

CHIRAL ANOMALIES IN CURVED SPACE

by

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### DECLARATION

Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by another person, except where due reference is made in the text of the thesis.

(J.J. GORDON)

## Abstract

The subject of this thesis is a combinatoric method developed by the author to calculate the gravitational contributions to the anomalies in the chiral currents of spin  $1/2$  and spin  $3/2$  fields in arbitrary space-time dimensions. Using general arguments it is possible to reduce the work involved in finding either of these contributions to the evaluation of a single one-loop Feynman diagram. It is a straightforward matter to calculate the loop momentum integral in this diagram, but one is then faced with the daunting task of summing the remaining function of the external momenta over all permutations of the external graviton legs. Chapter 4 outlines a notation by which the quantities relevant to this sum may be described. This notation is then used to show that recurrence relations exist between certain of the quantities in different dimensions. By solving the recurrence relations one finally arrives, with the aid of contour integral methods, at expressions for the spin  $1/2$  and spin  $3/2$  anomalies in terms of Bernoulli numbers.

The calculation of the spin  $3/2$  anomaly is complicated by the presence of gauge degrees of freedom in the Rarita-Schwinger tensor-spinor field. In the conventional Rarita-Schwinger formulation of spin  $3/2$  theory both the calculation of the spin  $3/2$  anomaly and a proof of its gauge independence are practically impossible due to the involved forms of the propagator and vertices. However, the Rarita-Schwinger formulation is not the only formulation of spin  $3/2$  field theory. In fact there exists a one-parameter family of possible formulations. As it happens, one of these formulations is particularly suited to a calculation of the spin  $3/2$  anomaly, while in another of them the gauge-independence of the anomaly is made manifest. I therefore adopted these two formulations when calculating the spin  $3/2$  anomaly and demonstrating its gauge-independence, and the work of this thesis is based on the assumption that the spin  $3/2$  anomaly remains the same in different formulations. Since all formulations within the one-parameter family may be reached from the Rarita-Schwinger formulation via linear transformations of the field variable, this assumption is entirely reasonable.

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**This Thesis is Dedicated**

**to my Family**

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## CHAPTER 1. Anomalies or Quantum Mechanical Symmetry Breaking

When fermions are coupled to gauge fields in a quantum field theory interesting and important effects may occur. These effects are tied to the presence in the theory of so-called anomalies. In this chapter I will attempt to summarize our present theoretical understanding of anomalies and review the methods of anomaly calculation. In order to introduce the subject let me briefly describe what anomalies are. As Jackiw has explained [1], a quantum theory is not a free-standing, self-contained structure. To create a quantum theory one begins with a classical theory and then quantizes it. The quantization may be effected either by imposing canonical commutation relations upon the field operators [2], or by following the path integral procedure [3]. Many of the properties of the classical theory are shared by the corresponding quantum theory. For instance the fields in each case carry the same number of dynamical degrees of freedom, and the structures of the interaction terms are identical.

Prior to the discovery of anomalies it was generally believed that the quantum theory also shares the symmetry properties and conservation laws of the classical theory. It was thought that if the classical theory is invariant under a set of symmetry transformations, and possesses a corresponding set of conserved currents, the quantum theory will admit the same invariances and will contain an identical set of conserved currents. We now know that this is not always so. Sometimes the quantization procedure does not respect the classical symmetries of a theory and the consequent symmetry breakdown manifests itself in the appearance of non-classical terms in the current conservation equations. These terms, which are thrown up in passage from a classical to a quantum theory, are called anomalies. They are invariably associated with divergences in the quantum theory, and consequently they do not become mathematically well defined until the infinities in the theory are tamed by means of some regularization procedure. In fact the position in which an anomaly occurs in a theory, and even the actual form of the anomaly, will be determined by the regularization procedure that is employed. In spite of this, the anomaly is not a product of the regularization process. Rather, the presence of an anomaly is signalled at a



calculational level by the fact that it is impossible to find a regularization scheme which respects all the classical symmetries of the theory.

The name "anomalies" suggests the surprise which attended the discovery of the sort of symmetry breaking described above. It was almost universally accepted in the past that classical symmetries would persist unchanged in the corresponding quantum theory. Nevertheless, anomalous symmetry breaking is not without its historical antecedents. We shall see below that anomalies occur in quantum field theories when the field operators do not form faithful representations of classical symmetry groups, but instead form representations only up to phase factors. Such representations are termed "projective" and quantum mechanics makes use of them as well. For instance, anomalous symmetry breaking in Yang-Mills theories is paralleled by the Bloch phenomenon in Schrodinger theory [4]. In the latter case a periodic Hamiltonian  $H(\mathbf{r}) = H(\mathbf{r}+\mathbf{a})$  gives rise to wave functions which are periodic only up to a phase  $\psi(\mathbf{r}) = \exp(-i\mathbf{k}\cdot\mathbf{a})\psi(\mathbf{r}+\mathbf{a})$ , and the energy spectrum exhibits a band structure. Examples such as this have prompted Jackiw to suggest [1] that a better name for anomalous symmetry breaking would be quantum mechanical symmetry breaking. This alternative term would better call to mind the close connection between anomalies and such effects as the Bloch phenomenon [4] and spontaneous magnetization [5] in quantum mechanics, and  $\theta$ -vacua and solitons [6,7] in quantum field theory.

Since their discovery by Adler [8] and Bell and Jackiw [9] in 1969 anomalies have assumed an increasingly important place in quantum field theories, and in gauge theories in particular. Though they began simply as a curious inconsistency of meson theory in the context of pion decay, their applications now extend to most areas of theoretical elementary particle physics. There areas include low energy theorems for QCD [10], renormalizability and unitarity of gauge theories [11], the U(1) [12] and CP [13] problems, composite models [14], solitons [15], baryon theory [16], monopoles [17] and superstrings [18]. This large catalogue of applications testifies to the enormous amount of research carried out into the nature of anomalous symmetry breaking. However, despite the effort expended, it is still not possible to satisfactorily answer the questions: why do anomalies occur? why does the quantization procedure sometimes break classical invariances? In

this introductory chapter of my thesis I have not attempted to compile a comprehensive listing of anomalies' applications in quantum field theory. Such listings may be found elsewhere [19]. Rather, I have tried to paint a picture of our present incomplete understanding of anomalies. In doing so I have found it convenient to treat the methods which are currently at our disposal for analysing anomalies under three headings, according to whether they were by nature algebraic, topological or calculational. In such a large subject these divisions tend to be artificial. However any others were even less attractive. I conclude the chapter by considering briefly the most important applications of anomalies within elementary particle physics.

### 1.1 Algebraic Theory

Anomalous symmetry breaking was originally discovered in perturbative calculations. In 1969 Alder [8] and Bell and Jackiw [9] separately encountered the anomalous behaviour of the famous triangle diagram, and calculated the value of the associated ABJ anomaly. They concluded that the anomaly was due to the divergent nature of the triangle diagram and that its value could be found by correctly regularizing the theory. Soon after this, Jackiw employed current algebra techniques to demonstrate that anomalous effects connected with the triangle diagram were also attributable to the presence of non-canonical Schwinger terms in current commutators [20]. It was possible to derive the Schwinger terms using general arguments which made no reference to regularization procedures. These two approaches to anomalies, the calculational and the algebraic, were subsequently complemented by a third approach which exploited topological ideas and which related anomalies to topological index theorems [21]. Since then, research into anomalies has progressed simultaneously along these three separate lines of enquiry. The perspectives offered by the three approaches are different, and each approach has yielded results which the others are not able to reproduce [23]. A future complete understanding of anomalies will no doubt be preceded by a merging of the three streams, and will allow one to translate fluently between global topological methods, local algebraic techniques and the mechanics of calculations.

This section is devoted to a review of the algebraic approach to anomalies. As mentioned above, this approach dates from Jackiw's use of current algebra techniques to find Schwinger terms in current commutators [20]. When current algebra methods declined in popularity in the seventies, the algebraic approach to anomalies languished for want of a wider theoretical framework into which Schwinger terms could be fitted, and because other approaches proved more immediately rewarding. Recently Zumino [22] and Stora [23] developed a new formalism for the description of Abelian and non-Abelian anomalies in theories of massless chiral fermions in external Yang-Mills fields. This formalism, which is based on ideas taken from cohomology theory, is essentially algebraic in character. It provides a coherent mathematical framework for anomalies and suggests ways in which this framework might be extended. Notably, in the theory of cocycles they have found the generalization of Jackiw's earlier work on Schwinger terms. I will now introduce the material of this section by posing the anomaly problem as Stora [23] would have it done.

STATEMENT OF THE PROBLEM Let  $S(\phi)$  be a classical action constructed from a matter field  $\phi$  which transforms linearly under some internal, compact Lie group  $G$ . Suppose that the theory based on  $S(\phi)$  is renormalizable according to the BPHZ criteria [24,25]. Then one can rigorously define a vertex functional  $\Gamma(\phi)$

$$\Gamma(\phi) = S(\phi) + \sum_{n=1}^{\infty} h^n \Gamma^n(\phi)$$

The statement that  $S(\phi)$  is invariant under  $G$  means that  $S(\phi)$  is invariant under the action of some representation of  $G$ 's Lie algebra,  $\text{Lie } G$ . This is expressed by the following two equations

$$W_{cl}(a)S(\phi) = 0 \quad \dots(1)$$

$$[W_{cl}(a), W_{cl}(b)] = W_{cl}([a, b]) \quad \dots(2)$$

Here  $a$  and  $b$  are elements of  $\text{Lie } G$ , and  $W_{cl}(a)$  is a classical functional differential operator which is linear in  $a$ . Equation (2) asserts that the operators  $W_{cl}(a)$  form a representation of  $\text{Lie } G$ , while equation (1) is a

Ward identity. Note that this set up covers the situation where the internal symmetry  $G$  is spontaneously broken [25]. It is possible to show [25] that the renormalized perturbation series representing  $\Gamma(\phi)$  can be defined, and that the classical operators  $W_{cl}(a)$  may be extended naturally to quantum operators  $W(a)$ , in such a way that

$$W(a) \Gamma(\phi) = 0$$

$$[W(a), W(b)] = W([a, b])$$

In other words, it is possible to define the quantum theory in such a way that it is invariant under a representation of the classical symmetry group  $G$ . Now let  $\hat{G}$  be the gauge group associated with  $G$ . In the simplest case elements of  $\hat{G}$  are maps from space-time into  $G$ . Using the customary minimum coupling prescription, it is a straightforward matter to extend  $S(\phi)$  into a classical gauge invariant action  $S(\phi, A)$ , and to find a classical representation  $W_{cl}(\hat{a})$  of Lie  $\hat{G}$  such that

$$W_{cl}(\hat{a}) S(\phi, A) = 0$$

Here the field  $A$  is a classical gauge field transforming under  $\hat{G}$  in the familiar manner and  $\hat{a}$ , of course, is an element of Lie  $\hat{G}$ . The anomaly problem may now be posed as follows. Can  $\Gamma(\phi)$  be extended into  $\Gamma(\phi, A)$ , and can quantum counterparts  $W(\hat{a})$  be found for the classical operators  $W_{cl}(\hat{a})$ , so that

$$W(\hat{a}) \Gamma(\phi, A) = 0 \tag{3}$$

$$[W(\hat{a}), W(\hat{b})] = W([\hat{a}, \hat{b}]) \tag{4}$$

That is, will the quantum gauge theory admit a representation of the classical gauge group  $\hat{G}$ ? If both equations (3) and (4) can be maintained in the quantum theory, then gauge invariance will be respected and the theory will be free of (gauge) anomalies. However this is generally not possible. Apparently, in the presence of chiral fermions equations (3) and (4) are modified as follows [1,23]

$$W(\hat{a})\Gamma(\phi, A) = \omega(A, \hat{a}) \quad \dots(5)$$

$$[W(\hat{a}), W(\hat{b})] = W([\hat{a}, \hat{b}]) + S(A, \hat{a}, \hat{b}) \quad \dots(6)$$

Evidently the Ward identity (5) has acquired an anomaly, and the commutator of the gauge operators (6) has picked up a Schwinger term [24]. We shall see below that the anomaly and the Schwinger term are closely related to cocycles.

**SCHWINGER TERMS** In papers published in 1967, Sutherland [26] and Veltman [27] applied the hitherto very successful methods of current algebra to the calculation of the amplitude for the decay of a neutral pion into two photons. In the spirit of PCAC [2] they replaced the pion field with the divergence of the axial current and arrived at a result which differed markedly from experiment. This seemed to represent the first failure for current algebra methods. In fact, the error in Sutherland and Veltman's treatment of  $\pi_0 \rightarrow 2\gamma$  decay lay not in the current algebra manipulations, but rather in the supposition that the axial current was divergenceless. In 1969 Adler [8] and Bell and Jackiw [9] cleared up the mystery by demonstrating that the triangle diagram alluded to at the start of this section contributes an extra term, the ABJ anomaly, to the divergence of the axial current. This term, of which people were previously ignorant, reconciled Sutherland and Veltman's theory with the results of experiment. Shortly after this Feynman diagrammatic resolution of the  $\pi_0 \rightarrow 2\gamma$  problem Jackiw [20] offered an alternative solution based on current algebra techniques themselves. He showed that the anomalous contribution to  $\pi_0 \rightarrow 2\gamma$  decay can also be traced to the occurrence of non-canonical Schwinger terms in current commutators. I will now briefly review Jackiw's analysis of these Schwinger terms.

Consider a theory described by a Lagrangian  $L(\phi)$  depending on fields  $\phi$  and their derivatives  $\phi_\mu \equiv \partial_\mu \phi$ . In Hamiltonian formalism the theory's canonical momenta are given by the formula  $\pi^\mu = \delta L / \delta \phi_\mu$ , and the Euler-Lagrange equations of motion are

$$\partial_\mu \pi^\mu = \frac{\delta L}{\delta \phi} \quad \dots(7)$$

The field operator representation of the quantum version of this theory rests on the following equal time commutation relations (ETCR) [20]

$$i[\pi^0(t, \mathbf{x}), \phi(t, \mathbf{y})] = \delta(\mathbf{x} - \mathbf{y}) \quad \dots(8)$$

$$i[\pi^0(t, \mathbf{x}), \pi^0(t, \mathbf{y})] = i[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = 0 \quad \dots(9)$$

These ETCR hold regardless of the particular structure of the Lagrangian. Now suppose that the Lagrangian is invariant under an  $r$ -parameter internal symmetry group  $G$  which acts upon the fields through the generators  $T^a$ ,  $a=1, \dots, r$ . In other words,  $L(\phi)$  is invariant under the infinitesimal transformation  $\phi \rightarrow \phi + \delta\phi$ ,  $\delta\phi = \sum_a C^a T^a \phi$ . Then Noether's theorems [28] inform us that there will be  $r$  functions  $J_\mu^a$  of the fields and their derivatives such that  $\partial^\mu J_\mu^a = 0$ . Using the equations of motion (7), it is not difficult to deduce that the currents  $J_\mu^a$  are given by

$$J_\mu^a = \pi_\mu T^a \phi \quad \dots(10)$$

Like relations (8) and (9), this expression for the currents is independent of the particular form of the Lagrangian. Because of this, the above ETCR can be used in conjunction with definition (10) to deduce model-independent results for the currents  $J^a$ . Current algebra techniques were evolved precisely to find and exploit such model-independent results at a time when quantum electrodynamics was the only generally accepted field theoretic model. For example, consider the ETCR for the time component of a current  $J^a$  and the spatial components of a current  $J^b$ . Assuming that  $J^a$  and  $J^b$  are conserved, and using only Lorentz covariance, equations (8) and (9), and the group property for the representation matrices  $[T^a, T^b] = if_{abc} T^c$ , it is possible to deduce [20] that the most general form for this ETCR is

$$[J_0^a(0, \mathbf{x}), J_1^b(0, \mathbf{y})] = -f_{abc} J_1^c(0, \mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) + S_{1j}^{ab}(0, \mathbf{x}) \partial^j \delta(\mathbf{x} - \mathbf{y}) \quad \dots(11)$$

The unusual feature of this commutation relation is that it contains a non-canonical Schwinger term  $S_{1j}^{ab}$  [29]. Such terms were discovered by Goto and Imamura [30] in 1955, and consequently predate anomalies by quite a few years. Because it was not possible to explicitly determine the forms of

Schwinger terms using current algebra methods, they represented a serious obstacle to the progress of current algebraic analysis. However people generally avoided this obstacle by adopting Feynman's conjecture [31]. Effectively this meant that they simply ignored the presence of Schwinger terms in relations such as (11). It was not until Jackiw [20] proved Feynman's conjecture wrong, and demonstrated the connection between Schwinger terms and anomalies, that their significance was finally understood. To finish off this treatment of Schwinger terms, I will now reproduce an argument of Schwinger's [20,29] which establishes that the term  $S_{ij}^{ab}$  in (11) must be non-zero. For simplicity's sake, consider the case where the internal symmetry is electromagnetic  $U(1)$  symmetry. Then there is no internal index  $a$  on the currents, and the first term on the right hand side of (11) vanishes. Writing  $C_i(\mathbf{x}, \mathbf{y}) \equiv [J_0(0, \mathbf{x}), J_i(0, \mathbf{y})]$  we have

$$\langle 0 | [J_0(0, \mathbf{x}), \partial^i J_i(0, \mathbf{y})] | 0 \rangle = \langle 0 | \frac{\partial}{\partial y_i} C_i(\mathbf{x}, \mathbf{y}) | 0 \rangle \quad ..(12)$$

The right hand side of (12) will be non-vanishing only to the extent that the relevant Schwinger terms are non-vanishing. Since the current  $J$  is assumed to be conserved, we can set  $\partial^i J_i = -\partial^0 J_0 = -i[H, J_0]$  where  $H$  is the Hamiltonian. Then, using the fact that the vacuum has zero energy, we find

$$\langle 0 | [J_0(0, \mathbf{x}), \partial^i J_i(0, \mathbf{y})] | 0 \rangle = -i \langle 0 | J_0(0, \mathbf{x}) H J_0(0, \mathbf{y}) + J_0(0, \mathbf{y}) H J_0(0, \mathbf{x}) | 0 \rangle$$

If we now multiply (12) by  $f(\mathbf{x})f(\mathbf{y})$ , where  $f(\mathbf{x})$  is an arbitrary real function, integrate over  $\mathbf{x}$  and  $\mathbf{y}$ , and use the last equation, we arrive at the following result

$$i \int d^3x d^3y \langle 0 | C_i(\mathbf{x}, \mathbf{y}) | 0 \rangle f(\mathbf{x}) \frac{\partial}{\partial y_i} f(\mathbf{y}) = 2 \langle 0 | F H F | 0 \rangle \quad ..(13)$$

where  $F = \int d^3x f(\mathbf{x}) J_0(0, \mathbf{x})$ . The right hand side of (13) is non-zero because, in general, the operator  $F$  possesses non-vanishing matrix elements between the vacuum and other states which necessarily carry positive energy.

$$\langle 0 | F H F | 0 \rangle = \sum_{n,m} \langle 0 | F | m \rangle \langle m | H | n \rangle \langle n | F | 0 \rangle = \sum_n E_n |\langle 0 | F | n \rangle|^2 > 0$$

Thus  $C_i(x,y)$  is non-zero, and the relevant Schwinger terms cannot vanish. Note that it is possible to extend this argument to non-conserved currents which transform under an arbitrary internal symmetry group [20]. Although Schwinger's reasoning tells us in this fashion that non-vanishing Schwinger terms will, in general, occur in ETCR such as (11), it is not possible to find the values of these terms using current algebraic methods. For this information one must turn instead to model-dependent calculations such as those used by Adler, Bell and Jackiw to evaluate the ABJ anomaly. Jackiw [20] did this in the case of quantum electrodynamics and, in the process, established that the ABJ anomaly is a direct consequence of the presence of Schwinger terms in electromagnetic current commutators. This work alerted people for the first time to the connection between anomalies and anomalous current commutators.

**COCYCLES** Jackiw's work on current commutators provided a different perspective on anomalies to the original calculational approach of Adler, Bell and Jackiw. Adler, Bell and Jackiw's treatment of the triangle diagram emphasized the breakdown of chiral symmetry which followed from the non-conservation of the axial current. That is, it was principally concerned with the sort of symmetry breaking which is described by an equation like (5). On the other hand, Jackiw's independent work centred on anomalous current commutators. He was more interested in equations such as (6). Of course, people understood that these two approaches to anomalies were connected, but the close relationship which exists between them was not properly appreciated until recently. Now, thanks to the work of Zumino [22], Stora [23] and others, we know that anomalous symmetry breaking and anomalous current commutators are both closely bound up with the theory of cocycles. The cocycle approach to anomalies has by no means been fully explored, but already a more comprehensive picture of anomalous symmetry breaking has begun to emerge from it. In the remainder of this section I will describe the theory of cocycles, and in the next section we will see that anomalies are related to certain cocycles which are formed from topological terms in quantum field theories. In covering both these topics I will rely heavily on Jackiw's review [1].



Suppose that we have a group  $G$  composed of elements  $g$  which satisfy a composition law  $g_1 g_2 = g_{12}$ . Further suppose that  $G$  acts on some variable  $q$  according to a definite rule  $g: q \rightarrow q^g$ . Then we can consider quantities  $\Omega_n(q; g_1, \dots, g_n)$  depending on  $q$  and on  $n$  group elements  $\{g_1, \dots, g_n\}$ . Such quantities are called  $n$ -cochains. An operation, called the coboundary operation and denoted by  $\Delta$ , inserts one more group element and thereby creates an  $(n+1)$ -cochain from an  $n$ -cochain:

$$\begin{aligned} \Delta \Omega_n \equiv & \Omega_n(q^{g_1}; g_2, g_3, \dots, g_n, g_{n+1}) - \Omega_n(q; g_{12}, g_3, \dots, g_n, g_{n+1}) \\ & + \Omega_n(q; g_1, g_{23}, \dots, g_n, g_{n+1}) - \dots + (-1)^n \Omega_n(q; g_1, g_2, \dots, g_{n-1}, g_{n+1}) \\ & + (-1)^{n+1} \Omega_n(q; g_1, g_2, \dots, g_{n-1}, g_n) \end{aligned} \quad \dots(14)$$

One may verify from (14) that  $\Delta^2 = 0$ . Quantities whose  $\Delta$  vanishes, modulo an integer, are called cocycles, and those that are  $\Delta$  of something are called coboundaries. Thus all coboundaries are cocycles. One distinguishes cocycles which are coboundaries from those which are not by calling the former trivial cocycles and the latter non-trivial cocycles. Let us now see how cocycles are relevant to representations of the group  $G$ . Consider functions  $F(q)$  of the variable  $q$  which are invariant under the action of  $G$

$$U(g)F(q) = F(q) \quad \dots(15)$$

If the operators  $U(g)$  satisfy the same composition law as the group elements

$$U(g_1)U(g_2) = U(g_{12}) \quad \dots(16)$$

they form a representation of  $G$ . Operator representations of space-time and internal symmetry groups are used extensively in quantum mechanics and quantum field theory [1,2]. The simple structure of group representations summarized in equations (15) and (16) can be complicated by the introduction of phases. The first generalization is to allow a phase in (15). Suppose we set

$$U(g)F(q) = e^{-2\pi i \omega_1(q; g)} F(q) \quad \dots(17)$$

Then it is an easy matter to check that the consistency of this equation with (16) imposes a constraint upon  $\omega_1(q;g)$ . We must have

$$\Delta\omega_1 = \omega_1(q^{g_1};g_2) - \omega_1(q;g_{12}) + \omega_1(q;g_1) = 0 \text{ (mod integer)} \quad ..(18)$$

Thus  $\omega_1$  is a 1-cocycle. It should be noted that, if  $\omega_1$  is a trivial cocycle, the function  $F(q)$  and the operators  $U(g)$  can be redefined so that they represent  $G$  according to equations (15) and (16). Thus a trivial cocycle can be removed. On the other hand, if  $\omega_1$  is non-trivial it cannot be removed and the operators  $U(g)$  form a projective, rather than a faithful, representation of  $G$ . In the next generalization of equations (15) and (16) a phase is introduced into the composition law (16)

$$U(g_1)U(g_2) = e^{-2\pi i\omega_2(q;g_1,g_2)}U(g_{12}) \quad ..(19)$$

A consistency condition on this phase follows from the assumed associativity of the composition law. If  $[U(g_1)U(g_2)]U(g_3) = U(g_1)[U(g_2)U(g_3)]$  one easily shows that  $\Delta\omega_2 = 0 \text{ (mod integer)}$ . Thus  $\omega_2$  is a 2-cocycle. Once again, a trivial 2-cocycle can be removed. It is possible to continue in this way progressively adding phases. The next step, for instance, is to abandon associativity. Then one has

$$[U(g_1)U(g_2)]U(g_3) = e^{-2\pi i\omega_3(q;g_1,g_2,g_3)}U(g_1)[U(g_2)U(g_3)] \quad ..(20)$$

By considering four-fold products of the operators  $U(g)$ , and associating in different ways, one can establish that  $\omega_3$  is a 3-cocycle. It is worth noting that 1-cocycles, 2-cocycles and 3-cocycles all occur in quantum mechanics, and are therefore of physical interest. In contrast, no physical role has so far been found for higher cocycles. We are now in a position to see how cocycles relate to anomalies. Suppose that  $G$  is a Lie group. Then the elements of  $G$  may be written as exponentials  $g = \exp(\theta^a T^a)$  where the matrices  $T^a$  generate  $G$ 's Lie algebra. Similarly, the operators  $U(g)$  may be expressed in terms of generators  $G^a$  :  $U(g) = \exp(i\theta^a G^a)$ . The properties of the generators  $T^a$  are summarized by the following three relations

$$T^a F(q) = 0 \quad \text{(invariance)} \quad ..(21)$$

$$[T^a, T^b] = f_{abc} T^c \quad (\text{structure relations}) \quad ..(22)$$

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \quad (\text{Jacobi identity}) \quad ..(23)$$

Were there no phases in equations (17), (19) and (20) the generators  $G^a$  would satisfy relations similar to (21), (22) and (23). Instead, the reader may check that they behave as follows

$$iG^a F(q) = \delta\omega_1 \quad ..(24)$$

$$i[G^a, G^b] = f_{abc} G^c + \delta\omega_2 \quad ..(25)$$

$$i[G^a, [G^b, G^c]] + i[G^b, [G^c, G^a]] + i[G^c, [G^a, G^b]] = \delta\omega_3 \quad ..(26)$$

Here  $\delta\omega_1$ ,  $\delta\omega_2$  and  $\delta\omega_3$  are respectively a 1-cocycle, a 2-cocycle and a 3-cocycle. They are related to the infinitesimal parts of the phases  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . Comparing equations (24) and (25) with (5) and (6) it is tempting to conclude that the anomaly in (5) is a 1-cocycle, while the Schwinger term in (6) is a 2-cocycle. Of course, any such conclusion must be tested by explicit calculation. Such calculations have now been performed and the relevant anomaly and Schwinger term have been shown to be cocycles which are constructed from topological functions of the gauge field [32]. Details of these topological functions will be provided in the next section. Thus, while much confirmatory work remains to be carried out on the connections between anomalies and cocycles, it seems safe to conclude that, at least in some cases, anomalous symmetry breaking can be traced to the fact that the generators of the internal symmetry group obey the anomalous algebraic relations (24), (25) and (26).

## 1.2 Topological Theory

As Jackiw [1] has explained, a field theoretic effect is called topological when it is insensitive to localized perturbations of the dynamical fields or of the parameters entering the dynamical description. This definition suggests that, when we are dealing with a problem defined on an open, infinite

space-time, topological effects will arise from behaviour "at infinity". Frequently, in a quantum field theory, one is interested in integrals of local quantities. The behaviour of the system at infinity typically determines the surface terms which contribute to these integrals. Thus, field theoretic topological effects emerge when attention is paid to surface terms at infinity which are mostly ignored in elementary discussions. For example, modifying a Lagrangian by a total derivative does not affect the equations of motion, but it can change the action. This has no significance in classical physics where all the dynamical information is contained in the equations of motion. However, a quantum effect may emerge because quantum theories are sensitive to total derivatives in the Lagrangian - the Heisenberg operator formalism requires identifying canonical momenta from the Lagrangian, while the path integral approach makes use of the action.

Topological effects enter into non-Abelian gauge theories because the gauge fields can assume topologically non-trivial configurations. The consequences of this fact for classical gauge theories were first explored by Belavin et.al. [33], who found a finite energy, pseudoparticle solution to the gauge field equations. Later 't Hooft [34], Callen, Dashen and Gross [35] and Jackiw and Rebbi [36] showed how to incorporate pseudoparticles into quantum processes. Since then, pseudoparticles and solitons have played a part in our understanding of several interesting problems in quantum field theories. These include CP non-conservation in the strong interactions [13,37,38,39], the U(1) problem [12,40,41], charge quantization [42,43], and anomalies. I now wish to demonstrate that the non-trivial topology of non-abelian gauge fields is closely connected with anomalous symmetry breaking in the corresponding gauge theories.

**Θ-VACUA** Perhaps the easiest way to expose the topological structure of non-Abelian gauge theories is to examine the structure of the vacuum [6,7]. To do this let us place our quantum system within the space-time box  $|t| < T$ ,  $|\mathbf{x}| < R$ . Then the vacuum condition

$$F^{\mu\nu} = 0 \quad \dots(27)$$

obtains outside the box. Here  $F^{\mu\nu}$  is the Lie algebra valued gauge field tensor. As usual it is related to the gauge field  $A^\mu$  as follows

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]$$

Under a gauge transformation  $\psi \rightarrow U\psi$  the field  $A$  goes into  $A'$  where

$$A'_\mu = U A_\mu U^{-1} + U^{-1} \partial_\mu U \quad \dots(28)$$

We are free to adopt the gauge fixing condition  $A_0=0$  for our system, but in order that this condition be consistent with (28) we must restrict ourselves to time-independent gauge transformations  $U(x) = U(\mathbf{x})$ . Then, since all matter fields must be zero outside the box, the vacuum will be described by a time-independent, pure gauge potential  $A_i(\mathbf{x}) = U^{-1}(\mathbf{x}) \partial_i U(\mathbf{x})$ . At initial time  $t=-T$  we can use the remaining gauge freedom to choose  $U(\mathbf{x})=1$  in which case  $A_i(\mathbf{x})=0$ . But the vacuum condition (27) implies that  $\partial_0 A_i=0$  for  $|t|>T$  or  $|\mathbf{x}|>R$ . Consequently  $A_i(\mathbf{x})$  is zero everywhere outside the box. This uniform vanishing of  $A_i(\mathbf{x})$  means that we can identify all points on the surface of the box. In particular for any given time  $t$ ,  $|t|<T$ , we can identify the points on the spatial edge of the box  $|\mathbf{x}|=R$ . Having brought about this situation using the device of a space-time box, we can now allow  $T, R \rightarrow \infty$ . Then the gauge fields may be regarded as maps from three-dimensional space with infinities identified (ie. from the three-dimensional sphere  $S^3$ ) into the gauge group  $G$ . A theorem due to Bott [6,44] states that

**THEOREM.** Any continuous mapping of  $S^3$  into a simple Lie group  $G$  can be continuously deformed into a mapping into an  $SU(2)$  subgroup of  $G$ .

We are therefore led to consider maps from  $S^3$  into  $SU(2)$ . Such maps can be divided into inequivalent homotopy classes on the basis of their (integer) winding numbers. The winding number  $n$  of a gauge field is defined to be [7]

$$n = \frac{1}{16\pi^2} \int d^4x \operatorname{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}) \quad \dots(29)$$

where  $\tilde{F}^{\mu\nu} = (1/2) \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}$  is the dual field tensor. The usual interpretation of this situation is that there is a multiplicity of vacuum states  $|n\rangle$

corresponding to pure gauge potentials having winding numbers  $n$ . These vacua are related to each other via "large" gauge transformations, and 't Hooft has demonstrated that pseudoparticle solutions of the gauge field equations tunnel between homologically distinct vacuum states. Because of this tunnelling phenomenon we expect the true, or physical, vacuum to be a suitable superposition of  $|n\rangle$  states. Suppose  $T$  is a large gauge transformation which maps  $|n\rangle$  into  $|n+1\rangle$ . If the true vacuum is to be an eigenstate of  $T$  then it must be constructed from a real parameter  $\theta$  as follows

$$|\theta\rangle = \sum_n e^{-in\theta} |n\rangle \quad \dots(30)$$

In this way we end up with a spectrum of  $\theta$ -vacua which are reminiscent of the multiple vacuum states that are a feature of the quantum mechanical Bloch phenomenon [4]. There is no passage between quantum states which are built on distinct  $\theta$ -vacua, and each value of  $\theta$  therefore characterizes a separate " $\theta$ -world". If our quantum system begins in one  $\theta$ -world then it will stay there for all time. Consider the vacuum to vacuum transition amplitude for our system in path integral formalism. In view of what has just been said, it must be of the form

$$\langle\theta'|e^{-iHt}|\theta\rangle_J = \delta(\theta-\theta') I_J(\theta) \quad \dots(31)$$

Writing the left hand side of (31) in terms of  $|n\rangle$  states one finds

$$\begin{aligned} \langle\theta'|e^{-iHt}|\theta\rangle_J &= \sum_{n,m} e^{im\theta'} e^{-in\theta} \langle m|e^{-iHt}|n\rangle \\ &= \sum_{v,m} e^{-iv\theta} e^{im(\theta'-\theta)} \int [dA]_v \exp[i\int (L+JA)d^4x] \end{aligned} \quad \dots(32)$$

In this last expression  $v=n-m$ , and the path integral is over all gauge fields  $A_\mu$  whose topology is such that they tunnel between the states  $|n\rangle$  and  $|n+v\rangle$ . Comparing (31) and (32), and using (29), we deduce that

$$I_J(\theta) = \sum_v \int [dA]_v \exp[i\int (L_{\text{eff}} + JA)d^4x]$$

where

$$L_{\text{eff}} = L + \frac{\theta}{16\pi^2} \text{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}) \quad \dots(33)$$

Thus  $\theta$  has turned up in the effective Lagrangian multiplying the same topological term  $(1/16\pi^2)\text{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu})$  that appears in equation (29). One might summarize the situation as follows. A quantized non-Abelian gauge theory contains a vacuum parameter  $\theta$ . According to (30),  $\theta$  determines the relationship between the vacua  $|n\rangle$ , which correspond to pure gauge configurations of definite winding number, and the  $\theta$ -vacua which, to the extent that they are eigenfunctions of gauge transformations  $T|\theta\rangle = e^{i\theta}|\theta\rangle$ , are the physical vacua. If one works with quantum states which are built on the non-physical vacua  $|n\rangle$ , then  $\theta$  remains buried within the gauge transformation rules of the theory's state vectors and operators. On the other hand, one can make  $\theta$  explicit within the theory's effective Lagrangian by working with quantum states which are built on the physical  $\theta$ -vacua. Naturally, in the latter case topological effects associated with  $\theta$  will be more readily accessible to analysis. We can convince ourselves of this by considering the example of a three-dimensional model gauge theory which was examined by Jackiw [1].

**THREE-DIMENSIONAL MODEL** The four-dimensional term  $(1/16\pi^2)\text{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu})$  which appears in equations (29) and (33) may justly be described as a topological quantity since its value is unaffected by localized variations of the gauge field  $A_\mu$ . In fact, formula (29) tells us that its (integer) value is altered only when  $A_\mu$  undergoes the sort of global transformation that brings about a change in winding number. In three-dimensional space-time there is an analogous topological term. By adding this term to the usual Yang-Mills Lagrangian one arrives at the following action [1]

$$I = \int d^3x \text{tr} \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - (m/4\pi) \epsilon^{\mu\nu\sigma} (A_\mu \partial_\nu A_\sigma + \frac{2}{3} A_\mu A_\nu A_\sigma) \right] \quad \dots(34)$$

In this expression the parameter  $m$  has the dimensions of mass, and an analysis of the Abelian  $U(1)$  case of the theory described by the action (34) shows that the "photon" is indeed massive. Consequently this model is called a topologically massive gauge theory. The action (34) leads to the following equation of motion

$$D_\mu F^{\mu\nu} + (m/8\pi) \epsilon^{\nu\sigma\rho} F_{\sigma\rho} = 0 \quad \dots(35)$$

Note that the parameter  $m$  appears explicitly in (35). By way of contrast, the parameter  $\theta$  which occurs in the four-dimensional effective Lagrangian (33) does not appear in the corresponding four-dimensional equation of motion. This fact is an indication that the character of topological effects varies with the space-time dimension. Generally speaking, there is a mismatch between the dimension of the Lagrangian and the dimension of the topological terms that one might add on to the Lagrangian. Therefore, if one wishes to include topological effects in the theory, one must couple the topological terms to dimensional parameters such as  $m$ . It is only in four dimensions that the relevant topological term is of exactly the same dimensions as the Lagrangian, and the coupling constant  $\theta$  is correspondingly dimensionless. Because of this, four-dimensional theories constitute a special case. Even so, it is still instructive to look at simpler lower-dimensional models such as the one presently under consideration.

When expressed in terms of the electric and magnetic fields, the Hamiltonian corresponding to the action (34) is conventional :  $H = (1/2) \int d^2x (E_a^2 + B_a^2)$ . However, the relationship between these fields and the canonical momenta is altered. One finds

$$B_a = -\frac{1}{2} \epsilon^{ij} F_a^{ij} \quad E_a^i = F_a^{i0} \quad \Pi_a^i = -E_a^i + (m/8\pi) \epsilon^{ij} A_a^j$$

Thus,  $H$  is non-standard when expressed in terms of canonical variables

$$H = \frac{1}{2} \int d^2x [(\Pi_a^i - (m/8\pi) \epsilon^{ij} A_a^j)^2 + B_a^2]$$

The generators of gauge transformations on the fields are the operators  $G_a$

$$G_a = [-(D \cdot E)_a + (m/4\pi) B_a]$$

Let us assume that an element of the gauge group is given by the expression  $g = \exp(\theta^a T^a)$  where the matrices  $T^a$  are the group generators. The corresponding finite operator on the fields is  $U(g) = \exp(i\theta^a G_a)$ . As Jackiw [1] has shown, the following two equations hold

$$U(g)\Psi(A) = e^{-2\pi i m \omega(A;g)} \Psi(A^g) \quad \dots(36)$$



$$[E_a^i(\mathbf{x}), E_b^j(\mathbf{y})] = -\frac{im}{4\pi} \delta_{ab} \epsilon^{ij} \delta(\mathbf{x}-\mathbf{y}) \quad ..(37)$$

Thus, among other effects, the topological parameter  $m$  has introduced a phase  $\omega(A;g)$  into the gauge transformation law (36), and has given rise to the anomalous current commutator relation (37). As the reader may suspect, the phase and anomalous commutator are related to cocycles. I will now explain where the cocycles come from.

**TOWERS OF COCYCLES** We have just seen that the non-trivial topology of non-Abelian gauge fields in four dimensions gives rise to the topological term  $P = (1/16\pi^2) \text{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu})$  in the effective Lagrangian (33). In the mathematical literature this term is called the Chern-Pontryagin density, and counterparts for it exist in any even-dimensional space-time. The density  $P$ , and terms of a similar topological nature, have been closely bound up in the recent past with progress in our understanding of several interesting phenomena in gauge field theories. Some of these have already been alluded to. For instance, up to a factor, the chiral anomaly in vector gauge theories is simply the Chern-Pontryagin density. In QCD the presence of this topologically non-trivial anomaly is thought to resolve the  $U(1)$  problem [39]. Likewise, when the Chern-Pontryagin density is included in the QCD Lagrangian, it induces CP non-conserving processes. So far as we can tell, the strong interactions are CP conserving. People have therefore been led to consider mechanisms based on axions as a means of having QCD respect CP invariance [38]. Lastly, as explained above, terms like  $P$  give rise to anomalous symmetry breaking effects in gauge field theories. I will now outline a little more of the relationship between these effects and Chern-Pontryagin densities in various dimensions.

The first thing to note about the four-dimensional Chern-Pontryagin density  $P$  is that it is invariant under small gauge transformations, but it changes by an integer under large gauge transformations. In other words,  $P$  is a 0-cocycle

$$\Delta P = 0 \quad ..(38)$$

Secondly, it is well-known that  $P$  can be expressed as a total derivative. By employing compact differential form notation [45] in which  $A \equiv A_\mu dx^\mu$  and  $F = dA + A^2$ ,  $P$  may be written as follows

$$P \equiv -\frac{1}{8\pi^2} \text{tr } F^2 = d\Omega_0 \quad \dots(39)$$

$$\Omega_0(A) = -\frac{1}{8\pi^2} \text{tr} \left( A dA + \frac{2}{3} A^3 \right)$$

$P$  and  $\Omega_0$  are four- and three-forms respectively. Strictly speaking, the result (39) is only locally valid. It may be extended to a global relation, however the extension involves certain subtleties which I do not wish to consider here [45,46,47]. Suffice it to say that, although these subtleties properly enter into the following discussion, they do not alter the conclusions which are detailed below. For the sake of simplicity, I will therefore ignore them.

The fact that  $P$  may be written as a total derivative is not unconnected with its topological nature. For example, if  $S$  denotes four-dimensional space-time and  $\partial S$  is the boundary of  $S$ , then (39) ensures that

$$\int_S P = \int_S d\Omega_0 = \int_{\partial S} \Omega_0$$

This means that, when  $P$  is included in the Lagrangian, its contribution to the action is a surface term. Hence  $P$ 's effect on the theory is global rather than local, and this is precisely what one would expect of a term which has been characterized as topological. Now consider the quantity  $\Delta\Omega_0$ . If one acts on  $\Delta\Omega_0$  with the differential operator  $d$ , and uses equation (38), one finds

$$d(\Delta\Omega_0) = \Delta(d\Omega_0) = \Delta P = 0$$

By Poincaré's lemma this result implies that  $\Delta\Omega_0$  may in turn be written as a total derivative

$$\Delta\Omega_0 = d\Omega_1 \quad \dots(40)$$

The object  $\Omega_1$  is clearly a 2-form. The same argument may be repeated once again. If we apply the operator  $d$  to  $\Delta\Omega_1$  and use the fact that  $\Delta^2 = 0$  we discover that  $d(\Delta\Omega_1) = \Delta(d\Omega_1) = \Delta^2\Omega_0 = 0$ . Thus

$$\Delta\Omega_1 = d\Omega_2 \quad \dots(41)$$

where  $\Omega_2$  is some 1-form. Equations (39), (40) and (41) are the start of a set of so-called "descent equations" which begin with the four-dimensional Chern-Pontryagin density. The whole process by which these equations are generated may be generalized to arbitrary even dimensions. If we now use  $P$  to denote the  $2n$ -dimensional Chern-Pontryagin density, then we have

$$P = \frac{i^n}{n!(2\pi)^n} \text{tr } F^n$$

Once again the  $2n$ -form  $P$  is both a 0-cocycle and a total derivative:  $\Delta P = 0$ ,  $P = d\Omega_{2n-1}^1$ . Here I have used a subscript  $2n-1$  to indicate that  $\Omega_{2n-1}^1$  is a  $(2n-1)$ -form. The superscript 1 on  $\Omega_{2n-1}^1$  corresponds to the subscript 1 on  $\Omega_1$  in (40). From the density  $P$  one obtains the following set of descent equations

$$P = d\Omega_{2n-1}^0$$

$$\Delta\Omega_{2n-1}^0 = d\Omega_{2n-2}^1$$

$$\Delta\Omega_{2n-2}^1 = d\Omega_{2n-3}^2$$

.

$$\Delta\Omega_1^{2n-2} = d\Omega_0^{2n-1}$$

$$\Delta\Omega_0^{2n-1} = 0 \quad \dots(42)$$

Note that the sequence ends with the 0-form  $\Omega_0^{2n-1}$ . The interesting thing about the set of equations (42) is that one can use them to generate a parallel sequence of cocycles. Suppose we set

$$\omega_p^q = \int_{S_p} \Omega_p^q \quad \dots(43)$$

where the integral in (43) is over infinite  $p$ -dimensional space  $S_p$ . By integrating the first of equations (42) over  $S_{2n}$  we find that

$$\int_{S_{2n}} P = \int_{S_{2n}} d\Omega_{2n-1}^0 = \int_{\partial S_{2n}} \Omega_{2n-1}^0 \quad \dots(44)$$

where  $\partial S_{2n}$  is the boundary of  $S_{2n}$ . Under certain reasonable assumptions [48] the boundary of  $S_{2n}$  may be taken to be  $S_{2n-1}$ . Thus (44) becomes

$$\int_{S_{2n}} P = \omega_{2n-1}^0$$

By similarly integrating each of equations (42) and replacing  $\partial S_p$  with  $S_{p-1}$  one arrives at the following set of relations

$$\omega_{2n-1}^0 = \int_{S_{2n}} P$$

$$\omega_{2n-2}^1 = \Delta \omega_{2n-1}^0$$

$$\omega_{2n-3}^2 = \Delta \omega_{2n-2}^1$$

.

.

$$\omega_0^{2n-1} = \Delta \omega_1^{2n-2}$$

From these relations, and the results  $\Delta P = \Delta^2 = 0$ , one can easily deduce that  $\Delta \omega_p^q = 0$ . Therefore  $\omega_p^q$  is a  $q$ -cocycle. In this way the descent equations (42) allow us to generate a tower of cocycles from the  $2n$ -dimensional Chern-Pontryagin density  $P$ . I acknowledge that the arguments by which I arrived at these cocycles are grossly over-simplified. However the mathematics can be made rigorous and the conclusions still hold [49]. The relevance of all this work to anomalies is not difficult to see. Knowing, as we do, that anomalies and anomalous current commutators are 1-cocycles and 2-cocycles respectively, it is tempting to identify them with the quantities  $\omega_{2n-2}^1$  and  $\omega_{2n-3}^2$ . Of course, in four dimensions the anomaly must be a 4-form and the current commutator a 3-form. Hence we must have  $n=3$ , in which case  $\omega_{2n-2}^1 = \omega_4^1$  and  $\omega_{2n-3}^2 = \omega_3^2$  are descended from the Chern-Pontryagin density in six dimensions. We are therefore led to ask the following question. Can the

four-dimensional gauge anomaly and associated anomalous current commutator be identified with the cocycles  $\omega_4^1$  and  $\omega_3^2$  which are descended from the six-dimensional Chern-Pontryagin density? This question can only be answered by carrying out the relevant perturbative calculations. These calculations have now been performed, and it has been ascertained that the four-dimensional anomaly and current commutator can indeed be identified with the cocycles  $\omega_4^1$  and  $\omega_3^2$  [32]. Clearly there is much which is yet to be understood about anomalies. However our recognition that anomalous symmetry breaking has its origins in cocycles which are descended from topological Chern-Pontryagin densities is a marked step forward in our knowledge of this subject.

### 1.3 Computational Methods

Having described, albeit briefly, the theoretical underpinnings of anomalous symmetry breaking, it is worth devoting some time to a review of the various methods by which anomalies may be calculated. Anomalies were discovered in 1969 by Adler, Bell and Jackiw [8,9], and it is only recently that they have been able to be fitted into some kind of unified theoretical framework. Because of the lack of a solid theoretical approach to the subject in the intervening seventeen years, most of the progress in our understanding of anomalies has been inspired by the methods that people devised to calculate them. For example, the Adler-Bardeen theorem, the relationship between anomalies and regularization, and anomaly cancellation mechanisms were all initially explored and understood from an almost exclusively calculational point of view. In this section I will briefly survey the various methods of anomaly calculation, commenting where appropriate on the strengths and failings of the different approaches. This task is simplified by dividing calculational methods as follows into three groups: Feynman diagrammatic methods, path integral methods, and differential geometric methods.

**FEYNMAN DIAGRAMMATIC METHODS** Under this heading come all those methods which rely upon the evaluation of one or more Feynman diagrams. These include the original analyses of the triangle diagram by Adler [8] and Bell and Jackiw [9], and also the method of anomaly calculation described later in this thesis. It is possible at the outset to make a couple of broad

generalizations about Feynman diagrammatic methods of calculation. Firstly, the only sort of diagram which will contribute to an anomaly is one of the form illustrated in figure 1. Such a diagram consists of a single fermion loop, and its legs are either external currents, gauge bosons or anti-symmetric tensor fields. Of course, any Feynman diagram which contains a sub-diagram of the form shown in figure 1 will contribute anomalous terms to the corresponding quantum amplitude. However, no exception has yet been found to the rule that anomalies themselves receive contributions only from diagrams of the form depicted below.

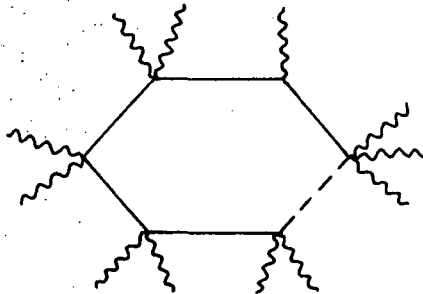


FIGURE 1

The second generalization about diagrammatic methods concerns the question of regularization. In the last section we saw that, in a three-dimensional model examined by Jackiw, anomalous symmetry breaking effects were tied to a topological parameter  $m$  which appeared explicitly in the theory's Lagrangian. This may suggest to the reader that anomalies and anomalous current commutators are straightforward consequences of the system's (non-standard) equations of motion. However, to see that this is not the case one need only look at four-dimensional non-Abelian gauge theories which are described by the effective Lagrangian (33). These theories contain anomalies even though the topological parameter  $\theta$  does not appear in the equations of motion. My point here is that anomalies are not simple consequences of the field equations. Instead, they are intimately bound up with the subtleties of regularization. In general, the position in which an anomaly occurs in a theory, and even the form of the anomaly, are determined by the way in which the theory is regularized. Not surprisingly, then, questions of regularization are an integral part of any Feynman diagrammatic method of anomaly calculation.

Let us now examine in a little more detail some of the considerations which lie behind the foregoing two generalizations about diagrammatic methods. The assertion that anomalies receive contributions only from Feynman diagrams of the form shown in figure 1 is known as the Adler-Bardeen theorem [50,51]. Adler and Bardeen's conjecture that anomalies are one-loop quantum effects was originally formulated in the context of QED. However their arguments appear to be more generally valid, and it is certainly the case that no anomaly has yet been found which receives contributions from Feynman diagrams containing two or more loops. A "proof" of the Adler-Bardeen theorem rests upon the observation that anomalies occur when a quantum theory cannot be regularized in such a way as to preserve all of its classical symmetries. Multi-loop Feynman diagrams necessarily contain internal boson lines. One can regularize these boson lines in a manner which respects all the usual symmetries by applying higher derivative regularization [52] to the boson propagators. (Fermion propagators cannot be effectively regularized using the higher derivative scheme. When one includes higher powers of gauge covariant derivatives in a fermion Lagrangian, one also introduces more gauge fields whose effect is to cancel the regularizing influence of the derivatives.) Since one can employ higher derivative terms of any order, it is possible to regularize any given multi-loop diagram in this fashion. It therefore follows that one-loop Feynman diagrams are the only diagrams which cannot be regularized in such a way as to preserve all of the theory's classical symmetries, and these diagrams are the only ones which will contribute to anomalies. In conclusion, it must be added that this proof is not altogether convincing and, particularly in supersymmetric theories, there are some aspects of the Adler-Bardeen theorem which remain controversial [53].

It should by now be clear that anomalies are the quantum corrections to classical current conservation equations. As such, they are potentially mathematically ill-defined until they are subjected to the dual processes of regularization and renormalization. This is, in fact, the case. Anomalies are invariably associated with divergent Feynman diagrams, and they do not assume a definite form until these diagrams are made well-defined through the application of some regularization procedure. One is free to use any number of regularization schemes, however people usually choose one of the

following four conventional methods: Pauli-Villars regularization [2], dimensional regularization [54], higher derivative regularization [52] and point-splitting [55]. These four methods all feature some sort of regularization parameter  $\Lambda$  which serves to measure the degree of divergence of quantities in the unregularized quantum theory. By allowing  $\Lambda$  to tend to some limiting value (usually 0 or  $\infty$ ) one can establish that a certain quantity diverges or converges as  $\ln\Lambda$ ,  $\Lambda^2$ ,  $1/\Lambda$  etc.. The role that the regularization parameter  $\Lambda$  plays in the emergence of an anomaly is, in its general aspects, common to all anomalies and to all of the above four regularization schemes. To give some idea of this role it therefore suffices to look at the way in which a chiral anomaly emerges under Pauli-Villars regularization. Consider massless QED in four dimensions. To the usual massless fermion field  $\psi$  and electromagnetic field  $A_\mu$  one adds a single fermion regulator field  $\psi_R$  of mass  $M$ . In this case the mass  $M$  is the regularization parameter, and the appropriate regulator limit is  $M \rightarrow \infty$ . In the unregularized theory the equations of motion can easily be used to show that the chiral current  $J_\mu = \bar{\psi} \gamma_5 \gamma_\mu \psi$  has zero divergence  $\partial \cdot J = 0$ . When the regulator field is taken into account one finds that the chiral current becomes  $J_{R\mu} = \bar{\psi} \gamma_5 \gamma_\mu \psi + \bar{\psi}_R \gamma_5 \gamma_\mu \psi_R$  and the equations of motion now imply

$$\partial^\mu J_{R\mu} = 2MJ_R \quad ..(45)$$

where  $J_R = i\bar{\psi}_R \gamma_5 \psi_R$ . Equation (45) is an operator identity, and should be checked by looking at the amplitudes of the operators between appropriate particle states. It turns out that, with one exception, the contributions of all of these amplitudes to the regulator parts of equation (45) contain inverse powers of  $M$ . Therefore in the limit  $M \rightarrow \infty$  they vanish. The one exception is the amplitude of  $J_R$  between the vacuum and the two photon state. Of course, the value of this amplitude is given by the ABJ triangle diagram. One finds that  $\langle 0 | J_R | 2\gamma \rangle = (1/16M\pi^2) F_{\mu\nu} \tilde{F}^{\mu\nu}$  and (45) becomes

$$\partial^\mu J_{R\mu} = \frac{M}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad ..(46)$$

Thus in the limit  $M \rightarrow \infty$  the ABJ triangle diagram gives rise to a non-vanishing residue in (45) and one ends up with the anomalous chiral current conservation equation  $\partial \cdot J = (1/8\pi^2) F_{\mu\nu} \tilde{F}^{\mu\nu}$ . The point I wish to make here is



that an anomaly occurs in (46) because a zero (a  $1/M$  in the regularized theory) comes up against an infinity ( $M$ ) and leaves a finite residue. The same is true of any anomaly when calculated with the help of a conventional regularization scheme. One always finds that the anomaly emerges from the theory when a positive power of the regularization parameter  $\Lambda$  is exactly cancelled by factors of  $1/\Lambda$ , leaving a finite residue. Clearly, even though it may be inaccurate to say that anomalies are caused by the regularization process, there is a close connection between the two.

This connection appears still closer when it is realized that the anomaly is not a passive bystander during the regularization of a theory. We have seen that the anomaly's presence signifies that there is no regularization scheme which is entirely satisfactory in the sense that it preserves all of the theory's classical symmetries. When regularizing an anomalous theory, one must therefore decide which of the classical symmetries is least desirable, or least painfully relinquished, and choose a regularization scheme accordingly. The anomaly will appear in the conservation equation of the current which corresponds to the symmetry that is broken by the regularization scheme, and in this sense the anomaly plays a part in determining which regularization scheme is used. For instance, suppose that one wishes to regularize an anomalous quantum theory whose classical counterpart is chirally and vector gauge invariant. Because the theory is anomalous one of these symmetries must be relinquished, and it is most likely that one would opt to give up chiral invariance and retain vector gauge invariance. In this case one would employ a regularization scheme like Pauli-Villars which respects gauge symmetry but breaks chiral symmetry, and the anomaly would appear in the chiral current conservation equation. On the other hand, one might wish to keep chiral invariance and give up vector gauge invariance. Then one would employ a scheme which respects chiral symmetry but not gauge symmetry, and the anomaly would appear in the conservation equations for the gauge currents. This aspect of anomalies has not been fully understood in the past. For instance, it was realized that significantly different regularization schemes could lead to anomalies in the conservation equations for entirely different currents. However, it was not appreciated that slightly different regularization schemes might give rise to anomalies in the same current conservation equation which are respectively gauge

covariant and non-gauge covariant [56]. It is precisely in the freedom to vary the regularization scheme that the difference between so-called consistent and covariant anomalies lies [56,57,58].

One final comment on anomalies must be made before I proceed with a discussion of functional integral methods of anomaly calculation. It was pointed out above that, like any field theoretic object, anomalies potentially require both regularization and renormalization. We have seen that they certainly must be regularized in some fashion. On the other hand it turns out that, because anomalies are one-loop phenomena, they do not have to be renormalized. To see that this is generally the case, one can use an argument due to Alvarez-Gaume and Witten [59]. They point out that, by using unitarity, one can uniquely reconstruct any one-loop amplitude from tree diagrams, up to a polynomial in the external momenta. Therefore, any of the diagrams which can contribute to an anomaly are well-defined modulo the ability to add such a polynomial. When one claims that a diagram is anomalous, one means that it is impossible to add a polynomial in the momenta so as to eliminate the anomaly and obtain an amplitude that respects all symmetries. It automatically follows from this that anomalies are always finite. After all, the infinite part of a diagram is always a polynomial in the external momenta. Our freedom to redefine an amplitude by adding a polynomial includes the freedom to throw away all infinite pieces. Hence anomalies are always finite and they do not need to be renormalized.

**PATH INTEGRAL METHODS** Once people had succeeded in calculating anomalies using Feynman diagrams and traditional field operator formalism, it was natural to ask whether the same could be done in the path integral formalism developed by Feynman and Hibbs [2,3]. In 1979 Fujikawa [60,61,62] answered this question positively by deriving the ABJ anomaly using path integral methods. His derivation goes as follows. One begins with the fermion effective action

$$e^{-\Gamma(A)} = \int d\psi d\bar{\psi} \exp[-\int d^4x \bar{\psi} i \not{D} \psi] \quad \dots(47)$$

where  $A$  is the electromagnetic field and  $D$  is the gauge covariant derivative. Note that  $A$  has not been integrated over in (47) since only external

electromagnetic fields enter into the ABJ anomaly. Consider how the various parts of the effective action transform under the chiral transformation

$$\psi \rightarrow \psi + i\alpha(x)\gamma_5\psi \quad ..(48)$$

For a start, one easily finds that the exponent in the integrand changes by

$$\int d^4x \bar{\psi} i \not{D} \psi \rightarrow \int d^4x \bar{\psi} i \not{D} \psi + \int d^4x \alpha(x) \partial^\mu J_\mu \quad ..(49)$$

where  $J_\mu = \bar{\psi} \gamma_5 \gamma_\mu \psi$  is the chiral current considered above. However, it is not quite such a simple matter to discover how the fermion determinant  $d\psi d\bar{\psi}$  behaves under chiral transformations. To do this Fujikawa expressed the field  $\psi$  in terms of eigenfunctions  $\psi_n$  of the Dirac operator  $i\not{D}$  :  $i\not{D}\psi_n = \lambda_n \psi_n$ . In this way he found that, for infinitesimal  $\alpha(x)$  in (48), the measure  $d\psi d\bar{\psi}$  transforms as follows

$$d\psi d\bar{\psi} \rightarrow \exp[-2i \int d^4x \alpha(x) i(x)] d\psi d\bar{\psi} \quad ..(50)$$

Here  $i(x) = \sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x)$ . The quantity  $i(x)$  is mathematically ill-defined. To calculate it, a gauge invariant cutoff  $M$  was introduced

$$\begin{aligned} i(x) &= \lim_{M \rightarrow \infty} \sum_n \psi_n^\dagger(x) \gamma_5 \exp[-(\lambda_n/M)^2] \psi_n(x) \\ &= \lim_{M \rightarrow \infty, y \rightarrow x} \text{tr} \gamma_5 \exp[-(\not{D}/M)^2] \delta(x-y) \\ &= \lim_{M \rightarrow \infty, y \rightarrow x} \int d^4k \text{tr} \gamma_5 \exp[-(D^2 + \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu})/M^2] e^{ik(x-y)} \\ &= \lim_{M \rightarrow \infty} (1/16) \text{tr} \gamma_5 ([\gamma^\mu, \gamma^\nu] F_{\mu\nu})^2 (1/2! M^4) \int d^4k \exp(-k^2/M^2) \\ &= - \frac{1}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad ..(51) \end{aligned}$$

Finally Fujikawa demanded that the effective action be invariant under the change of variable (48) :  $[\delta/\delta\alpha(x)] \exp[-\Gamma(A)]|_{\alpha=0} = 0$ . By substituting equations (49), (50) and (51) into this condition he arrived at the following anomalous chiral current conservation equation

$$\partial^\mu J_\mu = - \frac{i}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

This, of course, is none other than the Euclidean space version of the Minkowski space equation (46). Several comments might be made about this derivation of the ABJ anomaly. Firstly, Fujikawa's work suggests that anomalies arise in path integral formalism because the fermion measure is not invariant under the relevant symmetry transformation. This is indeed the case, not only for chiral anomalies but for all anomalies. Secondly, the above method of regulating the fermion measure is only one of an infinite number of possible methods. For example, Fujikawa has himself shown [62] that one may replace the higher derivative regulator factor  $\exp[-(\not{D}/M)^2]$  with  $f[-(\not{D}/M)^2]$  where  $f(z)$  is any smooth function that rapidly approaches zero as  $z \rightarrow \infty$ . Alternatively, one can regularize the fermion determinant simply by adding a standard massive Pauli-Villars regulator field to the Lagrangian [62]. In both these cases the end result is the same. My last comment on the above derivation of the ABJ anomaly concerns two methods of regularization which were not used by Fujikawa, but which are nevertheless very popular. In place of a Fujikawa-style higher derivative regularization of the effective action, one may employ either the heat kernel method [63] or zeta function regularization [64]. In both these cases the details of the anomaly calculation differ from those of Fujikawa. However, in its general features, the manner in which the anomaly emerges from the fermion determinant is the same.

Many anomalies have now been calculated using path integral methods. These methods enjoy certain advantages and certain disadvantages with respect to the Feynman diagrammatic techniques described earlier. In general, Feynman diagrammatic methods are preferable when there is some doubt as to the validity of an assumption or a procedure. For instance, Fujikawa's assumption that the effective action is invariant under chiral transformations is reasonable, and was justified a posteriori since it led to the correct anomaly. However, one might justly view this assumption with scepticism had it not been verified in old-fashioned diagrammatic calculations. Similarly, the Adler-Bardeen theorem and the non-renormalizability of anomalies are no better understood in path integral formalism than they are diagrammatically. Consequently one is probably still better off viewing these effects from a

diagrammatic point of view. On the other hand, when one is on firm ground the compact nature of path integral formalism makes it far more attractive and easily manipulated than complicated Feynman diagrammatic expressions. For instance, the connection between anomalies and regularization is much more transparent, and consequently much more easily explored, in path integral formalism than it is in terms of Feynman diagrams. Likewise, our understanding of the relationship between anomalies and topological index theorems [21] is a product of path integral methods. In fact, the subject of topological index theorems is one area where the path integral formalism has contributed insights into anomalies that diagrammatic methods could not reproduce. I will therefore devote the final part of this subsection to a brief explanation of the connection between the ABJ anomaly and the relevant index theorem. In Fujikawa's derivation of the ABJ anomaly we encountered the quantity  $\mathfrak{I}(x)$

$$\mathfrak{I}(x) = \sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x)$$

The fields  $\psi_n(x)$  are eigenfunctions of the Dirac operator :  $i\not{D}\psi_n = \lambda_n \psi_n$ . Consider those eigenfunctions  $\psi_n(x)$  whose eigenvalues  $\lambda_n$  are non-zero. Because  $\gamma_5$  anticommutes with  $\not{D}$  we have

$$\gamma_5 \psi_n(x) = \psi_{-n}(x)$$

and it is not difficult to see that, as a consequence of this, the only eigenfunctions  $\psi_n$  which contribute to  $\mathfrak{I}(x)$  are those with zero eigenvalues. When  $\lambda_n = 0$  we can always choose things so that  $\psi_n$  is of positive or negative chirality

$$\gamma_5 \psi_n = \pm \psi_n$$

I will denote positive chirality eigenfunctions by  $\psi_{n+}$ , and negative chirality eigenfunctions by  $\psi_{n-}$ . If the numbers of these states are respectively  $N_+$  and  $N_-$  then the integral of  $\mathfrak{I}(x)$  over space-time is equal to

$$\int d^4x \mathfrak{I}(x) = \int d^4x \left[ \sum_{n=1}^{N_+} \psi_{n+}^\dagger(x) \psi_{n+}(x) - \sum_{n=1}^{N_-} \psi_{n-}^\dagger(x) \psi_{n-}(x) \right] = N_+ - N_-$$

That is, the integral of  $\iota(x)$  is equal to the number of positive chirality zero modes minus the number of negative chirality zero modes. However, an index theorem for the Dirac operator [21] tells us that the difference  $N_+ - N_-$  is equal to the integral  $(-1/16\pi^2) \int d^4x (F_{\mu\nu} \tilde{F}^{\mu\nu})$ . We are therefore free to identify  $\iota(x)$  with  $(-1/16\pi^2) F_{\mu\nu} \tilde{F}^{\mu\nu}$ , which leads us to the ABJ anomaly (46). Fujikawa's method thus provides a straightforward means of showing the intimate connection between the ABJ anomaly and the index theorem, and establishes a framework for understanding the anomaly in topological terms.

**DIFFERENTIAL GEOMETRIC METHODS** The methods which, for want of a better term, I have decided to classify as differential geometric, date from a 1971 paper by Wess and Zumino [10]. Consequently, they predate the path integral methods which were developed by Fujikawa and others. Despite this chronological primacy, I have decided to deal with differential geometric methods last because their significance was not fully understood, and their potential not fully realized, until quite recently. In fact we shall soon see that the methods considered below are closely related to our newly acquired theoretical understanding of anomalies in terms of cocycles. The material of this section will therefore bring us full circle back to the algebraic considerations of the beginning of the chapter, and will neatly wrap up the subject matter of sections 1, 2 and 3. All that then remains for me to do in section 4 is to quickly review some of the applications of anomalies.

Differential geometric methods of anomaly calculation all depend upon some sort of Wess-Zumino type consistency condition. In 1971 Wess and Zumino [10] observed that, because gauge anomalies are equal to the variation of the vacuum functional under a gauge transformation (see equation (5)), they must obey certain consistency conditions. These conditions are direct consequences of the assumption that the generators of gauge transformations obey a regular composition rule as in equation (4). As we have seen, this assumption may not be valid. In general, the gauge generators will follow an anomalous composition rule of the sort shown in equation (6). Notwithstanding this fact, the methods developed by Wess and Zumino have been successfully employed to deduce the structure of various anomalies, and one can only assume that the Schwinger terms which creep into relations such as (6) do not affect the Wess-Zumino procedure. Let us now see how one would "solve"

the consistency conditions in the modern differential geometric notation which has become a hallmark of Wess-Zumino type methods [48,65]. Suppose that in some anomalous non-Abelian gauge theory the gauge transformations are effected by operators  $U(g)$  where  $g$  is an element of the gauge group. According to equation (5) the gauge anomaly is given by

$$\omega(g,A) = U(g)\Gamma(A)$$

If, following Wess and Zumino [10,48,65], we assume that the operators  $U(g)$  compose regularly, as in equation (16), then we are led to the conclusion that the anomaly  $\omega(g,A)$  satisfies none other than the 1-cocycle condition (18). That is, the anomaly must be a 1-cocycle. The question then arises as to whether one can solve the 1-cocycle condition to find the form of  $\omega(g,A)$ . In their original paper [10], Wess and Zumino found a particular solution of the 1-cocycle condition for the gauge group  $SU(3) \times SU(3)$ . They were then able to use this solution as an effective Lagrangian for the strong interactions. Nowadays, the so-called "solution" of the 1-cocycle condition depends upon recognizing that one can find a suitable 1-cocycle by applying the descent process described in section 2 to a Cherns-Simon term. In general, the  $2n$ -dimensional 1-cocycle  $\omega_{2n}^1(g,A)$ , constructed from the  $(2n+2)$ -dimensional Chern-Pontryagin density  $P$  according to the abstract algorithm given in equations (42), is a solution of the  $2n$ -dimensional Wess-Zumino conditions. The problem of finding the anomaly therefore becomes one of determining the form of  $\omega_{2n}^1(g,A)$ . Zumino, Wu and Zee [48] have shown how to do this in arbitrary even space-time dimensions  $d=2n$ . For interest's sake I reproduce their answer.

$$\omega_{2n}^1(g,A) = -n(n+1) \int dt(1-t) \text{tr}[g dP(A, F_t^{n-1})] \quad ..(52)$$

Here  $F_t$  is the 2-form  $F_t = t dA + t^2 A^2$ , and  $P(\lambda_1, \dots, \lambda_n)$  is the symmetrized product of the Lie algebra matrices  $\lambda_1, \dots, \lambda_n$ .

$$P(\lambda_1, \dots, \lambda_n) = (1/n!) \sum_{(i_1, \dots, i_n)} \lambda_{i_1} \dots \lambda_{i_n}$$

Only two questions about this procedure remain to be answered. Firstly, it is clear that the 1-cocycle condition can only determine  $\omega_{2n}^1(g,A)$  up to an

overall constant. How, then, can one determine this constant and normalize  $\omega_{2n}^1(g,A)$ ? Zumino, Wu and Zee [48] have suggested that this could be done by calculating the simpler Abelian anomaly in  $2n$ -dimensions using a path integral or diagrammatic method. The normalization of the Abelian anomaly would then fix the normalization of the non-Abelian anomaly. The second question regarding the above procedure concerns its generality. Is  $\omega_{2n}^1(g,A)$ , as given by (52), the most general solution of the 1-cocycle condition? To the author's knowledge no resolution of this question has so far emerged, though Alvarez-Gaume and Ginsparg [21] have suggested that one might make some progress with this problem by looking at topological index theorems of the sort mentioned above.

#### 1.4 Applications

As I explained at the beginning, this chapter was never destined to be a catalogue of the applications of anomalies within quantum field theories. Instead, I intended to devote it to a description of the theory of anomalies without reference to any of their particular uses. However, no-one can look at this subject without being impressed by the large number of specific problems in which anomalous symmetry breaking effects play some part. I therefore decided to use this final section of the chapter to redress the imbalance in outlook of the earlier sections by briefly listing the main applications of anomalies in theoretical elementary particle physics. In this task I will rely heavily on Bardeen's recent review of anomalies [19]. The obvious place to start is with the corrections to the  $\pi_0 \rightarrow 2\gamma$  decay amplitude that followed from the original ABJ triangle anomaly. As we saw in section 1, Sutherland and Veltman used current algebra methods to calculate the  $\pi_0 \rightarrow 2\gamma$  amplitude in 1967 [26,27]. Their result differed from the experimental value, and the discrepancy was not resolved until Adler [8] and Bell and Jackiw [9] discovered the anomaly in the conservation equation for the axial current. The anomalous corrections to the  $\pi_0 \rightarrow 2\gamma$  amplitude which were generated by the ABJ anomaly brought theory into line with experiment and restored faith in current algebra techniques. Moreover, Adler and others [66,67,68] subsequently showed that, given the amplitude for  $\pi_0 \rightarrow 2\gamma$  decay, one can also determine the amplitudes for  $\gamma \rightarrow 3\pi$  and  $2\gamma \rightarrow 3\pi$  by appealing to



gauge and chiral invariances. Thus the discovery of the ABJ anomaly also gave us correct rates for the processes  $\gamma \rightarrow 3\pi$  and  $2\gamma \rightarrow 3\pi$ .

Anomalies possess important implications for gauge field theories. These may be summarized as follows. Firstly, the process by which a gauge theory is perturbatively renormalized [2,25] involves the use of the theory's Ward identities. If the Ward identities contain anomalous terms, the renormalizability and the unitarity of the theory are threatened [11,69,70,71]. So far the only way of dealing with an anomalous gauge theory has been to adjust the fermion content so that the anomalies vanish. This method has the recommendation that it leads to attractive constraints upon gauge, grand unified and string theories. In particular, it forms the only theoretical basis for the physically inspired requirement that the number of quarks and leptons be the same. These successes have still not prevented Fadeev [23] asking whether there are other subtler mechanisms available for dealing with anomalous gauge theories, or whether anomalous gauge theories actually have a consistent interpretation at a non-perturbative level. Other consequences of anomalies for gauge theories are best illustrated by the standard model [6,7]. In the standard model there are flavour currents which are associated with  $SU(6)$  flavour symmetry, and dynamical currents which are connected with gauged  $SU(3)$  colour symmetry. At a naive level these currents obey certain commutation relations, and in particular the flavour currents commute with the dynamical currents. However, the presence of anomalies affects these relations. The flavour current conservation equations contain anomalies involving the dynamical gauge fields. These anomalies imply proton decay [34,72], lead to the resolution of the  $U(1)$  problem in QCD [12], and determine the structure of axion couplings in models which solve the strong CP problem [13,73,74]. Similarly, flavour currents may have anomalies which bring in other flavour fields. In composite models such anomalies impose constraints upon the structure of bound states which are summarized by the 't Hooft conditions [14,75,76]. The 't Hooft conditions provide practically the only firm information known about most bound state structures.

In the previous section I described something of the Wess-Zumino (1-cocycle) consistency conditions that must be satisfied by gauge anomalies. I also mentioned that in general one can solve these conditions using constructs

based on cochains and cocycles. A functional of the gauge fields which identically satisfies the Wess-Zumino consistency conditions may be used as an effective Lagrangian governing gauge field interactions. In this capacity it will generate the many particle vertices which describe the low energy consequences of anomalous terms in the Ward identities. For example, by solving the consistency conditions for the gauge group  $SU(3) \times SU(3)$ , Wess and Zumino [10] were able to determine the effective action for soft pion interactions, including the anomalous contributions which govern processes such as  $\pi_0 \rightarrow 2\gamma$ . The anomalous parts of effective actions which are deduced in this way are called Wess-Zumino terms. The significance of these terms has been emphasized by Witten [47], and they have recently found applications in many problems including skyrmions and superstrings. Skyrmions date from the original work of Skyrme [77] who found that pseudoparticle solutions of the chiral meson field equations are stabilized by certain higher derivative "Skyrme" terms contained within the relevant effective action. Interest in these pseudoparticle solutions, or skyrmions, was renewed [78,79] by the discovery of their anomaly induced charge and spin [15,47]. In the realm of QCD skyrmion theory has led to the interpretation of the observed baryons as solitons of the meson field, and preliminary attempts to develop a realistic phenomenology for baryons along these lines have enjoyed considerable success [16]. I shall deal with the relevance of anomalies to superstrings shortly. Suffice it to say here that Wess-Zumino type terms play an important role in anomaly cancellation mechanisms in superstring theories [18].

Jackiw and Rebbi [42] were the first to observe that the topological structure of gauge fields can lead to anomalous fractional charge for currents affected by the anomaly. Charge fractionalization was predicted to occur [80,81] in certain excitations of real solid state systems such as polyacetylene, and these anomalous effects have indeed been observed. The nature of topologically generated fractional charge has now been analyzed using the methods of index theory and spectral flow [43]. Anomalies have also had an impact on our understanding of the dynamics of the monopole. For instance, Rubakov [82] and Callan [83] have demonstrated that, in V-A fermion theories, anomalies cause a breakdown in fermion number conservation in the presence of magnetic monopoles. The consequence of this effect is that monopoles can catalyse proton decay in grand unified theories [84].

These discoveries by Rubakov and Callan stimulated a wide variety of work on the interactions of fermions with monopoles, including the conservation laws which follow from the anomalies related to the topological structure of the monopole field [17].

The final application of anomalies that I wish to mention concerns superstring theories. Such theories are currently thought to be the closest thing we have to a unified model of all particle interactions. The relevance of anomalies to superstrings has been concisely summarized by Bardeen [19], and I can do no better here than to simply repeat his analysis. There are two sorts of superstring, which are referred to respectively as being of types I and II. Through the study of gauge and gravitational anomalies in higher dimensions, Alvarez-Gaume and Witten [59] found that supergravity models based on type II superstrings were free of anomalies while those based on superstring I theories contained both gauge and gravitational anomalies. Unfortunately, only the type I theories seemed to contain the rich gauge structure needed to reproduce the known particle phenomenology, even when Kaluza-Klein effects were taken into account in the reduction from the natural ten dimensions of these theories to the physical four dimensions. A careful examination of type I theories by Schwarz and Green [18] proved that the loop anomalies could be cancelled by the introduction of additional anomalous terms involving the partners of the graviton field. These terms are similar to the Wess-Zumino terms of the chiral models and are actually already contained in the correct treatment of the superstring theory. This delicate cancellation mechanism works only for the gauge group  $SO(32)$ . Hence the anomaly structure demands an essentially unique unified fundamental theory of gauge and gravitational interactions.

Actually, the supergravity theory allows just one other gauge group,  $E_8 \times E_8$ . Gross, Harvey, Martinec and Rohm [85] exploited this possibility and invented an entirely new closed string theory, the heterotic string, which could incorporate both the  $SO(32)$  and  $E_8 \times E_8$  gauge groups. The analysis of the effective low energy theories produced by these superstring theories has been the subject of intense study. Although this analysis is quite complex, the heterotic string seems to produce all the elements of a physically correct low energy particle phenomenology [86], as well as a finite theory

of all interactions even beyond the Planck scale. As anomalies have played a crucial role in developing these superstring theories, there is every reason to believe that they will continue to provide an essential tool in their analysis.

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## CHAPTER 2. The Spin 1/2 and Spin 3/2 Fields

In chapter 1 anomalies, and associated anomalous symmetry breaking effects, were examined from a general point of view. In the remainder of the thesis, however, we will be concerned exclusively with the calculation and analysis of chiral anomalies. I will therefore introduce the material of this chapter with a few comments on the nature of chiral symmetry and chiral anomalies. Consider the Lorentz group  $O(1,d-1)$  in  $d$ -dimensional space-time. If  $d$  is odd, then there is only one type of spinor representation for  $O(1,d-1)$ . By way of contrast, if  $d$  is even,  $O(1,d-1)$  possesses two types of spinor representation which are variously referred to as being of opposite parity or chirality, or as being left and right handed respectively. The parity transformation maps left and right handed spinors into each other; so, if an even-dimensional fermion theory is to conserve parity, it must contain equal numbers of left and right handed spinors. In the following work I will be dealing with spin 1/2 and spin 3/2 field theories that are constructed around Dirac spinor, and Rarita-Schwinger vector-spinor, fields. These fields contain spinors or vector-spinors of both chiralities. For example, a Dirac spinor field in even dimensions belongs to the representation  $(0,1/2)+(1/2,0)$  of the Lorentz group. The irreducible representations  $(0,1/2)$  and  $(1/2,0)$  contain left and right handed spinors respectively. Since the parity transformation maps left handed spinors into right handed spinors, and vice versa, the representation  $(0,1/2)+(1/2,0)$  becomes irreducible when parity is included in the theory.

In an even-dimensional space-time, one can project out the left and right handed parts of a spinor or vector-spinor,  $\psi$ , using the projection operators  $P_1 = (1/2)[1 + \Gamma^{-1}]$  and  $P_2 = (1/2)[1 - \Gamma^{-1}]$ . Thus  $\psi = P_1\psi + P_2\psi$ . In these expressions,  $\Gamma^{-1}$  is the element of the  $d$ -dimensional Dirac gamma matrix algebra that corresponds to the four-dimensional matrix  $\gamma_5$ . The property  $(\Gamma^{-1})^2 = 1$  ensures that  $P_1^2 = P_1$ ,  $P_2^2 = P_2$ , and  $P_1P_2 = 0$ . Many familiar fermion theories are classically invariant under the following symmetry transformation :  $\psi \rightarrow \exp(i\theta\Gamma^{-1})\psi$ . Note that, in its infinitesimal form, this transformation changes the sign of the left handed part of  $\psi$ :  $\psi = P_1\psi + P_2\psi$ ,  $\Gamma^{-1}\psi = P_1\psi - P_2\psi$ . The corresponding symmetry is called chiral symmetry, and



one aspect of its anomalous breakdown is the appearance of an anomaly in the conservation equation for the associated chiral current. It is with the calculation of some of these chiral anomalies that I will be concerned in chapters 3 and 4. Note that, since chiral symmetry exists only in even-dimensional space-times, I lose no generality, in this chapter's review of spin 1/2 and spin 3/2 field theory, by working in a space-time whose "base" dimension is  $d=2n$ . Later in the thesis I will be using dimensional regularization, which will necessitate the analytic continuation of  $d$  away from its base value  $2n$ . When this is necessary, I will adopt the convention that  $d=2n$  is analytically continued to  $d=2\ell$ . In anticipation of the analytic continuation procedure, many of the results of this chapter are expressed in forms appropriate to a  $2\ell$ -dimensional, rather than a  $2n$ -dimensional, space-time.

## 2.1 The Spin 1/2 Field

The Lagrangian for a free massive spin 1/2 field is

$$L(\psi) = \bar{\psi}(i\partial - m)\psi \quad ..(1)$$

Variation of  $\bar{\psi}$  leads to the Euler-Lagrange equation of motion for  $\psi$

$$(i\partial - m)\psi = 0 \quad ..(2)$$

Canonical quantization of this theory is achieved by demanding that  $\psi$  satisfy the anticommutator relation [1]

$$\{\psi_{\xi}(t, \mathbf{x}), \psi_{\eta}^{\dagger}(t, \mathbf{y})\} = \delta_{\xi\eta} \delta^3(\mathbf{x} - \mathbf{y})$$

From (2) we deduce that the momentum space propagator for the quantum field  $\psi$  is given by

$$S(p) = \frac{1}{\not{p} - m} \quad ..(3)$$

As we are specifically interested in the contribution of the gravitational field to the spin 1/2 chiral anomaly we must investigate the coupling of the

gravitational field to  $\psi$ . It is well known that the form of this coupling is dictated by the gauge principle. The Lagrangian (1) is invariant under the following global Lorentz  $O(1,2n-1)$  transformation

$$\psi(x) \rightarrow e^{\omega \cdot \sigma / 2} \psi(x)$$

Here  $\omega \cdot \sigma \equiv \omega_{\alpha\beta} \sigma^{\alpha\beta}$  and  $\sigma^{\alpha\beta}$  is, of course, the antisymmetric product of two Dirac gamma matrices

$$\sigma^{\alpha\beta} = \frac{1}{4} [\gamma^\alpha, \gamma^\beta] \equiv \frac{1}{2} \gamma^{[\alpha} \gamma^{\beta]}$$

The constants  $\omega_{\alpha\beta}$  parametrize transformations within the Lorentz group. It is worth noting in passing that according to Noether's theorems [2] any invariance of a Lagrangian field theory under a continuous symmetry group is associated with conserved charges. In particular if the theory is invariant under an  $r$ -parameter Lie group of transformations there will be precisely  $r$  conserved charges. The Lorentz group  $O(1,2n-1)$  is parametrized by  $n(n-1)/2$  parameters and is therefore associated with  $n(n-1)/2$  conserved charges. These charges are the various components of angular momentum.

The gauge field of the Lorentz group is the gravitational (vielbein) field. The gauge principle tells us that its coupling to  $\psi$  is determined by requiring that the Lagrangian (1) remain invariant under the above transformation when the parameters  $\omega_{\alpha\beta}$  are allowed an arbitrary dependence on the space-time coordinates  $x$ . This coupling prescription leads to the following gravitationally covariant Lagrangian for  $\psi$ .

$$L(\psi, e) = -e \bar{\psi} (i \not{D} - m) \psi \quad \dots(4)$$

In this equation  $e$  is the inverse of the vielbein determinant :

$$e = (\det e^{\alpha\mu})^{-1}$$

The covariant derivative  $D_\rho$  is given by [3]

$$D_\rho \psi = e_\rho^\mu \left[ \partial_\mu + \frac{1}{2} \omega_{\mu\gamma\delta} \sigma^{\gamma\delta} \right] \psi \quad \dots(5)$$

where

$$\omega_{\mu\gamma\delta} = e_{\delta}^{\sigma} [e_{\gamma\lambda} \Gamma^{\lambda}_{\sigma\mu} - e_{\gamma\sigma,\mu}]$$

and

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} [g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa}]$$

As usual the space-time metric  $g_{\mu\nu}$  is related to the vielbein  $e_{\rho\mu}$  according to the relation  $g_{\mu\nu} = e_{\rho\mu} e^{\rho}_{\nu}$ . If space-time is flat then  $e_{\rho\mu} = \eta_{\rho\mu}$  and  $g_{\mu\nu} = \eta_{\mu\nu}$ . In a quantum theory of gravity flat space-time represents the vacuum and one would therefore expect that in flat space-time quantum fields would have zero expectation values. This is not the case with  $e_{\rho\mu}$  and in order to work with a field whose flat space expectation value is zero one usually decomposes the vielbein into its flat space expectation value  $\eta_{\rho\mu}$  and the quantum field  $h_{\rho\mu}$  [4]. The gravitational coupling constant  $\kappa$  features in the equation relating  $e_{\rho\mu}$  and  $h_{\rho\mu}$  :

$$e_{\rho\mu} = \eta_{\rho\mu} + \kappa h_{\rho\mu} \quad ..(6)$$

This decomposition of the vielbein corresponds to a change in the way space-time is treated mathematically. The indices carried by the field  $e_{\rho\mu}$  are different in character. One of them,  $\mu$ , is a curved space index while the other,  $\rho$ , is an internal Lorentz flat space index. Hence the vielbein  $e_{\rho\mu}$  possesses a dual curved space/flat space nature. On the other hand the quantities  $\eta_{\alpha\mu}$  and  $h_{\alpha\mu}$  clearly do not transform as curved space tensors in the index  $\mu$ . The decomposition (6) signifies that, at least locally, one has ceased to treat space-time as intrinsically curved. Instead, to facilitate the usual sort of perturbation expansion in the field operators, one has elected to regard space-time as being flat and the field  $h$  as being a small disturbance on this flat background. In this way the theory becomes accessible to the familiar methods of perturbation analysis and the field  $h_{\rho\mu}$  may be treated in the same manner as any other quantum field. In a weak field expansion in the coupling constant  $\kappa$  the curved space nature of the index  $\mu$  is naturally obscured and the indices  $\rho$  and  $\mu$  are effectively treated on the same footing as flat space indices. I will adopt equation (6)

as my definition of  $h_{\rho\mu}$  and adhere to the convention that indices on  $h_{\rho\mu}$  are raised and lowered using  $\eta$ 's. Thus  $h_{\mu}^{\rho} = \eta^{\rho\sigma} h_{\sigma\mu}$  etc.. It should be noted in this context that the field  $h_{\rho\mu}$  is determined only up to its invariance under local Lorentz transformations. This freedom can always be used to make  $h_{\rho\mu}$  symmetric in  $\rho$  and  $\mu$  and in the following work I will assume that this symmetrization has been effected.

Let us now expand the covariant Lagrangian (4) in powers of the coupling constant  $\kappa$ . It will become apparent in the next chapter that to calculate the gravitational contribution to the spin 1/2 chiral anomaly we need consider only Feynman diagrams containing external gravitational fields. It can therefore be assumed that  $h_{\rho\mu}$  satisfies the free field equations of motion  $\partial^{\rho} h_{\rho\mu} = h^{\rho}_{\rho} = 0$ . What is more, the diagrams in question involve only first order interaction vertices in  $\psi$  and  $h$ . Consequently it suffices to expand  $L$  up to first order  $\kappa$ . One finds that

$$L = L_0 + \kappa L_1 + O(\kappa^2) \quad \dots(7)$$

where  $L_0$  is the free Lagrangian of equation (1) and  $L_1$  is the first order part of the interaction Lagrangian. Using the results  $\partial^{\rho} h_{\rho\mu} = h^{\rho}_{\rho} = 0$  one can show that

$$L_1 = i h_{\rho\mu} \bar{\psi} \gamma^{\rho} \partial^{\mu} \psi$$

From this expression one can deduce the first order momentum space gravitational vertex. It is  $h_{\rho\mu}(k) \bar{\psi}(p) V^{\rho\mu}(p,k) \psi(p+k)$  where

$$V^{\rho\mu}(p,k) = i \kappa p^{\rho} \gamma^{\mu} \quad \dots(8)$$

Clearly, in view of  $h$ 's symmetry in its indices, only that part of  $V^{\rho\mu}$  which is symmetric in  $\rho$  and  $\mu$  will have any significance. However, rather than complicate the expression for  $V^{\rho\mu}$  by indicating this explicitly I have left the vertex in the simple form of equation (8). I will now finish this brief review of the theory of a spin 1/2 field with a derivation of the spin 1/2 chiral current conservation equation. As was explained above, Noether's theorems [2] tell us that if a Lagrangian is invariant under an  $r$ -parameter

Lie group of field transformations  $G_r$  then there will exist  $r$  conserved currents in the theory. To be specific there will be  $r$  linearly independent combinations  $J_\mu^a$  ( $a=1,\dots,r$ ) of the fields and their derivatives such that  $\partial^\mu J_\mu^a = 0$ . If the invariance under  $G_r$  is a global invariance the currents  $J_\mu^a$  will be conserved only on extremals of the Lagrangian, that is only on solutions of the equations of motion. On the other hand if the invariance under  $G_r$  is a local invariance the currents  $J_\mu^a$  will be conserved regardless of whether the fields satisfy their equations of motion or not. Suppose now that we have a Lagrangian depending on a set of independent fields  $\phi$  and on their derivatives  $\partial_\mu \phi \equiv \phi_\mu$ . Then the canonical momenta are defined as follows

$$\pi^\mu = \frac{\delta L}{\delta \phi_\mu}$$

and the Euler-Lagrange equations of motion for the theory are

$$\partial_\mu \pi^\mu = \frac{\delta L}{\delta \phi} \quad \dots(9)$$

If the Lagrangian  $L$  is invariant under a group of transformations, an infinitesimal element of which can be written as

$$\phi(x) \rightarrow \phi(x) + \delta\phi(x) \quad \dots(10)$$

then we have

$$\delta L = \frac{\delta L}{\delta \phi} \delta\phi + \frac{\delta L}{\delta \phi_\mu} \delta\phi_\mu = \frac{\delta L}{\delta \phi} \delta\phi + \pi_\mu \partial^\mu \delta\phi = 0 \quad \dots(11)$$

and the equations of motion (9) imply that

$$\partial_\mu [\pi^\mu \delta\phi] = 0$$

Hence Noether's conserved currents are given by

$$J_\mu = \pi_\mu \delta\phi \quad \dots(12)$$

where the explicit form of  $\delta\phi$  is assumed known. It may happen that the invariance of the theory under the transformations (10) is imperfect. For

instance some part of the Lagrangian might break the symmetry. In this case Noether's currents are no longer conserved under all circumstances. However, it is possible to deduce the circumstances under which they are conserved simply by inverting equations (11). In this way one arrives at the following conservation equations for the currents (12).

$$\partial_\mu J^\mu = \delta L \quad \dots(13)$$

Now let us apply these results in the case of the spin 1/2 field  $\psi$ . Chiral invariance of the above spin 1/2 theory is a global U(1) invariance. A finite chiral transformation of  $\psi$  takes the form

$$\psi(x) \rightarrow e^{i\theta\Gamma^{-1}} \psi(x)$$

where  $\theta$  is a space-time independent parameter, and  $\Gamma^{-1}$  is that element of the Dirac gamma matrix algebra in 2n dimensions corresponding to the four dimensional matrix  $\gamma_5$

$$\Gamma^{-1} = i^{n+1} \gamma_0 \gamma_1 \dots \gamma_{2n-1} \quad \dots(14)$$

Note that  $\Gamma^{-1}$  is hermitian. (My conventions are explained in appendix 1.) The infinitesimal version of this chiral transformation is

$$\psi(x) \rightarrow \psi(x) + i\theta\Gamma^{-1} \psi(x)$$

Applying the above formulae in this instance we deduce that the chiral current is

$$J_\mu^{1/2} = \bar{\psi} \gamma_\mu \Gamma^{-1} \psi \quad \dots(15)$$

Chiral invariance is not an exact invariance of the Lagrangian (4). It is broken by the mass term in L. Using (13) we find

$$\partial^\mu [e J_\mu^{1/2}] = e D^\mu J_\mu^{1/2} = 2iem \bar{\psi} \Gamma^{-1} \psi \quad \dots(16)$$

This is the chiral current conservation equation in the classical theory of a spin 1/2 field. It would be quite understandable were one to assume that the same equation holds as an operator identity in the corresponding quantum theory. After all equation (16) was derived using only equation (13), and equation (13) in turn follows directly from the equations of motion. Since the equations of motion are valid operator identities in the quantum spin 1/2 theory why should one suspect the validity of (16)?

However in reasoning this way one is overlooking an important aspect of quantum field theories. As was emphasized in chapter 1, a quantum field theory is not completely determined by a Lagrangian alone. In general such theories are beset by infinities and in order to deal with the infinities one has to apply some form of regularization. A theory will not be completely determined until both the Lagrangian and regularization scheme are specified. These ideas may be related to the case at hand as follows. When one passes from the classical to the quantum spin 1/2 theory the equations of motion remain valid when reinterpreted as operator equations. This is because they contain only finite quantities. On the other hand equations (13) and (16) involve fermion bilinears such as  $\bar{\psi}(x)\gamma_{\mu}\Gamma^{-1}\psi(x)$ . These bilinears are infinite and must be regularized. When a gauge invariant, chirally non-invariant regularization is applied to the theory the naive manipulations which led to equations (13) and (16) are no longer valid and an extra term, the anomaly, appears on the right hand side of (16). For example if Pauli-Villars regularization is used the contribution of the massive regulator fermion to equation (16) survives in the limit as the regulator mass tends to infinity and becomes the anomaly. In this way the correct quantum version of equation (16) is

$$\partial^{\mu}[e J_{\mu}^{1/2}] = 2ie m \bar{\psi}\Gamma^{-1}\psi + A^{1/2} \quad \dots(17)$$

One might object that the anomaly would be absent were one to use a regularization scheme that was both gauge and chirally invariant. However matters are not quite this simple. As was pointed out in chapter 1, the presence of an anomaly signifies that no regularization scheme will respect all the classical symmetries that are present in the theory. Consequently no regularization scheme is ever entirely satisfactory and the anomaly cannot

be removed or avoided. On the other hand by appropriately selecting one's regularization scheme one has the freedom to choose which symmetry of the classical theory is broken in the quantum theory. Vector gauge invariance is customarily regarded as being more important than chiral invariance. Consequently the above spin 1/2 theory is conventionally regularized using a gauge invariant, chirally non-invariant regularization scheme, and the anomaly  $A^{1/2}$  therefore appears in the chiral current conservation equation, not the gauge current conservation equation. When I calculate the spin 1/2 and spin 3/2 chiral anomalies in chapters 3 and 4 I will use dimensional regularization, a regularization scheme which is conventional in the sense that it preserves gauge invariance at the expense of chiral invariance.

One final comment is in order on the subject of the spin 1/2 chiral anomaly. Suppose that we wish to regularize an anomalous quantum field theory. If two regularization schemes respect exactly the same subset of invariances of the classical field theory then they will lead to identical anomalies. This is true for example of Pauli-Villars and dimensional regularization when applied to the spin 1/2 theory which is presently under consideration. Consequently it is not difficult to see that the chiral anomaly  $A^{1/2}$  is independent of the mass  $m$  of the spin 1/2 field  $\psi$ . This conclusion follows directly from the fact that under the Pauli-Villars scheme the anomaly arises out of regulator contributions and there is no way at all that  $m$  can enter into it. If  $A^{1/2}$  is independent of  $m$  under Pauli-Villars regularization then it must also be independent of  $m$  if the spin 1/2 theory is regularized using dimensional regularization. We are therefore free to set  $m=0$  when calculating  $A^{1/2}$  using dimensional regularization.

## 2.2 The Spin 3/2 Field

The Lagrangian theory of a classical spin 3/2 field is considerably more complicated than that of a spin 1/2 field. This is principally because the Rarita-Schwinger tensor-spinor field  $\psi_\alpha$  which is used to describe a spin 3/2 particle [5], carries not just a single spin 3/2 representation of the Lorentz group, but two spin 1/2 representations as well. In contrast to the Dirac spinor the Rarita-Schwinger tensor-spinor does not therefore form an



irreducible representation of the Lorentz group. The spin 1/2 and spin 3/2 parts of  $\psi_\alpha$  may be separated out using the projection operators  $\theta_{\alpha\beta}, P_\alpha, Q_\alpha$  [6] which of course satisfy a completeness relation  $\theta_{\alpha\beta} + P_\alpha P_\beta + Q_\alpha Q_\beta = \eta_{\alpha\beta}$ . Thus

$$\psi_\alpha = \theta_{\alpha\beta} \psi^\beta + P_\alpha P_\beta \psi^\beta + Q_\alpha Q_\beta \psi^\beta$$

$\theta_{\alpha\beta}$ , the projection operator for the spin 3/2 part of  $\psi_\alpha$ , is given in  $2\ell$ -dimensional momentum space by the expression

$$\begin{aligned} \theta_{\alpha\beta}(p) &= (\eta^{\alpha\beta} - \frac{1}{2\ell} \gamma^\alpha \gamma^\beta) + (\frac{2\ell}{1-2\ell}) (\frac{p^\alpha}{p} - \frac{\gamma^\alpha}{2\ell}) (\frac{p^\beta}{p} - \frac{\gamma^\beta}{2\ell}) \\ &= (\eta^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}) + (\frac{1}{1-2\ell}) (\frac{p^\alpha}{p} - \gamma^\alpha) (\frac{p^\beta}{p} - \gamma^\beta) \end{aligned}$$

This operator possesses the following properties

$$\gamma^\alpha \theta_{\alpha\beta} = \theta_{\alpha\beta} \gamma^\beta = p^\alpha \theta_{\alpha\beta} = \theta_{\alpha\beta} p^\beta = 0 \quad \dots(18)$$

$$p^\alpha \theta_{\alpha\beta} = \theta_{\alpha\beta} p^\beta$$

The projection operators  $P_\alpha$  and  $Q_\alpha$  for the spin 1/2 parts of  $\psi_\alpha$  are somewhat arbitrary [6]. They may be chosen to be any two linear combinations of  $p_\alpha$  and  $\gamma_\alpha$  which satisfy the conditions  $P \cdot Q = 0$ ,  $P \cdot P = Q \cdot Q = 1$ . Here I am using an abbreviated notation in which for example  $P \cdot Q \equiv P_\alpha Q^\alpha$ . Obviously, in view of (18), any two such operators will also satisfy the relations  $P \cdot \theta = \theta \cdot P = Q \cdot \theta = \theta \cdot Q = 0$ . Convenient choices for  $P_\alpha$  and  $Q_\alpha$  will be nominated below. In the following text I will suppress the indices on  $\psi_\alpha, P_\alpha, Q_\alpha$  and  $\theta_{\alpha\beta}$  unless this is likely to cause confusion.

There is no unique first order, hermitian Lagrangian for a spin 3/2 field. In fact Fronsdal [6] and Moldauer and Case [7] have shown that in four dimensions there is a one-parameter family of such Lagrangians. Appendix 2 contains a slight modification of Fronsdal's proof of this fact which reveals that the same is true in arbitrary dimension. In particular, in

dimension  $d=2\ell$  the one-parameter family of Lagrangians for a massive spin 3/2 field  $\psi$  is

$$L(\psi, A) = \bar{\psi}_\alpha [\eta^{\alpha\beta} (i\partial_\beta - m) + iA(\partial^\alpha \gamma^\beta + \gamma^\alpha \partial^\beta) + (\frac{1}{2\ell-2})((2\ell-1)A^2 + 2A + 1)\gamma^\alpha \partial^\beta \gamma^\beta] \psi_\beta \\ + \frac{m}{(2\ell-2)^2} (2\ell(2\ell-1)A^2 + 4(2\ell-1)A + 2\ell)\gamma^\alpha \gamma^\beta \psi_\beta$$

Massive spin 3/2 fields at present have no application in elementary particle physics. There is no experimental evidence to suggest that they exist in the elementary particle spectrum in nature, and no theoretical reason for supposing that they will one day be discovered there. In fact the quantum description of massive spin 3/2 particles would pose serious problems for Lagrangian field theory as massive spin 3/2 field theories are nonrenormalizable [8]. (Of course, a spin 3/2 particle might have dynamically generated mass, but this is a different question altogether.) On the other hand while massless spin 3/2 particles have likewise not been observed, supersymmetric theories [9] strongly suggest that they exist in nature as supersymmetric partners to gravitons, and consistent theories of interacting massless spin 3/2 fields are available in the guise of supergravity theories [10]. For these reasons I will restrict my attention from now on to massless spin 3/2 fields. The spin 3/2 anomaly like its spin 1/2 counterpart is in any case a mass independent effect and no generality is lost in adopting this restriction. The Lagrangians for the massless fields are the  $m \rightarrow 0$  limits of the above massive Lagrangians.

$$L(\psi, A) = i\bar{\psi}_\alpha [\eta^{\alpha\beta} \partial_\beta + A(\partial^\alpha \gamma^\beta + \gamma^\alpha \partial^\beta) + (\frac{1}{2\ell-2})((2\ell-1)A^2 + 2A + 1)\gamma^\alpha \partial^\beta \gamma^\beta] \psi_\beta \quad ..(19)$$

Classical massless, as opposed to massive, spin 3/2 field theories are complicated by the existence of gauge invariance. The Lagrangian (19) enjoys invariance under the gauge transformation

$$\psi_\alpha \rightarrow \psi_\alpha + [2(1+A)\partial_\alpha - (1+A)\gamma_\alpha \partial^\beta] \Lambda \quad ..(20)$$

where  $\Lambda(x)$  is an arbitrary spin 1/2 field. As I shall now show, this gauge invariance is related to the presence in  $\psi$  of the two spin 1/2 represent-

ations  $P \cdot \psi$  and  $Q \cdot \psi$ . Let us define the projection operators  $P$  and  $Q$  in momentum space as follows

$$P^\alpha = \sigma_P \left[ (2\ell-2) \frac{p^\alpha}{p} + ((2\ell-1)A+1) \gamma^\alpha \right] = \sigma_P \left[ (2\ell-2) \frac{p^\alpha}{p} + (2(1+\ell A) - (1+A)) \gamma^\alpha \right]$$

$$Q^\alpha = \sigma_Q \left[ 2(1+\ell A) \frac{p^\alpha}{p} - (1+A) \gamma^\alpha \right]$$

where

$$\sigma_Q = [2\ell+4(2\ell-1)A+2\ell(2\ell-1)A^2]^{-1/2}$$

and  $\sigma_P = (2\ell-1)^{-1/2} \sigma_Q$ . In momentum space the Lagrangian (19) can then be written

$$L = \bar{\psi}_\alpha [\not{p} \theta^{\alpha\beta} + \mu P^\alpha \not{p} P^\beta] \psi_\beta$$

where  $\mu = 1/[(2\ell-2)\sigma_Q^2]$ . The gauge transformation (20) likewise becomes

$$\psi_\alpha \rightarrow \psi_\alpha + Q_\alpha \Lambda \quad \dots(21)$$

This formulation of the  $A$ -dependent gauge transformation (20) emphasizes that the gauge invariance of the theory is just invariance under redefinition of that particular spin  $1/2$  component of  $\psi$  which does not figure in the Lagrangian, namely  $Q \cdot \psi$ . Consider now what happens when the massless Lagrangian (19) is reexpressed in terms of a new field  $\psi^-$  defined in momentum space by

$$\psi^{-\alpha} = \psi^\alpha + \left[ a \frac{p^\alpha}{p} + b \gamma^\alpha \right] \gamma \cdot \psi \quad \dots(22)$$

Here the numbers  $a$  and  $b$  are arbitrary real parameters and, of course,  $\gamma \cdot \psi \equiv \gamma^\alpha \psi_\alpha$ . As may be checked using the projection operators, the field  $\psi^-$  differs from  $\psi$  only in its spin  $1/2$  components. What effect would a change of variables such as (22) have upon the spin  $3/2$  theory? Any theory which satisfactorily describes spin  $3/2$  particles in terms of the Rarita-Schwinger tensor-spinor  $\psi$  will be such that the spin  $1/2$  components of  $\psi$  disappear

from the dynamics and have no effect upon physical quantities. Consequently, provided that the spin 3/2 theory described above is satisfactory in this sense, it should make no difference to the calculation of physical quantities such as the anomaly whether the field in terms of which the spin 3/2 theory is expressed is  $\psi$  or  $\psi'$ . I will assume in what follows that the spin 3/2 theory outlined above is indeed satisfactory from this point of view, and exploit the consequent freedom that this gives me to reexpress the theory via field redefinitions of the form (22). Note that these field redefinitions involve the nonlocal operator  $1/\not{p}$ .

Since the field redefinitions (22) alter the spin 1/2 components, but not the spin 3/2 component, of  $\psi$  it is perhaps not surprising that their effect on the Lagrangian (19) is to shift the parameter A. That is, changes of variables of the form (22) are completely equivalent to changes in the parameter A. In particular, even though if  $a \neq 0$  the transformation (22) involves the non-local operator  $1/\not{p}$ , the transformed Lagrangian is still local. For arbitrary A, a and b one finds

$$L(\psi, A) = L(\psi', A') \quad A' = \left( \frac{A-a-2b}{1+a+2\ell b} \right) \quad ..(23)$$

This relation tells us how to reexpress the A-dependent spin 3/2 theory under the change of variable (22). I will be interested below in formulations of the spin 3/2 theory corresponding to three particular choices of the parameter A :  $A=-1$ , 0 and  $-1/\ell$ . Conventional spin 3/2 theory is associated with the Rarita-Schwinger Lagrangian [5]. It corresponds to the choice  $A=-1$ :

$$L_{RS}(\psi) = i\bar{\psi}_\alpha [\eta^{\alpha\beta} \not{p} - (\partial^\alpha \gamma^\beta + \gamma^\alpha \partial^\beta) + \gamma^\alpha \not{p} \gamma^\beta] \psi_\beta = i\bar{\psi}_\alpha \gamma^{[\alpha \rho} \gamma^{\rho \beta]} \partial_\rho \psi_\beta \quad ..(24)$$

With this choice of A the gauge invariance is the familiar one

$$\psi_\alpha \rightarrow \psi_\alpha + \partial_\alpha \Lambda \quad ..(25)$$

Equation (23) assures us that the other two choices for A,  $A=0$  and  $A=-1/\ell$ , can both be reached (as can any other value of A) from  $A=-1$  using field redefinitions of the form (22). Consequently, for the reasons outlined

above, the anomaly will be the same whether calculated in the  $A=-1$ ,  $A=0$  or  $A=-1/\ell$  formulation of the theory. However, although the value of the anomaly will be the same, the way in which it emerges from the mathematics will be different in each case, and the freedom I have to choose the value of  $A$  can be exploited to advantage. As will become clear below, the calculation of the spin  $3/2$  anomaly is particularly simple in the  $A=0$  formulation of the theory, while in the  $A=-1/\ell$  formulation its gauge independence is manifest. I will now consider the gauge fixing and quantization of the spin  $3/2$  theory in the Rarita-Schwinger formulation. Then, because the two choices  $A=0$  and  $A=-1/\ell$  are so useful, I will devote the final section of this chapter to a brief look at the Lagrangians, propagators and vertices in each of these two cases.

Just as the classical theory of a spin  $3/2$  field is more complicated than its spin  $1/2$  counterpart, the quantization of a spin  $3/2$  field compared to that of a spin  $1/2$  field is relatively involved [11,12,13]. In order to quantize the spin  $3/2$  field theory described by the gauge invariant Rarita-Schwinger Lagrangian it is first of all necessary to add to the Lagrangian a gauge fixing term and a corresponding ghost term. Once this is done the quantization procedure itself is complicated by the presence of constraints within the spin  $3/2$  theory. Fronsdal and Hata have explained how to overcome these difficulties using the convenient B-field formalism developed by Nakanishi [14,15]. I choose to fix the gauge of the Rarita-Schwinger field in the customary way by adding to the Lagrangian (24) the gauge fixing term

$$L_{GF} = -\frac{1}{\alpha} \bar{\psi} \cdot \gamma \beta \gamma \cdot \psi \quad ..(26)$$

where  $\alpha$  is an arbitrary gauge parameter. Using Nakanishi's B-field, which plays the part of a Lagrange multiplier and in this case is a Dirac field obeying Fermi statistics, the gauge-fixing term may be rewritten

$$L_{GF} = \bar{B} \beta \gamma \cdot \psi - \bar{\psi} \cdot \gamma \beta B + i\alpha \bar{B} \beta B \quad ..(27)$$

This expression for  $L_{GF}$  can be shown to be equivalent to (26) simply by using the equations of motion for  $B$  to eliminate it from (27). The Fadeev-Popov ghost term corresponding to  $L_{GF}$  has been shown by Hata and Kugo [12] (see also Fronsdal and Hata [11]) to be

$$L_{FP} = i[\bar{c}\partial^2 c_* - \bar{c}_* \partial^2 c]$$

where  $c$  and  $c_*$  are two Dirac ghost fields obeying Bose statistics. In this formalism Hata and Kugo [12] went on to derive the appropriate four-dimensional canonical equal time anti-commutation relations for the fields  $\psi$ ,  $B$ ,  $c$  and  $c_*$ . I reproduce their results here for the sake of completeness.

$$\{\psi_\alpha(t, \mathbf{x}), \bar{\psi}_\beta(t, \mathbf{y})\} = \frac{1}{2} \gamma_\beta \gamma^0 \gamma_\alpha \delta^3(\mathbf{x}-\mathbf{y}) \quad \{B(t, \mathbf{x}), \bar{B}(t, \mathbf{y})\} = 0$$

$$\{\psi_\alpha(t, \mathbf{x}), \bar{B}(t, \mathbf{y})\} = i\eta_{\alpha 0} \delta^3(\mathbf{x}-\mathbf{y}) \quad \{\dot{c}(t, \mathbf{x}), \bar{c}_*(t, \mathbf{y})\} = i\delta^3(\mathbf{x}-\mathbf{y})$$

The total Lagrangian for this set of fields  $L = L_{RS} + L_{GF} + L_{FP}$  is given by

$$L = i\bar{\psi}_\alpha \gamma^{[\alpha} \gamma^\rho \gamma^{\beta]} \partial_\rho \psi_\beta + \bar{B} \not{\partial} \gamma \cdot \psi - \bar{\psi} \cdot \gamma \not{\partial} B + i\alpha \bar{B} \not{\partial} B + i[\bar{c}\partial^2 c_* - \bar{c}_* \partial^2 c]$$

This Lagrangian may be simplified by means of a device suggested by Endo and Kimura [16]. They rewrote the gauge fixing terms in  $L$  as follows

$$\bar{B} \not{\partial} \gamma \cdot \psi - \bar{\psi} \cdot \gamma \not{\partial} B + i\alpha \bar{B} \not{\partial} B = i\alpha \left[ \bar{B} + \frac{1}{\alpha} \bar{\psi} \cdot \gamma \right] \not{\partial} \left[ B - \frac{1}{\alpha} \gamma \cdot \psi \right] - \frac{1}{\alpha} \bar{\psi} \cdot \gamma \not{\partial} \gamma \cdot \psi$$

then defined a new field  $F$

$$F = \sqrt{\alpha} \left[ B - \frac{1}{\alpha} \gamma \cdot \psi \right]$$

in terms of which the total Lagrangian may be expressed as

$$L = i\bar{\psi}_\alpha \gamma^{[\alpha} \gamma^\rho \gamma^{\beta]} \partial_\rho \psi_\beta - \frac{1}{\alpha} \bar{\psi} \cdot \gamma \not{\partial} \gamma \cdot \psi + i\bar{F} \not{\partial} F + i[\bar{c}\partial^2 c_* - \bar{c}_* \partial^2 c] \quad ..(28)$$

This form of the gauge fixed Rarita-Schwinger Lagrangian is the one with which I will be working in the remainder of this section. Bear in mind that it corresponds to the choice  $A=-1$ . In the following section I will be

dealing with Lagrangians corresponding to the choices  $A=0$  and  $A=-1/\ell$ . These Lagrangians may be derived from (28) by transforming the purely  $\psi$  part of  $L$  under field redefinitions of the form (22) and using formula (23). Note that in all three of these Lagrangians the gauge fixing and ghost terms in  $F$ ,  $c$ , and  $c_*$  are the same.

There are several virtues associated with the structure of the gauge fixing and ghost parts of the Lagrangian (28). Firstly, as we shall see below, Hata and Kugo's ghosts  $c$  and  $c_*$  contribute equal but opposite terms to the spin  $3/2$  chiral anomaly. Therefore when calculating the anomaly using Hata and Kugo's formalism one can ignore ghost contributions. Secondly, Endo and Kimura's trick of rewriting the gauge fixing parts of  $L$  in terms of the field  $F$  neatly eliminates from the Lagrangian interaction terms in  $\psi$  and  $B$  and replaces them with a free field Lagrangian for  $F$ . Because  $\psi$  and  $F$  are non-interacting their contributions to the anomaly may be calculated independently. The field  $F$ 's contribution is merely that of a spin  $1/2$  field, while  $\psi$ 's may be found by considering the spin  $3/2$  part of  $L$  alone:

$$\begin{aligned} L_\psi &= i\bar{\psi}_\alpha \gamma^{[\alpha} \gamma^\rho \gamma^{\beta]} \partial_\rho \psi_\beta - \frac{1}{\alpha} \bar{\psi} \cdot \gamma \not{\partial} \gamma \cdot \psi \\ &= i\bar{\psi}_\alpha \left[ \eta^{\alpha\beta} \not{\partial} - (\gamma^\alpha \not{\partial} + \not{\partial} \gamma^\alpha) + \left(1 - \frac{1}{\alpha}\right) \gamma^\alpha \not{\partial} \gamma^\beta \right] \psi_\beta \end{aligned} \quad ..(29)$$

From (29) one can deduce the momentum space propagator for  $\psi$ . It is

$$S_\psi^{\alpha\beta}(p) = i \left[ \frac{\eta^{\alpha\beta}}{\not{p}} + \left( \frac{1}{2\ell-2} \right) \left( \frac{2\not{p}^\alpha}{\not{p}} - \gamma^\alpha \right) \frac{1}{\not{p}} \left( \frac{2\not{p}^\beta}{\not{p}} - \gamma^\beta \right) \right] - i\alpha \frac{\not{p}^\alpha \not{p}^\beta}{\not{p}^2 \not{p}}$$

Setting

$$\begin{aligned} \bar{S}^{\alpha\beta}(p) &= i \left[ \frac{\eta^{\alpha\beta}}{\not{p}} + \left( \frac{1}{2\ell-2} \right) \left( \frac{2\not{p}^\alpha}{\not{p}} - \gamma^\alpha \right) \frac{1}{\not{p}} \left( \frac{2\not{p}^\beta}{\not{p}} - \gamma^\beta \right) \right] \\ &= i \left[ \frac{\eta^{\alpha\beta}}{\not{p}} + \left( \frac{1}{2\ell-2} \right) \frac{1}{\not{p}} \gamma^\alpha \not{p} \gamma^\beta \frac{1}{\not{p}} \right] \end{aligned} \quad ..(30)$$

one then has

$$S_\psi^{\alpha\beta}(p) = \bar{S}^{\alpha\beta}(p) - i\alpha \frac{\not{p}^\alpha \not{p}^\beta}{\not{p}^2 \not{p}}$$

Note that the quantity  $\bar{S}^{\alpha\beta}(p)$  satisfies the equations

$$\gamma_\alpha \bar{S}^{\alpha\beta} = \bar{S}^{\alpha\beta} \gamma_\beta = 0 \quad ..(31)$$

The coupling of  $\psi$  to the gravitational field is engineered as before by replacing flat space derivatives with covariant derivatives. When this replacement is made in the Rarita-Schwinger Lagrangian (24) it becomes

$$L = -ie \bar{\psi}_\alpha \gamma^{[\alpha} \gamma^\rho \gamma^{\beta]} D_\rho \psi_\beta \quad ..(32)$$

Likewise the total Lagrangian (28) is transformed into

$$L = -ie [\bar{\psi}_\alpha \gamma^{[\alpha} \gamma^\rho \gamma^{\beta]} D_\rho \psi_\beta - \frac{1}{\alpha} \bar{\psi} \cdot \gamma \not{D} \psi + \bar{F} \not{D} F + \bar{c} \not{D}^2 c_* - \bar{c}_* \not{D}^2 c] \quad ..(33)$$

The form of the covariant derivative  $D_\rho$  when acting on spin 1/2 objects has already been described in equation (5) in terms of the quantities  $\omega_{\mu\gamma\delta}$  and  $\sigma_{\gamma\delta}$ . When acting on the spin 3/2 object  $\psi_\alpha$  it is given by

$$D_\rho \psi_\beta = e_\rho^\mu \left[ \partial_\mu + \frac{1}{2} \omega_{\mu\gamma\delta} (\sigma^{\gamma\delta} + \Sigma^{\gamma\delta}) \right] \psi_\beta \quad ..(34)$$

where

$$\Sigma^{\gamma\delta} \psi^\beta = \eta^{\gamma\beta} \psi^\delta - \eta^{\delta\beta} \psi^\gamma$$

Once again the field  $h_{\alpha\beta} = (e_{\alpha\beta} - \eta_{\alpha\beta})/\kappa$  may be taken to be symmetric in  $\alpha$  and  $\beta$  and to satisfy the equations of motion  $\partial_\alpha h^{\alpha\beta} = 0$ ,  $h_\alpha^\alpha = 0$ . When the gravitational field is introduced as above into the Rarita-Schwinger Lagrangian (32) gauge invariance of the combined gravitational spin 3/2 theory persists, but in a very complicated form [9,10]. This form can be radically simplified by adopting an assumption which in no way affects the spin 1/2 or spin 3/2 anomalies. Specifically, one can assume as Endo and Takao did [17] that the Ricci tensor is zero

$$R_{\mu\nu} = 0 \quad ..(35)$$



Once one assumes (35) one finds that in the A-parametrized spin 3/2 field theories described in this section covariant derivatives effectively commute

$$[D^\rho, D^\sigma] = 0 \quad \dots(36)$$

This result, which is derived and explained in appendix 3, may in turn be used to show that the covariant Rarita-Schwinger Lagrangian (32) is invariant under the following gauge transformation

$$\psi_\alpha \rightarrow \psi_\alpha + D_\alpha \Lambda$$

When gravity is added to the Rarita-Schwinger theory this transformation replaces the free field gauge transformation (25). Besides simplifying the form of gauge transformations within spin 3/2 field theory assumption (35) and its consequence (36) also reduce the complexity of the field transformations which mediate between various formulations of a gravitationally interacting spin 3/2 theory. More will be said of this in the next section. It only remains to be added that the reader will have to wait until section 3.2 to find out why condition (35) has no effect upon the spin 1/2 and spin 3/2 chiral anomalies.

The Lagrangian (32) may be expanded to first order in  $\kappa$  in which case one arrives at an expression of the same form as equation (7). From this expression one can deduce that in the above covariant, gauge fixed Rarita-Schwinger theory the first order momentum space gravitational vertex is given by  $h_{\rho\mu}(k)\bar{\psi}_\alpha(p)V^{\alpha\beta\rho\mu}(p,k)\psi_\beta(p+k)$  where

$$\begin{aligned} V^{\alpha\beta\rho\mu}(p,k) = i\kappa [ & (\eta^{\alpha\beta}p^\rho + \eta^{\alpha\rho}k^\beta - \eta^{\beta\rho}k^\alpha)\gamma^\mu - (p^\rho\eta^{\alpha\mu}\gamma^\beta + p^\rho\eta^{\beta\mu}\gamma^\alpha \\ & + \frac{1}{2}\eta^{\alpha\rho}\gamma^\mu k^\beta - \frac{1}{2}\eta^{\beta\rho}\gamma^\alpha k^\mu) + (1 - \frac{1}{\alpha})p^\rho\gamma^\alpha\gamma^\mu\gamma^\beta ] \end{aligned}$$

As in the spin 1/2 case I have used the fact that  $\partial^\rho h_{\rho\mu}$  and  $h^\rho_\rho$  are both zero to simplify the vertex. I will close this section by deriving the spin 3/2 chiral current conservation equation, the analogue of the spin 1/2 equation (17). The covariant, gauge-fixed Rarita-Schwinger Lagrangian (33) is classically invariant under the following chiral transformations

$$\psi^\alpha \rightarrow e^{i\theta\Gamma^{-1}} \psi^\alpha$$

$$F \rightarrow e^{-i\theta\Gamma^{-1}} F$$

$$c \rightarrow e^{i\theta\Gamma^{-1}} c$$

$$c_\star \rightarrow e^{-i\theta\Gamma^{-1}} c_\star$$

It has already been observed that the fields  $\psi$  and  $F$  decouple from each other and from  $c$  and  $c_\star$ . As was pointed out above, the virtue of the ghost formalism of Hata and Kugo, so far as the calculation of the anomaly is concerned, is that due to the opposite chiral charges of  $c$  and  $c_\star$  the contributions by these two fields to the anomaly are equal in magnitude but of opposite sign and consequently cancel. The field  $F$  contributes to the anomaly as a spin 1/2 field but with opposite sign since its transformation character is opposite to that of  $\psi$ . Thus the spin 3/2 anomaly is given by

$$A^{3/2} = A_\psi^{3/2} - A^{1/2} \quad \dots(37)$$

where  $A^{1/2}$  is the spin 1/2 anomaly and  $A_\psi^{3/2}$  is the contribution of the field  $\psi$ . To find  $A_\psi^{3/2}$  we need consider only the  $\psi$  part of the chiral current conservation equation. The chiral transformation rule for  $\psi$  leads us, via the procedure described in the last section, to the chiral current

$$J_\psi^\mu = \bar{\psi}_\alpha [\eta^{\alpha\beta} \gamma^\mu - (\eta^{\alpha\mu} \gamma^\beta + \eta^{\beta\mu} \gamma^\alpha) + (1 - \frac{1}{\alpha}) \gamma^\alpha \gamma^\mu \gamma^\beta] \Gamma^{-1} \psi_\beta \quad \dots(38)$$

and, since the divergence of this current is classically zero, to the equation

$$\partial_\mu [e J_\psi^\mu] = A_\psi^{3/2} \quad \dots(39)$$

This equation is the analogue in the Rarita-Schwinger formulation of the spin 1/2 equation (17).

### 2.3 The $A=0$ and $A=-1/\ell$ Formulations

In section 2.2 the Lagrangian and propagator for the spin 3/2 field, its first order gravitational vertex and its chiral current were all given in the Rarita-Schwinger ( $A=-1$ ) formulation of spin 3/2 theory. However when I

come to calculate the spin 3/2 anomaly and demonstrate its gauge independence I will be working not in the conventional Rarita-Schwinger formulation but in either the  $A=0$  or  $A=-1/\ell$  formulations. Consequently in this section I take the opportunity to translate the results of the previous section into forms appropriate to these two alternative choices of  $A$ .

### I. The $A=0$ Formulation.

To recast the Rarita-Schwinger theory described by the Lagrangian (24) in the form of an  $A=0$  theory one reexpresses  $\psi$  in terms of a field  $\phi$  defined by

$$\phi^\alpha = \psi^\alpha - \frac{1}{2} \gamma^\alpha \gamma \cdot \psi$$

Evidently  $\phi$  is related to  $\psi$  by an invertible transformation of type (22). Under this change of variable the covariant Rarita-Schwinger Lagrangian (32) goes into

$$L = i \bar{\phi}_\alpha [\eta^{\alpha\beta} \not{\partial} + (\frac{1}{2\ell-2}) \gamma^\alpha \not{\partial} \gamma^\beta] \phi_\beta$$

and the gauge fixed Lagrangian (33) becomes

$$L = -ie [\bar{\phi}_\alpha \not{\partial} \phi^\alpha + (\frac{1}{2\ell-2} - \frac{4}{\alpha(2\ell-2)^2}) \bar{\phi} \cdot \gamma \not{\partial} \gamma \cdot \phi + \bar{F} \not{\partial} F - \bar{c}_* \not{\partial}^2 c + \bar{c} \not{\partial}^2 c_*] \quad ..(40)$$

where the field  $F$  is given in terms of  $\phi$  by the expression

$$F = \sqrt{\alpha} [B - \frac{2i}{\alpha(2\ell-2)} \gamma \cdot \phi]$$

As mentioned above I will be using the  $A=-1/\ell$  formulation of spin 3/2 theory to establish the anomaly's gauge independence, and the  $A=0$  formulation to actually calculate the anomaly. Given that the anomaly can be shown not to depend on  $\alpha$ , the calculation of  $A^{3/2}$  in the  $A=0$  formulation is dramatically simplified by the gauge choice

$$\alpha = (\frac{4}{2\ell-2})$$

Then the total Lagrangian (40) reduces to

$$L = -e[\bar{\phi}_\alpha \not{p} \phi^\alpha + \bar{F} \not{p} F - \bar{c}_* \not{p}^2 c + \bar{c} \not{p}^2 c_*] \quad ..(41)$$

From (41) it is easy to deduce the propagator for  $\phi$  under this choice of gauge

$$S_\phi^{\alpha\beta}(p) = i \frac{\eta^{\alpha\beta}}{\not{p}} \quad ..(42)$$

Likewise it is a simple matter to find the first order gravitational vertex

$$V_{gr}^{\alpha\beta\rho\mu}(p,k) = i\kappa(\eta^{\alpha\beta}\not{p}^\rho + \eta^{\alpha\rho}\not{k}^\beta - \eta^{\beta\rho}\not{k}^\alpha)\gamma^\mu \quad ..(43)$$

As before the contributions to the anomaly by the two ghost fields cancel and the anomaly may be divided into independent contributions from  $F$  and  $\phi$

$$A^{3/2} = A_\phi^{3/2} - A^{1/2} \quad ..(44)$$

In this equation  $A^{1/2}$  is the spin 1/2 anomaly which appears in (17), while  $A_\phi^{3/2}$  is  $\phi$ 's contribution to  $A^{3/2}$ . Of course,  $A_\phi^{3/2}$  is equal to the anomalous divergence of the chiral current that is associated with the  $\phi$  part of the Lagrangian (41). This current is given by

$$J_\phi^\mu = \bar{\phi}_\alpha \gamma_\mu \Gamma^{-1} \phi^\alpha \quad ..(45)$$

so we have

$$A_\phi^{3/2} = \partial_\mu [e J_\phi^\mu] \quad ..(46)$$

The only other comment I wish to make before finishing this brief treatment of the  $A=0$  formulation concerns the parameter choices  $A=0$  and  $\alpha=[4/(2\ell-2)]$ . Suffice it to say here that no other combination of values for  $A$  and  $\alpha$  results in expressions for the spin 3/2 propagator and vertex which are as simple as (42) and (43). Clearly, it is for this reason that I adopt the values  $A=0$  and  $\alpha=[4/(2\ell-2)]$  when calculating  $A^{3/2}$  in chapters 3 and 4.

## II. The $A=-1/\ell$ Formulation.

The Rarita-Schwinger theory of a non-interacting spin 3/2 field may be transformed into the  $A=-1/\ell$  formulation by acting on the free field Lagrangian (24) with the momentum space field redefinition

$$\chi^\alpha = \psi^\alpha - \left( \frac{p^\alpha}{p} + \frac{\gamma^\alpha}{2} \right) \gamma \cdot \psi \quad ..(47)$$

in which case the free field Lagrangian becomes

$$L = i\bar{\chi}_\alpha \left[ \eta^{\alpha\beta} \not{p} - \frac{1}{\ell} (\gamma^\alpha \not{\partial}^\beta + \partial^\alpha \gamma^\beta) + \left( \frac{1+\ell}{2\ell^2} \right) \gamma^\alpha \not{p} \gamma^\beta \right] \chi_\beta \quad ..(48)$$

Setting

$$M^{\alpha\rho\beta} = \eta^{\alpha\beta} \gamma^\rho - \frac{1}{\ell} (\eta^{\alpha\rho} \gamma^\beta + \eta^{\beta\rho} \gamma^\alpha) + \left( \frac{1+\ell}{2\ell^2} \right) \gamma^\alpha \gamma^\rho \gamma^\beta \quad ..(49)$$

we then have

$$L = i\bar{\chi}_\alpha M^{\alpha\rho\beta} \partial_\rho \chi_\beta$$

This Lagrangian is invariant under the gauge transformation

$$\chi_\alpha \rightarrow \chi_\alpha + \gamma_\alpha \Lambda \quad ..(50)$$

Note that the above field redefinition involves the flat space operator  $\not{p}$ . Because of this it is by no means a simple matter to find a covariant version of the change of variable (47) which will take one from the covariant Rarita-Schwinger Lagrangian (32) to the corresponding  $A=-1/\ell$  Lagrangian. The difficulty in finding such a transformation lies in the fact that, whereas flat space derivatives commute, covariant derivatives do not. This problem can be remedied by adopting condition (35). As is explained in appendix 3, if the Ricci tensor is zero covariant derivatives effectively commute. I shall demonstrate in section 3.2 that the imposition of (35) in no way affects the anomalies  $A^{1/2}$  and  $A^{3/2}$ . We are therefore free to simplify matters by assuming that  $R^{\mu\nu}$  is equal to zero, in which case the desired covariant field redefinition is just

$$\chi_\alpha = \psi_\alpha - \left( \frac{D^\alpha}{\not{p}} + \frac{\gamma^\alpha}{2} \right) \gamma \cdot \psi$$

Under this transformation the complete Lagrangian (33) becomes

$$L = -ie \left[ \bar{\chi}_\alpha M^{\alpha\rho\beta} D_\rho \chi_\beta - \left( \frac{1}{\alpha\ell^2} \right) \bar{\chi} \cdot \gamma \not{p} \gamma \cdot \chi + \bar{F} \not{p} F - \bar{c}_* \not{p}^2 c + \bar{c} \not{p}^2 c_* \right] \quad ..(51)$$

where  $M^{\alpha\rho\beta}$  is as in (49). Of course gauge invariance no longer exists in the gauge fixed Lagrangian (51), however the existence of the free field gauge invariance persists in the fact that

$$\gamma_\alpha M^{\alpha\rho\beta} = M^{\alpha\rho\beta} \gamma_\beta = 0 \quad ..(52)$$

The propagator in this formulation is

$$S_\chi^{\alpha\beta}(p) = \bar{S}^{\alpha\beta}(p) - i \frac{\alpha}{4} \gamma^\alpha \frac{1}{\not{p}} \gamma^\beta \quad ..(53)$$

where  $\bar{S}^{\alpha\beta}$  is the quantity appearing in equation (30)

$$\bar{S}^{\alpha\beta}(p) = i \left[ \frac{\eta^{\alpha\beta}}{\not{p}} + \left( \frac{1}{2\ell-2} \right) \frac{1}{\not{p}} \gamma^\alpha \not{p} \gamma^\beta \frac{1}{\not{p}} \right]$$

The first order gravitational vertex is in turn given by

$$\begin{aligned} V^{\alpha\beta\rho\mu}(p,k) = i\kappa \left[ (\eta^{\alpha\beta} p^\rho + \eta^{\alpha\rho} k^\beta - \eta^{\beta\rho} k^\alpha) \gamma^\mu - \frac{1}{\ell} (p^\rho \eta^{\alpha\mu} \gamma^\beta + p^\rho \eta^{\beta\mu} \gamma^\alpha \right. \\ \left. + \frac{1}{2} \eta^{\alpha\rho} \gamma^\mu k^\beta - \frac{1}{2} \eta^{\beta\rho} \gamma^\alpha k^\mu) + \left( \frac{1+\ell}{2\ell^2} - \frac{1}{\alpha\ell^2} \right) p^\rho \gamma^\alpha \gamma^\mu \gamma^\beta \right] \end{aligned}$$

Setting

$$\begin{aligned} \bar{V}^{\alpha\beta\rho\mu}(p,k) = i\kappa \left[ (\eta^{\alpha\beta} p^\rho + \eta^{\alpha\rho} k^\beta - \eta^{\beta\rho} k^\alpha) \gamma^\mu - \frac{1}{\ell} (p^\rho \eta^{\alpha\mu} \gamma^\beta + p^\rho \eta^{\beta\mu} \gamma^\alpha \right. \\ \left. + \frac{1}{2} \eta^{\alpha\rho} \gamma^\mu k^\beta - \frac{1}{2} \eta^{\beta\rho} \gamma^\alpha k^\mu) + \left( \frac{1+\ell}{2\ell^2} \right) p^\rho \gamma^\alpha \gamma^\mu \gamma^\beta \right] \quad ..(54) \end{aligned}$$

we then have

$$V^{\alpha\beta\rho\mu}(p,k) = \bar{V}_{gr}^{\alpha\beta\rho\mu} - \left( \frac{i\kappa}{\alpha\ell^2} \right) p^\rho \gamma^\alpha \gamma^\mu \gamma^\beta \quad ..(55)$$

Observe that the quantity  $\bar{V}^{\alpha\beta\rho\mu}(p,k)$  satisfies relations similar to (52)

$$\gamma_\alpha \bar{V}^{\alpha\beta\rho\mu} = \bar{V}^{\alpha\beta\rho\mu} \gamma_\beta = 0 \quad ..(56)$$

Equations (56) and equations (31) and (52) together constitute a set of identities which are central to chapter 3's proof of the gauge independence of the spin 3/2 anomaly. It should not escape the reader that these identities are closely connected with the form (50) taken by the free field gauge invariance in this formulation of spin 3/2 theory. The only thing that now remains to be done is to give the  $A=-1/\ell$  versions of equations (38), (39), (45) and (46). They are

$$J_\chi^\mu = \bar{\chi}_\alpha [M^{\alpha\mu\beta} - (\frac{1}{\alpha\ell^2}) \gamma^\alpha \gamma^\mu \gamma^\beta] \Gamma^{-1} \chi_\beta$$

$$A_\chi^{3/2} = \partial_\mu [e J_\chi^\mu]$$

The quantity  $A_\chi^{3/2}$  is related to the full spin 3/2 anomaly  $A^{3/2}$  according to the following equation

$$A^{3/2} = A_\chi^{3/2} - A^{1/2} \quad ..(57)$$

Note that, by assumption, the value of  $A^{3/2}$  is the same in all formulations of spin 3/2 theory. In view of equations (37), (44) and (57) this means

$$A_\psi^{3/2} = A_\phi^{3/2} = A_\chi^{3/2} \quad ..(58)$$

A final comment on the  $A=-1/\ell$  formulation of spin 3/2 theory is in order. It was explained in appendix 2 that  $-1/\ell$  is the only value which the real parameter  $A$  cannot validly assume. In fact the  $A=-1/\ell$  Lagrangian (48) describes a theory which is not necessarily of purely spin 3/2 content, and in this sense it is unacceptable. These considerations would prove an obstacle to my proof of the gauge independence of the spin 3/2 anomaly were it not for the fact that I will not actually be working in the  $A=-1/\ell$  formulation. Instead I will be setting  $A=(-1/\ell)+\epsilon$ ,  $0<|\epsilon|<<1$  and working in formulations of spin 3/2 theory which are arbitrarily close to the  $A=-1/\ell$

formulation. According to the criteria set down in appendix 2 these  $\epsilon$ -formulations are entirely acceptable representations of spin 3/2 theory.

When  $A$  is shifted from  $-1/\ell$  to  $(-1/\ell)+\epsilon$  the propagator and vertex (53) and (55) receive  $o(\epsilon)$  corrections. In section 3.5 I demonstrate that these corrections are irrelevant, and that the anomaly is determined by an expression that one would get by naively working in the  $A=-1/\ell$  formulation. Since the gauge parameter  $\alpha$  drops out of this expression it is possible to show that the spin 3/2 anomaly is gauge independent. The point I wish to make here is that by arguing in this way I avoid the problems associated with the  $A=-1/\ell$  formulation of spin 3/2 theory, and my proof of the gauge independence of the spin 3/2 anomaly rests on quite as firm a basis as its actual  $A=0$  calculation. The reader should always interpret my statements about the  $A=-1/\ell$  formulation in terms of these  $\epsilon$ -type considerations. That concludes this chapter's review of spin 1/2 and spin 3/2 field theory. In the next chapter I will set up the formalism necessary for the anomaly calculations and single out the relevant Feynman diagrams.

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## CHAPTER 3. The Diagrammatic Method

This chapter begins the diagrammatic anomaly calculations which are the subject of the thesis. It therefore seems appropriate to start proceedings with a brief history of the diagrammatic method. A simple version of the method of anomaly calculation described below was first used by Delbourgo and Jarvis [1] to calculate the gravitational contribution to the spin  $1/2$  chiral anomaly in eight dimensions. However due to the amount of work involved the procedure of reference [1] proved inadequate for similar calculations in higher dimensions. The question then arose as to whether it could be extended in some way so as to permit one to calculate the higher dimensional anomalies, and perhaps even to calculate the spin  $1/2$  chiral anomaly in arbitrary dimensional space-times.

Such an extension was developed by R. Delbourgo and the author in references [2] and [3]. It exploited recurrence relations between anomalies in different dimensions and its outcome was the first explicit expression for the gravitational contribution to the spin  $1/2$  chiral anomaly in arbitrary dimensions. Previously the spin  $1/2$  anomaly had appeared only in A-genus form [5]. The work in references [2] and [3] was later adapted in reference [4] to the case of the gravitational contribution to the spin  $3/2$  chiral anomaly. Besides the more difficult spin  $3/2$  calculation, reference [4] also contains a proof of the gauge independence of the spin  $3/2$  chiral anomaly. In the remainder of this chapter, and in chapter 4, I will describe in detail the work which was carried out in references [2], [3] and [4].

### 3.1 Dimensional Regularization

Let us commence this treatment of the diagrammatic method by considering how the chiral anomaly emerges when dimensional regularization is applied to the interacting spin  $1/2$  and spin  $3/2$  theories of chapter 2. The object of any regularization scheme is to render finite in a well-defined way the infinite quantities that occur in quantum field theories. It is well known that the infinities present in these theories have their origin in divergent loop

momentum integrals. Dimensional regularization is based upon the observation that any loop momentum integral will become finite if the dimension of the space-time in which it is calculated is made small enough.

When using dimensional regularization one adopts the following procedure. Firstly the theory is established and quantities are formulated in terms of perturbation series in the usual fashion except that the dimension of space-time is left arbitrary. Expressions for quantities in the theory will then be analytic in this dimension. When all the manipulation and combination of these analytic quantities is done and a final result is desired one analytically continues back to the dimension in which the answer is sought. If there are quantities in the theory which diverge as the dimension is returned to its "true" value the theory will require renormalization. However this is never a problem with anomalies. In accordance with the arguments of Alvarez-Gaume and Witten [5] which were related in chapter 1, anomalies are always finite and so do not require renormalization.

At this point I would remind the reader that, for reasons discussed at the beginning of chapter 2, chiral anomalies occur only in even-dimensional space-times. As a consequence of this fact I restricted my attention in chapter 2 to space-times of dimensions  $d=2n$ ,  $n=1,2,\dots$ . Moreover I adopted the convention that if the space-time dimension were to be analytically continued away from  $d=2n$  then its continued value would be  $d=2\ell$ . In the remainder of the thesis I will continue to work in space-time whose true dimension is  $d=2n$  and whose analytically continued dimension is  $d=2\ell$ . The effect of dimensional continuation upon the interacting spin  $1/2$  and spin  $3/2$  field theories of chapter 2 is felt in the algebra of the Dirac gamma matrices. If the dimension is continued away from  $d=2n$  it becomes impossible [6] to maintain the familiar relations

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} \quad \dots(1)$$

$$\{\Gamma^{-1}, \gamma^\alpha\} = 0 \quad \dots(2)$$

Something must change and the choice conventionally falls upon  $\Gamma^{-1}$ , with the consequence that equation (2) is modified and equation (1) is left

unchanged. Note that in choosing to change (2) rather than (1) one is opting to preserve vector gauge invariance in the spin 1/2 and spin 3/2 theories at the expense of chiral invariance. In contrast to 2n dimensions where  $\Gamma^{-1}$  is defined by equation (2.14), in 2ℓ dimensions  $\Gamma^{-1}$  is given by the following expression [6]

$$\Gamma^{-1} = \frac{i^{n+1}}{(2n)!} \xi^{v_1 \dots v_{2n}} \gamma_{v_1} \dots \gamma_{v_{2n}} \quad ..(3)$$

$$\xi^{v_1 \dots v_{2n}} = \xi^{[v_1 \dots v_{2n}]} + \epsilon^{v_1 \dots v_{2n}} \quad ..(4)$$

The tensor  $\xi$  may be taken to be any totally antisymmetric tensor which reduces to the 2n-dimensional Levi-Civita tensor  $\epsilon$  as  $\ell \rightarrow n$  [6]. Of course, the expression (3) for  $\Gamma^{-1}$  degenerates into the familiar equation (2.14) in that limit. Using (3) it may be checked that after continuation from  $d=2n$  to  $d=2\ell$  the anticommutator relation (2) may be replaced by the following commutator relation

$$[\gamma^\alpha, \Gamma^{-1}] = \frac{2i^{n+1}}{(2n-1)!} \xi^{\alpha v_1 \dots v_{2n-1}} \gamma_{v_1} \dots \gamma_{v_{2n-1}} \quad ..(5)$$

The consequences of the definition (3) of  $\Gamma^{-1}$  for the spin 1/2 and spin 3/2 chiral currents are straightforward. Working in 2ℓ-dimensions and using Noether's prescription as described in detail in section 2.1 one arrives at the following expressions for the dimensionally continued chiral currents.

$$J_\mu^{1/2} = \frac{1}{2} \bar{\psi} [\gamma_\mu, \Gamma^{-1}] \psi \quad ..(6)$$

$$J_\mu^{3/2} = \frac{1}{2} \bar{\phi}_\alpha [\gamma_\mu, \Gamma^{-1}] \phi^\alpha \quad ..(7)$$

Note that the spin 3/2 current has been expressed in a form appropriate to the  $A=0$  formulation of spin 3/2 theory, and that for reasons described in section 2.3 the gauge choice  $\alpha=[4/(2\ell-2)]$  has been made. Since I will be employing the  $A=0$  formulation in combination with the choice of gauge  $\alpha=[4/(2\ell-2)]$  to calculate the spin 3/2 anomaly, all spin 3/2 formulae in this section and the following three will be expressed in a like manner. The reader who so desires can translate these formulae into forms appropriate to

the  $A=-1$  and  $A=-1/\ell$  formulations using the results of sections 2.2 and 2.3. In contrast, the work of section 5 will be carried out in the  $A=-1/\ell$  formulation since that formulation is the one in which the gauge independence of the spin 3/2 anomaly is best seen.

The dimensionally continued conservation equations for the currents (6) and (7) may be derived by making use of the fact that under a chiral transformation  $\partial \cdot J = \delta L$ . In this way one finds

$$\partial^\mu [e J_\mu^{1/2}] = 2ie m \bar{\psi} \Gamma^{-1} \psi + e \bar{\psi} \{\not{D}, \Gamma^{-1}\} \psi \quad ..(8)$$

$$\partial^\mu [e J_\mu^{3/2}] = e \bar{\phi}_\alpha \{\not{D}, \Gamma^{-1}\} \phi^\alpha \quad ..(9)$$

The terms on the right hand sides of equations (8) and (9) contain the anticommutator  $\{\not{D}, \Gamma^{-1}\}$  which is anomalous in the sense that it is zero for  $d=2n$  and becomes non-zero only when the dimension is continued away from  $d=2n$ . It is a product of the dimensional regularization procedure in the same way, for example, that massive regulator field contributions to the right hand sides of (8) and (9) would be products of the Pauli-Villars scheme. If Pauli-Villars regularization were employed, the spin 1/2 and spin 3/2 anomalies would emerge from the regulator field contributions to (8) and (9) in the limit as the regulator mass tends to infinity. In the present instance the anomalies will emerge from the anomalous anticommutator terms on the right hand sides of (8) and (9) in the limit  $\ell \rightarrow n$ . So one can write the operator equations

$$A^{1/2} = \lim_{\ell \rightarrow n} e \bar{\psi} \{\not{D}, \Gamma^{-1}\} \psi \quad ..(10)$$

$$A^{3/2} = \lim_{\ell \rightarrow n} e \bar{\phi}_\alpha \{\not{D}, \Gamma^{-1}\} \phi^\alpha \quad ..(11)$$

### 3.2 The Form of the Anomaly

A crucial step in the diagrammatic approach to anomaly calculation is the use of general considerations to deduce the form of the anomaly in a space-time of dimension  $d=2n$ . I will now show that its form may be derived using

arguments based on nothing more than dimensional analysis and the anomaly's transformation properties under general coordinate transformations. Consider the spin 1/2 and spin 3/2 anomalies  $A^{1/2}$  and  $A^{3/2}$  appearing in equations (10) and (11). In the remainder of this section it will sometimes be convenient not to distinguish between these two anomalies in which case I will simply refer to the anomaly  $A$  with the understanding that  $A$  might be either  $A^{1/2}$  or  $A^{3/2}$ . The starting point for a determination of the form of  $A$  is a consideration of the types of anomalous Feynman diagram that can possibly contribute to it.

We saw in chapter 1 that anomalous Feynman diagrams possess a unifying characteristic. They are all either single fermion loop diagrams of the form shown in figure 1.1, or they contain these fermion loops as sub-diagrams. Besides an axial current, the fields which emanate from the fermion loops are gauge fields, which in the present case is to say that they are gravitational fields. These facts taken together imply that any Feynman diagram which contributes to an anomalous amplitude of the axial current  $J^\mu$  will be of the form

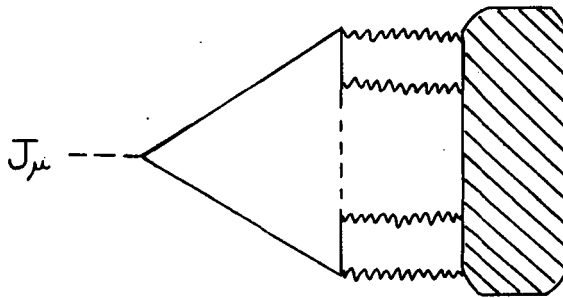


FIGURE 1

where the hatched region represents a sub-diagram whose structure is unimportant. The point here is that, in its anomalous interactions with other fields in the theory, the axial current is always "filtered through" a number of gravitational fields. It therefore follows that the anomaly itself, which is equal to the divergence of the axial current and which is a field operator, may be replaced by some polynomial in the gravitational fields. This provides us with our first hold on the form of the anomaly.

Once it has been settled that the anomaly is a polynomial in the gravitational field the chief determinant of its form is its transformation character under general coordinate transformations. Specifically, the anomaly is covariant under this type of transformation. This statement is not self-evident and requires some explanation. Bardeen and Zumino [7] have pointed out that the breakdown of symmetry associated with the appearance of an anomaly in a theory may extend to the anomaly itself. In particular this means that anomalies in gauge theories may not themselves be gauge covariant. This aspect of anomalous symmetry breaking was discussed in chapter 1 in connection with the distinction between consistent and covariant anomalies [7,8,9]. Fortunately the effect does not enter into the present calculation. General coordinate invariance is respected not only by the spin 1/2 and spin 3/2 theories with which we are dealing, but also by the dimensional regularization scheme which is being applied to these theories. Consequently the regularized theories, and in particular the spin 1/2 and spin 3/2 chiral anomalies, will be gravitationally covariant. On the other hand, while the unregularized spin 1/2 and spin 3/2 theories respect chiral invariance, the dimensional regularization scheme does not. Therefore we should allow for the possibility that  $A^{1/2}$  and  $A^{3/2}$  are chirally non-invariant. As it turns out,  $A^{1/2}$  and  $A^{3/2}$  are composed exclusively of graviton fields, and they are chirally invariant as well as gravitationally covariant.

Since  $A$  is covariant it will possess the tensor transformation properties which follow naively from equations (10) and (11), which is to say that it will be a pseudoscalar density. We are therefore faced with the problem of finding the most general polynomial in the gravitational field which transforms as a pseudoscalar density. Because  $A$  is gravitationally covariant it must of course be constructed from curved space tensors and densities. In fact it is not difficult to see that  $A$  may be regarded as being constructed from only three tensors together with their covariant derivatives. The three tensors are : the density  $e$ , the  $2n$ -dimensional Levi-Civita tensor density  $\epsilon^{\mu_1 \dots \mu_{2n}}$  and the Riemann tensor  $R^{\mu\nu\sigma\rho}$ . Other tensors such as the Ricci tensor  $R^{\mu\nu} = R^{\mu\sigma\nu}_{\sigma}$  need not be included in this group as they may be derived by combining the above four quantities in various ways. Thus we know the composition of  $A$  in general terms.

Several more bits of information may be deduced from the transformation characteristics of A by taking into account the properties of the  $\epsilon$ -tensor. The Levi-Civita  $\epsilon$ -tensor is a totally antisymmetric tensor density. Of the four objects  $e, \epsilon, R$  and the covariant derivative  $D$ , the  $\epsilon$ -tensor is the only one which is parity-odd. Since the anomaly too is parity-odd A must contain, in addition to an unknown number of Riemann tensors, covariant derivatives and possible factors of  $e$ , an odd number of  $\epsilon$ -tensors. Clearly, because A is a scalar the indices on the  $\epsilon$ -tensors and those on the Riemann tensors and covariant derivatives must all be contracted together in some fashion. Because  $\epsilon$  is totally antisymmetric the indices on any one  $\epsilon$ -tensor cannot be contracted against one another. Instead they must be contracted against the indices on other  $\epsilon$ -tensors or on the Riemann tensors and covariant derivatives. But the contraction of two  $\epsilon$ -tensors can always be expressed in terms of metric tensors  $g^{\mu\nu}$ . Consequently without loss of generality to this argument we may ignore the possibility that  $\epsilon$ -tensors are contracted together and assume that the indices on the  $\epsilon$ -tensors in A are all contracted against indices on Riemann tensors and covariant derivatives. This exhausts the information deducible from A's transformation properties alone. Further information may be obtained with the aid of dimensional analysis. The length dimensions of the quantities relevant to this discussion are as follows.

$$[e] = [\epsilon^{\mu_1 \dots \mu_{2n}}] = L^0 \quad [R^{\mu\nu\sigma\rho}] = L^{-2}$$

$$[D^\rho] = L^{-1} \quad [D \cdot J^{1/2}] = [D \cdot J^{3/2}] = L^{-2n}$$

We know from equations (8), (9), (10) and (11) that the dimension of the anomaly A is the same as those of the divergences  $D \cdot J^{1/2}$  and  $D \cdot J^{3/2}$ . On the other hand A is constructed from the objects  $e, \epsilon, R$  and  $D$ . Of these objects the only ones possessing length dimensions are  $R$  and  $D$ . (The gravitational coupling constant  $\kappa$ , which is a dimensional quantity, may be ignored as it remains buried within the Riemann tensor.) We see immediately that the number of Riemann tensors and covariant derivatives in A necessarily satisfies a certain relation. Specifically, in addition to  $\epsilon$ -tensors and factors of  $e$ , the anomaly A must consist of  $n_1$  Riemann tensors and  $2n_2$  covariant derivatives where  $n_1 + n_2 = n$ . Note that there must be an even number



of covariant derivatives. Let us now analyse the way in which the indices on Riemann tensors may be contracted against those on  $\epsilon$ -tensors. Of the four indices on any one Riemann tensor no more than two can be contracted against  $\epsilon$ -tensor indices due to the cyclic identity

$$R^{\mu\nu\rho\sigma} + R^{\mu\rho\nu\sigma} + R^{\mu\sigma\rho\nu} = 0 \quad \dots(12)$$

This means that overall there are no more than  $2n_1$  Riemann tensor indices which are available for contraction against  $\epsilon$ -tensor indices. Together with the  $2n_2$  indices carried by covariant derivatives there are therefore no more than  $2(n_1+n_2)=2n$  indices which can be contracted with  $\epsilon$ -tensor indices. But we have seen that all of the  $2n$   $\epsilon$ -tensor indices must be contracted against those on Riemann tensors and covariant derivatives. The conclusion is that A contains a single  $\epsilon$ -tensor and, further, that all of the indices on the covariant derivatives and two indices from each Riemann tensor are contracted against the indices on this  $\epsilon$ -tensor. But contractions of pairs of covariant derivatives against antisymmetric  $\epsilon$ -tensor indices will either vanish or be expressible in terms of more Riemann tensors. Thus without loss of generality we may ignore the possibility of covariant derivatives occurring in the anomaly and set  $n_2=0$ . It follows that A consists of a single  $\epsilon$ -tensor and a total of exactly  $n$  Riemann tensors. Two tensor indices from each Riemann tensor are contracted with indices of the  $\epsilon$ -tensor and no factors of  $e$  are needed since the resultant tensor is a density of the correct type. This arrangement neatly accounts for all indices on the  $\epsilon$ -tensor and leaves two free indices on each Riemann tensor which must be contracted amongst themselves. Note that by virtue of the cyclic identity (12) and the identities

$$R^{\mu\nu\rho\sigma} = R^{\sigma\rho\mu\nu} = -R^{\nu\mu\rho\sigma}$$

we lose no generality in assuming that it is the first two indices on each Riemann tensor which are contracted against  $\epsilon$ -tensor indices. The final step in this argument consists of showing how the remaining free Riemann tensor indices, the third and fourth on each tensor, may be contracted together. Since  $R^{\mu\nu\rho\sigma} = -R^{\nu\mu\rho\sigma}$  the third and fourth indices on any one Riemann tensor cannot be contracted against each other. We deduce that the Riemann tensors

must link up in chain-like fashion forming "loops" of different lengths. A loop of length  $L$  contains  $L$  Riemann tensors and is given by the expression

$$\langle R^L \rangle \equiv \langle R^L \rangle^{\mu_1 \nu_1 \dots \mu_L \nu_L} = R^{\mu_1 \nu_1 \sigma_1}_{\sigma_2} R^{\mu_2 \nu_2 \sigma_2}_{\sigma_3} \dots R^{\mu_L \nu_L \sigma_L}_{\sigma_1} \quad ..(13)$$

We need only consider loops of even length, that is loops containing even numbers of Riemann tensors, for the following reason. As  $R$  is antisymmetric in its third and fourth indices, the  $\sigma$ -indices on each of the tensors in (13) may be reversed with a change of sign. Reordering of the tensors then leads immediately to the result

$$\langle R^L \rangle = (-1)^L \langle R^L \rangle$$

Consequently if  $L$  is odd  $\langle R^L \rangle$  is zero. Therefore the Riemann tensors in  $A$  occur in loops of even length and the total number of such tensors is obviously also even. Since the total number of Riemann tensors in  $A$  is  $n$  it follows that pure gravitational contributions to the spin  $1/2$  and spin  $3/2$  chiral anomalies occur only in dimensions  $d=4N$ ,  $N=1,2,\dots$ . We are now in a position to write down the general form of the anomalies  $A^{1/2}$  and  $A^{3/2}$  in dimensions  $d=4N$ . Let me define an object  $T(n_1, \dots, n_N)$  according to the following equation.

$$T(n_1, \dots, n_N) = \epsilon \cdot \langle R^2 \rangle^{n_1} \langle R^4 \rangle^{n_2} \dots \langle R^{2N} \rangle^{n_N} \quad ..(14)$$

In this equation the  $n_i$  are any non-negative integers such that

$$n_1 + 2n_2 + 3n_3 + \dots + Nn_N = N \quad ..(15)$$

The abbreviated notation used in (14) is reasonably obvious. The " $\cdot$ " signifies contraction between the indices on the  $\epsilon$ -tensor and those on the Riemann tensors. For example in 16 dimensions

$$T(2,1,0,0) = \epsilon_{\mu_1 \nu_1 \dots \mu_8 \nu_8} \langle R^2 \rangle^{\mu_1 \nu_1 \mu_2 \nu_2} \langle R^2 \rangle^{\mu_3 \nu_3 \mu_4 \nu_4} \langle R^4 \rangle^{\mu_5 \nu_5 \dots \mu_8 \nu_8}$$

To shorten formulae I will sometimes use the following vector notation for the index  $(n_1, \dots, n_N)$

$$\mathbf{n} \equiv (n_1, \dots, n_N) \quad \dots(16)$$

and write  $T(n_1, \dots, n_N)$  as  $T(\mathbf{n})$ . Confusion between the boldfaced letter  $\mathbf{n}$ , which figures in equation (16), and the regular letter  $n$ , which is half the space-time dimension, is to be avoided. In view of condition (15) the integers  $n_1, \dots, n_N$  may be regarded as specifying a partition of  $N$  into  $n_1$  ones,  $n_2$  twos,  $\dots, n_N$   $N$ 's. Consequently I will sometimes refer to the admissible values of the index  $\mathbf{n}$  as partitions of  $N$ . Likewise, it will be convenient in the following to refer to the sum  $n_1 + 2n_2 + \dots + Nn_N$  as the modulus of  $\mathbf{n}$  and represent it by the symbol  $|\mathbf{n}|$ . In terms of this notation equation (15) may be re-expressed in the form

$$|\mathbf{n}| = N$$

and the anomalies  $A^{1/2}$  and  $A^{3/2}$  can finally be written as

$$A^{1/2} = \sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{1/2}(\mathbf{n}) T(\mathbf{n}) \quad \dots(17)$$

$$A^{3/2} = \sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{3/2}(\mathbf{n}) T(\mathbf{n}) \quad \dots(18)$$

where  $C^{1/2}(\mathbf{n})$  and  $C^{3/2}(\mathbf{n})$  are numerical coefficients, and the sums in (17) and (18) are over all partitions  $\mathbf{n}$  whose moduli  $|\mathbf{n}|$  are equal to  $N$ .

Before closing this section it is appropriate to comment on the condition (2.35). The gravitational field figures in expressions (17) and (18) for the anomalies  $A^{1/2}$  and  $A^{3/2}$  only in the form of the Riemann tensor. The Ricci tensor does not appear at all. Consequently the imposition of condition (2.35) brings about no formal degeneracies in the terms  $T(\mathbf{n})$  and does not prejudice the calculation of the anomalies. Since, as explained in sections 2.2 and 2.3, the imposition of this condition is in other respects desirable I will henceforth regard it as applying.

### 3.3 Perturbation Analysis

Combining the final results in each of sections 1 and 2 one arrives at the exact, that is nonperturbative, equations

$$\sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{1/2}(\mathbf{n})T(\mathbf{n}) = \lim_{\ell \rightarrow \infty} e^{-\bar{\psi}\{\not{p}, \Gamma^{-1}\}\psi} \quad \dots(19)$$

$$\sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{3/2}(\mathbf{n})T(\mathbf{n}) = \lim_{\ell \rightarrow \infty} e^{-\bar{\phi}_{\alpha}\{\not{p}, \Gamma^{-1}\}\phi^{\alpha}} \quad \dots(20)$$

As explained in the previous section these two equations involve the unknown numerical coefficients  $C^{1/2}(\mathbf{n})$  and  $C^{3/2}(\mathbf{n})$ . It will often be convenient in this section not to distinguish between  $C^{1/2}(\mathbf{n})$  and  $C^{3/2}(\mathbf{n})$ . In this event I will adopt the same convention that I employed in section 3. That is, just as I used  $A$  to stand for either of the anomalies  $A^{1/2}$  or  $A^{3/2}$  I will use  $C(\mathbf{n})$  to represent one or both of the coefficients  $C^{1/2}(\mathbf{n})$  and  $C^{3/2}(\mathbf{n})$ .

From equations (17) and (18) we know that finding the anomalies  $A^{1/2}$  and  $A^{3/2}$  is equivalent to finding the values of the coefficients  $C^{1/2}(\mathbf{n})$  and  $C^{3/2}(\mathbf{n})$  for all admissible partitions  $\mathbf{n}$ . In the diagrammatic method the starting point for the calculation of the  $C(\mathbf{n})$  is the observation that it should be possible to deduce their values by comparing coefficients on both sides of equations (19) and (20). Of course (19) and (20) are of no use to us as they stand because they are operator equations. The spin 1/2 and spin 3/2 field theories with which we are dealing are only perturbatively solvable. They are not solvable in closed form and consequently we cannot directly compare the operators occurring on both sides of (19) and (20). What we can do however is to compare the amplitudes of these operators between the vacuum and appropriate multiparticle states. Such amplitudes are expressible in the usual fashion as perturbation series, and one would hope in this way to derive expressions for the  $C(\mathbf{n})$  in terms of Feynman diagrams. In actual fact it is not necessary to consider all the terms in the operators on both sides of (19) and (20). In order to calculate the  $C(\mathbf{n})$  it suffices to consider only a small set of operator terms  $t(\mathbf{n})$  within the quantities  $T(\mathbf{n})$ . The  $t(\mathbf{n})$  are of a distinctive structure and it is easy to pick out their contributions to the amplitudes of the operators in (19) and

(20). By comparing these contributions one ends up with expressions for the  $C(n)$  in terms of a few single loop Feynman diagrams. These diagrams can then be evaluated to give the anomaly. I will now describe the structure and origins of the terms  $t(n)$ .

Consider the perturbation expansion of the terms  $T(n)$  in the gravitational coupling constant  $\kappa$ . The dependence of  $T(n)$  on  $\kappa$  is quite complicated. Using equation (2.6) one finds that  $T(n)$  contains terms of all orders in  $\kappa$  beginning with  $2N$  and extending to infinity. Among the numerous operator terms in the lowest order, that is  $\kappa^{2N}$ , part of  $T(n)$  there is one term  $t(n)$  which possesses a distinctive type of structure. In momentum space  $t(n)$  is given by

$$\begin{aligned} t(n) &= (2\kappa)^{2N} \epsilon(k) K(n) H(2N) \\ &= (2\kappa)^{2N} [\epsilon(k)]^{\mu_1 \dots \mu_{2N}} [K(n)]^{\rho_1 \dots \rho_{2N}} [H(2N)]_{\rho_1 \mu_1 \dots \rho_{2N} \mu_{2N}} \end{aligned} \quad ..(21)$$

where

$$\epsilon(k) = [\epsilon(k)]^{\mu_1 \dots \mu_{2N}} = \epsilon^{\mu_1 \nu_1 \dots \mu_{2N} \nu_{2N}} k_{1\nu_1} \dots k_{2N\nu_{2N}} \quad ..(22)$$

and

$$H(2N) = [H(2N)]_{\rho_1 \mu_1 \dots \rho_{2N} \mu_{2N}} = h_{\rho_1 \mu_1}(k_1) \dots h_{\rho_{2N} \mu_{2N}}(k_{2N}) \quad ..(23)$$

As indicated by my notation the forms of the  $t(n)$  and their sub-components are functions of the index  $n=(n_1, \dots, n_N)$  or the number  $N$ . In particular, the quantity  $K(n)$ , which is a product of the momenta  $k_i$ , is a function of the entire index  $n$ . In order to compactly describe its structure let me introduce the following notation.

$$(k_{s+r}^s) = k_s^{\rho_{s+1}} k_{s+1}^{\rho_{s+2}} k_{s+2}^{\rho_{s+3}} \dots k_{s+r-1}^{\rho_{s+r}} k_{s+r}^{\rho_s} \quad ..(24)$$

I will say that two momenta  $k_i^{\rho_j}$  and  $k_m^{\rho_n}$  are "linked" if  $i=n$  or  $j=m$ . The  $r$  momenta in the sequence (24) are all linked together in a single continuous

chain. I will therefore call  $(k_{s+r}^s)$  a momentum chain of length  $r$ , or an  $r$ -chain for short. In terms of these momentum chains,  $K(n)$  is given by

$$[K(n)]^{\rho_1 \dots \rho_{2N}} = (k_2^1)(k_4^3) \dots (k_{2n_1}^{2n_1-1})(k_{2n_1+4}^{2n_1+1})(k_{2n_1+8}^{2n_1+5}) \dots$$

$$\dots (k_{2n_1+4n_2}^{2n_1+4n_2-3})(k_{2n_1+4n_2+6}^{2n_1+4n_2+1}) \dots \quad \dots (25)$$

Observe that  $K(n)$  consists of  $n_1$  2-chains,  $n_2$  4-chains, ...,  $n_N$   $2N$ -chains. Of course it is no accident that I am using the same terminology for the above momentum chains as for the chains of Riemann tensors encountered in section 2. The sources of the momenta in  $K(n)$  are the derivative graviton couplings within the Riemann tensors in  $T(n)$ , and the reader may verify that the momenta in  $K(n)$  are linked together in chains in exactly the same pattern as the Riemann tensors in  $T(n)$ . Because of this fact it is obvious upon inspection just which  $t(n)$  comes from which  $T(n)$ , and it is clear that the  $t(n)$  are in one-to-one correspondence with both the  $T(n)$  and  $C(n)$ . As an example of the  $t(n)$  consider the sixteen-dimensional ( $N=4$ ) term  $t(2,1,0,0)$  :

$$t(2,1,0,0) = (2\kappa)^8 [\epsilon(k)]^{\mu_1 \dots \mu_8} [K(2,1,0,0)]^{\rho_1 \dots \rho_8} [H(8)]_{\rho_1 \mu_1 \dots \rho_8 \mu_8}$$

$$[\epsilon(k)]^{\mu_1 \dots \mu_8} = \epsilon^{\mu_1 \nu_1 \dots \mu_8 \nu_8} k_{1\nu_1} \dots k_{8\nu_8}$$

$$[H(8)]_{\rho_1 \mu_1 \dots \rho_8 \mu_8} = h_{\rho_1 \mu_1}(k_1) \dots h_{\rho_8 \mu_8}(k_8)$$

$$[K(2,1,0,0)]^{\rho_1 \dots \rho_8} = (k_1^{\rho_2} k_2^{\rho_1})(k_3^{\rho_4} k_4^{\rho_3})(k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_5})$$

The term  $t(n)$  is special among the operator terms in the perturbation expansion of  $T(n)$  because it is the only one which contains no dot products of the momenta  $k_i$  and no contractions of one gauge field with another. That is, it does not contain any factors  $k_i \cdot k_j$  or either of the combinations  $h^{\rho\mu} h_{\sigma\mu}$  and  $h^{\rho\mu} h_{\rho\mu}$ . In these respects  $t(n)$  is easily distinguishable from the other terms in  $T(n)$ . The  $t(n)$  as a group are also easily distinguishable from each other since the factor  $K(n)$  within each  $t(n)$  has a unique momentum chain structure. In this sense  $K(n)$  is a kind of signature for  $t(n)$ . Of course, the terms that I have labelled  $t(n)$  have been selected for

consideration precisely because they are readily identifiable. Recall that our general strategy involves taking the amplitudes of the operators in equations (19) and (20) between the vacuum and suitable multiparticle states, and then picking out of these amplitudes the contributions of the terms  $t(\mathbf{n})$ . We shall see below that the distinctive structure of the momentum products  $K(\mathbf{n})$  allows us to do this fairly easily.

However before proceeding any further it must be decided just which amplitudes of the operators in (19) and (20) are relevant to the present problem. Clearly we must require that the amplitudes of the terms  $t(\mathbf{n})$  between the vacuum and whichever multiparticle states are chosen be non-zero. Since  $t(\mathbf{n})$  contains  $2N$  graviton fields  $h^{\rho\mu}$  the obvious multiparticle state to consider is the one containing precisely  $2N$  gravitons. Denoting the amplitude of an operator  $O$  between the vacuum and such a state by  $\langle O \rangle_{2N}$ , one finds from equations (19) and (20) that

$$\sum_{|\mathbf{n}|=N}^n C^{1/2}(\mathbf{n}) \langle t(\mathbf{n}) \rangle_{2N} \subset \lim_{l \rightarrow n} \langle e \bar{\psi} \{ \not{D}, \Gamma^{-1} \} \psi \rangle_{2N} \quad ..(26)$$

$$\sum_{|\mathbf{n}|=N}^n C^{3/2}(\mathbf{n}) \langle t(\mathbf{n}) \rangle_{2N} \subset \lim_{l \rightarrow n} \langle e \bar{\phi}_\alpha \{ \not{D}, \Gamma^{-1} \} \phi^\alpha \rangle_{2N} \quad ..(27)$$

The signs " $\subset$ " indicate that the amplitudes of the  $t(\mathbf{n})$  are contained within those of the operators on the right hand sides of (26) and (27) along with the amplitudes of many other operator terms. Equations (26) and (27) may be simplified by making use of the expression (21) for  $t(\mathbf{n})$  :

$$\langle t(\mathbf{n}) \rangle_{2N} = (2\kappa)^{2N} \epsilon(k) K(\mathbf{n}) \langle H(2N) \rangle_{2N}$$

The expectation value  $\langle H(2N) \rangle_{2N}$  can be evaluated using the LSZ reduction formulae. In the present case the formulae tell us that the evaluation of  $\langle H(2N) \rangle_{2N}$  effectively reduces to a sum over the  $(2N)!$  permutations of the momenta  $k_1, \dots, k_{2N}$ . Not all of these permutations will leave  $K(\mathbf{n})$  invariant. In fact it is a simple matter to see that the number of permutations which do map  $K(\mathbf{n})$  into itself is

$$\tau(\mathbf{n}) = n_1! n_2! \dots n_N! (4)^{n_1} (8)^{n_2} \dots (4N)^{n_N} \quad ..(28)$$

Therefore

$$\langle t(n) \rangle_{2N} = (2\kappa)^{2N} \tau(n) \epsilon(k) K(n)$$

so equations (26) and (27) become

$$\epsilon(k)(2\kappa)^{2N} \sum_{|n|=N}^n C^{1/2}(n) \tau(n) K(n) \subset \lim_{l \rightarrow n} \langle e \bar{\psi} \{ \not{p}, \Gamma^{-1} \} \psi \rangle_{2N} \quad ..(29)$$

$$\epsilon(k)(2\kappa)^{2N} \sum_{|n|=N}^n C^{3/2}(n) \tau(n) K(n) \subset \lim_{l \rightarrow n} \langle e \bar{\phi}_\alpha \{ \not{p}, \Gamma^{-1} \} \phi^\alpha \rangle_{2N} \quad ..(30)$$

where  $\epsilon(k)$ ,  $K(n)$  and  $\tau(n)$  are as in (22), (25) and (28). As was explained above, the momentum product  $K(n)$  is unique to the term  $t(n)$ . It follows, then, that the problem of finding the  $C(n)$  reduces to the problem of finding the coefficients of the momentum factors  $K(n)$  within the amplitudes on the right hand sides of (29) and (30). Let us therefore single out which Feynman diagrams among those that contribute to these amplitudes could possibly contain the quantities  $K(n)$ . For a start we know that we need only consider single fermion loop diagrams since, according to Adler-Bardeen type arguments, these are the only diagrams which are potentially anomalous. Moreover we see from equations (29) and (30) that relevant diagrams must be of order  $\kappa^{2N}$ . The upshot of these considerations is that the only suitable Feynman diagrams are those of the form shown in figure 2 below.

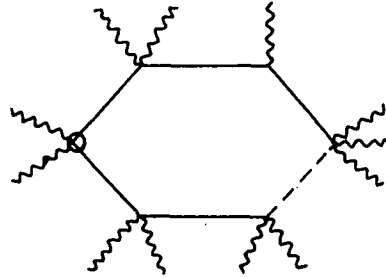


FIGURE 2

In this diagram there is one axial vertex, which is marked with a cross, and a number of regular vertices. Connected to the regular vertices and axial vertex are a total of  $2N$  gravitons. By requiring that there be exactly this



number of graviton fields one ensures that the diagram is of the correct order  $\kappa^{2N}$ . I now assert that diagrams which contain vertices to which more than one gauge field attaches may be ruled out of consideration. The reasons for this are as follows. The gauge fields that attach to a vertex, whether it be an axial or regular vertex, do so via the vector indices on quantities such as  $\gamma^\mu$ ,  $\sigma^{\gamma\delta}$  and  $\Sigma^{\gamma\delta}$  within the covariant derivative  $D^\rho$  [see equations (2.5) and (2.34)]. A simple count of available indices will convince one that if more than one gauge field attaches to a vertex then at least two of the gauge fields must be contracted together. In other words the gauge fields will form combinations such as  $h^{\rho\mu}h_{\sigma\mu}$  and  $h^{\rho\mu}h_{\rho\mu}$ . Correspondingly, the vertex will contain factors such as  $\eta_{\rho\mu}$ ,  $\eta_{\rho\rho}$  and  $\eta_{\mu\mu}$ . As we have seen, there are no contractions of gauge fields in  $t(n)$  and the factors  $\eta_{\rho\mu}$ ,  $\eta_{\rho\rho}$  and  $\eta_{\mu\mu}$  do not occur in the amplitude  $\langle t(n) \rangle$ . Therefore a diagram of the type shown in figure 2 will not contribute to  $\langle t(n) \rangle$  if more than one gauge field emerges from any given vertex. The only remaining diagrams which could possibly contribute to  $\langle t(n) \rangle$  are shown in figures 3 and 4.

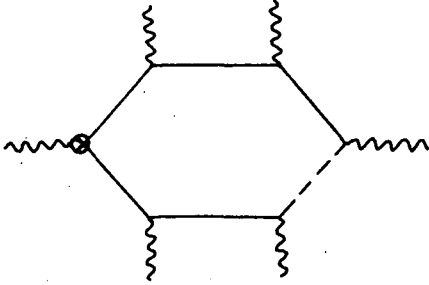


FIGURE 3

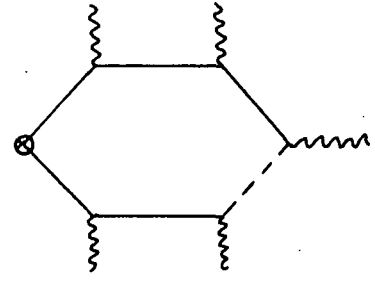


FIGURE 4

In the first of these diagrams one gauge field emerges from each vertex, whether it be axial or regular. Consequently in this diagram each vertex is of first order in  $\kappa$ . On the other hand in the second diagram there is one gauge field attached to each of the regular vertices but no gauge field attached to the axial vertex. In the diagram of figure 4, therefore, each of the regular vertices is of first order in  $\kappa$  while the axial vertex is of zeroth order. Now recall that each of diagrams 3 and 4 must have  $2N$  graviton legs. We deduce that the diagram of figure 3 contains one axial vertex and  $2N-1$  regular vertices, while the diagram of figure 4 contains one axial vertex and  $2N$  regular vertices. On the basis of this vertex count I can now

say that the Feynman diagram of figure 3 is zero. This conclusion follows from properties of the matrix  $\Gamma^{-1}$  that are dealt with in appendix 4. There are simply too few  $\gamma$ -matrices in the diagram's fermion loop to give a non-zero trace.

We are therefore left with the diagram of figure 4. This diagram, which is reproduced in more detail in figure 5, is the only Feynman diagram which can possibly contain the momentum factor  $K(n)$ . Consequently it is the only one which is relevant to the present anomaly calculation. In the spin 1/2 and spin 3/2 cases the loop particles in this diagram will be spin 1/2 and spin 3/2 fermions respectively. I will denote the values of the diagram in these two cases by  $D^{1/2}(N)$  and  $D^{3/2}(N)$ , thereby explicitly recognizing their dependence on the number  $N$ . If I do not wish to consider the spin 1/2 and spin 3/2 cases independently I will simply refer to the quantity  $D(N)$ .

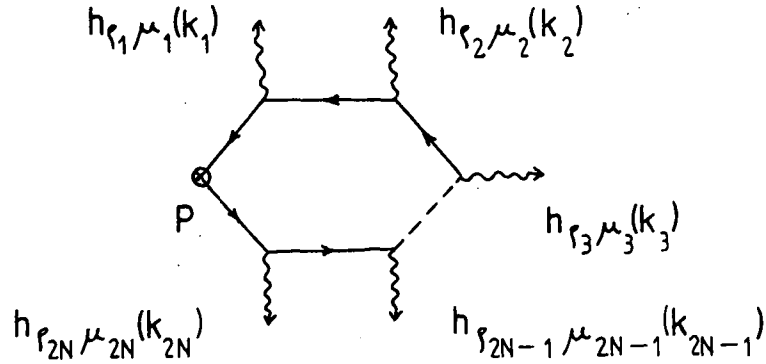


FIGURE 5

In terms of  $D^{1/2}(N)$  and  $D^{3/2}(N)$  equations (29) and (30) can be rewritten in the following form. The limits in (31) and (32) indicate that the values of the diagrams  $D^{1/2}(N)$  and  $D^{3/2}(N)$  are to be calculated in dimension  $d=2\ell$  and then continued back to  $d=2n$ . That is, one should take the limit  $\ell \rightarrow n$  only after the relevant loop momentum integrals have been done.

$$\epsilon(k)(2\kappa)^{2N} \sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{1/2}(\mathbf{n}) \tau(\mathbf{n}) K(\mathbf{n}) \subset \lim_{\ell \rightarrow n} D^{1/2}(N) \quad \dots(31)$$

$$\epsilon(k)(2\kappa)^{2N} \sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{3/2}(\mathbf{n}) \tau(\mathbf{n}) K(\mathbf{n}) \subset \lim_{\ell \rightarrow n} D^{3/2}(N) \quad \dots(32)$$

### 3.4 Anomaly Diagrams

As we have seen, the values of the anomalies  $A^{1/2}$  and  $A^{3/2}$  may be derived by considering just the two Feynman diagrams  $D^{1/2}(N)$  and  $D^{3/2}(N)$ . In the remainder of the thesis I will be concerned not so much with calculating  $D^{1/2}(N)$  and  $D^{3/2}(N)$  as with extracting the coefficients  $C^{1/2}(n)$  and  $C^{3/2}(n)$  from these quantities. Nevertheless  $C^{1/2}(n)$  and  $C^{3/2}(n)$  cannot be extracted directly from  $D^{1/2}(N)$  and  $D^{3/2}(N)$  as they stand, and some partial evaluation of the Feynman diagrams is necessary. In this section I will describe in some detail the steps in the partial evaluation of  $D^{1/2}(n)$ . The parallel manipulations of  $D^{3/2}(N)$  are almost identical and I have not wasted space by including them too. The reader should have no difficulty in adapting the spin 1/2 calculations to the spin 3/2 case. The section culminates in separate intermediate formulae for the two sets of coefficients  $C^{1/2}(n)$  and  $C^{3/2}(n)$ . These formulae then become the starting point for the calculations of chapter 4.

The propagator and regular vertices that occur in  $D^{1/2}(N)$  were described in chapter 2. The propagator  $S(p)$  for a massive spin 1/2 field is given by equation (2.3). For reasons discussed at the end of section 2.1, when calculating  $A^{1/2}$  we need consider only massless spin 1/2 fields. So  $m$  may be set to zero in  $S(p)$ . The first order spin 1/2 gravitational vertex  $V^{\rho\mu}(p)$  is likewise given by equation (2.8). The zeroth order spin 1/2 axial vertex must be extracted from the operator product on the right hand side of (26). It is  $\bar{\psi}(p)A(p,q)\psi(q)$  where

$$A(p,q) = \frac{1}{2} \{ \not{p} + \not{q}, \Gamma^{-1} \}$$

I am now in a position to write down an expression for  $D^{1/2}(N)$ . This expression is vastly simplified if the following notational convention is adopted.

$$p_i \equiv p - k_{i+1} - k_{i+2} - \dots - k_{2N} \quad \dots(33)$$

where  $i$  has the range  $0, 1, \dots, 2N$  and  $p_{2N} = p$ . In terms of the  $p_i$  and the above-mentioned propagator and vertices, the integral  $D^{1/2}(N)$  is given by

$$D^{1/2}(N) = \sum_P \int d^{2\ell} p (-1) \text{tr} [S(p_0) V^{p_1 \mu_1}(p_1) S(p_1) V^{p_2 \mu_2}(p_2) \dots \\ \dots S(p_{2N-1}) V^{p_{2N} \mu_{2N}}(p_{2N}) S(p_{2N}) A(p_{2N}, p_0)] \quad \dots (34)$$

$$= \frac{1}{2} \kappa^{2N} \sum_P \int d^{2\ell} p [p_1^{p_1} p_2^{p_2} \dots p_{2N}^{p_{2N}}] [p_0^2 p_1^2 \dots p_{2N}^2]^{-1} \\ \text{tr} [\not{p}_0 \gamma^{\mu_1} \not{p}_1 \gamma^{\mu_2} \dots \gamma^{\mu_{2N}} \not{p}_{2N} \{\not{p}_0 + \not{p}_{2N}, \Gamma^{-1}\}] \quad \dots (35)$$

In these equations the sum  $\Sigma$  is over the  $(2N)!$  Bose permutations of the momenta  $k_1, \dots, k_{2N}$ . The first simplification that can be made in the above expression for  $D^{1/2}(N)$  is effected by introducing Feynman parameters  $x_0, x_1, \dots, x_{2N}$  and rewriting the denominator  $[p_0^2 p_1^2 \dots p_{2N}^2]^{-1}$  using the formula

$$\prod_{k=0}^{2N} \frac{1}{A_k} = (2N)! \int_0^1 dx_0 \dots dx_{2N} \delta(1-x_0-\dots-x_{2N}) [x_0 A_0 + \dots + x_{2N} A_{2N}]^{-(2N+1)}$$

In this way we find that

$$[p_0^2 p_1^2 \dots p_{2N}^2]^{-1} = (2N)! \int_0^1 dx_0 \dots dx_{2N} \delta(1-x_0-\dots-x_{2N}) [p^2 - 2p \cdot X + q^2]^{-(2N+1)} \quad \dots (36)$$

On the right hand side of (36)  $q^2$  is some polynomial in both the Feynman parameters  $x_1, \dots, x_{2N}$  and dot products of the  $k$ 's. As we shall see, the exact form of  $q^2$  is unimportant. The quantity  $X$  is as follows.

$$X = x_1 k_1 + (x_1 + x_2) k_2 + (x_1 + x_2 + x_3) k_3 + \dots + (x_1 + \dots + x_{2N}) k_{2N} \quad \dots (37)$$

By changing variables

$$p \rightarrow p' = p - X$$

and integrating out the Feynman parameter  $x_0$  which appears in neither  $X$  nor  $q^2$ , one finally arrives at the following results

$$[p_0^2 p_1^2 \dots p_{2N}^2]^{-1} = [(p'_0 + X)^2 (p'_1 + X)^2 \dots (p'_{2N} + X)^2] \\ = (2N)! \int_0^1 dx_1 \dots dx_{2N} \theta(1-x_1-\dots-x_{2N}) [p'^2 - q'^2]^{-(2N+1)}$$

In these equations  $p'_i$  is given by the expression on the right hand side of (33) except that  $p$  is replaced by  $p'$ . The polynomial  $q'^2$  is simply equal to  $q^2 - X^2$ , and  $\theta$  is the familiar Heaviside theta function. If I now drop the primes on  $p'$  and  $q'$  then  $D^{1/2}(N)$  may be written

$$D^{1/2}(N) = i(2N)! 2^{\ell} (i\kappa)^{2N} \sum_P \int d^{2\ell} p \int dx \theta(1 - \Sigma x) T [(p_1 + X)^{\rho_1} \dots (p_{2N} + X)^{\rho_{2N}}] [p^2 - q^2]^{-(2N+1)} \quad ..(38)$$

where  $dx \equiv dx_1 \dots dx_{2N}$ ,  $\Sigma x \equiv x_1 + \dots + x_{2N}$  and  $T$  is the trace

$$T = -[i^{2N+1}/2^{\ell+1}] \text{tr}[(p_0 + X) \gamma^{\mu_1} (p_1 + X) \gamma^{\mu_2} \dots \gamma^{\mu_{2N}} (p_{2N} + X) \{\delta_0 + p_{2N} + 2X, \Gamma^{-1}\}]$$

$T$  may be calculated using some results from appendix 4. Firstly note that due to the form of the matrix  $\Gamma^{-1}$ , as given in equation (3), the  $\gamma$ -matrix trace  $\text{tr}[a_1 \dots a_{2M+1} \{\delta_{2M+2}, \Gamma^{-1}\}]$  is zero unless  $M > N$ . As a consequence the trace  $\text{tr}[a_1 \dots a_{2N+1} \{\delta_{2N+2}, \Gamma^{-1}\}]$  is antisymmetric under interchange of any two of the vectors  $a_1, \dots, a_{2N+1}$ . Applying this result to  $T$  one finds that

$$T = -[i^{2N+1}/2^{\ell+1}] \text{tr}[(p + X) \gamma^{\mu_1} k_1 \gamma^{\mu_2} k_2 \dots k_{2N-1} \gamma^{\mu_{2N}} k_{2N} \{2p + 2X - k_1 - \dots - k_{2N}, \Gamma^{-1}\}]$$

Having reduced  $T$  to this simplified form one then applies the following formula which is derived in appendix 3.

$$\text{tr}[a_1 \dots a_{4N+1} \{\delta_{4N+2}, \Gamma^{-1}\}] = 2^{\ell+1} (-1)^{2N+1} \sum_{k=0}^{4N+1} (-1)^k a_k \cdot a_{4N+2} \xi \cdot a_1 \dots a_{k-1} a_{k+1} \dots a_{4N+1} \quad ..(39)$$

The notation employed here is reasonably obvious. The tensor  $\xi$  is the totally antisymmetric tensor of equation (4), and

$$\xi \cdot b_1 \dots b_{4N} \equiv \xi^{v_1 \dots v_{4N}} b_{1v_1} \dots b_{4Nv_{4N}}$$

I will now extend this notation so that the vectors  $b_i$  may be replaced by tensor indices. In this way, for instance, one has

$$\xi \cdot \mu_1 k_1 \dots \mu_{2N} k_{2N} \equiv \xi^{\mu_1 v_1 \dots \mu_{2N} v_{2N}} k_{1v_1} \dots k_{2Nv_{2N}}$$

Using formula (39) and taking the antisymmetry of  $\xi$  into account one can decompose  $T$  into two groups of terms,  $T=T'+T''$ . The terms in  $T'$  contain one  $p$  while those in  $T''$  contain two. In arriving at the following expressions for  $T'$  and  $T''$  I have ignored any terms containing dot products of the  $k$ 's. As has been pointed out on previous occasions, such terms do not contain the momentum product  $K(n)$  and are therefore irrelevant to the anomaly.

$$\begin{aligned}
T' = & p \cdot (2X - k_1 - \dots - k_{2N}) \xi \cdot \mu_1 k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N} \\
& - (2X - k_1 - \dots - k_{2N})^{\mu_1} \xi \cdot p k_1 \mu_2 k_2 \mu_3 \dots \mu_{2N} k_{2N} \\
& - (2X - k_1 - \dots - k_{2N})^{\mu_2} \xi \cdot p \mu_1 k_1 k_2 \mu_3 \dots \mu_{2N} k_{2N} \\
& \cdot \\
& \cdot \\
& - (2X - k_1 - \dots - k_{2N})^{\mu_{2N}} \xi \cdot p \mu_1 k_1 \mu_2 k_2 \dots k_{2N-1} k_{2N}
\end{aligned}$$

$$\begin{aligned}
T'' = & 2p^2 \xi \cdot \mu_1 k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N} \\
& + 2p \cdot k_1 \xi \cdot p \mu_1 \mu_2 k_2 \mu_3 \dots \mu_{2N} k_{2N} - 2p^{\mu_1} \xi \cdot p k_1 \mu_2 k_2 \mu_3 \dots \mu_{2N} k_{2N} \\
& + 2p \cdot k_2 \xi \cdot p \mu_1 k_1 \mu_2 \mu_3 \dots \mu_{2N} k_{2N} - 2p^{\mu_2} \xi \cdot p \mu_1 k_1 k_2 \mu_3 \dots \mu_{2N} k_{2N} \\
& \cdot \\
& \cdot \\
& + 2p \cdot k_{2N} \xi \cdot p \mu_1 k_1 \mu_2 k_2 \dots k_{2N-1} \mu_{2N} - 2p^{\mu_{2N}} \xi \cdot p \mu_1 k_1 \mu_2 k_2 \dots k_{2N-1} k_{2N}
\end{aligned}$$

The numerator of the integrand in  $D^{1/2}(N)$  may now be written in the form

$$(p_1 + X)^{\rho_1} (p_2 + X)^{\rho_2} \dots (p_{2N} + X)^{\rho_{2N}} [T' + T'']$$

Of course, our principal concern is to find those terms in  $D^{1/2}(N)$  which contain the momentum factor  $K(n)$ , or more precisely the combination  $\epsilon(k)K(n)$ . Note that  $\epsilon(k)K(n)$  contains exactly  $4N$   $k$ 's, while the terms in the numerator of the integrand in  $D^{1/2}(N)$  contain  $4N+2$  momenta, some of which are  $p$ 's and some  $k$ 's. It follows that we must look for terms in the numerator of the integrand which contain  $4N$   $k$ 's and two  $p$ 's. There are two options here. In the first place both of the  $p$ 's in such a term could come from one of the terms in  $T''$ . Alternatively one of the  $p$ 's could come from a term in  $T'$  and the other from one of the factors  $(p_i + X)^{\rho_i}$ . Consider the

second option. Let us suppose that one  $p$  comes from the factor  $(p_1+X)^{\rho_1}$  and the other from within  $T'$ . The  $p$ -integration in (38) is even so within the product  $p^{\rho_1}T'$  we may replace the combination  $p^\alpha p^\beta$  with  $(p^2/2\ell)\eta^{\alpha\beta}$ . When this replacement is effected in the first and second terms in  $p^{\rho_1}T'$  they cancel due to the symmetry of  $h^{\rho_1\mu_1}$  in its indices :

$$p^{\rho_1}p \cdot (2X+k_1+\dots+k_{2N})\xi \cdot \mu_1 k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N} \rightarrow \frac{p^2}{2\ell} (2X+k_1+\dots+k_{2N})^{\rho_1} \xi \cdot \mu_1 k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N}$$

$$p^{\rho_1} (2X+k_1+\dots+k_{2N})^{\mu_1} \xi \cdot p k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N} \rightarrow \frac{p^2}{2\ell} (2X+k_1+\dots+k_{2N})^{\mu_1} \xi \cdot p k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N}$$

The other terms in  $p^{\rho_1}T'$  are zero when the same replacement is made because of the antisymmetry of  $\xi$  in its indices. Consequently  $p^{\rho_1}T'$  effectively vanishes. The same is true for  $p^{\rho_i}T', i=2,3,\dots,2N$  as similar cancellations occur in each case. Thus the terms in  $T'$  do not contribute to the anomalies and  $T$  is effectively equal to  $T''$ . Because each term in  $T''$  contains two  $p$ 's, the  $p$ 's in the factors  $(p_1+X)^{\rho_i}$  may be ignored. This means that the factor  $(p_1+X)^{\rho_i}$  can be replaced with the quantity  $X^{\rho_i}(1)$  where

$$X(1) = X - k_{1+1} - k_{1+2} - \dots - k_{2N} \quad \dots(40)$$

and  $X$  is as in equation (37). When the substitution  $p^\alpha p^\beta \rightarrow (p^2/2\ell)\eta^{\alpha\beta}$  is made throughout  $T=T''$  one finds that

$$T = (p^2/\ell)(2\ell-4N) \xi \cdot \mu_1 k_1 \mu_2 k_2 \dots \mu_{2N} k_{2N}$$

Therefore  $D^{1/2}(N)$  is equal to

$$D^{1/2}(N) = i(2N)! 2^\ell (i\kappa)^{2N} [(2\ell-4N)/\ell]$$

$$\xi \cdot \mu_1 k_1 \dots \mu_{2N} k_{2N} \sum_P I^{1/2}(N) \int d^{2\ell} p p^2 (p^2 - q^2)^{-(2N+1)} \quad \dots(41)$$

where

$$I^{1/2}(N) = \int_0^1 dx_1 \dots dx_{2N} \theta(1-x_1-\dots-x_{2N}) X^{\rho_1}(1) X^{\rho_2}(2) \dots X^{\rho_{2N}}(2N) \quad \dots(42)$$

Note that  $\xi \cdot \mu_1 k_1 \dots \mu_{2N} k_{2N}$  is invariant under permutations of the labels  $1, \dots, 2N$  so it has been taken outside the sum in (41). The momentum integral in our expression for  $D^{1/2}(N)$  can now be carried out with the help of the following formula [10]

$$\int d^{2\ell} p \, p^2 (p^2 - q^2)^{-(2N+1)} = i \frac{\Gamma(\ell+1)\Gamma(2N-\ell)}{\Gamma(\ell)\Gamma(2N+1)} (4\pi)^{-\ell} (q^2)^{\ell-2N}$$

Having calculated the loop momentum integral we are at last free to take the regulator limit  $\ell \rightarrow n$ , which in the present case becomes  $\ell \rightarrow 2N$ . Noting that in this limit  $(2N-\ell)\Gamma(2N-\ell) \rightarrow 1$ , and that in accordance with equation (4)  $\xi \rightarrow \epsilon$ , one arrives at the result

$$\lim_{\ell \rightarrow n} D^{1/2}(N) = 2 \left( \frac{i\kappa}{2\pi} \right)^{2N} \epsilon(k) \sum_P I^{1/2}(N) \quad ..(43)$$

Equations (42) and (43) are as far as I want to go with the analysis of the diagram  $D^{1/2}(N)$  in this chapter. The derivation of the spin 3/2 counterparts of these equations is almost identical to the above spin 1/2 derivation. One begins with an expression for  $D^{3/2}(N)$  similar to (34) except that the propagators and vertices both carry additional vector indices, and the vertices now depend upon the external momenta as well as the fermion loop momentum :

$$D^{3/2}(N) = \sum_P \int d^{2\ell} p \, (-1) \text{tr} [S_{\alpha_0 \beta_0}(p_0) V^{\beta_0 \alpha_1 \rho_1 \mu_1}(p_1, k_1) S_{\alpha_1 \beta_1}(p_1) \dots \\ \dots V^{\beta_{2N-1} \alpha_{2N} \rho_{2N} \mu_{2N}}(p_{2N}, k_{2N}) S_{\alpha_{2N} \beta_{2N}}(p_{2N}) A^{\beta_{2N} \alpha_0}(p_{2N}, p_0)] \quad ..(44)$$

Since I have elected to calculate the spin 3/2 anomaly in the  $A=0$  formulation, and with the gauge choice  $\alpha=[4/(2\ell-2)]$ , the appropriate propagator and regular vertex are given by equations (2.42) and (2.43). The zeroth order spin 3/2 axial vertex must be extracted from the operator product on the right hand side of (27). It is  $\bar{\phi}_\alpha(p) A^{\alpha\beta}(p, q) \phi_\beta(q)$  where

$$A^{\alpha\beta}(p, q) = \frac{1}{2} \eta^{\alpha\beta} \{ \not{p} + \not{q}, \Gamma^{-1} \}$$

As the reader will note, the expressions for the spin 3/2 propagator and vertices are very close to their spin 1/2 equivalents. Consequently there



are only two significant differences between the mathematics leading to equations (42) and (43) and the corresponding spin 3/2 mathematics. Firstly, the product  $p_1^{\rho 1} \dots p_{2N}^{\rho 2N}$  is replaced by the quantity

$$p_1^{\rho 1 \alpha_1} p_2^{\rho 2 \alpha_2} \dots p_{2N-1}^{\rho 2N-1 \alpha_{2N-1}} p_{2N}^{\rho 2N \alpha_{2N}} \alpha_1$$

where

$$p_i^{\rho \alpha \beta} = \eta^{\alpha \beta} p_i^{\rho} + \eta^{\alpha \rho} k_i^{\beta} - \eta^{\beta \rho} k_i^{\alpha}$$

and  $p_i$  is as in equation (33). Secondly, the product  $X^{\rho 1}(1) \dots X^{\rho 2N}(2N)$  is replaced by

$$Y^{\rho 1 \alpha_1} \alpha_2(1) Y^{\rho 2 \alpha_2} \alpha_3(2) \dots Y^{\rho 2N-1 \alpha_{2N-1}} \alpha_{2N}(2N-1) Y^{\rho 2N \alpha_{2N}} \alpha_1(2N)$$

where

$$Y^{\rho \alpha \beta}(i) = \eta^{\alpha \beta} X^{\rho}(i) + \eta^{\alpha \rho} k_i^{\beta} - \eta^{\beta \rho} k_i^{\alpha}$$

and  $X(i)$  is given by (40). When these substitutions are carried out one ends up with the following expression for  $D^{3/2}(N)$

$$\lim_{\ell \rightarrow \infty} D^{3/2}(N) = 2 \left( \frac{1}{2\pi} \right)^{2N} \epsilon(K) \int_P I^{3/2}(N) \quad \dots (45)$$

$$I^{3/2}(N) = \int_0^1 dx_1 \dots dx_{2N} \theta(1-x_1 \dots -x_{2N}) Y^{\rho 1 \alpha_1} \alpha_2(1) \dots Y^{\rho 2N \alpha_{2N}} \alpha_1(2N) \quad \dots (46)$$

From equations (43) and (45) we conclude that the problem of calculating the anomaly coefficients  $C(n)$  has been reduced to the problem of finding the coefficients of  $K(n)$  within the quantities  $\Sigma I^{1/2}(2N)$  and  $\Sigma I^{3/2}(2N)$ . Let me denote these latter coefficients by  $c^{1/2}(n)$  and  $c^{3/2}(n)$  respectively. By combining equations (28), (31) and (43) and equations (28), (32) and (45) one finds that

$$C^{1/2}(n) = 2(1/4\pi)^{2N} [n_1! n_2! \dots n_N! (4)^{n_1} (8)^{n_2} \dots (4N)^{n_N}]^{-1} c^{1/2}(n) \quad \dots (47)$$

$$c^{3/2}(\mathbf{n}) = 2(1/4\pi)^{2N} [n_1! n_2! \dots n_N! (4)^{n_1} (8)^{n_2} \dots (4N)^{n_N}]^{-1} c^{3/2}(\mathbf{n}) \quad \dots(48)$$

Equations (42) and (46) will be my starting points for the calculations of  $c^{1/2}(\mathbf{n})$  and  $c^{3/2}(\mathbf{n})$  in chapter 4.

### 3.5 Gauge Independence of the Spin 3/2 Anomaly

In chapter 2 I explained the relationship between the various formulations of spin 3/2 field theory. Working in the conventional Rarita-Schwinger formulation I showed that formulations of spin 3/2 theory characterized by values of  $A$  other than  $-1$  could be reached using local and non-local field redefinitions. Subsequently I made the not unreasonable assumption that physical quantities are invariant under such field redefinitions, and that correspondingly the spin 3/2 anomaly  $A^{3/2}$  is the same regardless of the formulation in which it is calculated. Once this assumption is made, and once the Feynman diagram of figure 5 is singled out as being the only one of relevance to the anomaly, a proof of the gauge independence of  $A^{3/2}$  is trivial.

Consider the Feynman diagram of figure 5 whose value in the spin 3/2 case I have denoted by  $D^{3/2}(N)$ . In the previous section I showed how to partially evaluate the quantity  $D^{3/2}(N)$  in the  $A=0$  formulation with the choice of gauge  $\alpha=[4/(2\ell-2)]$ . Let us now look at the value of  $D^{3/2}(N)$  in the  $A=-1/\ell$  formulation without fixing the gauge parameter  $\alpha$ . In this case  $D^{3/2}(N)$  is still formally given by equation (44), however the propagator and regular vertex are as in equations (2.53) and (2.55) rather than (2.42) and (2.43). The zeroth order spin 3/2 axial vertex appropriate to the  $A=-1/\ell$  formulation is  $\bar{\chi}_\alpha(p) A^{\alpha\beta}(p,q) \chi_\beta(q)$  where

$$A^{\alpha\beta}(p,q) = \frac{1}{2} (p+q)_\rho \{M^{\alpha\rho\beta}, \Gamma^{-1}\}$$

and  $M^{\alpha\rho\beta}$  is the quantity which appears in equation (2.49). The properties of the propagator (2.53), the vertex (2.55) and the quantity  $M^{\alpha\rho\beta}$  have been summarized in equations (2.31), (2.52) and (2.56). Using these properties it is a simple matter to verify that the four contractions

$$S_{\alpha\beta}^{\gamma\rho\mu}, V^{\alpha\beta\rho\mu}_{\beta\gamma}, S_{\alpha\beta}^{\beta\rho\gamma}, M^{\alpha\rho\beta}_{\beta\gamma} \quad ..(49)$$

are all independent of  $\alpha$ . For instance

$$\begin{aligned} V^{\alpha\beta\rho\mu}(p,k)S_{\beta\gamma}(p) &= [\bar{V}^{\alpha\beta\rho\mu}(p,k) - (\frac{i\kappa}{\alpha\ell^2})p^\rho\gamma^\alpha\gamma^\mu\gamma^\beta][\bar{S}_{\beta\gamma}(p) - i\frac{\alpha}{4}\gamma_\beta\frac{1}{p}\gamma_\gamma] \\ &= \bar{V}^{\alpha\beta\rho\mu}(p,k)\bar{S}_{\beta\gamma}(p) - (\frac{\kappa}{2\ell})p^\rho\gamma^\alpha\gamma^\mu\frac{1}{p}\gamma_\gamma \end{aligned}$$

Obviously if the four contractions (49) are independent of  $\alpha$  then so is the integral  $D^{3/2}(N)$  which is given by equation (44). But this implies that the spin 3/2 anomaly  $A^{3/2}$  is independent of  $\alpha$  in the  $A=-1/\ell$  formulation. Therefore it is independent of  $\alpha$  in all formulations.

There is one major problem with this proof as it stands - it employs the  $A=-1/\ell$  formulation of spin 3/2 theory. As is explained in appendix 2,  $-1/\ell$  is the only value that the parameter  $A$  cannot validly assume. This is because the  $A=-1/\ell$  Lagrangian does not describe a purely spin 3/2 theory. Instead it contains propagating spin 1/2 degrees of freedom in addition to the desired spin 3/2 degrees of freedom. Fortunately these problems can be avoided in the above proof by considering formulations of spin 3/2 field theory corresponding to the parameter choices  $A=(-1/\ell)+\epsilon$  where  $0<|\epsilon|<1$ . These formulations of spin 3/2 theory are entirely acceptable according to the criteria which are set down in appendix 2. Moreover, they can all be reached from the Rarita-Schwinger ( $A=-1$ ) formulation via local changes of the field variable. Consequently they are on exactly the same footing as the  $A=0$  formulation in which the anomaly calculation is being performed.

In chapter 2 I explained why it is reasonable to assume that the spin 3/2 anomaly is independent of the value of the parameter  $A$ . (The value  $-1/\ell$  is naturally excluded from consideration here.) In the present case this assumption tells us that the anomaly is the same in the  $A=-1$  and  $A=0$  formulations of spin 3/2 theory, as well as in the above  $\epsilon$ -formulations. In particular, when calculated in the  $\epsilon$ -formulations, the anomaly must be independent of the parameter  $\epsilon$ . Now consider the forms of the spin 3/2 propagator and vertices in these  $\epsilon$ -formulations. When one shifts  $A$  from  $-1/\ell$  to  $(-1/\ell)+\epsilon$  one finds that the propagator and vertices of equations (2.53),

(2.55) and (49) all receive corrections which are linear or quadratic in  $\epsilon$ . Note that none of these corrections involves negative powers of  $\epsilon$ . Consequently, when one substitutes the corrected propagator and vertices into the right hand side of (44) and expands in powers of  $\epsilon$ , one finds that the zeroth order part of  $D^{3/2}(N)$  is just the expression that one would obtain by naively working in the  $A=-1/\ell$  formulation. Of course,  $D^{3/2}(N)$  contains terms which are of higher orders in  $\epsilon$ . However, since the spin 3/2 anomaly has been assumed to be independent of  $\epsilon$ , these terms may be taken to vanish identically. Thus, in the (admissible)  $\epsilon$ -formulations of spin 3/2 field theory, the integral  $D^{3/2}(N)$  is exactly as one would expect it to be in the (inadmissible)  $A=-1/\ell$  formulation. In view of the above arguments, it is therefore clear that the spin 3/2 anomaly is independent of the gauge parameter  $\alpha$  in the  $\epsilon$ -formulations. This means that it is independent of  $\alpha$  in all formulations of spin 3/2 field theory, including the  $A=-1/\ell$  and  $A=0$  formulations.

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## CHAPTER 4. The Spin 1/2 and Spin 3/2 Anomalies

In this final chapter of the thesis I will complete the diagrammatic anomaly calculation begun in chapter 3. As we have seen, this task involves extracting the coefficients  $c^{1/2}(n)$  and  $c^{3/2}(n)$  from the combined sums and integrals  $\Sigma I^{1/2}$  and  $\Sigma I^{3/2}$ . My method of doing this depends upon expressing  $c^{1/2}(n)$  and  $c^{3/2}(n)$  in terms of sub-coefficients  $Q(m)$ , and then establishing and solving recurrence relations between the  $Q(m)$  in different dimensions. Once the general expression for the  $Q(m)$  is known, one can progress fairly easily to expressions for  $c^{1/2}(n)$  and  $c^{3/2}(n)$  for arbitrary values of  $N$ , and consequently to expressions for  $A^{1/2}$  and  $A^{3/2}$  in arbitrary dimension  $d=4N$ . I have found it convenient to commence the work of this chapter by considering the spin 1/2 and spin 3/2 cases separately in sections 1 and 2. Although the ideas behind the mathematical apparatus used here to analyse  $\Sigma I^{1/2}$  and  $\Sigma I^{3/2}$  are simple, the notation involved is quite complicated. For this reason sections 1 and 2 will largely be devoted to establishing notational conventions and terminology. In an attempt to rationalize and unify my cumbersome notation I have elected to use conventions which differ slightly from those of references [1], [2] and [3]. However the changes are not great and should cause the reader no difficulty.

Sections 1 and 2 culminate in expressions for  $c^{1/2}(n)$  and  $c^{3/2}(n)$  as weighted sums over the above-mentioned quantities  $Q(m)$ . In any given dimension both the  $Q(m)$  and their weights may be calculated according to well-defined procedures. These procedures are described in detail with the aid of examples in section 3. In section 4 I have tabulated the values of the  $Q(m)$  and their weights for  $N=1,2$  and, in order to illustrate how the formulae for  $c^{1/2}(n)$  and  $c^{3/2}(n)$  may be applied directly, I have used them to calculate  $A^{1/2}$  and  $A^{3/2}$  in each case. However deriving the anomalies by calculating the  $Q(m)$  and their weights quickly becomes impractical as  $N$  increases, and some alternative means must be found. In section 5 I firstly prove that it is possible to establish recurrence relations between the  $Q(m)$  in different dimensions, and secondly show that the tabulated low  $N$  values for these coefficients suggest a general formula for the  $Q(m)$ . In the event the recurrence relations bear out the postulated formula, and the end

product of this process is a general expression for the  $Q(m)$  in arbitrary dimensions. This general expression is substituted into the formulae for  $c^{1/2}(n)$  and  $c^{3/2}(n)$  in section 6, and the anomalies  $A^{1/2}$  and  $A^{3/2}$  are subsequently found in any dimension  $d=4N$ . All that remains to be done in section 7 is to contrast the diagrammatic method with other methods and to draw conclusions regarding its usefulness.

#### 4.1 Spin 1/2 Analysis

The problem that I will address in this section is how to extract the coefficient  $c^{1/2}(n)$  from  $\Sigma I^{1/2}$ . At this point it is convenient to reproduce the expression for  $\Sigma I^{1/2}$  occurring at the end of section 3.4.

$$\sum_P I^{1/2}(N) = \sum_P \int_0^1 dx \theta(1-\Sigma x) X^{\rho_1(1)} X^{\rho_2(2)} \dots X^{\rho_{2N}(2N)} \quad ..(1)$$

$$X(i) = X - k_{i+1} - k_{i+2} - \dots - k_{2N} \quad ..(2)$$

$$X = x_1 k_2 + (x_1 + x_2) k_2 + \dots + (x_1 + \dots + x_{2N}) k_{2N} \quad ..(3)$$

As explained in chapter 3, the sum in (1) is over all  $(2N)!$  permutations of the  $2N$  momenta  $k_1, \dots, k_{2N}$  and  $c^{1/2}(n)$  is the coefficient of the momentum product  $K(n)$  in  $\Sigma I^{1/2}$ . In the following I will designate by  $\Pi X$  the product  $X^{\rho_1(1)} X^{\rho_2(2)} \dots X^{\rho_{2N}(2N)}$  occurring in the integrand of  $I^{1/2}$ . To find  $c^{1/2}(n)$  our starting point is an analysis of the nature of the terms in  $\Pi X$ . Each of these terms consists of two parts : (i) a sequence of  $2N$  external momenta  $k_i$  which is multiplied by (ii) a product of some sub-group of the  $2N$  Feynman parameter (FP) factors  $x_1, (x_1 + x_2), \dots, (x_1 + \dots + x_{2N})$ . Our interest is exclusively in the  $k$ -sequences  $K(n)$ . Consequently from this point onwards when I refer to  $k$ -sequences I will mean only the  $k$ -sequences  $K(n)$  unless I indicate otherwise. To better describe both the  $k$ -sequences and products of FP factors occurring in  $\Pi X$  it is necessary to introduce some terminology.

Let us start by considering the  $k$ -sequences, each of which contains  $2N$   $k$ 's. There are two ways of describing or classifying a  $k$ -sequence. The first is according to its "form" and the second is according to its "type". I will

now explain what I mean by each of these terms. We have already encountered the notion of form. Specifying the form of a  $k$ -sequence is really a shorthand way of saying how the  $k$ 's in the sequence are "linked" together. As explained in section 3.3 two  $k$ 's,  $k_i^{\rho_j}$  and  $k_m^{\rho_n}$ , are linked if either  $j=m$  or  $i=n$ . Thus  $[k_1^{\rho_2} k_2^{\rho_3}]$  is a linked pair whereas  $[k_1^{\rho_2} k_3^{\rho_4}]$  is not. An  $n$ -chain, or a chain of length  $n$ , is a sequence of  $n$  linked  $k$ 's. It is either closed or open depending on whether or not the initial and final  $k$ 's of the chain are linked. For instance  $[k_1^{\rho_2} k_2^{\rho_3} k_3^{\rho_4} k_4^{\rho_5}]$  is an open 4-chain while the sequence  $[k_1^{\rho_2} k_2^{\rho_3} k_3^{\rho_4} k_4^{\rho_1}]$  is a closed 4-chain. To say that a  $k$ -sequence is of the form  $n=(n_1, \dots, n_N)$  will simply mean that it consists of  $n_1$  closed 2-chains,  $n_2$  closed 4-chains, ...,  $n_N$  closed  $2N$ -chains. Clearly the  $k$ -sequence  $K(n)$  is of form  $n$ .

The second way of characterizing a  $k$ -sequence is according to its type. The type of a sequence, which to a large extent is independent of its form, has to do with where the  $k$ 's in the sequence have come from in  $\Pi X$ . If they have come from one of the  $X$ 's within the factors  $X(i)$  in  $\Pi X$ , I will call them "bound"  $k$ 's. If they have not, I will call them "free"  $k$ 's. Each of the bound  $k$ 's is multiplied by one of the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$ . On the other hand, the free  $k$ 's occur by themselves in  $X(i)$  without any FP multipliers. Consequently, in a term in  $\Pi X$ , the number of factors in the FP product which multiplies the  $k$ -sequence is equal to the number of bound  $k$ 's in the sequence. For the purpose of this argument it will not be necessary to specify just which  $k$ 's in a given  $k$ -sequence are bound and which are free. Rather, it will suffice to specify more generally how the  $k$ 's in the sequence are partitioned into bound and free  $k$ 's. I will describe the type of a  $k$ -sequence in  $I^{1/2}$  using the index  $m=(m_1, \dots, m_{2N})$ , and I will define the related number  $m$  according to the following equation:  $m=m_1+m_2+\dots+m_{2N}$ . The statement that a  $k$ -sequence is of type  $(m_1, \dots, m_{2N})$  will mean that it consists of  $m$  bound  $k$ 's in addition to  $m_2$  open 1-chains of free  $k$ 's,  $m_3$  open 2-chains of free  $k$ 's, ...,  $m_{2N}$  open  $(2N-1)$ -chains of free  $k$ 's. Since there is a total of  $2N$   $k$ 's in any sequence we infer that the integers  $m_i$  satisfy the relation  $m_1+2m_2+\dots+2Nm_{2N}=2N$ . In this connection it is convenient to define the modulus of the index  $m$  to be the sum  $m_1+2m_2+\dots+2Nm_{2N}$  and to write it as  $|m|$ . Then  $m$  is a satisfactory spin  $1/2$  type specification only if  $|m|=2N$ .

A few things need to be explained about the type specification of  $k$ -sequences. Firstly, why are the free  $k$ 's specified as occurring only in open chains and not in closed chains? The answer to this question is simple. All of the free  $k$ 's in  $\Pi X$  are of the form  $k_i^{p_j}$  such that  $j < i$ . Consequently no closed chains of free  $k$ 's can form and no account need be taken of them in the type-specification. Secondly, the reader should verify for himself that if a  $k$ -sequence contains  $r$  open chains of free  $k$ 's it must also contain a minimum of  $r$  bound  $k$ 's. The reason for this is connected with the fact that the chains of free  $k$ 's are open. The  $r$  bound  $k$ 's, if you like, are needed to separate the open chains of free  $k$ 's. In terms of  $\mathbf{m}$  this condition implies that  $m_1 > 0$ . Thus an admissible type specification is provided by the index  $\mathbf{m} = (m_1, \dots, m_{2N})$  if and only if the  $m_i$  are non-negative integers such that  $|\mathbf{m}| = 2N$ . In this way admissible type specifications  $\mathbf{m}$  are in one-to-one correspondence with partitions of  $2N$ : the  $m_i$ ,  $i=1, 2, \dots, 2N$  may be considered as specifying a partition of  $2N$  into  $m_1$  ones,  $m_2$  twos, ...,  $m_{2N}$   $2N$ 's. This neat interpretation may help the reader remember which are the admissible values of  $\mathbf{m}$ .

One last comment should be made on the subject of  $k$ -sequences before we proceed on to a description of the products of FP factors that occur in  $\Pi X$ . It concerns the compatibility of the form and type specifications of a  $k$ -sequence. In general, the  $k$ -sequences that occur in  $\Pi X$  and its permutations possess all possible combinations of form and type subject to one restriction, which is that the type-associated partition of the sequence into bound and free  $k$ 's must be a sub-partition of the form-associated partition of the sequence into closed chains of even length. That is, the chains of free  $k$ 's must be sub-chains of the even length closed chains in  $K(\mathbf{n})$ . One implication of this condition, for instance, is that if  $\mathbf{n}$  is such that  $n_i = 0$ ,  $i > j$  for some  $j$ ,  $1 \leq j \leq N$ , then a  $k$ -sequence of form  $\mathbf{n}$  may only be of those types  $\mathbf{m}$  such that  $m_i = 0$ ,  $i > 2j$ . In other words, if the largest closed chain in  $K(\mathbf{n})$  is of length  $2j$  then the lengths of the chains of free  $k$ 's in  $K(\mathbf{n})$  may not exceed  $2j-1$ .

In order to illustrate these ideas I will now provide three examples of sixteen-dimensional ( $N=4$ )  $k$ -sequences. In each case I will commence by nominating the sequence's form and type. The reader can then verify whether



the way in which the  $k$ 's are linked together conforms with the form specification, and likewise whether the partition of the sequence into bound and free  $k$ 's conforms with its type specification. The first  $k$ -sequence is of form  $(1,0,1,0)$  and of type  $(1,2,1,0,0,0,0,0)$ :

$$K(1,0,1,0) = (k_1^{\rho_2} k_2^{\rho_1})(k_3^{\rho_4} k_4^{\rho_5} k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_3})$$

bound:  $k_2, k_4, k_7, k_8$  ; free:  $k_1, k_3, k_5, k_6$

The second  $k$ -sequence is of form  $(0,0,0,1)$  and of type  $(0,0,0,2,0,0,0,0)$ . Note that the two bound  $k$ 's in this sequence are "needed" to separate the two open 3-chains of free  $k$ 's:

$$K(0,0,0,1) = (k_1^{\rho_2} k_2^{\rho_3} k_3^{\rho_4} k_4^{\rho_5} k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_1}) \quad ..(4)$$

bound:  $k_4, k_8$  ; free:  $k_1, k_2, k_3, k_5, k_6, k_7$

The final example demonstrates that  $k$ -sequences of the same form and type may arise in different ways according to just which of the  $k$ 's are free and which bound. Like the last  $k$ -sequence it is of form  $(0,0,0,1)$  and of type  $(0,0,0,2,0,0,0,0)$ :

$$K(0,0,0,1) = (k_1^{\rho_2} k_2^{\rho_3} k_3^{\rho_4} k_4^{\rho_5} k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_1}) \quad ..(5)$$

bound:  $k_1, k_5$  ; free:  $k_2, k_3, k_4, k_6, k_7, k_8$

Having introduced the above notation and terminology for  $k$ -sequences, I will now similarly introduce notation to aid in the description of integrals over products of the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$ . For a start, let me associate with these factors the numbers  $1, 2, \dots, 2N$  respectively. Thus  $x_1$  corresponds to 1,  $(x_1+x_2)$  to 2, and so on. Next I will denote by  $(i_1, \dots, i_L)$  the product of the  $L$  factors associated with the numbers  $i_1, \dots, i_L$ . Obviously  $i_k \in \{1, 2, \dots, 2N\}$  and  $0 \leq L \leq 2N$ . Finally, I will employ  $S(i_1, \dots, i_L)$  to represent the following integral over the Feynman parameters  $x_1, x_2, \dots, x_{2N}$ .

$$S(i_1, \dots, i_L) = \int_0^1 dx_1 \dots dx_{2N} \Theta(1 - x_1 - \dots - x_{2N}) (i_1, \dots, i_L) \quad ..(6)$$

This, of course, is just the sort of integral that occurs in  $I^{1/2}$ . In analogy with the abbreviations  $\mathbf{n}$  and  $\mathbf{m}$  for the indices  $(n_1, \dots, n_N)$  and  $(m_1, \dots, m_{2N})$  I will sometimes, in the following work, abbreviate  $(i_1, \dots, i_L)$  to  $\mathbf{i}$ . There is one important thing to note about the integrals  $S(\mathbf{i})$ . The value of  $S(\mathbf{i})$  depends not on exactly which of the  $x_i$  are present in its integrand. Rather, it depends solely on how many factors there are in the integrand, and how many  $x_i$  there are in each factor. This is borne out by the following formula [4].

$$\int_0^1 dx_1 \dots dx_{2N} \Theta(1 - x_1 - \dots - x_{2N}) x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{2N}^{\alpha_{2N}} = \frac{\alpha_1! \alpha_2! \dots \alpha_{2N}!}{(2N + \alpha_1 + \alpha_2 + \dots + \alpha_{2N})!} \quad \dots(7)$$

Formula (7) will be used to good effect in section 5 in establishing recurrence relations. For the moment, however, the reader may employ it to verify the values of several eight-dimensional ( $N=2$ ) integrals which are listed below and which I have chosen to illustrate the  $\mathbf{i}$  notation.

$$S(1,2,3,4) = \int dx_1 \dots dx_4 \Theta(1 - x_1 - \dots - x_4) x_1(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_3+x_4) = \frac{1}{384}$$

$$S(1,3,4) = \int dx_1 \dots dx_4 \Theta(1 - x_1 - \dots - x_4) x_1(x_1+x_2+x_3)(x_1+x_2+x_3+x_4) = \frac{1}{210}$$

$$S(2,4) = \int dx_1 \dots dx_4 \Theta(1 - x_1 - \dots - x_4) (x_1+x_2)(x_1+x_2+x_3+x_4) = \frac{1}{72}$$

We are now in possession of almost all the terminology and notation that is needed to adequately describe the components of  $\Sigma I^{1/2}$ . The sole addition to the notation which remains to be made is that of a means of describing the effect on  $I^{1/2}$  of the permutations in the sum  $\Sigma$ . I will label these permutations according to their effect on the quantity  $X$  which is given by equation (3). When a permutation of the momenta  $k_1, \dots, k_{2N}$  acts on  $X$  its effect is to reorder the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$  which multiply these momenta. Of course such a permutation will act not only on  $X$ , but on the other parts of  $I^{1/2}$  too. However, for the purpose of labelling the permutation it suffices to concentrate on its effect on  $X$ . It is convenient to describe the permutation  $P$  which takes  $X$  to  $X'$  by specifying which of the FP factors multiplies each of the  $k_i$  in  $X'$ . This can be done using the label  $[s_1, \dots, s_{2N}]$  in which  $s_i \in \{1, \dots, 2N\}$  and  $s_i \neq s_j$  for  $i \neq j$ . In this label  $s_i$  is the

integer corresponding to whichever FP factor multiplies  $k_i$  in  $X'$ . The identity permutation is therefore represented by  $[1,2,\dots,2N]$ . Two examples of eight-dimensional ( $N=2$ ) permutations are given below.

$$X = x_1 k_1 + (x_1+x_2)k_2 + (x_1+x_2+x_3)k_3 + (x_1+x_2+x_3+x_4)k_4$$

$$[1,3,2,4] : X \rightarrow X', X' = x_1 k_1 + (x_1+x_2+x_3)k_2 + (x_1+x_2)k_3 + (x_1+x_2+x_3+x_4)k_4$$

$$[1,4,2,3] : X \rightarrow X'', X'' = x_1 k_1 + (x_1+x_2+x_3+x_4)k_2 + (x_1+x_2)k_3 + (x_1+x_2+x_3)k_4$$

Two features of this method of labelling permutations are of interest. For a start it is not difficult to see that acting on  $I^{1/2}$  with the permutation  $[s_1,\dots,s_{2N}]$  is equivalent to fixing the positions of the momenta  $k_1,\dots,k_{2N}$  in  $I^{1/2}$ , and applying instead a certain permutation to the indices  $i$  which are carried by both the  $k_i$  and the Feynman parameters  $x_i$ . Specifically, the action of  $[s_1,\dots,s_{2N}]$  on  $I^{1/2}$  is reproduced by making the substitution  $s_i \rightarrow i$  in these indices for all values of  $i : i=1,2,\dots,2N$ . In this connection note that, for reasons outlined above in the discussion preceding equation (7), permuting the Feynman parameters  $x_i$  will have no effect on the FP integrals  $S(i)$ . To illustrate this feature of the labels  $[s_1,\dots,s_{2N}]$  consider the eight-dimensional ( $N=2$ ) quantity  $I^{1/2}(4)$ . The reader should check that acting on  $I^{1/2}(4)$  with the permutation  $[1,4,2,3]$  is equivalent to making the replacements  $1 \rightarrow 1, 4 \rightarrow 2, 2 \rightarrow 3$  and  $3 \rightarrow 4$  in the indices carried by the  $k_i$  and  $x_i$ . Similarly, the effect of the permutation  $[1,3,2,4]$  on  $I^{1/2}(4)$  may be reproduced by making the substitutions  $1 \rightarrow 1, 3 \rightarrow 2, 2 \rightarrow 3, 4 \rightarrow 4$  in  $I^{1/2}(4)$ .

The second interesting feature of the labels  $[s_1,\dots,s_{2N}]$  has to do with which of the momenta  $k_1,\dots,k_{2N}$  occur as free  $k$ 's in a given permutation of  $I^{1/2}$ . I now assert that  $k_i^{\rho_j}$  occurs as a free  $k$  in the permutation  $[s_1,\dots,s_{2N}]$  of  $I^{1/2}$  if and only if  $s_j < s_i$ . This result may be proved by noting that  $k_i^{\rho_j}$  occurs as a free  $k$  in the unpermuted  $I^{1/2}$  of equation (1) if and only if  $j < i$ . Since acting on  $I^{1/2}$  with the permutation  $[s_1,\dots,s_{2N}]$  is equivalent to everywhere substituting  $i$  for  $s_i$  the assertion follows immediately. To illustrate this second property of the label  $[s_1,\dots,s_{2N}]$  I have reproduced below the values of the eight-dimensional ( $N=2$ ) quantity  $\Pi X$  after the two permutations  $[1,3,2,4]$  and  $[1,4,2,3]$ . In the case of the first

permutation [1,3,2,4] we have  $s_1=1$ ,  $s_2=3$ ,  $s_3=2$ , and  $s_4=4$  and so  $s_1 < s_2, s_3, s_4$ ;  $s_3 < s_2, s_4$ ; and  $s_2 < s_4$ . Correspondingly we see that the free  $k$ 's in  $\Pi X^-$  are  $k_2^{p_1}, k_3^{p_1}, k_4^{p_1}, k_2^{p_3}, k_4^{p_3}$  and  $k_4^{p_2}$ . Similarly, after the second permutation [1,4,2,3] the free  $k$ 's in  $\Pi X^{--}$  are  $k_2^{p_1}, k_3^{p_1}, k_4^{p_1}, k_2^{p_3}, k_4^{p_3}$  and  $k_2^{p_4}$  and we note that in this case  $s_1 < s_2, s_3, s_4$ ;  $s_3 < s_2, s_4$ ;  $s_4 < s_2$ .

$$\Pi X = [X - k_2 - k_3 - k_4]^{p_1} [X - k_3 - k_4]^{p_2} [X - k_4]^{p_3} X^{p_4}$$

$$[1,3,2,4] : \Pi X \rightarrow \Pi X^- , \Pi X^- = [X^- - k_2 - k_3 - k_4]^{p_1} [X^- - k_2 - k_4]^{p_3} [X^- - k_4]^{p_2} X^{-p_4}$$

$$[1,4,2,3] : \Pi X \rightarrow \Pi X^{--} , \Pi X^{--} = [X^{--} - k_2 - k_3 - k_4]^{p_1} [X^{--} - k_2 - k_4]^{p_3} [X^{--} - k_2]^{p_4} X^{--p_2}$$

At this point we are at last ready to consider the task of calculating the coefficient  $c^{1/2}(n)$ . It should now be clear that finding  $c^{1/2}(n)$  is entirely equivalent to finding, for all possible  $i$ , the number of terms in  $\Sigma I^{1/2}$  in which the  $k$ -sequence  $K(n)$  is multiplied by the integral  $S(i)$ . If this number is denoted by  $d^{1/2}(n, i)$  then we have

$$c^{1/2}(n) = \sum_{|i|=2N}^i (-1)^L d^{1/2}(n, i) S(i) \quad \dots(8)$$

The sum here is over all appropriate values of the index  $i=(i_1, \dots, i_L)$ . That is, it is over all those  $i$  such that  $i_k \in \{1, \dots, 2N\}$  and  $i_k \neq i_j$  if  $k \neq j$ . I will define the modulus of  $i$ ,  $|i|$ , to be equal to the largest of the integers  $i_k$  in  $i$ :  $|i| = \max i_k, k=1, 2, \dots, 2N$ . The condition  $|i|=2N$  therefore indicates that  $i_k \in \{1, \dots, 2N\}$ . The reason for the presence of the factor  $(-1)^L$  in (8) is as follows. Let us suppose that the sequence  $K(n)$  occurs somewhere in one of the permutations of  $I^{1/2}$  multiplied by the integral  $S(i)$ . As we have seen, the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$  multiply the bound  $k$ 's but not the free  $k$ 's in  $I^{1/2}$ . Consequently the number of bound  $k$ 's in  $K(n)$  is equal to the number of entries  $i_k$  in the index  $i$ . That is, the number of bound  $k$ 's in  $K(n)$  is equal to  $L$ . It follows that the number of free  $k$ 's in  $K(n)$  is  $2N-L$ . But each free  $k$  in  $I^{1/2}$  is multiplied by  $-1$ . Therefore whenever  $K(n)$  occurs in  $\Sigma I^{1/2}$  multiplied by the integral  $S(i)$  it will also be multiplied by a factor of  $(-1)^{2N-L} = (-1)^L$ . Hence the factor  $(-1)^L$  in equation (8).

The number  $d^{1/2}(n,i)$  may be defined alternatively. It is not difficult to see that the combination of a  $k$ -sequence of given form  $n$  with a particular integral multiplier  $S(i)$  will occur once or not at all in a specific permutation  $[s_1, \dots, s_{2N}]$  of  $I^{1/2}$ . In this connection note that once  $S(i)$  and  $[s_1, \dots, s_{2N}]$  are given it is possible to deduce which of the  $k$ 's are bound and which free :  $k_i$  is bound if  $s_i \in \{i_1, \dots, i_L\}$ , otherwise it is free. Now in the permutation  $[s_1, \dots, s_{2N}]$  of  $I^{1/2}$  there is either a unique way to match up bound and free  $k$ 's with the indices  $\rho_i$  appropriate to the form  $K(n)$ , or there is no way at all. Consequently  $d^{1/2}(n,i)$  is also equal to the number of permutations of  $I^{1/2}$  in which the  $k$ -sequence  $K(n)$  is multiplied by the integral  $S(i)$ . I will now state without proof that  $d^{1/2}(n,i)$  may be written as follows

$$d^{1/2}(n,i) = \sum_{|m|=2N}^m P(n,m)R(m,i) \quad \dots(9)$$

The sum here is over all indices  $m$  whose moduli are equal to  $2N$ . A proof of (9) will be postponed until section 3. For the moment I merely wish to explain the significance of the quantities  $P(n,m)$  and  $R(m,i)$ , and to emphasize the assumption implicit in (9). The integer  $P(n,m)$  is the number of ways of partitioning a  $k$ -sequence of form  $n$  into bound and free  $k$ 's of type  $m$ . Likewise, the integer  $R(m,i)$  is the number of permutations of  $I^{1/2}$  in which a particular  $k$ -sequence of form  $n$  and type  $m$  occurs multiplied by the integral  $S(i)$ . The assumption implicit in (9) is that it does not matter which  $k$ -sequence is used to compute  $R(m,i)$ , so long as it is of type  $m$ . That is,  $R(m,i)$  is independent not only of the form  $n$  of the  $k$ -sequence used in its computation, but also of the particular  $k$ 's which are chosen to be bound and free in this sequence, provided always that they are of type  $m$ .

Based on this assumption, which will be justified in section 4 where a detailed prescription for the calculation of  $R(m,i)$  is given,  $R(m,i)$  is endowed with functional dependence only on  $m$  and  $i$ . Once the assumption is made equation (9) is obviously true. It follows merely by grouping the instances where the sequence  $K(n)$  is multiplied by  $S(i)$  according to the type  $m$  of  $K(n)$ , and then summing over types. Note that there is no superscript  $1/2$  on either  $P(n,m)$  or  $R(m,i)$  on the right hand side of (9),

and consequently no explicit indication that they are spin 1/2 quantities. This is because the spin 1/2 and spin 3/2 versions of the sum in (9) are both over exactly the same coefficients  $P(\mathbf{n}, \mathbf{m})$  and  $R(\mathbf{m}, \mathbf{i})$ , and these coefficients are therefore common to the spin 1/2 and spin 3/2 cases. The main difference between the respective sums is that the spin 3/2 one is over a wider range of values of the index  $\mathbf{m}$ . This will be explained in the next section. I will finish this section by defining a new quantity  $Q(\mathbf{m})$  and rearranging equations (8) and (9) so that they may be written as follows.

$$c^{1/2}(\mathbf{n}) = \sum_{|\mathbf{m}|=2N}^{\mathbf{m}} (-1)^{\mathbf{m}} P(\mathbf{n}, \mathbf{m}) Q(\mathbf{m}) \quad \text{..(10)}$$

$$Q(\mathbf{m}) = \sum_{|\mathbf{i}|=2N}^{\mathbf{i}} R(\mathbf{m}, \mathbf{i}) S(\mathbf{i}) \quad \text{..(11)}$$

In (10) the factor  $(-1)^L$  has become  $(-1)^{\mathbf{m}}$  where  $\mathbf{m}$  is defined by the equation  $\mathbf{m} = m_1 + m_2 + \dots + m_{2N}$ . As explained above,  $L$  is equal to the number of bound  $k$ 's in  $K(\mathbf{n})$ . But if  $K(\mathbf{n})$  is of type  $\mathbf{m}$  then the number of bound  $k$ 's in the sequence is exactly  $\mathbf{m}$ . Consequently, in passing from (8) and (9) to (10) and (11),  $L$  may be replaced by  $\mathbf{m}$ .

#### 4.2 Spin 3/2 Analysis

The material of the preceding section may be adapted without trouble to the spin 3/2 case. Indeed much of it, including terminology and notation, remains unaltered. To extract  $c^{3/2}(\mathbf{n})$  from  $\Sigma I^{3/2}$  we must consider the following expressions

$$\sum_P I^{3/2}(N) = \sum_P \int_0^1 dx \theta(1-\Sigma x) Y^{\rho_1 \alpha_1}_{\alpha_2}(1) Y^{\rho_2 \alpha_2}_{\alpha_3}(2) \dots Y^{\rho_{2N} \alpha_{2N}}_{\alpha_1}(2N) \quad \text{..(12)}$$

$$Y^{\rho \alpha \beta}(i) = [\eta^{\alpha \beta} X^{\rho}(i) - \eta^{\alpha \rho} k^{\beta} + \eta^{\beta \rho} k^{\alpha}] \quad \text{..(13)}$$

These formulae occur at the end of section 3.4, and the quantities  $X$  and  $X(i)$  are as in equations (2) and (3). I will designate by  $\Pi Y$  the contraction of  $Y$ 's occurring in the integrand of  $I^{3/2}$ . Of course,  $c^{3/2}(\mathbf{n})$  is the

coefficient of  $K(n)$  in  $\Sigma I^{3/2}$ . To find it we start by considering the nature of the terms in  $\Pi Y$ . These terms, as before, consist of a  $k$ -sequence which is multiplied by a product of the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$ . The products of FP factors are the same as in the spin  $1/2$  case and may again be described by means of the label  $i=(i_1, \dots, i_L)$ . The  $k$ -sequences, however, are of slightly altered form. There are more  $k$ -containing terms in  $I^{3/2}$  than there are in  $I^{1/2}$ . The new terms are those of the form  $\eta^{\alpha\rho\beta} k_i^\beta$  or  $\eta^{\beta\rho\alpha} k_i^\alpha$  in  $Y^{\rho\alpha\beta}(i)$ . I will call the  $k$ 's occurring in these terms "rogue"  $k$ 's. They complement the bound and free  $k$ 's of section 1. Obviously, rogue  $k$ 's may be present in the  $k$ -sequences of  $I^{3/2}$  in addition to bound and free  $k$ 's. Just as free  $k$ 's in  $I^{1/2}$  or  $I^{3/2}$  can occur only in open chains, the rogue  $k$ 's in a  $k$ -sequence in  $I^{3/2}$  can occur only in a single closed chain. To see this, consider two adjacent factors in the product  $\Pi Y$ .

$$Y^{\rho_i\alpha\beta}(i)Y^{\rho_{i+1}\beta\gamma}(i+1) = [\eta^{\alpha\beta} X^{\rho_i}(i) - \eta^{\alpha\rho_i} k_i^\beta + \eta^{\rho_i\beta} k_i^\alpha] \\ [\eta^{\beta\gamma} X^{\rho_{i+1}}(i+1) - \eta^{\beta\rho_{i+1}} k_{i+1}^\gamma + \eta^{\rho_{i+1}\gamma} k_{i+1}^\beta]$$

As has been observed already, the  $k$ -sequences  $K(n)$  contain no dot products of  $k$ 's. Nor do they contain any  $\eta^{\rho_i\rho_j}$ 's. Consequently terms in  $\Pi Y$  containing these quantities may be ignored. This places restrictions upon the ways in which the terms in the above two factors may be contracted together. For instance,  $\eta^{\alpha\rho_i} k_i^\beta$  may be contracted with  $\eta^{\beta\gamma} X^{\rho_{i+1}}(i+1)$  or  $\eta^{\beta\rho_{i+1}} k_{i+1}^\gamma$  but not with  $\eta^{\rho_{i+1}\gamma} k_{i+1}^\beta$  since  $k_i \cdot k_{i+1}$  would result. Similarly  $\eta^{\rho_i\beta} k_i^\alpha$  may be contracted with  $\eta^{\beta\gamma} X^{\rho_{i+1}}(i+1)$  or  $\eta^{\rho_{i+1}\gamma} k_{i+1}^\beta$  but not with  $\eta^{\beta\rho_{i+1}} k_{i+1}^\gamma$  since  $\eta^{\rho_i\rho_{i+1}}$  would result. With these restrictions in mind it is not difficult to deduce from an examination of  $\Pi Y$  that the only way rogue  $k$ 's can link together is in a single closed chain. What is more, the rogue  $k$ 's in this single closed chain will link up so that their  $i$ -indices are in either strictly decreasing or strictly increasing order. Thus one will find the rogue chains  $[k_1^{\rho_3} k_3^{\rho_4} k_4^{\rho_7} k_7^{\rho_1}]$  and  $[k_6^{\rho_5} k_5^{\rho_2} k_2^{\rho_6}]$  in  $\Pi Y$  since the sequences 1,3,4,7 and 6,5,2 are strictly increasing and strictly decreasing respectively. On the other hand, one will not encounter the rogue chain  $[k_1^{\rho_5} k_5^{\rho_2} k_2^{\rho_4} k_4^{\rho_1}]$  in  $\Pi Y$  since the sequence 1,5,2,4 is neither monotonic increasing nor monotonic decreasing.

In section 1 I described k-sequences according to their forms  $n$  and their types  $m$ . In  $I^{3/2}$ , too, there will be k-sequences having the forms  $n$  in which we are interested. If not, there would be no spin 3/2 anomaly. However, although the form specification of k-sequences can be taken over unaltered from the spin 1/2 to the spin 3/2 case, the presence of rogue  $k$ 's in  $I^{3/2}$  necessitates a modification of the spin 1/2 type specification  $m$  if it is to be suitable for spin 3/2 k-sequences. I will specify the type of a k-sequence in  $I^{3/2}$  using the index  $m=(m_1, \dots, m_M)$  such that  $m_1+2m_2+\dots+Mm_M=M$ . With reference to this index I will define the number  $m$  to be the following sum :  $m=m_1+m_2+\dots+m_M$ . The statement that a spin 3/2 k-sequence is of type  $m$  will then mean that the sequence is composed of  $m$  bound  $k$ 's in addition to  $m_2$  open 1-chains of free  $k$ 's,  $m_3$  open 2-chains of free  $k$ 's,  $\dots$ ,  $m_M$  open  $(M-1)$ -chains of free  $k$ 's. Clearly the combined number of bound and free  $k$ 's is  $M$  and this scheme would be equivalent to the spin 1/2 one were I to demand that  $M=2N$ . However, in the spin 3/2 case I will let  $M \leq 2N$  and in this way allow for the fact that in addition to the above  $M$  bound and free  $k$ 's the sequence may contain  $2N-M$  rogue  $k$ 's. As in the spin 1/2 case, the constraint imposed upon the number of bound  $k$ 's by the fact that the free  $k$ 's occur in open chains translates into the condition  $m_1 \geq 0$ . An admissible type specification of a k-sequence in  $I^{3/2}$  is therefore provided by the index  $m$  if and only if the  $m_i$  are non-negative integers which satisfy the relation  $m_1+2m_2+\dots+Mm_M \leq 2N$ . In analogy with the spin 1/2 case, the  $m_i$  may correspondingly be interpreted as specifying a partition of  $M$  into  $m_1$  ones,  $m_2$  twos,  $\dots$ ,  $m_M$   $M$ 's. This neat interpretation of the  $m_i$  may help the reader to remember which are the admissible values of  $m$ . As before, it is useful to define a modulus of the index  $m$ . In the spin 3/2 case this modulus is given by  $|m|=m_1+2m_2+\dots+Mm_M$ . The admissible values of  $m$  are then succinctly specified by the equation  $|m| \leq M$ .

The compatibility of spin 1/2 form and type specifications was dealt with in section 2. The points made there, when suitably adjusted, are just as valid in the spin 3/2 case. The type-associated partition of the k-sequence  $K(n)$  into chains of bound, free and rogue  $k$ 's must be a sub-partition of its form-associated partition into closed chains of even length. Note in particular that type-associated closed chains necessarily correspond exactly to form-associated closed chains. In contrast, type-associated open chains



necessarily correspond to parts of form-associated closed chains. Consequently in a  $k$ -sequence  $K(n)$  of type  $m$  the closed  $(2N-M)$ -chain of rogue  $k$ 's must also be one of the form-associated even length closed chains. This means that when calculating the spin  $3/2$  anomaly we should consider only those types  $m$  which are admissible according to the above partition criterion  $|m|=M$ , and which are such that  $M$ , and consequently  $2N-M$ , are even. The last condition, that  $M$  be even, can be profitably overlooked in setting up and solving the recurrence relations of section 6. I will insist on it only when I come to calculate  $c^{3/2}(n)$  itself at the end of section 6.

To illustrate the use of the spin  $3/2$  type specification  $m$  I will now give three examples of sixteen-dimensional ( $N=4$ )  $k$ -sequences. The first is of form  $(1,0,1,0)$  and type  $(0,1,0,1,0,0)$  :

$$K(1,0,1,0) = (k_1^{\rho_2} k_2^{\rho_1})(k_3^{\rho_4} k_4^{\rho_5} k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_3})$$

bound:  $k_4, k_8$  ; free:  $k_3, k_5, k_6, k_7$  ; rogue:  $k_1, k_2$

The second is of form  $K(2,1,0,0)$  and type  $(2,1,0,0)$  :

$$K(2,1,0,0) = (k_1^{\rho_2} k_2^{\rho_1})(k_3^{\rho_4} k_4^{\rho_3})(k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_5})$$

bound:  $k_1, k_3, k_4$  ; free:  $k_2$  ; rogue:  $k_5, k_6, k_7, k_8$

In the final example the sequence is of form  $(0,0,0,1)$  and all of the  $k$ 's within it are rogue  $k$ 's. I will indicate this by saying that the sequence is of type  $(0)$  :

$$K(0,0,0,1) = (k_1^{\rho_2} k_2^{\rho_3} k_3^{\rho_4} k_4^{\rho_5} k_5^{\rho_6} k_6^{\rho_7} k_7^{\rho_8} k_8^{\rho_1})$$

bound: none ; free: none ; rogue:  $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$

In the modified spin  $3/2$  label  $m$  I now have a satisfactory means of specifying the types of  $k$ -sequences in  $I^{3/2}$ . As was mentioned at the start of this section, the products of FP factors occurring in  $\Pi Y$  are the same as those in  $\Pi X$ . Consequently the FP integrals occurring in  $I^{3/2}$  are the same as

those in  $I^{1/2}$ , and they may be described using the familiar notation  $S(i)$ . No modifications are needed here. Nor is it necessary to modify the method of labelling the permutations of  $k_1, \dots, k_{2N}$  that are associated with the sums in both equations (1) and (12). In the label  $[s_1, \dots, s_{2N}]$  the integer  $s_i$  here has the same significance with respect to the quantity  $X$  as it had in the previous section.

There is however one new feature of the labels  $[s_1, \dots, s_{2N}]$  which is of interest. We have seen that  $k_i$  occurs as a free  $k$  carrying the index  $\rho_j$  in the permutation  $[s_1, \dots, s_{2N}]$  of  $I^{1/2}$  if and only if  $s_j < s_i$ . The same is also true of free  $k$ 's in  $I^{3/2}$ . Moreover, a similar condition exists regarding the occurrence of chains of rogue  $k$ 's within permutations of  $I^{3/2}$ . Clearly the momentum chain  $k_i^{\rho_j} k_j^{\rho_l} \dots k_m^{\rho_l}$  will occur as a rogue chain in a permutation  $[s_1, \dots, s_{2N}]$  of the integral  $I^{3/2}$  if and only if the chain  $k_{s_i}^{\rho_{s_j}} k_{s_j}^{\rho_{s_l}} \dots k_{s_m}^{\rho_{s_l}}$  occurs as a rogue chain in  $I^{3/2}$  itself. But in view of the above comments on the structure of rogue chains in  $I^{3/2}$ , this implies that  $k_i^{\rho_j} k_j^{\rho_l} \dots k_m^{\rho_l}$  occurs as a rogue chain in the permutation  $[s_1, \dots, s_{2N}]$  of  $I^{3/2}$  if and only if the sequence  $s_i, s_j, \dots, s_m$  is a cyclic permutation of a strictly increasing or decreasing sequence of integers. Thus in the permutation  $[3, 2, 5, 4, 1, 6]$  of  $I^{3/2}(6)$  the sequences  $k_2^{\rho_4} k_4^{\rho_3} k_3^{\rho_6} k_6^{\rho_2}$  and  $k_1^{\rho_2} k_2^{\rho_4} k_4^{\rho_1}$  will occur as rogue chains since  $s_2, s_4, s_3, s_6 = 2, 4, 5, 6$  and  $s_4, s_1, s_2 = 4, 3, 2$  are respectively increasing and decreasing sequences of integers. However, one will not find a rogue chain  $k_6^{\rho_1} k_1^{\rho_3} k_3^{\rho_2} k_2^{\rho_6}$  since  $s_6, s_1, s_3, s_2 = 6, 3, 5, 2$  is neither strictly increasing nor strictly decreasing. This feature of the label  $[s_1, \dots, s_{2N}]$  will be useful when it comes to formulating a prescription for calculating the coefficients  $R(m, i)$ . I am now ready to introduce the spin 3/2 version of equation (8). It is

$$c^{3/2}(n) = \sum_{|i|=2N}^i (-1)^L d^{3/2}(n, i) S(i) \quad \dots(14)$$

The factor  $(-1)^L$  is included in (14) for the same reason that it was included in equation (8). It accounts for the negative signs that occur in front of the free  $k$ 's in the factors  $X(i)$  of equation (2). The reader may object that the negative signs in front of half of the rogue  $k$ 's in the quantities  $Y(i)$  of equation (13) should also appear in this formula. However

they can safely be ignored once it is recalled that rogue  $k$ 's always occur together in  $K(n)$  in chains of even length. Since there is always an even number of rogue  $k$ 's in  $K(n)$  the associated minus signs cancel.

Clearly, apart from the sign  $(-1)^L$ , the quantity  $d^{3/2}(n,i)$  is the combined coefficient which multiplies the product  $K(n)S(i)$  in  $\Sigma I^{3/2}$ . In section 1 we saw that the spin  $1/2$  counterpart of  $d^{3/2}(n,i)$ , namely  $d^{1/2}(n,i)$ , was simply equal to the number of permutations of  $I^{1/2}$  in which the  $k$ -sequence  $K(n)$  was multiplied by the integral  $S(i)$ . The coefficient  $d^{3/2}(n,i)$  is similarly related to the number of permutations of  $I^{3/2}$  in which the  $k$ -sequence  $K(n)$  is multiplied by  $S(i)$ . However, the relationship is not as straightforward as it was in the spin  $1/2$  case. To elucidate this relationship we must now determine two things. Firstly we must find how many of the terms in any given permutation of  $I^{3/2}$  contain the product  $K(n)S(i)$ . Secondly we must work out the factors which multiply  $K(n)S(i)$  in all such terms.

Both of these bits of information are easily found. For a start it is not difficult to see that, except in the special case when the  $k$ -sequence contains precisely two rogue  $k$ 's, the product  $K(n)S(i)$  occurs once or not at all in any given permutation  $[s_1, \dots, s_{2N}]$  of  $I^{3/2}$ . The proof of this fact is similar to the one in the spin  $1/2$  case. Once  $S(i)$  and  $[s_1, \dots, s_{2N}]$  are given, the bound  $k$ 's may be separated from the free and rogue  $k$ 's according to the following criterion : if  $s_i \in \{i_1, \dots, i_L\}$  then  $k_i$  is bound, otherwise it is either rogue or free. The rogue and free  $k$ 's are in turn easily separated from one another since the rogue  $k$ 's occur in a single closed chain while the free  $k$ 's occur in open chains. The reader may verify that there is then either a unique way to match up bound, free and rogue  $k$ 's with the  $\rho$ -indices appropriate to the form  $K(n)$ , or there is no way at all. Similar arguments reveal that, when the sequence  $K(n)$  contains precisely two rogue  $k$ 's, the product  $K(n)S(i)$  will occur twice or not at all in a given permutation of the integral  $I^{3/2}$ .

Next consider the factors that multiply the product  $K(n)S(i)$  in the terms in the permutations of  $I^{3/2}$ . Here we must differentiate between  $k$ -sequences  $K(n)$  which contain rogue  $k$ 's and those which do not. The part of  $I^{3/2}$  which contains no rogue  $k$ 's is exactly equal to  $I^{1/2}$  except for an overall factor

of  $\eta_{\alpha}^{\alpha} = 4N$ . Consequently if  $K(n)$  contains no rogue  $k$ 's then the product  $K(n)S(i)$  will be multiplied by a factor of  $4N$  wherever it occurs in the permutations of  $I^{3/2}$ . On the other hand, if  $K(n)$  contains rogue  $k$ 's then the product  $K(n)S(i)$  will occur in terms in the permutations of  $I^{3/2}$  without any additional factor. These considerations lead us to the conclusion that the coefficient  $d^{3/2}(n,i)$  may be written as follows.

$$d^{3/2}(n,i) = 4N \sum_{|m|=2N}^m P(n,m)R(m,i) + \sum_{L=1}^N \sum_{|m|=2N-2L}^m P(n,m)R(m,i) \quad ..(15)$$

The sum in the first term on the right hand side of (15) is over all spin  $1/2$  types  $m$ . Of course, these types correspond to  $k$ -sequences containing no rogue  $k$ 's. On the other hand, the sums in the second term on the right hand side of (15) are over spin  $3/2$  types  $m$ , and these characterize  $k$ -sequences containing non-zero numbers of rogue  $k$ 's. The reasons for considering only those types  $m$  for which  $|m|$  is even are explained above. The integer  $P(n,m)$  is the number of ways of partitioning a sequence of form  $n$  into bound, free and rogue  $k$ 's of type  $m$ . Similarly, except when  $M=2N-2$ ,  $R(m,i)$  is the number of permutations of  $I^{3/2}$  in which a particular  $k$ -sequence of form  $n$  and type  $m$  occurs multiplied by the integral  $S(i)$ . When  $M=2N-2$ ,  $R(m,i)$  is equal to twice the number of permutations of  $I^{3/2}$  in which a particular  $k$ -sequence of form  $n$  and type  $m$  is multiplied by the integral  $S(i)$ . The assumption implicit in (15) is that  $R(m,i)$  depends only on the type  $m$  of the  $k$ -sequence used to compute it, not on other features such as its form  $n$ . This assumption will be justified in the next section.

There is one important observation to be made about spin  $1/2$  and spin  $3/2$  quantities which are functionally dependent on type  $m$ , for example  $P(n,m)$  and  $R(m,i)$ . Obviously the set of spin  $3/2$  types  $m$  contains as a subset the smaller set of spin  $1/2$  types. Because of this the values of spin  $3/2$  type-dependent quantities such as  $P(n,m)$  and  $R(m,i)$  encompass as a subset the values of the equivalent spin  $1/2$  quantities. Thus the values of the coefficients  $R(m,i)$  in the sums (9) and (15) are the same provided  $M=2N$ . For this reason I have not distinguished between the spin  $1/2$  coefficients  $P(n,m)$ ,  $R(m,i)$  and  $Q(m)$  and their spin  $3/2$  counterparts. Clearly, when calculating the values of these coefficients, it suffices to consider those appropriate

to the spin 3/2 case. The spin 1/2 values are then just the subset for which  $M=2N$ . To finish this section I will present the spin 3/2 equivalents of equations (10) and (11).

$$c^{3/2}(n) = 4N \sum_{|m|=2N}^m (-1)^m P(n,m)Q(m) + \sum_{L=1}^N \sum_{|m|=2N-2L}^m (-1)^m P(n,m)Q(m) \quad ..(16)$$

$$Q(m) = \sum_{|i|=2N}^i R(m,i)S(i) \quad ..(17)$$

#### 4.3 Calculating $P(n,m)$ and $R(m,i)$

I now wish to dispel any mystery that might still surround the coefficients  $P(n,m)$  and  $R(m,i)$  by describing in detail how they may be calculated in specific instances, and giving some examples. The methods of this section may be employed to calculate the values of  $P(n,m)$  and  $R(m,i)$  once particular values of  $n, m$  and  $i$  are chosen. The relevant calculations are fairly painless for small  $N$  but, due to the amount of work involved, they rapidly become unmanageable as  $N$  increases. The shortcoming of the methods described below is that they cannot be generalized so as to give the general functional dependence of  $P(n,m)$  and  $R(m,i)$  on  $n, m$  and  $i$  for arbitrary  $N$ . This shortcoming is the motivation for looking to recurrence relations between quantities in different dimensions, rather than direct calculation, as a means of finding  $A^{1/2}$  and  $A^{3/2}$  in arbitrary dimensions. Nevertheless, although the recurrence relations of section 6 will form the real basis for calculating the anomalies, the material of this section should not be bypassed. The prescription given here for calculating the  $R(m,i)$  assumes a central role in the formulation of recurrence relations in section 6, and a certain subset of the  $P(n,m)$  whose values are computed below is essential to the later calculation of  $A^{1/2}$  and  $A^{3/2}$ .

It is easier to illustrate how to calculate  $P(n,m)$  than  $R(m,i)$ , so I will consider  $P(n,m)$  first. By definition  $P(n,m)$  is the number of ways of partitioning the  $k$ -sequence  $K(n)$  into bound, free and rogue  $k$ 's of type  $m$ . Two distinct type  $m$  partitions of  $K(n)$  are distinguished by the particular

$k$ 's which are bound, free and rogue in each case. For instance, the two distinct partitions of the sixteen-dimensional  $k$ -sequence  $K(0,0,0,1)$  in equations (4) and (5) are both of type  $m=(0,0,0,2,0,0,0,0)$ . The difference between them is that different  $k$ 's are bound and free in each case. To aid us in understanding how to calculate  $P(n,m)$  let us now look at some twelve-dimensional ( $N=3$ )  $k$ -sequences. For a start consider the choices  $n=(1,1,0)$  and  $m=(0,1,0,1,0,0)$ .  $K(1,1,0)$  is shown below. The type specification  $m=(0,1,0,1,0,0)$  indicates that there should be two bound  $k$ 's, a single open 1-chain of free  $k$ 's, a single open 3-chain of free  $k$ 's, and no rogue  $k$ 's. The eight ways of partitioning  $K(1,1,0)$  according to these requirements are listed below. (Note that the two chains of free  $k$ 's cannot both fit into the closed chain of length four. This is because open chains of free  $k$ 's must be separated from one another by at least one bound  $k$ .)

$$K(1,1,0) = (k_1^{p_2} k_2^{p_1})(k_3^{p_4} k_4^{p_5} k_5^{p_6} k_6^{p_3})$$

bound	free	bound	free
$k_1, k_3$	$k_2, k_4, k_5, k_6$	$k_2, k_3$	$k_1, k_4, k_5, k_6$
$k_1, k_4$	$k_2, k_3, k_5, k_6$	$k_2, k_4$	$k_1, k_3, k_5, k_6$
$k_1, k_5$	$k_2, k_3, k_4, k_6$	$k_2, k_5$	$k_1, k_3, k_4, k_6$
$k_1, k_6$	$k_2, k_3, k_4, k_5$	$k_2, k_6$	$k_1, k_3, k_4, k_5$

$$P[(1,1,0), (0,1,0,1,0,0)] = 8$$

Next consider the choices  $n=(0,0,1)$  and  $m=(0,1,0,1,0,0)$ . This time the  $k$ -sequence is of the same type but a different form. As shown below there are only six possible partitions and consequently  $P(n,m)=6$  in this case.

$$K(0,0,1) = (k_1^{p_2} k_2^{p_3} k_3^{p_4} k_4^{p_5} k_5^{p_6} k_6^{p_1})$$

bound	free	bound	free
$k_1, k_3$	$k_2, k_4, k_5, k_6$	$k_4, k_6$	$k_1, k_2, k_3, k_5$
$k_2, k_4$	$k_1, k_3, k_5, k_6$	$k_1, k_5$	$k_2, k_3, k_4, k_6$
$k_3, k_5$	$k_1, k_2, k_4, k_6$	$k_2, k_6$	$k_1, k_3, k_4, k_5$

$$P[(0,0,1), (0,1,0,1,0,0)] = 6$$

If the type specification  $m$  and the form specification  $n$  of a  $k$ -sequence are incompatible, then  $P(n,m)=0$ . For instance, it is impossible to partition  $K(3,0,0)$  into a sequence of type  $(3,0,1,0,0,0)$ . The type specification in this case calls for a single open chain of free  $k$ 's of length two. Since an open chain of length two will only fit into a closed chain whose length is greater than two, there are no possible partitions and we deduce as a consequence that  $P[(3,0,0),(3,0,1,0,0,0)]=0$ . Likewise the sequence  $K(1,1,0)$  cannot be partitioned into bound and free  $k$ 's of type  $(0,0,0,0,0,1)$ . The open 5-chain of free  $k$ 's which is demanded by the type specification is simply too large to be fitted into either the closed 2-chain or the closed 4-chain within  $K(1,1,0)$ .

The first two examples involved no rogue  $k$ 's. However the introduction of rogue  $k$ 's into this scheme presents no difficulties, as a last example will show. One simply chooses which of the closed chains of length  $2N-M$  in  $K(n)$  is to be the chain of rogue  $k$ 's, and then partitions the remainder of  $K(n)$  into bound and free  $k$ 's as described above. If there are no chains of length  $2N-M$  in  $K(n)$ , then  $P(n,m)=0$ . In this connection note that, as remarked earlier, we need consider only types  $m$  such that  $M$  is even. As an illustration of these points consider the choices  $n=(3,0,0)$  and  $m=(0,2,0,0)$ . In this case  $M=4$ , so the sequence contains  $2N-M=6-4=2$  rogue  $k$ 's.  $K(3,0,0)$  and its possible partitions are given below. Clearly  $P[(3,0,0),(0,2,0,0)]=12$ .

$$K(3,0,0) = (k_1^{p_2} k_2^{p_1})(k_3^{p_4} k_4^{p_3})(k_5^{p_6} k_6^{p_5})$$

bound	free	rogue	bound	free	rogue	bound	free	rogue
$k_1, k_3$	$k_2, k_4$	$k_5, k_6$	$k_1, k_5$	$k_2, k_6$	$k_3, k_4$	$k_3, k_5$	$k_4, k_6$	$k_1, k_2$
$k_1, k_4$	$k_2, k_3$	$k_5, k_6$	$k_1, k_6$	$k_2, k_5$	$k_3, k_4$	$k_3, k_6$	$k_4, k_5$	$k_1, k_2$
$k_2, k_3$	$k_1, k_4$	$k_5, k_6$	$k_2, k_5$	$k_1, k_6$	$k_3, k_4$	$k_4, k_5$	$k_3, k_6$	$k_1, k_2$
$k_2, k_4$	$k_1, k_3$	$k_5, k_6$	$k_2, k_6$	$k_1, k_5$	$k_3, k_4$	$k_4, k_6$	$k_3, k_5$	$k_1, k_2$

$$P[(3,0,0),(0,2,0,0)] = 12$$

Before we pass on to a description of the coefficients  $R(m,1)$  there is one point about the  $P(n,m)$  that I would like to make, and one special class of the  $P(n,m)$  whose values I want to examine. The point to be made concerns

those  $P(\mathbf{n}, \mathbf{m})$  for which  $M=2N-2L$  where  $L$  is an integer between 1 and  $N$  :  $1 \leq L \leq N$ . In this case the number  $P(\mathbf{n}, \mathbf{m})$  is clearly equal to the product of the number of ways of situating a rogue chain of length  $2L$  within  $K(\mathbf{n})$  and the number of ways of partitioning the remainder of  $K(\mathbf{n})$  into bound and free  $k$ 's of type  $\mathbf{m}$ . But the number of ways of situating a rogue chain of length  $2L$  within  $K(\mathbf{n})$  is simply  $n_L$ . Consequently, if  $0 < L < N-L$ ,

$$P_{2N}[(n_1, \dots, n_N), (m_1, \dots, m_{2N-2L})] = n_L P_{2N-2L}[(n_1, \dots, n_{L-1}, \dots, n_{N-L}), (m_1, \dots, m_{2N-2L})] \quad 0 < L < N-L \quad \dots(18)$$

while if  $0 < N-L < L$ ,

$$P_{2N}[(n_1, \dots, n_N), (m_1, \dots, m_{2N-2L})] = n_L P_{2N-2L}[(n_1, \dots, n_{N-L}), (m_1, \dots, m_{2N-2L})] \quad 0 < N-L < L \quad \dots(19)$$

Note that those integers  $n_i$  which do not appear on the right hand sides of the above two equations are necessarily zero due to the condition  $|\mathbf{n}|=N$ . In (18) and (19) I have emphasized the dimension to which the coefficients belong by supplying them with subscripts  $2N$  and  $2N-2L$ . Thus  $P_{2N}(\mathbf{n}, \mathbf{m})$  is a  $4N$ -dimensional quantity while  $P_{2N-2L}(\mathbf{n}, \mathbf{m})$  belongs in  $4N-4L$  dimensions. These equations therefore express  $4N$ -dimensional coefficients which involve rogue  $k$ 's in terms of  $(4N-4L)$ -dimensional coefficients which do not involve rogue  $k$ 's. Alternatively, they express spin  $3/2$  coefficients in terms of lower dimensional spin  $1/2$  coefficients. Because of this, (18) and (19) will be useful later on in section 6.

The special class of  $P(\mathbf{n}, \mathbf{m})$  whose values I want to examine consists of those  $P(\mathbf{n}, \mathbf{m})$  such that  $\mathbf{n}=(0, \dots, 0, 1)$  and  $M=2N$ . That is, all of the  $n_i$  are zero except for  $n$  which is one, and there are no rogue  $k$ 's. Within this class the  $P(\mathbf{n}, \mathbf{m})$  are functions only of the variable  $\mathbf{m}$  so I will write them as  $P(\mathbf{m})$

$$P(\mathbf{m}) \equiv P[(0, \dots, 0, 1), (\mathbf{m})] \quad , \quad M=2N \quad \dots(20)$$



$P(\mathbf{m})$  by definition is the number of ways of partitioning the  $k$ -sequence  $K(0, \dots, 0, 1)$  into bound and free  $k$ 's of type  $\mathbf{m}$ . Its value is not too difficult to work out. In a  $k$ -sequence of type  $\mathbf{m}$  there are  $m_2 + m_3 + \dots + m_{2N}$  chains of free  $k$ 's and  $m = m_1 + m_2 + \dots + m_{2N}$  bound  $k$ 's. The integer  $P(\mathbf{m})$  may be regarded as the number of ways of ordering the  $m_2 + m_3 + \dots + m_{2N}$  chains of free  $k$ 's and  $m_1 + m_2 + \dots + m_{2N}$  single bound  $k$ 's within the sequence  $K(0, \dots, 0, 1)$ . That is, one is looking at the number of ways of ordering  $m_1 + 2(m_2 + \dots + m_{2N})$  separate units within  $K(0, \dots, 0, 1)$ . Actually, this ordering problem is complicated by the fact that the chains of free  $k$ 's must be separated from each other by at least one bound  $k$ . It is easier to think in terms of "extended" chains of free  $k$ 's, each of which consists of a chain of free  $k$ 's with a single bound  $k$  attached to one end. The extended chains of free  $k$ 's may be shuffled around freely as their terminal bound  $k$ 's act as buffers between the real chains of free  $k$ 's. The ordering problem can therefore be posed in terms of the  $m_2 + m_3 + \dots + m_{2N}$  extended chains of free  $k$ 's and the remaining  $m_1$  bound  $k$ 's, a total of  $m_1 + m_2 + \dots + m_{2N}$  units. Suppose we choose one of these units. Any one of the  $2N$   $k$ 's in  $K(0, \dots, 0, 1)$  could be the initial  $k$  of this unit. Consequently there are  $2N$  ways of situating this first unit within  $K(0, \dots, 0, 1)$ . There are then  $(m_1 + m_2 + \dots + m_{2N} - 1)!$  ways of ordering the other  $m_1 + m_2 + \dots + m_{2N} - 1$  units around it. We deduce that the total number of ways of ordering the  $m_1 + m_2 + \dots + m_{2N}$  units within the  $k$ -sequence  $K(0, \dots, 0, 1)$  is  $2N(m_1 + m_2 + \dots + m_{2N} - 1)!$ . However, in so far as partitions of  $K(0, \dots, 0, 1)$  are concerned, the order of units of the same length is immaterial. So one should divide by the product  $m_1! m_2! \dots m_{2N}!$ . Thus  $P(\mathbf{m})$  has the value

$$P_{2N}(\mathbf{m}) = \frac{2N(m_1 + m_2 + \dots + m_{2N} - 1)!}{m_1! m_2! \dots m_{2N}!} \quad \dots (21)$$

Once again I have emphasized that  $P(\mathbf{m})$  belongs in  $4N$  dimensions by writing it as  $P_{2N}(\mathbf{m})$ . This completes my treatment of the coefficients  $P(\mathbf{n}, \mathbf{m})$ . Now consider  $R(\mathbf{m}, 1)$ . Before describing how this quantity may be calculated there are a couple of points to be made. The first concerns the manner in which  $k$ -sequences occur in permutations of  $I^{3/2}$ . Suppose that we have a  $k$ -sequence which is partitioned into bound, free and rogue  $k$ 's and that these  $k$ 's carry  $\rho$ -indices appropriate to some form  $\mathbf{n}$ . Further suppose that we wish to find a permutation of  $I^{3/2}$  which contains this  $k$ -sequence. Then it suffices to find

a permutation,  $I^{-3/2}$ , of  $I^{3/2}$  whose integrand  $\Pi Y'$  satisfies two criteria. Firstly, all of those  $k$ 's which have been designated free  $k$ 's in the chosen  $k$ -sequence must be present as free  $k$ 's with the correct  $\rho$ -indices in  $\Pi Y'$ . Secondly,  $\Pi Y'$  must also contain the rogue chain which appears in the chosen  $k$ -sequence. Provided that  $\Pi Y'$  conforms with these two criteria regarding rogue and free  $k$ 's, the desired  $k$ -sequence will appear in  $I^{3/2}$ . This is due to the fact that if a particular combination of rogue and free  $k$ 's occurs in a permutation of  $I^{3/2}$  it will occur together with every possible complement of bound  $k$ 's. This follows easily from the structure of  $\Pi Y$ .

The second point relevant to the coefficients  $R(m, i)$  has to do with the relationship between the  $k$ -sequence and its integral multiplier  $S(i)$  in a term in one of the permutations of  $I^{3/2}$ . Let me denote such a permutation by  $I^{-3/2}$  and the corresponding permuted version of  $X$  by  $X'$ . The  $k$ 's in the sequence are labelled bound, free or rogue according to their points of origin in  $I^{-3/2}$ . A bound  $k$  has originated in  $X'$  and carries with it one of the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$ . The free and rogue  $k$ 's, in contrast, have originated outside of  $X'$  and are not multiplied by FP factors. As usual I will label the permutation which takes  $I^{3/2}$  into  $I^{-3/2}$  by  $[s_1, \dots, s_{2N}]$ , where  $s_i$  is the number associated with the FP factor that multiplies  $k_i$  in  $X'$ . The point I wish to make is that, if  $k_i$  is one of the bound  $k$ 's in the sequence, then  $s_i \in \{i_1, \dots, i_L\}$ ; while if  $k_i$  is free or rogue,  $s_i \notin \{i_1, \dots, i_L\}$ . This fact has already been discussed in sections 2 and 3; however it is worth repeating here before commencing an analysis of  $R(m, i)$ .

$R(m, i)$  is equal to the number of permutations  $[s_1, \dots, s_{2N}]$  of  $I^{3/2}$  in which a given  $k$ -sequence of type  $m$  is multiplied by the integral  $S(i)$ . [In the special case  $M=2N-2$ ,  $R(m, i)$  is equal to twice the number of permutations of  $I^{3/2}$  in which a given  $k$ -sequence of type  $m$  is multiplied by the integral  $S(i)$ ]. To calculate it one first chooses a  $k$ -sequence of type  $m$ . As I shall soon prove, the value of  $R(m, i)$  is the same no matter which type  $m$   $k$ -sequence is chosen for the calculation. Nevertheless it is convenient for the moment to employ a  $k$ -sequence whose free  $k$ 's are all of the form  $k_i^{\rho_i+1}$ . It is always possible to find such a sequence. To make the description of the  $k$ 's in the sequence a little easier, I will use  $k_b$  to stand for any or all of the bound  $k$ 's. That is, if  $k_1, k_j, \dots, k_m$  are the bound  $k$ 's, then the

value of the index  $B$  ranges over the set  $\{i, j, \dots, m\}$ . Similarly I will use  $k_F$  and  $k_R$  to stand for the free and rogue  $k$ 's.

Now let us forget about the integral  $S(i)$  for the moment, and consider how we might single out the permutations of  $I^{3/2}$  in which the chosen  $k$ -sequence occurs. Given the comments in the second last paragraph, it suffices to find those permutations in which all the designated free  $k$ 's occur carrying the correct  $\rho$ -indices, and which contain the desired chain of rogue  $k$ 's. At this point recall that  $k_i^{\rho_j}$  occurs as a free  $k$  in the permutation  $[s_1, \dots, s_{2N}]$  of  $I^{3/2}$  if and only if  $s_j < s_i$ . This fact was demonstrated in section 2. Since the free  $k$ 's in our chosen sequence are all of the form  $k_i^{\rho_{i+1}}$  we are therefore interested in those permutations  $[s_1, \dots, s_{2N}]$  such that  $s_F > s_{F+1}$  for all  $F$ . Furthermore, to ensure that the permutations  $[s_1, \dots, s_{2N}]$  actually contain the rogue chain which appears in our chosen  $k$ -sequence we must also insist that the  $s_B$  satisfy a condition which was discussed in the previous section. Specifically, when ordered so that  $s_{B'}$  follows  $s_B$  if  $B' > B$ , the sequence of integers  $s_B$  must be a cyclic permutation of a strictly increasing or strictly decreasing sequence of integers.

From the permutations which satisfy these conditions on  $s_F$  and  $s_B$  we should then keep only those in which our chosen sequence is multiplied by the integral  $S(i)$ . The above comments on the relationship between  $k$ -sequences and their integral multipliers in terms in the permutations of  $I^{3/2}$  are helpful in this respect. They tell us that we need keep only those permutations which satisfy the additional conditions  $s_B \in \{i_1, \dots, i_L\}$  and  $s_F, s_R \notin \{i_1, \dots, i_L\}$ .  $R(m, i)$  is equal to the number of permutations which remain. Obviously  $R(m, i)$  will be zero unless the number of bound  $k$ 's is exactly  $L$  [remember that  $i = (i_1, \dots, i_L)$ ]. That is,  $R(m, i)$  will be zero unless  $m = m_1 + m_2 + \dots + m_M = L$ . If we tacitly exclude from consideration any combinations of the indices  $m$  and  $i$  for which this is not the case, and set  $L = m$ , then only one of the conditions  $s_F, s_R \notin \{i_1, \dots, i_L\}$  and  $s_B \in \{i_1, \dots, i_L\}$  need be kept, and the calculation of  $R(m, i)$  may be summarized in the formula

$$R(m, i) = \# [s_1, \dots, s_{2N}];$$

$$s_B \in \{i_1, \dots, i_L\}, \quad s_{F+1} < s_F, \quad s_R \text{ cyc. inc. dec.} \quad M \neq 2N-2 \quad \dots (22)$$

As I have indicated, this formula holds provided  $M \neq 2N-2$ . In the special case  $M=2N-2$  the coefficient  $R(m,i)$  is equal to twice the number of permutations of  $I^{3/2}$  in which a given  $k$ -sequence of type  $m$  is multiplied by the integral  $S(i)$ , and we have

$$R(m,i) = 2 \cdot \# [s_1, \dots, s_{2N}];$$

$$s_b \in \{i_1, \dots, i_L\}, s_{F+1} < s_F, s_R \text{ cyc. inc. dec.} \quad M=2N-2 \quad \dots (23)$$

Do not forget that (22) and (23) are based on the presupposition that all of the free  $k$ 's in the sequence used to compute  $R(m,i)$  are of the form  $k_i^{p_i+1}$ . To illustrate the above formulae I will now provide several examples of calculations of  $R(m,i)$  in twelve dimensions ( $N=3$ ). The first example involves the choices  $m=(1,1,1,0,0,0)$  and  $i=(1,3,5)$ . To calculate  $R(m,i)$  in this instance I will use the  $k$ -sequence  $K(0,0,1)$ , partitioned as below into bound and free  $k$ 's. [Note that all of the free  $k$ 's are of the correct form.] The suitable permutations  $[s_1, \dots, s_6]$  of  $I^{3/2}$  in this case are those such that  $s_2, s_5, s_6 \in \{1,3,5\}$  and  $s_1 > s_2$  and  $s_3 > s_4 > s_5$ . There are four of them and they are listed below. We therefore deduce that  $R[(1,1,1,0,0,0), (1,3,5)] = 4$ .

$$K(1,1,0) = (k_1^{p_2} k_2^{p_3} k_3^{p_4} k_4^{p_5} k_5^{p_6} k_6^{p_1})$$

bound:  $k_2, k_5, k_6$  ; free:  $k_1, k_3, k_4$  ; rogue: none

$[216435], [436215], [634215], [654213]$

$$R[(1,1,1,0,0,0), (1,3,5)] = 4$$

Now consider the choices  $m=(0,0,0,1)$  and  $i=(2)$ . In this case  $R(m,i)$  may be calculated using formula (23) and the sequence  $K(1,1,0)$ , partitioned as below into bound, free and rogue  $k$ 's. There are eight permutations which satisfy the relevant conditions  $s_6 \in \{2\}$  and  $s_3 > s_4 > s_5 > s_6$ , and which are such that the sequence  $s_1, s_2$  is a cyclic permutation of a strictly increasing or strictly decreasing sequence of integers. Consequently  $R[(0,0,0,1), (2)] = 16$ .

$$K(1,1,0) = (k_1^{p_2} k_2^{p_1})(k_3^{p_4} k_4^{p_5} k_5^{p_6} k_6^{p_3})$$

bound:  $k_6$  ; free:  $k_3, k_4, k_5$  ; rogue:  $k_1, k_2$

[165432],[615432],[156432],[516432],[146532],[416532],[136542],[316542]

$$R[(0,0,0,1),(2)] = 16$$

Lastly, let us suppose that the number of rogue  $k$ 's is four, and adopt the choices  $m=(0,1)$  and  $i=(5)$ . The sequence  $K(1,1,0)$  is compatible with the type specification  $m=(0,1)$  and we can therefore use it again to calculate  $R(m,i)$ , partitioning it as follows into bound, free and rogue  $k$ 's. From equation (22) we deduce that in this case suitable permutations  $[s_1, \dots, s_6]$  of  $I^{3/2}$  are such that  $s_2 \in \{5\}$  and  $s_1 > s_2$ . Moreover  $s_3, s_4, s_5, s_6$  must be a cyclic permutation of an increasing or decreasing sequence of integers. There are eight permutations which satisfy these conditions.

$$K(1,1,0) = (k_1^{p_2} k_2^{p_1})(k_3^{p_4} k_4^{p_5} k_5^{p_6} k_6^{p_3})$$

bound:  $k_2$  ; free:  $k_1$  ; rogue:  $k_3, k_4, k_5, k_6$

[654321],[651432],[652143],[653214],[651234],[654123],[653412],[652341]

$$R[(0,1),(5)] = 8$$

At this stage the reader should feel reasonably at home with the quantities  $P(n,m)$  and  $R(m,i)$  and the methods by which they are calculated in particular instances. The only thing that remains for me to do in this section is to prove that  $R(m,i)$  depends solely on the type of the  $k$ -sequence used in its computation, not on its form or on the particular  $k$ 's within the sequence that are chosen to be bound, free and rogue. The above prescription for the calculation of  $R(m,i)$  has already taken us a part of the way towards this proof. In fact the absence within the calculational procedure itself of any dependence on the form  $n$  which is chosen for the  $k$ -sequence used to calculate  $R(m,i)$  leads one directly to the conclusion that this choice has no bearing on the final result. Of course, the form that is chosen must, for

the sake of the calculation, be compatible with a partition of type  $m$ . Beyond this requirement, however, it does not matter which form  $n$  is chosen.

The second part of the proof is hardly less straightforward than the first.  $R(m, i)$  is the number of permutations of  $I^{3/2}$  in the sum  $\Sigma I^{3/2}$  in which a particular  $k$ -sequence of type  $m$  is multiplied by the integral  $S(i)$ . Suppose that we calculate  $R(m, i)$  twice using two distinct type  $m$   $k$ -sequences. As we have just seen, the forms of these  $k$ -sequences are irrelevant, so we are free to assume that they are of the same form. Now observe that two type  $m$   $k$ -sequences of the same form can always be transformed into each other using a single permutation of the momenta  $k_1, \dots, k_{2N}$ . Since  $\Sigma I^{3/2}$  itself must be invariant under such a permutation, we deduce that the values of  $R(m, i)$  obtained using the two type  $m$   $k$ -sequences must be identical. Therefore it does not matter which type  $m$   $k$ -sequence is used to compute  $R(m, i)$ . This completes the proof.

#### 4.4 Small N Calculations

This section is devoted to a tabulated presentation in tables 1 and 2 of values for all of the quantities in equations (10), (11), (16) and (17) in the cases  $N=1, 2$ . To convince the reader that equations (10), (11), (16) and (17) are in fact valid I have then used these values, together with equations (3.47) and (3.48), to find  $A^{1/2}$  and  $A^{3/2}$  in four and eight dimensions. The aim of this material is to allow the reader to further familiarize himself with the diagrammatic approach to anomaly calculation as described in the previous three sections. The whole of this section may be bypassed if such additional familiarization is felt to be unnecessary.

The figures in the accompanying tables are by and large self-explanatory. However, several things should be noted at the outset. Firstly, a full set of values has been given for those quantities which depend on the spin  $3/2$  index  $m$ . The spin  $1/2$  counterparts of these values form a subset of the spin  $3/2$  values. They are the subset corresponding to types  $m$  such that  $M=2N$ . Thus, while the coefficient  $c^{3/2}(n)$  is calculated according to equation (16) by summing over all types  $m$ ,  $c^{1/2}(n)$  is calculated by summing over only

TABLE 1 N=1

$$n_1 = 1 \Rightarrow n \in \{(1)\}$$

$$i_k \in \{1,2\} \Rightarrow i \in \{(0),(1),(2),(12)\}$$

$$m_1 + 2m_2 = 0,2 \Rightarrow m \in \{(0),(01),(20)\}$$

$S(0)=1/2$	$S(1)=1/6$	$S(2)=1/3$	$S(12)=1/8$
$R(0,0)=4$	$R(01,1)=1$	$R(01,2)=0$	$R(20,12)=2$
$Q(0)=2$	$Q(01)=1/6$	$Q(20)=1/4$	
$P(1,0)=1$	$P(1,01)=2$	$P(1,20)=1$	

$$c^{1/2}(1) = \frac{-1}{12} \qquad c^{3/2}(1) = \frac{5}{3}$$

$$A^{1/2} = \frac{1}{24} \left( \frac{1}{4\pi} \right)^2 \langle R^2 \rangle \qquad A^{3/2} = - \frac{21}{24} \left( \frac{1}{4\pi} \right)^2 \langle R^2 \rangle$$

TABLE 2 N=2

$$n_1 + 2n_2 = 2 \Rightarrow n \in \{(01), (20)\}$$

$$i_k \in \{1, 2, 3, 4\} \Rightarrow i \in \{(0), (1), (2), (3), (4), (12), (13),$$

$$(14), (23), (24), (34), (123), (124), (134), (234), (1234)\}$$

$$m_1 + 2m_2 + 3m_3 + 4m_4 = 0, 2, 4$$

$$\Rightarrow m \in \{(0), (01), (20), (0001), (1010), (0200), (2100), (4000)\}$$

$$S(0)=1/24$$

$$S(1)=1/120$$

$$S(2)=1/60$$

$$S(3)=1/40$$

$$S(4)=1/30$$

$$S(12)=1/240$$

$$S(13)=1/180$$

$$S(14)=1/144$$

$$S(23)=1/90$$

$$S(24)=1/72$$

$$S(34)=1/48$$

$$S(123)=1/336$$

$$S(124)=1/280$$

$$S(134)=1/210$$

$$S(234)=1/105$$

$$S(1234)=1/384$$

$$R(0,0)=8$$

$$R(01,1)=12$$

$$R(01,2)=8$$

$$R(01,3)=4$$

$$R(01,4)=0$$

$$R(0001,1)=1$$

$$R(0001,2)=0$$

$$R(0001,3)=0$$

$$R(0001,4)=0$$

$$R(20,12)=8$$

$$R(20,13)=8$$

$$R(20,14)=8$$

$$R(20,23)=8$$

$$R(20,24)=8$$

$$R(20,34)=8$$

$$R(1010,12)=2$$

$$R(1010,13)=1$$

$$R(1010,14)=1$$

$$R(1010,23)=0$$

$$R(1010,24)=0$$

$$R(1010,34)=0$$

$$R(0200,12)=4$$

$$R(0200,13)=2$$

$$R(0200,14)=0$$

$$R(0200,23)=0$$

$$R(0200,24)=0$$

$$R(0200,34)=0$$

$$R(2100,123)=6$$

$$R(2100,124)=4$$

$$R(2100,134)=2$$

$$R(2100,234)=0$$

$$R(4000,1234)=24$$

$$Q(0)=1/3$$

$$Q(01)=1/3$$

$$Q(20)=1/2$$

$$Q(0001)=1/120$$

$$Q(1010)=1/48$$

$$Q(0200)=1/36$$

$$Q(2100)=1/24$$

$$Q(4000)=1/16$$

$$P(01,0)=1$$

$$P(01,01)=0$$

$$P(01,20)=0$$

$$P(01,0001)=4$$

$$P(01,1010)=4$$

$$P(01,0200)=2$$

$$P(01,2100)=4$$

$$P(01,4000)=1$$

$$P(20,0)=0$$

$$P(20,01)=4$$

$$P(20,20)=2$$

$$P(20,0001)=0$$

$$P(20,1010)=0$$

$$P(20,0200)=4$$

$$P(20,2100)=4$$

$$P(20,4000)=1$$

$$c^{1/2}(20) = \frac{1}{144}$$

$$c^{1/2}(01) = \frac{1}{720}$$

$$c^{3/2}(20) = \frac{-5}{18}$$

$$c^{3/2}(01) = \frac{31}{90}$$

$$A^{1/2} = \left(\frac{1}{4\pi}\right)^4 \left[ \frac{1}{2304} \langle R^2 \rangle^2 + \frac{1}{2880} \langle R^4 \rangle \right]$$

$$A^{3/2} = \left(\frac{1}{4\pi}\right)^4 \left[ \frac{-41}{2304} \langle R^2 \rangle^2 + \frac{247}{2880} \langle R^4 \rangle \right]$$



those types  $m$  for which  $M=2N$ . Secondly, attention has been restricted to those  $P(n,m)$  and  $R(m,i)$  which are not a priori zero due to incompatibility of  $n$  and  $m$  or  $m$  and  $i$ . In a similar fashion only even values for  $M$  have been considered since they are the only ones relevant to the anomalies. Thirdly, for the sake of tidiness in the tables, I have omitted the commas between individual entries in  $n$ ,  $m$  and  $i$ . As all entries in these indices are single digits this should cause no difficulties. Lastly, remember that  $i=(0)$  stands for the product of none of the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$ . That is, the integrand (i) is simply equal to one.

#### 4.5 Recurrence Relations

The calculations of  $A^{1/2}$  and  $A^{3/2}$  in four and eight dimensions by means of the figures in tables 1 and 2 illustrate the way in which equations (10), (11), (16) and (17) may be used directly to find anomalies. However, as  $N$  increases this method of calculating  $A^{1/2}$  and  $A^{3/2}$  quickly becomes impractical due to the amount of work involved. For instance the same calculations in twelve dimensions ( $N=3$ ) entail finding 57  $P(n,m)$  and 1216  $R(m,i)$ , a considerable augmentation of the work involved in eight dimensions. The problem is that the prescriptions given in section 3 for calculating  $P(n,m)$  and  $R(m,i)$  for particular values of  $n$ ,  $m$ , and  $i$ , cannot be generalized to functional expressions for these quantities. Consequently, to calculate each new anomaly all the relevant coefficients must be laboriously computed. Clearly, when it comes to the task of calculating higher dimensional anomalies, or of finding  $A^{1/2}$  and  $A^{3/2}$  in arbitrary dimensional space-times, some other approach must be adopted.

The natural thing to do is to look for recurrence relations between anomalies in different dimensions. Such recurrence relations could perhaps be used to extrapolate from the known lower dimensional anomalies to expressions for  $A^{1/2}$  and  $A^{3/2}$  in any dimension. At first inspection, however, what struck me was not so much the possibility of recurrence relations between the anomalies themselves, but rather the possibility of such relations between the coefficients  $Q(m)$  in different dimensions. Consequently, it was the latter possibility which I followed up and whose

outcome I will describe in this section. As we shall see, the recurrence relations which emerge from this investigation lead to a functional form for the  $Q(m)$  which facilitates the direct calculation of both the coefficients  $c^{1/2}(n)$  and  $c^{3/2}(n)$  in arbitrary dimensions  $d=4N$  using equations (10) and (16). Once one knows  $c^{1/2}(n)$  and  $c^{3/2}(n)$  it is then a simple matter to find the chiral anomalies  $A^{1/2}$  and  $A^{3/2}$  in any dimension  $d=4N$ . Before proceeding, though, it must be pointed out that equations (10), (11), (16) and (17) carry some information that has not been made explicit. To ensure that the quantities with which I will be working in the remainder of the thesis are perfectly well-defined I will now rewrite (10), (11), (16) and (17) in forms which make explicit all their implicit information.

$$Q_{2N}(m) = \sum_{|i|=2N}^i R_{2N}(m,i) S_{2N}(i) \quad ..(24)$$

$$c^{1/2}(n) = \sum_{|m|=2N}^m (-1)^m P_{2N}(n,m) Q_{2N}(m) \quad ..(25)$$

$$c^{3/2}(n) = 4N \sum_{|m|=2N}^m (-1)^m P_{2N}(n,m) Q_{2N}(m) + \sum_{L=1}^N \sum_{|m|=2N-2L}^m (-1)^m P_{2N}(n,m) Q_{2N}(m) \quad ..(26)$$

These equations now explicitly recognize that the quantities  $Q(m)$ ,  $R(m,i)$  and  $S(i)$ , in addition to being functions of  $m$  and  $i$ , also depend upon the number  $2N$ . Let us see how this dependence is implicit in their definitions. The integral  $S(i)$  is defined by equation (6). Clearly, even though the integrand  $(i)$  is independent of  $2N$ , the integral  $S(i)$  itself is a function of  $2N$  because it is an integral over  $2N$  Feynman parameters. Consequently it is written  $S_{2N}(i)$  in (24). Likewise the coefficient  $R(m,i)$  is not just a function of  $m$  and  $i$ .  $R(m,i)$  is defined by equations (22) and (23), and implicit in this definition is the fact that the permutations being counted are permutations of the  $2N$  momenta  $k_1, \dots, k_{2N}$ . Thus  $R(m,i)$  is also a function of  $2N$ . Since  $S(i)$  and  $R(m,i)$  both depend upon  $2N$  it follows that  $Q(m)$  does too, and this has been indicated by writing it as  $Q_{2N}(m)$ . Although the coefficient  $P(n,m)$  has no implicit dependences, I have written it as  $P_{2N}(n,m)$  in (25) and (26) to emphasize that it belongs in  $4N$  dimensions.

Equations (24), (25), (26) and the quantities contained therein are now perfectly well-defined, and we are at last ready to proceed with a treatment of recurrence relations between the  $Q(m)$  in different dimensions.

The first step in such a treatment is to generalize our definition of the coefficients  $Q_{2N}(m)$  a little. The quantity  $Q_{2N}(m)$  is defined in terms of  $R_{2N}(m, i)$  and  $S_{2N}(i)$  by equation (24). The definitions of all three of these coefficients may be generalized from dimensions  $d=4N$  to arbitrary even dimensions  $d=2n$  simply by replacing the number  $2N$  with  $n$ . Thus the integral  $S_{2N}(i)$  over  $2N$  Feynman parameters  $x_1, \dots, x_{2N}$  generalizes to an integral  $S_n(i)$  over  $n$  Feynman parameters  $x_1, \dots, x_n$ . In keeping with this change, the integers  $i_k$  in the index  $i$  will come from the set  $\{1, \dots, n\}$  rather than  $\{1, \dots, 2N\}$ , and will represent the FP factors  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_n)$  rather than  $x_1, (x_1+x_2), \dots, (x_1+\dots+x_{2N})$ . In a similar fashion  $R_{2N}(m, i)$  generalizes to a  $2n$ -dimensional coefficient  $R_n(m, i)$  in which the index  $m$  is such that  $|m| \leq n$ .  $R_n(m, i)$  is calculated by applying equations (22) and (23) without reference to any accompanying  $k$ -sequence. In performing this calculation one must allow for the fact that there are now  $n!$  permutations  $[s_1, \dots, s_n]$  in place of the  $(2N)!$  permutations  $[s_1, \dots, s_{2N}]$ . In terms of  $R_n(m, i)$  and  $S_n(i)$ , the quantity  $Q_n(m)$  is defined by the generalization of equation (24)

$$Q_n(m) = \sum_{|i|=n}^i R_n(m, i) S_n(i) \quad \dots(27)$$

It is worth pointing out that, although it is possible to generalize the  $Q_{2N}(m)$  to even dimensions  $d=2n$ , it is not possible to similarly generalize the coefficients  $c^{1/2}(n)$  and  $c^{3/2}(n)$ . This is true simply because the anomalies  $A^{1/2}$  and  $A^{3/2}$  do not occur in dimensions  $d=4N+2$ . Since the  $(4N+2)$ -dimensional coefficients  $Q_{2N+1}(m)$  have nothing to do with the anomalies  $A^{1/2}$  and  $A^{3/2}$  there would seem to be little reason to consider them. However, when formulating recurrence relations between the  $Q(m)$  it is quite advantageous to consider the general class of coefficients  $Q_n(m)$ , because it is far easier to construct a relation between  $Q_n(m)$  and  $Q_{n+1}(m)$  than it is to construct one between  $Q_{2N}(m)$  and  $Q_{2N+2}(m)$ . Certainly we lose nothing by considering the enlarged class of coefficients  $Q_n(m)$  since, once we have solved the recurrence relations for these quantities, we are free to set  $n=2N$ .

Actually, when formulating recurrence relations it is convenient to deal not with the  $Q_n(m)$  themselves, but rather with quantities  $\bar{Q}_n(m)$  which are defined in terms of coefficients  $\bar{R}_n(m,i)$  by a formula which is nearly identical to (24).

$$\bar{Q}_n(m) = \sum_{|i|=n}^i \bar{R}_n(m,i) S_n(i) \quad ..(28)$$

The integers  $\bar{R}_n(m,i)$  are given by

$$\bar{R}_n(m,i) = \# [s_1, \dots, s_n]; s_B \in \{i_1, \dots, i_L\}, s_F > s_{F+1}, s_R \text{ cyc. inc.} \quad ..(29)$$

and are related to the  $R_n(m,i)$  as follows.

$$R_n(m,i) = \begin{cases} \bar{R}_n(m,i) & M=n, n-1 \\ 2\bar{R}_n(m,i) & M=n-2, n-3, \dots, 0 \end{cases} \quad ..(30)$$

Note that in calculating  $\bar{R}_n(m,i)$  according to formula (29) one counts only those permutations  $[s_1, \dots, s_n]$  for which the  $s_R$  are cyclic permutations of increasing (not decreasing) sequences of integers. The difference of a factor of two between  $R_n(m,i)$  and  $\bar{R}_n(m,i)$  when  $M=n-2, \dots, 0$  may be explained as follows. Suppose that we are dealing with an index  $m$  such that  $M=n, n-1$  or  $n-2$ . Then there are 0, 1 or 2 rogue integers  $s_R$  respectively. In these cases the number of ways of arranging the  $s_R$  into cyclic permutations of increasing and decreasing sequences of integers is exactly the same as the number of ways of arranging them into cyclic permutations of only increasing sequences of integers. That is, if the number of rogue  $s_R$  is 0, 1 or 2 there is no distinction between increasing and decreasing sequences of integers. Comparing (22) and (23) with (29) we therefore deduce that if  $M=n$  or  $n-1$  then  $R_n(m,i) = \bar{R}_n(m,i)$  while if  $M=n-2$  we have  $R_n(m,i) = 2\bar{R}_n(m,i)$ .

If there are three or more  $s_R$  then increasing and decreasing sequences of integers are distinct and the number of ways of arranging the  $s_R$  into cyclic permutations of increasing and decreasing sequences of integers is twice the number of ways of arranging them into cyclic permutations of only increasing

sequences of integers. This accounts for the factor of two in (30) in the cases  $M=n-3, \dots, 0$ . Clearly (30) implies that

$$Q_n(m) = \begin{cases} \bar{Q}_n(m) & M=n, n-1 \\ 2\bar{Q}_n(m) & M=n-2, n-3, \dots, 0 \end{cases} \quad \dots(31)$$

I will now set about establishing recurrence relations between the  $\bar{Q}_n(m)$  in different dimensions. The first obstacle that one faces when trying to express  $\bar{Q}_n(m)$  in terms of  $\bar{Q}_{n-1}(m)$  is that different FP integrals occur in each case. Let us divide the integrals  $S_n(i) = S_n(i_1, \dots, i_L)$  that occur in the sum in (27) into two groups according to whether or not  $i_k = n$  for some  $k$ . If none of the  $i_k$  is equal to  $n$  then the integrand  $(i_1, \dots, i_L)$  can be written as a sum of terms  $x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}}$  where  $\alpha_1 + \dots + \alpha_{n-1} = L$ . Thus

$$(i_1, \dots, i_L) = \sum_{\alpha} x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} \quad \alpha_1 + \dots + \alpha_{n-1} = L$$

Since  $S_n(i_1, \dots, i_L)$  is defined by

$$S_n(i_1, \dots, i_L) = \int_0^1 dx_1 \dots dx_n \theta(1-x_1-\dots-x_n) (i_1, \dots, i_L)$$

and  $x_n$  does not appear in the integrand, equation (7) tells us that

$$\begin{aligned} S_n(i_1, \dots, i_L) &= \sum_{\alpha} \frac{\alpha_1! \dots \alpha_{n-1}!}{(n+L)!} = \left(\frac{1}{n+L}\right) \sum_{\alpha} \frac{\alpha_1! \dots \alpha_{n-1}!}{(n-1+L)!} \\ &= \left(\frac{1}{n+L}\right) S_{n-1}(i_1, \dots, i_L) \end{aligned} \quad \dots(32)$$

It should not be too difficult to remember that equation (32) is valid only if  $i_k < n$  for all  $k$ , since we never deal with integrals  $S_{n-1}(i_1, \dots, i_L)$  for which  $i_k > n-1$ . Now suppose that one of the  $i_k$  in  $(i_1, \dots, i_L)$  is equal to  $n$ . In this event relevant integrands may be written in the form  $(i_1, \dots, i_{L-1}, n) = (i_1, \dots, i_{L-1}) \cdot (x_1 + \dots + x_n)$  where  $(i_1, \dots, i_{L-1})$  is a sum of terms  $x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}}$  such that  $\alpha_1 + \dots + \alpha_{n-1} = L-1$ :

$$(i_1, \dots, i_{L-1}, n) = (x_1 + \dots + x_n) \sum_{\alpha} x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} \quad \alpha_1 + \dots + \alpha_{n-1} = L-1$$

Consequently, using equation (7),

$$\begin{aligned}
 S_n(i_1, \dots, i_{L-1}, n) &= \sum_{\alpha} \frac{\alpha_1! \dots \alpha_{n-1}!}{(n+L)!} [(\alpha_1+1) + (\alpha_2+1) + \dots + (\alpha_{n-1}+1) + 1] \\
 &= \sum_{\alpha} \frac{\alpha_1! \dots \alpha_{n-1}!}{(n+L)!} [n+L-1] = \left(\frac{1}{n+L}\right) \sum_{\alpha} \frac{\alpha_1! \dots \alpha_{n-1}!}{[(n-1)+(L-1)]!} \\
 &= \left(\frac{1}{n+L}\right) S_{n-1}(i_1, \dots, i_{L-1}) \quad \dots(33)
 \end{aligned}$$

This completes the first step towards recurrence relations for the  $\bar{Q}_n(m)$ . In (32) and (33) we have expressions for the integrals  $S_n(i)$  in terms of integrals  $S_{n-1}(i)$ , and it now remains to be seen whether it is possible to formulate similar expressions for the coefficients  $\bar{R}_n(m, i)$ . Let us first of all consider only those coefficients such that  $M=n$ . Then there are no rogue  $k$ 's to worry about. Once again we can divide the  $\bar{R}_n(m, i)$  into two groups depending on whether or not  $i_k$  is equal to  $n$  for some  $k$ . If one of the  $i_k$  is equal to  $n$  then from equation (29) we have that

$$\begin{aligned}
 \bar{R}_n[(m_1, \dots, m_n), (i_1, \dots, i_{L-1}, n)] &= \\
 \# [s_1, \dots, s_n]; s_B \in \{i_1, \dots, i_{L-1}, n\}, s_F > s_{F+1} &\quad \dots(34)
 \end{aligned}$$

It is worth recalling a few things about this formula. Firstly, the integers  $s_i$  are divided into bound  $s_B$  and free  $s_F$  in conformity with the type specification  $m$ , and the index  $F$  takes values from the set  $\{1, \dots, n-1\}$  so that  $F+1$  is always contained in the set  $\{1, \dots, n\}$ . The  $s_F$  fall into chains, adjacent chains being separated from each other by at least one  $s_B$ . This means that if  $s_F$  is the last in a chain of free  $s_F$  then  $s_{F+1} = s_B$  for some  $B$ . That is, if  $F$  is the last free index in a chain then  $F+1 = B$  for some bound index  $B$ . I will call those numbers  $s_B$  in formula (34), which are such that  $B = F+1$  for some  $F$ , "terminal"  $s_B$ .

Consider how the numbers  $1, \dots, n$  starting with  $n$  may be assigned to the  $s_i$  in accordance with the conditions in formula (34). Clearly the value  $n$  must be assigned to one of the  $s_B$ , but it cannot be assigned to a terminal  $s_B$  since then we could not satisfy the condition  $s_F > s_{F+1}$ . Since there are  $m_2 + \dots + m_n = m - m_1$  chains of free  $k$ 's there are  $m - m_1$  terminal  $s_B$ . Consequently, out of a

total of  $m$  bound  $s_B$ , there are  $m_1$  which can be assigned the value  $n$ . Furthermore, it is a simple matter to check that, regardless of which of these  $m_1 s_B$  is set equal to  $n$ , the number of ways of assigning the remaining numbers  $1, \dots, n-1$  to the remaining  $s_1$  is exactly equal to the value of the coefficient  $\bar{R}_{n-1}[(m_1-1, m_2, \dots, m_{n-1}), (i_1, \dots, i_{L-1})]$ . We therefore deduce that

$$\bar{R}_n[(m_1, \dots, m_n), (i_1, \dots, i_{L-1}, n)] = m_1 \bar{R}_{n-1}[(m_1-1, m_2, \dots, m_{n-1}), (i_1, \dots, i_{L-1})] \quad ..(35)$$

There are two aspects of this equation which may seem questionable. Firstly one might be perturbed by the fact that if  $m_1=0$  the undefined coefficient  $\bar{R}_{n-1}[(-1, m_2, \dots, m_{n-1}), (i_1, \dots, i_{L-1})]$  appears on the right hand side of (35). However the value of this coefficient is not important because it is multiplied by  $m_1=0$ . If  $m_1=0$ , both sides of (35) vanish and the equation remains consistent. The second questionable feature of equation (35) is that, having thrown away the entry  $m_n$ , the reduced index  $(m_1-1, m_2, \dots, m_{n-1})$  may not be of standard form. That is, it may not satisfy the standard condition  $(m_1-1)+2m_2+\dots+(n-1)m_{n-1}=n-1$ . In this connection note that, due to the condition  $m_1+2m_2+\dots+nm_n=n$ , the only time that  $m_n$  will be non-zero is when  $m_1=\dots=m_{n-1}=0$  and  $m_n=1$ . If  $m_n$  is zero we see that  $(m_1-1)+2m_2+\dots+(n-1)m_{n-1}=n-1$ , so that the reduced index is of standard form. If  $m_n$  is non-zero the reduced index is of non-standard form, but in this case  $m_1=0$ , both sides of (35) vanish and the equation remains consistent.

Let us now suppose that none of the  $i_k$  in  $\bar{R}_n(m, i)$  is equal to  $n$  and again consider how the numbers  $1, \dots, n$  may be assigned to the  $s_1$  in accordance with the dictates of equation (29). In this case it is clear that the value  $n$  must be assigned to a free  $s_F$ . Moreover,  $n$  must be assigned to one of the  $s_F$  in such a way as to respect the condition  $s_F > s_{F+1}$ . This means that only the first  $s_F$  in a chain can possibly be equal to  $n$ . Consider the chains of  $s_F$  of length  $J-1$ . There are  $m_J$  of these chains and consequently there are  $m_J$  ways of assigning the value  $n$  to an  $s_F$  in a  $(J-1)$ -chain. Regardless of which of the leading  $s_F$  in these  $(J-1)$ -chains is set equal to  $n$ , the number of ways of assigning the numbers  $1, \dots, n-1$  to the remaining  $s_1$  is exactly  $\bar{R}_{n-1}[m_1, \dots, m_{J-1}+1, m_J-1, \dots, m_{n-1}), (i_1, \dots, i_L)]$ . This is true for  $J=2, \dots, n$  so

$$\bar{R}_n[(m_1, \dots, m_n), (i_1, \dots, i_L)] =$$

$$\sum_{j=2}^n m_j \bar{R}_{n-1}[(m_1, \dots, m_{j-1}+1, m_j-1, \dots, m_{n-1}), (i_1, \dots, i_L)] \quad \dots(36)$$

As with equation (35), any undefined coefficients which appear on the right hand side of (36) are multiplied by zero and so do not contribute to the sum. It should be remembered that equation (36) is valid only if none of the  $i_k$  is equal to  $n$ . We have now completed the second stage of our progress towards recurrence relations for those  $\bar{Q}_n(m)$  such that  $M=n$ . Using equations (35) and (36) we can express coefficients  $\bar{R}_n(m, i)$  in terms of the  $\bar{R}_{n-1}(m, i)$ . All that remains to be done is to combine equations (32) and (36) and equations (33) and (35) to arrive at an expression for  $\bar{Q}_n(m)$  in terms of  $\bar{Q}_{n-1}(m)$ . Noting that  $\bar{R}_n[(m_1, \dots, m_n), (i_1, \dots, i_L)]$  is zero unless  $L=m$  where  $m=m_1+m_2+\dots+m_n$ , we deduce that

$$\bar{Q}_n(m_1, \dots, m_n) = (n+m_1+\dots+m_n)^{-1} \left[ \sum_{j=1}^n m_j \bar{Q}_{n-1}(m_1, \dots, m_{j-1}+1, m_j-1, \dots, m_{n-1}) \right] \quad \dots(37)$$

The recurrence relation (37) can now be employed either to prove or disprove postulated functional expressions for those coefficients  $\bar{Q}_n(m)$  for which  $M=n$ . Such an expression may be deduced from the easily calculated small  $n$  values of  $\bar{Q}_n(m)$ . In table 3 I have provided a complete set of values for  $\bar{Q}_n(m)$  in the cases  $n=1, 2, 3, 4$ . Note that I have allowed  $M$  to take all values from 0 to  $n$ , not just the even ones. Moreover, since all the numbers in the index  $m$  are single digits for  $n=1, 2, 3, 4$  I have not bothered to separate them with commas. By staring at the values in table 4 long enough, the reader may conclude as I did that if  $M=n$  the coefficient  $\bar{Q}_n(m)$  is given by

$$\bar{Q}_n(m_1, \dots, m_n) = [1/2!]^{m_1} [1/3!]^{m_2} \dots [1/(n+1)!]^{m_n} \quad \dots(38)$$

This expression not only matches the figures in table 3, it is also consistent with the recurrence relation (37). These two facts in conjunction constitute an inductive proof that (38) is the general solution for  $\bar{Q}_n(m)$  when  $M=n$ . By setting  $n=2N$  and using (31) we see that



TABLE 3

$\bar{Q}_n(\mathbf{m}) \quad n = 1, 2, 3, 4$

n=1	$\bar{Q}(0)=1$	$\bar{Q}(1)=1/2$		
n=2	$\bar{Q}(0)=1$	$\bar{Q}(1)=1/2$	$\bar{Q}(01)=1/6$	$\bar{Q}(20)=1/4$
n=3	$\bar{Q}(0)=1/2$ $\bar{Q}(001)=1/24$	$\bar{Q}(1)=1/2$ $\bar{Q}(110)=1/12$	$\bar{Q}(01)=1/6$ $\bar{Q}(300)=1/8$	$\bar{Q}(20)=1/4$
n=4	$\bar{Q}(0)=1/6$ $\bar{Q}(001)=1/24$ $\bar{Q}(1010)=1/48$	$\bar{Q}(1)=1/4$ $\bar{Q}(110)=1/12$ $\bar{Q}(0200)=1/36$	$\bar{Q}(01)=1/6$ $\bar{Q}(300)=1/8$ $\bar{Q}(2100)=1/24$	$\bar{Q}(20)=1/4$ $\bar{Q}(0001)=1/120$ $\bar{Q}(4000)=1/16$

$$Q_{2N}(m_1, \dots, m_{2N}) = [1/2!]^{m_1} [1/3!]^{m_2} \dots [1/(2N+1)!]^{m_{2N}} \quad \dots(39)$$

What about the coefficients  $\bar{Q}_n(m)$  for which  $0 \leq M \leq n-1$ ? The arguments which led to the recurrence relation (37) may be extended to this case. In fact the arguments are virtually the same, except that when formulating an analogue of equation (36) one must allow for the fact that in formula (29) the value  $n$  may be assigned to one of the rogue  $s_R$ . In general there are  $n-M$  integers  $s_R$  and the number of ways of arranging them so that they are a cyclic permutation of an increasing sequence of integers is  $n-M$ . It is not difficult to see that the possibility of the value  $n$  being assigned to one of the  $s_R$  may be allowed for by including an extra term in equation (36). Thus, when  $0 \leq M \leq n-1$ , (36) becomes

$$\begin{aligned} \bar{R}_n[(m_1, \dots, m_M), (i_1, \dots, i_L)] &= \left(\frac{n-M}{n-M-1}\right) \bar{R}_{n-1}[(m_1, \dots, m_M), (i_1, \dots, i_L)] \\ &+ \sum_{J=2}^M m_J \bar{R}_{n-1}[(m_1, \dots, m_{J-1}+1, m_{J-1}, \dots, m_M), (i_1, \dots, i_L)] \quad \dots(40) \end{aligned}$$

If  $M=n-1$ , the factor  $[(n-M)/(n-M-1)]$  should be replaced by 1. When  $0 \leq M \leq n-1$ , formulae (32) and (33) are unchanged, and (35) is also unmodified except for the fact that the index  $(m_1, \dots, m_n)$  is replaced by  $(m_1, \dots, m_M)$  on the left hand side of the equation, while on the right hand side  $(m_1-1, m_2, \dots, m_{n-1})$  is replaced by  $(m_1-1, m_2, \dots, m_M)$ . Consequently, when  $0 \leq M \leq n-1$ , the recurrence relation (37) becomes

$$\begin{aligned} \bar{Q}_n(m_1, \dots, m_M) &= (n+m_1+\dots+m_M)^{-1} \left[ \left(\frac{n-M}{n-M-1}\right) \bar{Q}_{n-1}(m_1, \dots, m_M) \right. \\ &\quad \left. + \sum_{J=1}^M m_J \bar{Q}_{n-1}(m_1, \dots, m_{J-1}+1, m_{J-1}, \dots, m_M) \right] \quad \dots(41) \end{aligned}$$

Once again, by surveying the figures in table 3, one might guess that if  $0 \leq M \leq n-1$ ,  $\bar{Q}_n(m)$  is given by

$$\bar{Q}_n(m_1, \dots, m_M) = \frac{1}{(n-M-1)!} [1/2!]^{m_1} [1/3!]^{m_2} \dots [1/(M+1)!]^{m_M} \quad \dots(42)$$

This expression for  $\bar{Q}_n(\mathbf{m})$  may be shown to be correct using equation (41). Finally, if one concentrates on the special case  $n=2N$  and sets  $M=2N-2L$  where  $1 \leq L \leq N$ , one finds with the help of (31) that

$$Q_{2N}(m_1, \dots, m_{2N-2L}) = \frac{2}{(2L-1)!} [1/2!]^{m_1} [1/3!]^{m_2} \dots [1/(2N-2L+1)!]^{m_{2N-2L}} \dots (43)$$

#### 4.6 Evaluating $A^{1/2}$ and $A^{3/2}$

Now that we are in possession of a general functional expression for the coefficients  $Q_{2N}(\mathbf{m})$  it is natural to ask is whether  $c^{1/2}(\mathbf{n})$  and  $c^{3/2}(\mathbf{n})$  can be evaluated using equations (25) and (26). At first sight the answer to this question would appear to be no. The prescription for calculating the coefficients  $P(\mathbf{n}, \mathbf{m})$  which was given in section 3 cannot be generalized into a functional expression for these quantities, and without such an expression equations (25) and (26) are useless. However, closer inspection reveals that the situation is not quite so hopeless. As I shall now explain, the special form of the coefficients  $Q_{2N}(\mathbf{m})$  leads to simplifications of the sums in (25) and (26), and it is in fact possible to evaluate  $c^{1/2}(\mathbf{n})$  and  $c^{3/2}(\mathbf{n})$ . Let us consider  $c^{1/2}(\mathbf{n})$  first. When expression (39) for the spin 1/2 coefficients  $Q_{2N}(\mathbf{m})$  is substituted into (25) one has

$$c^{1/2}(\mathbf{n}) = \sum_{|\mathbf{m}|=2N}^{\mathbf{m}} P_{2N}(\mathbf{n}, \mathbf{m}) [-1/2!]^{m_1} [-1/3!]^{m_2} \dots [-1/(2N+1)!]^{m_{2N}} \dots (44)$$

Recall that  $m = m_1 + m_2 + \dots + m_{2N}$ . In (44) the factor  $(-1)^m$  which appears on the right hand side of (25) has simply been split up and absorbed into the factors in  $Q_{2N}(\mathbf{m})$ . The sum in (44) is over all possible partitions  $\mathbf{m}$  of spin 1/2  $k$ -sequences into bound and free  $k$ 's, and  $P_{2N}(\mathbf{n}, \mathbf{m})$  is equal to the number of type  $\mathbf{m}$  partitions of the  $k$ -sequence  $K(\mathbf{n})$ . As was pointed out above, any partition of  $K(\mathbf{n})$  into bound and free  $k$ 's must be a sub-partition of the form-associated partition of  $K(\mathbf{n})$  into even length closed chains. That is, if  $P_{2N}(\mathbf{n}, \mathbf{m})$  is to be non-zero then  $\mathbf{n}$  and  $\mathbf{m}$  must be compatible. Because of this, the sum in (44) may be regarded as a sum over type  $\mathbf{m}$  sub-partitions of the sequence  $K(\mathbf{n})$ . Furthermore, this sum over sub-partitions of  $K(\mathbf{n})$  may be expressed as a product of lesser sums, each of which is over the sub-partitions of a single closed chain in  $K(\mathbf{n})$ . Given the special

product form of the coefficients  $Q_{2N}(m)$  this line of thought leads one to the conclusion that  $c^{1/2}(n)$  factorizes as follows

$$c^{1/2}(n_1, \dots, n_N) = a_1^{n_1} a_2^{n_2} \dots a_N^{n_N} \quad ..(45)$$

The factors  $a_i$  are given by sums similar to the one in (44) except that they are over sub-partitions  $m$  of  $k$ -sequences consisting of single closed chains of momenta. Specifically, one has

$$a_I = \sum_{|m|=2I}^m P_{2I}(m) [-1/2!]^{m_1} [-1/3!]^{m_2} \dots [-1/(2I+1)!]^{m_{2I}} \quad ..(46)$$

The coefficients  $P_{2I}(m)$  are defined by equation (20) and their values are given by equation (21). When the expression on the right hand side of (21) is substituted into (46) we arrive at the following formula

$$a_I = 2I \sum_{|m|=2I}^m \frac{(m_1+m_2+\dots+m_{2I}-1)!}{m_1!m_2!\dots m_{2I}!} [-1/2!]^{m_1} [-1/3!]^{m_2} \dots [-1/(2I+1)!]^{m_{2I}} \quad ..(47)$$

The difficulty in evaluating (47) is that the integers  $m_i$  are constrained by the requirement that  $|m|=2I$ . This problem can be avoided if we insert into the summand the delta function

$$\delta(m_1+2m_2+\dots+2Im_{2I}-2I) = (1/2\pi) \int_0^{2\pi} d\theta e^{i\theta(m_1+2m_2+\dots+2Im_{2I}-2I)}$$

Then the integers  $m_i$  can be summed independently from zero to infinity

$$\begin{aligned} a_I &= \frac{1}{\pi} \int d\theta e^{-2iI\theta} \sum_{m_i=0}^{\infty} \frac{(m_1+m_2+\dots+m_{2I}-1)!}{m_1!m_2!\dots m_{2I}!} \left[ \frac{-e^{i\theta}}{2!} \right]^{m_1} \left[ \frac{-e^{2i\theta}}{3!} \right]^{m_2} \dots \left[ \frac{-e^{2iI\theta}}{(2I+1)!} \right]^{m_{2I}} \\ &= \frac{1}{\pi} \int d\theta e^{-2iI\theta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left[ \frac{e^{i\theta}}{2!} + \frac{e^{2i\theta}}{3!} + \dots + \frac{e^{2iI\theta}}{(2I+1)!} \right]^m \quad ..(48) \end{aligned}$$

Here, as always,  $m$  is equal to  $m_1+m_2+\dots+m_{2I}$ . Note that no generality is lost by starting the sum in (48) at  $m=1$  since the condition  $|m|=2I$  ensures that, in all of the terms which contribute to the right hand side of (47), at

least one of the  $m_i$  is non-zero. The last expression for  $a_I$  can be simplified by making the change of variable

$$e^{i\theta} \rightarrow \phi \quad \dots(49)$$

When (49) is adopted, the integral on the right hand side of (48) is transformed into the following contour integral around the unit circle in  $\phi$ -space

$$a_I = (-iI/\pi) \oint_{|\phi|=1} d\phi \phi^{-(2I+1)} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left[ \frac{\phi}{2!} + \frac{\phi^2}{3!} + \dots + \frac{\phi^{2I}}{(2I+1)!} \right]^m \quad \dots(50)$$

The square brackets in (50) contain the first  $2I$  terms in the power series expansion of  $[(e^\phi - 1)/\phi] - 1$ . No further residues are introduced into the integrand if the tail of the expansion is added on. Therefore

$$\begin{aligned} a_I &= (-iI/\pi) \oint_{|\phi|=1} d\phi \phi^{-(2I+1)} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left[ \left( \frac{e^\phi - 1}{\phi} \right) - 1 \right]^m \\ &= (iI/\pi) \oint_{|\phi|=1} d\phi \phi^{-(2I+1)} \ln \left( \frac{e^\phi - 1}{\phi} \right) \quad \dots(51) \end{aligned}$$

The factor  $a_I$  can now be evaluated by partially integrating the right hand side of (51), and using the defining equation for the Bernoulli numbers  $B_n$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

In this way one finds that

$$\begin{aligned} a_I &= (i/2\pi) \oint d\phi \phi^{-2I} \left[ \frac{d}{d\phi} \ln \left( \frac{e^\phi - 1}{\phi} \right) \right] = (i/2\pi) \oint d\phi \phi^{-2I} [e^\phi (e^\phi - 1)^{-1} - \phi^{-1}] \\ &= (i/2\pi) \oint d\phi \phi^{-2I} e^\phi (e^\phi - 1)^{-1} = (i/2\pi) \oint d\phi \phi^{-(2I+1)} [\phi(1 - e^{-\phi})^{-1}] \\ &= - \frac{B_{2I}}{(2I)!} \quad \dots(52) \end{aligned}$$

Finally, substituting (52) into (45), and setting  $n = n_1 + \dots + n_N$ , we arrive at the following expression for  $c^{1/2}(n)$

$$c^{1/2}(n) = (-1)^n \left(\frac{B_2}{2!}\right)^{n_1} \left(\frac{B_4}{4!}\right)^{n_2} \dots \left(\frac{B_{2N}}{(2N)!}\right)^{n_N} \quad \dots(53)$$

Now let us turn to the evaluation of  $c^{3/2}(n)$ . From equation (26) we have

$$c^{3/2}(n) = 4N \sum_{|m|=2N}^m (-1)^m P_{2N}(n, m) Q_{2N}(m) + \sum_{L=1}^N \sum_{|m|=2N-2L}^m (-1)^m P_{2N}(n, m) Q_{2N}(m) \quad \dots(54)$$

The first term on the right hand side of (54) involves only spin 1/2 coefficients  $P(n, m)$  and  $Q(m)$ , and is simply equal to  $4N c^{1/2}(n)$ . On the other hand, the second term involves spin 3/2 coefficients  $P(n, m)$  and  $Q(m)$ . The relevant spin 3/2 values of  $Q(m)$  are given by equation (43). Substituting these values into (54) we find

$$c^{3/2}(n) = 4N c^{1/2}(n) + \sum_{L=1}^N \frac{2}{(2L-1)!} \sum_{|m|=2N-2L}^m P_{2N}(n, m) [-1/2!]^{m_1} \dots [-1/(2N-2L+1)!]^{m_{2N-2L}}$$

The sum over types  $m$  on the right hand side of this last equation is exactly the sort of sum that would occur in the evaluation of the spin 1/2 coefficients  $c^{1/2}(n)$  in  $4N-4L$  dimensions, except for the fact that the coefficients  $P_{2N}(n, m)$  belong in  $4N$ , not  $4N-4L$ , dimensions. This problem can be remedied by using equations (18) and (19). One then finds that

$$c^{3/2}(n) = 4N c^{1/2}(n) + \sum_{L=1}^N \frac{2n_L}{(2L-1)!} c^{1/2}(n_1, \dots, n_{L-1}, \dots, n_{N-L}) \quad \dots(55)$$

Note that, for values of  $L$  for which  $0 < N-L < L$ , the index  $(n_1, \dots, n_{L-1}, \dots, n_{N-L})$  on the right hand side of this last equation simply becomes  $(n_1, \dots, n_{N-L})$ . Also, if  $n > 0$  then the condition  $|n|=N$  implies that  $n_{N-L+1} = \dots = n_N = 0$  in the full index  $n = (n_1, \dots, n_N)$ . Because of this we can use (53) to write the second term on the right hand side of (55) as follows

$$\begin{aligned}
& \sum_{L=1}^N \frac{2n_L}{(2L-1)!} c^{1/2}(n_1, \dots, n_{L-1}, \dots, n_{N-L}) \\
&= \sum_{L=1}^N \frac{2n_L}{(2L-1)!} (-1)^{n-1} \left(\frac{B_2}{2!}\right)^{n_1} \dots \left(\frac{B_{2L}}{(2L)!}\right)^{n_{L-1}} \dots \left(\frac{B_{2N}}{(2N)!}\right)^{n_N} \\
&= 4 \left( \sum_{L=1}^N \frac{n_L L}{B_{2L}} \right) (-1)^{n-1} \left(\frac{B_2}{2!}\right)^{n_1} \dots \left(\frac{B_{2N}}{(2N)!}\right)^{n_N} \\
&= -4 \left( \sum_{L=1}^N \frac{n_L L}{B_{2L}} \right) c^{1/2}(\mathbf{n})
\end{aligned}$$

where  $n = n_1 + \dots + n_N$ . Thus

$$c^{3/2}(\mathbf{n}) = [4N - 4 \left( \sum_{L=1}^N \frac{n_L L}{B_{2L}} \right)] c^{1/2}(\mathbf{n}) \quad \dots(56)$$

The final step in this work is to substitute expressions (53) and (56) into equations (3.47) and (3.48) to find the anomaly coefficients  $C(\mathbf{n})$ . One should not forget that the full spin 3/2 anomaly  $A^{3/2}$  receives contributions both from the spin 3/2 field  $\phi$  and from the spin 1/2 multiplier field  $F$  which appears in the Lagrangian (2.40). According to equation (2.44)

$$A^{3/2} = A_{\phi}^{3/2} - A^{1/2} = \sum_{|\mathbf{m}|=N}^{\mathbf{m}} [C^{3/2}(\mathbf{n}) - C^{1/2}(\mathbf{n})] T(\mathbf{n})$$

Bearing this in mind, we arrive at last at the following expressions for the anomalies  $A^{1/2}$  and  $A^{3/2}$

$$A^{1/2} = \sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{1/2}(\mathbf{n}) T(\mathbf{n})$$

$$C^{1/2}(\mathbf{n}) = 2(1/4\pi)^{2N} (-1)^n (n_1! n_2! \dots n_N!)^{-1}$$

$$\left(\frac{B_2}{4 \cdot 2!}\right)^{n_1} \left(\frac{B_4}{8 \cdot 4!}\right)^{n_2} \dots \left(\frac{B_{2N}}{4N(2N)!}\right)^{n_N} \quad \dots(57)$$

and

$$A^{3/2} = \sum_{|\mathbf{n}|=N}^{\mathbf{n}} C^{3/2}(\mathbf{n}) T(\mathbf{n})$$

$$C^{3/2}(\mathbf{n}) = 2(i/4\pi)^{2N} (-1)^n (n_1! n_2! \dots n_N!)^{-1}$$

$$[(4N-1) - 4 \left( \sum_{L=1}^N \frac{n_L L}{B_{2L}} \right)] \left( \frac{B_2}{4 \cdot 2!} \right)^{n_1} \dots \left( \frac{B_{2N}}{4N(2N)!} \right)^{n_N} \dots (58)$$

Note that in these equations  $n$  is not related to the space-time dimension; instead it is equal to the sum  $n = n_1 + \dots + n_N$ . The reader can verify that expressions (57) and (58) for the gravitational contributions to the spin 1/2 and spin 3/2 chiral anomalies correctly reproduce the results of specific calculations which have been carried out in the literature. In particular, equation (57) agrees with the results of Kimura [5] and Delbourgo [6] in the case  $N=1$ , and with those of Delbourgo and Jarvis [7] and Alvarez-Gaume and Ginsparg [8] in the case  $N=2$ . Similarly equation (58) agrees with the results of Nielsen et. al. [9] and Christensen and Duff [10] in the case  $N=1$ , and with the results of Alvarez-Gaume and Ginsparg [8] in the case  $N=2$ .

When equation (57) was first published [2] it was the only explicit expression for the gravitational contribution to the spin 1/2 chiral anomaly in arbitrary space-time dimensions. Prior to its appearance the value of this anomaly had been given only in implicit A-genus form [11]. Since then both equations (57) and (58) have been reproduced by Endo and Takao [12] using path integral methods, and Delbourgo and Matsuki [13] have also shown how to derive them from topological generating functions for the anomalies [14,15,16]. Note that, while my results agree exactly with those of Delbourgo and Matsuki [13], they differ slightly from the expressions obtained by Endo and Takao [12]. Specifically, Endo and Takao's expressions for the anomalies do not contain the factors  $(-1)^n$  that appear in equations (57) and (58).



#### 4.7 Conclusion

There are now many ways of calculating anomalies using either Feynman diagrammatic methods [11,17] or elegant path integral and topological techniques [8,12,18]. None of these approaches reduces the problem of calculating anomalies to a triviality, however the path integral and topological methods enjoy certain advantages. Firstly, these methods employ very compact notation. This renders the derivation of anomalies both clearer and simpler, and the connections between anomalies and symmetry breakdown or regularization are generally more easily seen. Of course, one does not avoid hard work altogether by using path integral or topological methods. In topological methods, for instance, one must still fix the constants that occur in front of differential geometric expressions, or calculate anomaly-associated topological indices. Similarly, in the heat kernel method, a path integral technique, one must calculate the heat kernel coefficients  $a_1$  and this is in general not a trivial problem. However, all things considered, path integral and topological techniques of anomaly calculation are much to be preferred to Feynman diagrammatic methods provided one is sure how to apply them, and provided one is confident of the answers that they give.

If, on the other hand, one is unsure how to apply these techniques, or in doubt as to whether the assumptions that they embody are valid, then one is forced to turn to diagrammatic methods of anomaly calculation. In the final analysis, diagrammatic methods are the only ones which are completely trustworthy. For example, it was pointed out in chapter 1 that Fujikawa's path integral derivation of the ABJ anomaly relied upon the assumption that the effective action was invariant under chiral transformations. This assumption was justified a posteriori when Fujikawa's analysis reproduced the results of earlier Feynman diagrammatic calculations. It was only after the validity of Fujikawa's procedure had been established in this fashion, by making contact with the results of diagrammatic calculations, that one could feel confident of extending his methods to other situations. Diagrammatic techniques played a similar role in establishing the truth of the postulate that anomalies are nothing but cocycles descended from higher dimensional Chern-Pontryagin densities [19]. The validity of this postulate could be verified only through explicit diagrammatic calculations. Other

methods of anomaly evaluation (especially, in this case, methods of the differential geometric variety [20]) were unacceptable as they all incorporated some untested assumption about the form of the anomaly. For these reasons diagrammatic methods of anomaly calculation, such as the one described in this thesis, cannot be neglected in favour of other more elegant techniques. Although, in comparison with these other techniques, they are often lengthy, cumbersome and ugly, they will in all probability continue to be an important investigatory tool so long as anomalies require investigating. In view of the importance of anomalies within many aspects of theoretical elementary particle physics, this should be for some time yet.

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## APPENDIX 1. Conventions

Throughout this thesis I work in Minkowski space of dimension  $d=2n$ . (When dimensional regularization is used, the dimension is continued from  $d=2n$  to  $d=2\ell$ .) The flat space metric of this space-time is as follows

$$\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$$

Derivatives with respect to contravariant ( $x^\mu$ ) or covariant ( $x_\mu$ ) coordinates are usually abbreviated

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \quad \partial^\mu = \frac{\partial}{\partial x_\mu}$$

and summation over repeated Lorentz (greek) or spatial (latin) indices is understood

$$V \cdot W = V_\mu W^\mu = V^\mu W_\mu = g^{\mu\nu} V_\mu W_\nu = V^0 W^0 - V^i W^i$$

Bold faced letters are reserved either for spatial vectors

$$V = (V^0, \mathbf{V})$$

or for the indices  $\mathbf{n} = (n_1, \dots, n_N)$ ,  $\mathbf{m} = (m_1, \dots, m_M)$  and  $\mathbf{i} = (i_1, \dots, i_L)$  which are described in chapters 3 and 4. When moving between coordinate and momentum spaces my convention will be that

$$p^\mu \leftrightarrow i\partial^\mu$$

The totally antisymmetric Levi-Civita tensor in  $2n$  dimensions is defined by the following expression

$$\epsilon^{\mu_1 \dots \mu_{2n}} = \begin{cases} +1 & \text{if } (\mu_1, \dots, \mu_{2n}) \text{ is an even permutation of } (0, 1, \dots, 2n-1) \\ -1 & \text{if } (\mu_1, \dots, \mu_{2n}) \text{ is an odd permutation of } (0, 1, \dots, 2n-1) \\ 0 & \text{otherwise} \end{cases}$$

In 2n-dimensional Minkowski space the algebra of Dirac gamma matrices is generated by the identity and 2n independent  $2^n \times 2^n$  matrices  $\gamma_0, \gamma_1, \dots, \gamma_{2n-1}$ . As usual, these matrices satisfy the anticommutator relation

$$\{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu}$$

Note that  $\gamma_0$  is hermitian, while the  $\gamma_i$  are anti-hermitian

$$\gamma_0^\dagger = \gamma_0 \quad \gamma_i^\dagger = -\gamma_i$$

This means that the chiral matrix  $\Gamma^{-1}$

$$\Gamma^{-1} = i^{n+1} \gamma_0 \gamma_1 \dots \gamma_{2n-1}$$

is hermitian. Also, we have

$$(\Gamma^{-1})^2 = 1$$

$$\{\gamma^\mu, \Gamma^{-1}\} = 0$$

The antisymmetric  $\sigma$ -matrices are defined as follows

$$\sigma^{\alpha\beta} = \frac{1}{4} [\gamma^\alpha, \gamma^\beta]$$

In general, I will signify that the antisymmetric product of a number of  $\gamma$ -matrices has been normalized by putting the commutator brackets around the space-time indices carried by the  $\gamma$ -matrices. Thus

$$\gamma^{[\alpha\beta]} = (1/2!)[\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha]$$

$$\gamma^{[\alpha\rho\beta]} = (1/3!)[\gamma^\alpha \gamma^\rho \gamma^\beta - \gamma^\alpha \gamma^\beta \gamma^\rho - \gamma^\rho \gamma^\alpha \gamma^\beta + \gamma^\rho \gamma^\beta \gamma^\alpha + \gamma^\beta \gamma^\alpha \gamma^\rho - \gamma^\beta \gamma^\rho \gamma^\alpha]$$

$$= \gamma^\alpha \gamma^\rho \gamma^\beta - \eta^{\alpha\rho} \gamma^\beta - \eta^{\rho\beta} \gamma^\alpha + \eta^{\alpha\beta} \gamma^\rho$$

## APPENDIX 2. Spin 3/2 Lagrangians

Consider a Rarita-Schwinger vector-spinor field  $\psi_\alpha$  of mass  $m$  [1]. The most general first order, hermitian Lagrangian which can be constructed for this field in  $d$ -dimensional momentum space is [2]

$$L = \bar{\psi}_\alpha [\eta^{\alpha\beta} (\not{p} - m) + A(\not{p} \gamma^\beta + \gamma^\alpha \not{p}) + B \gamma^\alpha \not{p} \gamma^\beta + C m \gamma^\alpha \gamma^\beta] \psi_\beta \quad ..(1)$$

The constants  $A, B, C$  in (1) are real numbers. (One can actually replace the term  $A(\not{p} \gamma^\beta + \gamma^\alpha \not{p})$  with  $A \not{p} \gamma^\beta + \bar{A} \gamma^\alpha \not{p}$  where  $A$  is now a complex number and  $\bar{A}$  is its complex conjugate. However I will not consider this possibility.) There are two things which we require of the above Lagrangian. Firstly,  $L$  must lead to the usual equation of motion for  $\psi_\alpha$  :  $(\not{p} - m)\psi_\alpha = 0$ . Secondly, the theory which  $L$  describes must be of purely spin 3/2 content. Observe that, in addition to a single spin 3/2 representation, the field  $\psi_\alpha$  carries two spin 1/2 representations of the Lorentz group. Our second condition on  $L$  therefore translates into the requirement that these two spin 1/2 representations drop out of the theory's dynamics. In effect, this means choosing the constants  $A, B, C$  so that the equations of motion imply that  $p \cdot \psi = \gamma \cdot \psi = 0$  ( $p \cdot \psi \equiv p_\alpha \psi^\alpha$  etc.) [1]. Let us now see how this may be done. By varying  $\bar{\psi}_\alpha$  in (1) we arrive at the following Euler-Lagrange field equation

$$[\eta^{\alpha\beta} (\not{p} - m) + A(\not{p} \gamma^\beta + \gamma^\alpha \not{p}) + B \gamma^\alpha \not{p} \gamma^\beta + C m \gamma^\alpha \gamma^\beta] \psi_\beta \quad ..(2)$$

Obviously, if we can arrange things so that  $p \cdot \psi = \gamma \cdot \psi = 0$  then equation (2) will lead directly to the desired equation of motion for  $\psi_\alpha$ . We need therefore concentrate on satisfying only the second of the above two conditions on  $L$ . Acting on equation (2) from the left with the two operators  $\not{p} \gamma^\alpha$  and  $\gamma^\alpha$  one finds that

$$[(A+1)\not{p} - m] \frac{1}{\not{p}} p \cdot \psi + [(A+B)\not{p} + cm] \gamma \cdot \psi = 0 \quad ..(3)$$

$$[(Ad+2)\not{p}] \frac{1}{\not{p}} p \cdot \psi + [(Bd+A-1)\not{p} + (Cd-1)m] \gamma \cdot \psi = 0 \quad ..(4)$$

In deriving equation (4) I have used the fact that in  $d$  dimensions  $\gamma^\mu \gamma_\mu = d$ . Clearly the relations  $p \cdot \psi = \gamma \cdot \psi = 0$  will emerge from (3) and (4) provided

$$[(A+1)p - m][(Bd+A-1)p + (Cd-1)m] - [(Ad+2)p][(A+B)p + Cm] \neq 0$$

That is, we require

$$[Cd-1]m^2 + [C(d-2)-Bd-2A]pm + [A^2(1-d)+B(d-2)-2A-1]p^2 \neq 0 \quad ..(5)$$

At this stage one runs into a difficulty. Generally speaking, regardless of the values of the constants  $A, B$  and  $C$ , there will be two values of  $m$  for which the quadratic expression on the left hand side of (5) vanishes. The usual response to this problem is to insist that things be contrived so that both of these values of  $m$  are equal to zero [2]. Then, provided  $(Cd-1)$  is non-zero, the only Lagrangians which will be troublesome, in the sense that they do not lead to the conditions  $p \cdot \psi = \gamma \cdot \psi = 0$ , will be the massless ones. Obviously, if both roots of the quadratic polynomial on the left hand side of (5) are to be zero, the coefficients of  $pm$  and  $p^2$  must vanish identically. This leads to the following expressions for  $B$  and  $C$  in terms of  $A$ .

$$B = \frac{1}{(d-2)} [(d-1)A^2 + 2A + 1] \quad ..(6)$$

$$C = \frac{1}{d(d-2)} [d(d-1)A^2 + 4(d-1)A + d] \quad ..(7)$$

Clearly the original freedom in the Lagrangian (1) has already been limited to the ability to arbitrarily fix the value of the single parameter  $A$ . We must still check to see whether condition (5) holds. When (6) and (7) are used to express  $B$  and  $C$  in terms of  $A$ , (5) becomes

$$(1-d)(2-d)^{-2}m^2[Ad+2]^2 \neq 0 \quad ..(8)$$

Thus the Lagrangian (1) is acceptable as long as equations (6) and (7), and the inequality (8), are all respected. This leaves us with the following one parameter set of Lagrangians for the field  $\psi_\alpha$  [2]

$$L = \bar{\psi}_\alpha [\eta^{\alpha\beta} (\not{p} - m) + A(\not{p} \gamma^\beta + \gamma^\alpha \not{p}) + \frac{1}{(d-2)}((d-1)A^2 + 2A + 1)\gamma^\alpha \not{p} \gamma^\beta + \frac{m}{(d-2)^2}(d(d-1)A^2 + 4(d-1)A + d)\gamma^\alpha \gamma^\beta] \psi_\beta \quad ..(9)$$

The parameter A can take on all real values except  $-2/d$ . For this one special value of A the Lagrangian (9) does not lead to the conditions  $p \cdot \psi = \gamma \cdot \psi = 0$ . This tends to suggest that if  $A = -2/d$ , the theory described by L is not a purely spin 3/2 theory and may contain propagating spin 1/2 degrees of freedom. One can find the  $2\ell$ -dimensional coordinate space versions of the Lagrangians (9) by setting  $d=2\ell$  and making the replacement  $p^\mu \rightarrow i\partial^\mu$ . They are

$$L = \bar{\psi}_\alpha [\eta^{\alpha\beta} (i\not{\partial} - m) + iA(\not{\partial} \gamma^\beta + \gamma^\alpha \not{\partial}) + \frac{1}{(2\ell-2)}((2\ell-1)A^2 + 2A + 1)\gamma^\alpha \not{\partial} \gamma^\beta + \frac{m}{(2\ell-2)^2}(2\ell(2\ell-1)A^2 + 4(2\ell-1)A + d)\gamma^\alpha \gamma^\beta] \psi_\beta$$

Once again, the parameter A can assume any real value except  $A = -1/\ell$ . The above  $2\ell$ -dimensional Lagrangians are introduced and examined in chapter 2.

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### APPENDIX 3. Ricci-Flat Space-Times

Space-times within which the Ricci tensor  $R_{\mu\nu}$  is uniformly zero are referred to as being Ricci-flat. Such space-times are worth considering in relation to the material of this thesis because the condition

$$R_{\mu\nu} = 0 \quad \dots(1)$$

simplifies the commutator of gravitationally covariant derivatives. In fact I shall now show that, if condition (1) holds, the commutator of two derivatives is zero provided that (i) the derivatives act upon a spin 1/2 field, and (ii) at least one of the derivatives is contracted with a gamma matrix. To see that this is so, consider a spin 1/2 field  $\Lambda$ . The form of a covariant derivative acting on  $\Lambda$  is detailed in equation (2.5). In general one can write [1]

$$[D_\rho, D_\kappa] \Lambda = R_{\rho\kappa\alpha\beta} \sigma^{\alpha\beta} \Lambda \quad \dots(2)$$

where  $\sigma^{\alpha\beta}$  is the antisymmetric product of two gamma matrices

$$\sigma^{\alpha\beta} = \frac{1}{4} [\gamma^\alpha, \gamma^\beta]$$

and  $R_{\rho\kappa\alpha\beta}$  is the Riemann tensor [1]. From (2) we have

$$\gamma^\rho [D_\rho, D_\kappa] \Lambda = \frac{1}{4} R_{\rho\kappa\alpha\beta} \gamma^\rho [\gamma^\alpha, \gamma^\beta] \Lambda \quad \dots(3)$$

Now note the following gamma matrix result

$$\gamma^\rho [\gamma^\alpha, \gamma^\beta] = 2\eta^{\alpha\rho} \gamma^\beta - 2\eta^{\beta\rho} \gamma^\alpha - 2\gamma^{[\alpha\rho} \gamma^{\beta]} \quad \dots(4)$$

In (4) the quantity  $\gamma^{[\alpha\rho} \gamma^{\beta]}$  is the normalized totally antisymmetric product of three gamma matrices

$$\gamma^{[\alpha\rho} \gamma^{\beta]} = \gamma^\alpha \gamma^\rho \gamma^\beta - \eta^{\alpha\rho} \gamma^\beta - \eta^{\beta\rho} \gamma^\alpha + \eta^{\alpha\beta} \gamma^\rho$$



Recall that the Ricci tensor is defined in terms of the Riemann tensor by the equation [1]

$$R_{\mu\nu} = R_{\mu}{}^{\lambda}{}_{\nu\lambda} \quad ..(5)$$

and that the Riemann tensor possesses the following symmetries

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta} \quad ..(6)$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \quad ..(7)$$

Substituting (4) into (3), and using (5) and (6), one finds that

$$\gamma^{\rho} [D_{\rho}, D_{\kappa}] \Lambda = [R_{\kappa\alpha} \gamma^{\alpha} + \frac{1}{2} R_{\kappa\rho\alpha\beta} \gamma^{[\alpha} \gamma^{\rho} \gamma^{\beta]}] \Lambda \quad ..(8)$$

In view of (7) the second term on the right hand side of (8) is zero, and if we take condition (1) to hold, then the first term vanishes too. Thus

$$\gamma^{\alpha} [D_{\rho}, D_{\kappa}] \Lambda = 0 \quad ..(9)$$

This result is relevant to the work contained in the thesis for the following reasons. In chapter 2 I argue that it is highly desirable to calculate the spin 3/2 anomaly and demonstrate its gauge independence in two non-standard formulations of spin 3/2 theory. Without going into details, the spin 3/2 calculation attains its simplest form in the  $A=0$  formulation, while in the  $A=-1/\ell$  formulation the gauge independence of the spin 3/2 anomaly becomes evident. In flat space the  $A=0$  and  $A=-1/\ell$  formulations of spin 3/2 theory are related to the conventional Rarita-Schwinger ( $A=0$ ) formulation by field transformations of the form (2.22). Fairly obviously, the work in this thesis is based on the assumption that the anomaly is the same in all formulations of spin 3/2 theory. This assumption is entirely reasonable provided the different formulations can be related to each other via simple transformations of the field variable. It therefore becomes important to show that the various formulations of spin 3/2 theory are so related, not only in flat space, but in curved space as well.

In general this task is a difficult one precisely because covariant derivatives do not commute. Were we able to assume that covariant derivatives do commute, the field transformations which connect the different formulations of spin 3/2 theory in curved space would be obtainable from their flat space counterparts simply by replacing flat space derivatives with covariant derivatives. The importance of equation (9) is that it tells us that, when the Ricci tensor vanishes, covariant derivatives do effectively commute. The reader may check for himself that, if a commutator of covariant derivatives arises when a spin 3/2 Lagrangian is transformed under a field transformation of the form (2.22), then the commutator always acts on a spin 1/2 quantity, and at least one of the derivatives is always contracted with a gamma matrix. Hence (9) applies and we can effectively set  $[D_\rho, D_\kappa] = 0$ .

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#### APPENDIX 4. Gamma Matrix Formulae

An algebra of Dirac gamma matrices exists in any  $d$ -dimensional space-time. However, the algebra contains an analogue of the four-dimensional chiral matrix  $\gamma_5$  only in even dimensions. In this appendix I will therefore restrict myself to space-times of dimension  $d=2n$ . In this case the gamma matrix algebra is generated by the identity and  $2n$  independent  $2^n \times 2^n$  matrices  $\gamma_0, \gamma_1, \dots, \gamma_{2n-1}$ . As usual these matrices satisfy the anticommutator relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad ..(1)$$

where  $\eta = \text{diag}(1, -1, \dots, -1)$ . Equation (1) may be used to show that the trace of the product of an odd number of gamma matrices is zero. On the other hand, it can also be used to show that if  $\not{a} \equiv a_\mu \gamma^\mu$  then

$$\text{tr}[\not{a}_1 \dots \not{a}_{2n}] = \sum_{i=2}^{2n} (-1)^i a_1 \cdot a_i \text{tr}[\not{a}_2 \dots \not{a}_{i-1} \not{a}_{i+1} \dots \not{a}_{2n}] \quad ..(2)$$

The  $2n$ -dimensional matrix  $\Gamma^{-1}$ , which corresponds to the four-dimensional matrix  $\gamma_5$ , is given by the product of all  $2n$   $\gamma_\mu$  :

$$\Gamma^{-1} = i^{n+1} \gamma_0 \gamma_1 \dots \gamma_{2n-1} \quad ..(3)$$

When applying dimensional regularization to a  $2n$ -dimensional theory, one analytically continues the dimension  $d$  from its "base" value  $d=2n$  to its continued value  $d=2\ell$ . In this case the correct analytically continued expression for  $\Gamma^{-1}$  has been shown [1] to be

$$\Gamma^{-1} = \frac{i^{n+1}}{(2n)!} \xi^{\mu_1 \dots \mu_{2n}} \gamma_{\mu_1} \dots \gamma_{\mu_{2n}} \quad ..(4)$$

where  $\xi$  is any totally antisymmetric tensor which tends to the  $2n$ -dimensional Levi-Civita  $\epsilon$ -tensor in the limit  $\ell \rightarrow n$

$$\xi^{\mu_1 \dots \mu_{2n}} = \xi^{[\mu_1 \dots \mu_{2n}]} \rightarrow \epsilon^{\mu_1 \dots \mu_{2n}}$$

Of course, the matrices  $\gamma_\mu$  in (4) are  $2\ell$ -dimensional, rather than  $2n$ -dimensional, gamma matrices. Consequently they satisfy the relation  $\gamma^\mu \gamma_\mu = 2\ell$ , not  $\gamma^\mu \gamma_\mu = 2n$ . Note that, as one would expect, the expression (4) becomes equal to (3) in the regulator limit  $\ell \rightarrow n$ . I now wish to derive some results for the analytically continued matrix  $\Gamma^{-1}$  in  $2\ell$ -dimensions. The first of these results follows directly from equation (2) and the antisymmetry of the tensor  $\xi$

$$\text{tr}[\not{a}_1 \dots \not{a}_{2m} \Gamma^{-1}] = 0 \quad m < n \quad \dots(5)$$

By again employing equation (2) one obtains a simple corollary to (5)

$$\text{tr}[\not{a}_1 \dots \not{a}_{2n} \Gamma^{-1}] = -(-1)^{n+1} 2^\ell \xi \cdot a_1 \dots a_{2n} \quad \dots(6)$$

Here I have adopted a fairly obvious dot notation in which, for example,

$$\xi \cdot a_1 \dots a_{2n} \equiv \xi_{\mu_1 \dots \mu_{2n}} a_1^{\mu_1} \dots a_{2n}^{\mu_{2n}}$$

A third result concerns the commutator  $[\gamma^\mu, \Gamma^{-1}]$ . In  $2n$  dimensions this commutator would be equal to  $2\gamma^\mu \Gamma^{-1}$ . However, in  $2\ell$  dimensions one can use equations (1) and (4) to show that

$$[\gamma^\alpha, \Gamma^{-1}] = \frac{2i^{n+1}}{(2n-1)!} \xi^{\alpha\mu_1 \dots \mu_{2n-1}} \gamma_{\mu_1} \dots \gamma_{\mu_{2n-1}} \quad \dots(7)$$

Once again arguments similar to those by which we obtained equations (5) and (6) lead us to the conclusion that

$$\text{tr}[\not{a}_2 \dots \not{a}_{2n} [\not{a}_1, \Gamma^{-1}]] = (-1)^{n+1} 2^{\ell+1} \xi \cdot a_1 \dots a_{2n} \quad \dots(8)$$

Finally we can use results (6) and (8) to progressively commute  $\not{a}_1$  through the matrices in the trace  $\text{tr}[\not{a}_1 \dots \not{a}_{2n+1} \{\not{a}_{2n+2}, \Gamma^{-1}\}]$ . In this way one arrives at the following useful formula. (Note that the anticommutator  $\{\gamma^\mu, \Gamma^{-1}\}$  is zero in  $2n$  dimensions, but acquires an anomalous non-zero value when the dimension is continued to  $d=2\ell$ .)

$$\begin{aligned}
& \text{tr}[\not{a}_1 \dots \not{a}_{2n+1} \{\not{a}_{2n+2}, \Gamma^{-1}\}] \\
&= \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \not{a}_1 \{\not{a}_{2n+2}, \Gamma^{-1}\}] \\
&= \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \not{a}_1 \not{a}_{2n+2} \Gamma^{-1}] + \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \not{a}_1 \Gamma^{-1} \not{a}_{2n+2}] \\
&= 4a_1 \cdot a_{2n+2} \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \Gamma^{-1}] - \text{tr}[\not{a}_2 \dots \not{a}_{2n+2} \{\not{a}_1, \Gamma^{-1}\}] \\
&\quad + \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \{\not{a}_1, \Gamma^{-1}\} \not{a}_{2n+2}] - \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \{\not{a}_{2n+2}, \Gamma^{-1}\} \not{a}_1] \\
&= 4a_1 \cdot a_{2n+2} \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \Gamma^{-1}] - \text{tr}[(\not{a}_2 \dots \not{a}_{2n+1}, \not{a}_{2n+2}) \{\not{a}_1, \Gamma^{-1}\}] \\
&\quad - \text{tr}[\not{a}_1 \dots \not{a}_{2n+1} \{\not{a}_{2n+2}, \Gamma^{-1}\}] \\
&= 2a_1 \cdot a_{2n+2} \text{tr}[\not{a}_2 \dots \not{a}_{2n+1} \Gamma^{-1}] \\
&\quad + \text{tr}\left[\left(\sum_{k=2}^{2n+1} (-1)^k a_k \cdot a_{2n+2} \not{a}_2 \dots \not{a}_{k-1} \not{a}_{k+1} \dots \not{a}_{2n+1}\right) \{\not{a}_1, \Gamma^{-1}\}\right] \\
&= -(-i)^{n+1} 2^{\ell+1} a_1 \cdot a_{2n+2} \xi \cdot a_2 \dots a_{2n+1} \\
&\quad + (-i)^{n+1} 2^{\ell+1} \sum_{k=2}^{2n+1} (-1)^k a_k \cdot a_{2n+2} \xi \cdot a_1 \dots a_{k-1} a_{k+1} \dots a_{2n+1} \\
&= (-i)^{n+1} 2^{\ell+1} \sum_{k=1}^{2n+1} (-1)^k a_k \cdot a_{2n+2} \xi \cdot a_1 \dots a_{k-1} a_{k+1} \dots a_{2n+1}
\end{aligned}$$

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