithm for numerical solution of the complete singular integral equation by giving a method for the evaluation of the modified moments

$$(14.1) \qquad d_i = \frac{1}{\pi}\int_{-1}^{1} \frac{T_i(\tau)}{r(\tau)Z(\tau)}\,d\tau \ , \quad d_i^* = \frac{1}{\pi}\int_{-1}^{1} \frac{Z(\tau)}{r(\tau)}\,T_i(\tau)d\tau \ , \qquad i = 0, 1, \ldots, n,$$

which are required in (12.12), (12.13) and (13.3).

We first give several alternative expressions for the canonical function X, defined by (4.14). We note that similar methods have been used by other authors; for example, see Cercignani [8].

Lemma 14.1 *Two alternative representations for* X *are*

$$(14.2) \qquad \begin{aligned} X(z) &= -\frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)Z(\tau)}{r(\tau)}\,\frac{d\tau}{\tau - z} + \chi_1(z) \\ X^{-1}(z) &= \frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)}{r(\tau)Z(\tau)}\,\frac{d\tau}{\tau - z} + \chi_2(z) \ , \end{aligned} \qquad z \notin [-1, 1]$$

where χ_1 *is given by* (4.22), *and* χ_2 *by* (7.1).

Proof From (4.19) we have

$$X^+(t) - X^-(t) = -\frac{2ib(t)Z(t)}{r(t)} \ , \quad \frac{1}{X^+(t)} - \frac{1}{X^-(t)} = \frac{2ib(t)}{r(t)Z(t)} \ , \qquad -1<t<1 \ ,$$

and applying Lemma 2.6, (14.2) follows. #

The value of this lemma is that it connects integrals of Θ (see (4.14)) and integrals involving the function Z.

Since

$$(14.3) \qquad \frac{1}{\tau - z} = -\frac{1}{z}\sum_{k=0}^{\infty} (\tau/z)^k \quad , \quad |\tau/z| < 1 \ ,$$

then by defining the moments

$$(14.4) \qquad E_k = \frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)}{r(\tau)Z(\tau)}\, \tau^k d\tau \ , \quad E_k^* = \frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau)Z(\tau)}{r(\tau)}\, \tau^k d\tau \ , \quad k = 0, 1, \ldots,$$

we can write (14.2) as

$$(14.5) \qquad \begin{aligned} X(z) &= \sum_{k=0}^{\infty} E_k^* z^{-k-1} + \chi_1(z) \\ X^{-1}(z) &= -\sum_{k=0}^{\infty} E_k z^{-k-1} + \chi_2(z) \ , \quad |z| > 1 \ . \end{aligned}$$

But, from (9.9) and (9.13), we have expansions of X and X^{-1} in powers of z, and so the moments E_k, E_k^* may be found in terms of the moments of θ. This result is sometimes useful, but it is only sufficient to give the modified moments d_i, d_i^* if b is a constant. Thus, we prove the following lemma.

<u>Lemma 14.2</u> *Let* $b(z) = \gamma \prod_{i=1}^{\mu} (z-\beta_i)^{\alpha_i}$ *, with* $m = \sum_{i=1}^{\mu} \alpha_i$.
Let R_1 *be a polynomial of degree* $m - 1$ *such that*

$$(14.6) \qquad \begin{aligned} &\frac{d^j}{dz^j}[R_1(z) - X(z)]_{z=\beta_i} = 0 \ , && \beta_i \notin [-1, 1] \\ &\frac{d^j}{dt^j}[R_1(t) - a(t)Z(t)/r(t)]_{t=\beta_i} = 0 \ , && \beta_i \in [-1, 1] \ , \end{aligned}$$

$j = 0, 1, \ldots, \alpha_i - 1 \quad , \quad i = 1, 2, \ldots, \mu$.

Let R_2 *be as in* (7.8), Ω_2 *as in* (7.7), *and let* Ω_1 *be a polynomial of degree* $-\kappa-m$ *such that*

(14.7) $$\lim_{z\to\infty} [\Omega_1(z) - X(z)/b(z)] = 0 \ . \quad (\textit{If } -\kappa-m<0, \textit{ then } \Omega_1 \equiv 0)$$

In order that R_1 *and* R_2 *exist, we assume that the first* $\alpha_i - 1$ *derivatives of* a *are Hölder continuous in a neighbourhood of the point* β_i, if $\beta_i \in [-1, 1]$, $i = 1, \mu$.

ital

Then the following alternative representations for X *are valid:*

(14.8) $$X(z) = b(z) \left(\Omega_1(z) - \frac{1}{\pi} \int_{-1}^{1} \frac{Z(\tau)}{r(\tau)} \frac{d\tau}{\tau - z} \right) + R_1(z) \ ,$$

$$X^{-1}(z) = b(z) \left(\Omega_2(z) + \frac{1}{\pi} \int_{-1}^{1} \frac{1}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - z} \right) + R_2(z)$$

$$z \notin [-1, 1] \ .$$

Proof Define

(14.9) $$\Phi_1(z) = \frac{X(z) - R_1(z)}{b(z)} - \Omega_1(z) \ , \quad \Phi_2(z) = \frac{X^{-1}(z) - R_2(z)}{b(z)} - \Omega_2(z) \ , \quad z \notin [-1, 1].$$

Then Φ_1 and Φ_2 are analytic in the complex plane excluding $[-1, 1]$, and in particular are analytic at the zeros β_i of b, because of the way R_1 and R_2 are chosen. Similarly, $\Phi_1^{\pm}, \Phi_2^{\pm} \in H^*$. By the definitions of Ω_1 and Ω_2, we see that Φ_1 and Φ_2 are zero at infinity. The limiting values of Φ_1, Φ_2 on $[-1, 1]$ can be obtained from (4.19), and applying Lemma 2.6 we obtain

$$\Phi_1(z) = -\frac{1}{\pi} \int_{-1}^{1} \frac{Z(\tau)}{r(\tau)} \frac{d\tau}{\tau - z} \ , \quad \Phi_2(z) = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{r(\tau)Z(\tau)} \frac{d\tau}{\tau - z} \ ,$$

$$z \notin [-1, 1] \ ,$$

and using (14.9), the lemma is proved. #

Using (14.3) and defining the moments

$$(14.10)\quad D_i = \frac{1}{\pi}\int_{-1}^{1} \frac{\tau^i}{r(\tau)Z(\tau)}\,d\tau \ , \quad D_i^* = \frac{1}{\pi}\int_{-1}^{1} \frac{Z(\tau)}{r(\tau)}\,\tau^i d\tau \ , \quad i = 0, 1, \ldots \ ,$$

we can write (14.8) as

$$(14.11)\quad \begin{aligned} X(z) &= b(z)\left(\Omega_1(z) + \sum_{k=0}^{\infty} D_k^* \, z^{-k-1}\right) + R_1(z) \\ X^{-1}(z) &= b(z)\left(\Omega_2(z) - \sum_{k=0}^{\infty} D_k \, z^{-k-1}\right) + R_2(z) \ , \quad |z| > 1 \ . \end{aligned}$$

Clearly if we can evaluate the moments D_k , D_k^* then it will be easy to obtain the modified moments d_i, d_i^* .

From (9.9) and (9.13) we have expansions for X, X^{-1}; equating these with (14.11), we have the identities

$$(14.12)\quad X(z) = (1-z)^{-\kappa} \sum_{k=0}^{\infty} e_k^* \, z^{-k} = b(z)\left(\Omega_1(z) + \sum_{k=0}^{\infty} D_k^* \, z^{-k-1}\right) + R_1(z) \ ,$$

$$(14.13)\quad X^{-1}(z) = (1-z)^{\kappa} \sum_{k=0}^{\infty} e_k \, z^{-k} = b(z)\left(\Omega_2(z) - \sum_{k=0}^{\infty} D_k \, z^{-k-1}\right) + R_2(z) \ ,$$

$$|z| > 1 \ .$$

These identities provide our method of evaluating the moments D_k, D_k^*, by relating them to the e_k, e_k^* which can be found using the methods of §9.

To find the D_k, D_k^* we proceed as follows. We assume that the e_k, e_k^* , $k = 0, 1, \ldots, N+1$ have been found for a given N. The polynomials R_1, R_2 can be constructed using the method of §10. If the index κ is non-negative, we multiply (14.12) by $(1-z)^{\kappa}$ and equate the powers z^0, $z^{-1}, \ldots, z^{-N-1}$, obtaining Ω_1 and then D_k^* ,

$k = 0, 1, \ldots, m+N+\kappa$. If the index is negative, we equate $z^{-\kappa}$, $z^{-\kappa-1}, \ldots, z^{-\kappa-N-1}$, obtaining Ω_1, and then D_k^*, $k = 0, 1, \ldots, m+N+\kappa$. In (14.13), if the index is positive, we equate the powers $z^{\kappa}, z^{\kappa-1}, \ldots, z^{\kappa-N-1}$ obtaining first Ω_2 and then D_k, $k = 0, 1, \ldots, m+N-\kappa$. If in (14.13) the index is negative, we multiply by $(1-z)^{-\kappa}$ and equate the powers $z^0, z^{-1}, \ldots, z^{-N-1}$ obtaining Ω_2, then D_k, $k = 0, 1, \ldots, m+N-\kappa$.

We note that it follows from Lemma 4.2 that Ω_2 is at most of degree one, and that if a is continuous (which we have been assuming) then $\Omega_1 \equiv 0$; however we have included Ω_1 for the purposes of Example 15.2.

Finally we show how the modified moments d_i, d_i^* are obtained from the moments D_k, D_k^*.

From Abramowitz and Stegun [1, §22.3.6], the expansion of T_n in powers is

$$(14.14) \qquad T_n(x) = \frac{n}{2} \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!} (2x)^{n-2m}, \quad n = 0, 1, \ldots,$$

which we write as $T_n(x) = \sum_{j=0}^{n} S_{nj} x^j$. Thus, from (14.1) and (14.10) we have

$$(14.15) \qquad \begin{aligned} d_i &= \sum_{j=0}^{i} S_{ij} D_j, \quad i = 0, 1, \ldots, m+N-\kappa \\ d_i^* &= \sum_{j=0}^{i} S_{ij} D_j^*, \quad i = 0, 1, \ldots, m+N+\kappa . \end{aligned}$$

We note that the S_{ij} grow rapidly with increasing i, (in fact $S_{ii} = 2^{i-1}$), and the d_i, d_i^* tend to zero, so that the growth of rounding errors will restrict the accuracy of the higher order modified moments. A number of different algorithms for the solution of the complete equation were tried, but each suffered a similar loss of accuracy. Therefore it is recommended in practice that the modified moments are calculated using double precision arithmetic, unless in a particular case it is known that only low order moments are required.

In Appendix A we derive a contour integral representation of the modified moments d_i^* .

Summary of the algorithm

To solve a given complete equation using this algorithm, we need to find the function θ, and provide methods of evaluating Z, the polynomials R_1 and R_2, and the moments c_i, $i = 0, 1, \ldots, N$. Then this section is used to give the polynomials Ω_1, Ω_2 and the modified moments d_i, d_i^* , $i = 0, 1, \ldots, m+N\pm\kappa$.

The d_i, d_i^* are needed for the quadrature rules of §13. The functions Γ_j are constructed using Ω_2, R_2 and the polynomials W_i, $i = 0, 1, \ldots, n-1$, which in turn depend on the d_i.

The system of equations (12.21) is then solved for the ξ_i, and then (12.22) and (12.17) are used to find the solution ϕ_n. If the index is negative, then the consistency condition (12.26) must be checked.

This completes the description of the algorithm for numerical solution of the complete singular integral equation. We will now illustrate the evaluation of the modified moments with two examples.

Example 14.1 With a, b as in Example 5.1, θ is given by $\theta(t) = -\alpha-\beta t$; thus the moments $c_k = \int_{-1}^{1} \theta(t)t^k\, dt$ (see 9.3) are

$$c_k = \begin{cases} -\dfrac{2\alpha}{k+1} & k \text{ even} \\[2ex] -\dfrac{2\beta}{k+2} & k \text{ odd} \end{cases}$$

Then from (9.10) we obtain

$$e_0^* = 1 \quad , \quad e_1^* = -2\alpha \quad , \quad e_2^* = 2\alpha^2 - 2\beta/3 \quad , \quad e_3^* = -4\alpha^3/3 + 4\alpha\beta/3 - 2\alpha/3 \ ,$$

etc., and so from (14.5) and (14.4) we obtain E_0^* , E_1^* , etc., which give (provided the integrals exist)

$$E_0^* = -e^{2\beta}\, \frac{1}{\pi} \int_{-1}^{1} \sin[\pi(\alpha+\beta t)] \left(\frac{1-t}{1+t}\right)^{\alpha+\beta t} dt = -2\alpha$$

$$E_1^* = -e^{2\beta}\, \frac{1}{\pi} \int_{-1}^{1} t \sin[\pi(\alpha+\beta t)] \left(\frac{1-t}{1+t}\right)^{\alpha+\beta t} dt = 2\alpha^2 - 2\beta/3$$

$$E_2^* = -e^{2\beta}\, \frac{1}{\pi} \int_{-1}^{1} t^2 \sin[\pi(\alpha+\beta t)] \left(\frac{1-t}{1+t}\right)^{\alpha+\beta t} dt = -4\alpha^3/3 + 4\alpha\beta/3 - 2\alpha/3 \ , \text{ etc.}$$

Example 14.2 We will consider the accuracy of the above method of evaluating the modified moments in more detail. Consider the equation

$$(14.16) \qquad -\sqrt{1-t^2}\ \phi(t) - \frac{1}{\pi} \int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t) \ , \ -1<t<1 \ .$$

Since b (= -1) has no zeros on [-1, 1] , the index is one; m = 0, $\Omega_1 \equiv R_1 \equiv R_2 \equiv 0$ and Ω_2 is of degree one. We note that (14.16) satisfies (4.10), i.e. $b(-1) \le 0$.

We have

(14.17) $$\theta(t) = -\tfrac{1}{2} - \frac{1}{\pi}\arctan\sqrt{1-t^2} \quad , \quad -1\le t\le 1 \ .$$

To find the modified moments, we need e_k , e_k^* , and thus require the moments

$$C_k = \int_{-1}^{1} \theta(t)t^k \, dt \ , \ k = 0, 1,\ldots \quad .$$

For k odd, $C_k = 0$. The even moments can be found by integrating by parts, giving

$$C_{2k} = -\frac{1}{2k+1} - \frac{1}{2k+1} c_{2k}^* \quad , \quad k = 0, 1,\ldots \ ,$$

where
$$c_{2k}^* = \frac{1}{\pi}\int_{-1}^{1} \frac{t^{2k+2}}{2-t^2} \frac{dt}{(1-t^2)^{\frac{1}{2}}}$$

$$= -2^{-2k}\frac{(2k)!}{(k!)^2} + 2c_{2k-2}^* \quad , \ k = 0, 1,\ldots \quad ,$$

and
$$c_{-2}^* = 1/\sqrt{2} \ .$$

The e_k, e_k^* can then be found using §9. From (14.12) we have $\sum_{k=0}^{\infty} e_k^* z^{-k} = (z-1)\sum_{k=0}^{\infty} D_k^* z^{-k-1}$, and equating powers of z, we obtain

(14.18) $$D_k^* = \sum_{j=0}^{k} e_j^* \quad , \quad k = 0, 1,\ldots$$

Finally the d_k^* are found using (14.15):

(14.19) $d_0^* = D_0^*$, $d_1^* = D_1^*$, $d_2^* = 2D_2^* - D_0^*$, $d_3^* = 4D_3^* - 3D_1^*$, etc.

Similarly, writing $\Omega_2(z) = A + Bz$, (14.13) is

$$(1-z)\sum_{k=0}^{\infty} e_k z^{-k} = \sum_{k=0}^{\infty} D_k z^{-k-1} - A - Bz \ ,$$

which gives $B = e_0$, $A = e_1 - e_0$ and

(14.20) $\quad D_k = e_{k+1} - e_{k+2}$, $k = 0, 1, \ldots$, the d_k being found as above.

Using these expressions, the modified moments d_k, d_k^* were calculated using single precision arithmetic (accurate to about 11 figures), and are given in the table below.

To check the accuracy of these figures we can also evaluate d_k and d_k^* more directly (in this case) as follows.

We have from (4.8), (4.18) and (14.17) that

$$r(t) = \sqrt{2 - t^2} \quad \text{and} \quad Z(t) = (1 - t^2)^{-\frac{1}{2}} \exp\left(\frac{1}{\pi} \int_{-1}^{1} \frac{\arctan\sqrt{1-\tau^2}}{\tau - t} \, d\tau \right) ,$$

The singular integral is evaluated in Appendix B, giving

$Z(t) = (1 - t^2)^{-\frac{1}{2}} \left(\frac{\sqrt{2} - t}{\sqrt{2} + t} \right)^{\frac{1}{2}}$. Then (14.1) gives

$$d_k = \frac{1}{\pi} \int_{-1}^{1} \frac{T_k(\tau)}{\sqrt{2} - t} (1-t^2)^{\frac{1}{2}} dt \ , \quad d_k^* = \frac{1}{\pi} \int_{-1}^{1} \frac{T_k(t)}{\sqrt{2} + t} \frac{dt}{(1-t^2)^{\frac{1}{2}}} \ , \quad k = 0, 1, \ldots,$$

and evaluating these integrals we obtain

$$d_k^* = (1 - \sqrt{2})^k \ , \quad k = 0, 1, \ldots$$

(14.21)

$$d_0 = \sqrt{2} - 1 \ , \ d_1 = \tfrac{1}{2}(3-2\sqrt{2}) \ , \ d_k = -(\sqrt{2} - 1)^k \ , \quad k = 2, 3, \ldots$$

In the following table, we give the d_k as found by the method of this section, i.e. (14.20), and the relative error = (exact - computed)/|exact| for both d_k, d_k^* using (14.21).

Table 14.1

k	Using (14.20) d_k	Relative error in d_k	d_k^*
0	0.4142 1356 237	8×10^{-12}	0
5	-0.0121 9330 900	-1×10^{-8}	1×10^{-8}
10	-0.0001 4867 967	-2×10^{-5}	-2×10^{-5}
12	-0.0000 2550 114	3×10^{-4}	-4×10^{-5}
15	-0.0000 0152 503	0.16	0.04
16	0.0000 0109 199	-0.4	-0.4
17	-0.0000 0228 033	-6	0.8

We observe that the relative error increases with k, and has reached serious proportions for k = 15.

§15. Examples - Complete Equation

Example 15.1 Equation with a known solution.

We choose the complete singular integral equation

$$(15.1)\quad -(1-t^2)^{\frac{1}{2}}\phi(t) + \frac{t-\beta}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau - t}\,d\tau + \frac{1}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau+t+\lambda}\,d\tau =$$

$$= -1 + (t^2 + 2\lambda t + \lambda^2 - 1)^{-\frac{1}{2}}\ , \quad -1<t<1\ ,$$

where β, λ are real, with $|\beta|<1$, $|\lambda|>2$. This singular integral equation has, as a particular solution,

$$(15.2)\qquad \phi(t) = (1-t^2)^{-\frac{1}{2}}\ , \quad -1<t<1\ .$$

To test the methods of §14, we will calculate the modified moments d_i, d_i^* from the moments C_i of θ, rather than substitute the explicit expression (11.26), for the function Z, in (14.1). However, for the calculation of Z at the zero of b (for R_1, R_2) and for use in (12.17), we will use the explicit expression (11.26).

From Definition 4.1 we obtain

$$(15.3)\quad \theta(t) = \begin{cases} -1 - \frac{1}{\pi}\arctan\,[(t-\beta)/(1-t^2)^{\frac{1}{2}}]\ , & -1<t<1 \\ -1/2 + \frac{1}{\pi}\arctan\,[(1-t^2)^{\frac{1}{2}}/(t-\beta)]\ , & -1\le t<\beta \\ -3/2 + \frac{1}{\pi}\arctan\,[(1-t^2)^{\frac{1}{2}}/(t-\beta)]\ , & \beta<t\le 1. \end{cases}$$

As in Example 11.3, the index is two, and by (4.8),

$$r^2(t) = 1 + \beta^2 - 2\beta t\ , \quad -1\le t\le 1\ .$$

To obtain the moments of θ, the following method was used. Define

$$(15.4) \qquad h(t) = \pi\,(1-t^2)^{\frac{1}{2}}\,[\theta(t) + 1 + (1/\pi)\arcsin t]\ , \ -1<t<1\ .$$

Then from (15.3) it follows that

$$(15.5) \qquad h(t) = (1-t^2)^{\frac{1}{2}}\arctan[\beta(1-t^2)^{\frac{1}{2}}/(1-\beta t)]\ , \ -1<t<1\ ,$$

so we obtain

$$C_n = \frac{1}{\pi}\int_{-1}^{1}\frac{h(t)t^n}{(1-t^2)^{\frac{1}{2}}}\,dt - \begin{cases} 2/(n+1) & , \ n \text{ even} \\ 1/(n+1) - 2^{-n-1}\, n!/\left[\left(\frac{n+1}{2}\right)!\right]^2 & , \ n \text{ odd} \end{cases} .$$

Since the nearest singularity of h to the arc [-1, 1] is at the point $-\frac{1}{2}(\beta + \beta^{-1})$, then provided β is not close to ± 1, the above integral can be evaluated accurately by M point Gauss - Chebyshev quadrature (see Hildebrand [18, §8.8]):

$$\frac{1}{\pi}\int_{-1}^{1}\frac{h(t)t^n\,dt}{(1-t^2)^{\frac{1}{2}}} = \frac{1}{M}\sum_{i=1}^{M} x_i^{(M)}\, h\,(x_i^{(M)}) + \text{remainder}\ ,$$

where $$x_i^{(M)} = \cos\left(\frac{\pi}{2}\ \frac{2i-1}{M}\right) \quad , \quad i = 1, 2, \ldots, M\ .$$

The method of §12-§14 was used to find the modified moments. The right side of (15.1) was approximated by a polynomial of degree n, and the integral $\int_{-1}^{1}\frac{\phi(\tau)}{\tau+t+\lambda}\,d\tau$ was evaluated using one of the quadrature formulae of §13, with $\nu = n$.

The approximate solution of (15.1) is given by (12.22) and (11.26) as

(15.6) $$\phi_n(t) = \frac{2\psi_n(t)}{(1+\beta^2-2\beta t)(1-t^2)^{\frac{1}{2}}} \quad , \quad -1<t<1 \ ,$$

where

(15.7) $$\psi_n(t) = v(t) - \rho_1 u_1(t) - \rho_2 u_2(t) \ .$$

The arbitrary constants ρ_1, ρ_2 were chosen by specifying that

(15.8) $$\phi_n(\pm 0.6) = 1.25 \ ,$$

so that (15.6) will be equal to the exact particular solution (15.2) at the points ±0.6.

In Table 15.1 below, we give the relative error $[\phi(t) - \phi_n(t)]\ r(t)/Z(t) = \frac{1}{2}\ r^2(t) - \psi_n(t)$ (ϕ given by (15.2)) for $\beta = 0.2$, $n = 4, 10$ and $\lambda = 2.1, 3$. All calculations were carried out in single length arithmetic, to 11 figures.

Table 15.1

Relative error of approximate solution of (15.1)

$[\phi(t) - \phi_n(t)]\, r(t)/Z(t)$, $t = -1.0(0.2)1.0$

t	$\lambda = 3$		$\lambda = 2.1$	
	n=4	n=10	n=4	n=10
-1.0	-0.0000 00505	absolute	-0.00446	-0.0000 178
-0.8	0.0000 00016	value	0.00086	-0.0000 044
-0.6	0.0	$<5 \times 10^{-11}$	0.0	0.0
-0.4	-0.0000 00087	"	-0.00123	-0.0000 030
-0.2	-0.0000 00101	"	-0.00126	-0.0000 007
0.0	-0.0000 00056	"	-0.00053	-0.0000 006
0.2	-0.0000 00010	"	0.00009	-0.0000 009
0.4	0.0000 00005	"	0.00018	0.0000 005
0.6	0.0	"	0.0	0.0
0.8	0.0000 00011	"	0.00017	0.0000 009
1.0	0.0000 00021	"	0.00028	0.0000 009

We also give values of the constants ρ_1, ρ_2 in (15.7) .

Table 15.2

Values of ρ_1 and ρ_2

	$\lambda = 3$		$\lambda = 2.1$	
	n=4	n=10	n=4	n=10
ρ_1	-0.1000 0010	-0.1	-0.1008 1311	-0.1000 0167
ρ_2	1.0000 001	1.0	1.0006 088	1.0000 009

The computer program used for obtaining these results is given in §16.

Using (11.26) and (14.1), it can be shown that

$$d_0 = \tfrac{1}{4}\ ,\ d_1 = 0\ ,\ d_2 = -1/8\ ,\ d_i = 0 \quad ,\quad i = 3, 4, 5, \ldots$$

$$d_i^* = 5^{2-i}/12 \qquad , \qquad i = 0, 1, \ldots$$

which are useful in checking the program.

The program was also run using these exact values for the modified moments, giving results identical with Table 15.1, indicating that in this example, single precision arithmetic was sufficient.

Example 15.2 Zelazny's Equation

In a problem of neutron transport theory, Zelazny [45] obtained the following singular integral equation:

$$(15.9) \qquad \left(1-\tfrac{1}{2}\, c\, \mu \log \frac{1+\mu}{1-\mu}\right) B(\mu) + \tfrac{1}{2}\, c \int_0^1 \frac{\nu B(\nu)}{\nu - \mu}\, d\nu \ + $$
$$+ \tfrac{1}{2}\, c \int_0^1 \frac{\nu\, e^{-2d/\nu}}{\nu + \mu} B(\nu) d\nu = \tfrac{1}{2} \frac{c\nu_0}{\nu_0 - \mu}\, e^{d/\nu_0} \ +$$
$$+ \tfrac{1}{2} \frac{c\nu_0}{\nu_0 + \mu}\, e^{-d/\nu_0} \quad , \quad 0<\mu<1\ ,$$

where ν_0 satisfies

$$(15.10) \qquad 1 = \tfrac{1}{2}\, c\, \nu \log \frac{\nu+1}{\nu-1} \quad ,$$

the branch of the logarithm being chosen so that $\log \frac{\nu+1}{\nu-1}$ is analytic in the complex ν plane excluding $[-1, 1]$, and is zero at infinity. Then for $0<c<1$, (15.10) has two real roots $\pm\nu_0$. For

$c>1$, (15.10) has two imaginary roots which we denote by $\nu = \pm i\rho$, $\rho>0$. Putting $\nu-1 = r_1 e^{i\theta_1}$, $\nu+1 = r_2 e^{i\theta_2}$, $-\pi<\theta_1<\pi$, $-\pi<\theta_2<\pi$ for $\nu = i\rho$ we have $r_1 = r_2$, $\theta_1 + \theta_2 = \pi$, and so (15.10) becomes

$$(15.11) \qquad 1 = c\,\rho \arctan (1/\rho) \ .$$

In the singular integral equation (15.9), the parameter d is the half-thickness of the slab, and the parameter c is related to the average number of neutrons produced per collision.

Physical considerations (see Case and Zweifel [6, §6.6]) indicate that (15.9) is not soluble for $0<c<1$, and for $c>1$ that (15.9) is not soluble if d is less than the "critical thickness", which will depend on c.

We note that the dominant part of equation (15.9) is the same as in Example 11.4, except for the change of sign in b. Thus the methods used by Chandrasekhar [9] and others for the solution of (11.34) may be applicable to (15.9). However, we shall solve (15.9) by the methods of this thesis.

We note again that a is unbounded at $t = 1$, and refer the reader to our comments made in Example 11.4.

To transform (15.9) into the form (1.1), we put $\tau = 2\nu-1$, $t = 2\mu-1$, $\phi(t) = \mu\, B(\mu)$ and after multiplying (15.9) by μ , obtain

$$(15.12) \qquad a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau + \int_{-1}^{1} K(t, \tau)\phi(\tau)d\tau = f(t) \ , \quad -1<t<1 \ ,$$

where

$$a(t) = 1 - \tfrac{1}{4}\, c(1+t)\log((3+t)/(1-t)) \;, \quad b(t) = \tfrac{1}{4}\, c\pi(1+t) \;,$$

(15.13)

$$K(t, \tau) = \tfrac{1}{4}\, c(1+t)\, \exp\, [-4d/(1+\tau)]/(2+t+\tau) \;, \; -1<t<1, \quad -1<\tau<1,$$

and

$$(15.14) \quad f(t) = \tfrac{1}{2}\, c\, \nu_0(1+t)[\exp\,(d/\nu_0)/(2\nu_0-1-t) + \exp(-d/\nu_0)/(2\nu_0+1+t)].$$

If $c>1$, then since $\nu_0 = i\rho$, (15.14) becomes

$$(15.15) \quad f(t) = \tfrac{1}{2}\, c\, \rho(1+t)[2\rho\, \cos(d/\rho) - (1+t)\sin(d/\rho)]/[4\rho^2 + (1+t)^2] \;, \quad -1<t<1 \;.$$

In (15.13), a has a zero in $[-1, 1]$, which we denote by t_0.

Case, Hoffmann and Placzek [7, fig 16, 17] give graphs of ν_0 $(= 1/\kappa_0)$ against c for $0<c<1$, and ρ $(= 1/k_0)$ against c for $c>1$. We note that the graphs have been reproduced in Case and Zweifel [6, fig 4.1, 4.2]; however fig. 4.2 is incorrect.

From Definition 4.1, we have

$$(15.16) \quad \theta(t) = \begin{cases} (1/\pi)\, \arctan\, (b(t)/a(t)) & , \; -1\le t<t_0 \\ 1/2 - (1/\pi)\, \arctan\, (a(t)/b(t)) \;, & -1<t<1 \\ 1 + (1/\pi)\, \arctan\, (b(t)/a(t)) & , \; t_0<t\le 1 \;. \end{cases}$$

Thus $\theta(1) = 1$, and so by (4.13), the index is minus one. We note that since a is not continuous at the point 1, Lemma 4.2 is not applicable unless we first divide (15.12) by $\log\,(1-t)/4)$, say.

Since the index is minus one, the function Z is given by (see (4.18)):

$$(15.17) \qquad Z(t) = (1-t)\exp\left(-\int_{-1}^{1}\frac{\theta(\tau)}{\tau - t}\,d\tau\right) \quad , \quad -1<t<1 .$$

We need a method of evaluating Z and the moments C_i of θ:

$$(15.18) \qquad C_i = \int_{-1}^{1} \theta(\tau)\tau^i \, d\tau \quad , \quad i = 0, 1, \ldots, N.$$

The logarithmic singularity of the function θ at the point 1 makes the accurate evaluation of (15.17) and (15.18) difficult.

To evaluate the moments of θ the following method was used. For brevity, write $g(\tau) = \theta(\tau)\,\tau^i$, and so (15.18) can be written

$$(15.19) \qquad C_i = \int_{-1}^{\epsilon} g(\tau)d\tau + \int_{\epsilon}^{1} g(\tau)d\tau \; ,$$

where $0<\epsilon<1$. The first integral was evaluated by M point Gaussian quadrature; suppose the quadrature formula is of the form

$$(15.20) \qquad \int_{-1}^{1} F(\tau)d\tau = \sum_{i=1}^{M} w_i \, F(t_i) + \text{remainder} .$$

Then after an elementary transformation, the first integral in (15.19) becomes

$$(15.21) \qquad \int_{-1}^{\epsilon} g(\tau)d\tau = \tfrac{1}{2}(1+\epsilon) \sum_{i=1}^{M} w_i \; g(\tfrac{1}{2}[(1+\epsilon)t_i + \epsilon-1]) + \text{remainder} .$$

In the second integral in (15.19), we put $\tau = 1-\exp(-1/u)$, $\sigma = -1/\log(1-\epsilon)$, so we have

$$\int_{\epsilon}^{1} g(\tau)d\tau = \int_{0}^{\sigma} g(1 - \exp[-1/u]) \exp(-1/u)u^{-2} \, du .$$

From (15.16) we have $\theta(1-\exp[-1/u]) = 1 - u + O(u^2)$, and thus the logarithmic singularity has been removed. Applying the quadrature formula (15.20), and defining

$h(u) = g(1-\exp[-1/u]) \exp(-1/u)\, u^{-2}$, $0<u<\sigma$, we have

$$\int_{\epsilon}^{1} g(\tau)d\tau = \tfrac{1}{2}\sigma \sum_{i=1}^{M} w_i\, h(\tfrac{1}{2}\sigma[t_i+1]) + \text{remainder} .$$

M was chosen to be 40.

The function Z was found by writing (15.17) as

$$Z(t) = (1-t)^{1-\theta(t)}(1+t)^{\theta(t)} \exp\left(- \int_{-1}^{1} \frac{\theta(\tau) - \theta(t)}{\tau - t}\, d\tau \right) \quad -1<t<1 ,$$

and evaluating the integral using the same method as above, that is putting $g(\tau) = [\theta(\tau) - \theta(t)]/(\tau-t)$, with t a parameter. This gave good results, except for t near 1.

Since the index of (15.12) is minus one, the integral euqation (15.12) will only be soluble if the consistency condition (12.26) is satisfied, i.e. if

$$\int_{-1}^{1} [f(t) - \int_{-1}^{1} K(t, \tau)\phi(\tau)d\tau] \frac{dt}{r(t)Z(t)} = 0 .$$

These integrals were evaluated numerically using the methods of §13, which gives the approximation δ_0:

(15.22) $$\delta_0 = \sum_{j=0}^{n} \omega_j\, [f(x_j) - \xi_j] + \text{remainder} .$$

Given $c>0$ and $d>0$, ν_0 or ρ was found from (15.10) or (15.11). The computer program of §16 was used to give ϕ_n, and δ_0 was found approximately from (15.22).

For $0<c<1$, several values of d were tried, but as δ_0 was always found to be positive, no solutions of (15.12) were found, in keeping with physical considerations.

For $c>1$, δ_0 was found to change sign as d was increased. Using a simple iterative method, these zeros of δ_0 were refined until $|\delta_0|<10^{-10}$, giving values of c and d for which (15.12) is soluble. These values of c and d are given in Table 15.3.

Using other methods, Case and Zweifel [6, Table 6.4] have given first and second order approximations to d, which we give in Table 15.3 for comparison.

Table 15.3

Critical slab half-thickness d

	Case and Zweifel		This thesis
c	First order	Second order	
1.1	2.1133	2.1132(5)	2.1132 7591
1.5	0.6076	0.6044	0.6040 4712
2.0	0.3268(5)	0.3198(5)	0.3089 5961

It was observed that (15.12) was also solvable for larger values of d (for $c>1$) due to the periodic nature of (15.15). The physical interpretation of this result is uncertain.

We also give the corresponding approximate solutions ϕ_n of equation (15.12) in Table 15.4.

Table 15.4

Approximate solutions of (15.12), (15.13), (15.14)

Values of $\phi_n(t)$, $t = -1.0(0.2)\ 0.8$

$n = 10$

t	c = 1.1	c = 1.5	c = 2.0
-1.0	0.0	0.0	0.0
-0.8	0.0192 8752	0.0451 6606	0.0667 4536
-0.6	0.0258 0374	0.0605 1289	0.0856 5003
-0.4	0.0288 8861	0.0673 8798	0.0901 4567
-0.2	0.0331 1037	0.0745 6001	0.0929 0176
0.0	0.0403 9662	0.0840 9360	0.0964 9583
0.2	0.0507 1523	0.0944 9368	0.0997 1506
0.4	0.0623 1702	0.1026 3817	0.1002 7167
0.6	0.0715 1067	0.1043 5845	0.0954 4905
0.8	0.0708 9022	0.0928 1006	0.0806 3954

§16. A computer program for the solution of the complete equation

In this section, we describe the computer program which was used to give the numerical results in several of the examples discussed above. The program is far from perfect, but may be useful to other workers in this field.

The program solves complete singular integral equations of the form (1.1), using the method of §12 - 14, and is written in Burroughs algol.

We first mention some assumptions made in the program. We require that (4.10) must be satisfied, that is

$$b(-1) \leq 0, \tag{16.1}$$

and that b is a polynomial of degree m, which has only real simple zeros which lie in $[-1, 1]$. This implies that $m = \mu$ (see (22.3)). The program may be extended to allow b to have complex or multiple zeros, or zeros which are not in $[-1, 1]$, provided that the vectors RONE, RTWO and PB are given, where

$$R_1(z) = \sum_{i=0}^{m-1} \text{RONE } [i]\, z^i \quad , \quad R_2(z) = \sum_{i=0}^{m-1} \text{RTWO } [i]\, z^i \quad ,$$

$$b(z) = \sum_{i=0}^{m} \text{PB } [i]\, z^i \quad .$$

We assume that the matrix A in (12.21) (MAT in the program) is nonsingular.

For a general purpose program, we recommend that the modified moments are calculated in double precision arithmetic, the remainder of the program being in single precision. (The program given below is in single precision). See the remarks on page 82 and Example 14.2.

To solve a singular integral equation, the user must provide the REAL PROCEDURES A, B, F, K, THETA and Z (which are of course a, b, f, K, θ and Z in the thesis). THETA will be unnecessary if not used for the moments C[i] and the function Z. The user must also provide the moments C[i], i.e.

$$C[i] = \int_{-1}^{1} \theta(\tau)\tau^{i}\, d\tau \ , \quad i = 0, 1, \ldots, N + |\kappa| - M \ .$$

The following data must be given:

N, M, MPOINT , INDEX, MODE, CONCON

X0, DEL

if MODE = 1 then Y0[i], PHI[i], i = 1, 2,..., κ

if MODE = 2 then RHO[i], i = 1, 2,..., κ

A heading card (any string of characters)

BETA[i] , i = 1, 2,..., m

C1

Explanation of symbols

We only explain the most important symbols. First, those which are mentioned in the thesis.

Program	Thesis
N	n = degree of polynomial approximation to f and number of nodes -1 in quadrature formulae of §13.
M	m = degree of b
INDEX	κ = the index of (1.1).
BETA[i]	β_i = position of the zeros of b.
C1	λ in Example 15.1.
C[i]	C_i = moments of θ

D[i, 1]	d_i = modified moments
D[i, 2]	d_i^* = " "
W[i]	ω_i = weights for quadrature rule of §13
WSTAR[i]	ω_i^* = " " " " " "
OMEGATWO[i]	Ω_2 ; $\Omega_2(z) = \sum_{i=1}^{\kappa-m}$ OMEGATWO[i] z^i

We now explain some symbols used only in the program.

Program	Explanation
MPOINT	number of nodes in the quadrature rule used for evaluating C[i].
MODE	If MODE = 1, then the user specifies YO[i] and PHI[i] , where ϕ(YO[i]) = PHI[i] , i = 1, 2,..., κ, and the program finds the constants RHO[i] , i = 1, 2,..., κ (ρ_{i-1} in (12.22)) and prints this particular solution. If MODE = 2 then the user specifies the constants RHO[i] , i = 1, 2,..., κ (ρ_{i-1} in (12.22)), and the computer prints the corresponding particular solution. If MODE = 0 , then the particular solution v, and κ homogenous solutions $u_1, u_2, \ldots, u_\kappa$ in (12.22) are printed.
DEL	The particular and homogenous solutions v(t) and $u_1(t), \ldots, u_\kappa(t)$ are printed for t = -1.0 (DEL) 1.0 .
CONCON	If CONCON = 1 then no solutions are printed, but the numbers δ_k, k = 0, 1,..., $-\kappa$-1 are found (see (13.6)), i.e. the consistency condition (12.27) is checked. For any other value of CONCON, the solution is printed.

XO May be chosen to be any number in [-1, 1] provided $b(XO) \neq 0$; is used to normalize the coefficients in the polynomial expansion of b.

The procedure GAMMA(t, CA, v)

Given the number t, the matrix $CA[i, j] = T_i(xx_j)$, $i, j = 0, 1, \ldots, n$ where $xx_j = \cos\left(\frac{\pi}{2}\frac{2j+1}{n+1}\right)$ and T_i are the Chebyshev polynomials of the first kind, and given N, D[i, 1] , OMEGATWO[i] , RTWO[i] , B, M, as global variables, then GAMMA gives the vector v such that $v[j] = \Gamma_j(t)$, $j = 0, 1, \ldots, n$, where Γ_j is defined by (12.15).

The procedures SINGULAR, DECOMPOSE, SOLVE and IMPROVE

The procedures SINGULAR, DECOMPOSE, SOLVE and IMPROVE are given in Forsythe and Moler [14, §16], although any library routine for solving linear systems of equations would suffice.

The procedures in the following program are those used for Example 15.1. We first give a typical set of data.

```
 10  1  30  2  1  0
  0.0
  0.2
 -0.6
  1.25
  0.6
  1.25
EXAMPLE  15.1
  0.2
  2.1
```

```
BEGIN
  INTEGER N,INDEX,M,EE,MPOINT,NN,IND,ABSIN,LL,I,J,S,L,II,
          NP,MODE,CONCON;
  REAL Y,P,G,Q,PI,DEL,T,CON,BXO,XJ,XO,BT,ERR,
       C1,C2,C3,C4,C5,C6,C7,C8,C9,C10;
COMMENT BETA ARE THE ZEROS OF B.
   LARGER BOUNDS MAY BE NEEDED;
  ARRAY YO,RHO,PHI,BETA[1:6];

COMMENT THE USER MUST SPECIFY THE FOLLOWING REAL PROCEDURES:
    A,B,F,K,THETA,Z;

  REAL PROCEDURE A(X); VALUE X; REAL X;
    BEGIN
      X:=1.0-X*X; A:=IF X>0.0 THEN -SQRT(X) ELSE 0.0;
    END;
  REAL PROCEDURE B(X); VALUE X; REAL X;
    B:=X-BETA[1];
  REAL PROCEDURE F(X); VALUE X; REAL X;
    F:=-1.0+1.0/SQRT(X*(X+2.0*C1)+C1**2-1.0);
  REAL PROCEDURE K(X,Y); VALUE X,Y; REAL X,Y;
    K:=1.0/(X+Y+C1)/PI;
REAL PROCEDURE THETA(X); VALUE X; REAL X;
    THETA:=IF X<C2 THEN -0.5-ARCTAN(A(X)/B(X))/PI
      ELSE IF X<C3 THEN -1.0+ARCTAN(B(X)/A(X))/PI
                   ELSE -1.5-ARCTAN(A(X)/B(X))/PI;

  REAL PROCEDURE R(X); VALUE X; REAL X;
    R:=SQRT(A(X)**2+B(X)**2);
  REAL PROCEDURE Z(X); VALUE X; REAL X;
  BEGIN
    REAL Y;
    Y:=1.0-X*X; Y:=IF Y>0.0 THEN SQRT(Y) ELSE @-30;
    Z:=2.0/Y/R(X);
  END;
  REAL PROCEDURE H(X); VALUE X; REAL X;
    H:=A(X)*ARCTAN(BETA[1]*A(X)/(1.0-BETA[1]*X));
  BOOLEAN PROCEDURE EVEN(I); VALUE I; INTEGER I;
    EVEN:=(I DIV 2)*2=I;
  INTEGER PROCEDURE MAX(I,J); VALUE I,J; INTEGER I,J;
    MAX:=IF I>J THEN I ELSE J;
  INTEGER PROCEDURE MIN(I,J); VALUE I,J; INTEGER I,J;
    MIN:=IF I<J THEN I ELSE J;
  EBCDIC ARRAY EA[0:79];
  FILE IN(KIND=READER),OUT(KIND=PRINTER);
FORMAT F8(I3,X2,8(E14.7,X2)),
       FG(F15.0);
  DEFINE RANGE=:=0 STEP 1 UNTIL N DO#,
         RAN=:=0 STEP 1 UNTIL INDEX-1 DO#;

COMMENT THE USER MUST PROVIDE DATA FOR THE
    FOLLOWING 5 READ STATEMENTS;

  READ(IN,<10I3>,N,M,MPOINT,INDEX,MODE,CONCON);
  READ(IN,FG,XO,DEL);
  IF MODE=1 THEN FOR I RAN READ(IN,FG,YO[I+1],PHI[I+1]);
  IF MODE=2 THEN FOR I RAN READ(IN,FG,RHO[I+1]);
  READ(IN,<A80>,EA[0]);
  WRITE(OUT,<X20,A80>,EA[0]);
  ABSIN:=ABS(INDEX);
  EE:=N+1;  NN:=N+ABSIN-M;
```

```
BEGIN
  ARRAY
  C[0:NN],                          % MOMENTS OF THETA
  XX,U,V,FF[0:MAX(NN+1,N)],CA[0:N,0:N],
  AA,BB[0:NN+1],  % EXPANSION OF X & 1/X
  PP[-NN-1:0],
  OMEGA,X[1:MPOINT],  % QUADRATURE FOR C
  OMEGATWO[0:1],  % PRINCIPAL PART OF 1/BX AT INFINITY
  PB[0:M+1],  % POLYNOMIAL COEFFICIENTS OF B
  RONE,RTWO[0:M], % INTERPOLATORS OF X & 1/X AT ZEROS OF B
  CC[0:M+ABSIN],
  E[1:N+1],
  D[0:N,1:2], % THE MODIFIED MOMENTS D,DSTAR
  PIND[0:ABS(INDEX)], % COEFFICIENTS OF (1-T)**ABS(INDEX)
  GAM,KAY[0:N,0:N],
  W,WSTAR[0:N],  % WEIGHTS FOR QUADRATURES OF SECTION 13
  KEEP[0:N,0:MAX(C,INDEX)],
  PS,RS,SOLN[1:N+1],
  MAT,LU[1:N+1,1:N+1],
  STWO[-ABSIN-NN-1:MAX(ABSIN,M-1)];
  LABEL OPTION,ERROR;

PROCEDURE GAMMA(T,CA,V);
VALUE T; ARRAY V[0],CA[0,0]; REAL T;
  BEGIN
    ARRAY U[0:N],W[-1:N-1];
    INTEGER I,J; REAL CON,G,OMEGA2,R2,SUM,BT;
    CON:=2.0/(N+1); U[0]:=1.0; W[0]:=D[0,1];
    U[1]:=T; G:=2.0*T;  V[0]:=V[1]:=W[-1]:=0.0;
    FOR J:=2 STEP 1 UNTIL N DO
      BEGIN
        V[J]:=0.0;
        U[J]:=G*U[J-1]-U[J-2];
        W[J-1]:=G*W[J-2]-W[J-3]+2.0*D[J-1,1];
      END;
    OMEGA2:=OMEGATWO[0]+OMEGATWO[1]*T; R2:=0.0; BT:=B(T);
    FOR J:=M-1 STEP -1 UNTIL 0 DO R2:=T*R2+RTWO[J];
    G:=R2+BT*OMEGA2;
    FOR I RANGE
      BEGIN
        SUM:=0.5*G;
        FOR J:=1 STEP 1 UNTIL N DO
          SUM:=CA[J,I]*(U[J]*G-BT*W[J-1])+SUM;
        V[I]:=CON*SUM;
      END;
  END GAMMA;

PROCEDURE CLEAR(A,N,M); VALUE N,M; INTEGER N,M; ARRAY A[*];
  BEGIN INTEGER I;
    FOR I:=N STEP 1 UNTIL M DO A[I]:=0.0;
  END CLEAR;


COMMENT THE PROCEDURES SING,DECOMPOSE,SOLVE & IMPROVE
SHOULD BE INCLUDED HERE. THEY ARE FROM FORSYTHE & MOLER;

IF B(-1.0) GEQ @-10 THEN
  BEGIN
    WRITE(OUT,<//"ERROR. B(-1.0) MUST NOT BE POSITIVE">);
    GO TO ERROR;
  END;
```

```
WRITE(OUT,</"N=",I3,X2,"M=",I3,X2,"MPOINT=",I3,X2,
  "INDEX=",I3,X2,"MODE=",I3,X2,"CONCON=",I3>,N,M,MPOINT,
  INDEX,MODE,CONCON);
IF MODE=1 THEN
    FOR I RAN WRITE(OUT,<"PHI(",F7.4,")=",F7.4>,YO[I+1],
        PHI[I+1]);
IF MODE=2 THEN
    FOR I RAN WRITE(OUT,<"RHO(",I2,")=",F7.4>,I,RHO[I+1]);
PI:=3.1415926536;

COMMENT IF M>0 THEN USER MUST PROVIDE ZEROS OF B AS DATA;

IF M>0 THEN
  BEGIN
    WRITE(OUT,<//X13,"ZEROS OF B">);
    FOR I:=1 STEP 1 UNTIL M DO
      BEGIN
        READ(IN,FG,BETA[I]);
        WRITE(OUT,<X11,F14.10>,BETA[I]);
      END;
    WRITE(OUT,<31("-")>);
  END;
COMMENT THE NEXT SECTION FINDS THE MOMENTS C;
C2:=(BETA[1]-1.0)*0.5; C3:=(BETA[1]+1.0)*0.5; BXO:=B(XO);
WRITE(OUT,</"B(",F7.4,")=",F7.4>,XO,BXO);
Q:=0.25;
FOR I:=0 STEP 1 UNTIL NN DO
  BEGIN
    C[I]:=IF EVEN(I) THEN -2.0/(I+1) ELSE -1.0/(I+1)+Q;
    IF NOT EVEN(I) THEN Q:=Q*(I+1)*(I+2)/(I+3)**2;
  END;
P:=PI*0.5/MPOINT;
FOR J:=1 STEP 1 UNTIL MPOINT DO
  BEGIN
    XJ:=COS((2.0*J-1.0)*P);
    Y:=H(XJ)/MPOINT; G:=1.0;
    FOR S:=0 STEP 1 UNTIL NN DO
      BEGIN
        C[S]:=C[S]+Y*G;
        G:=G*XJ;
      END;
  END; % C[S]=INT(-1,1)THETA(T)T**S*DT

COMMENT-NEXT, AA AND BB ARE FOUND USING EQUATIONS
   (9.6),(9.7),(9.12) IN THESIS;

U[0]:=V[0]:=1.0;  XJ:=C[0];
FOR S:=1 STEP 1 UNTIL NN+1 DO
  BEGIN
    G:=V[S]:=XJ*V[S-1]/S;
    U[S]:=G*(IF EVEN(S) THEN 1.0 ELSE -1.0);
  END;
FOR L:=2 STEP 1 UNTIL NN+1 DO
  BEGIN
    XJ:=C[L-1];
    FOR S:=0 STEP 1 UNTIL NN+1 DO
      BEGIN
        Y:=U[S];  Q:=V[S];  I:=S DIV L;
        IF I NEQ 0 THEN
          BEGIN
            G:=1.0;
```

```
          FOR J:=1 STEP 1 UNTIL I DO
             BEGIN
                II:=S-L*J; G:=XJ*G/J;
                Q:=Q+V[II]*G;
                Y:=Y+U[II]*G*
                   (IF EVEN(J) THEN 1.0 ELSE -1.0);
             END;
          END;
        AA[S]:=Y;   BB[S]:=Q;
      END S;
    IF L NEQ NN+1 THEN FOR S:=0 STEP 1 UNTIL NN+1 DO
      BEGIN
        U[S]:=AA[S];
        V[S]:=BB[S];
      END;
  END L;
WRITE(OUT,<//"  I",X7,"AA[I]",X11,"BB[I]",X12,"C[I-1]">);
WRITE(OUT,FB,0,AA[0],BB[0]);
FOR I:=1 STEP 1 UNTIL NN+1 DO
  WRITE(OUT,FB,I,AA[I],BB[I],C[I-1]);

COMMENT-THE POLYNOMIALS R1,R2 ARE FOUND.
ASSUMING THAT ALL ZEROS OF B ARE SIMPLE AND ARE IN [-1,1];

IF M>0 THEN
  BEGIN
    IF M=1 THEN
      BEGIN
        Q:=BETA[1]; P:=RONE[0]:=Z(Q)*SIGN(A(Q));
        RTWO[0]:=1.0/P;
      END ELSE
      BEGIN
        FOR I:=1 STEP 1 UNTIL M DO
          BEGIN
            P:=BETA[I]; RS[I]:=Z(P)*SIGN(A(P));
            FOR J:=1 STEP 1 UNTIL M DO MAT[I,J]:=P**(J-1);
          END;
        DECOMPOSE(M,MAT,LU); SOLVE(M,LU,RS,SOLN);
        IMPROVE(M,MAT,LU,RS,SOLN,G);
        FOR I:=1 STEP 1 UNTIL M DO
          BEGIN
            RONE[I-1]:=SOLN[I];
            RS[I]:=1.0/RS[I];
          END;
        SOLVE(M,LU,RS,SOLN); IMPROVE(M,MAT,LU,RS,SOLN,G);
        FOR I:=1 STEP 1 UNTIL M DO RTWO[I-1]:=SOLN[I];
      END;
    WRITE(OUT,<//"INTERPOLATORS OF Z AND 1/Z AT"
   ," ZEROS OF B"/"  I",X4,"RONE[I]",X9,"RTWO[I]">);
  FOR I:=0 STEP 1 UNTIL M-1 DO
    WRITE(OUT,FB,I,RONE[I],RTWO[I]);
  END M;
PIND[0]:=1.0; S:=ABS(INDEX);
FOR I:=1 STEP 1 UNTIL S DO PIND[I]:=-PIND[I-1]*(S-I+1)/I;
CLEAR(PB,0,M); PB[0]:=1.0; L:=0; % POLY COEFF OF B
IF M>0 THEN
  BEGIN
    FOR I:=1 STEP 1 UNTIL M DO
      BEGIN
        L:=L+1; G:=BETA[I];
        Y:=PB[0]; PB[0]:=-Y*G;
```

```
          FOR S:=1 STEP 1 UNTIL L DO
            BEGIN
              XJ:=PB[S]; PB[S]:=Y-G*XJ; Y:=XJ;
            END;
        END;
    END;
Y:=0.0; P:=1.0;
FOR I:=0 STEP 1 UNTIL M DO
    BEGIN
      Y:=PB[I]*P+Y;
      P:=P*X0;
    END;
G:=BX0/Y;
FOR I:=0 STEP 1 UNTIL M DO PB[I]:=PB[I]*G;
WRITE(OUT,<//"POLY COEFF OF B">);
FOR I:=0 STEP 1 UNTIL M DO WRITE(OUT,FB,I,PB[I]);

COMMENT NEXT,EQUATES POWERS OF Z TO GIVE C(I) FOR LL=1,
    D(I)STAR FOR LL=2.
    THE SAME METHOD IS USED FOR BOTH (14.12)&(14.13);

Y:=OMEGATWO[0]:=OMEGATWO[1]:=0.0;
FOR I:=0 STEP 1 UNTIL M DO
    Y:=PB[I]*(IF EVEN(I) THEN 1.0 ELSE -1.0)+Y;
IF Y>@-10 THEN
    BEGIN
      WRITE(OUT,<//"B(-1) MUST BE NONPOSITIVE">);
      GO TO ERROR;
    END;
FOR LL:=2,1 DO
    BEGIN
      IND:=(3-2*LL)*INDEX;  NN:=N+IND-M; % NN IS OVERWRITTEN
      IF LL=1 THEN
        BEGIN
          FOR I:=0 STEP 1 UNTIL NN+1 DO BB[I]:=AA[I];
          FOR I:=0 STEP 1 UNTIL M-1 DO RONE[I]:=RTWO[I];
      COMMENT BB,RONE ARE OVERWRITTEN;
        END;
      IF IND GEQ 0 THEN
        BEGIN
          FOR S:=0 STEP 1 UNTIL NN+1 DO
            BEGIN
             XJ:=0.0; L:=MAX(0,S-NN-1); II:=MIN(S,IND);
             FOR J:=L STEP 1 UNTIL II DO
               XJ:=*+PIND[IND-J]*BB[S-J];
             PP[-S]:=XJ;
           END;
         CLEAR(STWO,IND-NN-1,MAX(IND,M-1));
         FOR J:=0 STEP -1 UNTIL -NN-1 DO STWO[J+IND]:=PP[J];
         IF M>0 THEN FOR J:=0 STEP 1 UNTIL M-1 DO
           STWO[J]:=*-RONE[J];
         I:=IND-M;
         IF I GEQ 0 THEN
           BEGIN
             CASE I OF
               BEGIN
                 OMEGATWO[0]:=PP[0]/PB[M];
                 BEGIN
                   OMEGATWO[1]:=PP[0]/PB[M];
                   OMEGATWO[0]:=(PP[-1]-OMEGATWO[1]*
                     (IF M>0 THEN PB[M-1] ELSE 0.0))/PB[M];
```

```
                END;
              END;
            FOR J:=0 STEP 1 UNTIL M DO
              STWO[J]:=*-OMEGATWO[0]*PB[J];
            IF I=1 THEN FOR J:=1 STEP 1 UNTIL M+1 DO
            STWO[J]:=*-OMEGATWO[1]*PB[J-1];
          END;
        E[1]:=STWO[M-1]/PB[M];
        FOR S:=M-2 STEP -1 UNTIL IND-NN-1 DO
          BEGIN
            Y:=0.0; J:=MIN(M,M-S-1);
            IF J>0 THEN FOR I:=1 STEP 1 UNTIL J DO
              Y:=Y+PB[M-I]*E[M-S-I];
            E[M-S]:=(STWO[S]-Y)/PB[M];
          END;
    END  % IND GEQ 0
    ELSE % IND <0
    BEGIN
      S:=M-IND;
      CLEAR(STWO,IND-NN-1,MAX(IND,M-1));
      FOR I:=0 STEP 1 UNTIL S-1 DO
        BEGIN
          XJ:=Y:=0.0; L:=MAX(0,I-M); II:=MIN(I,-IND);
          FOR J:=L STEP 1 UNTIL II DO
            BEGIN
              XJ:=*+PIND[J]*RONE[I-J];
              Y:=*+PIND[J]*PB[I-J];
            END;
          STWO[I+IND]:=-XJ;  CC[I]:=Y;
        END;
      CC[M-IND]:=*+PIND[-IND]*PB[M];
      FOR I:=1 STEP 1 UNTIL NN+1 DO STWO[IND-I]:=BB[I];
      STWO[IND]:=*+BB[0];
      E[1]:=STWO[S-1+IND]/CC[S];
      FOR I:=2 STEP 1 UNTIL EE DO
        BEGIN
          Y:=0.0; L:=MIN(I-1,S);
          FOR J:=1 STEP 1 UNTIL L DO Y:=Y+CC[S-J]*E[I-J];
          E[I]:=(STWO[S-I+IND]-Y)/CC[S];
        END;
    END; %  IND<0
  L:=3-2*LL; D[0,LL]:=-E[1]*L; D[1,LL]:=-E[2]*L; XJ:=-1.0;
  FOR I:=2 STEP 1 UNTIL EE-1 DO
    BEGIN
      G:=XJ:=2.0*XJ; S:=I DIV 2; Y:=0.0;
      FOR J:=0 STEP 1 UNTIL S DO
        BEGIN
          Y:=Y+G*E[I-2*J+1];
          IF J NEQ S THEN
            G:=-G*(I-J*2)*(I-2*J-1)/((J+1)*(I-J-1)*4);
        END;
      D[I,LL]:=L*Y;

    END;
    END LOOP LL;
  WRITE(OUT,</X10,"REQUIRED MOMENTS"/" I",X4,"D[I]",X15,
    "DSTAR[I]">);
  FOR I:=0 STEP 1 UNTIL EE-1 DO
    WRITE(OUT,<I2,2(X2,E17.10)>,I,D[I,*]);
  Y:=PI*0.5/(N+1);
  FOR J RANGE
```

```
  BEGIN
    G:=XX[J]:=COS(Y*(J+J+1));
    CA[0,J]:=1.0; P:=2.0*G; CA[1,J]:=G;
    FOR I:=1 STEP 1 UNTIL N-1 DO
      CA[I+1,J]:=P*CA[I,J]-CA[I-1,J];
  END;
NP:=N+1; P:=2.0*PI/NP;
WRITE(OUT,<//" I",X8,"W[I]",X10,"WSTAR[I]",X9," XX[I]">);
FOR J RANGE
  BEGIN
    G:=-0.5*D[0,2]; Q:=-0.5*D[0,1]; T:=XX[J];
    GAMMA(T,CA,V);
    FOR I RANGE
      BEGIN
        G:=D[I,2]*CA[I,J]+G; Q:=D[I,1]*CA[I,J]+Q;
        GAM[I,J]:=V[I];
      END;
    WSTAR[J]:=P*G; W[J]:=P*Q;
    WRITE(OUT,FB,J,W[J],WSTAR[J],XX[J]);
  END;
WHILE NOT READ(IN,FG,C1) DO
  BEGIN
    FOR J RANGE
      BEGIN
        T:=XX[J]; FF[J]:=F(T);
        FOR I RANGE KAY[I,J]:=K(XX[I],T);
      END;
    WRITE(OUT,<//X10,"C1=",F15.10/>,C1);
FOR I RANGE
  BEGIN
    Y:=0.0; L:=I+1;
    FOR J RANGE
      BEGIN
        G:=0.0;
        FOR S RANGE G:=G+WSTAR[S]*KAY[I,S]*GAM[J,S];
        MAT[L,J+1]:=G;
        Y:=Y+G*FF[J];
      END;
    RS[L]:=Y;
    MAT[L,L]:=*+1.0;
  END;
DECOMPOSE(NP,MAT,LU); SOLVE(NP,LU,RS,SOLN);
IMPROVE(NP,MAT,LU,RS,SOLN,G);
WRITE(OUT,</"CONDITION OF EQUATIONS=",E10.3>,10**(11.0-G));
FOR I RANGE KEEP[I,0]:=FF[I]-SOLN[I+1]; %PARTICULAR SOLUTION
IF CONCON=1 THEN GO TO OPTION;
FOR S RAN
  BEGIN
    CLEAR(RS,1,NP);
    FOR J RANGE
      BEGIN
        G:=WSTAR[J]*B(XX[J])*CA[S,J];
        FOR I RANGE RS[I+1]:=*+G*KAY[I,J];
      END;
    SOLVE(NP,LU,RS,SOLN); IMPROVE(NP,MAT,LU,RS,SOLN,G);
    COMMENT HOMOGENOUS SOLUTIONS;
    FOR J RANGE KEEP[J,S+1]:=SOLN[J+1];
  END;
IF MODE=1 THEN
  BEGIN
    FOR I:=1 STEP 1 UNTIL INDEX DO
```

```
        BEGIN
          T:=YO[I];
          P:=0.0; G:=2.0*T; Q:=B(T);
          GAMMA(T,CA,V);
          CLEAR(U,O,MAX(O,INDEX-1));
          C[0]:=1.0; C[1]:=T; % C IS OVERWRITTEN
          FOR J:=2 STEP 1 UNTIL INDEX-1 DO
            C[J]:=G*C[J-1]-C[J-2];
          FOR S RANGE
            BEGIN
              XJ:=V[S];
              P:=P+XJ*KEEP[S,0];
              FOR J RAN U[J]:=U[J]+XJ*KEEP[S,J+1];
            END;
          FOR J RAN U[J]:=Q*C[J]-U[J];
          G:=Z(T)/R(T);
          RS[I]:=PHI[I]/G-P;
          FOR J RAN MAT[I,J+1]:=U[J];
        END;
      DECOMPOSE(INDEX,MAT,LU); SOLVE(INDEX,LU,RS,SOLN);
      IMPROVE(INDEX,MAT,LU,RS,SOLN,G);
      WRITE(OUT,<//"CONDITION OF EQUATIONS FOR ARBITRARY"
      ," CONSTANTS=",E10.3>,10**(11.0-G));
      WRITE(OUT,<"ARBITRARY CONSTANTS">);
      FOR I RAN
        BEGIN
          J:=I+1; WRITE(OUT,FB,J,SOLN[J]);
          RHO[J]:=SOLN[J];
        END;
    END;
  WRITE(OUT,<90("-")>);
  IF MODE=1 THEN
    WRITE(OUT,<//X16,"A",X29,"SOLUTION",/X3,"X",X7,
      "PARTICULAR",X8,"Z(X)/R(X)",X6,"WHICH EQUALS",X6,I3,X1,
      "HOMOGENOUS SOLUTIONS"/X12,"SOLUTION",X23,"PHI[I] AT"
      ," YO[I]">,MAX(O,INDEX))
  ELSE IF MODE=2 THEN
    WRITE(OUT,<//X16,"A",X29,"SOLUTION",/X3,"X",X7,
      "PARTICULAR",X8,"Z(X)/R(X)",X6,"WITH SPECIFIED",I3,X1,
      "HOMOGENOUS SOLUTIONS"/X12,"SOLUTION",X23,"CONSTANTS",
      " RHO[I]">,MAX(O,INDEX))
  ELSE
    WRITE(OUT,<//,X3,"X",X7,"PARTICULAR",X9,"Z(X)/R(X)",X6,
      I3,X1," HOMOGENOUS SOLUTIONS"/X13,"SOLUTION">,
      MAX(O,INDEX));
  FOR T:=-1.0 STEP DEL UNTIL 1.0001 DO
    BEGIN
      P:=0.0; G:=2.0*T; Q:=B(T);
      GAMMA(T,CA,V);
      CLEAR(U,O,MAX(O,INDEX-1));
      C[0]:=1.0; C[1]:=T;
      FOR J:=2 STEP 1 UNTIL INDEX-1 DO C[J]:=G*C[J-1]-C[J-2];
      FOR S RANGE
        BEGIN
          XJ:=V[S];
          P:=P+XJ*KEEP[S,0];
          FOR J RAN U[J]:=U[J]+XJ*KEEP[S,J+1];
        END;
      FOR J RAN U[J]:=Q*C[J]-U[J];
      G:=Z(T)/R(T);
      IF MODE=0 THEN
```

```
      WRITE(OUT,<F7.4,6(X2,F15.9)>,T,P*G,G,FCR L RAN U[L]*G)
     ELSE
       BEGIN
         Y:=P;
         FOR I RAN Y:=Y+RHO[I+1]*U[I];
         ERR:=0.5*R(T)**2-Y; Y:=Y*G;
         WRITE(OUT,<F7.4,6(X2,F15.9)>,T,P*G,G,Y,
           FOR L RAN U[L]*G,ERR);
COMMENT ERR=(EXACT MINUS APPROXIMATE SOLUTICN)*R/Z;
       END;
   END;
OPTION:IF INDEX<0 THEN
   BEGIN
     WRITE(OUT,</X20,"CONSISTENCY CONDITIONS"/"SINCE THE "
    ,"INDEX IS NEGATIVE,THE ABOVE IS A SOLUTION IF THE "
    ,"FOLLOWING NUMBERS DELTA ARE ZERO TO WITHIN "
     ,"APPROXIMATION ERROR"/"   S",X4,"DELTA[S]">);
     FOR S:=0 STEP 1 UNTIL -INDEX-1 DO AA[S]:=0.0;
     FOR J RANGE
       BEGIN
         G:=XX[J]; Q:=W[J]*KEEP[J,0];
         FOR S:=0 STEP 1 UNTIL -INDEX-1 DO
           BEGIN
             AA[S]:=AA[S]+G;
             Q:=Q*G;
           END;
       END;
     FOR S:=0 STEP 1 UNTIL -INDEX-1 DO WRITE(OUT,FB,S,AA[S]);
   END INDEX NEGATIVE;
   WRITE(OUT,<132("-")/>);
END;
ERROR:END;
END.
```

APPENDIX A

A contour integral for the modified moments

In §14, we derived implicit expressions for the modified moments d_j and d_j^* , by assuming that b was a polynomial. While this is the method we recommend for the evaluation of these moments, we can also derive an explicit expression for the modified moments in terms of a contour integral.

We assume that b is a polynomial, given by (12.3), and consider d_j^* ; d_j can be treated analogously.

In the proof of Lemma 14.2 , the function Φ_1 , which was defined by

$$(A.1)\qquad \Phi_1(z) = \frac{X(z) - R_1(z)}{b(z)} \quad , \quad z \notin [-1, 1] ,$$

was shown to be analytic in the complex plane excluding [-1, 1] , and zero at infinity. The alternative representation for Φ_1 was found:

$$(A.2)\qquad \Phi_1(z) = -\frac{1}{\pi}\int_{-1}^{1} \frac{Z(\tau)}{r(\tau)} \frac{d\tau}{\tau - z} \quad , \quad z \notin [-1, 1] .$$

To use this expression, we seek an expansion of $\frac{1}{\tau - z}$ of the form

$$(A.3)\qquad \frac{1}{\tau - z} = \sum_{k=0}^{\infty}{}' A_k(z)\, T_k(\tau) \quad , \quad z \neq \tau ,$$

where the A_k are to be found. Multiplying by $T_j(\tau)(1 - \tau^2)^{-\frac{1}{2}}$, integrating, and using (12.10), we have

$$(A.4)\qquad A_j(z) = \frac{2}{\pi}\int_{-1}^{1} \frac{T_j(\tau)}{(1 - \tau^2)^{\frac{1}{2}}} \frac{d\tau}{\tau - z} \quad , \quad j = 0, 1, \ldots ; \quad z \notin [-1, 1] .$$

To evaluate this integral, consider the function B_j, defined by

(A.5) $$B_j(z) = \frac{1}{(z^2-1)^{\frac{1}{2}}(z+\sqrt{z^2-1}\,)^j} .$$

To define the branch of the square root, put $z - 1 = r_1 e^{i\theta_1}$, $z + 1 = r_2 e^{i\theta_2}$, $-\pi<\theta_1<\pi$, $-\pi<\theta_2<\pi$. Then $(z^2-1)^{\frac{1}{2}} = (r_1 r_2)^{\frac{1}{2}} e^{\frac{1}{2}i(\theta_1+\theta_2)}$ is analytic in the complex plane excluding $[-1, 1]$. The limiting values of B_j on $[-1, 1]$ are

$$B_j^{\pm}(t) = \frac{1}{\pm i\sqrt{1-t^2}\,(t \pm i\sqrt{1-t^2})^j} , \quad -1<t<1 .$$

Putting $t = \cos\zeta$, $0 \le \zeta \le \pi$, then $t \pm i\sqrt{1-t^2} = e^{\pm i\zeta}$, and so using (12.10), we have

$$B_j^{+}(t) - B_j^{-}(t) = -\frac{2i\cos j\zeta}{(1-t^2)^{\frac{1}{2}}} = -\frac{2i\,T_j(t)}{(1-t^2)^{\frac{1}{2}}} , \quad -1<t<1 .$$

Applying Lemma 2.7, and noting that $\lim_{z\to\infty} B_j(z) = 0$, we have

(A.6) $$B_j(z) = -\frac{1}{\pi}\int_{-1}^{1} \frac{T_j(\tau)}{(1-\tau^2)^{\frac{1}{2}}}\,\frac{d\tau}{\tau - z} , \quad z \notin [-1, 1] .$$

Then, using this and (A.5), (A.4) becomes

(A.7) $$A_j(z) = -\frac{2}{(z^2-1)^{\frac{1}{2}}(z+\sqrt{z^2-1})^j} , \quad j = 0, 1\ldots; \quad z \notin [-1, 1] .$$

and so using (A.3) and (14.1), we can write (A.2) as

(A.8) $$\Phi_1(z) = \frac{2}{(z^2-1)^{\frac{1}{2}}} \sum_{k=0}^{\infty}{}' (z+\sqrt{z^2-1})^{-k}\, d_k^{*} , \quad z \notin [-1, 1] .$$

To obtain an explicit expression for the d_k^{*}, we define

$u = z + (z^2-1)^{\frac{1}{2}}$, choosing the branch as above; then $u^{-1} = z - (z^2-1)^{\frac{1}{2}}$ and so $z = \frac{1}{2}(u + u^{-1})$. Then (A.8) becomes

$$\tfrac{1}{2}(z^2-1)^{\frac{1}{2}} \Phi_1 (z) = \sum_{k=0}^{\infty}{}' d_k^* u^{-k} .$$

Multiplying by u^{j-1} and integrating in the positive direction around a contour c which encloses the unit circle, we obtain

$$\text{(A.9)} \qquad d_j^* = \frac{1}{4\pi i} \int_c u^{j-1} \tfrac{1}{2}(u - u^{-1}) \Phi_1 (\tfrac{1}{2}(u+u^{-1})) du \quad , j \neq 0 \quad ,$$

$$d_0^* = \frac{1}{2\pi i} \int_c u^{-1} \tfrac{1}{2}(u - u^{-1}) \Phi_1 (\tfrac{1}{2}(u+u^{-1})) du \quad .$$

Using (A.1) and changing variables, we have the desired result:

$$\text{(A.10)} \qquad d_j^* = \begin{cases} \dfrac{1}{4\pi i} \displaystyle\int_L (z + \sqrt{z^2-1})^j \dfrac{X(z) - R_1(z)}{b(z)} dz \ , & j = 1, 2, \ldots \\ \dfrac{1}{2\pi i} \displaystyle\int_L \dfrac{X(z) - R_1(z)}{b(z)} dz & , \ j = 0 \ , \end{cases}$$

where L is any contour enclosing [-1, 1] , taken in the positive direction.

If the contour L is shrunk down to the arc [-1, 1] , then the expression (14.1) can be obtained.

Using (A.9), good approximations to d_j^* were obtained by choosing the contour c to be a circle of radius greater than one, and using the quadrature formula of Lyness & Delves [30] . However, much computation was needed, since (A.10) is a double integral, the canonical function X being also given by an integral.

APPENDIX B

A singular integral

To evaluate the singular integral

$$\text{(B.1)} \qquad \psi(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{\arctan\sqrt{1-\tau^2}}{\tau - t}\, d\tau \quad , \quad -1<t<1 \ ,$$

which appeared in Example 14.2, we proceed as follows. Differentiating and using (7.13), we obtain

$$\text{(B.2)} \qquad \frac{d\psi}{dt} = -\frac{1}{\pi}\int_{-1}^{1} \frac{\tau}{2-\tau^2}\, \frac{1}{\sqrt{1-\tau^2}}\, \frac{d\tau}{\tau - t} \ .$$

Since $\int_{-1}^{1} \frac{1}{\sqrt{1-\tau^2}} \frac{d\tau}{\tau - t} \equiv 0$, $-1<t<1$, and using partial fractions, (B.2) can be written

$$\text{(B.3)} \qquad \frac{d\psi}{dt} = -\frac{1}{2\pi}\int_{-1}^{1}\left\{ \frac{1}{\sqrt{2}-\tau} - \frac{1}{\sqrt{2}+\tau} - \left(\frac{1}{\sqrt{2}-t} - \frac{1}{\sqrt{2}+t}\right)\right\} \frac{1}{\sqrt{1-\tau^2}}\, \frac{d\tau}{\tau - t} \ ,$$

and after a little algebra this becomes

$$\text{(B.4)} \qquad \frac{d\psi}{dt} = -\frac{C}{\sqrt{2}+t} - \frac{C}{\sqrt{2}-t} \ ,$$

where $\quad C = \frac{1}{2\pi}\int_{-1}^{1} \frac{d\tau}{(\sqrt{2}+\tau)\sqrt{1-\tau^2}} = \frac{1}{2}$.

Hence, integrating (B.4), we get

$$\psi(t) = \tfrac{1}{2}\log\frac{\sqrt{2}-t}{\sqrt{2}+t} + d \ .$$

From (B.1), we have $\psi(0) = 0$, and so the constant of integration $d = 0$, and

$$\text{(B.5)} \qquad \frac{1}{\pi}\int_{-1}^{1} \frac{\arctan\sqrt{1-\tau^2}}{\tau - t}\, d\tau = \tfrac{1}{2}\log\frac{\sqrt{2} - t}{\sqrt{2} + t} \quad , \quad -1<t<1 \ .$$

APPENDIX C

LIST OF IMPORTANT SYMBOLS

Symbol	Page	Symbol	Page
a	1	$P_{\kappa-1}$	18
$aa_k^{(m)}$	52	Q	31
b	1	r	22
b_m	38	R	31
$bb_k^{(m)}$	50	R_1	79
C_k	50	R_2	38
d_i	70	R_3	40
d_i^*	77	S	32
D_i	81	T_j	70
D_i^*	81	u_k	73
e_k	52	v	73
e_k^*	51	W_j	70
E_k	79	x_i	69
E_k^*	79	X	14, 23
f	1	$X^{\pm}$	24
f_n	40, 44	Z	24
g	14, 21		
G	14, 21	α_i	38
H	9	β_i	38
H^*	9	γ	38
K	1	Γ_j	71
m	38	δ_k	74
n	40, 44, 69	Δ_1	34
N	50	Δ_2	35

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