# SOME PROPERTIES OF TORSION CLASSES OF ABELIAN GROUPS

by

B.J. GARDNER, B.Sc. (Hons.)

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B.J.G.

Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any university, and to the best of my knowledge and belief, contains no copy or paraphrase of material previously published or written by another person except where duly acknowledged.

B. J. Gardner

B.J. Gardner.

# CONTENTS

# INTRODUCTION

The subject matter of this thesis has its origins in Dickson's generalization to certain abelian categories of the notion of torsion as applied to abelian groups [9] and the earlier work of Kurosh [27] and Amitsur [1] on a general theory of radicals for rings and algebras.

In [9] a *torsion theory* was defined as a couple $(\mathcal{J},\mathcal{F})$ of classes of objects such that

(i) $\mathcal{J} \cap \mathcal{F} = \{0\}$

(ii) $\mathcal{J}$ is closed under homomorphic images

(iii) $\mathcal{F}$ is closed under subobjects

and (iv) for every object K there is a short exact sequence

$$0 \longrightarrow T \longrightarrow K \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{J}$ and $F \in \mathcal{F}$.

In this situation $\mathcal{J}$ (resp. $\mathcal{F}$) was called a *torsion* (resp. *torsion-free*) *class*. In the motivating example, $\mathcal{J}$ (resp. $\mathcal{F}$) is the class of all torsion (resp. torsion-free) abelian groups.

Kurosh [27] defined a *radical class* of rings as a class $\mathcal{C}$ such that

(i) $\mathcal{C}$ is closed under conormal epimorphic images

(ii) every ring A has an ideal $\mathcal{C}A$ belonging to $\mathcal{C}$ and containing all other such ideals

and (iii) $\mathcal{C}(A/\mathcal{C}A) = 0$ for every ring A.

Kurosh's theory is applicable to quite general categories ("normal subobject" replaces "ideal" in (ii)), in particular to the abelian categories considered in [9], for which the concepts "torsion class" and "radical class" coincide. The function which assigns $\mathcal{C}A$ to A becomes a functor if its action on morphisms is defined by restriction. Such a functor is called an *idempotent radical*. (Some authors omit "idempotent").

A class $\mathcal{J}$ is a torsion class if and only if it is closed under homomorphic images, direct sums and extensions. In considering specific torsion theories for modules, it has been the practice of most authors to impose an extra condition: that $\mathcal{J}$ be closed under submodules, or *hereditary*, and indeed the analogy with abelian group torsion becomes a little tenuous in the absence of this condition. One expects, for example, that "torsion modules" be in some way characterized by the annihilators of their elements. Nevertheless the language of torsion classes and theories as described above, rather than that of radical classes, has become standard, at least among non-Russian authors writing about abelian categories.

In this thesis we are mainly concerned with torsion classes of abelian groups. In parts of Chapter 4, however, we work in abelian categories satisfying various sets of conditions, as the added generality involves no significant complication of the arguments.

Chapter 1 is essentially a catalogue of other people's results which are referred to in the text. The relation between Dickson's work and that of Kurosh and Amitsur is also briefly discussed.

The closure properties which characterize torsion classes ensure that any group or class of groups is contained in a smallest torsion class; this torsion class is said to be *determined* by the group or class in question.

Some examples of torsion classes determined by torsion-free groups are considered in Chapter 2. The behaviour of rational groups as members of torsion classes is investigated and the torsion classes determined by rational and torsion groups are completely described.

The principal results are to be found in Chapters 3 and 4, where additional closure properties for torsion classes are considered. It is shown in §1 of Chapter 3 that a torsion class is closed under countable direct products, (i.e. direct products of countable sets of groups) if and only if it is determined by torsion-free groups. The remainder of the chapter if devoted to closure under pure subgroups. The torsion classes with this property are characterized - such a torsion class either contains only torsion groups or is determined by a subring of the rationals and a set of primary cyclic groups - and the result generalized to obtain a description of those closed under S-pure subgroups, where S is a set of primes.

In Chapter 4 we consider torsion classes closed under certain generalized pure subgroups as defined by Carol Walker in [41]. The following question is investigated: if $\mathcal{U}$ and $\mathcal{J}$ are torsion classes, when is $\mathcal{J}$ closed under $\mathcal{U}$-pure subgroups? This question is actually a generalization of the one for ordinary pure subgroups answered in Chapter 3 - although purity is not

$\mathcal{U}$-purity for any torsion class $\mathcal{U}$, a torsion class is closed under pure subgroups if and only if it is closed under $\mathcal{J}_0$-pure subgroups, where $\mathcal{J}_0$ is the class of all torsion groups. Some special cases of the general question are answered, for example the case where each of $\mathcal{J}$ and $\mathcal{U}$ is determined by a single rational group. We also consider an approach to the general problem. A class of groups defined in terms of a rank function associated with a given $\mathcal{U}$ is described, whose members, with those of $\mathcal{U}$, determine all $\mathcal{J}$ closed under $\mathcal{U}$-pure subgroups. When $\mathcal{U} = \mathcal{J}_0$, the groups in question are the rational groups, so the results of Chapter 3 indicate that a smaller class will in general suffice for representations of the torsion classes $\mathcal{J}$. Some other examples are also given.

Chapter 5 has two brief sections. In the first we discuss the Amitsur radical construction [1] starting from a single rational group and in the second we characterize the torsion classes of abelian groups whose idempotent radicals r split, i.e. r(A) is always a direct summand of A.

Some of the results presented here have been published elsewhere ([18], [19]). The main results of [18] are contained in §2 of Chapter 3 while the theorem in [19] is a sort of *leit-motiv* for the theory of types in the present work: it is stated explicitly as COROLLARY 5.4, and the proof of THEOREM 5.3 is a generalization of the proof given in [19], the essential part of which is also contained in the proof of THEOREM 4.79; the result can also be obtained, in a rather different manner, as a joint corollary to THEOREMS 3.12 and 3.13.

In notation and terminology we generally follow Fuchs [15] or Mitchell [33], and most deviations have the sanction of popular usage, but for the reader's convenience we give the following table of notation.

| | |
|---|---|
| $\mathcal{A}\mathcal{b}$ | category of abelian groups |
| $Z$ | group of integers |
| $Q$ | group of rational numbers |
| $Q(p)$ | group (or ring) $\{\frac{m}{p^n} \mid m,n \in Z\}$ where p is a prime |
| $Q(S)$ | group (or ring) $\{\frac{m}{n} \mid m \in Z, n \in Z$ with prime factors in $S\}$ where S is a set of primes |
| $I(p)$ | group (or ring) of p-adic integers |
| $Z(n)$ | cyclic group of order n |
| $Z(p^\infty)$ | quasicyclic p-group (p prime) |
| $[\text{Ш}]$ | group generated by set $\text{Ш}$. |
| $[x_\lambda \mid \lambda \in \Lambda]$ | group generated by set $\{x_\lambda \mid \lambda \in \Lambda\}$ |
| $[x_1,\dots,x_n]$ | group generated by elements $x_1,\dots,x_n$ |
| $[x]_*$ | smallest pure subgroup containing x, where x is an element of a torsion-free group |
| $h(x)$ | height of an element x of a torsion-free group |
| $\tau(x)$ | type of an element x of a torsion-free group |
| $\tau(X)$ | type of a rational group X |
| $\tau(h_1,h_2,\dots)$ | type of a height $(h_1,h_2,\dots)$ |
| $G(\sigma)$ | subgroup $\{x \in G \mid \tau(x) \geq \sigma\}$ of a torsion-free group G |
| $A \oplus B$, $\bigoplus A_\lambda$ | direct sum (= coproduct = discrete direct sum) |

| | |
|---|---|
| $\prod A_\lambda$ | direct product (= product = complete direct sum) |
| $(a_\lambda)$ | element of $\bigoplus A_\lambda$ or $\prod A_\lambda$ |
| [A,B] | group of homomorphisms (morphisms) from A to B. |
| // | end of a proof |
| | Unless the contrary is stated explicitly, "GROUP" always means "ABELIAN GROUP". |

CHAPTER 1

# PRELIMINARIES

## 1. Torsion Theories

In this section we shall work in a *locally small* abelian category $\mathcal{K}$ which is *subcomplete* in the following sense:

*DEFINITION 1.1: A category is called subcomplete if for every set* $\{A'_\lambda \mid \lambda \in \Lambda\}$ *of subobjects of an object* A, *the direct sum (coproduct)* $\bigoplus A'_\lambda$ *and the direct product (product)* $\prod(A/A'_\lambda)$ *both exist.*

*DEFINITION 1.2: A torsion theory for* $\mathcal{K}$ *is an ordered pair* $(\mathcal{T}, \mathcal{F})$, *where* $\mathcal{T}$ *and* $\mathcal{F}$ *are classes of objects of* $\mathcal{K}$ *satisfying the following conditions:*

(i) $\mathcal{T} \cap \mathcal{F} = \{0\}$.

(ii) *If* $T \longrightarrow A \longrightarrow 0$ *is exact with* $T \in \mathcal{T}$, *then* $A \in \mathcal{T}$.

(iii) *If* $0 \longrightarrow A \longrightarrow F$ *is exact with* $F \in \mathcal{F}$, *then* $A \in \mathcal{F}$.

(iv) *For each object* K *of* $\mathcal{K}$ *there is a short exact sequence*

$$0 \longrightarrow T \longrightarrow K \longrightarrow F \longrightarrow 0$$

*with* $T \in \mathcal{T}$ *and* $F \in \mathcal{F}$.

*In this situation* $\mathcal{T}$ *is called a torsion class,* $\mathcal{F}$ *a torsion-free class.*

We shall describe (ii) and (iii) by saying that the relevant class is *closed under homomorphic images, subobjects* respectively. Other closure properties for classes will be described similarly. Note that both $\mathfrak{T}$ and $\mathfrak{F}$ are closed under isomorphisms.

*THEOREM 1.3: A non-empty class* $\mathfrak{T}$ *is a torsion class if and only if it is closed under homomorphic images, direct sums and extensions.* $\mathfrak{F}$ *is a torsion-free class if and only if it is non-empty and closed under subobjects, direct products and extensions.*

(Here *closure under extensions* means that in an exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

L belongs to the class if K and M do).

PROOF. [9] Theorem 2.3.//

*THEOREM 1.4: Let* $\mathfrak{T}$, $\mathfrak{F}$ *be classes of objects of* $\mathfrak{K}$. *Then* $(\mathfrak{T},\mathfrak{F})$ *is a torsion theory if and only if*

$$[T,K] = 0 \text{ for all } T \in \mathfrak{T} \Leftrightarrow K \in \mathfrak{F}$$

*and*

$$[K,F] = 0 \text{ for all } F \in \mathfrak{F} \Leftrightarrow K \in \mathfrak{T}.$$

PROOF. [9] Theorem 2.1.//

*THEOREM 1.5: A torsion class* $\mathfrak{T}$ *belongs to a unique torsion theory* $(\mathfrak{T},\mathfrak{F})$, *where*

$$\mathfrak{F} = \{B \mid [A,B] = 0 \text{ for all } A \in \mathfrak{T}\}.$$

*Similarly a torsion-free class* $\mathfrak{S}$ *belongs to a unique theory* $(\mathfrak{U},\mathfrak{S})$, *where*

$$\mathcal{U} = \{A \mid [A,B] = 0 \textit{ for all } B \in \mathcal{Y}\}.$$

PROOF. [9] Proposition 3.3.//

From THEOREM 1.3 it is clear that the intersection of any family of torsion (resp. torsion-free) classes is a torsion (resp. torsion-free) class.

*DEFINITION 1.6: For any class* $\mathcal{C}$ *of objects of* $\mathcal{K}$, $T(\mathcal{C})$ *is the smallest torsion class with* $\mathcal{C}$ *as a sub-class,* $F(\mathcal{C})$ *the smallest such torsion-free class. If* $\mathcal{C}$ *has a single member* $C$, $T(C)$ *and* $F(C)$ *will be used rather than* $T(\{C\})$ *and* $F(\{C\})$. $T(\mathcal{C})$ *will also be referred to as the torsion class determined by* $\mathcal{C}$.

*THEOREM 1.7: For any class* $\mathcal{C}$ *of objects of* $\mathcal{K}$,

$\{A \mid [A,C] = 0$ *for all* $C \in \mathcal{C}\}$ *is a torsion class,*

$\{B \mid [C,B] = 0$ *for all* $C \in \mathcal{C}\}$ *is a torsion-free class,*

$T(\mathcal{C}) = \{A \mid [C,B] = 0$ *for all* $C \in \mathcal{C} \Rightarrow [A,B] = 0\}$,

$F(\mathcal{C}) = \{B \mid [A,C] = 0$ *for all* $C \in \mathcal{C} \Rightarrow [A,B] = 0\}$.

*Hence* $T(\mathcal{A}) \subseteq T(\mathcal{B})$ *and* $F(\mathcal{A}) \subseteq F(\mathcal{B})$ *whenever* $\mathcal{A} \subseteq \mathcal{B}$ .

PROOF. [9] Propositions 3.1 - 3.3.//

## 2. Torsion Classes and Radicals

*DEFINITION 1.8: A functor* $r : \mathcal{K} \longrightarrow \mathcal{K}$ *is called a subfunctor of the identity if*

(i) $r(K) \subseteq K$ *for each object* $K$

*and* (ii) *for any morphism* $f : K \longrightarrow L$ *in* $\mathcal{K}$,

$r(f)$: $r(K) \longrightarrow r(L)$ *is the restriction of* $f$ *to* $r(K)$, *i.e. the diagram*

$$\begin{array}{ccc} r(K) & \xrightarrow{r(f)} & r(L) \\ \downarrow & & \downarrow \\ K & \xrightarrow{f} & L \end{array}$$

*commutes, where the vertical arrows represent inclusions.*
*If in addition*

(iii) $r(K/r(K)) = 0$,

r *is called a radical, and an idempotent radical if also*

(iv) $r^2 = r$, *ie.* $r(r(K)) = r(K)$ *for every object* K.

Conditions (ii), (iii), (iv) are independent for functions which assign subobjects to objects of $\mathcal{K}$, as is demonstrated by the following simple examples for $\mathcal{K} = \mathcal{Ab}$.

*EXAMPLE 1.9:* $r(A) = pA$ for every $A \in \mathcal{Ab}$, where p is prime. Since for any homomorphism $f: A \longrightarrow B$, we have $f(pA) \subseteq pB$, r can be made into a functor satisfying (ii). (iii) is also satisfied, but (iv) is not, as for example $p\,Z(p^2) \cong Z(p)$ and $p\,Z(p) = 0$.

*EXAMPLE 1.10:* Define

$$r(A) = \begin{cases} A \text{ if } A \cong Z \\ 0 \text{ if } A \ncong Z . \end{cases}$$

r satisfies (iii) and (iv), but if $0 \neq A \ncong Z$, there are non-zero homomorphisms f from Z to A, so $f(r(Z)) \nsubseteq r(A)$.

*EXAMPLE 1.11:* $r(A) = A[p] = \{a \in A \mid pa = 0\}$, where p is prime. (ii) and (iv) are satisfied, but for example $r(Z(p^2)) \cong Z(p)$ and $r(Z(p^2)/r(Z(p^2))) \cong r(Z(p)) = Z(p)$.

*PROPOSITION 1.12: Let* $(\mathfrak{T}, \mathfrak{F})$ *be a torsion theory for* $\mathcal{K}$. *Every object* K *has a unique largest subobject* $\mathfrak{T}K$ *belonging to* $\mathfrak{T}$. $\mathfrak{T}K$ *satisfies the following equalities*

(i) $\mathfrak{T}K = \bigcup\{K' \subseteq K \mid K' \in \mathfrak{T}\}$

(ii) $\mathfrak{T}K = \bigcap\{K' \subseteq K \mid K/K' \in \mathfrak{F}\}$ .

*Also,* $\mathfrak{T}(K/\mathfrak{T}K) = 0$ *for every* K.

PROOF. [9] Proposition 2.4.//

For any morphism $f: K \longrightarrow L$ in $\mathcal{K}$, we have $f(\mathfrak{T}K) \subseteq \mathfrak{T}L$, since $f(\mathfrak{T}K) \in \mathfrak{T}$. In addition $\mathfrak{T}(\mathfrak{T}K) = \mathfrak{T}K$ for every K, so as a consequence of PROPOSITION 1.12 we have

*COROLLARY 1.13: Let* $(\mathfrak{T}, \mathfrak{F})$ *be a torsion theory for* $\mathcal{K}$. *For any object* K *let* $r_{\mathfrak{T}}(K) = \mathfrak{T}K$ *and let* $r_{\mathfrak{T}}$ *act on morphisms by restriction. Then* $r_{\mathfrak{T}}$ *is an idempotent radical. Further,* $r_{\mathfrak{T}}(K) = K$ *if and only if* $K \in \mathfrak{T}$ *and* $r_{\mathfrak{T}}(K) = 0$ *if and only if* $K \in \mathfrak{F}$.//

Conversely, each idempotent radical determines a torsion class. Proof of this requires

*LEMMA 1.14: Let* $r: \mathcal{K} \longrightarrow \mathcal{K}$ *be a radical,* $\mathcal{C}$ *the class of objects* L *for which* $r(L) = 0$. *For any* $K \in \mathcal{K}$ *we have*

$$r(K) = \bigcap\{K' \subseteq K \mid K/K' \in \mathcal{C}\}$$

PROOF. As for Proposition 2.4 of [9].//

This result seems to be well-known. It appears without proof in [5]. It is clear that $r(K) = K$ if and only if $[K,L] = 0$ for each $L \in \mathcal{C}$. Using THEOREM 1.7 we therefore obtain

*COROLLARY 1.15: For any radical* $r: \mathcal{K} \longrightarrow \mathcal{K}$ *the class*

$$\{K \in \mathcal{K} \mid r(K) = K\}$$

*is a torsion class.//*

Thus in particular, if $r$ is an idempotent radical,

$$\mathcal{J}_r = \{K \in \mathcal{K} \mid r(K) = K\}$$

is a torsion class.

*LEMMA 1.16: If* $r : \mathcal{K} \longrightarrow \mathcal{K}$ *is a radical and* $K \subseteq r(L)$, *then* $r(L/K) = r(L)/K$.

PROOF. [32] p.110.//

For any idempotent radical $r$ and any $K \in \mathcal{K}$,

$$r(\mathcal{J}_r(K/r(K))) \subseteq r(K/r(K)) = 0 .$$

But as $r^2(K) = r(K)$, $r(K)$ belongs to $\mathcal{J}_r$, so $r(K) \subseteq \mathcal{J}_r K$ and by LEMMA 1.16,

$$(\mathcal{J}_r K)/r(K) = \mathcal{J}_r(K/r(K)) = 0$$

i.e. $r(K) = \mathcal{J}_r K$.

Also, for any torsion class $\mathcal{J}$, we have the identities

$$\mathcal{J} = \{K \in \mathcal{K} \mid \mathcal{J}K = K\} = \{K \in \mathcal{K} \mid r_{\mathcal{J}}(K) = K\} .$$

We have thus proved

*THEOREM 1.17: There is a one-to-one correspondence between torsion classes* $\mathcal{J}$ *of* $\mathcal{K}$ *and idempotent radicals* $r: \mathcal{K} \longrightarrow \mathcal{K}$, *defined by*

$$\mathcal{T} \longrightarrow r_{\mathcal{T}} \;;\; r \longrightarrow \mathcal{T}_r. \;//$$

In [9] a stronger assertion is made, namely that a subfunctor of the identity is an idempotent radical if and only if its class of fixed objects is a torsion class. The discussion above shows that this is false. A partial ordering $\leq$ of the radicals $\mathcal{K} \longrightarrow \mathcal{K}$ is defined by

$$r_1 \leq r_2 \Leftrightarrow r_1(K) \subseteq r_2(K) \text{ for all } K \in \mathcal{K}.$$

If $\mathcal{T}$ is a torsion class, its associated idempotent radical r is (in the sense of this relation) the smallest radical whose class of fixed objects is $\mathcal{T}$. This is proved in [32] (p.110).

## 3. The Kurosh Construction

We now discuss a construction, which while valid in more general categories, produces torsion classes in subcomplete, locally small abelian categories.

Let $\mathcal{A}$ be a category satisfying the following conditions:

(i) $\mathcal{A}$ has a zero object.

(ii) Every morphism of $\mathcal{A}$ has a kernel.

(iii) Every morphism has a conormal epimorphic image.

(iv) If $f:A \longrightarrow B$ is a conormal epimorphism and A' is a normal subobject of A, then f(A') is a normal subobject of B.

(v) Every infinite, well-ordered, strictly increasing chain of normal subobjects of an object has a normal union.

Given a class $\mathcal{C}$ of objects of $\mathcal{A}$, we define a class $\bar{\mathcal{C}}$ by transfinite induction as follows:

Let

$$\mathcal{C}_1 = \{A \in \mathcal{Q} \mid A \text{ is the image of a conormal epimorphism from some } C \in \mathcal{C}\}.$$

If $\mathcal{C}_\alpha$ has been defined for all ordinals $\alpha < \beta$, let

$$\mathcal{C}_\beta = \{A \in \mathcal{Q} \mid \text{For every non-zero conormal epimorphism } f:A \twoheadrightarrow B, B \text{ has a non-zero normal subobject belonging to some } \mathcal{C}_\alpha, \alpha < \beta\}.$$

Finally, let $\bar{\mathcal{C}} = \bigcup_\alpha \mathcal{C}_\alpha$. $\bar{\mathcal{C}}$ is called the *lower radical class* determined by $\mathcal{C}$.

This construction was first used by Kurosh [27] for rings and algebras. The possibility of generalizing it to categories satisfying conditions (i) - (v) was demonstrated by Shul'geifer [39]. It has since been used in the category of all (not necessarily abelian) groups (a brief discussion appears in [29]) and elsewhere. We have given here a slightly modified form due to Sulinski, Anderson and Divinsky [40].

For the categories in which we are interested, the construction takes a simpler form.

*PROPOSITION 1.18: If* $\mathcal{Q}$ *satisfies (i) - (v) and if in addition normality of subobjects is transitive in* $\mathcal{Q}$, *the lower radical class construction terminates at the second stage, i.e. for every class* $\mathcal{C}$, $\bar{\mathcal{C}} = \mathcal{C}_2$.

PROOF. We need only show that $\mathcal{C}_3 = \mathcal{C}_2$. To this end let $A \in \mathcal{C}_3$ have a conormal epimorphic image A", with $B \in \mathcal{C}_2$ a non-zero normal subobject of A". B has a non-zero normal subobject $B' \in \mathcal{C}_1$, and by assumption, B' is normal in A". Thus

$A \in \mathcal{C}_2$, so $\mathcal{C}_3 = \mathcal{C}_2$.//

*PROPOSITION 1.19: Let $\mathcal{K}$ be a subcomplete, locally small abelian category. For any class $\mathcal{C}$ of objects of $\mathcal{K}$,* $T(\mathcal{C})$ *is the lower radical class which $\mathcal{C}$ determines.*

PROOF. If K belongs to $T(\mathcal{C})$, so does any non-zero homomorphic image K", i.e. $[C,K''] \neq 0$ for some $C \in \mathcal{C}$. This means that $K \in \mathcal{C}_2$, so using PROPOSITION 1.18, we have $T(\mathcal{C}) \subseteq \mathcal{C}_2 = \bar{\bar{\mathcal{C}}}$. Conversely, if r is the idempotent radical associated with $T(\mathcal{C})$ and if $K \in \overline{\mathcal{C}} = \mathcal{C}_2$, then $[C,K/r(K)] = 0$ for all $C \in \mathcal{C}$, so $K/r(K) = 0$, i.e. $K \in T(\mathcal{C})$, i.e. $\bar{\mathcal{C}} = \mathcal{C}_2 \subseteq T(\mathcal{C})$.//

In some non-abelian categories upper bounds have been found for the number of steps required in the lower radical construction: $\omega$ (the first infinite ordinal) in the categories of all groups (Shchukin [38]) and associative rings [40], $\omega^2$ in the category of alternative rings [40]. In the case of associative rings, it has been shown that $\omega$ steps are sometimes needed [23].

## 4. The Amitsur Construction

Amitsur [1] has discussed the following construction in rings and in objects of certain categories:

Let $\pi$ be a property of normal subobjects. Given an object A, define

$$A_1 = \bigcup \{A' \subsetneq A \mid A' \text{ is normal and has property } \pi\}$$

and if $A_\alpha$ has been defined for all ordinals $\alpha < \beta$, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ if $\beta$ is a limit ordinal and otherwise let $A_\beta$ be defined by the exact sequence

$$(*) \quad 0 \longrightarrow A_{\beta-1} \longrightarrow A_\beta \longrightarrow (A/A_{\beta-1})_1 \longrightarrow 0 .$$

If the category is suitably chosen, there exists an ordinal $\lambda$ for which $A_\lambda = A_{\lambda+1}$. For such $\lambda$, Amitsur called the subobject $A_\lambda$ the *upper $\pi$-radical* $U(A,\pi)$. Extra conditions must be imposed on $\pi$ if the upper $\pi$-radical is to determine a radical in Kurosh's sense (what we have called an idempotent radical in the abelian case). However, if for a subcomplete locally small abelian category we take $\pi$ to mean membership of some class closed under homomorphic images, then all requirements are met and we have

PROPOSITION 1.20: *Let $\mathcal{C}$ be a class of objects in a subcomplete, locally small abelian category, $\mathcal{C}_1$ the class of all homomorphic images of members of $\mathcal{C}$, $\pi$ the property "membership of $\mathcal{C}_1$" and* r *the idempotent radical associated with* $T(\mathcal{C})$. *Then* $U(K,\pi) = r(K)$ *for any* $K \in \mathcal{K}$.

PROOF. We show firstly that $K_\beta \in T(\mathcal{C})$ for every $\beta$. If $\beta = 1$, this is clear from the closure properties of torsion classes. The same remark holds for $K_\beta$ when $\beta$ is a limit ordinal and $K_\alpha \in T(\mathcal{C})$ for all $\alpha < \beta$. In the remaining case it is clear from (*) that $K_\beta \in T(\mathcal{C})$ if $K_{\beta-1} \in T(\mathcal{C})$.

Now let $\lambda$ be any ordinal with $K_\lambda = K_{\lambda+1}$. Then $K_\lambda \in T(\mathcal{C})$, but $(K/K_\lambda)_1 = K_{\lambda+1}/K_\lambda = 0$, so $r(K/K_\lambda) = 0$ whence it follows that $r(K) = K_\lambda$.//

Even in abelian categories there is no finite upper bound on the number of steps that may be needed in the Amitsur construction. We shall consider some examples in Chapter 5.

## 5. Torsion Classes of Abelian Groups

For the remainder of this chapter we shall only consider torsion theories and classes for $\mathcal{Ab}$.

*DEFINITION 1.21: If* p *is a prime, a* p-*divisible group* G *is one for which* $pG = G$. *If* G *is* p-*divisible for all primes* p *in some set* P, *it is said to be* P-*divisible. A group is called* p-*reduced (resp.* P-*reduced) if it has no non-zero* p-*divisible (resp.* P-*divisible) subgroups.*

*DEFINITION 1.22: A* P-*group, where* P *is a set of primes, is a direct sum of* p-*groups, where* p *varies over* P.

The torsion theories described below make frequent appearances in the sequel; the notation given here will be preserved throughout.

*EXAMPLE 1.23:* $(\mathfrak{T}_0, \mathfrak{F}_0)$. $\mathfrak{T}_0$ (resp. $\mathfrak{F}_0$) is the class of all torsion (resp. torsion-free) groups. The maximum torsion subgroup of a group A will be denoted by $A_t$.

*EXAMPLE 1.24:* $(\mathfrak{T}_p, \mathfrak{F}_p)$. $\mathfrak{T}_p$ is the class of all p-groups, (p is a prime). The maximum p-subgroup of A will be denoted by $A_p$.

*EXAMPLE 1.25:* $(\mathfrak{T}_P, \mathfrak{F}_P)$. $\mathfrak{T}_P$ is the class of all P-groups, where P is a set of primes. The maximum P-subgroup of A will be denoted by $A_P$.

*EXAMPLE 1.26:* $(\mathfrak{D},\mathfrak{R})$. $\mathfrak{D}$ (resp. $\mathfrak{R}$) is the class of all divisible (resp. reduced) groups.

*EXAMPLE 1.27:* $(\mathfrak{D}_p, \mathfrak{R}_p)$. $\mathfrak{D}_p$ (resp. $\mathfrak{R}_p$) is the class of all p-divisible (resp. p-reduced) groups, where p is a prime.

*EXAMPLE 1.28:* $(\mathfrak{D}_P, \mathfrak{R}_P)$. $\mathfrak{D}_P$ (resp. $\mathfrak{R}_P$) is the class of all P-divisible (resp. P-reduced) groups, where P is a set of primes.

We conclude this chapter with a list of results from [8].

*PROPOSITION 1.29: For any prime* p, $T(Z(p)) = \mathfrak{T}_p$

PROOF. [8] Lemma 2.1.//

From this it is easy to deduce

*COROLLARY 1.30:* $\mathfrak{T}_p = T(A)$ *for any non-divisible* p-*group* A.//

*PROPOSITION 1.31: For any prime* p, $T(Z(p^\infty)) = \mathfrak{T}_p \cap \mathfrak{D}$.

PROOF. [8] Lemma 2.2.//

These results are used to obtain a complete description of all torsion classes $\mathfrak{T} \subseteq \mathfrak{T}_0$. We find it convenient to introduce a generic name for classes of this kind.

*DEFINITION 1.32: A torsion class containing only torsion groups is called a t-torsion class.*

*THEOREM 1.33: Let* $P_1$ *and* $P_2$ *be disjoint sets of primes and let* $\mathcal{T}$ *be the class of all groups of the form* $A_1 \oplus A_2$*, where* $A_1$ *is a* $P_1$*-group and* $A_2$ *a divisible* $P_2$*-group. Then*

$$\mathcal{T} = T(\{Z(p) \mid p \in P_1\} \cup \{Z(p^\infty) \mid p \in P_2\})$$

*Any t-torsion class is uniquely represented in this way.*

PROOF. [8] Theorem 2.6.//

This result was obtained earlier by Kurosh [28]. Kurosh actually proved that the classes described are the only radical classes for the full subcategory of abelian p-groups, but THEOREM 1.33 follows easily from this. If a category $\mathcal{A}$ satisfies conditions (i) - (v) of Section 3, then so does any full subcategory whose class of objects is closed under conormal epimorphic images and normal subobjects, and so lower radical classes can also be constructed in the subcategory. Such classes need not be radical classes in $\mathcal{A}$ , however. For example, if $\mathcal{A}$ is $\mathcal{Ab}$ and $\mathcal{B}$ is the category of countable groups, then $\mathcal{B}$, while a radical class in itself, is not a torsion class in $\mathcal{Ab}$ .

A torsion class is called *hereditary* if it is closed under subgroups; such a class is also called a *strongly complete Serre class.*

*THEOREM 1.34: The only non-trivial hereditary torsion classes are the classes* $\mathcal{T}_P$.

Proofs are given in [8], [37] and [42].//

*PROPOSITION 1.35: $T(Q(P)) = \mathfrak{D}_P$ for any set P of primes. In particular, $T(Q(p)) = \mathfrak{D}_p$ for any prime p and $T(Q) = \mathfrak{D}$.*

PROOF. The case of $Q(p)$ is treated in [8] (Proposition 4.1). The argument is easily adapted to cover the general situation.//

The results of this section provide examples of a method we shall find convenient for labelling torsion classes in the sequel.

*DEFINITION 1.36: A minimal representation of a torsion class $\mathfrak{J}$ is an equation $T(\mathcal{C}) = \mathfrak{J}$, where $T(\mathcal{C}') \subsetneqq \mathfrak{J}$ whenever $\mathcal{C}' \subsetneqq \mathcal{C}$.*

There is nothing unique about a minimal representation. For example

$$T(Z(p^m)) = \mathfrak{J}_p = T(Z(p^n))$$

for any prime p and positive integers m, n. We shall see further illustrations in the next chapter.

With the complete classification of the t-torsion classes, the following results show that the problem of classifying torsion classes in general reduces to that for torsion classes determined by torsion-free groups.

*PROPOSITION 1.37: Let $\mathfrak{J}$ be a torsion class, p a prime. Then either $Z(p) \in \mathfrak{J}$ or $\mathfrak{J} \subseteq \mathfrak{D}_p$.*

PROOF. [8] Lemma 5.1.//

*THEOREM 1.38: Let* $\mathfrak{T}$ *be a torsion class. A group* G *belongs to* $\mathfrak{T}$ *if and only if* $G_t$ *and* $G/G_t$ *do.*

PROOF. [8] Theorem 5.2.//

*COROLLARY 1.39: Any torsion class* $\mathfrak{T}$ *satisfies the equation*

$$\mathfrak{T} = T((\mathfrak{T} \cap \mathfrak{T}_0) \cup (\mathfrak{T} \cap \mathfrak{F}_0)).$$

PROOF. [8] Corollary 5.3.//

# CHAPTER 2

## SOME TORSION CLASSES DETERMINED BY TORSION-FREE GROUPS

As noted in §5 of Chapter 1, the problem of classifying torsion classes of groups has been effectively reduced to the problem for those which are determined by torsion-free groups. In Section 1 of this chapter, we examine the behaviour of the simplest torsion-free groups, the rational groups, as members of torsion classes. The torsion classes they determine are completely described and the classification is extended to torsion classes determined by rational and torsion groups.

Predictably, the situation with groups of rank greater than 1 is considerably more complicated. In Section 2 we consider some further examples of torsion classes determined by torsion-free groups and show that these may have quite distinct representations, for instance a group of rank $\geq 2$ can determine the same torsion class as a rational group and non-isomorphic indecomposable groups of equal rank $\geq 2$ may determine the same torsion class. (In contrast, rational groups determine the same torsion class if and only if they are isomorphic).

Section 3 is principally devoted to some torsion theories $(\mathcal{T},\mathcal{F})$ where $\mathcal{T}$ is "large".

All torsion theories and classes in this chapter are for $\mathcal{Ab}$ .

## 1. Rational Groups and Torsion Classes

*PROPOSITION 2.1: Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory with idempotent radical* r. *For every group* A, r(A) *is a pure subgroup.*

PROOF. Let p be a prime. If $Z(p) \notin \mathcal{T}$, then r(A) is always p-divisible (PROPOSITION 1.37) and therefore p-pure. If $Z(p) \in \mathcal{T}$, then A/r(A) has zero p-component, so again r(A) is p-pure.//

Since rational groups have no proper pure subgroups, we have

*COROLLARY 2.2: If* X *is a rational group and* $(\mathcal{T},\mathcal{F})$ *a torsion theory, then either* $X \in \mathcal{T}$ *or* $X \in \mathcal{F}$.//

*PROPOSITION 2.3: Let* X *and* Y *be rational groups. Then* $Y \in T(X)$ *if and only if* $\tau(X) \leq \tau(Y)$. *Thus in particular* $T(X) = T(Y)$ *if and only if* $X \cong Y$.

PROOF. If $\tau(X) \leq \tau(Y)$, then $[X,Y] \neq 0$, so by COROLLARY 2.2, $Y \in T(X)$. Conversely, if $Y \in T(X)$, then $[X,Y] \neq 0$ and every non-zero homomorphism from X to Y is a monomorphism.//

*COROLLARY 2.4: If* Y *and* $X_i$, $i \in I$ *are rational groups, then* $Y \in T(\{X_i | i \in I\})$ *if and only if* $\tau(X_j) \leq \tau(Y)$ *for some* $j \in I$.

PROOF. If $\tau(X_j) \leq \tau(Y)$, then $Y \in T(X_j) \subseteq T(\{X_i|i \in I\})$, while if $Y \in T(\{X_i|i \in I\})$, then for some j, $[X_j,Y] \neq 0$.//

*COROLLARY 2.5:* $\mathcal{D}$ *is the smallest torsion class containing torsion-free groups,* PROOF. Let $\mathcal{J}$ be a torsion class containing a torsion-free group $G$. Let $\{x_i | i \in I\}$ be a maximal linearly independent set of elements of $G$ and for some $j \in I$ let $G'$ be the smallest pure subgroup containing all $x_i$ with $i \neq j$. Then $G/G'$ is rational and belongs to $\mathcal{J}$. Hence $Q \in T(G/G') \subseteq \mathcal{J}$, and so $\mathcal{D} = T(Q) \subseteq \mathcal{J}$.//

Thus $Q$ is the only rational group which must belong to any torsion class containing torsion-free groups. On the other hand, it is clear that if a torsion class contains $Z$, it contains all groups.

*PROPOSITION 2.6: The following conditions on a group* $A$ *are equivalent:*

(i) $Z \in T(A)$.

(ii) $T(A) = \mathcal{Ab}$.

(iii) $A$ *has a homomorphic image (and therefore a direct summand) isomorphic to* $Z$.

PROOF. (i) $\Leftrightarrow$ (ii): If $Z \in T(A)$, then $T(A)$ contains all free groups and their homomorphic images, i.e. all groups.

(i) $\Leftrightarrow$ (iii): If $Z \in T(A)$, then $[A,Z] \neq 0$ and any non-zero homomorphism from $A$ to $Z$ has image isomorphic to $Z$.//

*COROLLARY 2.7: The class* $\mathcal{J}_\infty$ *of groups without free direct summands is the largest non-trivial torsion class.*

PROOF. A has a free direct summand if and only if it has a free direct summand of rank 1, i.e. $[A,Z] \neq 0$. Thus

$\mathcal{J}_\infty = \{A \mid [A,Z] = 0\}$ which is a torsion class by THEOREM 1.7. If $A \neq 0$, it can belong to a non-trivial torsion class if and only if $T(A)$ is non-trivial, i.e. $[A,Z] = 0$.//

*DEFINITION 2.8: A torsion class is called an r.t. torsion class if it is determined by a collection of rational and torsion groups.*

*DEFINITION 2.9: The type set of a torsion class $\mathcal{J}$ is the set of types of rational members of $\mathcal{J}$.*

It follows from COROLLARY 2.4 that a set $\Gamma$ of types is the type set of a torsion class if and only if it satisfies

$$(*) \qquad \gamma \in \Gamma,\ \chi \geq \gamma \Rightarrow \chi \in \Gamma\ .$$

*DEFINITION 2.10: Let $\Gamma$ be a set of types satisfying* (*), P *a set of primes such that* X *is* P-*divisible whenever* X *is rational with* $\tau(X) \in \Gamma$. $T(\Gamma,P)$ *is the torsion class*

$$T(\{X \text{ rational} \mid \tau(X) \in \Gamma\} \cup \{Z(p) \mid p \in P\}).$$

*THEOREM 2.11: A torsion class $\mathcal{J}$ is an r.t. torsion class if and only if it has the form* $T(\Gamma,P)$. *Such a representation is unique.*

PROOF. Let $\mathcal{J}$ be an r.t. torsion class. By THEOREM 1.33 we may assume that

$$\mathcal{J} = T(\{X_i \mid i \in I\} \cup \{Z(p) \mid p \in P_1\} \cup \{Z(p^\infty) \mid p \in P_2\}),$$

where $P_1$ and $P_2$ are disjoint sets of primes and the $X_i$ are rational. Let $\Gamma$ be the type set of $\mathcal{J}$ and

$$P = \{p \in P_1 \mid X \text{ rational},\ X \in \mathcal{J} \Rightarrow pX = X\},$$

We show that $\mathcal{J} = T(\Gamma,P)$.

Since $T(\{X_i | i \in I\})$ contains all divisible groups and all p-groups for $p \in P_1 - P$ (COROLLARY 2.5, PROPOSITION 1.37), we have

$$\{Z(p) | p \in P_1 - P\} \cup \{Z(p^\infty) | p \in P_2\}$$
$$\subseteq T(\{X_i | i \in I\})$$

whence

$$\mathcal{J} \subseteq T(T(\{X_i | i \in I\} \cup \{Z(p) | p \in P\}) \subseteq \mathcal{J}$$

A straightforward application of THEOREM 1.7 yields

$$\mathcal{J} = T(\{X_i | i \in I\} \cup \{Z(p) | p \in P\}) \subseteq T(\Gamma,P).$$

Thus $\mathcal{J} = T(\Gamma,P)$.

Now consider $T(\Gamma,P)$ and $T(\Sigma,S)$, where $\Gamma \nsubseteq \Sigma$, say $\gamma \notin \Sigma$ for some $\gamma \in \Gamma$. There is no $\sigma \in \Sigma$ with $\gamma \geq \sigma$, so if X and Y are rational, with $\tau(Y) = \gamma$ and $\tau(X) \in \Sigma$, we have $[X,Y] = 0$. Since also $[Z(p),Y] = 0$ for every p, Y cannot belong to $T(\Sigma,S)$, so $T(\Gamma,P) \nsubseteq T(\Sigma,S)$. Finally, suppose $T(\Gamma,P) = T(\Gamma,S)$, with $q \in P$, $q \notin S$. Then $[X,Z(q)] = 0$ whenever $\tau(X) \in \Gamma$ while $[Z(p), Z(q)] = 0$ for every $p \in S$, i.e. $Z(q) \notin T(\Gamma,S)$, and this contradiction completes the proof.//

It is not difficult to find torsion classes which are not r.t. classes.

*EXAMPLE 2.12.* Any torsion-free homomorphic image of I(p) is algebraically compact (e.g.[16]) so if countable must be divisible. Thus Q is the only rational group in T(I(p)). Since I(p) is reduced, it follows that T(I(p)) is not an r.t. class.

This example also shows that distinct torsion classes may have the same type set. On the other hand, distinct sets of rational groups may determine the same torsion class. It is an easy consequence of COROLLARY 2.4 that if $\Gamma$ and $\Sigma$ are sets of types, then

$$T(\{X \text{ rational} \mid \tau(X) \in \Gamma\}) = T(\{X \text{ rational} \mid \tau(X) \in \Sigma\})$$

if and only if for each $\gamma \in \Gamma$ there is a $\sigma' \in \Sigma$ with $\gamma \geq \sigma'$ and for each $\sigma \in \Sigma$ there is a $\gamma' \in \Gamma$ with $\sigma \geq \gamma'$. For this to happen it is not necessary that one of $\Gamma$, $\Sigma$ contain the other — they may be disjoint; (see EXAMPLE 2.14 below).

*PROPOSITION 2.13: If* $\Gamma$, $\Sigma$ *are sets of types for which*

$$T(\{X \textit{ rational} \mid \tau(X) \in \Gamma\}) = T(\{X \textit{ rational.} \mid \tau(X) \in \Sigma\}),$$

*and if* $\Gamma$ *has a subset* $\overline{\Gamma}$ *of types which are minimal in* $\Gamma$ *and which satisfy*

$$(**) \qquad \gamma \in \Gamma \Rightarrow \gamma \geq \overline{\gamma} \textit{ for some } \overline{\gamma} \in \overline{\Gamma},$$

*then* $\overline{\Gamma} \subseteq \Sigma$, $\overline{\Gamma}$ *is the set of all minimal types in* $\Sigma$ *and for every* $\sigma \in \Sigma$, $\sigma \geq \overline{\gamma}$ *for some* $\overline{\gamma} \in \overline{\Gamma}$.

PROOF. Let $\overline{\gamma}$ be any type in $\overline{\Gamma}$. Then $\overline{\gamma} \geq \sigma$ for some $\sigma \in \Sigma$ and $\sigma \geq \gamma$ for some $\gamma \in \Gamma$. Minimality of $\gamma$ then requires that $\overline{\gamma} = \sigma = \gamma$. Thus $\overline{\Gamma} \subseteq \Sigma$. Let $\overline{\Sigma}$ be the set of minimal elements in $\Sigma$. If $\overline{\gamma} \in \overline{\Gamma}$ and $\sigma \in \Sigma$ satisfy $\overline{\gamma} \geq \sigma$. then as above. $\overline{\gamma} = \sigma$. Hence $\overline{\Gamma} \subseteq \overline{\Sigma}$. For any $\overline{\sigma} \in \overline{\Sigma}$, there are types $\gamma \in \Gamma$, $\overline{\gamma} \in \overline{\Gamma} \subseteq \Sigma$ such that $\overline{\sigma} \geq \gamma \geq \overline{\gamma}$. But then $\overline{\sigma} = \overline{\gamma}$, so $\overline{\Gamma} = \overline{\Sigma}$. Finally, if $\sigma \in \Sigma$, then $\sigma \geq \gamma$ for some $\gamma \in \Gamma$, so $\sigma \geq \gamma \geq \overline{\gamma}$ for some $\overline{\gamma} \in \overline{\Gamma}$.//

Type sets of torsion classes do not necessarily have minimal elements:

*EXAMPLE 2.14:* Let heights $(h_1^{(n)}, h_2^{(n)}, \ldots)$ be defined for $n = 1,2,3,\ldots$ by

$$h_i^{(n)} = \begin{cases} 1 \text{ if } i = 2^n k,\ k = 1,2,3,\ \ldots \\ 0 \text{ otherwise.} \end{cases}$$

Writing $\tau_n$ for the type of $(h_1^{(n)}, h_2^{(n)}, \ldots)$, we have $\tau_n > \tau_{n+1}$ for each n. Let $X_n$ denote a rational group of type $\tau_n$ and $\mathfrak{J} = T(\{X_n | n = 1,2,3, \ldots\})$. The type set of $\mathfrak{J}$ is $\{\sigma | \sigma \geq \tau_n$, some $n\}$, which has no minimal element. Note also that

$$\mathfrak{J} = T(\{X_{2n} | n = 1,2,3,\ldots\}) = T(\{X_{2n-1} | n = 1,2,3,\ldots\}).$$

and that $\mathfrak{J}$ has no minimal representation by rational groups.

## 2. Further Examples

We now consider some examples of torsion classes having more than one minimal representation by torsion-free groups, beginning with some remarks on groups A for which T(A) is an r.t. torsion class. We first note that T(A) is an r.t. class if and only if $T(A) = T(\mathcal{C}_A)$ where $\mathcal{C}_A$ is the class of rational groups which are homomorphic images of A.

*PROPOSITION 2.15: Let* A *be torsion-free of rank 2. Then* T(A) *is an r.t. torsion class if and only if*

(i) $\tau(a) \geq \tau(X)$, *for some non-zero* $a \in A$, $X \in \mathcal{C}_A$.

*and* (ii) *if* p *is a prime for which* $pA \neq A$, *then* $pX \neq X$ *for some* $X \in \mathcal{C}_A$.

PROOF: If $T(A)$ is an r.t. class, i.e. if $T(A) = T(\mathcal{C}_A)$, then for some $X \in \mathcal{C}_A$, $[X,A] \neq 0$. Let $f: X \to A$ be non-zero; then any non-zero $a$ in the image of $f$ satisfies $\tau(a) \geq \tau(X)$. If $pA \neq A$, then $Z(p)$ belongs to $T(A) = T(\mathcal{C}_A)$, so $pX \neq X$ for at least one $X \in \mathcal{C}_A$.

Conversely, suppose A satisfies (i) and (ii) and let B be any one of its homomorphic images. If $B \ncong A$ but B is torsion-free, then $B \in \mathcal{C}_A$. If $B_t \neq 0$, then $B_t$ belongs to $T(A)$ (THEOREM 1.38), and so does $B_p$ for each prime p. If $B_p$ is divisible, it belongs to $T(\mathcal{C}_A)$ and if not, then $pA \neq A$, so $pX \neq X$ for some $X \in \mathcal{C}_A$, whence $B_p$ belongs to $T(X) \subseteq T(\mathcal{C}_A)$. Since also $[X,A] \neq 0$, for some $X \in \mathcal{C}_A$ (by (i)), it now follows from PROPOSITION 1.19 that $A \in T(\mathcal{C}_A)$, whence $T(A) = T(\mathcal{C}_A)$.//

*COROLLARY 2.16: Let* A *be a torsion-free group of rank 2. Then* $T(A) = T(X)$, *where* X *is rational, if and only if*

(i) $\tau(X)$ *is the least element among the types of groups in* $\mathcal{C}_A$

(ii) $\tau(a) \geq \tau(X)$ *for some non-zero* $a \in A$

*and* (iii) $pX \neq X$ *for every prime* p *for which* $pA \neq A$.//

These results are non-trivial: in §1 of Chapter 5 we shall construct indecomposable torsion-free groups A of arbitrary finite rank for which $T(A)$ is determined by a single rational group. Furthermore such groups A can be constructed for any rational group X which is not isomorphic to $Q(S)$ for any set S of primes. In the contrary case we have

*PROPOSITION 2.17: A torsion-free group* A *satisfies* $T(A) = T(Q(S))$ *if and only if* A *is* S*-divisible and has a direct summand isomorphic to* Q(S).

PROOF. If A satisfies the stated conditions, then Q(S) clearly belongs to T(A), so

$$\mathcal{D}_S = T(Q(S)) \subseteq T(A) \subseteq \mathcal{D}_S.$$

Conversely, if $T(A) = T(Q(S)) = \mathcal{D}_S$, A must be S-divisible, with $[A,Q(S)] \neq 0$. Any Z-homomorphism from A to Q(S) is a Q(S)-homomorphism (regarding Q(S) as a ring) and so must split.//

EXAMPLE 2.12 represents a special case of the following result:

*PROPOSITION 2.18: If* A *is a homogeneous, indecomposable torsion-free group of rank* > 1, *then* T(A) *is not an r.t. class.*

PROOF. Let A be homogeneous of type $\sigma$. If X is a rational homomorphic image of A, then $\tau(X) \geq \sigma$ and by a result of Baer (see [15] p.163), if A is indecomposable, $\tau(X) \neq \sigma$. But then $[X,A] = 0$.//

We have seen that two rational groups determine the same torsion class exactly when they are isomorphic. The corresponding statement for groups of rank 2 is false. If A has rank 2 and $T(A) = T(X)$, where X is rational, then $T(A) = T(X \oplus X)$, and A may be indecomposable.

That non-isomorphic indecomposable torsion-free groups of the same rank may determine the same torsion class is illustrated by the following example, essentially due to Jónsson [25].

*EXAMPLE 2.19:* Let P,S be infinite sets of primes such that $P \cap S = \emptyset$ and $5 \notin P \cup S$, U(resp.V) the set of square-free integers with prime factors in P(resp. in S). Let $\{x,y,z\}$ be a basis for a rational vector space and

$$A = [u^{-1}x \mid u \in U],\ B = [u^{-1}y, v^{-1}z, \tfrac{1}{5}(y+z) \mid u \in U,\ v \in V]$$

$$C = [u^{-1}y,\ v^{-1}z,\ \tfrac{1}{5}(3y+z) \mid u \in U,\ v \in V].$$

A homomorphism from B to A can be defined by $y \longrightarrow 5x$, $z \longrightarrow 0$, $\frac{1}{5}(y + z) \longrightarrow x$, so since A is rational, we have $A \in T(B)$. Similarly $A \in T(C)$. Also $A \oplus B \cong A \oplus C$, so $T(B) = T(\{A,B\}) = T(A \oplus B) = T(A \oplus C) = T(\{A,C\}) = T(C)$, but B and C are not isomorphic.

*PROPOSITION 2.20: Let* A *and* B *be torsion-free,* C *a torsion group,*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*an exact sequence. Then* $B \in T(A)$.

PROOF: If $B \notin T(A)$, then $C \notin T(A)$. Therefore $C_p \notin T(A)$ for some prime p, and thus $[A,Z(p)] = 0$. Let $x \in C$ have order p. There is induced a short exact sequence

$$0 \longrightarrow A \longrightarrow B' \longrightarrow [x] \longrightarrow 0.$$

If r denotes the idempotent radical associated with T(A), then since $r([x]) = 0$, we have $r(B') = A$. But this is impossible, since B' is torsion-free and r(B') is a pure subgroup (PROPOSITION 2.1).//

*COROLLARY 2.21: If* A *and* B *are quasi-isomorphic torsion-free groups, then* T(A) = T(B).//

(For *quasi-isomorphism* see, for example, [42]).

We conclude with another example, based on a result of Corner [7].

*EXAMPLE 2.22.* Let $\{x_n, y_n | n = 1,2,3,\ldots\}$ be a basis for a rational vector space, $p_n$, $q_n$, $t_n$ distinct primes, $n = 1,2,3,\ldots$,

$$A = [p_n^{-m} x_n,\ q_n^{-1}(x_n + x_{n+1}) | m,n, = 1,2,3,\ldots]$$

$$B = [p_n^{-m} y_n,\ t_n^{-1}(y_n + y_{n+1}) | m,n = 1,2,3,\ldots]$$

$$C_n = [p_n^{-m} x_n,\ p_{n+1}^{-m} y_{n+1},\ q_n^{-1}(x_n+y_{n+1}), t_n^{-1}(x_n+y_{n+1}) | m=1,2,3,\ldots],$$

$n = 1,2,3,\ldots$ . Clearly A and B have rank $\aleph_0$ and each $C_n$ is of rank 2. A monomorphism f from A to B can be defined by

$$(*) \qquad \begin{cases} f(x_1) = q_1\, y_1 \\ f(x_{n+1}) = q_n\, q_{n+1}\, y_{n+1} \end{cases}$$

In the resulting short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow B'' \longrightarrow 0$$

(*) implies that B" is a torsion group, whence by PROPOSITION 2.20, $B \in T(A)$. Similarly $A \in T(B)$, so $T(A) = T(A \oplus B) = T(B)$. But $A \oplus B \cong \bigoplus C_n, n=1,2,\ldots$ so $T(A) = T(\{C_n | n = 1,2,3,\ldots\})$. It is straightforward to show that $[C_m, C_n] = 0$ if $m \neq n$, so both representations are minimal.

## 3. Some Large Torsion Classes

An alternative method of describing a torsion class $\mathcal{J}$ is to specify a class $\mathcal{C}$ (preferably as simple as possible) for which in the corresponding torsion theory $(\mathcal{J},\mathcal{F})$ we have $\mathcal{F} = F(\mathcal{C})$. Thus for example consideration of classes $\mathcal{C}$ of rational groups provides further examples of torsion classes, none of which is an r.t. class.

*PROPOSITION 2.23: There is no torsion theory* $(\mathcal{J},\mathcal{F})$ *for which both* $\mathcal{J}$ *and* $\mathcal{F}$ *are determined by rational groups.*

PROOF. If both $\mathcal{J}$ and $\mathcal{F}$ are to contain rational groups, both must be non-trivial and $\mathcal{J}$ must contain Q. Let $X_\theta$ denote a rational group of type $\theta$. Suppose

$$\mathcal{J} = T(\{X_\gamma | \gamma \in \Gamma\}), \quad \mathcal{F} = F(\{X_\sigma | \sigma \in \Sigma\}).$$

For any $\sigma \in \Sigma$, there is a torsion-free group G of rank > 1, homogeneous of type $\sigma$, such that every rational homomorphic image is divisible. (See [17] p.21). For such a group G, $[G,X_{\sigma'}] = 0$ for every $\sigma' \in \Sigma$, so G belongs to $\mathcal{J}$. But for each $\gamma \in \Gamma$, $[X_\gamma,G] = 0$, so G is in $\mathcal{F}$. But then G = 0.//

Note that for PROPOSITION 2.23 it is sufficient to assume that $\mathcal{J}$ is an r.t. class.

Our next result characterizes some "large" torsion classes in terms of cardinal numbers.

*DEFINITION 2.24. A group* G *is called n-free, where* $n \geq 2$ *is a positive integer, if every subgroup of* G *with rank*

$< n$ *is free. If every countable subgroup is free,* G *is called* $\aleph_1$*-free.*

*PROPOSITION 2.25: Let*

$$\mathcal{T}_{m} = T(\{A \textit{ torsion-free} \mid [A,Z] = 0,\ A \textit{ has rank } < m\}),$$

*where* $m = 2,3,\ldots;\ \aleph_1$ *, and let* $(\mathcal{T}_m, \mathcal{F}_m)$ *be the corresponding torsion theory. Then* $\mathcal{F}_m$ *is the class of all* $m$*-free groups.*

PROOF. Each $\mathcal{T}_m$ contains p-reduced groups for every prime p and therefore all torsion groups, so only torsion-free groups need be considered.

(i) $m$ finite: Let $B \in \mathcal{F}_m$ have a subgroup $B'$ of rank $< m$. Then $[A,B'] = 0$ for each $A \in \mathcal{T}_m$. If $[B',Z] = 0$, then $B' \in \mathcal{T}_m$, so $B' = 0$. If not, then $B' = B_1' \oplus Z_1$, where $Z_1 \cong Z$, and $B_1'$ has rank $< m$, so as above, $B_1' = 0$ or $B_1' \cong B_2 \oplus Z_2$ $Z_2 \cong Z$. Since B has finite rank, repetitions of this argument show that $B'$ is free and thus that B is $m$-free. Conversely, if B is $m$-free, A has rank $< m$ and there is a non-zero homomorphism $f:A \longrightarrow B$, the image of f is free, so $[A,Z] \neq 0$. Hence B is in $\mathcal{F}_m$.

(ii) $m = \aleph_1$: If B belongs to $\mathcal{F}_{\aleph_1}$, then as in (i) B is n-free for n = 2,3,..., so by Pontryagin's Theorem (see [15] p.51), B is $\aleph_1$-free. The converse is proved in a similar way to that in (i).//

The result cannot be extended to arbitrary cardinal numbers: let $\mathcal{T}_m$ be defined as above,

$$\mathcal{F}_m = \{B \mid [A,B] = 0 \text{ for all } A \in \mathcal{T}_m\}.$$

Then $\mathcal{T}_{\mathfrak{m}}$ must contain $\prod_{i=1}^{\infty} Z_i$, each $Z_i = Z$, but this group is not $\mathfrak{m}$-free for $\mathfrak{m} > \aleph_1$. Note also that "2-free" means "homogeneous of type $\tau(Z)$".

All the torsion classes of PROPOSITION 2.25 are distinct. Before proving this we note more economical representation of them.

*PROPOSITION 2.26: For* $2 < \mathfrak{m} \leq \aleph_1$,

$$\mathcal{T}_{\mathfrak{m}} = T(\{A \textit{ torsion-free} \mid [A,Z] = 0,\ A \textit{ has rank } < \mathfrak{m} \textit{ and is homogeneous of type } \tau(Z)\}).$$

PROOF. Let r be the idempotent radical for the torsion theory $(\mathcal{T}_2, \mathcal{F}_2)$. Since $\mathcal{T}_2 \subseteq \mathcal{T}_{\mathfrak{m}}$, it follows from the exact sequence

$$0 \longrightarrow [A/r(A),\ B] \longrightarrow [A,B] \longrightarrow [r(A),\ B] = 0$$

that for any $B \in \mathcal{F}_{\mathfrak{m}}$, $[A,B] = 0$ if and only if $[A/r(A),\ B] = 0$, and that therefore

$$\mathcal{T}_{\mathfrak{m}} = T(\{A/r(A) \mid [A,Z] = 0,\ A \text{ is torsion-free with rank } < \mathfrak{m}\}).$$

Now if A has rank $< \mathfrak{m}$, then so does $A/r(A)$. Thus:

$$\{A/r(A) \mid [A,Z] = 0,\ A \text{ is torsion free with rank } < \mathfrak{m}\} \subseteq \{A \text{ torsion-free} \mid [A,Z] = 0,\ A \text{ has rank } < \mathfrak{m} \text{ and is homogeneous of type } \tau(Z)\} \subseteq \{A \text{ torsion-free} \mid [A,Z] = 0,\ A \text{ has rank } < \mathfrak{m}\}$$

This establishes the result.//

*PROPOSITION 2.27: The classes* $\mathcal{T}_2, \mathcal{T}_3, \ldots ; \mathcal{T}_{\aleph_1}$ *are all distinct.*

PROOF. It is clearly sufficient to show that $\mathcal{T}_n \subsetneqq \mathcal{T}_{n+1}$ for finite n.

$\mathcal{T}_2$ is determined by all rational groups X with $\tau(X) > \tau(Z)$, so by PROPOSITION 2.26, $\mathcal{T}_2 \neq \mathcal{T}_3$.

For each integer $n > 2$, there exists a torsion-free group $G_n$ of rank n such that

(i) $G_n$ is homogeneous of type $\tau(Z)$

(ii) every proper pure subgroup of $G_n$ is completely decomposable (and therefore free)

(iii) every rational homomorphic image X of $G_n$ has $\tau(X) > \tau(Z)$.

(This result is due to Corner, see [17]). Condition (ii) implies that $G_n$ is n-free, so $G_n \notin \mathcal{T}_n$, but by (iii), $G_n$ belongs to $\mathcal{T}_{n+1}$.//

The functor v considered by Chase in [6] is the idempotent radical corresponding to the torsion theory $(\mathcal{T}_{\aleph_1}, \mathcal{F}_{\aleph_1})$.

We have seen that the class $\mathcal{T}_\infty$ of all groups without free direct summands is the largest torsion class and that $(\mathcal{T}_\infty, F(Z))$ is the corresponding torsion theory. This theory does not have the form $(\mathcal{T}_m, \mathcal{F}_m)$: clearly it cannot be $(\mathcal{T}_n, \mathcal{F}_n)$ for finite n, and since there are indecomposable $\aleph_1$-free groups of uncountable rank (see [17] p.24), $\mathcal{T}_\infty \cap \mathcal{T}_{\aleph_1} \neq \{0\}$.

Finally we note a connection between F(Z) and the class $\mathcal{L}$ of torsionless groups. (For torsionless modules in general see [24] pp. 65-69).

*PROPOSITION 2.28:* $F(Z) = F(\mathcal{L})$

PROOF. $F(Z) \subseteq F(\mathcal{L})$, since $Z \in \mathcal{L}$. Also F(Z) contains all subgroups of direct products of copies of Z, i.e. all torsionless groups, so $\mathcal{L} \subseteq F(Z)$, whence $F(\mathcal{L}) \subseteq F(Z)$.//

The class of torsionless modules over any ring is closed under submodules and direct products so $\mathcal{L}$ has these properties. $\mathcal{L}$ does not coincide with F(Z), however, because of

*PROPOSITION 2.29: $\mathcal{L}$ is not closed under extensions.*

PROOF. If $\mathcal{L}$ is closed under extensions, then in every short exact sequence

$$e: \qquad 0 \longrightarrow Z \xrightarrow{f} A \longrightarrow \prod_{i=1}^{\infty} Z_i \longrightarrow 0,$$

where each $Z_i \cong Z$, A must be torsionless. Let $f(1) = a$. Then $g(a) \neq 0$ for some $g \in [A,Z]$. Let $g(a) = n$ and form the pushout corresponding to multiplication by n in Z:

$$\begin{tikzcd}
e: & 0 \arrow[r] & Z \arrow[r, "f"] \arrow[d, "n"'] & A \arrow[r] \arrow[d] \arrow[dl, dashed, "g"'] & \prod Z_i \arrow[r] \arrow[d, equal] & 0 \\
ne: & 0 \arrow[r] & Z \arrow[r] & B \arrow[r] & \prod Z_i \arrow[r] & 0 .
\end{tikzcd}$$

$gf(1) = n$ so ne splits (see [31]p.72). Since $\mathrm{Ext}(\prod_{i=1}^{\infty} Z_i, Z)$ is not a torsion group ([34] Theorem 8), the proposition is proved.//

By setting $r(A) = \bigcap \ker f$, $f \in [A,Z]$ we define a radical $r$ for which $r(A) = 0$ if and only if $A$ is torsionless. The last result shows that $r$ is not idempotent. Charles [5] points out that Fuchs has shown that $r^{\omega+1} \neq r^{\omega}$, where $\omega$ is the first infinite ordinal.

# CHAPTER 3.

## ADDITIONAL CLOSURE PROPERTIES FOR TORSION CLASSES I: DIRECT PRODUCTS, PURE SUBGROUPS

We begin the discussion of closure properties for torsion classes by considering closure under direct products and pure subgroups, one section of the present chapter being devoted to each of these two properties. There is some interrelation between the material of the two sections; in particular, the theory of algebraically compact groups has a central role in each.

In the first section we obtain a complete description of the torsion classes closed under countable direct products (i.e. direct products of countable sets of groups) and in the second we characterize those which are closed under pure subgroups. The latter result is then generalized to cover closure under S-pure subgroups, where S is a set of primes.

In this chapter all torsion classes are understood to be torsion classes of abelian groups.

### 1. Direct Products

The following result will be used several times in this chapter.

*LEMMA 3.1: Let* $\mathfrak{J}$ *be a torsion class containing a torsion-free group* A *which is not* p*-divisible, for some prime* p*. Then* $I(p) \in \mathfrak{J}$ .

PROOF. $[A, I(p)] \neq 0$ ([11] p.52) so let $f: A \longrightarrow I(p)$ be non-zero and consider $B = I(p)/\mathrm{Im}(f)$. $B/B_t$ as a torsion-free proper homomorphic image of $I(p)$ is divisible, (see [11]) and so belongs to $\mathcal{J}$ (COROLLARY 2.5). $T(I(p))$ contains $B$ and therefore $B_t$ (THEOREM 1.38), whence $B_q$ is divisible for all primes $q \neq p$. Since in addition $B_p$ belongs to $T(A) \subseteq \mathcal{J}$ (PROPOSITION 1.37), $\mathcal{J}$ contains $B_t$ and therefore $B$. Since also $\mathrm{Im}(f)$ belongs to $\mathcal{J}$, so does $I(p)$.//

The principal result of this section is

*THEOREM 3.2: A torsion class $\mathcal{J}$ is closed under countable direct products if and only if it is determined by torsion-free groups.*

Most of the proof of THEOREM 3.2 is contained in the proofs of the next two results.

*PROPOSITION 3.3: Let $A_n$, $n = 1,2,3,\ldots$ be torsion-free groups. Then*

$$T(\{A_n | n = 1,2,3,\ldots\}) = T(\bigoplus_{n=1}^{\infty} A_n) = T(\prod_{n=1}^{\infty} A_n).$$

PROOF. The first equality obviously holds; since also $A_m \in T(\prod A_n)$ for each $m$, we have $T(\bigoplus A_n) \subseteq T(\prod A_n)$.

Let $f: \prod A_n \longrightarrow Y$ be a non-zero epimorphism.

If $Y_p \neq 0$ for some prime $p$, then if $Y_p$ is reduced, we have $p \prod A_n \neq \prod A_n$ so $pA_m \neq A_m$, for some $m$, and thus $Y_p \in T(A_m) \subseteq \mathcal{J}$, while if $Y_p$ is not reduced, then $[A_n, Y_p] \neq 0$ for each $n$.

If $Y$ is torsion-free, then either $f(A_m) \neq 0$ for some m or $f(\bigoplus A_n) = 0$, in which case f factorizes as

$$\begin{tikzcd} \prod A_n \arrow[r, "f"] \arrow[d] & Y \\ \prod A_n / \bigoplus A_n \arrow[ur] & \end{tikzcd}$$

where all maps are epimorphisms. $\prod A_n / \bigoplus A_n$ is algebraically compact (see [23]). Thus $\prod A_n / \bigoplus A_n$ is the direct sum of a divisible group and a (reduced) cotorsion group [16]; so, therefore, is Y, which being torsion-free is algebraically compact [16]. Thus $Y = D \oplus \prod R(p)$, where D is divisible and $R(p)$ is *inter alia* a reduced $I(p)$-module. If $D \neq 0$, then $[A_n, Y] \neq 0$ for each n. If $D = 0$, let $R(p) \neq 0$. Then $p\prod A_n \neq \prod A_n$ and thus $pA_m \neq A_m$ for some value of m. By LEMMA 3.1, $I(p) \in T(A_m)$. Since there is an epimorphism (actually an $I(p)$-epimorphism) from a direct sum of copies of $I(p)$ to $R(p)$, we have $R(p) \in T(A_m)$.

Thus in all cases $[A_m, Y] \neq 0$ for at least one value of m, whence by PROPOSITION 1.19, $\prod A_n$ belongs to $T(\bigoplus A_n)$. This completes the proof.//

*PROPOSITION 3.4: Let* $\mathfrak{J} = T(\{A_\lambda \mid \lambda \in \Lambda\})$, *where each* $A_\lambda$ *is torsion-free and let* $B_n$, $n = 1,2,3,\ldots$ *be torsion groups in* $\mathfrak{J}$. *Then* $\mathfrak{J}$ *contains* $\prod_{n=1}^{\infty} B_n$.

PROOF. Let $f: \prod B_n \to G$ be a non-zero epimorphism. If for some prime p, $G_p$ is non-zero and divisible, then $[A_\lambda, G_p] \neq 0$ for each $\lambda \in \Lambda$, while if $G_p$ is non-zero but not divisible, then $p\prod B_n \neq \prod B_n$, so $pB_m \neq B_m$ for some m which means that $p(B_m)_p \neq (B_m)_p$. Since $(B_m)_p$ belongs to $\mathcal{T}$, so do all p-groups; in particular $G_p$ is in $\mathcal{T}$.

If G is torsion-free, then $f(\bigoplus B_n) = 0$, so f factorizes as

$$\begin{tikzcd} \prod B_n \arrow[r, "f"] \arrow[d] & G \\ \prod B_n / \bigoplus B_n \arrow[ur] & \end{tikzcd}$$

where all maps are epimorphisms. As in PROPOSITION 3.3, $G = D \oplus \prod R(p)$, p prime, and we need only consider the case where $D = 0$. If this is so, and $R(p) \neq 0$, then $p\prod B_n \neq \prod B_n$, and as in the first part of the proof, $\mathcal{T}$ contains all p-groups. Hence at least one $A_\lambda$ is not p-divisible, so as in PROPOSITION 3.3, I(p) belongs to $\mathcal{T}$ whence R(p) does also. This proves that $\prod B_n \in \mathcal{T}$.//

PROOF OF THEOREM 3.2. If $\mathcal{T}$ is determined by torsion-free groups and if $\{A_n | n = 1,2,3,\ldots\} \subseteq \mathcal{T}$, then $(A_n)_t$ and $A_n/(A_n)_t \in \mathcal{T}$ for each n. By PROPOSITION 3.3, $\prod A_n/(A_n)_t \in T(\bigoplus A_n/(A_n)_t) \subseteq \mathcal{T}$ and by PROPOSITION 3.4, $\prod (A_n)_t \in \mathcal{T}$, so from the short exact sequence

$$0 \longrightarrow \prod (A_n)_t \longrightarrow \prod A_n \longrightarrow \prod A_n/(A_n)_t \longrightarrow 0$$

clearly $\prod A_n \in \mathcal{T}$.

Conversely, suppose $\mathcal{T}$ is closed under countable direct products. Clearly $\mathcal{T}$ is not a t-torsion class. If it is not determined by torsion-free groups, then for some prime p,

$Z(p) \in \mathcal{J}$ but all groups in $\mathcal{J} \cap \mathcal{F}_0$ are p-divisible. Let $[x_n] \cong Z(p^n)$, $n = 1,2,\ldots$ . Then $\prod [x_n] \in \mathcal{J}$, so $\prod [x_n]/(\prod [x_n])_t \in \mathcal{J} \cap \mathcal{F}_0$. Suppose $p(a_n x_n) - (x_n) \in (\prod[x_n])_t$, $a_n \in Z$. Then for some positive $k \in Z$, $p^k(p(a_n x_n)-(x_n)) = 0$, so $p^k(pa_n - 1)\, x_n = 0$ for all n, i.e. $p^n | p^k(pa_n - 1)$. For $n > k$, this means that $p^{n-k}|(pa_n - 1)$, which is impossible. Thus $(x_n) + (\prod[x_n])_t$ has zero p-height in $\prod[x_n]/(\prod[x_n])_t$, contradicting the required p-divisibility of $\prod [x_n]/(\prod[x_n])_t$.//

If $\{A_\lambda | \lambda \in \Lambda\} \subseteq \mathcal{D}_P$ for any set P of primes, then $\prod_{\lambda\in\Lambda} A_\lambda \in \mathcal{D}_P$, without any restriction on the size of $\Lambda$. Whether any other torsion classes have this property, or the corresponding one for $|\Lambda| < \mathfrak{m}$, where $\mathfrak{m} > \aleph_0$, is not known. A related result is

*PROPOSITION 3.5 Let $\mathcal{C}$ be a class of slender groups and*

$$\mathcal{J} = \{G | [G,C] = 0 \text{ for all } C \in \mathcal{C}\}$$

*Then $\mathcal{J} \cap \mathcal{F}_0$ is closed under direct products for which the number of components does not exceed the first cardinal number of non-zero measure.*

PROOF. Let $\{G_\lambda | \lambda \in \Lambda\} \subseteq \mathcal{J} \cap \mathcal{F}_0$, where $\Lambda$ has appropriate cardinality. Then for any $C \in \mathcal{C}$, $[\bigoplus G_\lambda, C] = 0$ and consequently for any homomorphism $f: \prod G_\lambda \longrightarrow C \in \mathcal{C}$, $f(\bigoplus G_\lambda) = 0$. By a theorem of Łoś ([15]p.170), $f = 0$, so $\prod G_\lambda \in \mathcal{J}$.//

In particular, $\mathcal{C}$ may consist of a single reduced countable torsion-free group [36]. Additional information concerning the structure of direct products modulo direct sums of torsion groups, or direct products modulo their torsion subgroups, together with PROPOSITION 3.5 should provide at least partial answers to the questions raised above.

*PROPOSITION 3.6: Let* P *be a set of primes and* $\mathcal{A}_P$ *the class of* P*-divisible algebraically compact groups. Then*

$$T(\mathcal{A}_P) = T(\{I(p) \mid p \notin P\}).$$

PROOF. $T(\mathcal{A}_P) \supseteq T(\{I(p) \mid p \notin P\})$, since each I(p) is algebraically compact. Conversely, if G belongs to $\mathcal{A}_P$, then $G = D \oplus \prod R(p)$, $p \notin P$, where R(p) is an I(p)-module. For each $q \notin P$, $R(q) \in T(I(q))$, so since also $D \in T(I(q))$, THEOREM 3.2 implies that G belongs to $T(\{I(p) \mid p \notin P\})$, whence $T(\mathcal{A}_P) \subseteq T(\{I(p) \mid p \notin P\})$.//

In view of this result and Nunke's characterization [34] of slender groups as reduced torsion-free groups containing no copy of any I(p) and no direct product of infinitely many infinite cyclic groups, there seem to be good grounds for the following

CONJECTURE: $(T(\mathcal{A}), F(\mathcal{S}))$ is a torsion theory, where $\mathcal{A}$ is the class of all algebraically compact groups, $\mathcal{S}$ the class of all slender groups.

If $(T(\mathcal{A}), \mathcal{F})$ denotes the torsion theory for $T(\mathcal{A})$, then by PROPOSITION 3.6, $\mathcal{F}$ consists of those reduced torsion-free groups containing no copy of any I(p). Note also that $\mathcal{S} \subsetneqq F(\mathcal{S})$, since while $\mathcal{S}$ is closed under extensions and subgroups, it is not closed for products — Z is slender but direct products of copies of Z are not.

## 2. Pure Subgroups

As a first step in the discussion of closure under pure subgroups, we show that every torsion class with this property is either a t-torsion class or an r.t. torsion class.

*PROPOSITION 3.7: All t-torsion classes are closed under pure subgroups.*

PROOF. Let $S_1$, $S_2$ be disjoint sets of primes. If $A_1$ is an $S_1$-group and $A_2$ a divisible $S_2$-group, then clearly any pure subgroup of $A_1 \oplus A_2$ is the direct sum of an $S_1$-group and a divisible $S_2$-group.//

*THEOREM 3.8: A torsion class $\mathcal{T}$ is closed under pure subgroups if and only if $\mathcal{T} \cap \mathcal{F}_0$ is.*

PROOF. Let A' be a pure subgroup of $A \in \mathcal{T}$, and consider the induced diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A'_t \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A'/A'_t \arrow[r] \arrow[d, "g"'] & 0 \\
0 \arrow[r] & A_t \arrow[r] & A \arrow[r] & A/A_t \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns, where g is defined by $g(a'+A'_t) = a'+A_t$. $A'_t$ is pure in $A'$ and hence in $A$. Therefore $A'_t$ is pure in $A_t$, so by PROPOSITION 3.7, $A'_t \in \mathfrak{J} \cap \mathfrak{J}_0$.

The kernel of g is $A' \cap A_t/A'_t = 0$. If, for some non-zero $n \in Z$, $a' \in A'$ and $a \in A$ we have $g(a'+A'_t) = n(a+A_t)$, then $m(a'-na) = 0$ for some non-zero $m \in Z$, i.e. $ma' = mna$. Since $A'$ is pure in $A$, there exists $a'' \in A'$ with $ma' = mna''$. But then $g(a'+A'_t) = ng(a''+A'_t)$, so that g is a pure monomorphism. Thus if $\mathfrak{J} \cap \mathfrak{F}_0$ is closed under pure subgroups, $A'/A'_t \in \mathfrak{J} \cap \mathfrak{F}_0$, so $A' \in \mathfrak{J}$ and $\mathfrak{J}$ is therefore closed under pure subgroups.

The converse is obvious.//

*THEOREM 3.9: If a torsion class $\mathfrak{J}$ is closed under pure subgroups, then*

$$\mathfrak{J} = T((\mathfrak{J} \cap \mathfrak{J}_0) \cup \underline{\mathfrak{J}})$$

*where $\underline{\mathfrak{J}}$ is the class of rational groups in $\mathfrak{J}$.*

The proof uses the following lemmas:

*LEMMA 3.10: For $\mathfrak{J}$ and $\underline{\mathfrak{J}}$ as in THEOREM 3.9,*

$$\mathfrak{J} \cap \mathfrak{F}_0 = T(\underline{\mathfrak{J}}) \cap \mathfrak{F}_0.$$

PROOF. Clearly $T(\underline{\mathfrak{J}}) \cap \mathfrak{F}_0 \subseteq \mathfrak{J} \cap \mathfrak{F}_0$. Let A be any group in $\mathfrak{J} \cap \mathfrak{F}_0$. Then A is a homomorphic image of $\bigoplus [a]_*$ where the direct sum extends over all $a \in A$. Thus $A \in T(\underline{\mathfrak{J}})$, since each $[a]_* \in \underline{\mathfrak{J}}$.//

*LEMMA 3.11: For any two classes $\mathcal{C}_1$, $\mathcal{C}_2$ of groups,*

$$T(\mathcal{C}_1 \cup \mathcal{C}_2) = T(T(\mathcal{C}_1) \cup T(\mathcal{C}_2)).//$$

We omit the proof of this result, which consists of a simple application of THEOREM 1.7.

To complete the proof of THEOREM 3.9, we observe that

$$\mathcal{J} = T[(\mathcal{J} \cap \mathcal{J}_0) \cup (\mathcal{J} \cap \mathcal{F}_0)] = T[(\mathcal{J} \cap \mathcal{J}_0) \cup (T(\underline{\mathcal{J}}) \cap \mathcal{F}_0)] \subseteq T[(\mathcal{J} \cap \mathcal{J}_0) \cup T(\underline{\mathcal{J}})] = T[(\mathcal{J} \cap \mathcal{J}_0) \cup \underline{\mathcal{J}}] \subseteq \mathcal{J}.//$$

Let $X_1, \ldots, X_n$ be rational groups with types $\chi_1, \ldots, \chi_n$ respectively. Then any $(x_1, \ldots, x_n) \in X_1 \oplus \ldots \oplus X_n$, with $x_1, \ldots, x_n \neq 0$ has type $\chi_1 \cap \ldots \cap \chi_n$. Thus one requirement if an r.t. torsion class $T(\Gamma,P)$ is to be closed under pure subgroups is that $\Gamma$ satisfy

$$(*) \qquad \gamma_1, \ldots, \gamma_n \in \Gamma \Rightarrow \gamma_1 \cap \ldots \cap \gamma_n \in \Gamma$$

so if X and Y are rational groups with incomparable types, $T(\{X,Y\})$ is not closed under pure subgroups.

Now if $T(\Gamma,P)$ is an r.t. torsion class for which $\Gamma$ satisfies (*), then for every torsion-free group G,

$$G(\Gamma) = \{x \in G | \tau(x) \in \Gamma\}$$

is a pure subgroup, since if $x, y \in G(\Gamma)$ we have $\tau(x-y) > \tau(x) \cap \tau(y) \in \Gamma$ (0 is regarded as having a type greater than all others) and $\tau(nx) = \tau(x)$ for $n \in Z$. $G(\Gamma)$ is also functorial in an obvious way. The following theorem describes a connection between the functor $(\ )(\Gamma)$ and the idempotent radical associated with $T(\Gamma,P)$.

*THEOREM 3.12: Let* $T(\Gamma,P)$ *be an r.t. torsion class for which* $\Gamma$ *satisfies (*),* r *its idempotent radical and*

$$\mathcal{C}(\Gamma) = \{G \textit{ torsion-free} | G(\Gamma) = G\},$$

*Then the following conditions are equivalent:*

(i) $\mathcal{C}(\Gamma)$ *is closed under extensions*

(ii) $G(\Gamma) = r(G)$ *for all torsion-free groups* $G$

(iii) $T(\Gamma,P)$ *is closed under pure subgroups.*

PROOF. (i) => (ii): Let $G$ be torsion-free and let $G'$ be the subgroup of $G$ defined by the short exact sequence

$$0 \longrightarrow G(\Gamma) \longrightarrow G' \longrightarrow (G/G(\Gamma))(\Gamma) \longrightarrow 0.$$

If $\mathcal{C}(\Gamma)$ is closed under extensions, then $G' \in \mathcal{C}(\Gamma)$ and since for $x \in G'$ we have $\tau_{G'}(x) \leq \tau_G(x)$, it follows that $G' \subseteq G(\Gamma)$, whence $(G/G(\Gamma))(\Gamma) = 0$, and $[X,G/G(\Gamma)] = 0$ whenever $X$ is rational with $\tau(X) \in \Gamma$. Also $G/G(\Gamma)$ is torsion-free, so $r(G/G(\Gamma)) = 0$. Since $G(\Gamma) \in T(\Gamma,P)$, therefore, $G(\Gamma)$ must be $r(G)$.

(ii) => (iii): If $G(\Gamma) = r(G)$ whenever $G$ is torsion-free, then $T(\Gamma,P) \cap \mathcal{F}_0 = \mathcal{C}(\Gamma)$, which is closed under pure subgroups. By THEOREM 3.8 $T(\Gamma,P)$ is also.

(iii) => (i): Let $T(\Gamma,P)$ be closed under pure subgroups with $G \in T(\Gamma,P)$ torsion-free. Then $[x]_* \in T(\Gamma,P)$ so $\tau(x) = \tau([x]_*) \in \Gamma$ for all $x \in G$, whence $G \in \mathcal{C}(\Gamma)$, i.e. $T(\Gamma,P) \cap \mathcal{F}_0 \subseteq \mathcal{C}(\Gamma)$. Since the reverse inclusion also holds, $\mathcal{C}(\Gamma) = T(\Gamma,P) \cap \mathcal{F}_0$ is closed under extensions.//

*THEOREM 3.13: A torsion class* $\mathcal{T}$ *is closed under pure subgroups if and only if either*

(i) $\mathcal{T}$ *is a t-torsion class*

*or* (ii) $\mathcal{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$, *where* $P$ *and* $S$ *are sets of primes with* $P \subseteq S$.

For the proof we need

*LEMMA 3.14: Let* $\{X_\lambda | \lambda \in \Lambda\}$ *be a set of rational groups and* $S = \{p \text{ prime} | pX_\lambda = X_\lambda \text{ for each } \lambda \in \Lambda\}$. *If* $T(\{X_\lambda|\lambda \in \Lambda\})$ *is closed under pure subgroups, then it contains* $Q(S)$.

PROOF OF LEMMA. Let $p_1, p_2, p_3, \ldots$ be the natural enumeration of the primes, let $J = \{j | p_j \notin S\}$ and denote $\bigoplus X_\lambda$ by A. For each $j \in J$, choose $a_j \in A$ with $h_j(a_j) = 0$, where $h_j$ denotes height at $p_j$. For example, let $a_j = (x_{j\lambda})$ with $x_{j\lambda} \in X_\lambda$ satisfying the following conditions: (i) $x_{j\mu} \neq 0$ for some $\mu \in \Lambda$ for which $X_\mu$ is $p_j$-reduced; (ii) $h_j(x_{j\mu}) = 0$; (iii) $x_{j\lambda} = 0$ for $\lambda \neq \mu$. For a natural number $i \notin J$, let $a_i$ be an arbitrary element of A, and regard the resulting $(a_i)$ as an element of $\prod A_i$, $i = 1,2,3,\ldots$, where each $A_i = A$. $h((a_i)) = \bigcap_{i=1}^{\infty} h(a_i)$, the former height being taken in $\prod A_i$, the latter in A. In particular, $h_j((a_i)) = 0$ for $j \in J$. Therefore, since $\prod A_i$ is S-divisible, the height of $(a_i)$ at a prime p is infinite if $p \in S$ and zero otherwise, i.e. $\tau((a_i)) = \tau(Q(S))$ and $\prod A_i$ has a pure subgroup isomorphic to $Q(S)$. By THEOREM 3.2, $\prod A_i \in T(\{X_\lambda | \lambda \in \Lambda\})$, which if closed under pure subgroups must therefore contain $Q(S)$.//

Since each $X_\lambda$ is S-divisible and $T(Q(S))$ is the class of all S-divisible groups (PROPOSITION 1.35) we have

*COROLLARY 3.15: With the notation of LEMMA 3.14, if* $T(\{X_\lambda|\lambda \in \Lambda\})$ *is closed under pure subgroups, it is the class of all* S-*divisible groups.*//

PROOF OF THEOREM 3.13. Let $\mathcal{T}$ be a torsion class closed under pure subgroups. If $\mathcal{T}$ is not a t-torsion class, let $\Gamma$ be its type set and for each $\gamma \in \Gamma$ let $X_\gamma$ be a rational group of type $\gamma$. Then

$$\mathcal{T} = T((\mathcal{T} \cap \mathcal{T}_0) \cup \{X_\gamma | \gamma \in \Gamma\}) \qquad \text{(THEOREM 3.9)}$$

and

$$\mathcal{T} \cap \mathcal{F}_0 = T(\{X_\gamma | \gamma \in \Gamma\}) \cap \mathcal{F}_0 \qquad \text{(LEMMA 3.10)}$$

By THEOREM 3.8, $T(\{X_\gamma | \gamma \in \Gamma\})$ is closed under pure subgroups and therefore, by COROLLARY 3.15, is the class of all S-divisible groups, where S is the set of primes dividing $\bigoplus X_\gamma$. Thus

$$\mathcal{T} = T((\mathcal{T} \cap \mathcal{T}_0) \cup \{Q(S)\}).$$

Let $P = \{p \in S | Z(p) \in \mathcal{T}\}$. Since $T(Q(S)) \subseteq \mathcal{T}$, $\mathcal{T}$ contains the groups $Z(p^\infty)$ for all primes p as well as $Z(p)$ for primes $p \notin S$. Thus by THEOREM 1.33 and LEMMA 3.11,

$$\begin{aligned}\mathcal{T} &= T(\{Z(p) | p \notin S\} \cup \{Z(p) | p \in P\} \cup \{Z(p^\infty) | \text{ all } p\} \cup \{Q(S)\}) \\ &= T(\{Z(p) | p \in P\} \cup \{Q(S)\}).\end{aligned}$$

Conversely, that any class

$$\mathcal{T} = T(\{Z(p) | p \in P\} \cup \{Q(S)\})$$

with $P \subseteq S$ is closed under pure subgroups follows from THEOREM 3.8, LEMMA 3.10 and the observation that $T(Q(S))$ is closed under pure subgroups. By PROPOSITION 3.7, the proof is now complete.//

Note that by THEOREM 1.7, for a torsion class $\mathcal{T}$ which is not a t-torsion class, the representation

$$\mathcal{T} = T(\{Z(p) | p \in P\} \cup \{Q(S)\})$$

is unique. Our next result characterizes the groups in such a class.

*PROPOSITION 3.16: A group* G *belongs to*

$$\mathcal{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$$

*where* P *and* S *are sets of primes with* $P \subseteq S$*, if and only if there is a short exact sequence*

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

*where* G' *is a* P*-group and* G'' *is* S*-divisible.*

PROOF. Let $G \in \mathcal{T}$ and $G' = \bigoplus G_p$ where the direct sum extends over all $p \in P$, $G'' = G/G'$. Then $G''_t$ has no P-component and belongs to $\mathcal{T}$ (THEOREM 1.38) so therefore has divisible S-component. Thus $G''_t$ is S-divisible. $G''/G''_t$ is torsion-free and belongs to $\mathcal{T}$. If not S-divisible, it has a non-zero S-reduced torsion-free homomorphic image B. But then $B \in \mathcal{T}$ and $[Q(S),B] = 0 = [Z(p),B]$ for each $p \in P$ and this contradicts THEOREM 1.7, so $G''/G''_t$ is S-divisible, whence G'' is also. The converse is obvious.//

*DEFINITION 3.17: A subgroup* G' *of a group* G *is* S*-pure, where* S *is a set of primes, if* $G' \cap nG = nG'$ *for every* n *in* S**, *the multiplicative semigroup generated by* S*. If* S *has a single element* p*,* S*-purity is called* p*-purity.*

THEOREM 3.13 can be generalized fairly easily to describe the torsion classes which are closed under S-pure subgroups. We shall need

*LEMMA 3.18: Let* P *be a set of primes. Then* $T(Q(P)) = \mathcal{D}_P$ *is closed under* S*-pure subgroups if and only if* $P \subseteq S$.

PROOF. If $P \nsubseteq S$, then $Q(P \cap S)$ is an S-pure subgroup of $Q(P)$, but $Q(P \cap S) \notin \mathcal{D}_P$. The converse follows from the fact that S-purity implies P-purity if $P \subseteq S$.//

*THEOREM 3.19: A torsion class* $\mathcal{J}$ *is closed under* S*-pure subgroups if and only if either*

(i) $\mathcal{J}$ *is a t-torsion class such that* $\mathcal{J} \cap \mathcal{J}_p$ *is hereditary for* $p \notin S$

*or* (ii) $\mathcal{J} = T(\{Q(P)\} \cup \{Z(p) \mid p \in R\})$, *where* $R \subseteq P \subseteq S$.

PROOF. Since pure subgroups are S-pure, only the classes described in THEOREM 3.13 need be considered.

If $\mathcal{J}$ is a t-torsion class, then clearly $\mathcal{J}$ is closed under S-pure subgroups if and only if $\mathcal{J} \cap \mathcal{J}_p$ has this property for every prime p. Now $\mathcal{J} \cap \mathcal{J}_p$ is either $\{0\}$, $\mathcal{J}_p$ or $\mathcal{D} \cap \mathcal{J}_p$ so interest is centred on $\mathcal{D} \cap \mathcal{J}_p$. If $p \in S$, then in $\mathcal{D} \cap \mathcal{J}_p$, S-purity is equivalent to purity whence $\mathcal{D} \cap \mathcal{J}_p$ is closed. If $p \notin S$, any exact sequence

$$0 \longrightarrow Z(p) \longrightarrow Z(p^\infty) \longrightarrow Z(p^\infty) \longrightarrow 0$$

is S-pure, so $\mathcal{D} \cap \mathcal{J}_p$ is not closed.

If $\mathcal{J} = T(\{Q(P)\} \cup \{Z(p) \mid p \in R\})$ where $R \subseteq P$, then as in the proof of THEOREM 3.13, $\mathcal{J} \cap \mathcal{F}_0 = T(Q(P)) \cap \mathcal{F}_0$, so by LEMMA 3.18 we may assume $P \subseteq S$. If $A'$ is S-pure in $A \in \mathcal{J}$, then $A'_S$ is S-pure in $A_S$ which, as a direct summand of $A_t$, is in $\mathcal{J} \cap \mathcal{J}_S$ by THEOREM 1.38. Thus $A'_S \in \mathcal{J} \cap \mathcal{J}_S$. Since $Z(p) \in T(Q(P))$ for every $p \notin S$, we have $\mathcal{J} \cap \mathcal{F}_S = T(Q(P)) \cap \mathcal{F}_S$. Analogously with THEOREM 3.8, it can now be shown that $A'/A'_S$ has a natural S-pure embedding in $A/A_S$ and so belongs to $\mathcal{J} \cap \mathcal{F}_S$. Thus $A' \in \mathcal{J}$.//

# CHAPTER 4

## ADDITIONAL CLOSURE PROPERTIES FOR TORSION CLASSES II: GENERALIZED PURE SUBGROUPS

In this chapter we consider torsion classes closed under generalized pure subobjects in the sense of [41]. Specifically, we consider $\mathcal{U}$-purity where $\mathcal{U}$ itself is a torsion class. The first section is a summary of the relevant results from [41]. In the second we obtain some results concerning the idempotence of products and intersections of idempotent radicals. The question of idempotence of products has some relevance to the material of Section 3 in which a generalization of THEOREM 3.8 is obtained for certain abelian categories with global dimension 1, purity being replaced by $\mathcal{U}$-purity.

The remainder of the chapter deals with torsion classes for $\mathcal{Ab}$ only, though many results have obvious generalizations to module categories.

A prerequisite for a generalization of THEOREM 3.9 is a class of groups to take on the role played by the rational groups in Chapter 3, i.e. given a torsion theory $(\mathcal{U},\mathcal{G})$, we need a class of groups in $\mathcal{G}$ whose members, together with those of $\mathcal{U}$, determine all torsion classes closed under $\mathcal{U}$-pure subgroups. Such a class of groups is introduced in Section 4; the groups are described in terms of a rank function associated with $(\mathcal{U},\mathcal{G})$ which coincides with the standard (torsion-free) rank in the case of $(\mathcal{J}_0, \mathcal{F}_0)$. The groups we consider are those of generalized rank ($\mathfrak{U}$-rank) 1.

In Section 5 we investigate the structure of groups of $\mathfrak{U}$-rank 1. Because of the purpose for which these groups have been chosen, some emphasis is placed on the kinds of $\mathfrak{U}$-pure subgroups they can possess. It is shown that they cannot be mixed and cannot be direct sums of infinitely many subgroups. They can however be infinite direct products.

In Section 6 we consider some examples. The groups of $\mathfrak{D}_p$-rank 1 (p prime) are completely described and this leads to a representation of the groups of $\mathfrak{D}_S$-rank 1, where S is a set of primes. The groups of $\mathfrak{U}$-rank 1 are also characterized when $\mathfrak{U}$ is determined by divisible torsion groups. Some additional similarities between rational groups and groups of $\mathfrak{D}_p$-rank 1 are also noted.

In the final section we solve the following special case of our general problem: to find conditions on rational groups X and Y, necessary and sufficient for the closure of T(X) under T(Y)-pure subgroups.

## 1. Generalized Purity

Of the several well-known characterizations of pure subgroups, the one given by the next proposition has the advantage of being element-free and of therefore suggesting generalizations of the notion of purity to categories.

*PROPOSITION 4.1: A subgroup* A *of a group* B *is pure if and only if for every group* G *with* $A \subseteq G \subseteq B$ *and* G/A *finite,* A *is a direct summand of* G.

PROOF. [15] p.82.//

PROPOSITION 4.1 may be paraphrased as follows: a short exact sequence

$$(*)\qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of groups is pure if and only if for every finite subgroup C' of C the pullback induced by the inclusion $C' \longrightarrow C$ gives a commutative diagram

$$(**)\qquad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus C' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

in which the top row represents the natural splitting. The obvious generalization is to substitute some class $\mathcal{C}$ of groups (or of objects in a suitable category) for the class of finite groups and define a short exact sequence (*) to be $\mathcal{C}$-pure if every subobject C' of C which belongs to $\mathcal{C}$ gives a diagram (**).

In this section we shall discuss the theory of generalized purity due to Walker [41] in the setting of a subcomplete locally small abelian category $\mathcal{K}$ which has enough projectives and for which $\mathrm{Ext}^n(A,B)$ is a set for all objects $A,B \in \mathcal{K}$.

*DEFINITION 4.2: Let* $\mathcal{C}$ *be a class of objects of* $\mathcal{K}$ *closed under homomorphic images. A subobject* A *of an object* B *in* $\mathcal{K}$ *is said to be* $\mathcal{C}$*-pure if* A *is a direct summand of every subobject* B' *of* B *with* $A \subseteq B'$ *and* $B'/A \in \mathcal{C}$. *A short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

*in* $\mathcal{K}$ *is* $\mathcal{C}$ *-pure if the image of* f *is* $\mathcal{C}$ *-pure in* B.

*THEOREM 4.3: The class of* $\mathcal{C}$ *-pure short exact sequences is a proper class.*

PROOF. [41] Theorem 2.1.//

For a discussion of *proper classes* see [31] pp. 367 ff.

*DEFINITION 4.4: For objects* A,C *of* $\mathcal{K}$, $\text{Pext}_{\mathcal{C}}(C,A)$ *is the group of equivalence classes of* $\mathcal{C}$ *-pure short exact sequences*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*DEFINITION 4.5: An object* K *of* $\mathcal{K}$ *is* $\mathcal{C}$ *-pure projective if for every* $\mathcal{C}$ *-pure short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*the induced sequence*

$$0 \longrightarrow [K,A] \longrightarrow [K,B] \longrightarrow [K,C] \longrightarrow 0$$

*is exact.*

*THEOREM 4.6: Let* $\mathcal{P}$ *be the class of projective objects of* $\mathcal{K}$. *Then* K *is* $\mathcal{C}$ *-pure projective if and only if it is a direct summand of a direct sum of members of* $\mathcal{P} \cup \mathcal{C}$.

PROOF. [41] Theorem 2.5.//

If has global dimension 1 (and thus in particular if $\mathcal{K} = \mathcal{Ab}$), the $\mathcal{C}$ -pure projectives have an alternative description:

*THEOREM 4.7: If* $\mathcal{K}$ *has global dimension 1, an object* K *is* $\mathcal{C}$ *-pure projective if and only if* $K = L \oplus M$, *where* L *is* *projective and* M *is a direct summand of a direct sum of objects in* $\mathcal{C}$.

PROOF. [41] Theorem 2.6.//

*THEOREM 4.8: A short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*is* $\mathcal{C}$ *-pure if and only if the induced sequence*

$$0 \longrightarrow [K,A] \longrightarrow [K,B] \longrightarrow [K,C] \longrightarrow 0$$

*is exact for every* $\mathcal{C}$*-pure projective* K.

PROOF. [41] Theorem 2.7.//

As examples of generalized purity in $\mathcal{Ab}$, we have S-purity for a set S of primes, in which case $\mathcal{C}$ is the class of finite S-groups, and for an infinite cardinal number $\mathfrak{m}$, the $\mathfrak{m}$-purity introduced by Fuchs [14] where $\mathcal{C}$ is the class of groups G with $|G| < \mathfrak{m}$.

The generalized notion of purity has a dual:

*DEFINITION 4.9: Let* $\mathcal{B}$ *be a class of objects of* $\mathcal{K}$ *which is closed under subobjects. A subobject* A *of* B *is said to be* $\mathcal{B}$*-copure if* $A/A'$ *is a direct summand of* $B/A'$ *whenever* $A' \subseteq A$ *and* $A/A' \in \mathcal{B}$. *A short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

*is called* $\mathcal{B}$*-copure if the image of* f *is* $\mathcal{B}$*-copure in* B.

The results given above for $\mathcal{C}$-purity can be dualized, with $\mathcal{B}$-copurity replacing $\mathcal{C}$-purity. In particular, since the postulates for a proper class are self-dual, we have

*THEOREM 4.10: The class of $\mathfrak{B}$-copure short exact sequences is a proper class.//*

*DEFINITION 4.11: For* A,C *in* $\mathcal{K}$, $\mathrm{Copext}_{\mathfrak{B}}(C,A)$ *is the group of equivalence classes of* $\mathfrak{B}$ *-copure short exact sequences*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

In the sequel we shall be principally concerned with the case where $\mathcal{C}$ is a torsion class. The pure and copure short exact sequences associated with a torsion theory are described by

*THEOREM 4.12: Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory for* $\mathcal{K}$ *and let* r *be the associated idempotent radical. The short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*is* $\mathcal{T}$ *-pure (resp.* $\mathcal{F}$ *-copure) if and only if the induced sequence*

$$0 \longrightarrow r(A) \longrightarrow r(B) \longrightarrow r(C) \longrightarrow 0$$

*(resp.*

$$0 \longrightarrow A/r(A) \longrightarrow B/r(B) \longrightarrow C/r(C) \longrightarrow 0)$$

*is splitting exact. As a consequence, we have*

(i) $r(A) = A \cap r(B)$

*and* (ii) $r(B/A) = (r(B)+A)/A$

*whenever* A *is either* $\mathcal{T}$ *-pure or* $\mathcal{F}$ *-copure in* B.

PROOF. [41] Theorems 3.4, 3.9 and Corollaries.//

*DEFINITION 4.13: For a torsion theory* $(\mathcal{T},\mathcal{F})$, *a short exact sequence which is both* $\mathcal{T}$ *-pure and* $\mathcal{F}$ *-copure is said to be*

$(\mathcal{T},\mathcal{F})$*-bipure and for any* A,C, $\mathrm{Pext}_{\mathcal{T}}(C,A) \cap \mathrm{Copext}_{\mathcal{F}}(C,A)$ *is denoted* $\mathrm{Bipext}_{(\mathcal{T},\mathcal{F})}(C,A)$.

Since the intersection of two proper classes is a proper class, it is clear that the $(\mathcal{T},\mathcal{F})$-bipure short exact sequences make up a proper class.

## 2. Idempotence of Composite Radicals

As noted in §5 of Chapter 1, a group A belongs to a torsion class $\mathcal{T}$ if and only if both $A_t$ and $A/A_t$ do. This result suggests the problem of determining whether there are any other torsion classes besides $\mathcal{T}_0$ for which the corresponding statement is true for all $\mathcal{T}$ or for some given $\mathcal{T}$. For this section we shall work in a subcomplete locally small abelian category $\mathcal{K}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ will denote torsion classes in $\mathcal{K}$, $r_1$ and $r_2$ their associated idempotent radicals.

*DEFINITION 4.14: Let* u *and* v *be subfunctors of the identity. The subfunctor* $u \cap v$ *is defined by*

$$(u \cap v)(K) = u(K) \cap v(K)$$

*with action on morphisms being determined by restriction.*

*PROPOSITION 4.15: If* u *and* v *are radicals, then so are* uv *and* $u \cap v$.

PROOF. [32] p.110.//

Thus in particular $r_1 r_2$ is a radical, but as we shall see, not necessarily idempotent.

*PROPOSITION 4.16: The statement*

$$(*) \qquad K \in \mathfrak{T}_2 \Leftrightarrow r_1(K),\ K/r_1(K) \in \mathfrak{T}_2$$

*holds for every* $K \in \mathcal{K}$ *if and only if* $r_1\ r_2$ *is idempotent.*

PROOF. If (*) holds, then for every $K \in \mathcal{K}$, $r_2(K)$ belongs to $\mathfrak{T}_2$ so $r_1\ r_2(K)$ does also, i.e. $r_2\ r_1\ r_2(K) = r_1\ r_2(K)$, or since K is arbitrary, $r_2\ r_1\ r_2 = r_1\ r_2$, so that

$$r_1\ (r_2\ r_1\ r_2) = r_1\ (r_1\ r_2) = r_1\ r_2,$$

i.e. $(r_1\ r_2)^2 = r_1\ r_2$. Conversely, let $(r_1\ r_2)^2 = r_1\ r_2$. Then for any $K \in \mathcal{K}$,

$$r_1\ r_2\ (K) = r_1\ r_2\ r_1\ r_2(K) \subseteq r_2\ r_1\ r_2(K) \subseteq r_1\ r_2(K)$$

i.e. $r_1\ r_2 = r_2\ r_1\ r_2$. Thus if $K \in \mathfrak{T}_2$, we have

$$r_1\ (K) = r_1\ r_2(K) = r_2\ r_1\ r_2\ (K)$$

which is also in $\mathfrak{T}_2$. Since $\mathfrak{T}_2$ is closed under homomorphic images and extensions, the proof is complete.//

*COROLLARY 4.17: If* $r_1\ r_2 = r_2\ r_1$ *then*

$$K \in \mathfrak{T}_2 \Leftrightarrow r_1(K),\ K/r_1(K) \in \mathfrak{T}_2$$

*and*

$$K \in \mathfrak{T}_1 \Leftrightarrow r_2(K),\ K/r_2(K) \in \mathfrak{T}_1.$$

PROOF. If $r_1\ r_2 = r_2\ r_1$, then

$$(r_1 r_2)^2 = r_1(r_2 r_1) r_2 = r_1(r_1 r_2) r_2 = (r_1 r_1)(r_2 r_2) = r_1 r_2$$

and similarly $(r_2 r_1)^2 = r_2 r_1$.//

The problem is therefore, in part, that of finding commuting pairs of idempotent radicals, and in particular of finding those idempotent radicals which commute with all others. The next two propositions give some examples for $\mathcal{K} = \mathfrak{Ab}$.

*PROPOSITION 4.18: Let $\mathcal{K} = \mathfrak{Ab}$ and let $\mathcal{T}_1$ be hereditary. Then for any $\mathcal{T}_2$, $r_1 r_2 = r_2 r_1 = r_1 \cap r_2$.*

PROOF. For any group A, we have

(i) $r_1 r_2(A) = r_1(A) \cap r_2(A)$

and (ii) $r_2 r_1(A) \subseteq r_1(A) \cap r_2(A)$

so $r_2 r_1(A) = r_2^2 r_1(A) \subseteq r_2 r_1 r_2(A)$. Also, $r_1 r_2(A) \subseteq r_1(A)$, whence $r_2 r_1 r_2(A) \subseteq r_2 r_1(A)$. Thus $r_2 r_1 r_2 = r_2 r_1$.

Now $r_1$ is either the identity functor, in which case there is nothing to prove, or $r_1$ assigns to each group A the subgroup $\bigoplus_{p \in S} A_p$, where S is a fixed set of primes. Since $(r_2 A)_t \in \mathcal{T}_2$ (THEOREM 1.38), its direct summand $r_1 r_2(A)$ is also. Thus $r_2 r_1 r_2(A) = r_1 r_2(A)$, so

$$r_2 r_1 = r_2 r_1 r_2 = r_1 r_2 = r_1 \cap r_2.//$$

$\mathcal{T}_1$ need not be hereditary, however:

*PROPOSITION 4.19: Let $\mathcal{K} = \mathfrak{Ab}$, $\mathcal{T}_1 = \mathcal{D}$. Then for any $\mathcal{T}_2$, $r_1 r_2 = r_2 r_1$.*

PROOF. Case (1): $\mathcal{T}_2$ contains torsion-free groups. In this case $\mathcal{T}_1 \subseteq \mathcal{T}_2$, so for any group A, we have $r_2 r_1(A) = r_1(A)$, i.e. $r_2 r_1 = r_1$. Also $r_1(A) \subseteq r_2(A)$ and $r_1 r_2(A) \subseteq r_1(A)$, whence

$$r_1(A) = r_1^2(A) \subseteq r_1 r_2(A) \subseteq r_1(A)$$

so that $r_1 r_2 = r_1 = r_2 r_1$.

Case (2): $\mathcal{T}_2$ is a t-torsion class. $r_2(A)$ has the form $\bigoplus_{p \in S} r_p(A)$, where S is a fixed set of primes and for each $p \in S$ $r_p(A)$ is either $A_p$ or its divisible part. Thus $r_1 r_2(A) = \bigoplus_{p \in S} r_1 r_p(A)$. If $r_p(A) = A_p$, then $r_1 r_p(A) = r_p r_1(A)$ as in

PROPOSITION 4.18, and if not then $r_1r_p(A) = r_pr_1(A) = r_p(A)$ as in Case (1). Thus

$$r_1r_2(A) = \bigoplus_{p\in S} r_1r_p(A) = \bigoplus_{p\in S} r_pr_1(A) = r_2r_1(A)$$

i.e. $r_1r_2 = r_2r_1$.//

Let r be the idempotent radical associated with the torsion class $\mathcal{T}_1 \cap \mathcal{T}_2$.

*PROPOSITION 4.20:* $r_1r_2$ *is idempotent if and only if* $r_1r_2 = r$.

PROOF. Let $r_1r_2$ be idempotent with torsion class $\mathcal{U}$. Then for every $K \in \mathcal{T}_1 \cap \mathcal{T}_2$, we have $r_1r_2(K) = r_1(K) = K$, i.e. $K \in \mathcal{U}$, so $\mathcal{T}_1 \cap \mathcal{T}_2 \subseteq \mathcal{U}$. Since

$$r_1r_2(L) = r_1r_2r_1r_2(L) \leq r_2r_1r_2(L) \subseteq r_1r_2(L)$$

for any L, we have $r_2r_1r_2 = r_1r_2$, so in particular $r_2(L) = L$ if $L \in \mathcal{U}$, i.e. $\mathcal{U} \subseteq \mathcal{T}_2$. Since also for every $L \in \mathcal{U}$,

$$r_1(L) = r_1(r_1r_2(L)) = r_1^2r_2(L) = r_1r_2(L) = L,$$

we have $\mathcal{U} \subseteq \mathcal{T}_1$, so $\mathcal{U} = \mathcal{T}_1 \cap \mathcal{T}_2$ and $r_1r_2 = r$.

The converse is obvious.//

Using COROLLARY 4.17, we obtain

*COROLLARY 4.21:* $r_1r_2 = r_2r_1$ *if and only if* $r_1r_2$ *and* $r_2r_1$ *are both idempotent, in which case* $r_1r_2 = r_2r_1 = r$.//

*PROPOSITION 4.22:* $r_1 \cap r_2$ *is idempotent if and only if* $r_1 \cap r_2 = r$.

PROOF. If $r_1 \cap r_2$ is idempotent, then its torsion class is $\mathcal{J}_1 \cap \mathcal{J}_2$ since $r_1(K) \cap r_2(K) = K$ if and only if $r_1(K) = K = r_2(K)$. The converse is obvious.//

Although an object is left fixed by $r_1 \cap r_2$ exactly when it belongs to $\mathcal{J}_1 \cap \mathcal{J}_2$, this does not mean that $r_1 \cap r_2$ must be idempotent (cf. §2 of Chapter 1).

*LEMMA 4.23: If* $r_1r_2 = r_1 \cap r_2$, *then* $r_2r_1$ *is idempotent.*

PROOF. $r_2\, r_1\, r_2\, r_1 = r_2((r_1r_2)(r_1)) = r_2((r_1r_1) \cap (r_2r_1)) = r_2(r_1 \cap (r_2r_1)) = r_2(r_2r_1) = r_2r_1$.//

*PROPOSITION 4.24: If any two of* $r_1r_2$, $r_2r_1$, $r_2 \cap r_1$ *are idempotent, then* $r_1r_2 = r_2r_1$.

PROOF. If $r_1r_2$ and $r_2r_1$ are idempotent, then $r_1r_2 = r = r_2r_1$ (COROLLARY 4.21), while if $r_1r_2$ and $r_1 \cap r_2$ are idempotent, then by PROPOSITIONS 4.20 and 4.22, $r_1r_2 = r_1 \cap r_2$, so by LEMMA 4.23, $r_2r_1$ is idempotent.//

We now give an example to show that $r_1r_2$ and $r_2r_1$ need not be equal. Note that by COROLLARY 4.21 this is sufficient to show that idempotence is not preserved by products in general.

*EXAMPLE 4.25:* We consider a group which has been discussed by Erdős [12] and de Groot [20], [21]. Let $\{x,y\}$ be a basis for a 2-dimensional rational vector space, and let

$$G = [p^{-n}x,\ q^{-n}y,\ t^{-n}(x+y) \mid n = 1,2,3,\dots ]$$

where p, q and t are distinct primes. Let $r_1$ and $r_2$ be the idempotent radicals for $\mathcal{D}_p$ and T(G) respectively. From an examination of the type set of G (see [20] p.295), it is clear that $r_1(G) = [p^{-n}x | n = 1,2,3,\dots] \cong Q(p)$. Let f be any homomorphism from G to $r_1(G)$. Since $r_1(G)$ has no non-zero elements of infinite q-height or t-height, we have $f(y) = 0 = f(x+y) = f(x)+f(y)$. But then $f(x) = 0$ and thus $f = 0$. Hence $[G,r_1(G)] = 0$, so $r_2 r_1(G) = 0$. But $r_1 r_2(G) = r_1(G) \cong Q(p)$.

This example also shows that idempotence need not be preserved by intersections. Let r be the idempotent radical associated with $\mathcal{D}_p \cap T(G)$. Then $r_1(G) \cong Q(p) \notin \mathcal{D}_p \cap T(G)$ so $r\, r_1(G) = 0$. Since $r(G) \in \mathcal{D}_p$, we have $r(G) \subseteq r_1(G)$, so $r(G) = r^2(G) \subseteq r\, r_1(G) = 0$. Therefore $r(G) = 0$, while $r_1(G) \cap r_2(G) = r_1(G) \cong Q(p)$. By PROPOSITION 4.22, idempotence of $r_1 \cap r_2$ would require $r_1 \cap r_2 = r$.

To conclude this part of the discussion we give a "local" construction, using transfinite induction, of the radical associated with the intersection of a set of torsion classes. This construction was used by Leavitt [30] for radical classes of associative rings. In an abelian category the fact that all subobjects are normal allows a minor simplification of the original argument.

Let $\{\mathcal{J}_i | i \in I\}$ be a set of torsion classes in $\mathcal{K}$, $\{r_i | i \in I\}$ the associated set of idempotent radicals and $K \in \mathcal{K}$. Define $L_1 = K$ and for an ordinal number $\beta$, assuming $L_\alpha$ has been defined for all $\alpha < \beta$, define

$$(**) \quad L_\beta = \begin{cases} \bigcap_{\alpha<\beta} L_\alpha \text{ if } \beta \text{ is a limit ordinal. Otherwise} \\ r_i(L_{\beta-1}) \text{ for some } i \in I \text{ (if such exists)} \\ \text{such that } r_i(L_{\beta-1}) \neq L_{\beta-1}. \end{cases}$$

Since the $L_\alpha$'s form a set, there exists an ordinal number $\gamma$ such that $r_i(L_\gamma) = L_\gamma$ for each $i \in I$.

*THEOREM 4.26: For any* $K \in \mathcal{K}$, *let* $L_\beta$ *be defined by* (**). *Then there exists an ordinal number* $\gamma$ *such that* $r(K) = L_\gamma$, *where* $r$ *is the idempotent radical for* $\bigcap_{i \in I} \mathcal{J}_i$.

PROOF. As noted above, there exists $\gamma$ such that for each $i \in I$, $r_i(L_\gamma) = L_\gamma$, i.e. $L_\gamma \in \mathcal{J}_i$. Thus $r(L_\gamma) = L_\gamma$, so $L_\gamma \subseteq r(K)$. We show, by transfinite induction, that $r(K) \subseteq L_\gamma$. Trivially $r(K) \subseteq L_1$, so assume $r(K) \subseteq L_\alpha$ for each $\alpha < \beta$. If $\beta$ is a limit ordinal it is clear from (**) that $r(K) \subseteq L_\beta$. If $\beta$ is a successor, then by assumption $r(K) \subseteq L_{\beta-1}$, while since $r(K) \in \mathcal{J}_i$ we have $r(K) \subseteq r_i(L_{\beta-1}) = L_\beta$. Thus $r(K) \subseteq L_\beta$ for every $\beta$, so $r(K) \subseteq L_\gamma$.//

It is not clear whether the construction can always be achieved in a finite number of steps in $\mathcal{K}$ even when the number of torsion classes involved is finite. With $r_1$ and $r_2$ as in EXAMPLE 4.25, at least three steps are needed: for G we have

$$G \supsetneqq r_1(G) \supsetneqq r_2\, r_1(G) = 0 = r(G).$$

## 3. A Simplification of the Problem

In this section we shall work in a subcomplete abelian category $\mathcal{K}$ satisfying the same conditions as in Section 1 and in addition having global dimension 1.

*PROPOSITION 4.27: Let* $(\mathcal{J},\mathcal{F})$ *and* $(\mathcal{U},\mathcal{G})$ *be torsion theories for* $\mathcal{K}$, *with associated idempotent radicals* $r$, $s$

*respectively such that* sr *is idempotent. Then* $\mathcal{J}$ *is closed under* $(\mathfrak{U},\mathcal{G})$*-bipure subobjects.*

PROOF. Let K' be $(\mathfrak{U},\mathcal{G})$-bipure in $K \in \mathcal{J}$. Then there are split exact sequences

$$0 \longrightarrow s(K') \longrightarrow s(K) \longrightarrow s(K/K') \longrightarrow 0$$

and
$$0 \longrightarrow K'/s(K') \longrightarrow K/s(K) \longrightarrow (K/K')/s(K/K') \longrightarrow 0.$$

By PROPOSITION 4.16, both s(K) and K/s(K) belong to $\mathcal{J}$ and so their direct summands s(K') and K'/s(K') do also. Hence $K' \in \mathcal{J}$.//

Note that this proposition is true without any restriction on the global dimension of $\mathcal{K}$.

For $\mathfrak{U}$-pure subobjects there is a result analogous to THEOREM 3.8. Proof of this requires

*LEMMA 4.28: A short exact sequence*

$$(*) \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*is* $\mathfrak{U}$*-pure, for a torsion class* $\mathfrak{U}$ *if and only if the induced sequence*

$$(**) \quad 0 \longrightarrow [K,A] \longrightarrow [K,B] \longrightarrow [K,C] \longrightarrow 0$$

*is exact for every* $K \in \mathfrak{U}$.

PROOF. By assumptions on $\mathcal{K}$, THEOREM 4.7 and the properties of torsion classes, an object L is $\mathfrak{U}$-pure projective exactly when it has the form $M \oplus K$ where M is projective and $K \in \mathfrak{U}$. The sequence (*) induces a morphism.

$$[L,B] \cong [M,B] \oplus [K,B] \xrightarrow{f \oplus g} [M,C] \oplus [K,C] \cong [L,C].$$

Since M is projective, f is an epimorphism, so if (**) is assumed exact for every $K \in \mathfrak{U}$, $f \oplus g$ is an epimorphism, so by THEOREM 4.8, (*) is $\mathcal{U}$-pure.

The converse is obvious.//

*COROLLARY 4.29: Let* $(\mathcal{U},\mathcal{G})$ *be a torsion theory for* $\mathcal{K}$, $K' \subseteq K \in \mathcal{G}$. *Then* $K'$ *is* $\mathcal{U}$*-pure if and only if* $K/K' \in \mathcal{G}$.

PROOF. For $U \in \mathcal{U}$ the induced sequence

$$0 = [U,K] \longrightarrow [U,K/K'] \longrightarrow 0$$

is exact if and only if $[U,K/K'] = 0$. By LEMMA 4.28, $K'$ is $\mathcal{U}$-pure in $K$ if and only if $[U,K/K'] = 0$ for all $U \in \mathcal{U}$.//

In a similar way we can prove

*COROLLARY 4.30: Let* $s$ *be the idempotent radical for the torsion class* $\mathcal{U}$. *Then* $s(K)$ *is* $\mathcal{U}$*-pure in* $K$ *for all* $K \in \mathcal{K}$.//

*PROPOSITION 4.31: Let* $(\mathcal{J},\mathcal{F})$ *and* $(\mathcal{U},\mathcal{G})$ *be torsion theories for* $\mathcal{K}$, *with associated idempotent radicals* $r$ *and* $s$. *If* $\mathcal{J}$ *is closed under* $\mathcal{U}$*-pure subobjects, then* $sr$ *is idempotent.*

PROOF. Since for any $K \in \mathcal{K}$ $s(K)$ is a $\mathcal{U}$-pure subobject, in particular $sr(K)$ is always $\mathcal{U}$-pure in $r(K)$. By assumption on $\mathcal{J}$, therefore, we have $rsr(K) = sr(K)$ for each $K$. But then

$$(sr)^2 = s(rsr) = s(sr) = s^2r = sr.//$$

*THEOREM 4.32: Let* $(\mathcal{J},\mathcal{F})$ *and* $(\mathcal{U},\mathcal{G})$ *be torsion theories for* $\mathcal{K}$ *with associated idempotent radicals* $r,s$ *respectively. Then* $\mathcal{J}$ *is closed under* $\mathcal{U}$*-pure subobjects if and only if* $sr$ *is idempotent and* $\mathcal{J} \cap \mathcal{G}$ *is closed under* $\mathcal{U}$*-pure subobjects.*

PROOF. We first show that $\mathcal{T} \cap \mathcal{U}$ is always closed under $\mathcal{U}$-pure subobjects. If K' is $\mathcal{U}$-pure in K $\varepsilon\ \mathcal{T} \cap \mathcal{U}$, then K/K' $\varepsilon\ \mathcal{U}$ and the sequence

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K/K' \longrightarrow 0$$

is split, by definition of $\mathcal{U}$-purity. Hence K' $\varepsilon\ \mathcal{T} \cap \mathcal{U}$.

Now suppose $\mathcal{T} \cap \mathcal{G}$ is closed under $\mathcal{U}$-pure subobjects and sr is idempotent. If M' is $\mathcal{U}$-pure in M $\varepsilon\ \mathcal{T}$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & s(M') \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M'/s(M') \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & s(M) \arrow[r] & M \arrow[r] & M/s(M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

f is a monomorphism, having kernel $M'/s(M') \cap N$, where N is the kernel of the natural map from $M/s(M')$ to $M/s(M)$, i.e. $N = s(M)/s(M')$ and thus

$$(M'/s(M')) \cap N = (M' \cap s(M))/s(M') = 0,$$

by THEOREM 4.12.

THEOREM 4.12 also says that $(s(M) + M')/M' = s(M/M')$, so

$$M/(s(M)+M') \cong (M/M')/((s(M)+M')/M') = (M/M')/s(M/M') \ \varepsilon\ \mathcal{G}.$$

Hence the sequence

$$0 \longrightarrow (s(M)+M')/s(M) \longrightarrow M/s(M) \longrightarrow M/(s(M)+M') \longrightarrow 0$$

is $\mathcal{U}$-pure exact, whence as $M/s(M)\ \varepsilon\ \mathcal{T} \cap \mathcal{G}$, it follows that $(s(M)+M')/s(M)\ \varepsilon\ \mathcal{T} \cap \mathcal{G}$. But as

$$M'/s(M') = M'/(M' \cap s(M)) \cong (M'+s(M))/s(M),$$

this means that $M'/s(M')\ \varepsilon\ \mathcal{T} \cap \mathcal{G}$.

Also, $s(M)\ \varepsilon\ \mathcal{T} \cap \mathcal{U}$ (PROPOSITION 4.16). Since s(M') is $\mathcal{U}$-pure in M' and the $\mathcal{U}$-pure short exact sequences form a proper class (THEOREM 4.3), s(M') is $\mathcal{U}$-pure in M and hence in s(M), so s(M') $\varepsilon$ $\mathcal{J} \cap \mathcal{U}$. M' is therefore in $\mathcal{J}$, as both s(M') and M'/s(M') are, i.e. $\mathcal{J}$ is closed under $\mathcal{U}$-pure subobjects.

By PROPOSITION 4.31 the converse is obvious.//

COROLLARY 4.29 shows that in $\mathcal{F}_0$, purity and $\mathcal{J}_0$-purity coincide, so as a consequence of THEOREMS 3.8 and 4.32, we see that in $\mathcal{Ab}$, a torsion class is closed under $\mathcal{J}_0$-pure subgroups exactly when it is closed under pure subgroups, which raises the question: if $\mathcal{C}$ is homomorphically closed, when is closure of a torsion class under $\mathcal{C}$-pure subobjects equivalent to that for $\mathcal{J}$-pure subobjects for a torsion class $\mathcal{J}$? This question is related to the problem of determining projective closures, for $\mathcal{C}$ and $\mathcal{J}$ satisfy the condition in particular when $\mathcal{C}$-purity coincides with $\mathcal{J}$-purity, i.e. $\mathcal{C}$ and $\mathcal{J}$ have the same projective closure, e.g. if $\mathcal{C}$ is the class of homomorphic images of Q and $\mathcal{J} = \mathcal{D}$ (see [35] or [41]).

In Chapter 3 we used implicitly the fact that if $\mathcal{J}$ is a torsion class (in $\mathcal{Ab}$) and $\mathcal{C} \subseteq \mathcal{J}_0$, then $T(\mathcal{J} \cup \mathcal{C}) \cap \mathcal{F}_0 = \mathcal{J} \cap \mathcal{F}_0$, whence if $\mathcal{J}$ is closed under pure subgroups, the same is true of $T(\mathcal{J} \cup \mathcal{C})$. THEOREM 4.32 raises the question whether, given a torsion class $\mathcal{J}$, a torsion theory $(\mathcal{U}, \mathcal{G})$ and a subclass $\mathcal{C}$ of $\mathcal{U}$, it is possible for $T(\mathcal{J} \cup \mathcal{C})$ to contain objects of $\mathcal{G}$ which do not belong to $\mathcal{J}$. By LEMMA 3.11, which obviously holds also in $\mathcal{K}$, it may be assumed that $\mathcal{C}$ is a torsion class.

*PROPOSITION 4.33: Let* $(\mathcal{U}, \mathcal{G})$ *and* $(\mathcal{J}, \mathcal{F})$ *be torsion theories for* $\mathcal{K}$*, the latter with idempotent radical* r. *If* $\mathcal{C}$ *is a torsion subclass of* $\mathcal{U}$*, then* $\mathcal{J} \cap \mathcal{G} = T(\mathcal{J} \cup \mathcal{C}) \cap \mathcal{G}$ *if and only if* r(G) *is a* $\mathcal{C}$*-pure subobject of* G *for every* $G \in \mathcal{G}$.

PROOF. An element K of $T(\mathcal{J} \cup \mathcal{C}) \cap \mathcal{G}$ belongs to $\mathcal{J}$ if and only if $K/r(K) = 0$. Since $K/r(K) \in T(\mathcal{J} \cup \mathcal{C})$, this is equivalent to $s(K/r(K)) = 0$, where s is the idempotent radical for $\mathcal{C}$. But $s(K) = 0$ so by COROLLARY 4.29, $s(K/r(K)) = 0$ if and only if r(K) is $\mathcal{C}$-pure in K. Thus if r(G) is $\mathcal{C}$-pure in G for every $G \in \mathcal{G}$, then $\mathcal{J} \cap \mathcal{G} = T(\mathcal{J} \cup \mathcal{C}) \cap \mathcal{G}$. Conversely, if this equality is satisfied, then for any $G \in \mathcal{G}$, r(G) is the largest subobject belonging to $T(\mathcal{J} \cup \mathcal{C}) \cap \mathcal{G}$, so $s(G/r(G)) = 0$.//

The conditions of PROPOSITION 4.33 are not always satisfied, as the following example shows.

*EXAMPLE 4.34:* For distinct primes q, t, consider the torsion theory $(T(Q(\{q,t\})), \mathcal{G})$ and the group G of EXAMPLE 4.25:

$$G = [p^{-n}x,\ q^{-n}y,\ t^{-n}(x+y) \mid n = 1,2,3,\ldots]$$

where p is a prime other than q, t and x, y are linearly independent. $G \in \mathcal{G}$, but since $pG \neq G$, $G \notin T(Q(p)) \cap \mathcal{G}$. However, there is a short exact sequence

$$0 \longrightarrow Q(p) \cong [x]_* \longrightarrow G \longrightarrow G/[x]_* \cong Q(\{q,t\}) \longrightarrow 0$$

which shows that G belongs to $T(T(Q(p)) \cup T(Q(\{q,t\}))) \cap \mathcal{G}$.

Another problem suggested by results in Chapter 3 is that of determining when the class of extensions of objects in a torsion class $\mathcal{T}_1$ by members of a torsion class $\mathcal{T}_2$ is itself a torsion class. (cf. PROPOSITION 3.16). The conditions of PROPOSITION 4.33 are sufficient. Before showing this we prove

*PROPOSITION 4.35: Let* $(\mathcal{T}_1, \mathcal{F}_1)$ *and* $(\mathcal{T}_2, \mathcal{F}_2)$ *be torsion theories for* $\mathcal{K}$. *The following conditions on* $K \in \mathcal{K}$ *are equivalent.*

(i) *There exists a short exact sequence*

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0$$

*with* $K' \in \mathcal{T}_1$ *and* $K'' \in \mathcal{T}_2$.

(ii) *There exists a short exact sequence*

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0$$

*with* $K' \in \mathcal{T}_1$ *and* $K'' \in \mathcal{T}_2 \cap \mathcal{F}_1$.

PROOF. Let $(\mathcal{T}_1, \mathcal{F}_1)$ have idempotent radical $r_1$.

(i) $\Rightarrow$ (ii): There is a short exact sequence

$$0 \longrightarrow K' \longrightarrow L \longrightarrow r_1(K'') \longrightarrow 0$$

where $L \subsetneq K$ and necessarily $L \in \mathcal{T}_1$. The resulting exact sequence

$$0 \longrightarrow L \longrightarrow K \longrightarrow K''/r_1(K'') \longrightarrow 0$$

satisfies (ii). Obviously (ii) $\Rightarrow$ (i).//

*PROPOSITION 4.36: Let* $(\mathcal{T}_1, \mathcal{F}_1)$, $(\mathcal{T}_2, \mathcal{F}_2)$ *be torsion theories for* $\mathcal{K}$ *with idempotent radicals* $r_1$, $r_2$. *If* $r_2(K)$ *is* $\mathcal{T}_1$*-pure in* $K$, *for every* $K \in \mathcal{F}_1$, *then the following conditions are equivalent:*

(i) $K \in T(\mathcal{T}_1 \cup \mathcal{T}_2)$

(ii) *and* (iii) *as in PROPOSITION 4.35.*

PROOF. Clearly (ii) => (i) without restriction on $\mathcal{J}_1$ and $\mathcal{J}_2$. By PROPOSITION 4.35 it suffices to show (i) => (iii). For every $K \in T(\mathcal{J}_1 \cup \mathcal{J}_2)$, there are short exact sequences

$$0 \longrightarrow r_1(K) \longrightarrow K \longrightarrow K/r_1(K) = K'' \longrightarrow 0$$

and
$$0 \longrightarrow r_2(K'') \longrightarrow K'' \longrightarrow K''/r_2(K'') \longrightarrow 0,$$

the latter being $\mathcal{J}_1$-pure. But then $K''/r_2(K'') \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap T(\mathcal{J}_1 \cup \mathcal{J}_2)$, so $K''/r_2(K'') = 0$, i.e. $K'' \in \mathcal{J}_2 \cap \mathcal{F}_1$.//

## 4. Generalized Rank

For the remainder of this chapter we shall work in $\mathcal{A}b$. Many results however can be generalized to modules over hereditary rings (at least).

The torsion classes closed under subgroups and pure subgroups have been classified by minimal representations. In attempting a similar classification of torsion classes closed under generalized pure subgroups, we therefore begin by searching for groups which give simple representations of such classes.

The hereditary torsion classes are determined by groups of the form Z(p) and Z, which have no non-isomorphic proper subgroups and for any torsion theory $(\mathcal{J},\mathcal{F})$ belong to either $\mathcal{J}$ or $\mathcal{F}$. Groups of the form Q(P), Z(p) and $Z(p^{\infty})$ give representations of all torsion classes closed under pure subgroups, and a group of this kind has no proper pure subgroups and belongs to $\mathcal{J}$ or $\mathcal{F}$ for any torsion theory $(\mathcal{J},\mathcal{F})$. Some information about torsion classes closed under $\mathcal{C}$-pure subgroups might therefore come from the study of groups which belong to

$\mathcal{T}$ or $\mathcal{F}$ for any $(\mathcal{T},\mathcal{F})$ and which have no proper $\mathcal{C}$ -pure subgroups.

When $\mathcal{C}$ is a torsion class, however, we can specify groups determining torsion classes closed under $\mathcal{C}$ -pure subgroups in terms of a generalized rank function which we now introduce.

To justify the definition of generalized rank the following result is needed:

*PROPOSITION 4.37: Let* $(\mathcal{U},\mathcal{G})$ *be a torsion theory. If* $G \in \mathcal{G}$*, then the intersection of any family of* $\mathcal{U}$*-pure subgroups of* $G$ *is* $\mathcal{U}$*-pure.*

PROOF. By COROLLARY 4.29, it suffices to show that $G/\bigcap_{\lambda\in\Lambda} G_\lambda \in \mathcal{G}$ for any set $\{G_\lambda | \lambda \in \Lambda\}$ of $\mathcal{U}$ -pure subgroups of G. Suppose $\bigcap_{\lambda\in\Lambda} G_\lambda \subseteq G' \subseteq G$ and $G'/\bigcap_{\lambda\in\Lambda} G_\lambda \in \mathcal{U}$. Then for each $\mu \in \Lambda$, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & (G' \cap G_\mu)/\bigcap_{\lambda\in\Lambda} G_\lambda \arrow[r] & G'/\bigcap_{\lambda\in\Lambda} G_\lambda \arrow[r] & G'/(G' \cap G_\mu) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
 & & 0 \arrow[r] & (G'+G_\mu)/G_\mu \arrow[r] & G/G_\mu
\end{tikzcd}$$

with exact rows. $(G'+G_\mu)/G_\mu \in \mathcal{G}$, but by assumption on $G'/\bigcap_{\lambda\in\Lambda} G_\lambda$, $G'/(G' \cap G_\mu) \in \mathcal{U}$. Hence $G'/(G' \cap G_\mu) = 0$. But this means that $G' \subseteq G_\mu$ for each $\mu$, so $G'/\bigcap_{\lambda\in\Lambda} G_\lambda = 0$, i.e. $G/\bigcap_{\lambda\in\Lambda} G_\lambda \in \mathcal{G}$.//

Every element or subset of a group $G \in \mathcal{G}$ is therefore contained in a smallest $\mathcal{U}$ -pure subgroup.

The generalized rank for a torsion theory $(\mathcal{U},\mathcal{G})$ is introduced in the following definitions.

*DEFINITION 4.38: If* $\mathfrak{W}$ *is a subset of the elements of a group* $G \in \mathcal{G}$, $[\mathfrak{W}]_{\mathcal{U}}^{G}$ *denotes the smallest* $\mathcal{U}$*-pure subgroup of* $G$ *containing* $\mathfrak{W}$. *If* $\mathfrak{W}$ *is a finite set* $\{x_1,\ldots,x_n\}$ *or a countable set* $\{x_1,x_2,x_3,\ldots\}$; $[\mathfrak{W}]_{\mathcal{U}}^{G}$ *is denoted by* $[x_1,\ldots,x_n]_{\mathcal{U}}^{G}$ *or* $[x_1,x_2,\ldots]_{\mathcal{U}}^{G}$ *respectively. When there is no confusion about the containing group the superscript* $G$ *will be omitted.*

*DEFINITION 4.39: A non-zero group* $G \in \mathcal{G}$ *has* $\mathcal{U}$*-rank* $\mathfrak{m}$ *if it has a subset* $\mathfrak{S}$ *with* $[\mathfrak{S}]_{\mathcal{U}} = G$ *and* $|\mathfrak{S}| = \mathfrak{m}$ *and if* $\mathfrak{m}$ *is the least cardinal number for which such a set exists. We denote this by writing* $\mathcal{U}$-rank $(G) = \mathfrak{m}$. $\mathfrak{S}$ *is called a* $\mathcal{U}$*-basis for* $G$.

If $(\mathcal{U},\mathcal{G}) = (\mathcal{J}_0, \mathcal{F}_0)$ this definition gives the standard (torsion-free) rank, since in $\mathcal{F}_0$ purity coincides with $\mathcal{J}_0$-purity. Note that $\mathcal{U}$-rank is defined only on non-zero groups in $\mathcal{G}$.

Obviously for every non-zero $x \in G \in \mathcal{G}$, $[x]_{\mathcal{U}}$ has $\mathcal{U}$-rank 1, so since $G$ is generated by such subgroups, we have

*PROPOSITION 4.40: If* $(\mathcal{U},\mathcal{G})$ *is a torsion theory then every* $G \in \mathcal{G}$ *is a homomorphic image of a direct sum of groups in* $\mathcal{G}$ *with* $\mathcal{U}$*-rank* 1.//

Using THEOREM 4.32 and PROPOSITION 4.40 and reasoning as in the proof of THEOREM 3.9, we obtain

*THEOREM 4.41: Let* $(\mathcal{J},\mathcal{F})$ *and* $(\mathcal{U},\mathcal{G})$ *be torsion theories such that* $\mathcal{J}$ *is closed under* $\mathcal{U}$*-pure subgroups. Then*

$$\mathcal{J} = T((\mathcal{J} \cap \mathcal{U}) \cup \{G \in \mathcal{J} \cap \mathcal{G} \mid \mathcal{U}\text{-rank}(G) = 1\}).//$$

The groups with $\mathcal{J}_0$-rank 1 are the rational groups, and we have seen that a torsion class is closed under $\mathcal{J}_0$-pure subgroups if and only if it is closed under pure subgroups. Also there is a rank function ($\{0\}$-rank) corresponding to the trivial torsion theory $(\{0\},\mathfrak{Ol})$, with $\{0\}$-rank $(A) = 1$ if and only if A is non-zero cyclic and cyclic groups determine the hereditary torsion classes (closed under $\{0\}$-pure subgroups). Thus THEOREM 4.41 is a generalization of THEOREM 1.34 and THEOREM 3.9. Although the theory of types, divisibility and algebraic compactness is too directly involved in the discussion leading up to THEOREM 3.13 for any more detailed generalizations to appear likely, the groups of generalized rank 1 nevertheless seem to provide a convenient point of departure in the search for representations for torsion classes with additional subgroup closure properties.

An alternative description of generalized rank is given by

*PROPOSITION 4.42:* $G \in \mathfrak{G}$ *has* $\mathfrak{U}$*-rank* $\mathfrak{m}$ *if and only if it has a subset* $\mathfrak{S}$ *with* $|\mathfrak{S}| = \mathfrak{m}$, *satisfying the following equivalent conditions*

(i) $G/[\mathfrak{S}] \in \mathfrak{U}$,

(ii) If $H \in \mathfrak{G}$ *and* $f:G \longrightarrow H$ *satisfies* $f(\mathfrak{S}) = 0$, *then* $f = 0$,

*and* $\mathfrak{m}$ *is the smallest such cardinal number.*

PROOF. We first verify the equivalence of (i) and (ii). If (i) is satisfied and $f:G \longrightarrow H \in \mathfrak{G}$ satisfies $f(\mathfrak{S}) = 0$, then there is a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "f"'] & G/[\mathfrak{B}] \arrow[dl, "g"] \\ H & \end{tikzcd}$$

where necessarily $g = 0$, so $f = 0$. If $G/[\mathfrak{B}] \notin \mathfrak{U}$, it has a non-zero homomorphic image $H$ in $\mathfrak{F}$. There results a non-zero homomorphism $G \longrightarrow G/[\mathfrak{B}] \longrightarrow H$, whose kernel includes $\mathfrak{B}$.

Now let $\mathfrak{W}$ be any subset of $G$ with $G = [\mathfrak{W}]_{\mathfrak{U}}$. For any non-zero epimorphism $f:G/[\mathfrak{W}] \longrightarrow H \in \mathfrak{F}$, $f\,g(\mathfrak{W}) = 0$, where $g$ is the natural map $G \longrightarrow G/[\mathfrak{W}]$. But $f\,g$ has a $\mathfrak{U}$-pure kernel, which must contain $[\mathfrak{W}]_{\mathfrak{U}} = G$, i.e. $fg = 0$, so $f = 0$ and $G/[\mathfrak{W}] \in \mathfrak{U}$. Conversely, if $G/[\mathfrak{W}] \in \mathfrak{U}$, then $G/[\mathfrak{W}]_{\mathfrak{U}}$ is both a homomorphic image of $G/[\mathfrak{W}]$ and a member of $\mathfrak{F}$, i.e. $G/[\mathfrak{W}]_{\mathfrak{U}} = 0$. Thus $G = [\mathfrak{W}]_{\mathfrak{U}}$ exactly when $G/[\mathfrak{W}] \in \mathfrak{U}$. In particular this is so when $\mathfrak{W}$ is replaced by a set $\mathfrak{B}$ of minimal cardinality.//

*COROLLARY 4.43:* $G \in \mathfrak{F}$ *has finite* $\mathfrak{U}$*-rank* $n$ *if and only if there exist linearly independent elements* $x_1,\dots,x_n \in G$ *with* $G/[x_1] \oplus \dots \oplus [x_n] \in \mathfrak{U}$ *and* $n$ *is the least such integer.//*

*PROPOSITION 4.44: Let* $(\mathfrak{U}_1, \mathfrak{F}_1)$ *and* $(\mathfrak{U}_2, \mathfrak{F}_2)$ *be torsion theories with* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. *Then for any group* $G \in \mathfrak{F}_1$,

$$\mathfrak{U}_1\text{-rank}(G) \leq \mathfrak{U}_2\text{-rank}(G).$$

PROOF. Let $\mathfrak{B}$ be a $\mathfrak{U}_2$-basis for $G$. Then $G/[\mathfrak{B}] \in \mathfrak{U}_2 \subseteq \mathfrak{U}_1$, so $\mathfrak{U}_1$-rank $(G) \leq |\mathfrak{B}|$.//

*COROLLARY 4.45: If $(\mathfrak{U},\mathfrak{G})$ is a torsion theory with $\mathfrak{G} \subseteq \mathfrak{F}_0$, then the $\mathfrak{U}$-rank of a group in $\mathfrak{G}$ cannot exceed its rank.//*

*COROLLARY 4.46: Let* X *be rational and $\mathfrak{U}$ the torsion class* $\{A \mid [A,X] = 0\}$*; then* $\mathfrak{U}$-rank(X) = 1.

PROOF. $\mathfrak{T}_0 \subsetneq \mathfrak{U}$, so

$$0 < \mathfrak{U}\text{-rank}(X) \le \text{rank}(X) = 1.//$$

*COROLLARY 4.47: For any torsion theory $(\mathfrak{U},\mathfrak{G})$ with* $\mathfrak{U} \nsubseteq \mathfrak{T}_0$, $\mathfrak{U}$-rank(G) $\le \mathfrak{D}$-rank(G) *for every group* $G \in \mathfrak{G}$ *.//*

One can introduce a notion of $(\mathfrak{U},\mathfrak{G})$-independence in groups in $\mathfrak{G}$ , which coincides with the standard linear independence in torsion-free groups for $(\mathfrak{U},\mathfrak{G}) = (\mathfrak{T}_0,\mathfrak{F}_0)$: call an element x of G $(\mathfrak{U},\mathfrak{G})$-dependent on $\{x_1,\ldots,x_n\} \subsetneq G$ if $x \in [x_1,\ldots,x_n]_{\mathfrak{U}}$. It would be interesting to know of conditions on torsion theories (for $\mathfrak{Ab}$ or for module categories) under which this notion of dependence gives an abstract dependence structure of the kind studied by Kertész [26], Dlab [10] and others. We shall not study this question explicitly, though several examples of pathology are to be found in the subsequent discussion.

For any torsion theory $(\mathfrak{U},\mathfrak{G})$, the class of groups in $\mathfrak{G}$ with finite $\mathfrak{U}$-rank is closed under homomorphic images in $\mathfrak{G}$ , and under extensions, but not under subgroups in general.

*PROPOSITION 4.48: Let* $G \in \mathfrak{G}$ *have finite* $\mathfrak{U}$*-rank* n. *Then any non-zero homomorphic image of* G *which belongs to* $\mathfrak{G}$ *has* $\mathfrak{U}$*-rank* $\le$ n.

PROOF. Let $\{x_1,\ldots,x_n\}$ be a $\mathcal{U}$-basis for $G, G/G' \in \mathcal{G}$, $\{y_1,\ldots,y_r\}$ the set of distinct cosets of $x_1,\ldots,x_n$ mod. $G'$ and $\hat{G}/G' = [y_1,\ldots,y_r]_{\mathcal{U}}$. Then $\hat{G}/G'$ is $\mathcal{U}$-pure in $G/G'$ and since $G/G' \in \mathcal{G}$, $G'$ is $\mathcal{U}$-pure in $G$. Since $\mathcal{U}$-purity determines a proper class, $\hat{G}$ is therefore $\mathcal{U}$-pure in $G$ and since it contains $x_1,\ldots,x_n$, $\hat{G} = G$, i.e. $G/G' = [y_1,\ldots,y_r]_{\mathcal{U}}$, so $\mathcal{U}$-rank $(G/G') \leq r \leq n$.//

*COROLLARY 4.49: If* $G \in \mathcal{G}$ *has* $\mathcal{U}$*-rank 1, so does any of its non-zero homomorphic images in* $\mathcal{G}$ *.//*

*PROPOSITION 4.50: If* $G'$ *is* $\mathcal{U}$*-pure in* $G \in \mathcal{G}$*, and if* $G'$ *and* $G/G'$ *have finite* $\mathcal{U}$*-rank, so does* $G$*, and*

$$\mathcal{U}\text{-rank}(G) \leq \mathcal{U}\text{-rank}(G') + \mathcal{U}\text{-rank}(G/G')$$

(For an instance of strict inequality, see EXAMPLE 4.58 below).

PROOF. We choose $\mathcal{U}$-bases $\{x_1,\ldots,x_n\}$ for $G'$, $\{z_1,\ldots,z_m\}$ for $G/G'$ and representatives $y_i$ for $z_i$ in $G$, $i = 1,\ldots,m$, and define groups

$$\hat{G} = [x_1,\ldots,x_n]; \quad \overline{G} = [x_1,\ldots,x_n,y_1,\ldots,y_m].$$

For any homomorphism $g: G \longrightarrow K \in \mathcal{G}$ with $g(\overline{G}) = 0$, we have a commutative diagram

$$\begin{array}{ccccc} \hat{G} & \xrightarrow{f'} & G' & & \\ \big\downarrow h & & \big\downarrow k & & \\ \overline{G} & \xrightarrow{f} & G & \xrightarrow{g} & K \end{array}$$

where all other maps are inclusions. Since $g\,k\,f' = g\,f\,h = 0$, we infer from PROPOSITION 4.42 that $g\,k = 0$ so there is a homomorphism $v: G/G' \longrightarrow K$ such that $vu = g$, where $u$ is the natural map from G to G/G' and a commutative diagram

$$\begin{tikzcd}
 & G' \arrow[d, "k"] & \\
\overline{G} \arrow[r, "f"] \arrow[d, "t"] & G \arrow[r, "g"] \arrow[d, "u"] & K \\
(G'+\overline{G})/G' \arrow[r, "f''"] & G/G' \arrow[ur, "v"'] &
\end{tikzcd}$$

where $f''$ is inclusion and $t$ the natural map. From this, $v\, f''\, t = v\, u\, f = g\, f = 0$, so $v\, f'' = 0$, since $t$ is an epimorphism. Also, $(G'+\overline{G})/G' = [z_1,\ldots,z_m]$ so by PROPOSITION 4.42, $v = 0$. Thus $g = v\, u = 0$, so as in the proof of PROPOSITION 4.42, we have

$$[x_1,\ldots,x_n,y_1,\ldots,y_m]_{\mathcal{U}} = G$$

so $\mathcal{U}$-rank$(G) \leq m + n = \mathcal{U}$-rank$(G') + \mathcal{U}$-rank$(G/G')$.//

The next example shows that a group with a generalized rank 1 may have a subgroup for which the corresponding rank is infinite.

*EXAMPLE 4.51:* Let $p_1$, $p_2$, ... be the natural enumeration of the primes, $Y = Q(\{p_{2n} | n = 1,2,\ldots\})$ and $X = [\ \frac{1}{p_{2n}}\ | n = 1,2,\ldots]$. Y has $\mathcal{D}$-rank 1, since any element with zero height at all primes $p_{2n-1}$ gives a $\mathcal{D}$-basis. Any finite subset $\{x_1, \ldots, x_m\}$ of X generates a cyclic subgroup [x], and $X/[x] \cong \bigoplus Z(p_n^{\alpha(n)})$, $n = 1,2,\ldots$, where $\alpha(n) \in Z$. Thus $X/[x] \in \mathcal{R}$, so $\mathcal{D}$-rank$(X) = \aleph_0$.

## 5. Groups of $\mathcal{U}$-rank 1

Among other things we wish to describe the groups G of $\mathcal{U}$-rank 1 ($(\mathcal{U},\mathcal{F})$ is a torsion theory throughout this section) for which T(G) is closed under $\mathcal{U}$-pure subgroups. This requires in particular that $G' \in T(G)$ for every $\mathcal{U}$-pure subgroup $G'$ of G. The most obvious way in which this can be satisfied is for G to have no proper $\mathcal{U}$-pure subgroups at all, and the possible relevance of this property has already been noted (§4). This section is largely devoted to the structure of groups of $\mathcal{U}$-rank 1. We begin by discussing groups G of $\mathcal{U}$-rank 1 which have no proper $\mathcal{U}$-pure subgroups and then consider G satisfying a weaker condition: every proper $\mathcal{U}$-pure subgroup G' has finite index. An example shows that even the weaker condition is not universally satisfied and points to the difficulty in obtaining an analogue of the type set to describe the groups of $\mathcal{U}$-rank 1 in a torsion class (cf.§1 of Chapter 2). This difficulty is also apparent in THEOREM 4.63, which partly generalizes THEOREM 3.12.

The other main result on groups of $\mathcal{U}$-rank 1 asserts that they cannot be mixed.

*DEFINITION 4.52: A group* G *is said to be* $\mathcal{U}$*-pure simple if it has no proper* $\mathcal{U}$*-pure subgroups.*

*PROPOSITION 4.53: A group* $G \in \mathcal{G}$ *with* $\mathcal{U}$*-rank 1 is* $\mathcal{U}$*-pure simple if and only if for every non-zero* $x \in G$, $\{x\}$ *is a* $\mathcal{U}$*-basis.*

PROOF. If every non-zero element gives a $\mathcal{U}$-basis and $0 \neq G' \subsetneq G$, then for any non-zero $x \in G'$,

$$G/G' \cong (G/[x])/(G'/[x]) \in \mathcal{U}.$$

so G' is not $\mathcal{U}$-pure. Conversely, if G is $\mathcal{U}$-pure simple $[x]_{\mathcal{U}} = G$ for every non-zero $x \in G$.//

A torsion-free group is $\mathcal{J}_0$-pure simple if and only if it has $\mathcal{J}_0$-rank 1. It is not necessary that $\mathcal{U}$-rank(G) = 1 implies G is $\mathcal{U}$-pure simple in general, although the last proof shows that the converse implication always holds.

*PROPOSITION 4.54: If for some prime* p, $Z(p) \in \mathcal{G}$, *then every* $\mathcal{U}$*-pure simple group in* $\mathcal{G}$ *is either* p*-divisible or isomorphic to* Z(p).

PROOF. pG is $\mathcal{U}$-pure in G for every $G \in \mathcal{G}$, since $G/pG \in \mathcal{G}$. Thus a $\mathcal{U}$-pure simple group $G \in \mathcal{G}$ which is not p-divisible is p-elementary, and then necessarily cyclic.//

Every non-trivial $\mathcal{G}$ contains Z which has $\mathcal{U}$-basis {1}. If for some prime p, $Z(p) \in \mathcal{G}$, then by the previous result, Z is not $\mathcal{U}$-pure simple. Thus we have

*PROPOSITION 4.55: If every* $G \in \mathcal{G}$ *with* $\mathcal{U}$*-rank 1 is* $\mathcal{U}$*-pure simple, then* $\mathcal{G} \subseteq \mathcal{F}_0$.//

The converse of PROPOSITION 4.55 is false (cf. EXAMPLE 4.58 below).

In using groups with $\mathcal{U}$-rank 1 to classify torsion classes closed under $\mathcal{U}$-pure subgroups, the fact that such a group G may have a proper $\mathcal{U}$-pure subgroup p G presents no great difficulty, since we are essentially concerned only with torsion-free G, and for such groups, $pG \cong G$, so $T(pG) = T(G)$, We shall see in EXAMPLE 4.58 that much greater complexity of $\mathcal{U}$-pure subgroup structure of G is possible. First however, we note a connection between single-element $\mathcal{U}$-bases and a property which may be regarded as a generalization of $\mathcal{U}$-pure simplicity.

*PROPOSITION 4.56: If* $\mathcal{U}$-rank(G) = 1, *then the following conditions are equivalent:*

(i) *For every non-zero* $x \in G$ *and for every* $\mathcal{U}$*-basis* $\{y\}$ *of* G *there exists a non-zero integer* n *such that* $n\,y \in [x]_{\mathcal{U}}$.

(ii) *Every proper* $\mathcal{U}$*-pure subgroup of* G *has finite index.*

PROOF. (i) $\Longrightarrow$ (ii): Let G' be a proper $\mathcal{U}$-pure subgroup of G, $x \in G'$, $x \neq 0$, $\{y\}$ a $\mathcal{U}$-basis for G with $n\,y \in [x]_{\mathcal{U}} \subseteq G'$, and $H = [G',y]$. Then H/G' is cyclic with order $m = \min\{k \in Z \mid k > 0,\ k\,y \in G'\}$. In the resulting short exact sequence

$$0 \longrightarrow Z(m) \cong H/G' \longrightarrow G/G' \longrightarrow G/H \longrightarrow 0$$

we have $H/G' \in \mathcal{G}$ and $G/H \in \mathcal{U}$. If $m = p_1^{i_1}\, p_2^{i_2} \cdots p_r^{i_r}$, where $p_1,\dots,p_r$ are primes, then $H/G' = Z(p_1^{i_1}) \oplus \cdots \oplus Z(p_r^{i_r})$, and $(G/H)_{p_j}$ is divisible for $j = 1,2,\dots r$, since no $Z(p_j)$ belongs to $\mathcal{U}$. If for some j, $Z(p_j^{i_j})$ is embedded (in G/G') in a subgroup isomorphic to $Z(p_j^{\infty})$, then $Z(p_j^{\infty})$ belongs to both $\mathcal{U}$ and $\mathcal{G}$, which is impossible. It follows that $Z(p_j^{i_j}) \cong (G/G')_{p_j}$ for $j = 1,2,\dots,r$, so that H/G' is a finite pure subgroup and hence a direct summand of G/G' (see for example [15] p.80). But since $G/G' \in \mathcal{G}$ and $G/H \in \mathcal{U}$, we then have $H/G' = G/G'$, so G' has finite index m.

(ii) $\Rightarrow$ (i): If every proper $\mathcal{U}$-pure subgroup has finite index and $x \neq 0$, then either $\{x\}$ is a $\mathcal{U}$-basis or $[x]_{\mathcal{U}}$ has finite index, in which case for every $\mathcal{U}$-basis $\{y\}$, we have $n\, y \in [x]_{\mathcal{U}}$ for some $n \in Z$, $n \neq 0$.//

*COROLLARY 4.57: If* G *has* $\mathcal{U}$ *-rank 1, is torsion-free and satisfies the conditions of PROPOSITION 4.56 and if* T(G) *is closed under* $\mathcal{U}$ *-pure subgroups, then* T(G) = T(G') *for every proper* $\mathcal{U}$ *-pure subgroup* G' *of* G.

PROOF. Since G/G' is finite, PROPOSITION 2.20 says that $G \in T(G')$. By assumption, $G' \in T(G)$.//

If every group with $\mathcal{U}$-rank 1 satisfies the conditions of PROPOSITION 4.56, and if in addition $\mathcal{G} \subseteq \mathcal{F}_0$, then the groups with $\mathcal{U}$-rank 1 are all $\mathcal{U}$-pure simple.

The next example shows among other things that the conditions of PROPOSITION 4.56 need not be satisfied when $\mathcal{G} \subseteq \mathcal{F}_0$.

*EXAMPLE 4.58:* Let

$$\mathcal{U} = T(\{Q(\{2,3\})\} \cup \{Z(p) \mid \text{all primes } p\})$$

and $$G = [2^{-n}x,\ 3^{-n}y \mid n = 1,2,\dots]$$

where x and y are linearly independent. Then $Q(2) \oplus Q(3) \cong G \in \mathcal{G}$. Denoting the cosets of x,y modulo [x+y] by $\bar{x},\bar{y}$ respectively, we have $\bar{x}$ and $\bar{y}$ with the same type, with infinite 2-height, 3-height respectively. Thus $G/[x+y] \in \mathcal{U}$ and G has $\mathcal{U}$-rank 1.

Clearly every $\mathcal{U}$-basis of G must have the form $\{a+b\}$ with $a \in [x]_*$, $b \in [y]_*$ and $a \neq 0$, $b \neq 0$, and $a+b$ can have no non-zero multiple in either $[x]_{\mathcal{U}} = [x]_*$ or $[y]_{\mathcal{U}} = [y]_*$.

If a group of $\mathcal{U}$-rank 1 is directly decomposable, then every direct summand has $\mathcal{U}$-rank 1, by COROLLARY 4.49. Such a group cannot be a direct sum of infinitely many non-zero subgroups, for factoring out a cyclic subgroup leaves almost all summands intact. It is however possible for direct products of infinitely many groups to have generalized rank 1. Wiegold [43] has shown, in effect, that $\prod I(p)$ has $\mathcal{D}$-rank 1, where the product is taken over all primes p. By COROLLARY 4.49 the corresponding statement is true for any set of primes. For the group G of EXAMPLE 4.58, we have $T(G) = T(\{Q(2),Q(3)\})$, while by PROPOSITION 3.3, $T(\prod_{\text{all } p} I(p)) = T(\{I(p) \mid \text{all } p\})$. We are led to ask whether, in general, the discussion of groups of $\mathcal{U}$-rank 1 in torsion classes can be reduced to consideration of indecomposable groups. We must leave this question unanswered.

The conditions under which a rational group can have generalized rank 1 are given by

*PROPOSITION 4.59: Let* X *be a rational group belonging to* $\mathcal{G}$. $\mathcal{U}$-rank(X) = 1 *if and only if* $\tau(X) = \tau(n_1,n_2,\ldots)$ *where* $n_i = 0$ *if* $Z(p_i^\infty) \in \mathcal{G}$, 0 *or* $\infty$ *if* $Z(p_i) \in \mathcal{G}$ *but* $Z(p_i^\infty) \notin \mathcal{G}$.

PROOF. If $\tau(X)$ is as described, let $x \in X$ have height $(n_1,n_2,\ldots)$. Then $X/[x] \cong \bigoplus Z(p_i^{n_i})$, $i = 1,2,\ldots$ . If $Z(p_i) \notin \mathcal{G}$, then $Z(p_i^{n_i}) \in \mathcal{U}$. If $Z(p_i) \in \mathcal{G}$ but $Z(p_i^\infty) \notin \mathcal{G}$, then $Z(p_i^{n_i}) = 0$ or $Z(p_i^\infty)$, and so belongs to $\mathcal{U}$, and if $Z(p_i^\infty) \in \mathcal{G}$

then $Z(p_i^{n_i}) = 0$. Thus $X/[x] \in \mathcal{U}$ and $\mathcal{U}$-rank$(X) = 1$. Conversely, if $X$ has $\mathcal{U}$-basis $\{x\}$, then $X/[x]$ is a direct sum of primary groups, each belonging to $\mathcal{U}$ and it is a simple matter to show from the restriction this places on their orders, that the height of $x$ is as required.//

A non-zero direct summand of a group of $\mathcal{U}$-rank 1 also has $\mathcal{U}$-rank 1 (COROLLARY 4.49). Thus together with the remarks following EXAMPLE 4.58, PROPOSITION 4.59 gives a necessary condition for a completely decomposable group $G \in \mathcal{G}$ to have $\mathcal{U}$-rank 1: $G$ must have finite rank and the types of its direct summands must be as in PROPOSITION 4.59. This condition is not sufficient, however. For example let $A$ and $B$ be isomorphic rational groups of $\mathcal{U}$-rank 1. If $\{x\}$ is a $\mathcal{U}$-basis for $A \oplus B$, then clearly $A \oplus B/[x]_* \in \mathcal{U}$. But $[x]_*$ is a direct summand (see for example [15]p.166), so $A \oplus B/[x]_*$, which is rational, also belongs to $\mathcal{G}$. This clearly is impossible.

Although in the investigation of torsion classes only torsion and torsion-free groups need be considered, there is some interest in the fact that mixed groups cannot have generalized rank 1. As a first step in showing this we prove

*PROPOSITION 4.60: A torsion group with $\mathcal{U}$-rank 1 is cyclic.*

PROOF. Let $G \in \mathcal{T}_0 \cap \mathcal{G}$ have $\mathcal{U}$-basis $\{x\}$. Then since $G/[x]$ has no more non-zero primary components than $G$, and belongs to $\mathcal{U}$, it follows that $G/[x]$ is divisible and has zero p-component if $Z(p^{\infty}) \in \mathcal{G}$. Taking primary decompositions of $G_t$ and $[x]$, we have $G_t/[x] = (\bigoplus G_p)/(\bigoplus [x_p])$, where $x_p = 0$ for almost all values of p so that for such p, $G_p = G_p/[x_p]$ is in both $\mathcal{G}$ and $\mathcal{U}$, i.e. $G_p = 0$. All the remaining $G_p$'s are reduced, (since factoring out a cyclic subgroup cannot eliminate a non-zero divisible subgroup) and have $G_p/[x_p]$ divisible. If $x_p$ has finite height (in $G_p$) it is contained in a cyclic direct summand (see for example [15] p.80), whose complementary summand must vanish as it is divisible. There remains the case where $x_p$ has infinite height. In this case, if $y \in G_p$, then for any positive integer n, there exists $y' \in G_p$ such that $y - p^n y' = m\, x_p$, for some $m \in Z$, since $G_p/[x_p]$ is divisible. But then $x_p = p^n x'$ for some $x' \in G_p$, whence it follows that y has infinite height and $G_p$ is divisible. With this contradiction the proof is complete.//

*PROPOSITION 4.61: A group with $\mathcal{U}$-rank 1 is either torsion or torsion-free.*

PROOF. Let $G \in \mathcal{G}$ be mixed and have a $\mathcal{U}$-basis $\{x\}$. If $x \in G_t$, then $G_t/[x]$ is pure in $(G/[x])_t$, so by THEOREM 1.38 and PROPOSITION 3.7 $G_t/[x] \in \mathcal{U}$. But then $\mathcal{U}$-rank$(G_t) = 1$, so by PROPOSITION 4.60, $G_t$ is cyclic which means that G splits (see for example [15] p.80) and this is not possible since a summand complementary to $G_t$ is not affected by the factoring out of $[x]$. Thus x must have infinite order. As in the proof of PROPOSITION 4.60 it can be seen that $G_t$ is reduced whence it follows that G has a direct summand of the form $Z(p^n)$ ([15] p.80).

Let such a summand be generated by y. Then $x = k\,y + z$, where $k \in Z$, $k\,y \neq 0$ and z has infinite order. Since $\mathcal{U}$ contains no reduced p-groups, $[y] \subseteq [x]$, i.e. for some $m \in Z$, $y = mx = m\,k\,y + m\,z$. But then linear independence of y and z requires that $m\,z = 0$, i.e. $m = 0$ and $y = m\,k\,y = 0$. Again we have a contradiction.//

The next theorem is a partial analogue of THEOREM 3.12. Its proof makes use of the following lemma.

*LEMMA 4.62: If* $G'$ *is* $\mathcal{U}$*-pure in* $G \in \mathcal{G}$ *then* $[x]_{\mathcal{U}}^{G'} = [x]_{\mathcal{U}}^{G}$ *for any* $x \in G'$.

PROOF. $[x]_{\mathcal{U}}^{G'}$ is $\mathcal{U}$-pure in G, so $[x]_{\mathcal{U}}^{G} \subseteq [x]_{\mathcal{U}}^{G'}$. But $x \in [x]_{\mathcal{U}}^{G} \cap G'$ which is $\mathcal{U}$-pure in G', so $[x]_{\mathcal{U}}^{G'} \subseteq [x]_{\mathcal{U}}^{G} \cap G' \subseteq [x]_{\mathcal{U}}^{G}$ .//

*THEOREM 4.63: Let* $\mathcal{C} \subseteq \mathcal{G}$ *be a class of groups with* $\mathcal{U}$*-rank 1,* $\vec{\mathcal{C}}$ *the class of homomorphic images in* $\mathcal{G}$ *of direct sums of copies of groups in* $\mathcal{C}$, r, s *the idempotent radicals for* $T(\mathcal{C})$, $\mathcal{U}$. *If* $\vec{\mathcal{C}}$ *is closed under extensions and satisfies*

$$(*)\quad C_1,\ldots,C_n \in \mathcal{C},\ C\ \mathcal{U}\text{-pure in } C_1 \oplus \ldots \oplus C_n,\ \mathcal{U}\text{-rank}(C) = 1 \Rightarrow C \in \mathcal{C}$$

*then* $T(\mathcal{C})$ *is closed under* $\mathcal{U}$*-pure subgroups if and only if* s r *is idempotent.*

PROOF. For a group $A \in \mathcal{G}$, we define a subgroup $\bar{A}$ to be generated by all elements of A which belong to the images of homomorphisms from groups $C \in \mathcal{C}$ . Clearly $\bar{A} \in T(\mathcal{C})$ for any

$A \varepsilon \mathcal{G}$. If $\overline{(A/\overline{A})} = A'/\overline{A}$, we have an exact sequence

$$0 \longrightarrow \overline{A} \longrightarrow A' \longrightarrow \overline{(A/\overline{A})} \longrightarrow 0$$

whence $A' \varepsilon \vec{\mathcal{C}}$, so $A' = \overline{A}$, $\overline{(A/\overline{A})} = 0$ and $\overline{A} = r(A)$. In other words, $T(\mathcal{C}) \cap \mathcal{G} = \vec{\mathcal{C}}$. Thus for any $A \varepsilon T(\mathcal{C}) \cap \mathcal{G}$, with $\mathcal{U}$-pure subgroup A' we have an exact diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & \bigoplus\limits_{\lambda\varepsilon\Lambda} C_\lambda \arrow[r] & A \arrow[r] & 0 \\ & & & A' \cong K/B \arrow[u] & \\ & & & 0 \arrow[u] & \end{tikzcd}$$

where each $C_\lambda \varepsilon \mathcal{C}$ and K is the inverse image of A' in $\bigoplus C_\lambda$. B is $\mathcal{U}$-pure in $\bigoplus C_\lambda$, since $A \varepsilon \mathcal{G}$; also K/B is $\mathcal{U}$-pure in $\bigoplus C_\lambda/B$. Therefore K is $\mathcal{U}$-pure in $\bigoplus C_\lambda$. For any $y \varepsilon K$, we have, after suitably re-labelling, $y \varepsilon C_1 \oplus \dots \oplus C_n$, which is $\mathcal{U}$-pure in $\bigoplus C_\lambda$, so $[y]^K_{\mathcal{U}} = [y]^{\oplus}_{\mathcal{U}}$ is $\mathcal{U}$-pure in $C_1 \oplus \dots \oplus C_n$ and therefore belongs to $\mathcal{C}$. Thus $K \varepsilon T(\mathcal{C})$, so $A' \varepsilon T(\mathcal{C})$. $T(\mathcal{C}) \cap \mathcal{G}$ is therefore closed under $\mathcal{U}$-pure subgroups. The result now follows from THEOREM 4.32.//

## 6. Groups of $\mathcal{U}$-rank 1 (continued)

We commence this section by characterizing the groups of $\mathcal{D}_p$-rank 1, where p is a prime. As a first step we prove

*PROPOSITION 4.64: Let* G *be* p*-reduced with* $\mathcal{D}_p$*-rank 1. Then for any* $\mathcal{D}_p$*-basis* {x} *of* G, [x] *is a* p*-pure subgroup.*

PROOF. It clearly suffices to show that G/[x] has no direct summand of the form $Z(p^\infty)$. By PROPOSITION 4.61 it may be assumed that G is torsion or torsion-free, and in the former case G = [x], by PROPOSITION 4.60. There remains only the torsion-free case. If $G/[x]$ has a summand $Z(p^{\infty})$ then $G$ has a subgroup $G'$, containing $x$, for which the sequence

$$0 \longrightarrow [x] \longrightarrow G' \longrightarrow Z(p^{\infty}) \longrightarrow 0$$

is exact. But then $G' \cong Q(p)$, ([15], p.149) contrary to the assumption that $G$ is p-reduced.//

This result will enable us to give a complete description of the groups with $\mathcal{D}_p$-rank 1. We first recall a definition introduced by Fuchs [16]:

*DEFINITION 4.65: A subgroup* B *of a group* A *is called a* p-*basic subgroup, where* p *is a prime, if* B *is a direct sum of cyclic groups of infinite and/or* p-*power order,* B *is* p-*pure in* A *and* A/B *is* p-*divisible.*

We have shown in PROPOSITION 4.64 that if $G \in \mathcal{R}_p$ has a $\mathcal{D}_p$-basis $\{x\}$, then $[x]$ is a p-basic subgroup. On the other hand, if a p-reduced group has a cyclic p-basic subgroup $[y]$, then $\{y\}$ is clearly a $\mathcal{D}_p$-basis.

*PROPOSITION 4.66: If* $\mathcal{D}_p$-rank(G) = 1 *and* $x \in G$, *then* $\{x\}$ *is a* $\mathcal{D}_p$*-basis if and only if* $[x]$ *is a* p-*basic subgroup.//*

A p-reduced torsion group must be a p-group and it is shown in [3] that a torsion-free p-reduced group has a cyclic p-basic subgroup if and only if it is isomorphic to a p-pure subgroup of I(p). These observations, with PROPOSITION 4.66, give a proof of

*THEOREM 4.67: A group* $G \in \mathcal{R}_p$ *has* $\mathcal{D}_p$*-rank 1 if and only if it is isomorphic to either a non-zero* p*-pure subgroup of* $I(p)$ *or* $Z(p^n)$ *for some finite* n.//

If a torsion group G has $\mathcal{D}_p$-rank 1, only a generator can give a $\mathcal{D}_p$-basis. In the case of torsion-free G, the $\mathcal{D}_p$-bases are described by

*PROPOSITION 4.68: Let* G *be a torsion-free,* p*-reduced group with* $\mathcal{D}_p$*-rank 1, viewed as a* p*-pure subgroup of* $I(p)$*. The following conditions are equivalent for* $x \in G$:

(i) x *has* p*-height* 0 *(in* G *and hence in* $I(p)$*).*

(ii) x *is a* p*-adic unit.*

(iii) $\{x\}$ *is a* $\mathcal{D}_p$*-basis for* G.

PROOF. The equivalence of (i) and (ii) is well-known. By a theorem in [2], $G/[x]$ is p-divisible if and only if $[x]$ contains a p-adic unit, and this is so precisely when x itself is a unit.//

Thus when G is torsion-free of $\mathcal{D}_p$-rank 1, every non-zero element has the form $p^n x$, where $\{x\}$ is a $\mathcal{D}_p$-basis.

The results obtained thus far enable us to give a rough description of the groups of $\mathcal{D}_S$-rank 1, where S is a set of primes. In the uninteresting torsion case they are the cyclic S-groups; in the torsion-free case they are described by

*PROPOSITION 4.69: A non-zero* S*-reduced, torsion-free group* G *has* $\mathcal{D}_S$*-rank 1 if and only if*

(i) *for each* $p \in S$ *there is an exact sequence*

$$0 \longrightarrow G'(p) \longrightarrow G \longrightarrow G''(p) \longrightarrow 0$$

*where* $G'(p)$ *is* p*-divisible and* $G''(p)$ *is isomorphic to a* p*-pure subgroup of* $I(p)$*, and*

(ii) *there exists an element* x *of* G *such that for every* $p \in S$ *with* $pG \neq G$*, the image of* x *in* $G''(p)$ *has* p*-height* 0.

PROOF. Let $\mathcal{D}_S$-rank$(G) = 1$ and let $\{x\}$ be a $\mathcal{D}_S$-basis for G. For $p \in S$ such that G is p-reduced, $\{x\}$ is a $\mathcal{D}_p$-basis, since $G/[x]$ is S-divisible and hence p-divisible. Thus G is isomorphic to a p-pure subgroup of $I(p)$ and we let $G'(p) = 0$, $G''(p) = G$. If G is p-divisible, we let $G'(p) = G$ and $G''(p) = 0$. Finally we consider $p \in S$ such that G is neither p-divisible nor p-reduced. Let $G'(p)$ be the maximal p-divisible subgroups of G and $G''(p) = G/G'(p)$. Denoting the coset of x mod.$G'(p)$ by $\bar{x}$, we have a commutative, exact diagram

$$\begin{tikzcd}
0 \arrow[r] & {[x]} \arrow[r] & G \arrow[rr] \arrow[dr] & & G/[x] \arrow[r] \arrow[dr] & 0 & \\
0 \arrow[r] & \overline{[x]} = ([x] + G'(p))/(G'(p)) \arrow[rr] & & G/G'(p) \arrow[rr] \arrow[dr] & & G/([x]+G'(p)) \arrow[r] \arrow[dr] & 0. \\
& & & & 0 & & 0
\end{tikzcd}$$

$G/[x]$ is S-divisible and hence p-divisible, so the same is true of $G/([x] + G'(p))$. It therefore follows that $G/G'(p) = G''(p)$ has $\mathcal{D}_p$-rank 1 and thus has a p-pure embedding in $I(p)$. As defined, $G''(p) \neq 0$ exactly when $pG \neq G$ and in such cases it is clear from the proof so far that the image of x in $G''(p)$ has p-height 0.

Conversely, if G satisfies (i) and (ii), we shall show that for x as described in (ii), x is a $\mathfrak{D}_S$-basis. If $G''(p) = G$, then x has p-height 0, so $G/[x]$ is p-divisible and if $G''(p) = 0$, G itself is p-divisible. In the remaining case, $G''(p)/[\bar{x}]$ is p-divisible, where again $\bar{x}$ denotes the image of x in $G''(p)$. But

$$G''(p)/[\bar{x}] \cong (G/G'(p))/(([x]+G'(p))/G'(p)) \cong G/([x]+G'(p))$$

($G'(p)$ has been treated as a subgroup of G), and $([x]+G'(p))/[x]$ is also p-divisible, so from the exact sequence

$$0 \longrightarrow ([x]+G'(p))/[x] \longrightarrow G/[x] \longrightarrow G/([x]+G'(p)) \longrightarrow 0$$

it then follows that $G/[x]$ is p-divisible. Thus $G/[x]$ is p-divisible for every $p \in S$, so $\{x\}$ is a $\mathfrak{D}_S$-basis and the proof is complete.//

If the groups $G'(p)$ are regarded as subgroups of G, they are all pure, so their intersection is also pure in G and hence in each $G'(p)$. Being therefore S-divisible, $\bigcap_{p \in S} G'(p) = 0$. Thus we have

*COROLLARY 4.70: A torsion free group with* $\mathfrak{D}_S$*-rank 1 is a subdirect product of torsion-free groups with* $\mathfrak{D}_p$*-rank 1, at most one for each* $p \in S$.//

In our discussion of groups of generalized rank 1, we have not so far examined the question when such groups G satisfy

(*) $\quad G \in \mathfrak{J}$ or $\mathfrak{F}$ for any torsion theory $(\mathfrak{J}, \mathfrak{F})$.

(This condition is satisfied by groups of $\mathfrak{J}_0$-rank 1). PROPOSITION 4.69 provides a characterization of the groups of $\mathfrak{D}$-rank 1 which satisfy (*):

*PROPOSITION 4.71: With the notation of PROPOSITION 4.69, and* S *the set of all primes, if* G *is torsion-free and* $\mathcal{D}$-rank(G) = 1, *then* G *satisfies* (*) *if and only if for every* p, G = G'(p) or G"(p).

PROOF. By definition of the groups G'(p) and G"(p), the stated condition is necessary for (*). Conversely, if G satisfies this condition it is *cohesive* in the sense of [11], and so G/r(G) is divisible for any idempotent radical r with r(G) $\neq$ 0 (r(G) is a pure subgroup). Thus if r(G) $\neq$ 0, r is associated with a torsion class containing torsion-free groups, so that G/r(G) = r(G/r(G)) = 0, i.e. G = r(G).//

There are groups of $\mathcal{D}_S$-rank 1 which do not satisfy (*), e.g. $\prod$ I(p), p $\in$ S if S has at least two elements.

To obtain a description of the class of groups of $\mathcal{U}$-rank 1 in a torsion class $\mathcal{J}$, analogous to that of the type set given by THEOREM 2.11, it is first necessary to find conditions on groups A,B with $\mathcal{U}$-rank(A) = $\mathcal{U}$-rank(B) = 1 which ensure that B $\in$ T(A). For a really close analogy with §1 of Chapter 2, we should restrict our attention to those torsion theories ($\mathcal{U}$,$\mathcal{G}$) for which all non-zero homomorphisms between groups with $\mathcal{U}$-rank 1 are monomorphisms. Such is the case when the groups with $\mathcal{U}$-rank 1 are all $\mathcal{U}$-pure simple, though as we shall see in the case of ($\mathcal{D}_p$,$\mathcal{R}_p$), this condition is not necessary. For such a ($\mathcal{U}$,$\mathcal{G}$) let A and B have $\mathcal{U}$-rank 1, with [A,B] $\neq$ 0. Then we may assume that A $\subseteq$ B, and it follows that B belongs to T(A) if and only if B/A does. If ($\mathcal{U}$,$\mathcal{G}$) = ($\mathcal{J}_0$,$\mathcal{F}_0$), then in every case $B \in T(A)$ and $B/A \in \mathcal{J}_0$. If every group with $\mathcal{U}$-rank 1 is $\mathcal{U}$-pure simple, then $B/A \in \mathcal{U}$ always, but EXAMPLE 4.34 raises doubts as to whether $B \in T(A)$, though certainly $B \in T(\{A\} \cup \mathcal{C})$ for some subclass $\mathcal{C}$ of $\mathcal{U}$, and in particular when $\mathcal{C} = \mathcal{U}$.

A complete set of conditions under which $B \in T(\{A\} \cup \mathcal{U})$ may be regarded as a partial generalization of THEOREM 2.11, since in §1 of Chapter 2 the class $T(X)$, $X$ rational, could as well have been replaced by $T(\{X\} \cup \mathcal{C})$ for any $\mathcal{C} \subseteq \mathcal{J}_0$, as the inclusion of extra torsion groups in a torsion class $\mathcal{J}$ does not enlarge $\mathcal{J} \cap \mathcal{F}_0$. Similarly, if $\mathcal{J}$ has type set $\Gamma$, so does $T(\mathcal{J} \cup \mathcal{C})$ if $\mathcal{C} \subseteq \mathcal{J}_0$.

We shall investigate the question for $(\mathcal{U}, \mathcal{G}) = (\mathcal{D}_p, \mathcal{R}_p)$, where p is a prime. We begin with a description of homomorphic images of groups with $\mathcal{D}_p$-rank 1.

*PROPOSITION 4.72: Let B be torsion-free of $\mathcal{D}_p$-rank 1. Then any proper homomorphic image of B is the direct sum of a p-divisible group and a bounded p-group.*

PROOF. If $0 \neq A \subseteq B$ and $n = \min\{\text{p-height in } B \text{ of } a \mid a \in A\}$, then $A \subseteq p^nB$ and if $a \in A$ has p-height n in B, it has zero p-height in $p^nB$, since otherwise $a = p^{n+1}b$ for some $b \in B$. Being isomorphic to B, $p^nB$ has $\mathcal{D}_p$-rank 1, so by PROPOSITION 4.68, $\{a\}$ is a $\mathcal{D}_p$-basis for $p^nB$. Since $p^nB/[a]$ belongs to $\mathcal{D}_p$, so does $p^nB/A$, and thus $\mathrm{Ext}(B/p^nB, p^nB/A) = 0$, so the natural exact sequence

$$0 \longrightarrow p^nB/A \longrightarrow B/A \longrightarrow B/p^nB \longrightarrow 0$$

is split.//

*COROLLARY 4.73: If* B *is torsion-free with* $\mathcal{D}_p$*-rank 1, then* B/A *is a bounded* p*-group for any proper* $\mathcal{D}_p$*-pure subgroup* A.//

*COROLLARY 4.74: If* A, B *are torsion-free with* $\mathcal{D}_p$*-rank 1, then any non-zero homomorphism* $f:A \longrightarrow B$ *is a monomorphism.*

PROOF. If f has non-zero kernel, its image is both a subgroup of B and the direct sum of a p-divisible group and a bounded p-group, so f = 0.//

If $A \subseteq B$ and both groups are torsion-free with $\mathcal{D}_p$-rank 1, then since T(A) contains all p-groups, we have $B \in T(\{A\} \cup \mathcal{D}_p)$. This fact with COROLLARY 4.74 gives

*PROPOSITION 4.75: The following conditions are equivalent for torsion-free groups* A *and* B *with* $\mathcal{D}_p$*-rank* 1:

(i) $B \in T(\{A\} \cup \mathcal{D}_p)$

(ii) $[A,B] \neq 0$

(iii) A *is isomorphic to a subgroup of* B.//

I(p) has $\mathcal{D}_p$-rank 1 and has subgroups isomorphic to all other torsion-free groups with $\mathcal{D}_p$-rank 1. In the case of $\mathcal{T}_0$-rank, Q plays a similar role. A further similarity between the two groups is noted in the following proposition (cf. COROLLARY 2.5).

*PROPOSITION 4.76: The following conditions are equivalent for a torsion class* $\mathcal{T}$

(i) $\mathcal{T}$ *contains a non-zero torsion-free* p*-reduced group.*

(ii) $\mathcal{J}$ *contains a non-zero torsion-free group which is not* p-*divisible.*

(iii) $I(p) \in \mathcal{J}$.

PROOF. Obviously (i) => (ii) and (iii) => (i); (ii) => (iii) is just LEMMA 3.1.//

To conclude the discussion of groups with generalized rank 1 we describe some torsion classes closed under $\mathcal{U}$-pure subgroups when $\mathcal{U}$ is determined by groups $Z(p^\infty)$ (for various primes p).

*PROPOSITION 4.77: A group has* $(\mathcal{J}_p \cap \mathcal{D})$*-rank 1 if and only if it is either a non-zero cyclic torsion group or isomorphic to* $Q(S)$, *where* $S \subseteq P$.

PROOF. Clearly the groups indicated have $(\mathcal{J}_p \cap \mathcal{D})$-rank 1, and for the converse, by PROPOSITIONS 4.60 and 4.61, we need only consider torsion-free groups. If such a group G has $(\mathcal{J}_p \cap \mathcal{D})$-rank 1, there is an exact sequence

$$0 \longrightarrow Z \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

with $G'' \in \mathcal{J}_p \cap \mathcal{D}$. Thus G has rank 1 and $G'' = \bigoplus G''_p$, $p \in P$ where $G''_p \cong 0$ or $Z(p^\infty)$. It follows that $G \cong Q(S)$, where $S = \{p \in P \mid G''_p \cong Z(p^\infty)\}$.//

*THEOREM 4.78: If* $(\mathcal{J}_p \cap \mathcal{D})$-rank(G) = 1, *then* T(G) *is closed under* $(\mathcal{J}_p \cap \mathcal{D})$*-pure subgroups.*

PROOF. If G is a torsion group, then T(G) is hereditary; if not, then $T(G) = \mathcal{D}_S$, where $S \subseteq P$. The idempotent radical of $\mathcal{J}_p \cap \mathcal{D}$ commutes with all others (cf. the proof of PROPOSITION 4.19) so in this case, by THEOREM 4.32, it suffices to consider S-divisible groups with no direct summands $Z(p^\infty)$, for $p \in \mathbf{P}$. If

$$(**) \qquad 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is exact, with $A, A'' \in \mathcal{D}_S$ without direct summands $Z(p^\infty)$, $p \in \mathbf{P}$, then $A''_S = 0$. But then (**) is S-pure, so $A' \in \mathcal{D}_S$ and has no direct summand $Z(p^\infty)$, $p \in \mathbf{P}$.//

## 7. An Example

To conclude this chapter, we find necessary and sufficient conditions on rational groups X and Y for the closure of T(X) under T(Y)-pure subgroups.

Our notation for heights, types etc. largely conforms to that of [15], Chapter VII. In particular, $p_1, p_2, \ldots$ is the natural enumeration of the primes, and in a height $(h_1, h_2, \ldots, h_n, \ldots)$, $h_n$ denotes height at $p_n$.

*THEOREM 4.79: Let* X,Y *be rational such that* $\tau(X)$ *is the type of a height* $(h_1, h_2, \ldots, h_n, \ldots)$ *with* $0 < h_n < \infty$ *for infinitely many values of* n. *Then* T(X) *is closed under* T(Y)*-pure subgroups if and only if* $\tau(Y) \leq \tau(X)$.

PROOF. Let $(T(Y), \mathcal{F})$ be the torsion theory for T(Y) and let $(g_1, g_2, \ldots, g_n, \ldots)$ be a height with the same type as Y.

If $\tau(Y) \leq \tau(X)$, then $X \in T(Y)$, so T(Y)-pure subgroups are T(X)-pure. For groups in T(X), such subgroups are direct summands, and so belong to T(X) themselves.

For the converse we need to consider two cases:

(i) $\tau(Y) \nleq \tau(h_1+1, h_2+1, \ldots, h_n+1, \ldots)$. Let $M = \{n \mid h_n = \infty\}$. Let $(k_1, k_2, \ldots, k_n, \ldots)$ be the subsequence of positive finite terms of $(h_1, h_2, \ldots, h_n, \ldots)$ and re-label the associated primes as $q_1, q_2, \ldots$ . Let $\{x, y\}$ be a basis for a 2-dimensional rational vector space and

$$G = [p^{-n}x, p^{-n}y, q_n^{-k_n}x, q_n^{-k_n}(q_n^{-1}x+y) \mid p \in M,\ n = 1,2,\ldots].$$

A routine argument using the linear independence of x and y shows that x has height $(h_1, h_2, \ldots, h_n, \ldots)$ in G. Suppose y is divisible by $q_n^{k_n}$ for some n. Since the same is true of $q_n^{-1}x+y$, x has $q_n$-height $k_n+1$ at least, which is impossible. Thus $\tau(y) < \tau(x) = \tau(X)$ (in G). Denoting the coset of y mod. $[x]_*$ by $\bar{y}$, we have

$$G/[x]_* = [p^{-n}\,\bar{y},\ q_n^{-k_n}\,\bar{y} \mid p \in M,\ n = 1,2,\ldots]$$

so $G/[x]_*$ is rational with type $\tau(X)$. From the exact sequence

$$0 \longrightarrow X \cong [x]_* \longrightarrow G \longrightarrow G/[x]_* \cong X \longrightarrow 0$$

it is clear that $G \in T(X)$.

Observing that $[y]_* \notin T(X)$, we now show that $[y]_*$ is $T(Y)$-pure in G. Let $\bar{x}$ denote the coset of x mod. $[y]_*$. Then

$$G/[y]_* = [p^{-n}\bar{x},\ q_n^{-(k_n+1)}\,\bar{x} \mid p \in M,\ n = 1,2,\ldots\ ]$$

which is rational of type $\leq \tau(h_1 + 1, h_2 + 1, \ldots, h_n + 1, \ldots)$, so $G/[y]_* \in \mathcal{J}$ and $[y]_*$ is $T(Y)$-pure in G.

(ii) $\tau(Y) \leq \tau(h_1+1, h_2+1, \ldots, h_n+1, \ldots)$. Let

$$U = \{p_n \mid p_n\, Y = Y\} \text{ and } S = \{p_n \mid h_n < g_n\}.$$

Note that our assumption concerning $\tau(Y)$ requires that $p\,X = X$ for all $p \in U$, S is infinite and $g_n$ is finite for each $p_n \in S$.

Let

$$V = \{p_n \mid h_n \geq g_n \mid g_n < \infty\}.$$

and re-label the entries of $(h_1, h_2, \ldots, h_n, \ldots)$ as follows: denote the primes $p_n \in S$ by $s_1, s_2, \ldots$, their heights by $k_1, k_2, \ldots$ and denote the primes in $V$ by $v_1, v_2, \ldots$ with heights $j_1, j_2, \ldots$ . Finally let

$$H = [p^{-n}x,\ s_n^{-k_n}x,\ v_n^{-j_n}x,\ p^{-n}y,\ s_{2n-1}^{-k_{2n-1}}y,\ v_n^{-j_n}y,\ s_{2n}^{-k_{2n}}(s_{2n}^{-1}x+y)$$
$$|\, p \in U,\ n = 1,2,\ldots].$$

As in case (i), $\tau(y) < \tau(x) = \tau(X)$, $[x]_* \cong X \cong H/[x]_*$ and $H \in T(X)$. Also,

$$H/[y]_* = [p^{-n}\bar{x},\ s_{2n-1}^{-k_{2n-1}}\bar{x},\ s_{2n}^{-(k_{2n}+1)}\bar{x},\ v_n^{-j_n}\bar{x} \mid p \in U,\ n = 1,2,\ldots]$$

which is rational with type $\not\geq \tau(Y)$, since it has lower height at infinitely many primes, namely $s_{2n-1}$, $n = 1,2,\ldots$ . Hence $[y]_*$ is $T(Y)$-pure in $H$, but $[y]_* \notin T(X)$.//

THEOREM 4.79 has some obvious minor generalizations: If $T(Y)$ is replaced by an r.t. torsion class whose type set has a least element, the theorem remains true. If $\mathcal{J}$ is an r.t. torsion class whose type set $\Gamma$ has a subset $\Gamma'$ of minimal elements such that for every $\gamma \in \Gamma$ there is $\gamma' \in \Gamma'$ with $\gamma \geq \gamma'$ and if in addition $\gamma' \not\leq \tau(h_1+1, h_2+1, \ldots, h_n+1, \ldots)$ for every $\gamma' \in \Gamma'$, then the argument in case (i) of the proof of THEOREM 4.79 can be easily modified to show that $T(X)$ is not closed under $\mathcal{J}$-pure subgroups.

Only the case $X \cong Q(P)$ now remains. Here we prove a more general result.

*THEOREM 4.80:* $\mathcal{D}_P = T(Q(P))$ *is closed under* $\mathcal{J}$ *-pure subgroups, for a torsion class* $\mathcal{J}$ *, if and only if* $Z(p^\infty) \in \mathcal{J}$ *for every* $p \in P$.

PROOF. If $Z(p^\infty) \in \mathcal{J}$ for each $p \in P$, then by PROPOSITION 4.77 and THEOREM 4.78, $\mathcal{D}_P$ is closed under $(\mathcal{J}_P \cap \mathcal{D})$-pure subgroups and in particular, $\mathcal{J}$ -pure subgroups.

Conversely, if $Z(p^\infty) \notin \mathcal{J}$ for some $p \in P$, then $\mathcal{J}$ is a t-torsion class, and $Z(p^\infty) \in \mathcal{F}$, where $(\mathcal{J}, \mathcal{F})$ is the torsion theory associated with $\mathcal{J}$. The natural exact sequence

$$0 \longrightarrow Q(P-\{p\}) \longrightarrow Q(P) \longrightarrow Z(p^\infty) \longrightarrow 0$$

is accordingly $\mathcal{J}$ -pure, but $Q(P-\{p\}) \notin \mathcal{D}_P$.//

*COROLLARY 4.81: If* $\mathcal{J}$ *is not a t-torsion class, (in particular if* $\mathcal{J} = T(Y)$ *for some rational* $Y$*), then* $\mathcal{D}_P$ *is closed under* $\mathcal{J}$ *-pure subgroups.//*

# CHAPTER 5

## MISCELLANEOUS TOPICS

### 1. Rational Groups and the Amitsur Construction

The Amitsur radical construction described in Chapter 1 does not always terminate after a finite number of steps, even in abelian categories. If for example we begin with the class $\{Z(p)\}$ where p is prime, then for any group G, we have $G_n = G[p^n]$, so in this case there is no finite upper bound on the number of steps which may be required. In this section we shall discuss the Amitsur construction for the idempotent radical r corresponding to T(X), where X is rational, starting from $\{X\}$. Thus for any group G, we define $G_1$ to be the subgroup generated by the images of all homomorphisms from X to G, and $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ or $(G/G_{\beta-1})_1 = G_\beta/G_{\beta-1}$ according as $\beta$ is a limit ordinal or not.

*PROPOSITION 5.1: If* G *is a torsion group, then* $r(G) = G_1$ *for every rational group* X.

PROOF. For any prime p with $pX \neq X$, let $y \in G$ have order $p^n$; then $X/p^nX \cong [y]$ so $G_p \subseteq G_1$. If $pX = X$, let $G_p = D \oplus R$, where D is divisible, R reduced. Then $D \subseteq G_1$ but $[X,R] = 0$. It follows that $G_1 = \bigoplus G^{(p)}$, where $G^{(p)}$ is $G_p$ if $pX \neq X$ and otherwise its divisible part. This clearly is r(G).//

*PROPOSITION 5.2: If* $X = Q(P)$ *for some set* P *of primes, then* $r(G) = G_1$ *for every group* G.

PROOF. Since $r(G)$ is the maximum P-divisible subgroup, its maximum P-subgroup is divisible. In a complementary direct summand H of $r(G)$, divisibility by powers of primes in P is uniquely defined, so H, as a $Q(P)$-module, is a homomorphic image of a direct sum of copies of X. Hence $r(G) \subseteq G_1$, so the two subgroups coincide.//

*THEOREM 5.3: If* $X = Q(P)$ *for some set* P *of primes, then* $r(G) = G_1$ *for all torsion-free groups* G. *Otherwise there exists, for each positive integer* k, *a torsion-free group* $G^{(k)}$ *of rank* k *such that*

$$r(G^{(k)}) = G^{(k)} = G_k^{(k)}.$$

PROOF. Only the case $X \not\cong Q(P)$ needs to be considered. Let $\tau(X) = \sigma$ and let $(h_1,h_2,\ldots,h_n,\ldots)$ be a height of type $\sigma$, $(j_1,j_2,\ldots,j_n,\ldots)$ the subsequence of finite non-zero terms of $(h_1,h_2,\ldots,h_n,\ldots)$, $q_1,q_2,\ldots,q_n,\ldots$ the associated primes. The groups $G^{(k)}$ are defined by

$$G^{(k)} = [p^{-n}x_1,\ \ldots,\ p^{-n}x_k,\ q_n^{-j_n}x_1,\ q_n^{-j_n}(q_n^{-1}x_1 + x_2),\ \ldots,\\ q_n^{-j_n}(q_n^{-1}x_{k-1} + x_k) \mid pX = X,\ (p \text{ prime}),\ n = 1,2,\ldots]$$

where $\{x_1,x_2,\ldots,x_n,\ldots\}$ is a linearly independent set.

We first show, by induction, that

$$G_1^{(k)} = [x_1]_*,\ k = 1,2,\ldots .$$

Note that $A_1 = A(\sigma)$ for any torsion-free group A.

Now

$$G^{(1)} = [p^{-n}x_1,\ q_n^{-j_n}x_1 \mid pX = X,\ n = 1,2,\ldots] \cong X$$

so $G^{(1)}(\sigma) = G^{(1)} = [x_1]_*$. Assume $G^{(k-1)}(\sigma) = [x_1]_*$. Denoting the coset of an element $x$ by $\bar{x}$, we have

$$G^{(k)}/[x_1]_* = [p^{-n}\bar{x}_2,\dots, p^{-n}\bar{x}_k,\ q_n^{-j_n}\bar{x}_2,\ q_n^{-j_n}(q_n^{-1}\bar{x}_2+\bar{x}_3),\ \dots,$$
$$q_n^{-j_n}(q_n^{-1}\bar{x}_{k-1}+\bar{x}_k) \mid pX = X,\ n = 1,2,\dots\ ].$$

$G^{(k)}/[x_1]_*$ is thus isomorphic, in an obvious way, to $G^{(k-1)}$. For any $y \in G^{(k)}(\sigma)$, $\bar{y}$ belongs to $[\bar{x}_2]_*$, so $y$ belongs to $[x_1,x_2]_*$. Let $my = m_1x_1+m_2x_2$ where $m,m_1,m_2 \in Z$. From the definition of $G^{(k)}$, it is clear that $x_1$ has height $(h_1,h_2,\dots,h_n,\dots)$ and therefore type $\sigma$. Suppose $x_2$ is divisible by $q_n^{j_n}$ for some n. Since the same is true of $q_n^{-1}x_1+x_2$, $x_1$ is divisible by $q_n^{j_n+1}$, and this is impossible. Thus $\tau(x_2) < \sigma$. But $m_2x_2 = my-m_1x_1$ and $\tau(y)$, $\tau(x_1) \geq \sigma$, so $m_2 = 0$ and $y \in [x_1]_*$. This proves the assertion.

Now for any k, again denoting the coset of x by $\bar{x}$, we have $(G^{(k)}/[x_1]_*)(\sigma) = [\bar{x}_2]_*$, so from the exact sequence

$$0 \to G^{(k)}(\sigma) = G_1^{(k)} \to G_2^{(k)} \to (G^{(k)}/G_1^{(k)})_1 = (G^{(k)}/G^{(k)}(\sigma))(\sigma) \to 0$$

we deduce that $G_2^{(k)} = [x_1,x_2]_* \cong G^{(2)}$ and repetitions of this argument give an ascending series

$$\begin{array}{ccccccc} G_1^{(k)} & \subsetneqq & G_2^{(k)} & \subsetneqq & \cdots & \subsetneqq & G_k^{(k)} \\ \wr\| & & \wr\| & & & & \| \\ G^{(1)} & & G^{(2)} & & & & G^{(k)} \end{array}$$

so that $r(G^{(k)}) = G^{(k)}$.//

By taking k = 2 in THEOREM 5.3, we obtain

*COROLLARY 5.4:* $(G/G(\sigma))(\sigma) = 0$ *for all torsion-free* *groups* G *if and only if* $\sigma = \tau(Q(P))$ *for some set* P *of primes*.//

The groups $G^{(k)}$ of THEOREM 5.3 are indecomposable. For suppose $G^{(k)} = H \oplus K$; then $G \in T(X)$ and

$$X \cong G^{(k)}(\sigma) = H(\sigma) \oplus K(\sigma)$$

so $H(\sigma) = 0$ or $K(\sigma) = 0$. But since H and K both belong to T(X), this means that one of them must be zero. In addition, $G^{(k)}/G^{(k-1)} \cong X$ (for $k > 1$). Thus (cf. Chapter 2) we have proved

*COROLLARY 5.5: If a rational group* X *does not have the form* Q(P) *and if* k *is any positive integer, there exists an indecomposable torsion-free group* A *of rank* k *such that* T(A) = T(X).//

## 2. Splitting Idempotent Radicals

In Chapter 2 we saw that r(A) is a pure subgroup of A for every group A and idempotent radical r. The cases in which r(A) is always a direct summand are described by the next proposition.

*PROPOSITION 5.6: Let* $(\mathcal{J}, \mathcal{F})$ *be a torsion theory for* $\mathfrak{Ab}$ *with idempotent radical* r *such that* r(A) *is a direct summand for every* A. *If* r *is non-trivial, then* $\mathcal{J} \subseteq \mathfrak{D}$ *(and thus* r(A) *is always the divisible subgroup or its* S*-component for some fixed set* S *of primes).*

PROOF. If $Z(p) \in \mathcal{F}$ for every prime p, then all groups in $\mathcal{J}$ are divisible. If for some prime p, Z(p) belongs to $\mathcal{J}$, then so do all p-groups. If $\mathcal{F}$ contains non-zero groups, it contains in particular $\prod Z_n$, $n = 1,2,\ldots$, where $Z_n \cong Z$. A theorem of Baer [4], Erdős [13] and Sąsiada (see [15] p.190) asserts that $\mathrm{Ext}(\prod Z_n, G) = 0$ for a p-group G if and only if G is the direct sum of a bounded group and a divisible group. Let A' be a reduced, unbounded p-group and consider a non-split short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow \prod Z_n \longrightarrow 0.$$

Since $A' \in \mathfrak{J}$ and $\prod Z_n \in \mathfrak{F}$, we have $r(A) \cong A'$.//

# I N D E X *



* Containing new or uncommon terms and symbols only. See also the table of notation on pp. *(ix)-(x)*.

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