The Middle Years of Schooling: A Critical Time for Developing Mental Computation

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Declaration

This thesis contains no material that has been accepted for a degree or diploma by the University or any other institution. To the best of my knowledge no material has been previously published or written by another person except where due acknowledgement is made in the text of the thesis.

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Annaliese Caney

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Abstract

In the field of mathematics education, this thesis is intended to make a contribution to the literature on the teaching and learning of mental computation. The aim of the thesis is to explore the role and potential of mental computation in strengthening numeracy practices across the middle years of schooling by providing a detailed analysis of the mental computation experiences of both middle years teachers and their students (Grades 5 to 8). A focus of the study is mental computation with part-whole numbers including fractions, decimals, and percents, extending previous research that has focused almost exclusively on mental computation with whole numbers. Given the emphasis of the middle years mathematics curriculum on part-whole numbers, it is argued that this period of schooling is a critical time for developing mental computation.

The seminal work of Shulman (1986, 1987) in relation to seven domains of teacher knowledge is the theoretical framework underpinning the design of the study that was conducted through four phases. Phase 1 considers how teachers in middle years classrooms are addressing mental computation. The responses of 34 teachers (16 primary and 18 secondary) to a questionnaire are analysed using the work of Shulman as a framework. Phases 2 and 3 focus on one aspect of Shulman's work – knowledge of learners' and their characteristics – as evidenced by the students' experiences. In the second phase, data were collected from three instruments: a mental computation test, a comparison test (with pairs of fractions and decimals), and a questionnaire. A total of 172 middle years students participated from eight classes. In the third phase, 46 students participated in a task-based interview to investigate the mental computation strategies students use to solve non-contextual fraction, decimal, and percent problems. Finally, in the fourth phase of the study, seven key teachers participated in an interview session to investigate how teachers position fractions, decimals, and percents in relation to mental computation.

Outcomes associated with the teachers are presented in relation to each aspect of interest to Shulman (1987): general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of educational contexts; knowledge of educational ends, purposes and values; content knowledge; and knowledge of learners and their characteristics. Student outcomes are presented through the construction of a profile of mental computation based on three levels of student performance on the mental computation test (High, Middle, and Low). Part-whole mental computation strategies used by the students are described and discussed in relation to strategies observed for whole number. Additionally, mental computation competence is also considered in relation to working procedurally and conceptually, with the majority of student responses classified as working conceptually. A set of recommendations regarding professional development are provided based on the findings, with suggestions for future research.

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Chapter 1

Introduction

1.1 School Experience: Background to the Research

In September 1999 I started working at a school in South London. My position as a Learning Support Assistant involved working with two boys, one in Grade 5 and the other in Grade 6. My time with them was spent both in and out of the classroom; generally morning sessions were allocated to literacy and numeracy and afternoon sessions allocated to other areas of the curriculum. It was in these two classrooms, working closely with the teachers that I first became aware of some of the changes and new directions that mathematics education was embracing. Mathematics had not featured particularly on my life's radar, as the churning out of long, repetitious written computations at school had far from inspired me.

My position at Elm Court coincided with the arrival of the new National Numeracy Strategy that was being implemented in schools across the United Kingdom and as a staff member I was required to participate in a program of professional learning focusing on numeracy. Much of the time in these sessions was spent revisiting estimation, solving problems mentally, and then as a group, discussing our solutions. There were many discussions as to who had the best strategies, which were the quickest strategies, and which strategies we imagined the students might come up with. Few of us could remember being asked *how* we had solved a problem and few of us had entertained the idea that we might all be doing it differently. Right answers came from the correct application of a written algorithm or a very good memory. What was clear from these sessions was that here was a group of primary teachers and their support staff being interested, becoming involved, and perhaps even a little excited about working with number.

My return to Tasmania in 2001 proved timely. I visited my Honours supervisor in Psychology to share my experiences and discovered that a project on developing mental computation was being advertised in the Faculty of Education. I soon discovered that there was much happening in the state in relation to the teaching and learning of mental computation.

1.2 Mental Computation Research in Tasmania: A Collaborative Environment

In 1999 the Department *of* Education Tasmania (D*o*ET) and the Tasmanian Catholic Education Office (CEO), together with the Department of Education and Training (DEAT) in the Australian Capital Territory (ACT) approached the University of Tasmania to develop a collaborative numeracy project. A priority for these three educational systems was the development and implementation of numeracy policy, with any new policies being founded on current and innovative research activities. In 2000 the pilot project *Enhancing Numeracy OutcomeS* (*ENOS*) was conducted in six primary schools (Grades K – 6) in Tasmania and the ACT. This project explored the development of numeracy through mental computation.

The positive outcomes of the ENOS project motivated the mathematics education research team at the University of Tasmania and the three educational systems (now Industry Partners) to extend the research and develop a large-scale mental computation project to be conducted over a three-year period. In 2001 the project, *Assessing and Improving the Mental Computation of School-Aged Students*, was successful in receiving federal funding through Strategic Partnerships with Industry – Research and Training (SPIRT). With the completion of the project in 2004, the two main outcomes were:

1. Describing levels of achievement in mental computation by providing a theoretical framework of mental computation ability, including descriptions of sequential competency levels for students; and
2. Developing curriculum resources and materials to support teachers in developing programs and assessing mental computation, based on the

sequential approach for improving students' mental computation ability through Grades 3 to 10.

1.3 Aim and Objectives of the Study

This PhD was developed as a companion project to *Assessing and Improving the Mental Computation of School-Aged Students*. It was supported by one of the Industry Partners – the D*o*ET – as an Australian Postgraduate Award Industry (APAI) scholarship. The D*o*ET specified an area of concern within its educational system, namely the need to strengthen continuity of numeracy and approaches to teaching and learning mathematics at the point of transfer from primary school to secondary school. As this study is related to *Assessing and Improving the Mental Computation of School-aged Students*, it was proposed that mental computation would be the vehicle for exploring the development of numeracy at this level of schooling – the middle years – with both teachers and their students. Both mental computation projects were supported by the Australian Research Council (Grant No. C00107187).

The aim of the study is to explore the potential role of mental computation in strengthening numeracy across the middle years of schooling. Facility in working mentally with fractions, decimals, and percents is the avenue through which this aim is explored. Two objectives underpin the research activity:

- First, an educational objective, to provide the D*o*ET with a set of recommendations to assist the on-going development and evaluation of numeracy targets for mental computation.
- Second, a research objective, to profile a number of aspects of mental computation at the middle years level, including the experiences of teachers and students, as well as students' mental computation skills and strategies.

The following set of research questions are posed for the study:

1. How is mental computation being addressed by teachers in middle mathematics years classrooms?

2. How is mental computation being experienced by middle years students?
3. What strategies do students use to solve mental computation problems with fractions, decimals, and percents?
4. How do teachers position the teaching and learning of fractions, decimals, and percents in relation to mental computation?

1.4 Defining Mental Computation: The Case Against Mental Arithmetic

The definition of mental computation adopted for the study is provided by Reys, Reys, and Hope (1993): "the ability to derive exact numerical answers without the aid of calculating or recording devices" (p. 306). The use of the word *exact* in this definition effectively distinguishes mental computation from estimation – a skill closely related but involving approximation. Another common definition of mental computation is offered by Sowder (1988, p. 182): "the process of carrying out arithmetic calculations without the aid of external devices." The use of the word *arithmetic*, however, may draw attention away from the emphasis on students' individual thinking strategies suggested by McIntosh, Reys, and Reys (1997).

The meanings attached to the terms mental computation and mental arithmetic, are quite different in light of the agendas of the educational climates that they represent. The term mental computation appears to have been coined by Barbara and Robert Reys in the late 1970s in relation to their research on computational estimation. During the early 1980s the United States adopted the term mental computation as educators were encouraged to support new approaches to teaching and learning number, in particular a more balanced approach to computation (Reys & Nohda, 1994). In the United Kingdom the term mental arithmetic was still in circulation during the 1990s, due to its "air of respectability and tradition" (Thompson, 1999a, p. 147). It has since been replaced by the term mental calculation, which according to Thompson encompasses more than just the ability to recall number facts from memory – an emphasis of mental arithmetic – and extends to include and stress the importance of mental thinking strategies. Leading the way in Australia, prominent researchers in the field, have embraced *mental computation* and this is now the term that is favoured in Australian curriculum standards and documents.

In trying to encapsulate what mental computation *is*, one approach is to start by considering what mental computation *is not.* Anghileri (1999) offers a simple but powerful description of mental computation as not merely calculating *in* the head but rather calculating *with* the head (p. 186). The message that permeates the curriculum is that mental computation is not mental arithmetic, at least not in the old sense. Morgan (2000, p. 2) characterises traditional mental arithmetic in five ways:

1. Answers are of paramount importance;
2. Answers are often obtained by applying memorised rules, with little concern for the mathematical processes involved;
3. Lessons are characterised by a series of short, low-level, unrelated questions;
4. Time is emphasised with answers being quickly calculated, recorded, and marked; and
5. Sessions are effectively focussed on testing and not teaching.

Only remnants of this description of mental arithmetic are useful in constructing a picture of mental computation.

In relation to Morgan's first point, it is perhaps the word *paramount* that is problematic, as answers to mathematical problems are always important. This might be because mental arithmetic has long been associated with "a collection of facts not with networks of relationships" (McIntosh, 1990, p. 25). Mental computation, however, involves a more holistic approach to calculating, where questions "How?" and "Why?" are equally as important as "What?" Working mentally might involve manipulating the calculation process, for example, encouraging students to develop their own questions when provided with an answer as the starting point (McIntosh, De Nardi, & Swan, 1994). Teachers can address this in the classroom by "indicating to students that developing and using thinking strategies is a valued process" (Green, 1999, p. 141).

The mechanical application of rules is a feature Morgan associates with mental arithmetic. Anghileri (2000) agrees, maintaining that "the meaning of arithmetic has over time become limited to performance of standard algorithms without an underlying understanding" (p. 1). Mental computation is also more than memorised number facts. The emphasis on understanding the workings of the number system, including the relationships between numbers and operations, is now espoused as the foundation for developing mental computation skills.

Mental computation activities may take on almost any form that is appropriate at a given time in a classroom; they do not resemble the restrictive nature of mental arithmetic activities – Morgan's third point. With a focus on understanding, mental computation has taken on a "less is more" approach. Mental computation might involve, for example, just one or two mathematical problems: the difference being the depth of investigation as facilitated by the teachers.

An emphasis on reasoning and justification as facilitated through investigation and discussion is central to mental computation. Morgan also equates time in mental arithmetic with speed. Emphasising speed in relation to computation is a practice that McIntosh (1998) strongly suggests must stop. He writes, "If children are given time, they try – often with success – to invent an algorithm. If we emphasise speed, we remove this possibility" (p. 47). It seems the emphasis on time should actually be to give students a chance to be creative, think deeply, and consider the strategies that others use or devise their own.

Morgan's final point is that mental arithmetic sessions are simply test orientated. The focus of mental computation, however, can be quite different, with sessions being the basis for investigations of more depth to develop understanding of essential mathematical concepts. Importantly, the students' experience of mental computation should emphasise "supporting and encouraging their attempts to think for themselves" (McIntosh, 1998, p. 47).

It could also be argued that mental arithmetic is a fairly isolated, individual activity, a feature that Morgan does not include in his discussion. Potentially mental computation investigations might involve one to one discussions between a teacher and a student, between the students themselves, or larger whole-class investigations. Mental computation is not a restrictive activity and this interpretation of mental computation is very much in alignment with the current understanding of numeracy in Australia.

1.5 Numeracy - 50 Years Young

Having first appeared in the United Kingdom in the Crowther Report (1959), the term *numeracy* is approaching its 50th birthday. The term may have evolved somewhat over the years but this does not necessarily mean it has aged. Crowther imparted a sophisticated view of numeracy that encompassed both "understanding of the scientific approach to the study of phenomena – observation, hypothesis, experiment, verification" and the need "to think quantitatively" (quoted in Cockcroft, 1982). Over two decades later, the term numeracy reappeared in the Cockcroft Report (1982) representing a "culture of utility" (Noss, 1998). Cockcroft put forward a broader view of numeracy, stressing the practicalities of mathematics education in relation to the workplace and adult life, specifically including "appreciation and understanding of information which is presented in mathematical terms" (p. 11). Australia and New Zealand essentially inherited the term numeracy from the United Kingdom. Although Australia and the United Kingdom share a functional view of numeracy that emphasises the value of individuals having mathematical skills to cope with their everyday life experiences, Australia has moved away from a solely number-based conception of numeracy that educators in the United Kingdom have adopted (Doig, 2000). Numeracy in Australia is perhaps more comparable to the tenets of "quantitative literacy" as proposed in the United States (e.g., Steen, 2001) and "mathematical literacy" as defined by the Organisation for Economic Co-operation and Development (OECD, 2006) in Europe. Across the different education systems – both public and private – within Australia, most educators share a common definition of numeracy (Australian Association of Mathematics Teachers (AAMT), 1998):

> To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life. In school education, numeracy is a fundamental component of learning, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:
>
> - Underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical, and algebraic);
> - Mathematical thinking and strategies;
> - General thinking skills; and
> - Grounded appreciation of context. (p. 2)

Although there are variations in how numeracy is conceptualised, notably numeracy has forged an identity of its own having been overshadowed in education practice by literacy definitions and interventions (Luke, Elkins, Weir, Land, Carrington, Dole et al. 2003). How numeracy is conceptualised affects not only the school mathematics curriculum but also the relationship between mathematical content knowledge and pedagogy. Noss (1998) reinforces the view that new numeracies may continue to evolve to represent the ever changing social and economic needs of society.

1.5.1 Numeracy and the Tasmanian curriculum

In Tasmania the established definition of numeracy weaves together the five strands of the mathematics curriculum as outlined in the *Mathematics Guidelines K – 8* (Department of Education and the Arts Tasmania (DEAT, 1992):

> To be numerate is to have and be able to use appropriate mathematical knowledge, understanding, skills, intuition, and experience whenever they are needed in everyday life. Numeracy is more than just being able to manipulate numbers. The content of numeracy is derived from five strands of the mathematics curriculum – space, number, measurement, chance and data, and (pattern and) algebra – as

described in the National Statement and Profile. (*Numerate Students, Numerate Adults* (DEAT, 1995, p. 6)

Tasmania is one of several of Australian states implementing curriculum reform founded on a values-based philosophy (e.g., Education Queensland, 2000; South Australia Curriculum Standards and Accountability (SACSA), 2001). In Tasmania, the *Essential Learnings* framework (D*o*ET, 2002, 2003) details five curriculum organisers, which are considered the areas of essential learning for students Grades K – 10: Thinking, Communicating, Personal Futures, Social Responsibility and World Futures. These are guided by a core set of values and purposes and are followed by a set of principles to direct learning, teaching, and assessment. Although innovative, the *Essential Learnings* raises many questions and challenges for educators involved in the more traditional curriculum areas such as mathematics and science. Reference to *being numerate* is listed as a key element of one of the curriculum organisers, *communicating*. The first part of the description associated with *being numerate* links mathematical concepts and skills to "everyday problems" and the "demands of everyday life." It upholds that:

> Being truly numerate requires the knowledge and disposition to think and act mathematically and the confidence and intuition to apply particular mathematical principles to everyday problems. (p. 21)

The second part of the *being numerate* description moves to a cross-curricular focus:

> Being numerate not only includes numeracy skills and understandings, but it also involves the critical and life-related aspects of being able to interpret information thoughtfully and accurately when it is represented in numerical and graphic form. This aspect of numeracy is akin to critical literacy – being able to recognise that information can be constructed to influence the reader or viewer. (p. 21)

School-based mathematics is a foundation learning area for numeracy but not in an exclusive sense. Developing the desired skills and competencies for being numerate – justifying, reasoning, communicating – becomes a responsibility across the curriculum. The *Essential Learnings* message is "teaching for numeracy" as opposed to "teaching numeracy" (D. Neal, personal communication, March, 2003).

1.5.2 Situating mental computation

For those who favour a definition of numeracy that privileges number over other aspects of mathematics, mental computation is valued in terms of developing sound conceptual understanding of number properties, operations, and fostering number relationships. Yet within a broader definition of numeracy that encompasses a wider range of skills, mental computation is valued as the means to engage in interpreting, communicating, and applying mathematical knowledge. Mental computation then, can be accommodated within either perspective and provides a foundation for exploring numeracy at the middle school level.

1.6 Numeracy in the Middle Years

For students the middle school years are marked by the transition from the final years of primary school to the early years of secondary school. In Australia this involves, for the most part, moving through Grade 5 to Grade 8, in some states incorporating Grade 9. Students moving through the middle years of schooling are typically between the ages of 10 to 15. Some schools cater for all compulsory years of schooling including primary and secondary. Other schools offer a primary or secondary education only. Therefore, the experience of the middle years transition may involve remaining at the same school, or relocating to a new secondary school site. What drives the interest in this period of schooling for all educators is the "unique developmental and educational needs of young adolescent learners" (Barber, 1999). The perceived lack of alignment between the developmental characteristics of students in adolescence and the school organisation along with its programs continues to be the platform for the middle school reform agenda, particularly in the United States (Beane & Brodhagen, 2001).

The advent of the new millennium has brought about fundamental social and economic changes worldwide. In Australia, educators are faced with the task of considering the learning needs of students and how best to prepare students to be competent, global citizens. The values-based curriculum reforms are one outcome of this preparation. In the report *Beyond the Middle*, Luke et al. (2003) write, "middle years education has become a clear motivational force for reform and for the framing and focusing of teachers' and students' work in schools and classrooms" (p. 12). Accordingly there are a number of significant issues of teaching and learning that challenge the area of numeracy.

The middle years are arguably a critical time in addressing the essential needs of numeracy. Willis (1998) identifies three critical aspects of numeracy: mathematical knowledge, contextual knowledge, and strategic knowledge. Balancing these three aspects becomes more challenging in the middle years as the dimensions of numeracy expand; contextual knowledge and strategic knowledge become more integrated in the actual experiences of the students (Siemon, Virgona, & Corneille, 2001). In the mathematics curriculum there is an explosion of key mathematical ideas associated with each of the curriculum strands, along with number, these include pattern and algebra, space, measurement, and chance and data. Within the number strand alone, the conceptual basis shifts from additive reasoning to multiplicative reasoning, and the number system expands to include part-whole numbers (or rational numbers). An important contribution of this study is the bringing of fractions, decimals, and percents into the domain of mental computation, responding to the call of McIntosh (2002a) who advocated that it is a matter of urgency to "find effective ways to ensure that well developed approaches to number find their way particularly into the majority of middle school classrooms" (p. 463).

In terms of students' outcomes during the middle school years, there is some indication that students regress or level out in terms of academic performance standards. This is referred to as a "performance dip" or the "plateau effect." The phenomenon often seems to be included in discussions as an anecdotal overtone, a shared general understanding. There is, however, some evidence based research emerging to support this claim. In the report on the *Victorian*

Quality School Project, Hill, Rowe, Holmes-Smith, and Russell (1996, p. 32) report a "flattening out of the growth trajectory" beginning in Year 4 and continuing until Year 9. This pattern is described for three strands of the English profile but is not as marked for the mathematics strands. Similarly, in the document *Middle Years Numeracy Research Project: 5 – 9* the authors report a "performance dip" in numeracy between Year 6 and Year 7 (Siemon et al. 2001). There are many challenges that impact on the teaching and learning of numeracy at the middle school level: some of these will be covered in the following chapter.

1.7 Thesis Overview

Following the Introduction, the thesis is divided into seven chapters. Chapter 2 contains a review of the literature in the field of mental computation, drawing on work from related fields to support the objectives and the set of research questions which are outlined at the close of the chapter. In Chapter 3, a discussion of mixed methodology research with an emphasis on mathematics education is presented. As well, the design of the study – which was conducted through four phases – is introduced along with the theoretical framework. The life of the project is also detailed in Chapter 3 including details on the participants, instruments, procedures, ethical considerations, data analysis, and limitations related to the study.

The results of the study are presented over four chapters. In Chapter 4 the results of a questionnaire completed by 34 middle years teachers are analysed (Phase 1). Chapter 5 and Chapter 6 focus on the students' experiences of mental computation and comprise Phases 2 and 3 of the study respectively. In Chapter 5, data were collected from three instruments: a mental computation test, a comparison test (with pairs of fractions and decimals), and a questionnaire. A total of 172 middle years students participated from eight classes. This is followed, in Chapter 6, by an analysis of 46 task-based interviews, investigating the mental computation strategies students use to solve non-contextual fraction, decimal, and percent problems. In the final results chapter – Chapter 7 – the responses of seven key teachers who participated in an interview session are presented (Phase 4). Finally, in Chapter 8, the findings in relation to the research literature along with the implications of the observed outcomes for the research questions are discussed.

Chapter 2

Literature Review

2.1 Introduction

In this chapter, the review of the literature spans five main areas.

The review starts by unpacking mental computation in terms of its value as reflected in current national and international curriculum and in relation to theoretical thought, both historical and current. The second main area of the review considers some of the pedagogical issues for teachers with the current emphasis on mental computation and the impact on learning environments. The review then proceeds to examine the two fields of research that frame the study: a) mental computation and b) fractions, decimals, and percents. It argues that the links between the two fields have not been well established. In terms of mental computation, research activities have been conducted fairly consistently since the 1980s. Methodological approaches have been situated both in the quantitative and qualitative domains, with valuable contributions from each discussed. In the fourth section on fractions, decimals, and percents, the three concepts are examined selectively, from a mental computation perspective. In the fifth section the concepts of working procedurally and working conceptually are briefly reviewed. The literature review then closes with the aim of the current study and the objectives and research questions proposed for the study.

2.2 Unpacking Mental Computation

Scenario. For the problem 10% of 45, Daniel was able to arrive at his answer via a rule he had learned, "Four point five – I'm just moving the tens down into the units and the units into the tenths." For solving 10% of 45 it worked. He extended this strategy to solve 20% of 15, "I think that's point one five," although he did not sound sure about his answer. His explanation, "Well, umm, ten percent is one point five so I thought twenty percent would just do down again." He continued to employ this strategy again, oblivious to his error, to solve 30% of 80, "point eight - I was thinking ten percent would make it into units, twenty percent would make it into tenths and then thirty percent would take it into the hundredths?"

If the answers – either correct or incorrect – were the only point of interest to a teacher or a researcher then much of above scenario would be superfluous. It is the value of unpacking a student response that underpins the current emphasis on mental computation.

2.2.1 The value of working mentally

The value of mental computation is evident in terms of its links to mathematics education reform efforts, particularly in emphasising sense-making and conceptual understanding, which are two aspects that drive mathematics reform initiatives (Parker & Leinhardt, 1995). Mental computation is considered to facilitate and strengthen the development of understanding associated with the workings of the number system (Reys, 1984). This includes the properties of numbers and operations, and the relationships between them. Mental computation is also considered to support the development of number sense (Markovits & Sowder, 1994; McIntosh, Reys, & Reys, 1992; Sowder, 1988). An important element of number sense according to McIntosh et al. (1992) is the motivation of the learner in developing and choosing computational strategies. Number sense then underpins mental computation in terms of making decisions about the effectiveness of particular mental strategies and also in determining the reasonableness of an answer. Working mentally assists students to develop problem solving skills as students develop a critical perspective as to why one strategy might be considered more efficient for a particular problem and how to use the mathematical knowledge they have to work through a problem. It is also purported that mental computation fosters creative and independent thinking around number concepts (Reys, 1984).

Reys (1984) argued that mental computation promotes later success in the transition to written computation, particularly in terms of algorithmic procedures being taught with firm conceptual understanding rather than students relying solely on learned sets of rules and procedures (Kamii & Dominick, 1998). Additionally, mental computation is a basis for developing estimation skills (Reys, 1984). It is conceivable that success in mental computation can be achieved without computational estimation skills, as learned procedures are carried out in a mechanistic manner (Sowder, 1992). The reverse, however, does not necessarily apply. Reys (1988) suggests that many students follow a misconceived idea that estimation is about finding an exact answer and then rounding it to produce an estimate. But really in terms of its value as a skill there are many situations in real life that only require an estimate. Sowder (1988) suggests that "researchers do not appreciate the potential power of estimation, particularly as a unifying theme throughout the study of rational numbers" (p. 189).

A key idea that supports numeracy is the application of mathematics both formally and informally in everyday contexts. Appropriately, mental computation is highly valued for its practicality and immediate social utility (McIntosh, Nohda, Reys, & Reys, 1995). There is a strong case for the utilitarian value of mental computation. Northcote and McIntosh (1999) argued that mental computation is a critical adult skill. The authors conducted survey research and found that for the most part, everyday calculations performed by adults were done mentally (85%). These calculations predominantly involved the calculation of time and calculations during shopping activities. Addition and subtraction featured as the most commonly used operations. Their research supports the earlier work of Wandt and Brown (1957) who reported that calculating mentally accounted for three quarters of the calculations completed by adults. It follows that being able to compute mentally is often the simplest way to calculate, particularly in everyday situations where applications of written techniques can be laborious and simply inappropriate (McIntosh, 1998). Mental computation is also a universally valued skill due to its applicability for solving problems encountered in everyday situations, such as totalling amounts and working out discounts. New technologies are also evolving at an ever-increasing rate and students need mathematical skills that are flexible and support technical competency.

2.2.2 Mental computation: Its place in the curriculum

A gauge of the value of mental computation is the level of recognition and emphasis it receives in current mathematics curriculum and policy documents. At a basic level, curriculum as *subject matter* provides answers to questions such as what to teach, how to teach, and when to teach (Print, 1993). For mental computation these questions need to be considered in relation to written computation and the use of calculators. Within the Australian mathematics curriculum, importance is generally placed on students being able to *choose* an appropriate calculative method – either mental, written, or calculator (Curriculum Corporation, 2000). This reflects the more balanced approach to computation that has developed consistently during the 1980s as part of mathematics education reform efforts (Hope, 1987; McIntosh, 1990; Reys & Nohda, 1994).

2.2.2.1 National curriculum emphasis

Within Australian curriculum and standards documents, mental computation is embedded in the description of learning and outcomes associated with Number. Experiences in computation and estimation, along with experiences in number and numeration in the Number Strand, are detailed in *A National Statement on Mathematics for Australian Schools* (Australia Education Council (AEC), 1991). In choosing an appropriate method for either an exact or an approximate calculation, mental computation is described in the following fashion:

> People need to carry out straightforward calculations mentally, and students should regard mental arithmetic as a first resort in many situations where a calculation is needed. Strategies associated with mental computation should be developed explicitly throughout the school years, and should not be restricted to the recall of basic facts. People who are competent in mental computation tend to use a range of personal methods which are adapted to suit the particular numbers and situation. Therefore, students should be encouraged to develop personal mental computation strategies, to experiment with and compare strategies used by others, and to choose from amongst their available

strategies to suit their own strengths and the particular context. (ACE, 1991, p. 109)

Although the emphasis on mental computation is to be welcomed, the welcome appears to weaken beyond the whole number boundary. *Mathematics – A Curriculum Profile for Australian Schools* (AEC, 1994) allocates one of seven strand organisers within the Number Strand to mental computation. In the six level outcomes described specifically for mental computation, the first four levels are concerned solely with whole numbers. At the fifth level, "simple fractions" are mentioned and at level six, the following outcome is described for students: "Estimates and calculates mentally with whole and fractional numbers, including finding frequently used fractions and percentages of amounts" (AEC, 1994, p. 104). Of the individual states and territories in Australia some have placed a greater degree of emphasis on mental computation than others. In Tasmania, for example, mental computation has featured in curriculum materials since early in the 1990s (DEAT, 1992). In contrast, it is only recently that curriculum review in Queensland has explicitly addressed mental computation, in particular the issue of how best to teach mental computation – juxtaposing teacher taught strategies with strategies invented by the students (Heirdsfield, 2003a).

2.2.2.2 International curriculum emphasis

At the lower levels of the New Zealand curriculum, facility with whole numbers is emphasised for mental computation. At Level 3, covering the middle school years, "mental methods" is one of three approaches to finding "fractions of whole numbers and decimal amounts (including money and measurements)" (Ministry of Education, 1992, p. 41). At Level 5, which covers the later secondary school years, mental strategies are to be developed for operations with "positive and negative numbers using a calculator, a variety of methods, and other approaches" (p. 49). A similar emphasis on mental computation exists in the latest *Standards* document released by the National Council of Teachers of Mathematics (NCTM, 2000) in the United States. Whereas the focus in the Numbers and Operations Standard of the early grades is on whole numbers, in the standard for Grades 6 – 8, fractions and decimals are mentioned in relation to choosing appropriate calculation methods. This is expanded with a rationale for mental computation and estimation.

> Students should also develop and adapt procedures for mental calculation and computational estimation with fractions, decimals, and integers. Mental computation and estimation are also useful in many calculations involving percents. Because these methods often require flexibility in moving from one representation to another, they are useful in deepening students' understanding of rational numbers and helping them think flexibly about these numbers. (NCTM, 2000, pp. 220-221)

In the United Kingdom mental calculation has assumed a place "at the heart of numeracy" and, as such, is one of the key principles underpinning the approach to teaching numeracy as recommended in the *National Numeracy Strategy* (DfEE, 1998, 1999). This strategy was introduced in 1998 as a government initiative to support higher numeracy performance for primary and secondary school students. The mathematics program in the United Kingdom is relatively prescriptive, with guidelines for expected student achievement outlined in terms of the types and size of numbers students should be able to work with mentally at different ages (Threlfall, 2000). Again, the British focus is largely based on whole numbers. There are sketchy directions in the *Primary Framework for Literacy and Mathematics: Year 6 progression to Year 7* outlining that students should be able to draw on their personal collections of strategies for solving whole number problems: "Consolidate and extend mental methods of calculation to include decimals, fractions and percentages" (DfES, 2006). Guidelines for working with whole numbers, however, are much more specific than those outlined for fractions, decimals, and percents, for example; "multiply a two-digit by a one-digit number."

In the Netherlands mental computation has a clear role in the realistic mathematics education movement, particularly in the lower grades (Blote, Klein, & Beishuizen, 2000; Neuman, 1995; Treffers & Beishuizen, 1999). As a foundation for developing number sense the Dutch have embraced mental representations for developing mental processes, for example, based on the number line (Beishuizen, 1997). Heirdsfield (2003a) suggests that an emphasis on developing mental representations is one of the main differences between the Dutch approach to mental computation and that of Australia, New Zealand, and the United States.

2.2.3 A historical perspective

The current emphasis on mental computation is not necessarily a new phenomenon within the field of mathematics education. The "mental" aspect of mathematics has been a fundamental part of teaching number over the last century, although interest has periodically waxed and waned. Trends are described briefly by Reys (1984), Reys and Barger (1994), and Pepper (1997). Discussions tend to start at the end of 19th century: an era in education marked by the strong hold of the mentalist philosophy of education that was dominant in the United States and to a lesser extent in the United Kingdom (Thompson, 1999a). Mental discipline theory likened the human mind to a muscle, which by its very nature required exercise to promote development and increase strength (Stanic, 1986a). A basic premise of the mentalist view of learning was grounded in the physicality of strengthening the faculties of the mind, particularly the intellect, the senses, and the will. Mathematical activity that involved the repetition and rehearsal of number facts, particularly multiplication tables, was considered the "perfect technique for developing the faculties of the mind" (Reys, 1984, p. 549). The fundamental mentalist argument for the inclusion of mental arithmetic was based upon the nature and perceived benefits of the activity, with little regard for the mathematical content.

Not surprisingly, mental computation has fluctuated with the impact of different theories of learning. Goldin (2000), for example, argues that it is hard to overestimate the impact of behaviourism on research and school practices. He refers to "an exclusive emphasis on discrete, rule-based, easily testable skills, and the explicit de-emphasis of understanding as an educational goal" (p. 536). Associatism and behaviourism were dominant in the early part of the 20th century. These approaches were fundamentally mechanistic and centred on the notion that "learning is largely a matter of habit formation" (Schoenfeld, 2002, p. 437). The work of E. L. Thorndike (1874-1949) was extremely influential in matters of educational pedagogy, including instruction and assessment. His basic learning theory involved developing and strengthening the associations between stimuli and responses. In his discussion Schoenfeld (2002) uses the example of "5 x 3" and "15" to demonstrate the mental bond that exists between them. To Thorndike, bonds such as multiplication facts should be taught together to reinforce and strengthen the bonds between isolated instances of mathematical knowledge. This period marks the birth of "drill and practice" in mathematics.

Reys and Nohda (1994) argue that mental computation is a "higher-order thinking process" and that this position moves mental computation into the realm of "thinking strategies" (p. 12). Reys and Barger (1994) write, "Students are encouraged to generate thinking strategies based on their prior experience and knowledge" (p. 39). This view of mental computation is embedded in constructivist thought. From a constructivist perspective:

> We construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge. Learning is the process by which human beings adapt to their experiential world." (Simon, 1995, p. 115)

For the learning of mathematics, students as learners construct their own knowledge both individually and collectively.

Underpinned by constructivist thought, educators have been encouraged to conduct research and increase professional development activity to support the objectives of mental computation as set out in educational policies. Just as new social and economic conditions prompted educators to rethink the needs of students for the 20th century (Stanic, 1986b), educators worldwide have been scrutinising current curricula, endeavouring to anticipate the skills that students of the 21st century will need to become competent members of society (Steen, 2001). This thesis argues that mental computation is one of those skills.

2.3 Pedagogical Issues: Mental Computation in the Classroom

Advice to teachers on developing mental computation is generally embedded within pedagogy associated with teaching numeracy (Askew, Brown, Rhodes, Johnson & Wiliam, 1997; Askew, Denvir, Rhodes, & Brown, 2000) and also in developing number sense (Anghileri, 2000). Number sense and mental computation are frequently paired in the literature in the manner of a harmonious, symbiotic relationship. Number sense is less easy to define than mental computation, which suggests it is a multifaceted construct (Case 1989; Maclellan, 2001; Resnick, 1989). There is general acceptance that number sense is not about the numerical knowledge as such but rather it is about how mathematical knowledge is manipulated with the emphasis on flexibility, reasoning, and thoughtfulness (Maclellan, 2001). McIntosh, Reys, & Reys (1992) define number sense as:

> ...a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communication, processing and interpreting information. (p. 3)

Heirdsfield (2003a) points out that, whereas many teachers readily acknowledge and support the emphasis on mental computation in the curriculum, they actually fail to see it as part of a larger picture in terms of developing number sense. This observation has pedagogical implications for classroom practice, including how mental computation is developed and what assessment practices are used by teachers. There is no research to date that considers the teachers' views on mental computation, including reports of how teachers are addressing mental computation in the classroom.

2.3.1 A balancing act

Traditionally a large amount time in the mathematics classroom has been devoted to written computation instruction in the form of standard written algorithms; McIntosh, Reys, and Reys (1997) suggest approximately 85% to

95% of time was spent in this manner. Advocates of mental computation, however, argue that mental computation should be the main form of computation in schools (e.g., Willis, 1990, 1992). McIntosh (1990) painted a picture of three methods of computation and it is not hard to detect where his loyalties lie:

> So there we have the picture: great amounts of time and energy dedicated to written calculation which is little used or trusted by people out of school. Little or no time devoted to improving mental computation which is used daily by everyone. Little or no time devoted to calculator use, though everyone would agree that the calculator could, indeed does, make everyone able to compute. (p. 25)

If mental computation were to overshadow written computation in the curriculum, the question still remains, when should written algorithms be introduced to students during their school years? There is growing support for the view that written algorithms should be introduced at a much later stage of schooling (from Grade 4) than is traditionally the case (from about Grade 2) (McIntosh, 2002b, 2005; Thompson, 1999b). This would mirror the approach to written algorithms espoused in both the Netherlands and Germany (Beishuizen, 1997). Research has shown that vertical written algorithms for addition can interfere with the development of children's natural (or invented) strategies for solving problems (Cooper, Heirdsfield, & Irons, 1995; Ginsburg, Posner, & Russell, 1981). Heirdsfield, Cooper, and Irons (1999) conducted a case study on a competent student and found that he could compute quite competently before written algorithms were introduced, thus demonstrating number sense and flexibility. The authors question what written algorithms can offer this student. Additionally, invented strategies – that often involve calculating from left to right – are more accurately used than those mental strategies that move from right to left mirroring written algorithmic procedures (Carraher, Carraher, & Schliemann, 1987; Kamii, 1989; Kamii, Lewis, & Jones, 1991).

De-emphasising written computation is a considerable change for mathematics teaching. Plunkett (1979) was one of the first to capture and contrast the fundamentals of mental algorithms with written algorithms. This is summarised in Table 2.1.

Table 2.1

The Characteristics of Written and Mental Algorithms

Written as	Mental as
Written: Calculation is permanent and correctible.	*Fleeting*: Calculations can be momentary and passing.
Standardised: Calculation procedure is always the same.	*Variable*: There are many different ways to calculate.
Contracted: Calculations consist of summarised lines of equations.	*Flexible*: Calculations are adapted to suit the numbers involved.
Efficient: Every column in a calculation (working from the right) is treated as ones.	*Active:* Calculation choice is often controlled by the user.
Automatic: Calculation can be executed with little or no understanding.	*Holistic*: Calculations often involve working with complete numbers.
Symbolic: Calculations are based on the manipulation of symbols.	*Constructive*: Calculation follows a pathway from one part of the question to the answer.
General: A method will work for any combination of numbers for an appropriate operation.	*Not designed for recording*: Recording of calculations is generally not predicable.
Analytic: Numbers are broken up and digits are dealt with separately.	*Requiring understanding*: Understanding demonstrated through the flexible construction of calculations.
Not easily internalised: Generally do not reflect the ways people think naturally about numbers.	*Iconic*: Calculation may be supported by a visual tool or image.
Encourage cognitive passivity: Calculation requires limited decision making.	*Early approximation* to correct answer: as demonstrated through the practice of working left to right.
	Limited: Do not always suit difficult calculations or every problem.

With the advent of the technological age there surfaced another compounding element that educators had to consider alongside mental and written computation – the introduction of calculators in classrooms. Educators were forced to consider what might be needed to support students in using technology effectively and efficiently. Those giving standing to mental computation were quick to recognise that mental computation was important in "the efficient use of technology" (Reys, 1984, p. 549) and needed in order to check calculator results (Cockcroft, 1982), so that students did not simply develop unquestionable faith in the = sign.

2.3.2 Developing mental computation: Teacher-taught versus student-invented

For teachers, instructional emphasis in relation to mental computation is commonly presented as a dichotomous position: teacher-taught strategies versus student-invented strategies. Managing this tension is perhaps one of the biggest challenges teachers face in developing mental computation with their students. Traditional pedagogy suggests that teachers look for the best mental computation procedures and teach them. "Best" might be defined as "the most common" or the methods that appear to be most easily understood by the majority of students, or possibly methods best understood by the teacher. Threlfall (2000) expresses concern that explicit instruction from teachers compromises the strategic and flexible elements of students' own thinking processes. He writes, "in a structured teaching situation there is a decision about how to calculate, but it is made for the child by the teacher, in effect, through the teacher's intention to practise particular approaches" (p. 81). This model of teaching mental computation is aligned with behaviourist theory of learning – where students are given specific strategies to learn and their ability to incorporate the strategies in computation problem solving forms the basis of assessment. There is always the possibility that learning strategies and then executing them, directed solely by the teacher, will not be any more successful than written algorithms (McIntosh, 1991).

A prevailing issue regarding the assessment of mental computation concerns the use of traditional testing in classrooms. This involves pencil and paper tests where the students record only an answer: thus the focus is on what knowledge students have acquired and can recall in the given testing situation. As the sole mental computation activity this form of assessment is reminiscent of the era of mental arithmetic (Heirdsfield, 2003a) and her concern is that the emphasis on student understanding, that now drives the push for mental computation, is not reflected in assessment practices that are comprised solely of traditional testing.

Callingham and McIntosh (2002) consider that in terms of documented outcomes for mental computation, there are generally too few expectations for whole numbers other than for basic number facts. They also note that there is "none at all for decimals, percentages, and fractions" (p. 423). Some expectations and outcomes are noted for the fraction, decimal, and percent content domains, yet specific goals are rarely addressed for mental computation.

2.4 Mental Computation: The Research Domain

There are two features of mental computation research that are of particular relevance to the current study. First, mental computation involving whole numbers has dominated the research field and this provides the focus for discussing the literature in this section. Second, qualitative research has rarely extended beyond the upper primary grades to incorporate the early years of secondary school. Quantitative studies that do incorporate the middle years do so within a range of grades, providing only snapshots of how students at this level of schooling perform (Caney, 2002).

Quantitative research contributions have tended to come from large student sample sizes where data have been collected using pencil and paper tests of mental computation ability. This type of methodological approach does not appear to reflect a contemporary, constructivist view of mental computation that emphasises individual thinking strategies. Generally, however, researchers do not advocate pencil and paper tests of mental computation as an appropriate testing and assessment tool for teachers and use in the classroom. Mental computation tests are largely research tools and in this way research in this area has provided some valuable contributions. Quantitative research in the field of mental computation has provided a perspective on three aspects: levels of mental computation performance, error patterns, and comparative international performance.

2.4.1 Levels of mental computation performance

In the context of performing calculations mentally, how to monitor students' progress was the motivation for the work of Callingham and McIntosh (2001, 2002). The authors argued that, although there is an accepted hierarchy of development to support written computations, teachers are left largely to their own devices in developing, implementing, and assessing mental computation programs. Using Rasch modelling as the theoretical framework, the authors considered the mental computation performance of students across Grades 3 to 10 ($N = 1452$) and constructed a developmental scale of mental computation ability in which they described eight levels of performance. These levels represented an increasing complexity in the type of problems students could solve successfully according to the type of numbers, both whole numbers and part-whole numbers, and across the four operations (Callingham & McIntosh, 2001).

In extending the research, Callingham and McIntosh (2002) used the levels of mental computation performance to report on two aspects of student performance: patterns of student ability across the school years and growth for individual year groups. Grades 3 and 4, for example, were the grades where students exhibited the greatest period of growth in mental computation competence. Across Grades 6 and 7, however, the growth rate plateaued before increasing again between Grades 7 and 8. Callingham and Watson (2004) considered the four operations with part-whole numbers only and further extended the work by identifying six levels of increasing complexity across fraction, decimal, and percent problems.

This body of research represents a substantial quantitative contribution to mental computation research, as the data on student performance were based on rigorously designed tests of mental computation ability and analysed with a complex statistical model. Although McIntosh et al. (1995) developed a set of mental computation tests with some link items, comparisons across grades were limited, as the individual grade tests were different. Similarly, Bana and Korbosky (1995) demonstrated increasing performance across the primary grades for basic number facts, but the work did not expand on types of computation problems that might be appropriate for students working at different levels. The work of Callingham and McIntosh (2002) has important implications for classroom teachers in providing a research base from which to sequence activities that develop mental computation and support assessment of ability. Resources developed from the work of Callingham and McIntosh (2001, 2002) support teachers in making decisions about their students learning and in confidently making judgements as to what types of problems might be appropriate and when (McIntosh, 2004).

2.4.2 Errors in mental computation

In investigating students' mathematical ideas, describing common errors that students make, the sources of these errors, and the associated underlying misunderstandings is a popular line of research (Even & Tirosh, 2002). In the field of mental computation, quantitative studies have contributed to research of this nature (Bana, Farrell, & McIntosh, 1995; McIntosh, 2002; Watson, Kelly, & Callingham, 2004). Generally, the errors that students make in mental computation appear to be different qualitatively from those described for written computation (McIntosh, 1998).

Bana et al. (1995) selected 12 non-contextual number problems from a mental computation test to investigate errors across Grades 3, 5, 7, and 9. The test items reflected key mathematical content areas and were chosen to illustrate interesting error patterns. They reported on specific error percentages associated with each item; for example, 190 was given as the answer to 38 x 50 and this was recorded by 4%, 8%, and 7% of students in Grades 5, 7, and 9 respectively. The authors went on to suggest that this type of answer demonstrated a lack of understanding related to the order of magnitude of numbers or place value understanding. It was also possible to see that for many of the items the number of correct responses increased consistently over Grade 3 to Grade 9 and the number of students not attempting problems decreased in a similar fashion.

Building on these earlier studies McIntosh (2001) sought to increase the sample sizes and number of questions, analysing the most common errors made at each grade level (3 – 10). Again, the incorrect responses from pencil and paper tests of mental computation were clustered and the most common incorrect responses for the different number types described. McIntosh suggested that in working mentally at the most basic level, errors can be identified as either procedural errors or conceptual errors. From the same data set Watson et al. (2004) adopted a developmental approach and completed a more fine-grained error analysis that focussed on items from one of the eight development levels previously described (Callingham & McIntosh, 2001, 2002).

2.4.3 International student performance

A third research area involves comparative studies of mental computation performance at an international level. During the 1990s Australia joined Japan and the United States to conduct research on mental computation performance that could be compared internationally. McIntosh et al. (1995) were able to present some general trends from data collected from Grades 2 to 9. For example, initially the performance of the Japanese students at Grades 2/3 was much higher than for students from Australia and the United States. The difference was minimised, however, by Grade 8/9 with the performance of the Australian students exceeding that of the Japanese students. The individual results for the Japanese students are reported in Reys, Reys, Nohda, and Emoir (1995). There has also been interest in the related field of number sense in relation to comparative performance on an international level (McIntosh, Bana, & Farrell, 1997; McIntosh, Reys, Reys, Bana, & Farrell, 1997). These researchers also conducted an investigation into student attitudes to mental computation and the types of problems students would prefer to do mentally (McIntosh, Bana, & Farrell, 1995).

2.4.4 Mental computation strategies

Studies in the field of mental computation that have used a qualitative methodology have sought to capture the mathematical thinking in which students engage and investigate the development and use of mathematical concepts. Task-based interviews have been the main method of inquiry (Goldin, 2000; Heirdsfield, 2002b; Hunting, 1997). Several key findings associated with mental computation have emerged from qualitative studies. First, mental strategies that students report often do not reflect the algorithmic procedures that school mathematics has emphasised through written computation. From the 1980s the literature is alive with quotes from students explaining the bizarre and wonderful ways that they solved problems. Accordingly, many of the strategies were self developed or self taught and often with limited knowledge of formal algorithms (Carroll, 1997; Kamii, Lewis, & Livingston, 1993). The explanations provided by students were not ways of thinking that were explicitly taught or expected of students in the classroom.

A second important finding is that the introduction of written computation can have a negative effect on the continued development of mental computation strategies (Cooper, Heirdsfield, & Irons, 1996; Ginsburg, Posner, & Russel, 1981; Heirdsfield & Cooper, 1996; Kamii & Dominick, 1998). Heirdsfield and Cooper (1996) found that although young children were inventive in solving unfamiliar problems, they tended to make use of written algorithms for mental computation once taught them.

There is also evidence that mathematical computations embedded in a context tended to elicit invented strategies, whereas noncontextual computations tended to elicit mental versions of written algorithms (Carraher, Carraher, & Schliemann, 1987). Cooper et al. (1995) reported similar findings after comparing children's mental strategies for algorithmic exercises and word problems in Grades 2 and 3.

The literature includes a large assortment of descriptions of solution sequences or strategies employed by students for working with multi-digit whole numbers. Essentially, working mentally involves "a wider range of strategies than traditional written procedures" (Heirdsfield, 2002a). Research that investigates and describes the strategies that students use to solve problems mentally beyond basic number facts, has predominantly involved the addition and subtraction of two-digit whole numbers, as either non-contextual problems or word problems. Threlfall (2000) suggests that this level of calculation is a reasonable expectation for all students to achieve and problems are unlikely to be solved solely by recall. It follows that the studies focus on students in the middle to upper years of primary school. Of the studies that describe the variety of mental strategies that students use, the underlying interest is in how these strategies initially develop.

2.4.4.1 Whole number strategies: Addition and subtraction

In relation to mental computation, McIntosh, de Nardi, and Swan (1994) document two types of strategies: changing the operation and using commutativity. The former is likely to involve changing the operation of a subtraction problem to addition. The latter involves changing the order of the numbers in the problem. For addition this requires the student to understand that the order of the operands does not affect the final outcome, for example, $4 + 6 = 6 + 4$, following the rule of commutativity. The strategy, however, is not mirrored for subtraction. These two strategies are called initial strategies, as this conceptual "rearrangement" of the problem appears to precede any computational activity.

Mental computation strategies for solving basic number fact problems (single digits numbers to 20) have been well documented. The summary presented in Table 2.2 is collated from the work of Carpenter and Moser (1984), Thompson (1999), and McIntosh et al. (1994), and provides a comprehensive description of counting strategies.

Table 2.2

Summary of Strategies for Basic Number Facts

Strategy	Addition		Subtraction	
	Description	Example	Description	Example
Count all	Count out each operand & count altogether	3 + 5 Count out "1, 2, 3" and "1, 2, 3, 4, 5" and count all, "1, 2, 3, 4, 5, 6, 7, 8"	Count out first operand, count down to the second operand and recount remainder	8 – 3 Count out "1, 2, 3, 4, 5, 6, 7, 8," count down "1, 2, 3" and recount "1, 2, 3, 4, 5"
Count on from first number	Start with one number & count on the second number (count on from the larger number involves commutativity)	3 + 5 "3 plus 4, 5, 6, 7, 8" or "5 plus 6, 7, 8"	a) Count back from first operand b) Count back to second operand c) Count up from operand changing the problem into one involving addition	8 – 3 a) 8, count back "8, 7, 6, 5" b) 8, count back "8, 7, 6, 5, 4," keep tally of 5 c) "3. 4, 5, 6, 7, 8" – keep tally of 5
Use known fact	Use known number facts, number bonds, doubles, or near doubles	3 + 5 "3 plus 3 is 6, plus 2. 8" or "5 plus 5 is 10, take 2. 8"	Use known number facts, number bonds, doubles, or near doubles	8 – 3 "8 take 4 is 4 [know 4 plus 4 is 8] so add one to get 5"
Bridge to 10	Use known number bonds or doubles to make 10 first & then work with the remainder	8 + 5 "8 plus 2 is 10, add on another 3. 13"	Use known number bonds or doubles to make 10 first and then work with the remainder	13 – 5 "13 take 3 is 10, take 2 is 8"

Beyond problems involving single digits many of the strategies in Table 2.2 can be extended. McIntosh et al. (1994) distinguish between elementary counting (counting in ones) and counting in larger units. The latter is generally a more sophisticated approach to counting that might involve, for example, any of the counting strategies listed above but using twos, fives, or tens, as well as strategies such as repeated addition (and subtraction) and skip counting.

As students' mathematical thinking develops students move beyond basic counting strategies to strategies that are considered more efficient for working with numbers larger than 20 (McIntosh, 1998). Although the literature is replete with terminological variations, researchers appear to distinguish between three types of invented strategies. First, Fuson, Wearne, Hiebert, Murray, Human, Olivier, et al. (1997) discuss strategies that involve combining units separately, or collections-based solutions. Sequential strategies are a second category (Fuson et al. 1997) and are also referred to as counting-based solutions. The former involves separating and recombining the numbers. The latter involves keeping a running total during the calculation. A third type of strategy documented involves a wholistic approach (Cooper et al. 1996). In using this type of strategy, Carpenter, Franke, Jacobs, Fennema, and Empson (1997) report, "the numbers are adjusted to simplify the calculation" (p. 4). In the following summary, whole number strategies involving *collections* are considered first with a particular emphasis on the treatment of place value. The examples used to illustrate particular strategies are from the student interview data collected as part of the current study.

In Figure 2.1 the student separates both operands and then regroups the numbers according to place value. This calculation process can involve either working first from the left (with the tens) or from the right (with the units), the latter reflecting the formal written algorithm (Cooper et al. 1995). The strategy is known as *regrouping* (Ginsberg et al. 1981), *separated place value* (Cooper et al. 1996), *1010* (Beishuizen, 1993; Klein & Beishuizen, 1998), and the *split-method* (Thompson, 1999). McIntosh et al. (1994) simply refer to this strategy as *used tens/hundreds*.

$$58 + 34$$

"92. 50 plus 30 is 80 and then 8 plus 4 is 12, 80 plus 12 is 92"

$$50 + 30 = 80$$
$$8 + 4 = 12$$
$$80 + 12 = 92$$

Figure 2.1. A Grade 6 student using a collection based strategy for two-digit addition.

A different strategy for the same problem, 58 + 34, is presented in Figure 2.2 and in this case it still involves manipulation of the numbers based on place value. The difference is, however, that the student keeps or preserves one of the numbers (often the largest), but splits the second number by place value (or other quantity). The numbers are added progressively in parts as a mechanism for keeping track of the answer during the process. This is referred to as a counting (or sequence) based strategy (Carpenter et al., 1997). Again, in adding the second number, the calculation may involve working with the tens first or with the units first. This strategy is known as *aggregation* (Cooper et al. 1996), *N10* (Beishuizen, 1993; Klein & Beishuizen, 1998), *jump-method* (Thompson, 1999) and *worked with parts of a second number* (McIntosh et al. 1994). There is some evidence that working from the left using the tens is more natural for students in their early number development (Cooper et al. 1996).

58 + 34

"92. Well with the 58 I just added the 30 so ended up with 88 and then added the 4"

58 + 30 = 88
88 + 4 = 92

Figure 2.2. A Grade 7 student using a counting based strategy for two-digit addition.

The third category of strategies is broadly identified as wholistic strategies (Cooper et al. 1996). Carpenter et al. (1997) refer to *compensating*: an example using 58 + 34 is presented in Figure 2.3. The first operand, 58, is recognised by the student, as being close to 60, which transforms the problem into what is arguably an easier computation. This strategy is known as *levelling* or *compensation* (Cooper et al. 1996), or *over-jump* (Thompson, 1999), and is also considered a form of *bridging* (McIntosh et al. 1994). Adjusting both operators in the same problem, for example, "60 + 32" is an example of *levelling* (Cooper et al. 1996). This final class of strategies is argued to show a deeper level of understanding. Askew (2003) suggests that the use of the first two types of strategy is a good indicator of conceptual understanding of how numbers work; the third category shows a level of understanding that could be described as strategic. This group of strategies tends to be dictated by the properties of operands involved in a computation problem with both operands involved considered in relation to each other.

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58 + 34

"92. 58 is nearly 60 so I just add 60 and 34 which is 94 and take 2"

$58 + 2 = 60$
$60 + 34 = 94$
$94 - 2 = 92$

Figure 2.3. A Grade 6 student using a levelling strategy for two-digit addition.

Importantly, within any of these three different categories of strategies, the more elementary counting strategies may be used during the calculation process.

2.4.4.2 Whole number strategies: Multiplication and division

Mental computation involving the operations of multiplication and division has received less research attention than addition and subtraction. Drawing on the wider literature base, however, invented strategies for solving word problems contribute to the discussion for single digits (e.g., Anghileri, 1999; Kouba, 1989; Mulligan & Mitchelmore, 1997) and for larger numbers of digits (e.g., Murray, Olivier, & Human, 1994). These studies all discuss the fundamental conceptual understandings associated with multiplication and division that are a platform from which students go on to develop strategies to work with problems of larger number combinations. These are referred to in the literature as the multiplication "laws." Along with the commutative law (described in relation to addition) they include: the associative law (e.g., 3 x 6 = 3 x (3 x 2)), and the distributive law (e.g., 13 x 6 = (10 x 6) + (3 x 6)). Typically these laws develop first through working with smaller numbers or within the confines of the multiplication tables (Anghileri, 1999).

In working with basic number facts for single-digit multiplication and related division facts, McIntosh (2005) describes four strategies, including *commutativity*. A summary is provided in Table 2.3. The first strategy, *doubling*, is considered the springboard from which students are introduced to the concept of multiplication. Doubling is such a powerful strategy that "they [students] appear to gain control of this long before they can perform other multiplications" (McIntosh, 2005, p. 6). The second strategy is based on doubling with the additional step of *adding one more lot* for problems involving multiplication by three. *Skip counting* – the third strategy – involves students using familiar number patterns to count in groups.

Table 2.3

Summary of Strategies for Basic Number Facts (Multiplication)

Strategy	Description	Example
Doubling	For problems involving multiples of 2 (and later 4).	2 x 6 "double 6 is 12"
Adding one more lot	Based on doubling for problems involving multiples of 3.	3 x 6 "double 6 is 12 and add one more 6, 18"
Skip counting	Use of number patterns.	3 x 6 "6, 12, 18"
Commutativity	Multiplication pairs can be reversed with the same answer.	2 x 6 "2 times 6 is the same as 6 times 2, 12"

These four strategies, described in relation to basic number facts involving multiplication, are also applicable for problems involving division. Students commonly approach division by changing a division problem into one involving multiplication. For example 18 ÷ 3 is changed to 3 x ? = 18, with students reforming the problem to be "how many 3's make 18?" to which a number of familiar strategies apply such as doubling and skip counting.

Contrary to addition and subtraction, many of the strategies for solving multi-digit multiplication and division problems are basically extensions of those used for single-digit problems. Doubling and halving, for example, remain extremely important strategies: two variations are shown in Figure 2.4 and Figure 2.5. For the problem 24 x 3, the first student doubles 24 and then adds another 24. This is a version of adding one more lot as identified by McIntosh

(2005) but with a larger number. The second example involves the simultaneous act of doubling and halving as the problem is translated from 24 x 3 to 12 x 6.

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24 x 3

"I went double 24 is 48 and then added the other 24. 72."

24 x 2 = 48

48 + 24 = 72

Figure 2.4. A Grade 8 student using a doubling/add one more lot strategy.

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24 x 3

"72. I worked that out as 12 x 6...halving to 12 first and then doubling the 3."

12 x 6 = 72 = 24 x 3

Figure 2.5. A Grade 8 student using a doubling/halving strategy.

As well as skip counting (refer to Table 2.3), a number of the basic counting strategies that students become familiar with for addition and subtraction are used for solving problems involving multiplication. Heirdsfield, Cooper, Mulligan, and Irons (1999) reported the mental strategies that students used to solve word problems involving combinations of single-digit and multi-digit numbers and the operations of multiplication and division. They devised a typology of five strategies, the first of which is *counting strategies*. This strategy is described as "any form of counting, skip counting, forwards and backwards, repeated addition and subtraction, and halving and doubling strategies" (p. 91). Essentially this category includes many of the strategies for basic facts listed by McIntosh et al. (1994). It also condenses the calculation strategies Mulligan and Mitchelmore (1997) described, including direct counting, rhythmic counting, skip counting, and additive calculation.

Figure 2.6 details an example of a *separation strategy* where the student separates the numbers by place value (Heirdsfield et al., 1999). This strategy is underpinned by the distributive law and parallels a strategy used for addition. In their work, Heirdsfield et al. (1999) distinguished between working from the left and working from the right as individual strategies. Being able to identify those students using a separation strategy starting from the right was important for the authors, who were interested in a potential instructional effect following the introduction of formal written algorithms for multiplication. For larger combinations of numbers Heirdsfield et al. (1999) also reported examples of students working from the left and right to solve 100 ÷ 5.

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24 x 3

"72. I did 20 times 3 and then 4 three times, add them together."

20 x 3 = 60
4 x 3 = 12
60 + 12 = 72

Figure 2.6. A Grade 8 student using a separation strategy.

Wholistic strategies in relation to addition and subtraction were described in Section 2.4.4.1. Heirdsfield et al. (1999) report a parallel strategy for problems involving multiplication and division where numbers are treated in a wholistic fashion. The problem 24 x 3 is shown in Figure 2.7 and the strategy involves changing the first operand "24" to "25": the student also uses skip counting, i.e., "25, 50, 75." This was the fourth strategy described by the authors. For larger combinations of numbers Heirdsfield et al. (1999) also reported examples of students working from the left and right to solve 100 ÷ 5 using a separation strategy.

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24 x 3

"72. 24 is close to 25 so I went 25, 50, 75, and then took 3 off."

25 x 3 = 75
75 – (1 x 3) = 72

Figure 2.7. A Grade 6 student using a wholistic strategy.

Finally, the importance of a student's individual store of basic number facts was discussed for addition and subtraction. *Basic facts* is the fifth strategy described by Heirdsfield et al. (1999) in relation to multiplication and division. Number facts are not necessarily just common table facts but may also be familiar sets of doubles, number bonds, or virtually any number relationship that is meaningful to the individual. Anghileri (2000) writes, "making connections among the facts will not only minimize the number of facts to be learned but will encourage strategies that will reduce the working in later calculations" (pp. 78-79). In this way many single-digit problems where students initially use a counting strategy later become part of a pool of basic facts from which the student can draw upon (Heirdsfield et al. 1999). Multi-digit problems are commonly solved by splitting the numbers by place value, however, students also use other quantities, for example, 24 x 3 as 15 x 3 and 9 x 3, if it is meaningful for them. Building a store of known facts to use for mental computation is a far cry from the repetitious nature of learning tables by rote.

2.4.5 Mental computation: Its associated links

The question, "Why are some children better than others at working mentally?" has motivated some consideration by mathematics educators. American researcher Hope (1985) examined the literature to furnish a profile of expert mental calculation. Hope and Sherrill (1987) went on to study the characteristics of skilled and unskilled senior secondary students and their ability to calculate multiplication problems mentally. One of the main findings they reported was that unskilled students tended to use versions of written algorithms that involved working digit by digit and strictly from right to left. The authors noted that these calculations were often accompanied by the use of "imaginary writing instruments" (p. 106). Skilled students, however, adapted strategies to suit the number properties of the given task and, in particular, discarded "carrying," a feature adopted from written computation.

Heirdsfield and Cooper (1997) looked at the issue of student competence from the perspective of proficiency, observing that some students employed one strategy consistently to solve a selection of problems whereas other students employed a variety of strategies. On this basis students were categorised as being *unistrategy* or *multistrategy*. Like their predecessors Hope and Sherrill (1987), Heirdsfield and Cooper report overdependence on right to left strategies, reflecting written algorithmic procedures for unistrategy students. It is argued that these students were operating with little number sense, and blindly applying a strategy with little attention to the numbers involved in a given problem. Heirdsfield (2001, 2002, 2003) and Heirdsfield and Cooper (2002) went on to examine accuracy and flexibility in some detail. Heirdsfield

(2001) examined accuracy in mental computation, identifying that proficient students have a much larger and stronger set of mathematical connections available to them. Complex interactions were reported between knowledge bases including aspects of number sense, and metacognitve components of mental computation, such as the students' perceptions of their ability. Alternatively, students who demonstrated accuracy in their mental computation work, but were not flexible in choosing efficient strategies, were limited in their mathematical knowledge connections.

Threlfall (2002) poses the question: "Is children's mental calculation strategic?" He touches on a complex issue. Threlfall questions the appropriateness of the term "strategy," arguing that this implies that such decisions are strategic and choice based. He maintains that it is misleading in the sense that it implies students are conscious about the choices they make to solve problems. The issue as to how conscious students are in their strategy choice is difficult.

2.5 Fractions, Decimals, and Percents

Fractions, decimals, and percents are versatile mathematical concepts that feature in the everyday mathematical experiences of both children and adults. One form of this experience involves working mentally yet the role of mental computation in developing these quite sophisticated concepts is not clear. Research investigating those mental strategies students use to solve fraction, decimal, and percent problems and how these strategies develop has received considerably less attention than its whole number counterpart (Caney & Watson, 2003). Over time some researchers have posed questions that allude to important directions for future research in this area; for example, Reys and Barger (1994, p. 45) ask, "How self-generated mental thinking strategies apply to the study of non-whole number work (e.g., fractions, percent)?" In terms of combining these areas of research, however, little progress has been made. The general literature surrounding each of the three concepts under consideration in this section – fractions, decimals, and percents – is vast. Therefore, work is selectively reviewed from the perspective of mental computation to support the research questions proposed for the study.

Fractions, decimals, and percents as related concepts, are integral parts of middle school curricula across the globe, building on whole number concepts and students' intuitive ideas and informal experiences with rational number. These concepts are, however, repeatedly reported as being difficult for students (Behr, Harel, Post, & Lesh, 1992; Kieren, 1988; Siemon, Virgona, & Corneille, 2001), and are associated with low standards of performance in studies comparing students' performance on an international level (TIMSS). Parker and Leinhardt (1995) completed a review of percent literature that opens with the question: "Why is percent, a ubiquitous mathematical concept, so hard to learn?" (p. 421). The same question applied to fractions and decimals has motivated much discussion and research in the mathematics community.

Fractions, decimals, and percents feature different notational (symbolic) systems, although in many ways this is a surface difference as all three areas share founding concepts built on multiplicative structures. The complexity of rational numbers has been captured through the semantic analysis of rational number subconstructs (Jones, Langrall, Thornton, & Nisbet, 2000), which include: decimals, equivalent fractions, ratio, multiplicative operators, quotients, and measures on a number line (Behr et al, 1992, 1993; Kieran, 1988, 1992; Sowder, Bezuk, & Sowder, 1993). Others position rational number itself as a subconstruct of proportional or multiplicative reasoning (Lamon, 1999; Thompson & Saldanha, 2003; Vergnaud, 1988). Neither position dismisses or reduces the intricate conceptual links, which is why this is such a difficult area for students.

The mere presence of a decimal point, the fraction bar, or the percent symbol, is one aspect that students find exceedingly difficult to integrate into their mathematical thinking. Research has shown that students find it very difficult to discard their whole number thinking, which can be the start of misconceptions that endure throughout the school years (Behr, Wachsmuth, Post, & Lesh, 1984; Hart, 1981; Stephens & Pearn, 2003). The links between rational number representations may never become apparent for some students

(Markovits & Sowder, 1994). Many researchers have also noted that errors in the most basic fraction, decimal, and percent problems are often incorrect applications of written algorithms, and accordingly, are classed as procedural errors. A cause for concern across the three domains is the demonstrated absence of number sense when students engage in computation problems (Hiebert & Wearne, 1985).

For each of the three concepts – fractions, decimals, and percents – there is a well established body of literature that focuses on where students go wrong, the types of errors they make, and importantly, what we can infer from this in terms of unearthing underlying conceptual difficulties and misunderstandings. Traditionally the interest of researchers has favoured documenting the thinking and knowledge of those students struggling to advance in their mathematical understanding, over those defined broadly as successful. More recently, however, the general interest in students' thinking strategies has emphasised that it is equally important to focus on student success and ask what is it that students understand in relation to particular mathematical ideas? How do they use these understandings? What mathematical connections are fundamental? This perspective is suited to the study and practice of mental computation. The only study to date that has considered fractions, decimals, and percents solely from the perspective of mental computation strategies is Caney and Watson (2003). The authors began to document the strategies that students use to solve problems involving fractions, decimals, and percents noting the replication of mental strategies from the whole number domain. The data set for this preliminary work is part of the SPIRT project – *Assessing and Improving the Mental Computation of School-Aged Students* – to which the current study is related.

In this section, examples of research concerning fractions, decimals, and percents are reviewed, highlighting work more specifically related to the field of mental computation. Although mental computation strategies are generally not the focus of these studies, descriptions of such strategies are embedded in many tasks.

2.5.1 Fractions

Examples of computations with fractions tend to highlight the fractional misunderstandings that students carry through schooling, such as treating the numerator and denominator as separate whole numbers and adding accordingly, for example, $\frac{1}{2} + \frac{3}{4} = \frac{4}{6}$ (Stephens & Pearn, 2003) and $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ (Silver, 1983). Such examples are often used as evidence of procedural thinking. Examples of students successfully solving fraction mental computation problems are less common in the research literature. Hart (1981) documented responses to some fraction problems in her discussion of two tests that formed part of a research program – *Concepts in Secondary Mathematics and Science*. For the "easiest" computation problem, $10\frac{1}{2} \times 3$, she suspects that the students could be employing a repeated addition strategy based on (3×10) and $(3 \times \frac{1}{2})$, but comments that this strategy would perhaps fail a student in attempting the harder problems such as $3\frac{1}{2} \times 2\frac{1}{2}$. In a study investigating students' informal fraction knowledge, Mack (1990) recorded that students invented their own algorithms for some of the fraction subtraction problems. For example, the problem $4\frac{1}{8} - 1\frac{5}{8}$, Mack (1990) describes the following strategy based on regrouping: "First subtract one from four to get three, next subtract $\frac{5}{8}$ from three ("because you can't subtract $\frac{5}{8}$ from $\frac{1}{8}$") to get $2\frac{3}{8}$, then add $\frac{1}{8}$ to $2\frac{3}{8}$ ("because that's still left from what you started with") to get $2\frac{4}{8}$ or $2\frac{1}{2}$." (p. 26).

Weber (1999) studied the impact of a series of lessons designed to strengthen students' conceptual understanding of mental computation procedures with fractions. Several examples of students working mentally with fractions (post-interview) are presented. For the problem $\frac{5}{8} + \frac{1}{2}$ a student gave the following response starting with knowledge of equivalent fractions: "Four eighths is a half, so one half plus one half is one and that one eight is left over." In a division problem, $5 \div \frac{1}{3}$, two examples of mental strategies were recorded. The first student responded: "15. Basically it is like saying how many one thirds are there in five? Say ten divided by two is five...so five divided by one third, there are fifteen sets of one third in five" (p. 56). Initially this student was able to engage in the problem by changing the expression of the operation. The second student responded, "15. There are three parts in each whole then three times five would be fifteen" (p. 55). Weber reports that these strategies were based on a representation of division based on how many of the divisors were contained in the dividend.

Caney and Watson (2003) observed a number of strategies that students used to solve fraction problems mentally. Some of these strategies included changing the representation of problems, for example, changing ¾ – ½ to its equivalent decimals representation and changing the operation of a problem from division to multiplication. A repeated addition strategy was also reported for the problem 4 x ¾ whereby students described progressively adding ¾.

2.5.2 Decimals

How students work mentally with decimals and the four operations has not featured extensively in recent research activity. It seems plausible, however, that this is an area where mental computation strategies will mirror those used to solve whole number problems due to the explicit links within the place value system. Perhaps the area that has received the most attention is the development of decimal understanding and misunderstandings through decimal comparison tasks (e.g., Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Stacey & Steinle, 1998; Steinle & Stacey, 2003, 2004).

Hiebert and Wearne (1985) outlined a model for students' decimal computation procedures and tested the model on a sample largely comprised of middle years students (Grades 5, 6, 7, and 9). Part of the research involved interviewing students to substantiate how closely the model predicted the processes students actually used to solve the decimal problems. Many of the responses were procedural in nature and were based on students explaining how they worked through a problem after completing a written item. For the problem 0.23 + 0.41, for example, a student response is recorded: "Cause you just add it ... you go to line up the decimals first, then you add the problem like any other addition problem, then you just bring the decimal straight down" (p. 198). Successful solutions were not discussed in the scope of this study but the implication was that the student sample, having been introduced to written algorithms, would use procedural strategies.

Weber (1999) also provided examples of students working through decimal problems, for example, 0.07 + 0.2 (post-interview):

> "Two tenths and seven hundredths. Twenty-seven hundredths. Just added a zero at the end of the two and then zero and your seven is seven and zero and two is two. [Why do you add the zero and the two and not the seven and the two?] You can't add like hundredths and tenths together." (p. 54)

Another example for the problem 4 – 0.9: "Three and one tenth. I just rounded the nine tenths to one and then I subtracted and got three and added the tenth that I took away" (p. 54). This example would seem to demonstrate an element of number sense and aligns with the idea of *bridging* as a mental computation strategy (Caney & Watson, 2003). Caney and Watson suggest that students use ideas such as bridging to a whole or a reference point in a similar way to bridging to 10 with whole numbers. From a number sense perspective, Anghileri (2000) advises that making links between decimal representations and percents (e.g., $^{1}/_{10}$ = 10% and $^{1}/_{100}$ = 1%), and also developing decimal benchmarks, can help to "establish more meaningful calculation and flexibility" (p. 114).

2.5.3 Percents

The key skills behind solving percent application problems involve a variety of arithmetic procedures such as common fraction, decimal-fraction, percent conversions, whole number multiplication and division, and decimal-fraction multiplication and division (Parker & Leinhardt, 1995). What role might mental computation play? In an investigation into middle school students' understanding and knowledge of percent, Dole, Cooper, Baturo, and Conoplia (1997) reported that a characteristic of proficient students was strong mental computation skills. Lembke and Reys (1994) investigated the strategies students use to solve percent problems at different levels of mathematical development. The authors were interested in the role of intuitive percent knowledge and how this interacts with school-taught ideas about percent. This research involved an element of mental computation but students also had a variety of aids available to support their work, including calculators, paper and pen, and concrete aids.

Students' use of percent benchmarks involves an association with the fractional parts of a whole (Parker & Leinhardt, 1995), and particularly concerns interpreting the common fractions ½ as 50%, ¼ as 25%, and ¾ as 75%. Lembke and Reys (1994) attach the following explanation to the benchmark strategy for percent, describing it as, "Uses of common reference points to establish boundaries or initial values when estimating or finding exact values" (p. 243). The ability to use percent benchmarks intuitively appears to make sense to young students before formal instruction occurs (Risacher, 1992).

Parker and Leinhardt (1995) caution that emphasis on benchmarks, to any great extent, may hinder students' progress. They argue that benchmarks do not help students work with non-benchmark values (e.g., 32%) and that the concept of percent is reduced to a mere association with familiar fractions or a basic divisional process (e.g., divide by 2). Importantly, however, the value in understanding benchmark percents does seem to feature when estimation and checking of answers is required. This is what Lembke and Reys (1994) contended as they reported the benchmark strategy being used by students to justify their answers. The authors report, "this solution, although not exact, reflects conceptual understanding and the invention of a useful approach to approximating answers" (p. 247). Caney and Watson (2003) also reported the use of benchmarks but within the context of changing the representation of the problem from percents to fractions in mental computation. They provided an example of a student working in the following way, "25% of 80, that's 20. That's a quarter, just like a quarter." Multiple representations for percent were recorded as a separate strategy by Lembke and Reys (1994).

The role of number sense in relation to understanding percent was investigated by Gay and Aichele (1997). The authors reported the use of benchmark percents in making comparisons in several number sense tasks. They observed that students performed better on problems involving 50%, 100% and 25%. They also reported the use of fractional relationships by some students, although they later commented that the students could often quote relationships between fractions, decimals and percent but "did not seem to use the interrelationships among numerical equivalents with confidence" (p. 33).

Although computation was not a specific requirement of the problems used by Gay and Aichele (1997), some instances of strategies were detailed. In solving 65% of 35, for example, "one seventh-grade student noted that one-half of 35 was 17 and that 15% more was needed" (p. 32). Moss and Case (1999) described several strategies reported by students for the problem 65% of 160 that involved splitting the operator into parts, for example, 60% and 5% or 50% and 15%. Moss (2002) described an invented algorithm for the problem: calculate 75% of the length of an 80cm desktop? Students started by finding 50%, then worked out 25% before adding the total parts. Similarly for the problem 25% of 15, Gay and Aichele (1997) described a strategy: "50% of 15 was one-half of 15 which was 7.5, and one-half of 7.5 was 3.75" (p. 32). Although the use of a benchmark value was referred to in the latter strategy, it resembles a repeated halving strategy reported by Caney and Watson (2003, p. 6).

A further strategy reported by Lembke and Reys (1994) is a ratio (or proportion) strategy, providing the following definition of a procedure: "Sets up a comparison or a proportion to solve the problem or finds a proportionality constant" (p. 243). This strategy was reported for problems that involved quantities larger than 100, for example 21% of 400, but also in terms of working with 75% in a problem that essentially asked about 25%.

As a final comment, these examples – described for fractions, decimals, and percents – are generally not reported within a mental computation context. They appear in studies focussing on number sense and improving students' conceptual understanding of these domains, and hint at the possibility of documenting the ways for solving problems mentally that are invented or self-generated strategies.

2.6 Working Procedurally, Working Conceptually

The theme of working procedurally and working conceptually underpinned Section 2.5, as research involving part-whole numbers is frequently used to illustrate the two types of knowledge in relation to student performance. Procedural knowledge relates to the connections between the system of symbols that represent mathematical ideas and the rules for which the symbols can be manipulated to solve mathematical problems. Conceptual knowledge relates to the connections between pieces of information, with relationships creating rich networks to support conceptual understanding (Hiebert & LeFevre, 1986). Both types of knowledge have endured over time in the field of mathematics education. There have been many attempts to capture the characteristics of these two domains and their contribution to mathematical understanding; for example, Skemp (1976, 1986) distinguishes between *instrumental* and *relational* understanding, and Baroody and Ginsburg (1986) discuss *meaningful* knowledge and *mechanical* knowledge.

For mental computation, working procedurally is associated with the teaching of formal procedures for solving written computations. Weber (1999) classified pre- and post-interview responses using the procedural/conceptual distinction when exploring the outcomes of a mental computation instructional program for a Grade 8 class. His study suggested that when the teaching of computation centres on written procedures (algorithms), students' mental computation competence is likely to be restricted to mental versions of written algorithms with little demonstration of understanding the number system in which they are working. With an interest in the middle years, Weber pointed out that for the whole number problems in particular, learned procedures were often exclusively used, although this was not the case for rational number problems where students were perhaps less familiar with traditional written procedures and seemed to benefit most from instruction emphasising conceptual knowledge. These results are aligned with a study by McIntosh (2002) in which common errors in mental computation were also classified as procedural or conceptual. Overall, the errors recorded for whole numbers were more often than not associated with procedural workings, and the errors made on problems involving fractions and decimals were largely attributed to conceptual misconstructions. In a similar manner, Caney and Watson (2003) applied the instrumental/conceptual distinction in describing mental computation responses for solving fraction, decimal, and percent problems. Responses classified as working instrumentally were classified as reflecting learned procedures, such as written algorithms or rules. Students' responses classified as working conceptually, however, involved the use of their knowledge of part-whole quantities and operations. Importantly, these studies suggested that the use of written procedures and rules as mental strategies are indicators that students are working procedurally, rather than conceptually.

2.7 Summary and Research Questions

The current emphasis on mental computation is situated within constructivist thought and places value on students' thinking strategies as avenues for which to develop conceptual understanding. The real life applicability of mental computation adds weight to its inclusion in current curricula. As such its importance is recognised both nationally and internationally in relation to teaching numeracy and has generated much advice for teachers. Pedagogical issues include de-emphasising written computation, focussing on teacher-taught strategies versus student-invented strategies, and assessing mental computation. The research base and resulting discussion, however, has been generated in relation to the primary school years and also focuses on whole number mental computation, particularly for the operations of addition and subtraction. Middle and secondary levels of schooling are considered in relation to levels of mental computation performance and errors in mental computation; research that is quantitative in nature. In middle years classrooms the number system expands significantly to include part-whole numbers, yet little consideration of the role mental computation might play in developing these important concepts has been documented. At the forefront of researchers' interest concerning fractions, decimals, and percents, the themes of working procedurally and working conceptually are commonly explored and these themes also surface in relation to mental computation.

This chapter has outlined the literature relevant to the aim of this study: to explore the potential role of mental computation in strengthening numeracy across the middle years of schooling. Two objectives frame the research activity:

- First, an educational objective, to provide the D*o*ET with a set of recommendations to assist the on-going development and evaluation of numeracy targets for mental computation.
- Second, a research objective, to profile a number of aspects of mental computation at the middle years level, including the experiences of teachers and students, as well as students' mental computation skills and strategies.

Based on the foregoing literature review of the field of mental computation and related areas, the following research questions are posed for the current study:

1. How is mental computation being addressed by teachers in middle years mathematics classrooms?
2. How is mental computation being experienced by middle years students?
3. What strategies do students use to solve mental computation problems with fractions, decimals, and percents?
4. How do teachers position the teaching and learning of fractions, decimals, and percents in relation to mental computation?

Each of the four research questions is addressed individually in one of the four phases of the study, the design of which comprises Chapter 3.

Chapter 3
Research Methodology and Design

3.1 Introduction

This chapter comprises five main sections. The first section explores some of the research perspectives and methodological issues that impact on mixed method designs. The intention of this section is not to evaluate one paradigm (quantitative or qualitative) by contrasting it with another but rather to acknowledge the qualities of each in relation to the study. The second section outlines the research design and introduces Shulman's framework (1987) in relation to the research questions. The methods of inquiry that the design encompasses are discussed in the third section. The works of several authors are referred to in outlining and discussing the design of the study from a mixed methodology perspective. Although several of the references post date the design, they provide validation tools and have been useful in describing the design of the study. The fourth section details the life of the project including those who participated, the instruments used, the procedures followed and the necessary ethical considerations for conducting research in an educational setting. The fifth and final section of the chapter outlines the limitations associated with the study described in terms of *generality* and *trustworthiness* (Schoenfeld, 2002).

3.2 Research Perspective and Methodology

Mixed method approaches encompass "collecting and analysing both qualitative and quantitative forms of data in a single study" (Creswell, 2003, p. 15). Key concepts associated with this approach include: pluralism, integration, and synthesis. Originating in the United States, many fields under the umbrella of the social and behavioural sciences have embraced mixed methods. Education, evaluative nursing, public health, sociology, clinical research, administrative sciences, and community psychology are some of the broad fields of research identified by Tashakkori and Teddlie (1998).

3.2.1 Keeping the peace: The pragmatist position

Approaching research using mixed methods is associated with the paradigm of pragmatism. Paradigms, as worldviews or belief systems (Guba & Lincoln, 1994), guide the work that a researcher undertakes by providing a set of "interlocking philosophical assumptions" (Greene & Caracelli, 1997, p. 101) including knowledge claims (epistemologies), strategies of inquiry (methodologies) and methods to conduct the research (Creswell, 2003). In the social and behavioural sciences the dominant paradigms have been the traditional positivist position with epistemologies that lend themselves to quantitative methods of inquiry and the interpretivist position where methods of inquiry are essentially more qualitative in nature. For many decades advocates of these two paradigms have competed for some sort of supremacy across many of the fields of the social and behavioural sciences. A critical question for advocates of mixed methodologies is, "where are mixed methods situated in relation to the dominant paradigms?"

The sometimes heated debate between supporters of the positivist and interpretivist paradigms is often described by applying an analogy of war. In extending the war analogy, Tashakkori and Teddlie (1998) position the pragmatists as the *pacifists* or the *peace keepers* in the social science paradigm wars. This is because pragmatists accept that the two approaches are compatible and that it is possible, and even advantageous, to combine elements associated with both. Although this stand is the foundation of mixed methodology, it is also the point at which critics of the position censure pragmatism, for disregarding the irrevocable link between epistemology and methods of inquiry.

Pragmatists firmly place the research question(s) or the research problem(s) at the heart of an investigation. In this sense "researchers have adopted the tenets of paradigm relativism or the use of whatever philosophical and methodological approach works for the particular research problem under study" (Tashakkori & Teddlie, 1998, p. 5). This is an important distinction from the positivist and interpretivist paradigms, which have in the past emphasised the pre-eminence of the research methods employed or an epistemological position.

A motivation for using mixed methods approaches is described by Greene and Caracelli (1997): "to generate deeper and broader insights, to develop important knowledge claims that respect a wide range of interests and perspectives" (p. 97). The notion of *triangulation* is central, where findings from different data sources are corroborated, with the unique perspectives of each source used to overcome the deficiencies of the other. Triangulation was the key idea that sparked interest in the seminal work of Campbell and Fiske (1959). Using what has been described as a *within methods triangulation* (Jick, 1979), Campbell and Fiske used different quantitative techniques to study a psychological trait. Jick (1979) used the term *across methods triangulation* to illustrate the application of quantitative and qualitative methods to study phenomena. Clearly working within a pragmatic paradigm is attractive to researchers in terms of its emphasis on multiplicity and practicality.

One issue that affects all paradigms equally – no matter how diverse – is that of quality through rigorous research design. Essentially the literature in the area of mixed methods has moved from discussions of viability to issues of design and quality. Authors such as Tashakkori and Teddlie (1998) and Creswell (2003) are working to develop typologies of mixed method designs. Miles and Huberman (1994) make the comment that, "The question, then, is not whether the two sorts of data and associated methods can be linked during study design, but whether it should be done, how it will be done, and for what purposes" (p. 41). These aspects are addressed in relation to the research design that is detailed in Section 3.3.

3.2.2 Mixed methods research in the field of mathematics education

The stronghold of positivism, and its associated quantitative methods, has in more recent times weakened across many of the social science fields (Simon, 2004) and this includes mathematics education. The contributions of qualitative approaches are now more widely recognised. Inquiry that combines the two methodologies is an approach welcomed by the mathematics education community (Lesh, 2002) in conducting research "aimed at making a difference in theory or in practice" (p. 32).

In a four-yearly review of research in mathematics education in Australasia produced by the *Mathematics Education Research Group of Australasia (MERGA)*, Walshaw and Anthony (2004) report on methods of data collection and analysis associated with the field through a review of publications from 2000 to 2003. The authors report that three of the most commonly used methods – task assessments, observations, and questionnaires – were used individually or collectively with other methods. It seems that researchers working in the field of mathematics education often do not explicitly acknowledge working within a mixed methods framework, although the use of several different methods within a study suggests a multifaceted look at the phenomenon of interest. This is demonstrated by a study design that moves from the quantitative to the qualitative, where, for example, students independently solve problems and document their mathematical responses (survey methods) and then later talk about and explain their thinking, perhaps attempting to apply it to a different but related problem situation (interview methods). The work on students' understanding of decimal notation (Stacey & Steinle, 1998; Steinle & Stacey, 2003; 2004) is one example of this. By employing quantitative data collection techniques the authors document several mathematical behaviours as inferred through students' written responses. Through interviews the authors are able to describe the students' thinking and thus confirm and further elucidate their initial findings. From the field of chance and data, investigations into students' levels of understanding associated with concepts such as random, average, and probability provide other examples, as data are generated and collated from both survey and interview methods (Watson & Caney, 2005; Watson & Kelly, 2004). Watson and Caney (2005) further report higher levels of response to an interview item than an identical item presented in surveys. These works provide convincing examples of where quantitative and qualitative methods intertwined have produced results that explore the complexity of the area of interest. In addition, the findings such as those reported by Watson and Caney have implications in terms of influencing the decisions teachers make in assessing their students' performance.

Datta (1994) writes from an evaluative background and perspective and lists four persuasive and practical reasons in support of mixed methodologies. First, for decades paradigms associated with quantitative and qualitative research have been used to frame research activity. Second, this use has been supported and encouraged by many evaluators and researchers. Third, both paradigms have received and continue to receive funding. Fourth, as a consequence of the support both quantitative research and qualitative research have influenced decision making and policy. These points seem applicable to the field of mathematics education. Large-scale research projects that are privileged to have extended timeframes and funding frequently conduct inquiry through both quantitative and qualitative methods. The Effective Teachers of Numeracy project (Askew et al. 1997), for example, used questionnaires, interviews, and observations of teachers, along with a measure of students' numeracy performance, to investigate the complexity of what constitutes effective mathematics teaching. The work of the Trends in International Mathematics and Science Study (TIMSS) group provides an international perspective on issues of mathematics education, employing quantitative methods to explore teaching practice and student performance for comparative benchmarking. The group has also conducted studies of classrooms using video analysis to gather rich descriptions of teaching and evidence of how the curriculum is being implemented (e.g., Givvin, Hiebert, Jacobs, Hollingsworth, & Gallimore, 2005; Stigler & Hiebert, 1997). Among a number of benefits listed, the descriptive data enables the study of complex processes and facilitates the integration of qualitative and quantitative information (http://nces.ed.gov/timss/faqvideo.asp?FAQType=2).

3.3 Research Design

3.3.1 Teacher knowledge framework

The seminal work of Shulman (1986, 1987) in relation to domains of teacher knowledge is the theoretical framework that underpins the design of the study. Originally Shulman (1986) proposed a framework for analysing teachers' knowledge differentiating subject-matter knowledge, pedagogical content knowledge and curricular knowledge. He expanded this work in 1987 to specify seven domains of knowledge: a) content; b) general pedagogical; c) curricular; d) pedagogical content; e) learners and their characteristics; f) educational contexts; and g) educational ends, purposes and values. Generally, Shulman does not appear to make any claims about the exclusiveness and parameters of each knowledge domain. His work was motivated by an era when there was extensive interest in questions of effective teaching, teaching expertise, and the professionalism of teaching, particularly in the United States.

Although Shulman's approach originated in the 1980s, recent research in mathematics education has employed Shulman's domains of teacher knowledge in a variety of contexts for assessing teachers. Watson (2001) used all seven knowledge domains as the framework for a profile for detailing teacher competence in relation to a particular mathematics curriculum strand, in this case probability and statistics. Shulman's knowledge domains have also been used individually; for example, Kanes and Nisbet (1996) employed the content knowledge, pedagogical content knowledge, and curriculum knowledge categories to explore the knowledge bases of mathematics teachers. Mayer and Marland (1997) also explored teachers' knowledge of students.

The important aspects of teacher knowledge for mental computation in association with Shulman's knowledge domains include the following:

a) Content knowledge: Mental computation in relation to whole numbers, fractions, decimals, and percents.
b) General pedagogical knowledge: Professional teaching background.

c) Curriculum knowledge: The place of mental computation in relation to other topics in the curriculum.

d) Pedagogical content knowledge: How to develop mental computation through activities and in relation to class organisation, time and assessment.

e) Knowledge of learners and their characteristics: Likely student responses to mental computation tasks and perception of the students' attitude to mental computation.

f) Knowledge of educational contexts: Understanding of the primary and secondary school contexts.

g) Knowledge of educational ends, purposes, and values: How mental computation fits within the broader context of mathematics and numeracy; and the alignment of mental computation in terms of a skill that students need to be competent members of society.

3.3.2 Four phase research design

This study combines elements of quantitative and qualitative inquiry in a mixed method design. Miles and Huberman (1994) maintain that "qualitative data are useful when one needs to supplement, validate, explain, illuminate, or reinterpret quantitative data gathered from the same setting" (p. 10). Essentially this is the premise from which the design of this study originates and it is illustrated in Figure 3.1. The four phases of the study follow a sequential implementation, as opposed to a concurrent one. This is because each phase influences the next in terms of sampling or content (Creswell, 2005). The work of Shulman (1986,1987) which underpins the design is also represented in the Figure.

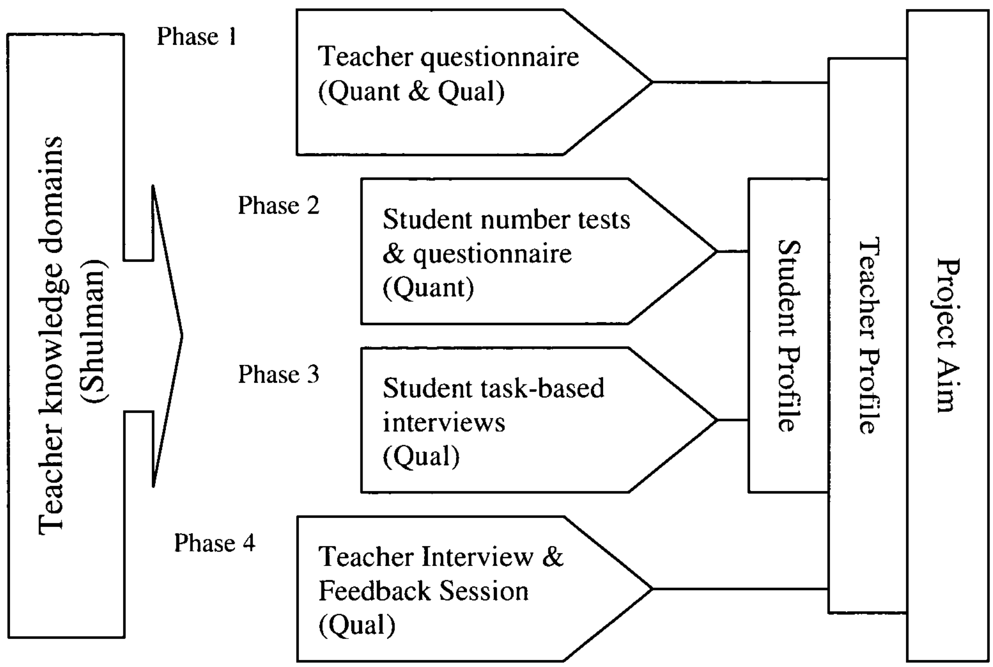

Figure 3.1. Four phase research design.

Phase 1. The first phase of the study addresses the question, how is mental computation being addressed by teachers in middle years mathematics classrooms? A range of data is collected on multiple aspects of teachers' experience and responses encompass all seven of Shulman's knowledge domains through a questionnaire instrument.

Phase 2 and 3. In the second and third phases of the study, the focus is on documenting the characteristics of students as learners in relation to mental computation (Shulman, 1987). Within the context of this study, this does not entail an investigation of the teachers' own understanding of the characteristics of their students but rather an exploration of the students' own understanding and experience of mental computation (Phase 2) and students' thinking in relation to fraction, decimal, and percent mental computation (Phase 3).

Phase 4. This phase was guided by the research question: how do teachers position the teaching and learning of fractions, decimals, and percent in relation to mental computation? Like Phase 1, the fourth phase of the study addresses the majority of Shulman's teacher knowledge domains with a presentation of the discussion generated by the responses of teachers in Phase

1, including their perspectives on the interview data gathered from the students during Phase 3.

3.4 Methods of Inquiry

As Figure 3.1 illustrates, the four phases of the research design culminate in two profiles: a student profile of mental computation and a teacher profile of mental computation. This section provides an account of the methods and techniques utilised in the generation of the study's data to construct the two profiles. In particular, the discussion focuses on the use of profiling through survey and interview methods as applicable to the study. According to Tashakkori and Teddlie (1998) both these methods of inquiry belong to the category "asking individuals for information and/or experiences" (p. 100). They are fundamentally self-report techniques.

3.4.1 Profiling

Watson (2001), who developed a single profiling instrument, describes the term *profile* as "a framework for reporting teachers' achievements and competencies" and identifies the context for which the profile was being implemented as "teaching the topics chance and data in the mathematics curriculum" (p. 306). In the context of this study, the term is not restricted to a single instrument but is concerned with the process of investigating the attributes and experiences of teachers and students in relation to mental computation. The profiling approach is well aligned with a mixed methods study. An example is the work of McIntosh, Bana, and Farrell (1995) who surveyed students in relation to three aspects of mental computation: the types of computational problems students prefer to do, attitudes towards mental computation, and an assessment of mental computation performance. Although the authors do not employ the term *profiling*, their work provided a snapshot of students across Grades 3, 5, 7, and 9.

3.4.2 Survey methods

Survey methodologies in general have long served the research needs of the social sciences (Sarantakos, 1993) and are commonly associated with a quantitative approach to research. Information can be collected through oral and written questioning techniques and essentially "they provide a quantitative or numeric description of trends, attitudes, or opinions of a population by studying a sample of that population" (Creswell, 2003, p. 153).

Among the strengths associated with survey methods, one of the primary features is that it is possible to collect large quantities of information within a relatively short period (Thomas, 2003). In addition to the time-saving aspect, this can be a relatively inexpensive endeavour. Second, respondents can participate at their convenience. Third, a wide variety of information can be gathered, with an assortment of research questions (Muijs, 2004). Fourth, the information generated is likely to reveal the present status of the selected characteristics, with a considered and objective view of the issues.

There are, however, a number of limitations associated with survey methods, one of which is the restricted opportunity for respondents to clarify or expand their responses (Thomas, 2003). As survey questions tend to be standardised, the instrument itself is likely to limit the length and depth of responses (Muijs, 2004). Sarantakos (1993) points out that there is usually no opportunity to motivate respondents. These issues give rise to another concern – the limitations associated with self reported data. Four issues are raised by Sarantakos (1993) in relation to the priories that researchers using survey methods are obliged to relinquish control over: (a) the order in which respondents address the questions, (b) the true identity of the respondents, (c) the conditions under which the questionnaire is answered by the respondents, and (d) the provision of partial responses. In relation to the final point, non-responses to questionnaire instruments can be extremely problematic for researchers. Tashakkori and Teddlie (1998) caution that non-responses can affect the generalizability and inference quality of a study. It can be very difficult, even impossible, to control or predict the total number of responses. Attention to the length and presentation of a questionnaire instrument then is important in terms of encouraging participation.

In mathematics education, survey methodologies have been used widely to investigate aspects of classroom and teaching practice. The study of teacher beliefs is an area that commonly employs closed scale survey instruments (e.g., Beswick, 2002). Students' understanding of mathematical content associated with specific curriculum areas, for example chance and data, have also been investigated using survey methodologies (Watson, 1994).

3.4.3 Interviews

Universally, interviewing is one of the most accepted methods for conducting research. Indeed Kvale (1996) points out that "conversation is an ancient form of obtaining knowledge" (p. 8) and introduces the term "professional conversation" to describe the style of communication that takes place during a research interview. In many respects "conversations" have traditionally been an implicit and peripheral part of research conducted by psychologists and social scientists. They are now central to many of the methodological approaches associated with qualitative research. Cannold (2001) suggests that interviews are "conversations between researcher and participant in which the researcher seeks to elicit the participant's subjective point of view on a topic of interest to the researcher" (p. 179). The appeal of interviewing is the experience of one-to-one communication and interaction between the researcher and the participant(s).

Generally interviews tend to be described in terms of whether they are structured or unstructured. In considering the continuum between these two forms there is, however, as much variation in the style of interviews as there is in the terminology used to describe them. Structured interviews are associated with a formal setting, pre-established questions, and limited response categories. The interviewer retains a neutral and unobtrusive role (Denzin & Lincoln, 2004). In effect these interviews lend themselves to quantitative methods of data collection and analysis. Conversely, unstructured interviews feature non-directive, open-ended questions, and encourage dialogue and interaction between the interviewer and interviewee (Denzin & Lincoln, 2004). In general unstructured interviews are aligned with qualitative research. Much of the time this polarization of structured versus unstructured is only helpful to the extent that it delineates the boundaries for researchers. Tashakkori and Teddlie (1998) point out the qualitative/quantitative distinction here is not especially useful as many data collection procedures contain elements of both approaches. Perhaps the idea of a continuum is more beneficial, giving researchers the freedom to combine features as they best suit – consequently many researchers favour a semi-structured interview format.

In mathematics education, a revival of interest in the clinical interview in the 1980s coincided with an increased emphasis on learning as conceptualised by constructivism, in particular with the rediscovering of Piaget's clinical interview techniques (Hunting, 1997). Essentially the impetus was to provide a more informed view of students' mathematical understanding and development. Heirdsfield (2002b) lists a number of Australian projects that essentially focus on a diagnostic interview. These include Count Me In Too (CMIT, Bobis & Gould, 1999) and the Early Numeracy Research Project (ENRP, Clarke, Rowley, Gervasoni, Horne, McDonough, & Cheeseman 2001).

Structured, task-based interviews are a method of qualitative inquiry that has taken on distinctive characteristics suited for research in mathematics education (Goldin, 2000). Goldin's descriptor – *task-based* – is important in terms of reinforcing that the participants' interactions are not merely with the interviewer but also with the task environment (p. 519). During interviews mathematical dialogue between interviewers and the participants is obviously central. The process of observing and engaging with participants, however, is perhaps what brings the interviewer a little closer to what Kvale (1996) refers to as, "a construction site of knowledge" (p. 2). Through these interactions the interviewer endeavours to make inferences about the phenomenon observed; in the context of the current study this is the mathematical thinking and learning that participants engage in during mental computation activities. These inferences contribute to a shared understanding of the field of mathematics education. This type of interview is particularly suited to the study of students' mental computation performance. Without an interview, investigating mental computation is reduced to a form of answer-only mental arithmetic questions.

3.5 The Study

The fourth section of this chapter provides a detailed description of how the methods of inquiry were developed and implemented at each phase of the study in terms of collecting and analysing the data. This includes details of the schools and the participants. Associated appendices are at Appendix A.

3.5.1 Participants: Sample and selection

Nonprobability sampling (Creswell, 2005) was employed in the study as the original sample of teacher participants was chosen from one Australian state to reflect the target population – teachers of middle years students. Teachers selected for interview were chosen for their experience and involvement in the teaching and learning of mental computation as the central phenomena (*purposeful sampling*, p. 204). The student sample was selected due to their association with a key teacher (*convenience sampling*, p. 204), although the selection of individual students for the interviews was guided by a set of criteria (*purposeful sampling*).

3.5.1.1 Schools

Eighteen state government schools were initially approached to participate in the study. The sample included six primary schools (Grades K – 6), six district schools (Grades K – 10) and six secondary schools (Grades 7 – 10). In the Tasmanian education system primary teachers teach up to Grade 6, whereas secondary teachers teach from Grade 7. Schools were selected in consultation with the Department *of* Education Tasmania, the Industry Partner supporting the study. Schools were identified in consultation with the State-wide Co-ordinator for Numeracy on the basis of having an interest in developing numeracy across the school. This interest was reflected in either a school numeracy policy or a staff commitment to the Department to participate in numeracy professional development programs. Of the schools suggested by the Department, one primary school and one district school were also participating in the project – *Assessing and Improving the Mental Computation of School-Aged Students* (see Section 1.2).

Of the eighteen schools initially approached, twelve Principals supported Phase 1 of the study (four primary, four district, and four secondary) with teachers from ten of these schools responding (three primary, four district, and three secondary). Unexpectedly, from the initial round of schools contacted, more secondary teachers responded to the questionnaire than primary teachers. Accordingly, a further six primary schools and three district schools were approached to be involved in Phase 1 during the following school term which succeeded in providing a more balanced sample of primary and secondary teachers. At the time of the study, the state was divided into six districts that provided support services to children in Tasmanian government schools: four of these districts were represented in the study. The demographic details of the schools involved in the study are presented in Table 3.1.

Table 3.1

Demographic Details of Participating Schools

School District	School ID	Number of students at Gr 5 and Gr 6	Number of students at Gr 7 and Gr 8	Educational Needs Index (ENI) %	Number of teachers (N = 34)
A	1*	82	–	32.99	1
	2	24	–	56.92	1
	3	104	–	38.77	1
	4*	–	305	39.03	6
	5	39	39	70.56	3
	6*	38	57	63.39	3
B	7*	102	–	39.80	1
	8	94	–	36.35	2
	9*	344	–	44.94	2
	10	79	120	68.56	4
	11*	81	81	49.58	3
C	12*	101	–	51.87	1
	13	159	–	38.66	1
	14	66	–	35.04	1
	15*	135	173	54.18	1
D	16	–	305	39.03	2
	17	92	90	45.07	1

Note. * Denotes those schools from which key teachers were from and whom participated in all phases of the study, not just Phase 1.

3.5.1.2 Phases 1 and 4: Teacher questionnaire and interview

In total 34 middle school teachers (Grades 5 – 8) participated in Phase 1 of the study by completing a mental computation questionnaire. The sample of teachers included 16 primary teachers (Grades 5 – 6) with more female teachers at the primary level (n = 13) than males (n = 3). At the secondary level 18 teachers (Grades 7 – 8) participated in the study with equal numbers of male and female teachers (n = 9).

The primary teachers (n = 16) either taught a single Grade 5 or Grade 6 class, or a composite Grade 5/6 class. One primary teacher taught a composite Grade 4/5 class. For the teachers at the secondary level (n = 18), many combinations of classes taught were noted. Some teachers reported taking mathematics with a single Grade 7 or Grade 8 class, others taught mathematics to several Grades 7 and/or Grade 8 classes, and some teachers were mathematics teachers across Grades 7 to 10.

Of the 34 teachers who completed the questionnaire in Phase 1, eight teachers were asked to participate further in the study as *key teachers*. This sample of teachers included four primary teachers and four secondary teachers and each teacher came from a different primary, secondary, or district school. The eight key teachers were chosen based on their responses to the questionnaire that indicated they were actively developing a culture of mental computation in their classrooms. Importantly these teachers had to be willing to share their classroom with a researcher and involve one of their classes in Phase 2 and Phase 3 of the study. For the primary teachers this involved the class taught on a daily basis. If teachers had several mathematics classes of the same grade level, as was the case with some of the secondary teachers, teachers were asked to choose a class that consisted of middle to high ability students. Unfortunately one key teacher was unable to attend an interview session; therefore the final interviews were conducted with seven key teachers.

3.5.1.3 Phase 2: Student number tests and questionnaire

Associated with the eight key teachers, eight classes of students in Grades

5 – 8 participated in the study by completing a student mental computation questionnaire and two number tests (N = 172). The sample of students comprised 83 primary students in Grades 5 or 6 (aged 10 to 12), and 89 secondary students in Grades 7 or 8 (aged 12 to 14). Details of the distribution of students across grades are presented in Table 3.2.

Table 3.2

Number and Distribution of Students in Phase 2 Across Grades 5 – 8

Classes	Primary		Secondary		Total
	Grade 5	Grade 6	Grade 7	Grade 8	
Class 1D	13	5	–	–	18
Class 2P	–	21	–	–	21
Class 3P	8	17	–	–	25
Class 4P	3	16	–	–	19
Class 5D	–	–	–	20	20
Class 6S	–	–	–	24	24
Class 7S	–	–	23	–	23
Class 8D	–	–	–	22	22
Subtotal	24	59	23	66	
Total	83		89		172

Note. Classes are specified as being from district schools (D); primary schools (P); and secondary schools (S).

3.5.1.4 Phase 3: Student interviews

From the eight classes of students that participated in Phase 2 of the study, 55 students were selected to participate in an individual task-based interview: six to eight students per class as described in Table 3.3. Students were selected for interview based on achievement on the two classroom tests: a mental computation test and a comparison test involving pairs of fractions and decimals. As well, the teachers of those students selected by the researcher were asked to judge whether the students were articulate and would be willing to discuss their ideas with a researcher. There were several cases where the teacher did not recommend a student selected by the researcher.

As the study intended to investigate the successful mathematical thinking strategies used by students, particularly for part-whole mental computation, initially selecting middle to high ability students was important because these were the students most successful with the target content. Given that the classes were accessed through the key teachers, however, it was not possible to have such an exclusive sample. During the data analysis stage of the study, nine students were excluded from further in-depth analysis. These students were generally not very successful with the part-whole interview questions and solved few of the questions overall. Data collected from these students were therefore extremely limited and did not significantly contribute to the relevant research question for Phase 3: what strategies do students use to solve mental computation problems with fractions, decimals, and percents? For that reason, the final number of student interviews used for Phase 3 was 46.

Table 3.3

Number and Distribution of Students in Phase 3 Across Grades 5 – 8

Classes	Primary		Secondary	
	Grade 5	Grade 6	Grade 7	Grade 8
Class 1D	6	2	–	–
Class 2P	1	7	–	–
Class 3P	–	7	–	–
Class 4P	2	4	–	–
Class 5D	–	–	–	6
Class 6S	–	–	–	6
Class 7S	–	–	8	–
Class 8D	–	–	–	6
Subtotal	9	20	8	18
Total	29		26	

Note. Classes are specified as being from district schools (D); primary schools (P); and secondary schools (S).

3.5.2 Data collection instruments

3.5.2.1 Phase 1: Teacher questionnaire

A teacher questionnaire was developed to explore how teachers in the middle years are addressing mental computation and investigate what pedagogical practices might be needed to support mental computation at this level. The questions explicitly covered six of Shulman's knowledge domains, and are outlined in Table 3.4. The order of the questions in Table 3.4 reflects the order in which the questions are presented in the results. The order was different in the actual instrument completed by the teachers. The full questionnaire is presented in Appendix A.1.

Questions were multiple choice, multi-part Likert type items or open-ended questions. The questionnaire instrument was designed so that three of the open-ended questions were at the beginning of the questionnaire. The intention was to encourage the teachers to respond according to their own understanding of mental computation and reflect on their practices before being exposed to ideas and situations embedded in the questionnaire. It was anticipated that these questions would encourage a reflective attitude that would be sustained throughout the questionnaire (Watson, 2001).

The Likert-type questions required teachers to respond to individual statements related to an overarching main question. Teachers were asked to rate each component of the statement based on a five point Likert scale, for example, Always (1), Frequently (2), Sometimes (3), Rarely (4), and Never (5). These scales were altered depending on the nature of the items involved. It was important to offer teachers the chance to shed light on their responses by explaining the conditions that affected their replies. For this reason at the end of each question space was provided for teachers to record additional comments regarding their responses and experiences. This feature also addressed a criticism of questionnaire design, referred to earlier in the chapter, that the nature of questionnaire instruments seldom provides respondents the opportunity to clarify or elaborate on their responses (Thomas, 2003).

Table 3.4

Description and Design of the Teacher Questionnaire

General pedagogical knowledge	
Question 15	Years of teaching experience
Question 16	Current year groups
Question 17	Previous year groups
Question 18*	Mathematical expertise
Question 19	Related professional development
Educational ends, purposes, and values	
Question 1	Valuing mental computation
Knowledge of educational contexts	
Question 3	Exploring issues in relation to mental computation at the primary and secondary school levels
Curriculum knowledge	
Question 4	Whole numbers, part-whole numbers, and related activities
Question 7	Time devoted to computation
Pedagogical content knowledge	
Question 2	Mental computation activities (Part A)
Question 5	Mental computation activities (Part B)
Question 6	Classroom organisation
Question 8	Assessing mental computation
Question 9	Associated mathematical competencies
Knowledge of learners' characteristics	
Question 10**	Mental computation strategies
Question 11	Enjoyment and challenges
Question 12	Student attitudes

Note. * Question 13 was later excluded from the analysis (see Appendix A.1). Question 14, relating to the sex of the teachers does not appear in this list. **Denotes those questions that also address Shulman's *content knowledge*, which was the only knowledge domain not explicitly addressed.

The design of the questionnaire, including the layout, was influenced by the teacher questionnaire used in the Leverhulme Numeracy Research Programme (LNRP). Several of the questions developed for the section of the questionnaire – Pedagogical Content Knowledge – were sourced from the LNRP instrument: examples are presented in Table 3.5. For each question the wording was changed to reflect an emphasis on mental computation as the original instrument focused on teachers' effective numeracy practices.

Table 3.5

Use of the LNRP in Questionnaire Design

Teacher mental computation questionnaire	LNRP teacher questionnaire
Question 5. In developing mental computation, how often do you use the following activities? • strategy discussion • quick recall questions	*Question 11.* Please indicate approximately how often your mathematics teaching involves each of the following: • pencil/paper calculations • mental calculation-rapid recall of bonds
Question 6. How do you organise your class for mental computation? • whole class (daily) • small groups (ability)	*Question 1.* Please indicate approximately how often your mathematics teaching involves organising this class for all or part of a lesson: • whole class • ability/attainment groups
Question 8. In assessing mental computation, how often do you use each of the following types of assessment? • teacher-made tests (timed) • observation of students	*Question 15.* In assessing the work of the pupils in your mathematics class, how often do you use each of the following types of assessment? • teacher-made tests • non-scheme standardised tests
Question 9. To be good at mental computation, how important do you think it is for students to... • recall number bonds and tables? • use knowledge of written algorithms?	*Question 13.* To be good at mathematics in school, how important do you think it is for pupils to... • recall number bonds and tables? • recall methods (e.g. subtracting 3 digit numbers)

Note. For each question just two examples from the list following the main question have been included in this table. All questions appear in full in Appendix A1.

Before the final questionnaire was distributed, two primary and two secondary teachers not otherwise involved in the study were asked to review the questionnaire. The four teaches were experienced teachers and had been extensively involved in the mathematics education research community, a perspective that was considered important given the nature of the task. The teachers were informed about the purpose of the questionnaire and asked to comment on: content, structure, layout, clarity, and wording. These comments helped to refine the questionnaire particularly in terms of language consistency and clarity with minor changes being made regarding the content of some questions.

3.5.2.2 Phase 2: Student number tests and questionnaire

Three instruments – two number tests and a questionnaire – were developed to provide a comprehensive perspective on teachers' knowledge of learners' characteristics. The data from the three instruments contributed to profiling middle years students in relation to mental computation.

Mental computation test. The mental computation test was developed as a short version of a mental computation test developed for Grades 3 – 10 as part of the SPIRT project (refer to Section 1.2). The original test comprised 50 items for students at Grade 3/4 and 65 items for students at Grades 5/6, 7/8, and 9/10 (Callingham & McIntosh, 2001, 2002). For the primary students, 20 items were selected and for the secondary students 25 items were selected from the pool of test items. The mental computation test items used in this study are presented in Table 3.5. The items were selected using the hierarchical levels of mental computation associated with types of items (Callingham & McIntosh, 2001, 2002). The levels represent a hierarchical progression of the difficulty of items as determined from students' performances on the test of mental computation. These levels provided the framework for designing the tests used in the study.

Table 3.5

Distribution of Mental Computation Items by Level

Level	Whole number				Part-whole number		
	Add.	Sub.	Mult.	Div.	Fractions	Decimals	Percents
8					$\frac{1}{2} + \frac{1}{3}$*		30% of 80
7					$3 \div \frac{1}{2}$ $4 \times \frac{3}{4}$*	0.6 x 10 0.5 + 0.75 3 ÷ 0.5* 1.25 – 0.5*	75% of 200* 10% of 45
6			24 x 3		$1 - \frac{1}{3}$ $1\frac{1}{4} - \frac{1}{2}$	1 – 0.4	25% of 80
5	37 + 24	52 – 25		70 ÷ 5	$\frac{1}{2} + \frac{3}{4}$ $\frac{2}{7} + \frac{3}{7}$	0.25 + 0.25	50% of 24
4				21 ÷ 3			
3		17 – 8					
2	9 + 8		5 x 6				
1							

Note: * denotes items for secondary students only

The whole number items included a lower level item (Levels 1 to 4) and a higher level item (Levels 5 to 8) for each of the four operations. The lower level whole number items were selected as a basic number facts (for addition and subtraction) and numbers less than 20 (for multiplication and division). The higher level whole number items comprised double digits (for addition and subtraction) and one double digit operand (for multiplication and division). In the original tests developed by Callingham and McIntosh (2001) some items involving halves and quarters appeared at Level 5, however, part-whole number items were clustered around Level 6 and Level 7. At least one problem at Level 5 was chosen for each of fractions, decimals, and percents. The rest of the items involving part-whole numbers were chosen to include a selection of the operations. The percent items included an increase in difficulty of the percentage from 50% (at Level 5) to 30% (at Level 8).

The original tests developed by Callingham and McIntosh (2001) distinguished between short items (five second response time) and long items (fifteen second response time). For the current study, however, a 10 second response time per item was allocated, as the study was not investigating the affect of response time on test performance.

Decimal and fraction comparison tests. Although mental computation was central to the student profile, an additional task using part-whole numbers was included to assist in understanding the target mathematical content. Furthermore, both mental computation and number comparisons are paired skills contributing to developing students' number sense (Sowder, 1988).

Decimal comparison tasks have been well researched to investigate how students interpret decimal notation and make comparisons about the magnitude of decimal numbers (Stacey & Steinle, 1998; Steinle & Stacey, 2003; Steinle & Stacey, 2004). Based on their research, Steinle, Stacey, and Chambers (2002) developed three tests for teachers to use in their classrooms: the tests ranged from ten to thirty items. The decimal comparison test used in this study contained twelve pairs of decimals from two of the tests. The first ten pairs were chosen from the Quick Comparison Test and the last two pairs from the

Zero Comparison Test (Steinle et al. 2002). In comparing the pairs of decimals, students were instructed to circle the largest decimal in each pair or record an = sign if the two decimals were of equal value.

The fraction comparison task was designed to replicate the decimal comparison test and involved students' circling the largest fraction in eight pairs of fractions. Item selection was influenced by Stephens and Pearn (2003). The comparison tests are presented Figure 3.3.

For each pair, circle the **largest fraction**.

a)	$^2/_4$	$^3/_4$	e)	$^3/_4$	$^3/_9$
b)	$^3/_8$	$^6/_8$	f)	$^1/_5$	$^1/_8$
c)	$^4/_8$	$^4/_{12}$	g)	$^5/_6$	$^3/_4$
d)	$^9/_{10}$	$^2/_3$	h)	$^1/_9$	$^2/_{12}$

For each pair, circle the **largest decimal** OR write = if they are the same.

a)	4.67	4.8	g)	0.8	0.0008
b)	4.2	4.67	h)	8.41237	8.41
c)	0.80	0.8	i)	3.77	3.7777
d)	0.45	0.450	j)	2.543	2.5431
e)	0.731	0.73100	k)	3.0	3
f)	0.86	1.3	l)	0.5	0.36

Figure 3.3. Comparison tests involving pairs of fractions and decimals.

Questionnaire. The questionnaire was developed to explore the characteristics of middle years students in relation to their experiences of mental computation. The content of the questionnaire is outlined in Table 3.6. Most of the questions were Likert type questions that required students to respond to individual statements that were based on a five-point scale. For two questions set responses were ordered by the students, for example, students were asked to order calculator, mental computation, and written computation 1 – 3 against descriptors most (1), some (2), and least (3). Some questions were adapted from a survey instrument designed by McIntosh, Bana, and Farrell (1995). For Question 8, students simply had to indicate their preference by choosing either *yes* or *no*. This particular question involved 12 mental computations items: the students were not required to provide answers but simply indicate whether they would consider doing the problems mentally. This question was adapted from

McIntosh et al. (1995). The student questionnaire and instructions are presented in full in Appendix A.1.

Table 3.6

Overview of the Student Mental Computation Questionnaire

Question 1	Your views*
Question 2	Use of computation in class*
Question 3	Use of computation outside school
Question 4	Whole numbers, part-whole numbers, and related activities*
Question 5	Classroom organisation*
Question 6	Using mental maths
Question 7	Mathematical competencies*
Question 8	Mental computation preference test
Question 9	Self assessment
Question 10	Attitudes*
Question 11	Activities*

Note. * Denotes those questions that are linked to the teacher questionnaire.

As well as contributing to the mental computation profile of middle years students, responses to the questionnaire provided by the students are considered in relation to the responses provided by the teachers to similar questions. In this case the perspective of the students contributes to the discussion of how teachers are addressing mental computation as framed by Shulman's teacher knowledge domains, the framework for the study.

3.5.2.3 Phase 3: Student interviews

The student interviews were a third component in constructing a mental computation profile of students along with the number tests and the questionnaire. This constituted an important mathematical content focus in understanding middle years students as learners (Shulman, 1987). An individual task-based interview (Goldin, 2000) was developed to explore the strategies students used to solve fraction, decimal, and percent problems mentally. A second part of the interview schedule is not considered within the context of this study, details of questions asked are included in Appendix A.1.

At the beginning of each interview students were presented with three multi-digit whole number mental computation problems: 24 x 3, 54 + 38, and 52 –25. The purpose of the whole number questions was for students to become familiar with the interview protocol and what was expected of them in working through calculations in the interview setting, as well as to establish a rapport with the researcher. It was anticipated that students would have more experience working mentally with problems involving whole numbers, thus questions of this type seemed an appropriate way to start the interview session. After the introductory whole number problems, the interview continued with students solving mental computation problems for fractions, decimals, and percents. The full set of interview questions are detailed in Appendix A.1.

The core set of mental computation problems was based on the classroom mental computation test that students completed as part of Phase 2 of the study. These problems were chosen directly from the mental computation tests developed by Callingham and McIntosh (2001, 2002). Additional items were chosen from a second version of the test and were asked if time permitted.

The interview schedule included a series of appropriate follow up questions for students depending on their initial responses (Goldin, 2000). Three types of questioning techniques were used to explore the responses provided by the students and these are outlined with examples in Table 3.7.

Table 3.7

Questioning Techniques Used During Student Interviews

Questioning Techniques	Examples
Nondirective	"What did you do for that one?" "How did you work it out?"
Suggestive (minimal)	"What's the first thing that you tried?" "Can you tell a little bit more about what you did with the ½?"
Guided	"4 x ¾ is hard. Can you think about 2 x ¾ instead?" "Now how does that help you work out 4 x ¾?"

3.5.2.4 Phase 4: Teacher interviews

The interview sessions with the key teachers were semi-structured in that each of the teachers was asked the same basic questions: these questions are presented in Appendix A.1. They included three components designed to generate discussion. First, teachers were asked to respond to a set of general questions as generated from the combined set of earlier teacher questionnaire responses (Phase 1). In particular the questions were designed to address Shulman's teacher knowledge domains of general pedagogical knowledge, knowledge of educational contexts, and knowledge of educational ends, purposes and values, in relation to fractions, decimals, and percents. Second, for some individual teachers there were a few clarification questions regarding their own responses to the initial questionnaire. The interviews also afforded the opportunity for teachers to express opinions or raise related issues that the questionnaire did not address directly. Third, teachers were asked to comment on some examples of student work collected during the student interviews. Given that the interviews post-dated the student interviews it was considered important to include an element of feedback for the teachers involved in the study.

3.5.3 Procedures

The study received ethical approval from the *Southern Tasmania Social Sciences Human Research Ethics Committee* at the University of Tasmania in 2003. The committee abides by the guidelines outlined in the *National Statement on Ethical Conduct in Research Involving Humans* (National Health and Medical Research Council, 1999). The study also had permission and approval of the D*o*ET, and satisfied department criteria for *Conducting Research in Tasmanian Government Schools*. For all appendices relating to this section, see Appendix A.2.

All student data were collected over the three school terms that make up a full academic calendar year in Tasmania. Table 3.8 outlines in full the data collection timeline for the study, including ethical considerations. For the teachers the questionnaire data were collected during the first and second terms of Year 2. The final interview sessions, however, with the key teachers were conducted during the second school term of Year 3. These final teacher sessions were scheduled at a later time so that the interview questions could be developed as a direct result of the questionnaire analysis, and also to allow time for the researcher to analyse aspects of the student data to provide feedback to the key teachers.

Table 3.8

Timeline for Ethical Requirements and Data Collection

	Year 1	Year 2			Year 3		
	Term 3	Term 1	Term 2	Term 3	Term 1	Term 2	Term 3
Submit ethics – University and D*o*ET	➽						
Ethics approval granted		➽					
Identification of project schools		➽					
Communicate with principal and teachers		➽					
Administer voluntary teacher questionnaire		➽	➽				
Select and contact teachers for further project involvement			➽				
Notify parents of class involvement			➽				
Administer classroom student questionnaire			➽	➽			
Select students for interview and obtain parental consent			➽	➽			
Conduct student interviews			➽	➽			
Conduct follow up teacher sessions						➽	
Notify University and D*o*ET that data collection phase completed							➽

3.5.3.1 Phase 1: Teacher questionnaire

School Principals were informed about the study with a letter of invitation and accompanying information sheet. These letters were followed by a telephone call from the researcher. Those Principals who agreed to support the study gave permission for the questionnaire to be distributed to the teachers in their schools who were currently teaching mathematics at the middle school level (Grades 5 – 8). This included Grade 5/6 classroom teachers in the primary schools and all teachers who taught mathematics at Grade 7 and Grade 8 in district or high schools. Teachers were given a three-week period to complete the questionnaire independently and then return it by post to the researcher. To increase and balance the teacher sample size for Phase 1, two rounds of the questionnaire took place, as noted in Section 3.5.1.1.

Ethical considerations. Teacher questionnaires were accompanied by a detailed information sheet, which included an overview of the purpose and aims of the study, appropriate contacts (for example, where to direct concerns or complaints), and statements on the treatment of confidentiality and withdrawal of participants. This information also stated that a small number of teachers would be asked to involve their students and further participate in a semi-structured interview and feedback session. The cover page of the questionnaire included a consent form containing a statement of informed consent to be completed by the teachers and returned with the questionnaire. All teachers were sent a letter of appreciation to convey the gratitude of the researcher for their time, effort, and support of research activities in the field of mathematics education.

3.5.3.2 Phase 2: Student number tests and questionnaire

The student instruments were administered by the researcher during a mathematics session as nominated by the primary teacher or as scheduled in the secondary timetables. The questionnaire and the tests were independent tasks; however it was made clear to the students that they could ask questions at any time. The secondary students took approximately 20 minutes to complete the questionnaire, including the written comparison tests. Generally, the primary students took a little longer due to the reading demands. The mental computation test was read aloud to the class by the researcher. At no time did the students see the written form of the mental computation questions.

Ethical considerations. Parents/Guardians received a detailed information sheet and a 'Withdrawal of Participation' form. This process of notifying Parents/Guardians was administered by the individual schools as requested by the Research Ethics Committee at the University. Full parental consent was not an ethical requirement for the class data collection but parents were given a period of time to withdraw their children if they did not wish them to participate in the classroom activities. Students were also asked to sign a consent form at the time the questionnaire was administered: this was included on the questionnaire cover page.

3.5.3.3 Phase 3: Student interviews

The students selected for an interview were interviewed individually in a separate room in their respective schools. The interview sessions were approximately 30 to 40 minutes for both the primary and the secondary school students. During the interviews the students were videotaped with full parental permission. Participation in these interviews was voluntary and the students were told they could conclude the session any time they desired. No students stopped or asked to leave early. The students were not provided with any other computational tools, such as a calculator, or permitted to use pencil and paper, however students were not stopped from using their fingers. At the beginning of the interviews students were told they could work through the problems out loud or work out their answer and then discuss the strategies used with the researcher. There were no constraints on the students in terms of appropriate language, for example, the students were free to use any decimal, fraction, or percent representation regardless of the nature of the problem statement.

Ethical considerations. For students to participate in the interviews full consent was an ethical requirement and this involved Parents/Guardians returning a signed consent form to the school giving permission for their child to be interviewed. At the time of interviewing, students were also asked to sign an interview consent form and were given access to an information sheet. As an expression of gratitude, students who participated in an interview were later given a Certificate of Appreciation for their involvement.

3.5.3.4 Phase 4: Teacher interviews

The key teachers were invited to take part in a final follow up session with the researcher. These sessions were organised individually at a time and place nominated by each teacher. The sessions were either recorded or transcribed.

Ethical considerations. At the beginning of the interview session, key teachers were asked to sign a consent form that allowed the researcher to use any of the information collected, provided that the teachers or their schools could not be identified. Examples of the students' work were shown to the teachers. This included examples from students in their own class and from other classes, although at no time was the identity of any of the students disclosed.

3.5.4 Data analysis and presentation

The data sets for each of the four phases of the study are analysed and presented across four results chapters. The links between the data sets are considered in the final discussion (Chapter 8).

3.5.4.1 Phase 1: Teacher questionnaire

Phase 1 of the study generated questionnaire data for 34 middle years teachers. The Likert-type items on the teacher questionnaire were analysed descriptively using frequencies of responses. Differences between the two groups of teachers, primary and secondary, and also level of participation in professional development are described where appropriate. The open-ended questions included in the teacher questionnaire were analysed using a clustering procedure (Miles & Huberman, 1994) and organised using Shulman's teacher knowledge domains. This is an iterative qualitative clustering process used to identify categories of responses and emergent themes.

3.5.4.2 Phase 2: Student number tests and questionnaire

Phase 2 of the study generated three data sets from the 172 middle years students. The analyses associated with each data set are described in this section.

Mental computation test. In the first place, students' responses for each item on the mental computation test were scored as correct or incorrect (including items not attempted). Data were organised by total score. Students were then grouped according to the criteria outlined in Table 3.9, which includes links to the levels of mental computation performance described by Callingham and McIntosh (2001). Given the small number of primary and secondary students answering 1 – 4 items correctly, these students were combined with the students answering 5 – 11 items correctly. The same process applied to the secondary students with the four students answering 23 – 25 items correctly combined with students answering 17 – 22 items. Students were assigned to one of three groups: Group H(igh) aligned with Level 7, Group M(iddle) with Level 6, and Group L(ow) with Level 5 as described by Callingham and McIntosh (2002). The spread of student performance on the mental computation test is discussed in Section 5.2.

Table 3.9

Mental Computation Test Scoring Criteria

School level	Number of items correct	Level of mental computation performance	Number of students	Group
Primary	1-4	4	3	Low
	5-11	5	34	
	12-16	6	28	Middle
	17-20	7	18	High
Secondary	1-4	4	5	Low
	5-11	5	29	
	12-16	6	28	Middle
	17-22	7	23	High
	23-25	8	4	

Decimal and fraction comparison tests. Performance on the fraction comparison and decimal comparison tests was considered across the three groups of students – Group H, Group M, and Group L – as determined by mental computation performance. For each of the three groups, students with similar numbers of correct responses were clustered to enable the researcher to look for patterns of responses across the comparison items.

Questionnaire. Responses to the Likert-type items on the student questionnaire were analysed descriptively using frequencies, as was used with the teacher questionnaire data. Like the comparison tasks, performance is considered across the three groups of students (Group H, Group M, and Group L), as determined by mental computation performance. As the student sample size was larger than for the teachers, comparisons of means for responses across the Likert indicators using *one-way ANOVA* were conducted between the three student groups. Effects are considered significant at $p < 0.05$.

3.5.4.3 Phase 3: Student interviews

Phase 3 of the study generated interview data for 55 middle years students. The first task undertaken in analysing the student interviews involved transcribing the digital videotapes of the interview sessions. This was a lengthy process due to the number of hours recorded – approximately 30 hours in total. As well it was intended that the hard copy transcripts would reflect the video data as closely as possible. This attention to detail included recording details of the sessions such as pauses, additional questions, hand movements and if necessary notes on the body language of the students. Examples of the students working during the interviews using direct quotes are presented whenever appropriate. At times this includes hesitations, pauses (although "umms" have been removed), interviewer questions and incorrect use of language (although interpretations are provided as considered necessary). The interview questions are written in numerical form, however, student quotes are written in full to assist in interpreting the numerical language used.

When the transcription process was complete, the first analysis was undertaken on the interview data from the mental computation items for fractions, decimals, and percents. Like responses were grouped according to three types of responses generated during the interviews: *successful responses* (correct answer followed by a discernable strategy), *guided responses* (success achieved with interviewer intervention), and *unsuccessful responses* (demonstration of misunderstandings through error). The second analysis involved classifying responses as procedural or conceptual. In line with Caney and Watson (2003), *working procedurally* involved responses that appeared to be learned by rote and have no accompanying explanation that displays conceptual understanding of the processes taking place. *Working conceptually* involved responses in which students do appear to connect their knowledge of part-whole quantities and operations to solve problems mentally. These two classifications essentially relate to the manner in which the strategies were employed, not necessarily a strategy as such. Descriptive analyses for both interview analyses are presented in Sections 6.2 to 6.4.

3.5.4.4 Phase 4: Teacher interviews

Phase 4 of the study generated interview data for seven key teachers. Transcripts were prepared for each of the interview sessions. Like the questionnaire data (Phase 1) each interview question was aligned with one of Shulman's teacher knowledge domains. Five were addressed in this phase: knowledge of educational ends, purposes, and values; knowledge of contexts; curriculum knowledge; pedagogical content knowledge; and knowledge of learners. Across each of the questions, the teachers' responses were pooled and then coded to identify themes – generally a code word was assigned using a key word highlighted in the teacher's response. This enabled like responses to be clustered or unique responses to be isolated for discussion (Miles & Huberman, 1994). Data are organised using Shulman's teacher knowledge domains as headings and direct quotes are used where appropriate.

3.6 Limitations of the Research

The limitations of the study are described in terms of *generality* and *trustworthiness* (Schoenfeld, 2002). In the first place generality (or scope) concerns the "the set of circumstances in which the author(s) of a study claim that the findings of the study apply" (pp. 466). In this study, the sample size for the teacher questionnaire is moderately small with data collected from 34 primary and secondary teachers. This is not necessarily an uncommon phenomenon when collecting information of a voluntary nature. It does, however, have consequences in terms of the generalizability of the results and conclusions. The sample of schools was chosen to have an interest in numeracy development and in some cases specific links with mental computation programs or research. Additionally the focus on mental computation may have encouraged or attracted only those teachers with a specific interest in the topic – either personal or professional – to respond. In this sense, data may unintentionally be skewed to encompass, for example, more favourable beliefs or more frequent reports of mental computation activities. There is also the possibility that this type of bias may affect the responses of students, however, it is considered that the larger student sample size would negate this.

Schoenfeld (2002) also regards trustworthiness as an attribute to judge (mathematics) research. He suggests researchers consider trustworthiness in terms of "How well substantiated is the claimed generality of the study? How solid are the warrants for the claims? Do they truly apply in the circumstance in which the author(s) assert that the results hold?" (p. 467). In this study the data set is largely composed of data collected via self report techniques. Although this is a very common form of data collection across many fields in the social sciences, it is the responsibility of the researcher to acknowledge the shortcomings of such techniques. With the teachers it can be argued, for example, that a divergence exists between reported pedagogical beliefs and experiences and the reality of how teachers actually conduct their teaching activities. In this sense information is "filtered through the views of the interviewees" (Creswell, 2003, p. 187) and it is not always possible to distinguish objective and subjective perspectives.

It is also acknowledged that the occasional Likert-type statement contained either a qualifier or a double meaning. Even with the six pilot readers, these did not get picked up.

The thesis now continues to a presentation and discussion of the results of the study over four chapters. These chapters comprise data from the four phases of the study, as outlined in Figure 3.1. A brief summary section is provided at the end of each chapter with the general discussion of results in relation to current literature, beginning in Chapter 8.

Chapter 4

Results (Phase 1): Reports From Middle Years Teachers

4.1 Introduction

The review of the literature highlighted that although there has been considerable advice to teachers to support students' numeracy development by fostering mental computation, little research focuses specifically on teachers' knowledge of mental computation and how they are working to develop mental computation in the classroom. This chapter focuses exclusively on this area and uses a framework of teacher knowledge (Shulman, 1986, 1987) to address the research question: how is mental computation being addressed by teachers in middle years mathematics classrooms? This chapter reports on Phase 1 of the study, which involved a teacher questionnaire completed by 34 middle years teachers: the sample included 16 primary and 18 secondary teachers. The full details of the analyses associated with each section of the questionnaire are presented in Appendix B.

4.2 General Pedagogical Knowledge

In this study the professional backgrounds of the middle years teachers are considered as a measure of the teachers' general pedagogical knowledge. This included current and previous teaching experience, mathematical expertise, and details of professional development related to mental computation that teachers had participated in during the last five years. The information provides a setting for examining the teachers' responses across the rest of Shulman's teacher knowledge categories. Analyses are detailed in Appendix B.1.

4.2.1 Current teaching experience

The sample of middle years teachers that participated in the study was made up of relatively experienced teaching professionals. At the start of the school year in which the study was conducted, approximately two thirds (N = 34) of the teachers reported that they had been in the teaching profession for ten or more years. Of the 16 primary teachers, three quarters of the teachers had ten or more years of teaching experience. The remaining primary teachers reported a minimum of two years experience. Of the 18 secondary teachers, half reported ten or more years of teaching experience. Just less than half reported a minimum of two years experience with two secondary teachers indicating that it was their first year of teaching.

4.2.2 Previous teaching experience

The teachers were also asked to list the details of grades they had taught in past years. The majority of the primary teachers (n = 16) indicated experience with either the upper primary grades (Grades 4 – 6) or across all primary grades (Grades 1 – 6). Additionally, three teachers indicated experience across a number of secondary grades. Of the 18 secondary teachers, the majority indicated experience across all the secondary grades (Grades 7 – 10), and for two teachers this also included senior secondary grades (Grades 11 – 12). Five of the secondary teachers also indicated experience in the upper primary grades (Grades 5 – 6). There were two secondary teachers who also had experience in the early to middle primary grades (Grades 1 – 4).

4.2.3 Formal mathematical expertise

Mathematics teachers at the secondary level (n = 18) were asked to indicate if mathematics was their main area of teaching expertise. The teachers were not required to list specific courses and qualifications but some chose to include this information. Of the secondary teachers, one third had trained or reported backgrounds in science and/or mathematics. One teacher with a science background commented that mathematics was his "main area of enjoyment and preferred teaching area" (Teacher 4). Two thirds of the teachers simply stated that mathematics was not their area of expertise. One teacher in the sample of secondary teachers indicated that she was primary trained but now teaching Grade 8 as part of a middle school program. It was noted in Section 3.5.1.2 that for the teachers at the secondary level many combinations of classes were reported with few teachers taking only mathematics classes.

4.2.4 Professional development related to mental computation

Teachers were asked if they had undertaken any professional development related to mental computation in the last five years. Those teachers who answered *yes* (n = 25) were encouraged to list the details of these sessions including: by whom it was organised, by whom it was led, and the number and length of the session(s). Many of the teachers recorded more than one session. Accordingly teachers were classified as having *extensive*, *moderate*, *limited* or *no* involvement in professional development related to mental computation. The spread of primary and secondary teachers across the four classifications is shown in Table 4.1.

Table 4.1
Participation of Teachers in Professional Development Related to Mental Computation

Level of participation	Primary Teachers	Secondary Teachers
Extensive	5	3
Moderate	8	2
Limited	2	5
None	1	8
Total number of teachers	16	18

In terms of the collective professional development that this sample of teachers had undertaken, approximately half had participated in multiple individual sessions or had participated in more than one extended mental computation numeracy program or research project, which reflects the interest and emphasis on mental computation within the educational community in Tasmania. Accordingly, eight teachers were classified with extensive involvement in professional development, with participation in the extended programs generally voluntary. Teachers classified as being moderately involved in professional development involving mental computation ($n = 10$) had participated in either a larger extended program or research project that consisted of a number of related sessions or had been involved in several single sessions. Those teachers with limited involvement in professional development ($n = 7$) had participated in either single District numeracy sessions organised by the D*o*ET or single sessions organised by their own schools and led by a colleague. Two examples of the latter provided by the teachers involved a session organised by the school mathematics committee and a session organised as part of a school mathematics investigation day. Nine teachers had not participated in any such sessions, the majority of whom were secondary teachers.

Overall, primary teachers had participated in more professional development sessions or programs related to mental computation than the secondary teachers, with the secondary teachers classed mostly as having limited professional development or none at all. The teachers' level of professional development and the primary/secondary distinction are two variables that will be used to examine the remaining sections of the questionnaire (Sections 4.3 – 4.7).

4.3 Knowledge of Educational Ends, Purposes, and Values

The teachers were asked to record up to three values they associated with mental computation, with the question addressing the educational ends, purposes, and values aspect of Shulman's teacher knowledge framework. Four main values emerged from the responses of teachers: a mathematical understanding value, the value of real life applicability, an affective value, and a teaching value. These are detailed, with examples, in Table 4.2. The 34 teachers generated 91 responses in total, although two responses were classed as *undefined,* as they did not sufficiently address the question. Analyses are detailed in Appendix B.2.

Mental computation was valued primarily by the teachers for its contribution to developing mathematical understanding. Encompassing several sub-categories, the greatest numbers of comments, approximately half of the responses, were made in relation to the contribution of mental computation to mathematical understanding. In unpacking mathematical understanding the teachers placed importance on mathematical thinking, problem solving, sense making, and improving efficiency (including an emphasis on speed and being able to recall number facts).

Table 4.2

Values Teachers Associate with Mental Computation

Values	Example response	Number of responses
Mathematical Understanding		
Encouraging mathematical thinking	"They encourage children to think through simple strategies."	9 (10.1%)
Avenue for developing problem solving skills	"Mental computation can help to identify strategies they can use in problem solving."	6 (6.7%)
Sense-making	"Can demonstrate and/or reinforce understanding of processes [&] connections."	12 (13.5%)
Improving efficiency	"Allows quicker solution of problems – answers at fingertips."	13 (14.6%)
Real life applicability	"Mental computation skills make up the bulk of maths used on a daily basis in the real world."	23 (25.8%)
Affective		
Avenue for developing confidence and enjoyment	"Feeling confident to quickly estimate and then accurately compute is powerful."	9 (10.1%)
Independence	"Independence from physical aids, e.g. calculator, pen & paper."	9 (10.1%)
Teaching	"They are usually short activities or practices not requiring too much formal writing."	8 (9.0%)
	Total number of responses	89

The second value that emerged concerned the notion that mental computation was valuable in terms of its applicability as a real life skill. Teachers put emphasis on mental computation skills being "fundamental life skills," "life long skills," and being useful on a "daily basis."

The teachers also positioned mental computation in terms of having an affective value that encompassed two sub-categories of responses. The first involved ideas of developing the confidence of students and also the students' enjoyment of mental computation. Several responses referred to accessibility and the benefits for "each child" and "all students." The second idea concerned the notion of independence, including not only independence from aids, for example, "electronic devices," but also independence in terms of "being able to understand and solve problems independently" (Teacher 13) One teacher reported mental computation as being valued in terms of its empowering qualities for students (Teacher 1).

The fourth value identified by the teachers concerned a general teaching value. Responses were grouped together primarily in that the link between mental computation and an associated teaching domain were identified. For example, teachers referred to mental computation as assisting in "assessing understanding" (Teacher 31), and "as the starting point of maths work" leading into written maths (Teacher 14). Several teachers referred specifically to the structure of the activities as being short and that it was often "easier and quicker to get an answer to a *why* (or *how*) question, than to ask for it in writing" (Teacher 8).

Space was provided on the questionnaire for the teachers to provide three responses. Some teachers provided multiple responses (two or three) that were all attributed to a single value and some teachers provided multiple responses each of which represented a different value. In either case, responses were counted individually. A summary of the spread of responses across the teachers is presented in Appendix B.2. The summary indicated that 20.6% of the teachers provided three responses with each of these responses representing one of the three main values. Just over half of the teachers (58.8%) provided responses that represented two of the main values. These teachers may have provided either three responses, two of which represented one value and one response that corresponded to a different value, or two different responses that represented two values. The remaining teachers (20.6%) responded in relation to only one of the main values. In a similar manner, some teachers provided two or three responses that all corresponded to the same value, or a single response that represented one of the values.

Overall, nearly every teacher provided at least one idea that represented the value of mental computation in association with an element of mathematical understanding. For most of the primary teachers this was coupled with at least one of the other values. The pattern of responses was similar for the secondary teachers, although there were slightly more secondary teachers who only focused on the value of mathematical understanding. The level of professional development did not differentiate the teachers' knowledge of the educational purposes, and values associated with mental computation.

4.4 Knowledge of Educational Contexts

Teachers were asked the question, in what ways does the emphasis on mental computation change as students move through primary school and into secondary school? This question addressed the teachers' knowledge of educational contexts, in this case, mental computation in the middle years of schooling. Analyses are detailed in Appendix B.3.

All teachers concurred that mental computation should be emphasised throughout primary school with many of the teachers indicating *strong agreement* (70.6%). Although responses were similar when considering the role of mental computation in secondary school, fewer teachers were in *strong agreement* (55.9%) and a small number of teachers indicated that they were *uncertain* (12.5%).

Teachers were then asked an open-ended question concerning what they actually thought happened with mental computation in the transition from primary to secondary school. Overall, just over a third of the teachers (35.3%) indicated that mental computation decreased as students progressed through the school system; comments were directed at the secondary level and are further explored in Table 4.3 The comments of just under a third of the teachers (32.4%) suggested that an emphasis on mental computation was generally limited but did not specify a particular school level. Just under a quarter of the teachers did not make a comment, indicated they were unsure, or did not significantly address the question, for example, "It depends on the students/the teachers/schools" (Teacher 32). One primary teacher suggested mental computation was improving at both levels particularly for schools involved in professional development programs and projects.

Many of the teachers made suggestions as to why mental computation was generally limited or why it decreased as students progressed through the school grades. The teachers generated twenty-seven suggestions that were clustered into four emergent themes: curriculum-related, environment-related, teacher-related, and student-related. Examples are presented in Table 4.3.

Of the themes identified by teachers as influencing a decline in mental computation over the middle years, comments were weighted overall towards content-related themes and environment-related themes. Curriculum-related responses included the predominance of other methods of computation such as written computation and calculators in relation to mental computation. As well the influence of particular teaching activities was raised, for example, "The use of 'Speed maths' [where you answer as many Q's as possible in a give time limit] predominates" (Teacher 3). Environment-related factors relating to a decreased emphasis on mental computation stressed time restraints for teachers, particularly in relation to the crowded mathematics curriculum, resulting in limited time for mental computation. There were also two comments directed at parents and their expectations of teachers.

Table 4.3

Themes Identified by Teachers as Affecting a Decline in Mental Computation over the Middle Years

Theme	Examples	Number of responses
Curriculum-related		
Emphasis on other methods of computation	"An increasing reliance on aids as students progress through the schools, e.g. paper/pen, calculators, charts."	7 (25.9%)
Teaching activities	"At times it's 'taught' by mental maths 10 questions rather than discussing different strategies & using purposeful activities."	3 (11.1%)
Environment-related		
Constrained by time and large content demand	"The secondary classroom becomes content/curriculum driven and there is less time for mental computation activities."	7 (26.0%)
Parental expectations	"Parents like work in books [to] know all children are taking part."	2 (7.4%)
Teacher-related		
Teacher preference, assumptions and competence	"A large number of teachers are challenged to run with student thinking strategies."	4 (14.8%)
Student-related		2 (7.4%)
Perception of students	"Secondary school students need to be motivated and confident within peer groups to 'perform' or participate."	
Behaviour management	"I think it gets ignored or not used ... behaviour management in classrooms."	2 (7.4%)
	Total number of responses	27

Fewer responses overall were teacher- or student-related. Teacher-related responses concerned both the competence of teachers in this area and also the personal assumptions that teachers might hold, for example, "Mental computation by high school is seen as something most kids should already have! As such its importance is downgraded in years 7-10" (Teacher 3). One secondary teacher suggested that teachers might possibly perceive mental computation as an educational fad, commenting, "A new idea, approach, technique, etc, becomes the *flavour of the month* and slips by the way when teachers take on *new* ideas" (Teacher 33). Both behavioural management issues and comments on the teachers' perceptions of the students were classed as student-related. These comments included ideas about confidence and peer groups. A Grade 8 teacher noted, "A lot of secondary school students are not as keen to participate in mental computation in case they are embarrassed in front of others, or else they've developed a negative approach to number" (Teacher 22).

4.5 Curriculum Knowledge

This section addresses two aspects associated with teachers' curriculum knowledge. The first aspect of interest is the position of mental computation in relation to calculator and written computation. The second aspect considered, is the emphasis of mental computation in developing computation skills with whole and part-whole numbers as well as estimation and calculator skills. Analyses are detailed in Appendix B.4.

4.5.1 Time devoted to developing written, mental, and calculator computation in the classroom

Teachers were asked to estimate the comparative amount of time (in percent) devoted to developing written, calculator, and mental computation skills. Responses originally provided as percentages were categorised according to descriptors – most, some, least, and even. Patterns of responses are reported in Table 4.4 with the method of computation featured *most* in each of the pattern, highlighted.

Approximately half of the teachers reported spending time developing written computation skills more than for mental or calculator computation. Pattern 1 was the most common pattern of how teachers' divided time among the three methods of computation (36.4%) and was predominantly a response given by the primary teachers. An example of a typical distribution of time was written work – most (60%), calculator work – least (10%), and mental work – some

(30%). Patterns 2 and 3, where teachers also reported the *most* time being spent on written computation skills, accounted for a further 15.2% of teachers.

Table 4.4

Estimated Time Devoted to Developing Written, Calculator, and Mental Computation Skills in the Classroom

Pattern	Written	Mental	Calculator	Number of responses	
				Primary	Secondary
1	**Most**	Some	Least	10 (30.3%)	2 (6.1%)
2	**Most**	Least	Some	0 (0.0%)	2 (6.1%)
3	**Most**	Even	Even	0 (0.0%)	3 (9.1%)
4	Some	**Most**	Least	5 (15.2%)	2 (6.1%)
5	Least	**Most**	Some	1 (3.0%)	1 (3.0%)
6	Even	**Most**	Even	0 (0.0%)	1 (3.0%)
7	Some	Least	**Most**	0 (0.0%)	1 (3.0%)
8	Even	Even	Least	1 (3.0%)	1 (3.0%)
9	Even	Least	Even	1 (3.0%)	2 (6.1%)
		Total number of responses		33	

The second most common pattern overall featured mental computation (21.2%). Seven teachers divided their time to according to Pattern 4, emphasising developing skills with mental computation over written computation, with a small amount of time devoted to working with calculators, for example written work – some (40%), calculator work – least (10%), and mental work – most (50%). Calculator computation received limited attention in comparison with written and mental computation with just one secondary teacher indicating that calculator work accounted for *most* of the time spent developing computation skills (Pattern 7).

Overall, responses from secondary teachers were more varied than those reported by primary teachers with four of the patterns described (2, 3, 6, and 7), being exclusively reported by secondary teachers. The extent of the variation is captured in two examples. One secondary teacher recorded just 5% for written, 25% for calculator, and 70% for mental work (Pattern 5, Teacher 20), another indicated 70% for written, 10% for calculator, and 20% for mental work (Pattern 1, Teacher 23). Overall, the responses of primary teachers were relatively consistent, with the majority of teachers reporting either Pattern 1 or Pattern 4. When grouped by level of professional development, those teachers with limited or moderate involvement largely reported patterns that emphasised written computation. For those teachers with extensive involvement in professional development, however, the emphasis on written computation or mental computation was more even, and this was similar for those teachers with no involvement in professional development related to mental computation. The teacher who emphasised calculator computation fell into this final group.

4.5.2 Developing mental computation with whole numbers, part-whole numbers and related activities

Whole number mental computation. Overall, the majority of both primary and secondary teachers indicated that developing mental computation skills *always* or *frequently* occurred when working with basic whole number facts (42.4% and 45.5% respectively). Responses were identical for basic number facts with addition and subtraction and also for basic number facts with multiplication and division (refer to Figures 4.1 and 4.2). For both sets of operations, the four teachers who marked *sometimes* and *rarely* were from the secondary level.

The teachers were also asked to report how frequently they worked to develop mental computation skills with multi-digit whole numbers. For the operations of addition and subtraction with multi-digit numbers more than half of the teachers indicated this occurred *frequently* in class (54.5%). The most noticeable change in the teachers' responses was that fewer teachers marked *always* for multi-digit numbers (15.2%) than for basic number facts (42.4%) (refer to Figure 4.1). For the operations of multiplication and division, a greater number of teachers also indicated that developing mental computation skills *always* occurred for basic facts (42.4%) than for multi-digit numbers (6.1%). Consequently more teachers reported *sometimes* (30.3%) or *rarely* (15.2%).

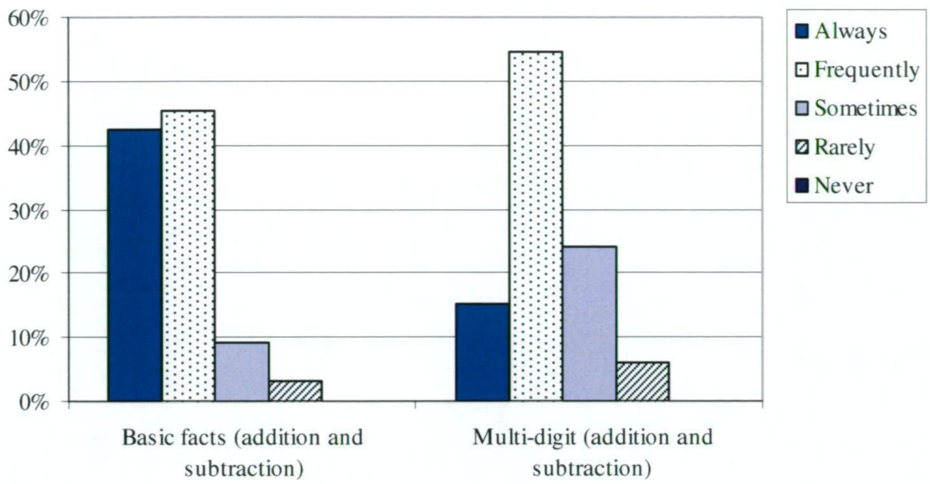

Figure 4.1. Developing whole number mental computation (addition and subtraction).

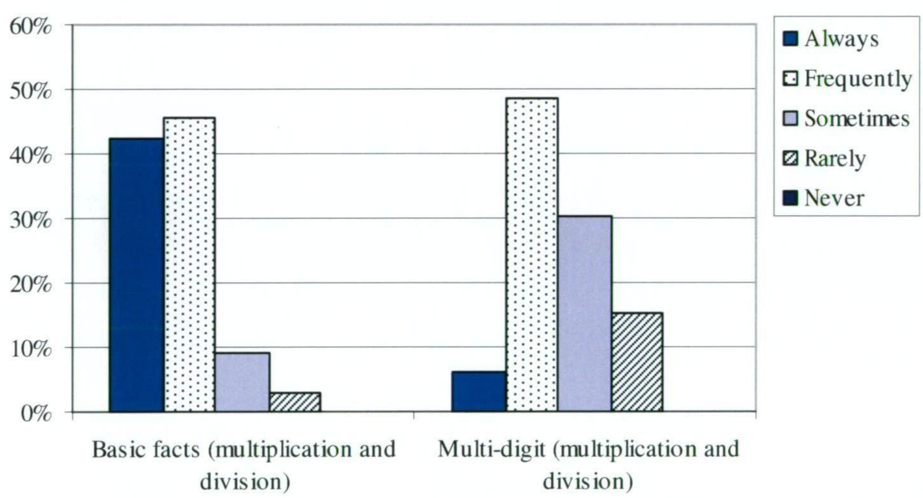

Figure 4.2. Developing whole number mental computation (multiplication and division).

Although there was a trend for teachers with moderate or extensive involvement in professional development related to mental computation to develop mental computation with the operations of addition and subtraction with whole numbers over those with limited to no involvement, this distinction was not apparent with multiplication and division with whole numbers.

Fractions, decimals, and percents. Just under half of the teachers indicated that they would *sometimes* work to develop mental computation skills with fractions (48.5%) (refer to Figure 4.3). Although many of the teachers indicated developing mental computation with fractions occurred *frequently* (27.3%) or *always* (12.1%), several primary and secondary teachers indicated that they *rarely* developed mental computation with fractions (12.1%). The pattern of response shifted slightly, however, when teachers considered working mentally with decimal numbers. Here, more teachers indicated that they *frequently* developed mental computation skills (45.5%). Although percent was a topic where most teachers appeared to develop mental computation skills *frequently* or *sometimes* (36.4% each), when compared to fractions and decimals, percent also had a slightly higher number of teachers who marked *rarely* (18.2%).

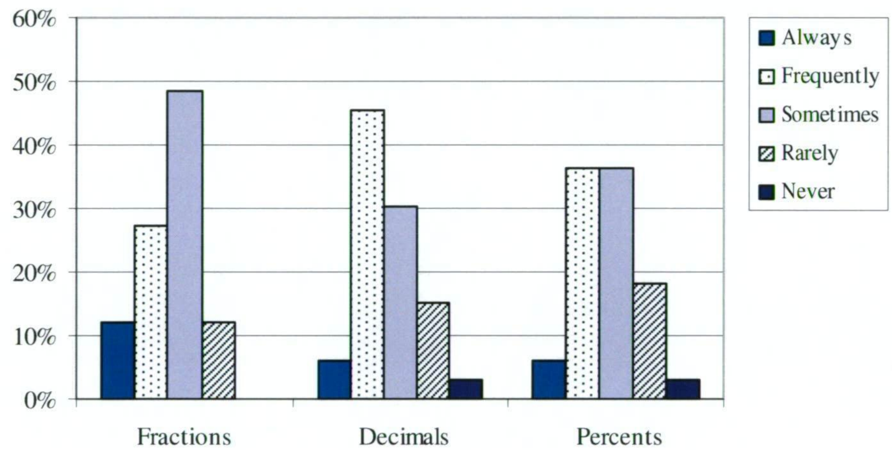

Figure 4.3. Developing mental computation with fractions, decimals, and percents.

Developing mental computation skills with fractions and percents appeared to be emphasised less frequently by teachers than with whole numbers. There was some indication that teachers developed mental skills with decimals more frequently than with fractions and percents: the more traditional link with whole number place value and the four operations might account for this. Developing fractions with mental computation, however, was reported more frequently by those teachers with greater involvement in relevant professional development, although this trend did not extend to decimals or percents.

Estimation and calculator activities. The majority of teachers indicated that developing mental computation skills *always* or *frequently* occurred with estimation activities (36.4% and 39.4% respectively), as shown in Figure 4.4. Responses for these two categories dropped, however, for calculator activities (*always* (12.1%) and *frequently* (24.2%)) and a greater number of teachers indicated mental computation skills were *sometimes* (42.4%) or *rarely* (18.2%) developed with calculator activities.

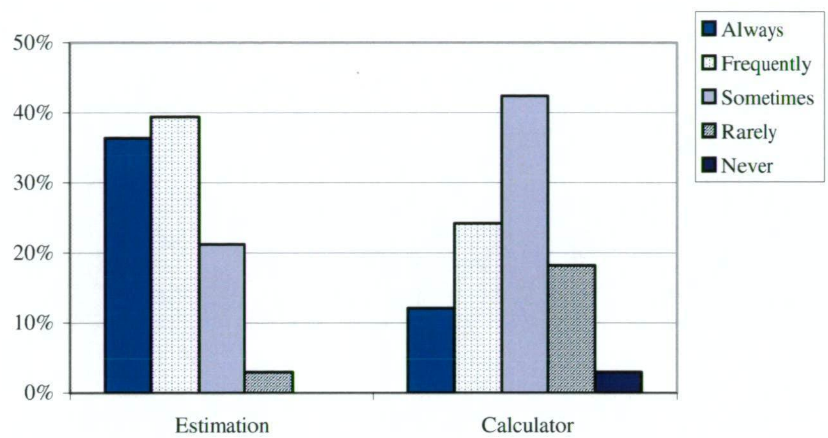

Figure 4.4. Developing mental computation with estimation and calculator activities.

4.6 Pedagogical Content Knowledge

This section of the questionnaire asked teachers to report some of their teaching practices associated with mental computation. These practices included teaching activities, classroom organisation, assessment, as well as related mathematical competency associated with mental computation. Collectively, these four aspects are considered to address teachers' pedagogical content knowledge. Like Section 4.5, the figures used through this section represent data associated with all the teachers although differences between primary and secondary teachers are described where appropriate, as well as participation in professional development. Analyses are detailed in Appendix B.5.

4.6.1 Mental computation activities

Part A: Initially, teachers were asked to describe a common mental computation activity or session they used to develop mental computation skills. Descriptions of the sessions or activities provided by teachers were classified as *traditional* or *non-traditional* and are summarised in Table 4.5. In total 52 responses were provided by 33 teachers, as many provided more than one response. Five responses, however, were classed as *undefined* with teachers providing very little description of the session, for example, "Mental maths at the start of each lesson" (Teacher 6).

Table 4.5

Traditional and Non-traditional Mental Computation Activities

Mental computation activity	Example responses	Number of responses
Traditional		12 (25.5%)
Test-based (no discussion)	"Automatic response - 1 minute per column, facts to 10, 20, 30, 50, 100, 144 on column (+, -, x, ÷)."	
Test-based (with discussion)	"We use mental computation each day in the morning as a warm up session which involves answering 20 tables, sums/money problems etc. We then discuss strategies to solve them."	4 (8.5%)
Answer-based (with class)	"Game to learn/reinforce tables, i.e. standing behind chair of a student; multiplication posed by teacher; winner moves on, loser sits in chair. A variation of above using a foam ball passed the student who gets the multiplication correct; that student then asks a question & passes the ball to the person who is first to put hand up (and get it correct)."	4 (8.5%)
Team-based	"Noughts & Crosses - I have my own grid made up with 9 questions, each team takes turn choosing a square. If the answer is correct, they get a nought or cross (on board). If incorrect, question can go the other team. Winner is the team with 3 noughts/crosses in a row."	1 (2.1%)
Non-traditional		7 (14.9%)
Strategy discussion	"Addition of 2-digit no's e.g. 25+36; Chn have time to think and then answer together at the same time; Discuss strategies, write on whiteboard; Practise a series of adding 2 digit numbers, assist chn who might be struggling; continue practicing for the week, introduce difference strategies e.g. 0-99 chart; Discuss which strategies chn find useful."	
Conceptual number work	"Today's Number is (82) - Students share everything they know about this number."	15 (31.9%)
With concrete aids	"Nominating the numbers which would normally reside in a 10 x 10 grid (blank) e.g. third row middle? It requires visualisation of filled in grid."	4 (8.5%)
	Total number of responses	47

The traditional activities were predominantly test-based activities involving versions of individual automatic response. Four teachers, however, reported some degree of discussion as follow up of the test-based activities. Two other types of activities were classed as traditional. The first involved an answer-focused activity that, although similar to the testing-based activities, is a more "public" activity involving a whole class. The second, of which there was only one example, described a team-based activity. A feature of all the traditional activities was that essentially they appeared to involve little or no discussion.

Descriptions of activities that were classed as non-traditional, however, involved mainly activities based around discussion. This predominantly involved activities involving conceptual number work that contributes to foundation number work as a basis for developing strategies. As well, seven teachers described activities that specifically involved discussion of strategies. Four teachers also described activities that involved using concrete aids; an example of a 0 – 99 grid is given in Table 4.5. Cards, dice, and number boards were among the other examples of concrete aids provided.

Overall the responses provided by the teachers showed a relatively even distribution of traditional and non-traditional activities. There were, however, a higher percentage of more traditional types of activities reported by the secondary teachers, with only three reports of conceptual number work and no reports of strategy discussion. Across the four levels of professional development participation (see Table 4.6), those teachers with none or limited experience provided more traditional responses overall than did those teachers with moderate or extensive participation.

Table 4.6

Distribution of Traditional, Non-traditional, and General Mental Computation Activities by Teachers' Level of Participation in Professional Development

	Level of participation in professional development			
	None	Limited	Moderate	Extensive
No response	0	1	0	0
General	4	0	1	0
Traditional	5	6	5	6
Non-traditional	1	2	11	12

Part B: Teachers were also provided with a list of specific teaching activities later in the questionnaire, and asked to report the frequency with which they used the activities to develop mental computation skills. The activities are listed in Figure 4.5. Overall, games and strategy discussion were the activities reported in conjunction with mental computation most *frequently* (47.1% and 50.0% respectively). More primary teachers reported using games *frequently* than did secondary teachers. Similarly, the primary teachers reported discussing strategies slightly more frequently than secondary teachers. In looking at the results across the teachers' level of professional development, there was a trend for teachers with a greater level of involvement to use discussion of strategies more frequently than those with a lower level of involvement in professional development. In using games, however, this pattern was not apparent.

The activity for which the reports of teachers were the least consistent overall was for 20 quick recall questions. Responses were spread relatively evenly over the indicators *always* to *rarely*. Only one primary teacher indicated this was an activity *never* used for developing mental computation in class. These results suggest that the teachers held more contrasting views on the value of recall questions as an appropriate activity for developing mental computation. This item, however, elicited the highest number of teachers who marked this as something they *always* did (26.5%). There were no differences between the responses of primary and secondary teachers. Activities involving memorizing

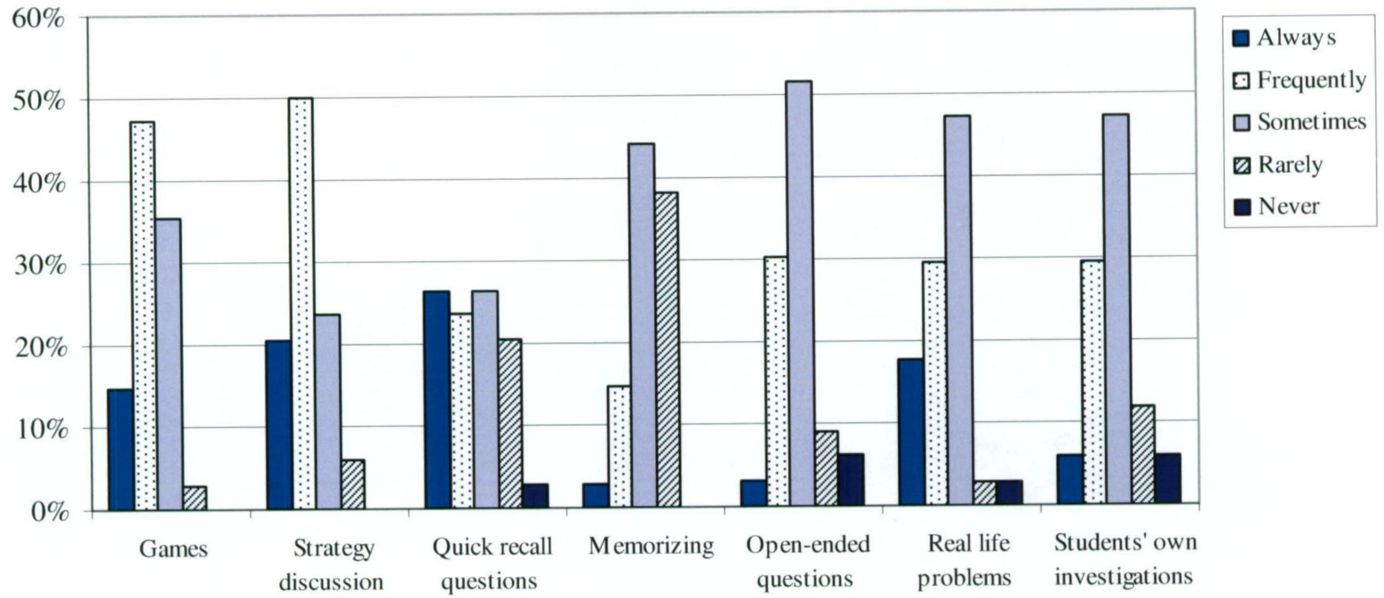

Figure 4.5. Activities teachers use to develop mental computation skills.

were not frequently reported by the teachers, with the majority marking *sometimes* (44.1%) and *rarely* (38.2%). The pattern of responses for primary teachers and secondary teachers was similar although of the 14.7% of teachers who indicated they used activities involving memorizing *frequently*, most were secondary.

Teachers reported using open-ended questions with more than one answer, real life problems and the students' own investigations on occasion, with *sometimes* being the modal response across all three activities. It is conceivable that without the provision of examples the teachers interpreted these items differently: of the list provided to teachers these activities are perhaps less well defined than some of the others. In using open-ended questions most of the *sometimes* responses were from secondary teachers. Primary teachers, however, indicated that open-ended questions were an activity they *frequently* used in class. Smaller numbers of teachers marked *rarely* and *never* and these responses were mainly reported by teachers at the secondary level. Primary and secondary teachers did not differ in their responses to students' own investigations or in using real life problems. Possibly there is an association between this question and the previous question involving students' own investigation, as responses for *frequently* and *sometimes* were virtually identical.

There were some inconsistencies in the responses of teachers for Part A and Part B. In Part B, where teachers responded to a list of activities, many reported employing games and engaging in the discussion of strategies. These were the most frequently reported types of mental computation activities conducted in the classroom. Interestingly, however, in Part A where teachers provided their own descriptions of activities, a number of activity descriptions essentially involved a game but were classed as *traditional* given the lack of apparent discussion. Additionally in Part A, many of these "typical activities" also involved descriptions of automatic response even though overall teachers indicated discussions of strategies was a more common practice than automatic response.

4.6.2 Assessment of mental computation

The assessment practices reported by teachers for mental computation are depicted in Figure 4.6 based on how frequently teachers reported using the particular methods listed. The items pertaining to the use of teacher-made tests – both timed and un-timed – as forms of mental computation assessment generated the most wide-ranging variations in responses amongst teachers. Responses were spread relatively evenly over the indicators *always* to *never* than for the other items. Generally, the primary and secondary teachers responded similarly; the only small difference was that secondary teachers tended to indicate using testing more frequently than did primary teachers. The level of teachers' professional development did not affect responses. Just under half of the teachers indicated that timed mental computation testing was *rarely* or *never* used (43.8% combined). Interestingly, from the list of mental computation activities presented to teachers (Section 4.6.1), the pattern of teachers' responses to quick recall questions was also distributed evenly over the five indicators. More teachers, however, marked *never* for the use testing than for quick recall questions. Possibly some teachers consider the two activities to be quite different.

The limited use of tests from commercial schemes suggests that when teachers do use testing as a form of assessment, they develop their own materials as the majority of teachers indicated they *rarely* or *never* (34.4% each) used tests from commercial schemes.

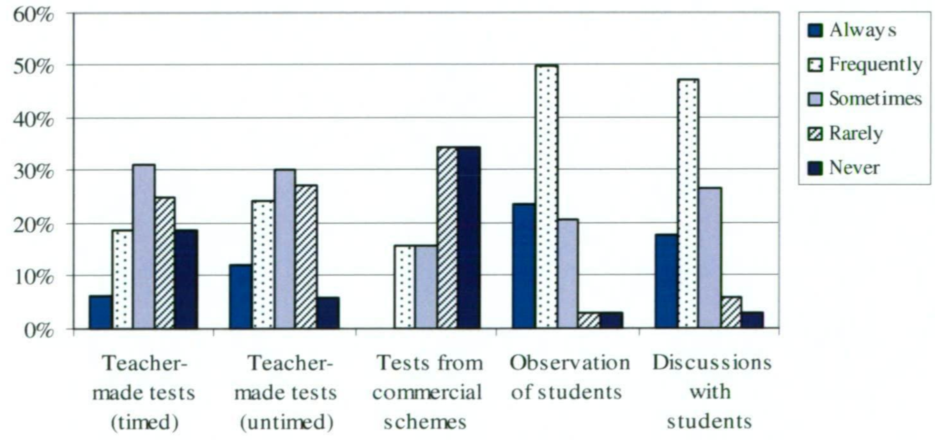

Figure 4.6. Mental computation assessment techniques reported by teachers.

Observation of students and discussions with students were rated similarly overall, with approximately half of the teachers indicating these were frequent forms of mental computation assessment (50% and 47.1% respectively). A consensus on using observation and discussion as methods of assessment was stronger for the primary teachers, with secondary teachers overall reporting these methods less frequently. Observation of students was also reported more frequently for those teachers with a higher level of involvement in professional development, although there was no difference regarding using discussions with students. "A lot of listening!" was an additional comment made be a Grade 5 teacher (Teacher 4), and "Explain how you got that Josh" was an explicit example provided by a Grade 6/7 teacher (Teacher 7).

4.6.3 Classroom organisation

Teachers were asked to report how they grouped students for mental computation: Figure 4.7 presents the results of the relative frequencies of the responses by teachers. Overall responses showed that mental computation was largely structured around working with the whole class and was integrated into classroom work as a daily activity, with just over half of the teachers indicating that mental computation was *frequently* organised as a daily whole class activity (55.9%). The responses of the secondary teachers largely contributed to the *sometimes* to *never* categories. Teachers seemed to prefer working with the whole class in daily sessions, as whole class weekly sessions were reported less frequently.

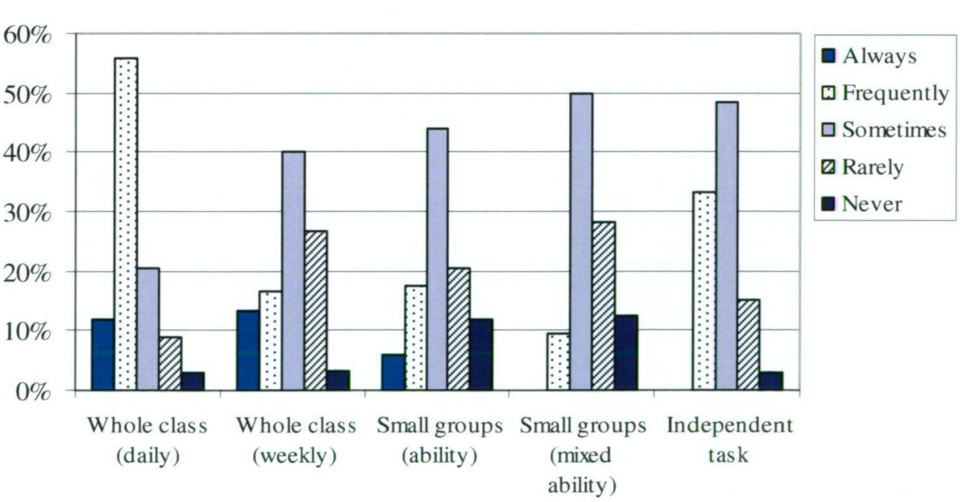

Figure 4.7. Classroom organisation for mental computation reported by teachers.

Teachers also rated their use of homogenous small groups when teaching mental computation. The secondary teachers provided responses that encompassed all five indicators with six teachers indicating it was a form of classroom organisation that they *rarely* used and four teachers indicating it was *never* used. The primary teachers, however, more consistently marked *sometimes*, with four teachers indicating small ability groups were used *frequently*. The patterns of response were not markedly different when teachers were asked about small student groups assembled by mixed ability.

Just under half of the teachers reported that in their class, mental computation was *sometimes* structured as an independent task (48.5%). More of the secondary teachers indicated that this was *frequent* than did primary teachers and consequently more of the primary teachers indicated this was *rare*. An additional comment made a secondary teacher pointed out that for mental computation there was "much opportunity for incidental learning" (Teacher 20).

4.6.4 Mental computation and associated mathematical competency

The teachers were asked to consider the association of nine mathematical competencies with the development of mental computation skills. These competencies included: recalling number bonds and tables, using knowledge of written algorithms, thinking logically, thinking creatively, being accurate and quick, having a selection of strategies, being able to estimate, providing reason to support answers, and checking answers. The teachers reported eight of the nine competencies presented as being *essential* or *important* in supporting mental computation, as shown in Figure 4.8. The modal responses dropped slightly for using knowledge of written algorithms to *somewhat important* (47.1%): the majority of primary teachers indicated this. Additionally, using knowledge of written algorithms and being accurate and quick were the two competencies which a small number of teachers did not positively support in relation to mental computation. Generally, these teachers were those with a higher level of involvement in professional development related to mental computation. In fact all but one of the teachers with no

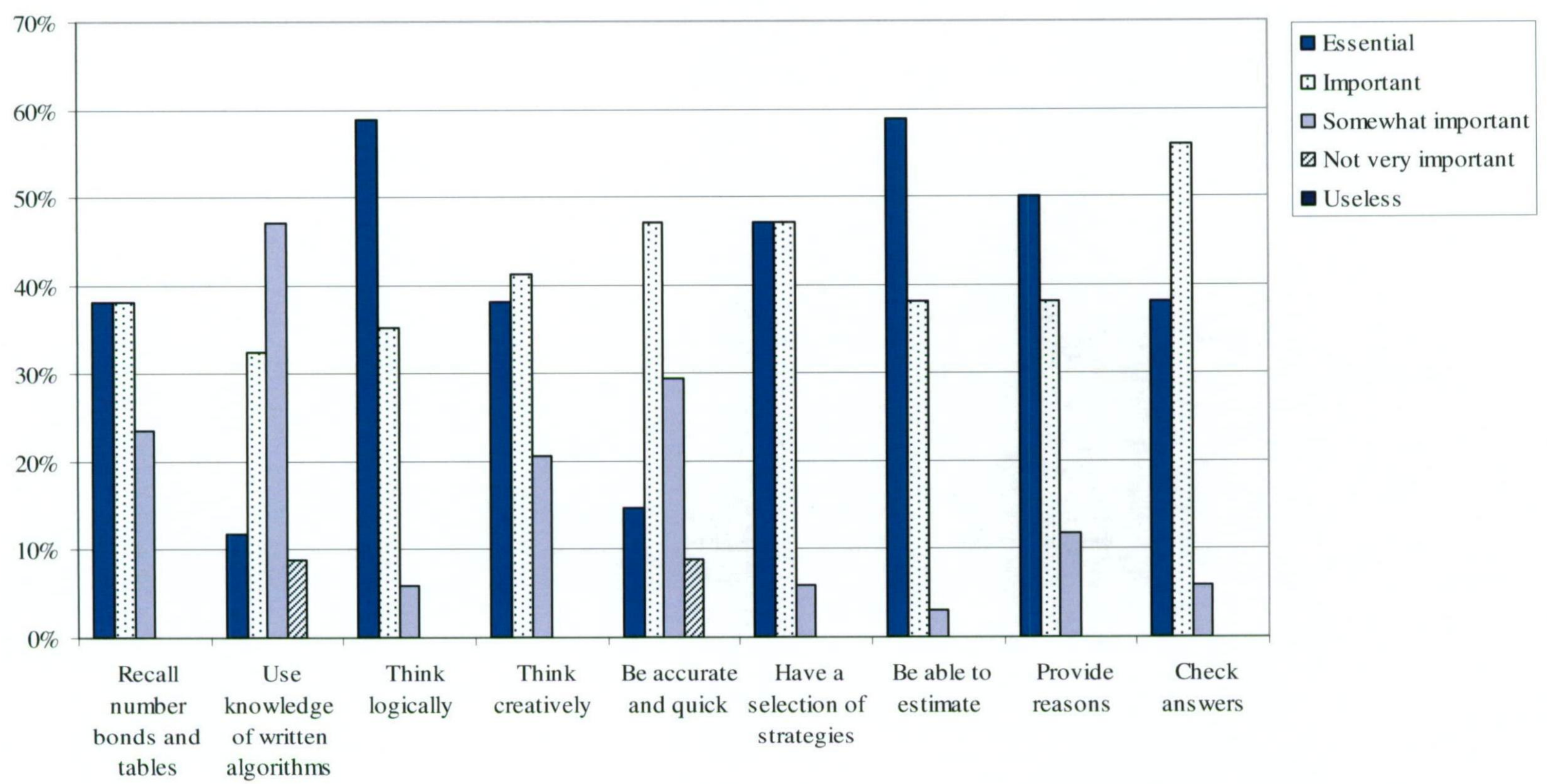

Figure 4.8. Mathematical competencies teachers associate with mental computation.

professional development involvement marked using knowledge of written algorithms as *essential* or *important.* There were no instances of the teachers marking the *useless* indicator.

4.7 Knowledge of Learners' Characteristics

Three aspects of mental computation are considered in this section that relates to Shulman's knowledge of learners' characteristics. First, the teachers provided comment on what they thought students might enjoy (Part A) or find demanding (Part B) about mental computation. Second, the teachers considered a series of attitudinal statements related to mental computation and rated how common they believed the attitudes to be amongst their students. For analysis these statements have been separated into those that present a more positive attitude to mental computation and those that are more negative. Third, the teachers were presented with an opportunity to document the mental computation strategies their students might use in solving six problems mentally. Associated analyses are presented in Appendix B.6.

4.7.1 Students' enjoyment of and challenges associated with mental computation

Part A: Teachers' perception of students' enjoyment of mental computation. Twenty-eight teachers provided responses to Part A and many teachers indicated that their classes enjoyed mental computation, providing a range of reasons as to why. Five teachers indicated that their students enjoyed mental computation by simply responding with, "yes," but offered no further explanation. Four teachers responded more indifferently, for example, "Some children thrive on it, some find it very challenging" (Teacher 7). The responses of seventeen teachers that were more detailed were assigned to categories that represented four emerging themes. These themes are described in Table 4.9. Although no primary teachers reported a lack of enjoyment, two secondary teachers gave more negative reports: "They don't like it and find it difficult, there is a lack of memorisation of common computation rotes" (Teacher 9) and "They don't enjoy having to try, i.e. working hard to have to learn, remember strategies" (Teacher 17).

Table 4.9

Reasons Teachers Attributed to Students' Enjoyment of Mental Computation

Themes	Examples	Number of responses
Enjoyment attributed to the mental computation activities	"Short, sharp not threatening activities that go down well."	9 (52.9%)
Enjoyment attributed to success	"Some of my class find it most enjoyable because they are able to calculate quickly mentally."	4 (23.5%)
Enjoyment attributed to being challenged	"Yes. The more challenging the more they like it."	3 (17.6%)
Enjoyment attributed to understanding	"Those who understand the patterns & links find it easy."	1 (5.9%)
	Total number of responses	17

Responses clustered around the notion that students enjoyed the characteristics and structure of the activities teachers used to develop mental computation were assigned to the first theme. One teacher made the following comment in relation to his students, "Enjoy when seen as unstructured – e.g. Today's number is, or What's my number? Follow Me cards" (Teacher 3). This was the most common theme that emerged from the teachers' reports – particularly the primary teachers – and represents a pedagogical emphasis.

A second theme that emerged was that of enjoyment associated with success, although these comments appeared to be directed at "more able students." Three secondary teachers proposed that their students enjoyed the challenge of working mentally; however, it is difficult to know if students liked the "challenge" of the content or the structure of the activity. It is possible that being challenged was associated with timed activities. One secondary teacher referred to the understanding behind mental computation as promoting enjoyment; the fourth theme.

Part B: Teachers' perception of challenges students might associate with mental computation. The challenging aspects of mental computation that 17 teachers associated with their students are described across five themes. The themes are presented with examples in Table 4.10. Responses assigned to the first theme included a number of different mathematical skills that when lacking, can make mental computation a demanding task, for example, visualisation and checking the reasonableness of an answer. Generally, this was a response provided by the secondary teachers.

Table 4.10

Challenges Teachers Associate with their Students and Mental Computation

Themes	Examples	Number of responses
Lack of associated skills	"Many of my students are not 'flash' so are not confident with tables and simple calculations"	5 (29.4%)
Speed	"They feel challenged if they are timed or speed is required"	4 (23.5%)
Language and discussion	"If they have trouble expressing themselves, they often find it hard to work on mental computations"	3 (17.6%)
Expanding mental computation	"Accommodating new strategies to mentally work with bigger numbers or other e.g. percents"	3 (17.6%)
Dependence	"They are too used to using pen and paper or calculator, instead of using their brain to estimate or work out the answer"	2 (11.8%)
	Total number of responses	17

Four teachers reported that being able to work quickly was a challenge for their students, which suggests that for mental computation speed is emphasised in some classrooms, although generally being quick was not rated highly on the list of competencies associated with mental computation. Limited literacy skills including language and ability to participate in discussions, were reported by three teachers as a factor that presents as challenging for some students in relation to mental computation. Three primary teachers posited that expanding mental computation strategies to, for example, work with double-digit numbers or to explore multiple ways for solving problems was challenging for some students. A final theme reported by two teachers was that mental computation was challenging due to the dependence of students on other methods of computation, for example, written computation. This dependence on other methods of computation was one of the primary factors suggested by teachers that contributed to a perceived decline in the use of mental computation activities as students moved into secondary school (as reported in Section 4.4).

4.7.2 Student attitudes towards mental computation

Positive attitudes. The relative frequencies of the teachers' responses across four positive views are reported in Figure 4.9. Overall, few of the more positive attitudes associated with mental computation received strong support or strong disagreement from the teachers. The teachers did report, however, that students would support mental computation as being useful outside of school. This was the only attitude to receive favourable support from both primary and secondary teachers, with the modal response being *frequent* (39.4%). Across the other three items, the modal response was *occasional.*

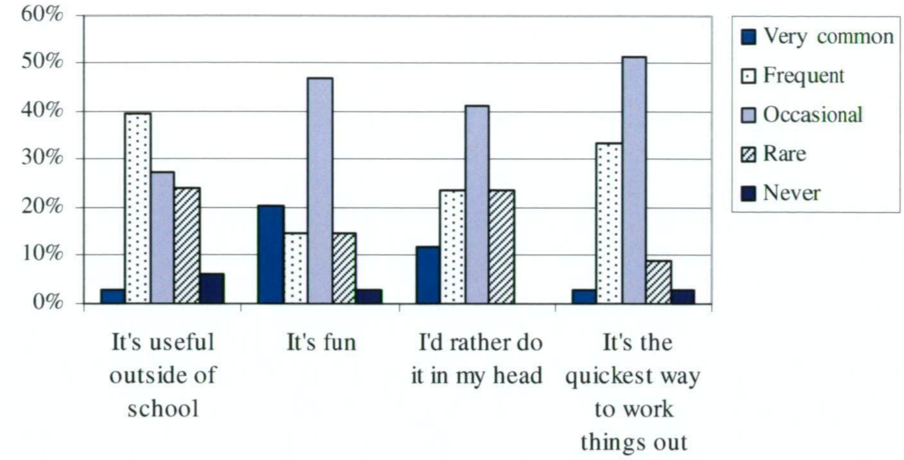

Figure 4.9. Teachers' perception of the students' positive attitudes to mental computation.

The responses to the view "It's fun" were not restricted to either the primary or secondary teachers across the *very common* to *occasional* indicators. Of the six teachers who indicated this view was *rare* or marked *never*, however, five were from the secondary level. For the view, "I'd rather do it in my head than write it down" the modal response was *occasional* (41.2%), with a greater contribution by secondary teachers. The primary teachers were generally more supportive indicating this was a *frequent* view amongst their students than the secondary teachers. There was no difference between the responses of the primary and secondary teachers to the view "It's the quickest way to work things out." It is likely that the level of mental computation being developed in the classroom would influence how teachers responded to this item. For example, teachers emphasising mental computation with basic number facts might view this more positively than teachers working with larger combinations of digits.

Two teachers contributed their own comments. A primary teacher listed an alternative view, "It's faster to write it down" (Teacher 7) but did not indicate if this was a common view amongst students. A secondary teacher commented, "We rarely talk *about* mental maths" (Teacher 34). It would appear that for this teacher her responses were based solely on observations of students and not on discussions with the students.

Negative attitudes. The frequencies of the teachers' responses to five more negative views are shown in Figure 4.10. From the teachers' perspective, students would not tend to view mental computation as something "below" their year level or describe mental computation as stressful – two of the more negative attitudes. In response to the view of mental computation being "for the younger kids," the majority of responses from the primary teachers (approximately three quarters) were *rarely* or *never*, with more of the secondary teachers indicating their students *frequently* or *occasionally* expressed this view. Possibly teachers believed that mental computation for their students was something they thought they "had done" in primary school, particularly if the work involved revisiting number facts.

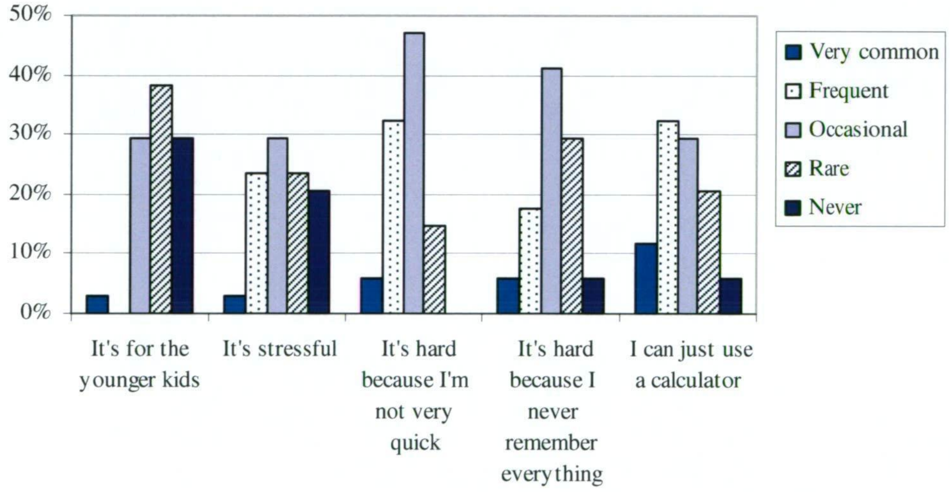

Figure 4.10. Teachers' perception of the students' negative attitudes to mental computation.

The view, "It's stressful" was included to represent a general performance anxiety that could be associated with mental computation. Responses across the indicators, *frequent* to *never*, were relatively uniform overall for teachers, however, primary and secondary teachers responded quite differently. The majority of responses from the primary teachers indicated that the view "It's stressful" was not a common view amongst their students. The exceptions were one teacher who marked *very common* and one teacher who marked *frequent*. For secondary teachers however, the modal response was *frequent*. This attitude may be related to the types of activities teachers are using to develop mental computation skills and what skills they emphasise.

There was some consensus among the teachers that their students might negatively associate speed with mental computation performance as just under a third of teachers indicated that the view "It's hard because I'm not very quick" was *frequent* amongst their students (32.4%), although the responses of primary and secondary teachers did not differ over this view. A related view was "It's hard because I never remember everything," emphasised memory. Overall, more teachers marked this as *rare* (29.4%) amongst their students than with the previous view emphasising speed. For the primary teachers in particular this was a *rare* attitude, with two teachers indicating *never*.

There was also some consensus among the teachers that their students would prefer to use a calculator instead of working mentally. The view was more commonly perceived amongst secondary teachers with all of the *very common* responses and just over half of the *frequent* responses being from secondary teachers. Although fewer teachers rated this view as *rare* or *never* overall, more primary teachers marked *rare* than any other response. It could be that access to calculators is more common in the secondary classrooms and therefore students are thought to be more likely to want to engage with them.

Overall, there was more difference between the responses of the primary and secondary teachers to the negative views than the positive ones. Generally the secondary teachers indicated that the relatively negative views were more common amongst students than did the primary teachers.

4.7.3 Mental computation strategy use

Teachers were asked to list the mental strategies they would expect from their students for six mental computation problems: a) 58 + 34, b) 52 – 25, c) 24 x 3, d) 0.5 + 0.75, e) 4 x ¾, and f) 10% of 80. This was primarily an opportunity to explore the teachers' knowledge of students as learners in terms of appreciating the strategies the students might use in solving mental computation problems. It is possible, however, for the question to be a secondary measure of the teachers' content knowledge. For each problem, the common strategies provided by the teachers are categorised. Additionally, the percentages of teachers providing more than one response per question are reported. A few examples of undefined strategies were identified; details are provided in Appendix B.6.

Whole number addition (two-digit). For the problem 58 + 34, 32 teachers generated 63 responses, with one response categorised as undefined: these are summarised in Table 4.11. Splitting both numbers according to place value was the most common response recorded by the teachers. The majority of examples showed working with the tens first followed by the units. The second most frequently reported response involved a process of levelling or compensating, whereby the 58 is made into 60 by taking 2 off the 34, and the problem becomes 60 + 32. Teachers also reported a strategy whereby students preserve one number and then sequentially add on the second number in parts, in this case using place value. Six teachers reported a strategy that mirrored a vertical written algorithm. The final strategy involved a form of bridging first the 58 to 60, adding 34, and then taking 2 was the main example provided.

Table 4.11

Summary of Strategies Associated with 58 + 34

Strategy	Description	Examples	Number of responses
Place value split	Split 58 and 34 by place value	(50 + 30) + (8 + 4)	29
Levelling	Change 58 and 34	60 + 32	11
Worked with parts of a second number	Keep 58 and split 34 by place value	58 + (30 + 4)	9
Used written algorithm	Work through a mental version of a vertical algorithm	58 + 34	6
Bridging	Keep 34 and change 58	(60 + 34) – 2	6
	Change 34 and 58	(60 + 35) – 3	1
		Total number of examples	62*

*1 response categorised as undefined

Whole number subtraction (two-digit). Strategies reported by 31 teachers for the problem 52 – 25 are reported in order of frequency in Table 4.13. Overall the teachers generated 57 responses, although four were categorised as undefined. The most frequently reported strategy for the problem 52 – 25 involved working with parts of a second number using to place value. Two types of examples were recorded. In splitting the 25 into parts, one teacher wrote this as "52 –10 = 42, – 10 = 32, – 5." The second example involved responses of teachers explaining the link with 25 being half of 50. Ten teachers provided examples that involved splitting both numbers by place value. This is an interesting choice for this particular problem in that moving through the problem in the same manner as reading a sentence, students might find working with the 2 and the 5 difficult to manage as units. Some teachers gave examples of additive strategies and three teachers noted a written vertical algorithm. Single examples of bridging and levelling were also recorded.

Table 4.12

Summary of Strategies Associated with 52 – 25

Strategy	Description	Examples	Number of responses
Worked with parts of a second number	Keep 52 and split 25 by place value	(52 – 20) – 5	6
	Split 52 and keep 25 by place value	(50 – 25) + 2	24
Place value split	Split 52 and 25 by place value	(50 – 20) – 5 + 2	10
Additive strategy	Start with 25, count up	25, 35, 45 + 7	1
		25, 25, + 2	3
		Counted up in 5s from 25, then added 2	1
Used written algorithm	Worked through a mental version of a vertical algorithm	52 –25	4
Bridging	Keep 25 and change 52	(55 – 25) – 3	1
	Keep 52 and change 25	(52 – 30) + 5	1
Used visual tool	A visual picture is described	Number line	1
Levelling	Change 52 and 25	50 – 23	1
		Total number of examples	53

*4 responses categorised as undefined

Whole number multiplication (two-digit). For the problem 24 x 3, 32 teachers generated 68 responses, although six responses were categorised as undefined: responses are summarised in Table 4.13. The most frequently reported strategy involved a distributed split, the main group of examples involving place value. A much smaller number of teachers split the 24 by a quantity not related to place value, working with 3 x 12 and 3 x 12. As well two teachers described adding 24 in succession or multiplied two 24's, adding the final 24. The second most frequently reported strategy by the teachers involved bridging from 24 to 25 (25 x 3) and taking 3 away as the final step in the calculation. Eight teachers noted a written vertical algorithm and six teachers provided examples of doubling/halving where students would change the problem from 24 x 3 to 12 x 6. Finally, five teachers detailed an additive strategy of repeated addition.

Table 4.13

Summary of Strategies Associated with 24 x 3

Strategy	Description	Examples	Number of responses
Distributed split	Keep 3 and split 24 by place value	(3 x 20) + (3 x 4)	24
	Keep 3 and split 24 by other quantity	(3 x 12) + (3 x 12)	4
	Keep 24 and split 3	(24 x 2) + 24	2
Bridging	Keep 3 and change 24	(25 x 3) – 3	13
Used written algorithm	Work through a mental version of a vertical algorithm	24 x 3	8
Doubling/halving	Change 24 and 3	12 x 6	6
Repeated addition	Keep 24 and split 3 by other quantity	24 + 24 + 24	5
		Total number of strategies	62

*4 responses categorised as undefined and 2 were incorrectly recorded

Decimal addition. Twenty-eight teachers generated 42 responses for the problem 0.5 + 0.75: these are summarised in Table 4.14. The most commonly reported strategy involved changing the representation of the decimal to the fraction equivalent with 11 teachers noting this. Some teachers also went further explaining how students would actually add the fractions together. Splitting the 0.75 into the quantities 0.5 and 0.25 was more frequently reported than the place value split, 0.7 and 0.05. Seven examples involving a written algorithm and five examples involving whole number knowledge were also recorded.

Table 4.14

Summary of Strategies Associated with 0.5 + 0.75

Strategy	Description	Examples	Number of responses
Change representation	Simple change to fraction equivalents	½ + ¾	6
	Change to ½ + ¾ and add parts	½ + ½ + ¼	5
Split by other quantity	Keep 0.5 and split 0.75 by other quantity	0.5 + 0.5 + 0.25	9
Split by place value	Keep 0.5 and split 0.75 by place value	0.5 + 0.7 + 0.05	3
Written algorithm	Work through a mental version of a vertical algorithm	$\begin{array}{r} 0.75 \\ \underline{+\,0.5} \end{array}$	7
Whole number	Describe in whole number context	"50 + 75"	3
	Used a money context	50c + 75c	2
		Total number of strategies	35

*2 responses categorised as undefined and 4 were incorrectly recorded

In addition to the 35 correct responses provided by the teachers for the problem 0.5 + 0.75, four responses were classed as *undefined.* Responses included general comments, for example, "using knowledge of tenths and whole numbers." Two errors were also recorded, for example, "$0.50 + $0.75 = 80c, = 0.80." It was not clear, however, whether the error was made inadvertently by the teachers or was intentional in terms of suggesting an example of an incorrect strategy that students might use.

Fraction multiplication. Twenty-nine teachers recorded 49 strategies for the problem 4 x ¾: these are summarised in Table 4.15. There was a wide array of strategies reported although individually they were reported by small numbers of teachers. The most common strategy reported involved using quarters in an algorithmic fashion. Repeated addition was described by six teachers and a further six teachers reported a strategy whereby students would, in the first place simplify the problem to 2 x ¾, effectively a multiplicative/distributive split. Six teachers described a visual picture that students might use solving this problem mentally, examples included circles, number lines and apple piles. Four teachers reported a form of bridging. For this problem students bridge from ¾ to a whole and then subtract as a final step in the calculation. Two teachers identified a rule that could be applied to the problem. Doubling/halving was identified as a separate strategy; however, it could be that the strategy represents a later version of the multiplicative/distributive split. The final set of strategies involved individual examples of changing the representation of the fraction ¾ to 75% or 0.75. A third possibility was included in this group of strategies whereby two teachers identified that students may interpret the operation of multiplication with the use "of."

Table 4.15

Summary of Strategies Associated with 4 x ¾

Strategy	Description	Examples	Number of responses
Algorithms with quarters	Multiply quarters	4 x 3 = 12 quarters, 12/4 = 3	11
Repeated addition	Add ¾ successively	¾ + ¾ + ¾ + ¾	6
Multiplicative/distributive split	Split the 4	Work out 2 lots of ¾, then double answer	6
Split by other quantity	Keep 4 and split ¾ by ¼	¼ x 4 = 1; 1 x 3 = 3	6
Visualisations	Use of diagram/ number line	Imagine 4 pies each with ¾	6
Bridging	Bridging to the closest whole	4 x 1= 4; 4 x ¼=1; 4-1=3	4
Rule	Learned rule	Cancel 4s = 3	2
Doubling/halving	Change 4 and ¾	2 x 1½	1
Change representation	Change to percent equivalent	¾ is 75% so 75% of 4 is 3	1
	Change to decimal equivalent	0.75 x 4	1
	Change operation to 'of'	Some know 'x' can mean 'of'	2
		Total number of strategies	46

*3 responses categorised as undefined

For 4 x ¾, three responses were classed as *undefined.* Responses included general comments, for example, “break into stages.”

Percent. Twenty-nine teachers recorded 41 strategies for the problem 10% of 80: these are summarised in Table 4.16. The most frequently reported strategy involved the use of related number knowledge, which is separated into three possibilities. Changing the representation of the 10% to its fraction equivalent of $^1/_{10}$ was reported by 12 teachers, although this strategy was more common than changing the representation to 0.1, which was reported by just one teacher. There were nine reports of a rule associated with the problem 10% of 80. Most descriptions of strategies involved, for example, removing the zero, with one strategy reflecting a version of a written algorithm.

Table 4.16

Summary of Strategies Associated with 10% of 80

Strategy	Description	Examples	Number of responses
Related number knowledge	Use of division	80 ÷ 10	10
	Use of multiplication	How many times 10 goes into 80	5
	Knowledge base of 10% in relation to 100%	10% means 10 out of 100 so answer must be <10 since 80 is <100	3
Changed representation	Simple change to fraction equivalent	$^1/_{10}$ of 80	11
	Simple change to decimal equivalent	0.1 of 80	1
Rule	Place value rule associated with zero or procedure	Take off zero = 8 or 10/100 x 80	9
		Total number of strategies	39

*2 responses categorised as undefined

Table 4.17 presents the number of strategies provided by the 34 teachers for each of the six mental computation problems. A majority of teachers were able to detail at least one or two strategies for each of the problems. Generally, there was a reduction in the number of strategies detailed for solving part-whole number problems compared to whole number problems. This is perhaps an indication that mental strategies for the fraction, decimal, and percent problems were not as widely known to the teachers. The number of teachers who did not report any strategies for the part-whole number problems is also higher than for the whole number problems.

Table 4.17

Number of Strategies Provided for Six Mental Computation Problems by Teachers

Mental computation problems	No strategies reported	1 or 2 strategies	3 or 4 strategies	5 or 6 strategies
58 + 34	5.9%	73.5%	14.7%	5.9%
52 – 25	8.8%	76.5%	17.6%	2.9%
24 x 3	5.9%	64.7%	29.4%	0.0%
0.5 + 0.75	17.6%	76.5%	5.9%	0.0%
4 x ¾	14.7%	73.5%	11.8%	0.0%
10% of 80	14.7%	82.4	2.9%	0.0%

Primary teachers provided more examples for the first two whole number problems 58 + 34 and 52 – 25, and also for the decimal addition problem, 0.5 + 0.75. Teachers did not differ in their responses across the other three problems and responses for all problems were relatively even across the different levels of professional development.

The number of teachers reporting strategies involving versions of written algorithms was generally quite low. Overall there were fewer algorithmic strategies reported for the part-whole number problems; perhaps the teachers perceived that the application of written algorithms is more difficult for students in transferring written algorithms to the part-whole domains. Additionally, teachers may not have reported algorithmic strategies because their students had yet encountered formal written algorithms.

4.8 Chapter Summary

As a starting point for the study, the results of the teacher questionnaire present a collective overview of how middle years teachers are addressing mental computation. Using Shulman's teacher knowledge domains as a framework for organising the teachers' experiences, initially a number of characteristics emerged. Representing the teachers' knowledge of the educational ends, purposes, and values, nearly every teacher attributed the value of mental computation to developing mathematical understanding. This is perhaps not necessarily surprising given that mental computation is generally detailed in the current mathematics curriculum in Tasmania. For a number of the secondary teachers, however, this was the only value they ascribed to mental computation. Many teachers also acknowledged the real life applicability of mental computation.

In relation to the teachers' knowledge of educational contexts, the teachers strongly supported an emphasis on mental computation in primary school. The view, however, was not as strongly supported in relation to secondary school. The teachers indicated that the emphasis on mental computation declined as students moved from primary school into the secondary school, with reasons emphasising curriculum-related factors (predominantly other methods of computation) and environment-related factors (crowded curriculum and parental expectations). Accordingly the teachers reported that time devoted to written computation out-weighed that spent developing mental and calculator computation – an element of the teachers' curriculum knowledge. The teachers displayed an inconsistent approach to developing mental computation across different types of numbers. In the first place, the teachers' reported developing mental computation with multi-digit whole numbers less often than with basic number facts. As well, mental computation in relation to fractions and percents was reported less often in relation to multi-digit whole numbers. Working with decimals, however, was reported at a similar level to multi-digit numbers, particularly for the operations of multiplication and division.

Representing the teachers' pedagogical content knowledge, middle years teachers reported organising mental computation as a daily activity involving the whole class. They also acknowledged the importance of a range of mathematical competencies in terms of contributing to the students' mental computation development. When asked to describe their own mental computation activities, the teachers' provided an even distribution of both traditional activities (test and answer based) and non-traditional activities (discussion and conceptual number work). Few actual descriptions of the non-traditional activities, however, were provided by the secondary teachers. In responding to a list of classroom activities the teachers reported that games and strategy discussion were the activities most frequently used to develop mental computation; subsequently observation of and discussions with students were the most frequently reported forms of assessment. Overall, the teachers were more evenly divided in their responses to quick recall questions as a mental computation activity and this was also apparent in their responses to using testing (both timed and untimed) as an assessment technique. Both the activities used to develop mental computation and the types of assessment were considered as aspects of the teachers' pedagogical content knowledge.

In relation to the teachers' knowledge of learners' characteristics three aspects associated with mental computation were considered. First, the teachers conveyed that the students' enjoyment of mental computation was largely due to the characteristics and structure of the activities used in the classroom. Lack of skills and speed in relation to mental computation were considered the challenging aspects for students. Second, in relation to attitudes displayed by students, the teachers felt the students would support the of view mental computation as being useful outside of school – a positive view. Of the more negative views, which were more strongly supported by the secondary teachers, teachers felt that students would report mental computation as stressful, hard due to speed, and would consider using a calculator over mental computation. Third, in describing the strategies students might use, across three whole number problems (58 + 34, 52 – 25, and 24 x 3), splitting the numbers by place value was the most commonly reported strategy provided by the teachers. The teachers did not consider the same strategy in relation to the decimal problem (0.5 + 0.75). Alternatively, the teachers indicated that students were more likely either to change the representation of the problem to fractions or to split by a different quantity (0.5 + 0.5 + 0.25). The most frequently reported strategy by the teachers for the problem 4 x ¾ involved an algorithm with quarters, and finally for the problem 10% of 80 the teachers reported the use of division associated with the strategy of using related number knowledge.

As a general comment, the primary teachers were more unified in reporting how they were addressing mental computation than the secondary teachers. For example, in recording time devoted to developing mental computation skills (Section 4.5.1), the majority of the primary teachers reported just two response patterns. The secondary teachers, however, described nine different patterns emphasising mental, written, and calculator computation. Although the teachers' levels of professional development did not have a strong impact on the responses provided by teachers, pedagogical content knowledge was one area where there were differences. Closer inspection of the data revealed that generally it was the same group of teachers creating the differences, that is secondary teachers with no or limited professional development. Pedagogical content knowledge is, in a sense, a more practical area related to the implementation of mental computation in classrooms. Differences were reported, for example, for the distribution of traditional and non-traditional activities. The other areas where no differences were noted between teachers with different levels of professional development – knowledge of educational ends, purposes, and values and also knowledge of educational contexts – are more to do with teachers' own beliefs and understandings, which are likely to be more deep-rooted in the teachers' professional experiences.

Many of the teachers who participated by completing the questionnaire phase of the study exhibited an encouraging attitude to mental computation, although still in the shadow of written computation. The sample of middle years teachers who responded to the questionnaire did so voluntarily and it is possible that only those teachers with a particular interest in developing mental computation were motivated to return the questionnaire. The results in relation to Shulman's knowledge domains are addressed in full in the final discussion in Chapter 8.

In Chapter 5 the presentation of the results continues with the results and initial discussion of the three student instruments. The results generated from the mental computation test, number comparison tests, and student questionnaire and are presented and discussed in terms of developing a mental computation profile of middle years students. The profile represents an investigation of students' characteristics as learners based on evidence collected from the students themselves, and comprises the second phase of the study.

Chapter 5

Results (Phase 2): Profiling Middle Years Students

5.1 Introduction

One of the aims of the current study is to profile students in the middle years in relation to mental computation. The profiling approach addresses one of Shulman's teacher knowledge categories discussed in Chapter 4 – understanding the characteristics of learners. This chapter is guided by the question: how is mental computation being experienced by middle years students? Profiling of the students is achieved using three data sets, collected from three instruments administered at the same classroom session to eight classes and a total of 172 students. In the first place the results of a mental computation test are analysed to determine the students' level of mental computation performance (Callingham & McIntosh, 2001, 2002). Based on the students' total test scores, three groups of student mental computation performance are established and described, with students assigned to either Group H (high performance), Group M (middle performance) or Group L (low performance). The performance of students on two written comparisons tasks, one with pairs of fractions and one with pairs of decimals, are then analysed across the three groups to which all students were assigned. The third and final stage involved analysing perspectives of students on a number of aspects related to mental computation, expanding the profile from a sole emphasis on mathematical performance to include for example, attitudes, beliefs, and self assessment. Data was collected through a student questionnaire and are also considered across the three established groups of student mental computation performance. The full details of analyses associated with each of the data sets are available in Appendix C.

In the last section of the chapter, data are reported for two additional questions from the questionnaire completed by the students. The questions focus on classroom activity related to mental computation and are of interest in understanding the classroom environment, but do not directly contribute to the profile. Data are reported for the total student sample with differences between primary and secondary students highlighted.

The middle years students appeared to take the task of completing the questionnaire and number tasks seriously. This is suggested by the very small number of non-responses overall and there were no questionnaires returned with disparaging comments. The perceived level of concentration and engagement during the administration of the questionnaire in the classrooms was very high.

5.2 Performance on the Mental Computation Test

The students' total mental computation test scores were assigned to one of three groups based on performance on the mental computation test, Group H (high performance), Group M (middle performance) and Group L (low performance). The three groups of students provided a base from which to build a profile of middle years students in relation to mental computation and were associated with the mental computation performance levels described by Callingham and McIntosh (2001, 2002).

Of the eight levels (1 – 8) of mental computation performance described by Callingham and McIntosh (2002), in this study Group H is aligned with Level 7, Group M with Level 6, and Group L with Level 5. Callingham and McIntosh (2002) recorded the percentage of students in Grades 3 to 10 at each of the eight levels. Collectively, Level 5, Level 6, and Level 7 accounted for the largest groups of students in Grade 5 to 8. From the content analysis performed by the authors, whole number items from Level 5 onwards expand to include more sophisticated combinations of multi-digit numbers across the four operations. Few items involving part-whole numbers appear before Level 5.

The overall performance of the students is presented in Table 5.1 with the students' performance on individual mental computation items reported in Appendix C.1. The process for determining students' levels was described in Section 3.5.4.2.

Table 5.1

Three Groups of Student Performance on the Mental Computation Test

Groups	Primary students	Secondary students	Number of students
High (H)	18 (21.7%)	27 (30.3%)	45 (26.2%)
Middle (M)	28 (33.7%)	28 (31.5%)	56 (32.6%)
Low (L)	37 (44.6%)	34 (38.2%)	71 (41.3%)
Number of students	83	89	172

Just over a quarter of the students were assigned to Group H (26.2%), the highest level of mental computation performance. Overall these students made only a few (if any) errors on the test. Just three instances of errors on whole number items were recorded for students in Group H and involved multi-digit addition and subtraction of items at Level 5 and Level 6. For the primary students all errors were on part-whole items; the two items that stood out as being the most difficult were two Level 7 items, 10% of 45 and 0.5 + 0.75 (primary students were not given items higher than Level 7). For the secondary students in Group H few were successful in answering 30% of 80 and ½ + $^{1}/3$, both Level 8 items. Errors across Group H were mainly spread over the Level 7 items, for example, approximately a third of the secondary students made errors on 10% of 45, 3 ÷ ½, and 4 x ¾.

Students assigned to Group M (32.6%), the middle group, were less successful on the whole number items than students in Group H and additionally the number of students making errors on the part-whole items increased. Primary students in the middle group were largely successful in answering the whole number items, although 24 x 3 (at Level 6) was the item with the highest number of errors. Not one primary student successfully answered 10% of 45 and few were successful in answering 0.6 x 10 and 0.5 + 0.75, all Level 7 items. For secondary students the hardest whole number item was 52 – 25 (at Level 5), with very few students answering the Level 7 part-whole items correctly. There was a decrease in performance for all students in the middle group across the Level 5 and Level 6 part-whole items, compared to students in the highest group.

Students assigned to the lowest group of mental computation performance, Group L (41.3%), were not successful in solving the whole number items at Level 5 and Level 6. More than half of the primary students in the lowest group, for example, did not attempt or were unsuccessful in solving 52 – 25 and 24 x 3. These students were, however, generally successful in solving the whole number questions at Level 2 to Level 4, including 9 + 8, 17 – 8, 5 x 6, and 21 ÷ 7. Overall, students in Group L answered few (if any) part-whole items. Primary students were most successful in answering items 50% of 24 and ½ + ¼. Secondary students were most successful in answering items 50% of 24, 25% of 80 and 0.25 + 0.25. Interestingly, half of the primary students in Group L were successful in answering the item 3 ÷ ½, although no secondary students assigned to the lowest group provided a correct answer for the same item.

5.3 Performance on the Comparison Tests

Overall, 166 students attempted the fraction comparison task and 171 attempted the decimal comparison test (N = 172). Students' performance on individual fraction and decimal comparison items is reported in Appendix C.2.

5.3.1 Performance on the fraction comparison test

The spread of student performance on the fraction comparison task across the three groups of students is presented in Table 5.2. Two thirds of the students assigned to the highest group for mental computation performance ($n = 45$), correctly identified the largest fraction in all eight pairs (33.3%) or made a single error (33.3%). The error for most students involved the last item on the test, with students indicating that $^1/_9$ was a larger fraction than $^2/_{12}$.

Table 5.2

Student Performance on the Fraction Comparison Test

Number of items answered correctly	Group H	Group M	Group L	Number of students
8	15 (33.3%)	2 (3.6%)	2 (2.8%)	19 (11.0%)
7	15 (33.3%)	9 (16.1%)	3 (4.2%)	27 (15.7%)
6	1 (2.2%)	9 (16.1%)	8 (11.3%)	18 (10.5%)
5	8 (17.8%)	21 (37.5%)	37 (52.1%)	66 (38.4%)
4	0 (0.0%)	3 (5.4%)	8 (11.3%)	11 (6.4%)
3	3 (6.7%)	9 (16.1%)	8 (11.3%)	20 (11.6%)
2	2 (3.6%)	2 (3.6%)	2 (2.8%)	6 (3.5%)
1	1 (2.2%)	2 (3.6%)	2 (3.6%)	5 (2.9%)
0	2 (4.4%)	1 (1.8%)	3 (4.2%)	6 (3.5%)
Number of students	45	56	71	172
	172			

Just two students in Group M (middle performance) correctly answered all eight items with a further nine students making a single error. Like the students in the Group H, the error most commonly involved comparing the fractions $^1/_9$ and $^2/_{12}$. The largest number of students in Group M correctly identified the largest fraction in five pairs (37.5%). Approximately half of these students, however, made errors on the same three items: those where the pair of fractions involved the same numerator, for example, $^4/_8$ and $^4/_{12}$, $^3/_4$ and $^3/_9$, and $^1/_5$ and $^1/_8$. For these same students, the pattern of response across all eight pairs of items showed that students consistently chose those fractions with the larger denominator as being "larger," suggesting the students were not seeing the fraction as a composite relational entity. Generally, however, most students were successful in identifying the larger of two items where the comparison involved fractions with the same denominators: $^2/_4$ and $^3/_4$, and $^3/_8$ and $^6/_8$. A different pattern of response was identified for students who correctly identified just three of the larger fractions (16.1% in Group M). In comparing fractions with the same denominator, these students chose the fraction with the smaller numerator as being "larger," for example, $^3/_8$ over $^6/_8$, and also choose "larger" fractions based on the size of the denominators, for example, $^2/_3$ over $^9/_{10}$. These students persistently choose fractions with smaller numerators and denominators as being "larger."

In the lowest group, Group L, just five students correctly identified the largest fraction in all eight pairs, or made a single error. Like students in Group M, the largest cluster of students correctly identified the largest fraction in five pairs (52.1%). The tendency for students to choose fractions with larger denominators as being larger overall was stronger for students in Group L, than for Group M. As well, there was also a small number of students (11.3%) who appeared to be associating larger fractions with smaller denominators, although student numbers were similar to students in Group M.

Across each of the three groups there were small numbers of students who marked some items as being the *same* despite being asked to mark the larger fraction. It seems likely that students were comparing these fractions based on the number of parts missing from the whole, which in this case is one part for both fractions. For example, for the item comprising $^9/_{10}$ and $^2/_3$ there is $^1/_{10}$ missing from the whole (to leave $^9/_{10}$) and also $^1/_3$ missing from the whole (to leave $^2/_3$). There were also small numbers of students answering just one or two items correctly (9.9% of the total number of students); in most cases these were the only items attempted.

5.3.2 Performance on the decimal comparison test

The spread of student performance on the decimal comparison test across the three groups of students defined by mental computation performance is presented in Table 5.3. Approximately half of the students assigned to Group H for mental computation performance correctly identified the largest decimal in all twelve pairs (51.1%). Small numbers of students (five or fewer) were spread over the other possible scores, with no students identifying less than four correct pairs overall.

Table 5.3

Student Performance on the Decimal Comparison Test

Number of items answered correctly	Group H	Group M	Group L	Number of students
12	23 (51.1%)	7 (12.5%)	6 (8.5%)	36 (20.9%)
11	5 (11.1%)	10 (17.9)	2 (2.8%)	17 (9.9%)
10	4 (8.9%)	4 (7.1%)	6 (8.5%)	14 (8.1%)
9	4 (8.9%)	7 (12.5%)	3 (4.2)	14 (8.1%)
8	2 (4.4%)	4 (7.1%)	5 (7.0%)	11 (6.4%)
7	2 (4.4%)	3 (5.4%)	9 (12.7%)	14 (8.1%)
6	0 (0.0%)	8 (14.3%)	11 (15.5%)	19 (11.0%)
5	2 (4.4%)	5 (8.9%)	12 (16.9%)	19 (11.0%)
4	1 (2.2%)	3 (5.4%)	11 (15.5%)	15 (8.7%)
3	0 (0.0%)	3 (5.4%)	4 (5.6%)	7 (4.1%)
2	0 (0.0%)	0 (0.0%)	2 (2.8%)	2 (1.2%)
1	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
0	0 (0.0%)	0 (0.0%)	1 (1.4%)	1 (0.6%)
Number of students	45	56	71	172
	172			

For the students assigned to Group M for mental computation performance, the spread of responses was more evenly distributed across the twelve possible scores than for Group H. The number of students correctly identifying ten or more larger decimals decreased overall in comparison with the students in the Group H. For students in the middle group who correctly identified the larger of four to seven decimal pairs, there was some indication that students were choosing the longer decimals as being larger, although not always consistently across all 12 items. Many students experienced difficulty with items that involved comparing decimal values containing zeros: 0.8 and 0.80, 0.450 and 0.45, 0.731 and 0.73100, and 3 and 3.0. Similarly the choices made by students tended to be related to a certain type of decimal such as the truncated decimals, for example, 3.77 and 3.7777. Performance on some individual items, however, was quite high, for example, most students in Group M chose 4.8 as being larger than 4.67; similarly students generally chose 0.5 as being larger than 0.36.

In Group L, the majority of students correctly identified four to seven of the larger decimals. Decimals with more digits featured consistently in the pattern of responses; for example, on individual items more students in Group L marked 4.67 as being larger than 4.8, than in the Group M. More than half also experienced difficulty with items that involved comparing decimal values containing zeros, with similar results across the truncated decimals.

A different pattern was identified for a smaller group of students in Group L. These students consistently chose the decimals with the least number of zeros as being larger, and also identified the truncated decimals as being larger, a pattern of response that indicated students were engaging in reasoning based on the idea that shorter decimals represent a larger value. Less students in Group L (19.8%) were able to identify ten or more of the larger decimals, compared to either Group M (or Group H).

5.4 Student Mental Computation Questionnaire

In expanding the profile from a sole emphasis on mathematical performance, data from the student questionnaire are also considered across the three groups of students (Group H, Group M, and Group L) and constitutes the third stage in building a student profile of mental computation. The combined data for all students are reported and then analysed across the three student groups based on mental computation performance. For individual questions, comparisons of means between the three groups are reported using *one-way ANOVA* with effects considered significant at $p < 0.05$. Only significant differences are reported in this section although the full details of analyses associated with each question are available in Appendix C.3.

5.4.1 Student beliefs: The importance of mental computation

Five beliefs statements were presented to the students for their consideration; relative frequencies of responses across the total number of students are presented in Figure 5.1. The majority of the students communicated that they considered mental computation important at both the primary and secondary school levels. Two opposing statements were also presented to students regarding the relative importance of mental computation and written computation compared to each other. For both statements there were high levels of responses that indicated the students were *uncertain*; generally the students more widely supported the statement emphasising mental computation over written computation. The statement concerning the importance of mental computation in relation to its use by adults provoked a more varied range of responses amongst the students. Half of the students *agreed* or *strongly agreed* with the statement, with just under a quarter of the students expressing *disagreement* or *strong disagreement.*

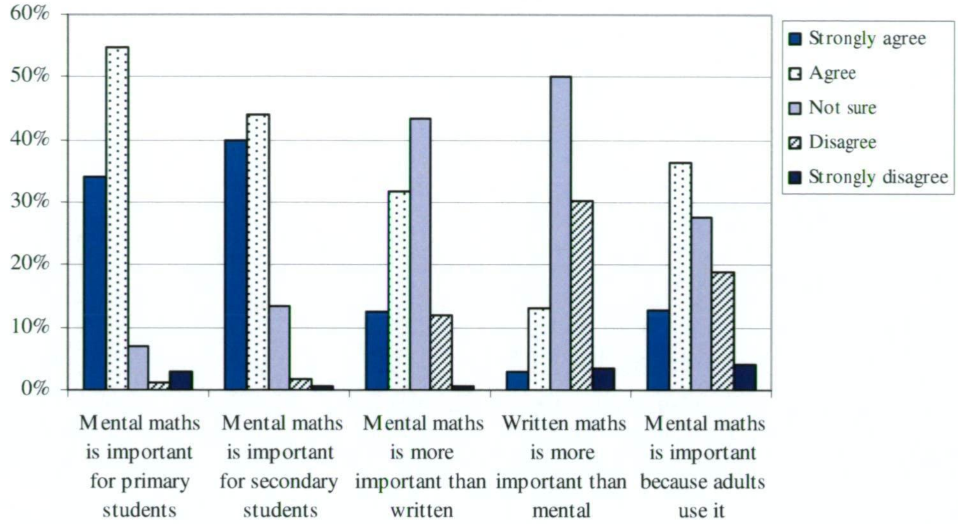

Figure 5.1. Students' views on the importance of mental computation.

The responses of students across the three groups did not differ significantly across the five views between the Group H and Group M, or Group M and Group L. Between the highest and lowest groups, however, there were differences. Students in Group H indicated stronger agreement for the importance of mental computation at the secondary level than did students in Group L, $F(2,167) = 4.424$, $p < 0.05$. Similar results were obtained when students considered mental computation at the primary level, although this was less pronounced at $F(2,167) = 2.902$, $p = 0.058$. Students in Group L, however, expressed more support for the importance of mental computation as associated with adult use than did students in Group H or Group M, $F(2, 167) = 3.160$, $p < 0.05$.

5.4.2 Student self assessment

Students completed a self assessment over five statements relating to aspects of their mental computation ability. A summary of responses is provided in Figure 5.2 and differences between the three student groups are described.

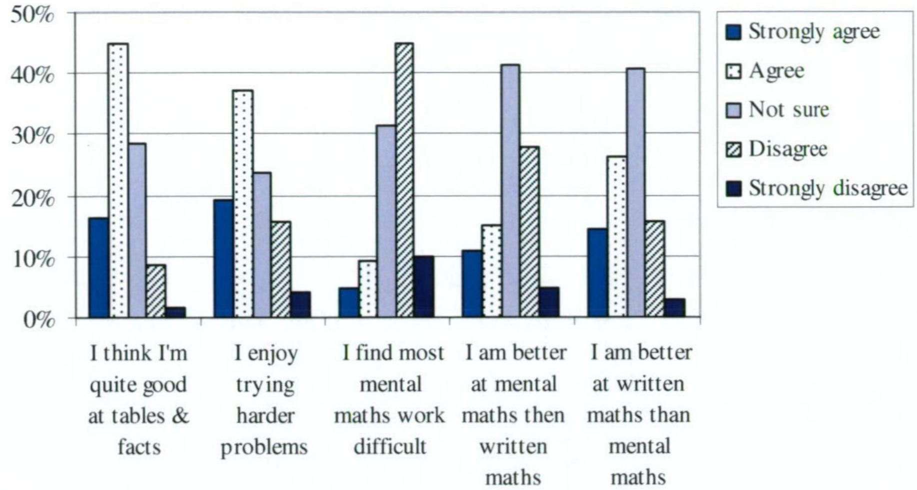

Figure 5.2. Students' reported self assessment on aspects of computation.

More than half of the students *strongly agreed* or *agreed* that they were "quite good at tables and number facts," with a similar pattern of responses reported for enjoying "harder maths problems." Additionally, over half of the students expressed *disagreement* (or *strong disagreement*) with the idea that mental computation on the whole was difficult. As a group, many of the students were indecisive about their mental computation ability in relation to written computation and vice versa. Those who offered a stronger opinion appeared to nominate written computation over mental computation, although there were no significant differences between groups on this latter pair of statements.

Students in both Group H and Group M differed significantly from students in Group L in assessing whether they were "quite good at tables and number facts" $F(2,169) = 8.383$, $p < 0.001$. Students in Group L reported a much lower opinion of their ability. Students also significantly differed in their reported enjoyment of "harder maths problems," $F(2,169) = 9.335$, $p < 0.001$. Students in Group H, expressed higher agreement with the statement than students in either Group M or Group L. Similarly students in Group H differed significantly from Group M or Group L in their agreement with the statement "I find most mental maths work difficult" $F(2,169) = 7.767$, $p < 0.001$, with students in the highest group less likely to disagree.

5.4.3 Attitudes towards mental computation

Students were asked to consider a series of nine attitudinal statements, with data used to answer the question, what attitudes do students in the middle years hold towards mental computation? The data were separated into those statements that presented a more positive attitude to mental computation (Figure 5.3) and those that were more negative (Figure 5.4).

Positive Attitudes. The students displayed a relatively positive attitude in considering the usefulness of mental computation outside of school and supported the view that mental computation can be the quickest way to work through a problem. Generally students indicated that mental computation was "fun" although responses were more varied across the five indicators. The students also reported that working mentally was not always preferable to "being able to write it down," with the largest number of students *disagreeing* with the view. There were no significant differences across the more positive attitudes among the three groups of students.

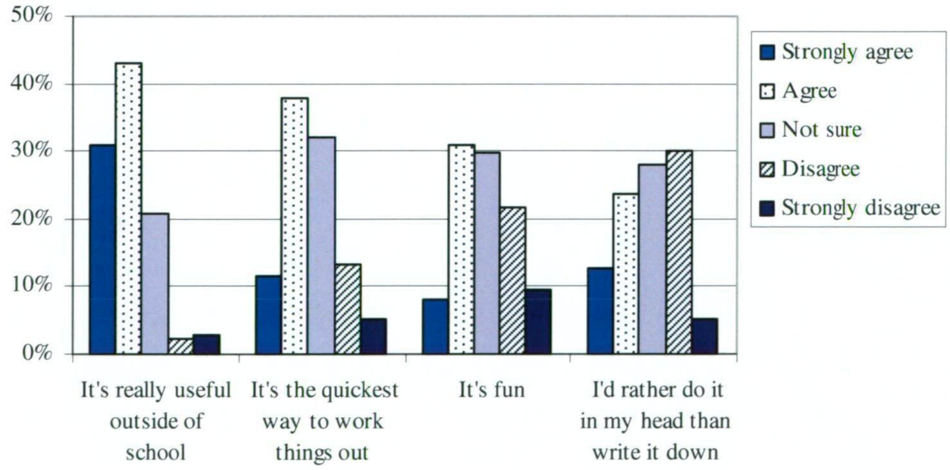

Figure 5.3. Students' responses to positive attitudes to mental computation.

Negative Attitudes. The students did not support the idea that mental computation was for students in lower grades. As well, the students did not appear to associate mental computation with speed, although responses across the three groups differed ($F(2,169) = 8.755$, $p < 0.001$). Students in Group L reported more support for the view than did students in Group H. Similarly, students in Group L also reported more support for the belief in mental computation as "being hard" because of having to "remember everything" than students in Group M and Group H, $F(2,169) = 6.574$, $p = 0.05$.

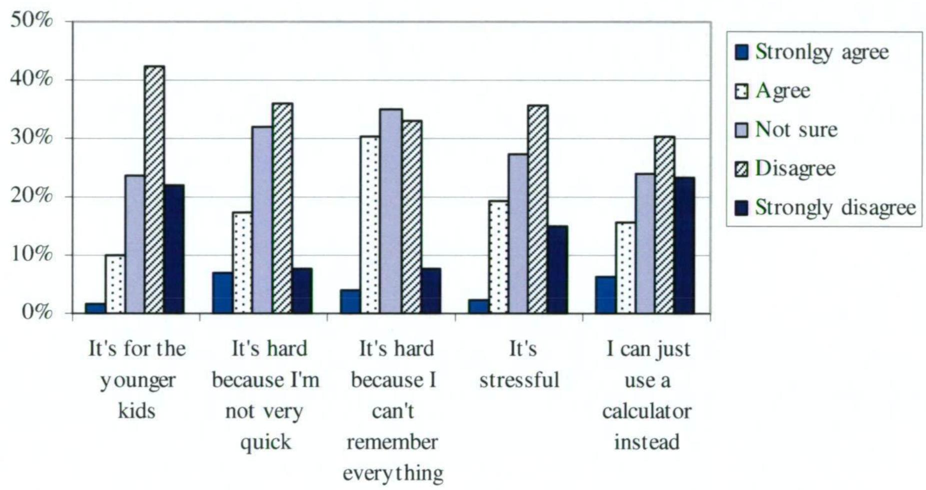

Figure 5.4. Students' responses to negative attitudes to mental computation.

Students were varied in their responses to the suggestion of mental computation as being "stressful," although generally it was not supported with *disagree* the modal response. Additionally students in Group L were also more inclined to support the view relating to the ease of using calculators instead of mental computation than students in Group H, $F(2,168) = 3.427$, $p < 0.05$, although generally calculator use was not strongly supported by the students as a whole.

5.4.4 Students' use of written, calculator, and mental computation

Students were asked to report on their use of written, calculator, and mental computation work in two settings: during their class mathematics time and outside of school. The students were asked to rate the two settings using descriptors: 1 = most, 2 = some, and 3 = least. For this question a summary of the patterns of student responses is provided in Table 5.4. In some cases, for example Pattern 10, students did not rate all three computation categories. As well some students marked two out of the three categories with descriptors *same* or *even*, for example Pattern 7.

Table 5.4

Students' Reported Use of Written, Calculator, and Mental Skills

Pattern	Written	Calculator	Mental	Number of students (in class)	Number of students (outside of school)
1	**Most**	Least	Some	84 (49.1%)	15 (8.7%)
2	**Most**	Some	Least	18 (10.5%)	21 (12.2%)
3	**Most**	Even	Even	3 (1.8%)	1 (0.6%)
4	Some	**Most**	Least	1 (0.6%)	9 (5.2%)
5	Some	Least	**Most**	35 (20.5%)	51 (29.7%)
6	Least	Some	**Most**	5 (2.9%)	40 (23.3%)
7	Even	Even	**Most**	0 (0.0%)	3 (1.7%)
8	Even	Least	Even	5 (2.9%)	0 (0.0%)
9	Even	Even	Least	1 (0.6%)	0 (0.0%)
10	**Most**	–	–	9 (5.3%)	2 (1.2%)
11	–	–	**Most**	6 (3.5%)	8 (4.7%)
12	Least	**Most**	Some	1 (0.6%)	13 (7.6%)
13	Even	Even	Even	3 (1.8%)	2 (1.2%)
14	–	**Most**	–	–	4 (2.3%)
15	Even	**Most**	Even	–	1 (0.6%)
16	Least	Even	Even	–	2 (1.2%)
		Total number of responses		171	172

In considering first what methods of computation students used most in their *mathematics class*, Patterns 1, 2, and 5 were the most commonly reported by the students. Two thirds of the students overall (66.7%) reported a pattern whereby written work featured (shown in Table 5.4 as *most*) and a further 26.9% of students reported working mentally *most*.

In reporting what methods of computation students used most *outside of school*, the responses were more varied than those reported for mathematics class time. The use of the most common pattern reported for in class – Pattern 1 (49.1%) – which ordered written work, mental work, and then calculator work, decreased considerably with only 8.7% of students indicating this was what they used outside of school. Students reporting Pattern 5 in class (20.5%), which ordered mental work first, followed by written work and calculator work, increased slightly to 29.7% in use outside school. Noticeably, Pattern 6, which was reported in class by only 2.9% of students, increased to 23.3% for outside of school again featuring mental computation. Calculator computation in class was reported as *most* by only two students. In reporting calculator use outside of school, however, 15.7% of students marked a pattern that specified calculator use as *most*. Overall, only 14.6% of students reported the same computation pattern in both the class setting and outside of school, with most students associating the use of mental computation outside of the school environment.

Table 5.5 presents the responses of students across the three groups of students defined by mental computation performance in Section 5.2. Across the three methods of computation used *in class*, approximately three quarters of the students in Group H and Group M reported a pattern that featured written computation. For students in Group L written computation was reported for just over half of the students with an increase in students reporting a pattern that featured mental computation. For students in each of the three groups, the emphasis changed from written computation to mental computation when considering computation *outside of school*. It was highest for students in Group H with approximately three quarters reporting a pattern that featured mental computation outside of school, compared to a quarter for use in class. As well, a small number of students in each group reported a pattern that featured using calculators for computation outside of school.

Table 5.5

Computational Use Across the Three Student Groups Defined by Mental Computation Performance

Computation used in class			
	Group H	Group M	Group L
Mental	11 (24.4%)	9 (16.1%)	28 (39.4%)
Written	32 (71.1%)	42 (75.0%)	40 (56.3%)
Calculator	0 (0.0%)	1 (1.8%)	0 (0.0%)
Other*	2 (4.4%)	4 (7.1%)	3 (4.2%)
Totals	45 (100%)	56 (100%)	71 (100%)
Computation used outside of class			
	Group H	Group M	Group L
Mental	32 (71.1%)	30 (53.6%)	38 (53.5%)
Written	7 (15.6%)	14 (25.0%)	18 (25.4%)
Calculator	6 (13.3%)	10 (17.9%)	12 (16.9%)
Other*	0 (0.0%)	2 (3.6%)	3 (4.2%)
Totals	45 (100%)	56 (100%)	71 (100%)

Note: * totals refer to those patterns reported in Table 5.4 where students did not nominate a computational preference (most) instead marking the choices even and least.

5.4.5 Students' mental computation preferences

As part of the questionnaire, students were presented with 12 computation items and asked to indicate which items they would choose to do mentally by indicating *yes* or *no*. Students were not asked to record answers. The data are reported across the three student groups defined by mental computation performance (see Figure 5.5, 5.6, and 5.7) and there were differences between the types of items students would consider using mental computation to solve, indicating that the students had limits associated with what they considered reasonable or felt comfortable with for mental computation.

The responses of students in Group H were relatively consistent across the items (refer to Figure 5.5), with the majority of students indicating they would attempt most of the items mentally. Across the addition and subtraction items the proportion of students marking *yes* decreased across the items: 58 + 34, 47 + 54 + 23, 165 + 98, and 264 – 99. The pattern of response was similar across the multiplication items from 7 x 25, 60 x 70, 945 x 100, and 14 x 83, the last of which was the only item to provoke a response whereby most students indicated they would not attempt the item mentally. The other item that provoked a different response pattern was 10% of 45 with half of the students indicating they would attempt the item mentally and half indicating they would not. More than three quarters of the students indicated they would choose to do $1 - {}^1/_3$, ½ + ¾, and 6.0 + 4.5 mentally.

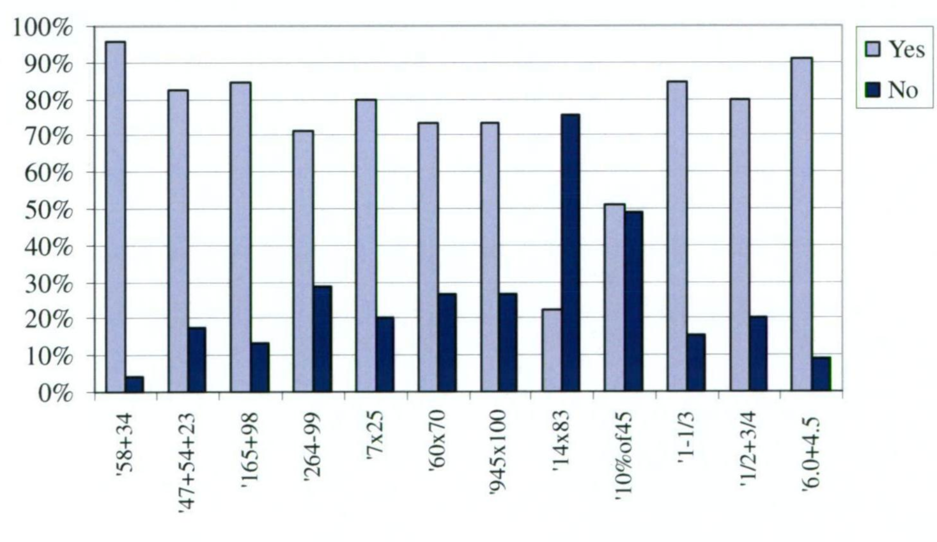

Figure 5.5. Group H students preferences for mental computation items (n = 45).

For students in Group M responses across the addition items were similar to students in Group H, although generally the numbers of students marking *yes* progressively decreased (refer to Figure 5.6). For the item, 264 – 99, however, more than half of the students indicated this was not an item they would choose to do mentally. Across the multiplication items 7 x 25 and 60 x 70, responses that favoured mental computation were only slightly higher than those than did not. For the item 945 x 100, however, more students indicated that this was not an item they would choose for mental computation and like students in Group H, most students indicated they would not choose to do 14 x 83 as a mental problem. Apart from 10% of 45, the majority of students in Group M indicated they would attempt the fraction and decimal items, although again the proportion of students marking *yes* was less than in Group H.

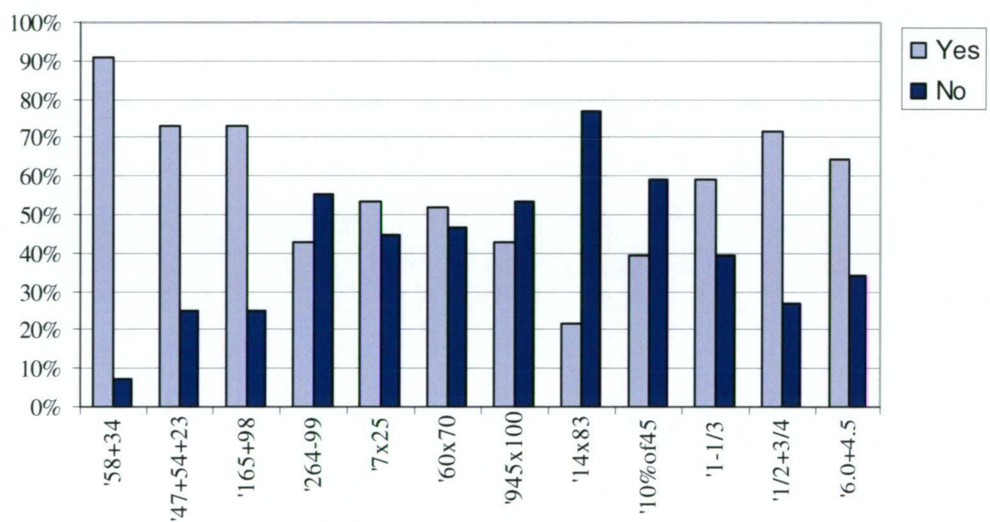

Figure 5.6. Group M students preferences for mental computation items (n = 56).

For the students in Group L responses across the addition items did not vary from the responses provided by students in Group M (see Figure 5.7). More than half of the students in Group L, however, would choose to do the problem 264 – 99 mentally, unlike students in Group M. The proportion of students indicating they would choose the multiplication items to do mentally decreased progressively. Across the fraction, decimal, and percent items, students in Group L were relatively even in their preferences. More students indicated they would not attempt 10% of 45 or $1 - {}^{1}/_{3}$ mentally, although more students indicated they would attempt ½ + ¾ and 6.0 – 4.5.

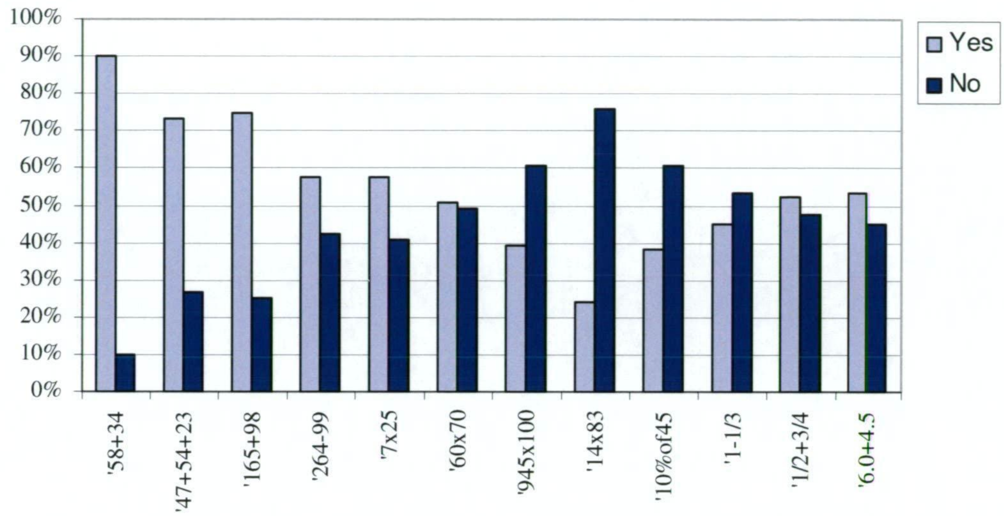

Figure 5.7. Group L students preferences for mental computation items (n = 71).

5.4.6 Students' use of mental computation with whole numbers, part-whole numbers and related activities

The different types of numbers and activities are considered in three groups of similar items in relation to mental computation. A summary of the responses of students is presented in Figure 5.8. Overall students reported using mental computation to solve fraction, decimal, or percent items less often than they did for operations with whole numbers.

Whole numbers. Half of the students (50.3%) indicated that they *frequently* used mental computation to help them "add and subtract numbers," with a further 20.5% indicating that they *always* did. The frequency levels of the same indicators decreased, however, for responses to multiplication and division. Here, 32.7% of students indicated that they *frequently* used mental computation to "multiply and divide numbers" and 12.3% indicated that they *always* did. The percentage of students who marked *rarely* also increased to 17.0% for the operations of multiplication and division compared to 0.6% for addition and subtraction. Student responses to the question regarding using mental computation to "work out tables you can't remember" were very similar to reports related to multiplication and division.

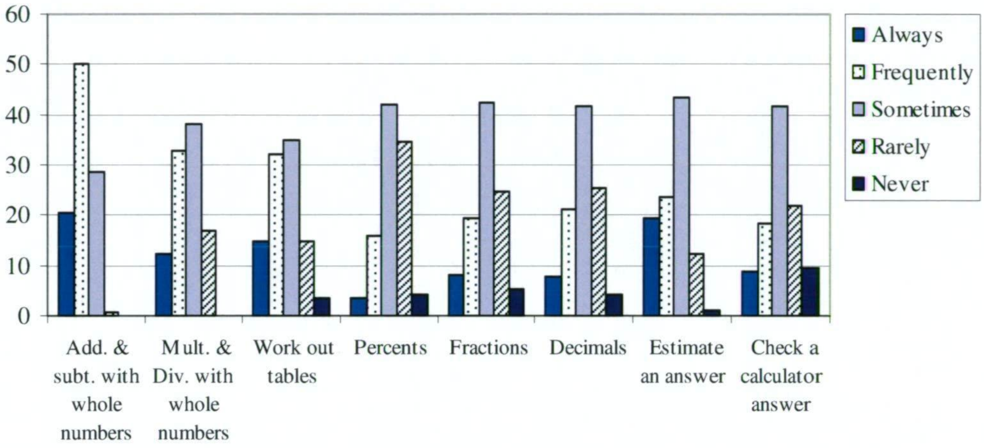

Figure 5.8. Mental computation with whole numbers, part-whole numbers and related activities.

In using mental computation for whole number problems there were significant differences between students in Group H and students in both Group M and Group L, $F(2,168) = 5.600, p < 0.05$), with students in Group H, indicating they more frequently used mental computation with whole number problems for addition and subtraction. Students did not significantly differ in their reported use of mental computation for whole number problems with multiplication and division or to work out tables.

Fractions, Decimals, and Percents. In considering how often students used mental computation to work out fractions, decimals, and percents, the responses were relatively consistent across all three types of numbers, in particular the number of students marking *sometimes* was almost identical (42.4%, 41.8%, and 42.1%). There was a small increase in *rarely* responses for the area of percents (34.5%) compared to fractions (24.7%) and decimals (25.3%). There was also a small number of students who indicated they *never* use mental computation across these topics compared to none for whole numbers.

The students did not differ significantly in their reported use of mental computation with either percents or fractions; there were differences, however for the use of decimals ($F(2, 167) = 4.455, p < 0.05$). It was the students in Group H who indicated more frequent use of mental computation with decimals than did students in Group M or Group L.

Estimation and Calculator Activities. Students indicated that they *sometimes* used mental computation in estimation activities (43.5%), with indicators *always* and *frequently* narrowly accounting for the largest group of students overall (19.4% and 23.5% respectively). Students were also asked about using mental computation to check a calculator answer and the majority also indicated this occurred *sometimes* (41.8%). Differences between the three groups of students in their reported use of mental computation with estimation activities or calculator activities were not significant.

5.4.7 Mathematical competencies associated with mental computation

The students were asked to report on the importance of nine mathematical competencies associated with mental computation, including: remember tables, be able to work things out on paper, think logically, be creative, get the right answer, have a range of ways to work things out, be able to estimate, give reasons for answers, and be able to answer quickly. A summary of responses is provided in Figure 5.9. Across all nine mathematical competencies, the modal response reported by the students was *important*. In combining the *essential* and *important* indicators, an emphasis on remembering tables was supported by 83.6% of students. The lowest cumulative percentages were for "be creative" (54.7%) and "be able to answer quickly" (51.5%). There were no significant differences across

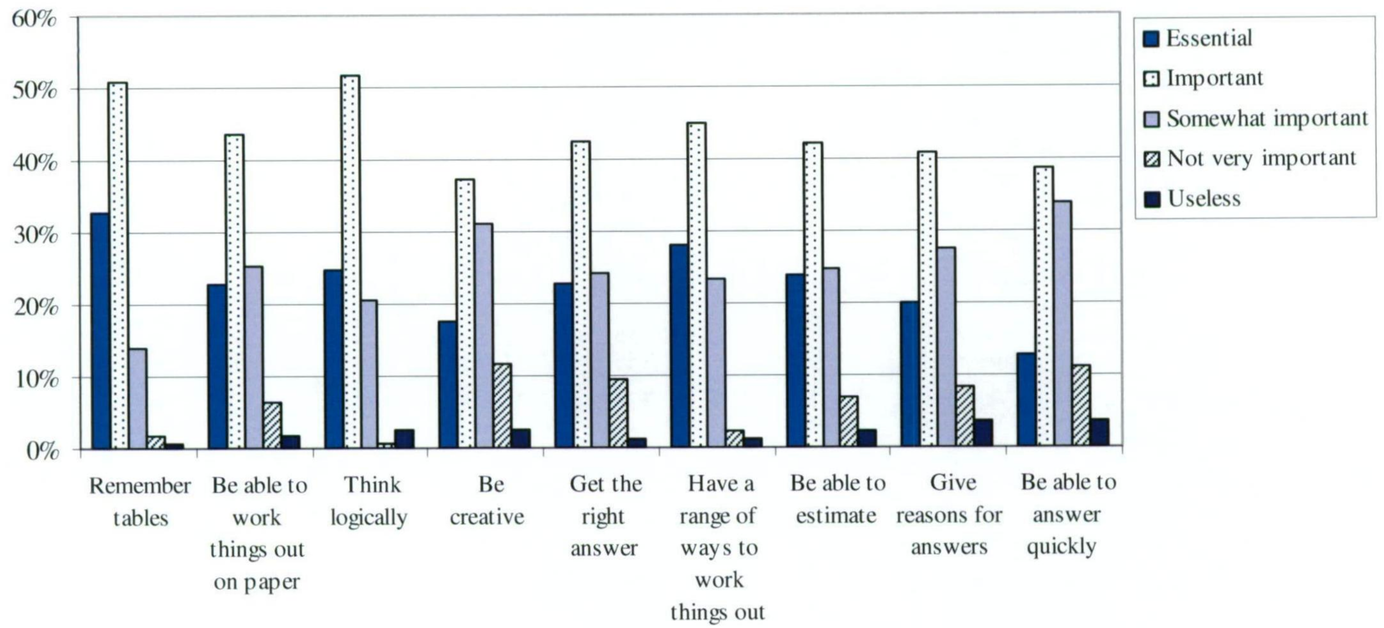

Figure 5.9. Students association of mathematical competencies with mental computation.

the competencies associated with mental computation for the three groups of students as defined by mental computation performance.

5.5 Questions related to classroom activity

5.5.1 Classroom organisation

In terms of the experience of mental computation in the classroom, students reported working on their own as the most frequent way for the class to be organised, as shown in Figure 5.10. Students reporting working with the whole class or with friends less frequently. Additionally, working in small groups was not reported as a frequent activity for these students, with half indicating *sometimes* and just over a third of the students indicating *rarely* or *never*.

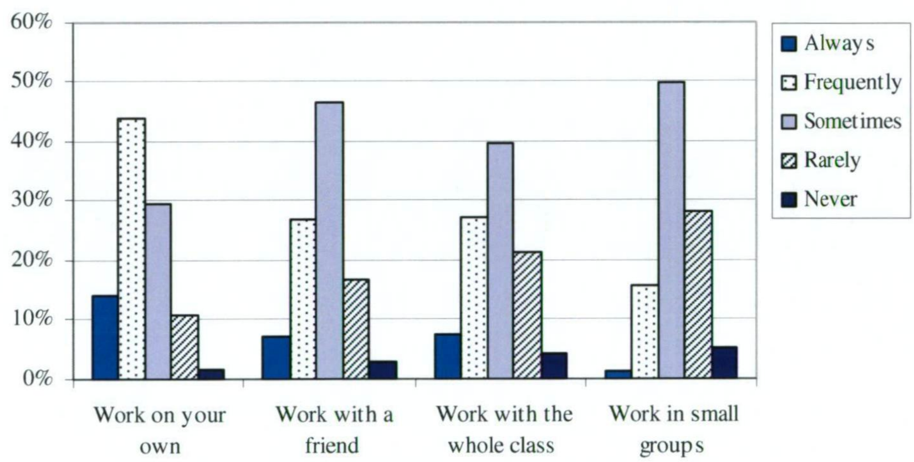

Figure 5.10. Classroom organisation for mental computation reported by students.

5.5.2 Mental computation activities

Students were asked to report which of the mental computation activities listed in the questionnaire were conducted in their classes. A summary of the activities and student responses are presented in Figure 5.11. The activity that the students most frequently associated with mental computation in the classroom involved "discussing different solutions," with students responding similarly to the item concerning "real life problems." The use of "20 quick questions" was the activity

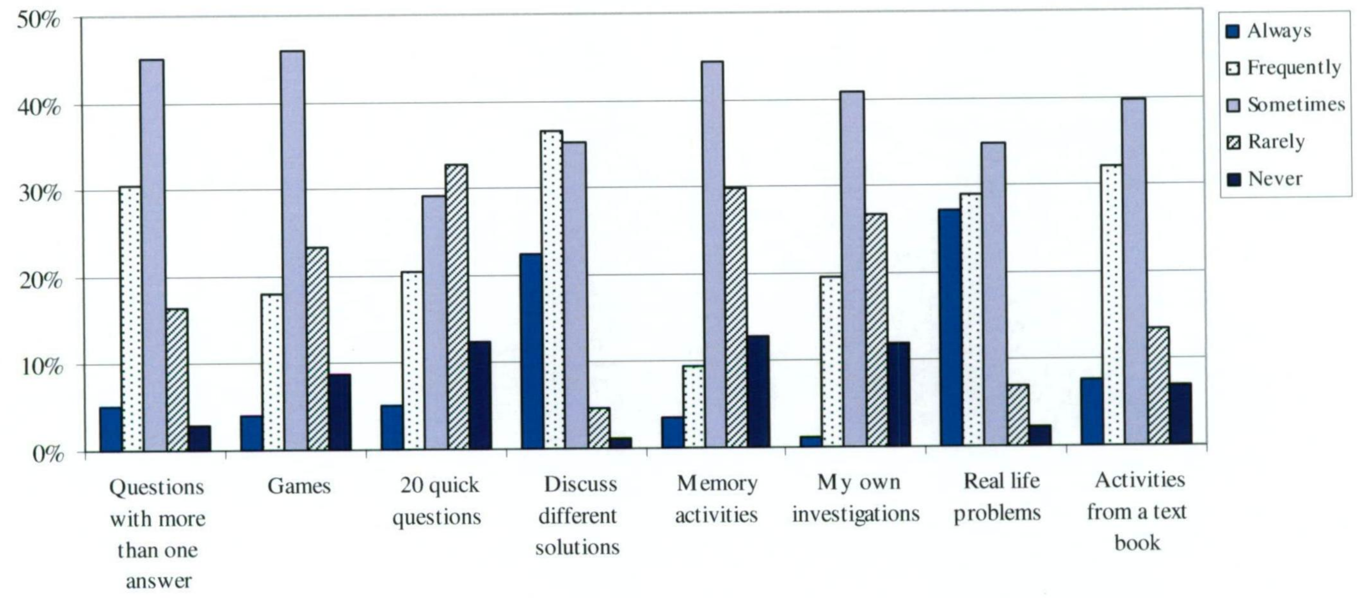

Figure 5.11. Activities students associate with mental computation.

that the students reported less frequently than other activities. Reported by approximately a third of the students, this was the only item where the modal response was *rarely*. Secondary students were less likely to report "games" as a classroom activity than primary students ($F(1,170) = 24.640$, $p < 0.001$), although they were more likely to report "memory activities" ($F(1,169) = 5.338$, $p < 0.05$) and "textbook activities" than primary students ($F(1,168) = 4.401$, $p < 0.05$).

5.6 Chapter Summary

In the second phase of the study, mental computation performance is described in relation to three groups of students defined by the mental computation performance levels described by Callingham and McIntosh (2001, 2002). Across the three groups, the numbers of students in the lowest group, Group L, were slightly higher than for Group M or Group H. This is perhaps due to the fact that there were limited items on the mental computation test to distinguish students at lower levels effectively. It is likely that students in Group L represent at least Level 5 and Level 4, as the lower levels are comprised of only whole number items. It is also a reasonable expectation that some of the students answering only one or two items may not provide a complete representation of student performance associated with Level 5.

The test items used in the current study are a sample from the original tests used to develop the mental computation levels (Callingham and McIntosh, 2001). The original tests were constructed using a much larger number of items and were administered over a considerable sample size. Even with fewer items than the original tests, there appears to be a high level of consistency between Callingham and McIntosh's mental computation performance levels and the three groups identified in this study. For each of the three groups, mental computation competence increased for both whole and part-whole numbers. For the whole number items the students increased in their competence across the items involving multi-digit numbers but not so much for the items involving single digits. For the items involving fractions, decimals, and percents students in the lowest group, Group L, achieved some success with items that involved 50% and 25% and the equivalents representations in decimals and fractions. Students in the highest group, Group H, however, demonstrated a consistently high level of competence across the range of items involving part-whole numbers and operations.

Taking each of the three groups of students, as determined by mental computation performance, it was then possible to look at the consistency of performance on a different type of task – comparing fractions and decimals – but one closely related to number sense like mental computation. Two thirds of the students assigned to Group H were competent in successfully identifying the largest fraction in eight pairs or making a single error. The results for the decimal comparison task were similar, indicating that the students at the higher level of mental computation performance could demonstrate a quite good understanding of the magnitude of fraction and decimals numbers.

Students at the lowest mental computation performance and assigned to Group L, did not perform as strongly as students in Group H. Just over half of the students, for example, correctly identified five of the larger fractions with analysis of the pattern of response showing students were relatively consistent in choosing fractions with larger/longer denominators as being larger. It is likely students were influenced by the traditional association of increasing number of digits with size, a familiar whole number concept. Furthermore, when choosing decimals it appeared that the same principle was being practiced. Over Group M and Group H, however, the number of students reasoning by size decreased, implying that students' understanding of rational number concepts was perhaps evolving.

The student responses to the mental computation questionnaire were also analysed across the three groups defined by mental computation performance. The question regarding the students' perception of the importance of various mathematical competencies in relation to mental computation (Section 5.4.7) was the only question over which the three groups of students did not differ significantly on any part of the question.

Across the five beliefs regarding the importance of mental computation (Section 5.4.1), only students in the highest and lowest groups differed significantly on some of their responses. Students in Group H expressed a higher level of agreement for the importance of mental computation over the primary and the secondary years than students in Group L. A higher level of association with mental computation as important because of adult use, however, was reported by students in Group L over students in Group H.

Students completed a self assessment relating to some of aspects of mental computation ability (Section 5.4.2). Students in the lowest mental computation group were less inclined to support their ability to work with tables and number facts and also in enjoying harder problems than students in the highest and middle groups. Accordingly, students in the lower group were more likely to agree with the perception of mental computation as difficult. There was very little difference between the students reported ability with written computation over mental computation and vice versa. Slightly more students overall were inclined to nominate written computation although this choice was not different for the three groups of students.

The students also rated their level of agreement with a set of attitudes related to mental computation (Section 5.4.3). Overall, the students across the three groups did not differ in their responses to the attitudes described as more positive. Of the more negative attitudes, however, more students in the lowest mental computation group compared to the other two groups, agreed with the association of mental computation as "hard" due to "having to remember everything." Interestingly, more students in the same group disagreed with the association of mental computation as "hard" due to speed and having to work quickly, than was reported by the other two groups.

In comparing the use of written, mental, and calculator computation (Section 5.4.4), all three groups of students reported a higher level of use of written computation in the mathematics classroom. Although the percentage of students in Group L was less than in Group M and Group H, more Group L students reported a pattern emphasising mental computation than did students in Group M and Group H, perhaps through encouragement from the teacher or perhaps through an unwillingness to record their mathematics. In reporting the comparative use of the three methods of computation outside of school, however, students in Group H reported a much higher use of mental computation than students in Group L and Group M. Overall the difference in the use of computation between the school and non-school environments was more extreme for students at the highest level of mental computation.

The size of the numbers involved in a problem influenced the students' choices as to whether they would consider using mental computation, particularly for whole numbers (Section 5.4.5). Across the three groups, there was a steady decline in the proportions of students indicating they would choose to do a problem mentally, particularly between Group H and the other two groups. Between Group M and Group L the decline was related to specific items.

In the final question (Section 5.4.6), there were no significant differences between the three groups for reported use of mental computation with fractions, percents, for estimation activities or for calculator activities. Students at the highest mental computation level differed from the other two groups by indicating they more frequently used mental computation with whole number problems for addition/subtraction. There were no differences between the groups, however, for the operations of multiplication and division. For decimals there was a higher use of mental computation reported by students in Group H than for Group M and Group L.

In Chapter 6 the study advances to consider mental computation competence at the middle school level through task-based interviews with 55 students. The avenue through which to achieve this is a mathematical content focus on working mentally with fractions, decimals, and percents. This comprises the third phase of the study.

Chapter 6

Results (Phase 3): Strategies for working mentally with percents fractions, and decimals

6.1 Introduction

Like Chapter 5, this chapter builds on one of Shulman's domains of teacher knowledge – understanding learners' and their characteristics. It comprises Phase 3 of the study. In exploring the potential role of mental computation in strengthening numeracy across the middle years of schooling, this chapter presents the results of 46 student task-based interviews, which have a mathematical content focus devoted to fractions, decimals, and percents. Individually the three conceptual domains under consideration – fractions, decimals, and percents – have received extensive attention from mathematics educators. As discussed in the Chapter 2, however, the role of mental computation within these areas has not been investigated.

Across three sections, the level of each mental computation problem on the mental computation scale developed by Callingham and McIntosh (2002) is identified. Then the strategies that students used to solve mental computation problems with percents, fractions, and decimals with consideration of the characteristics of the mental computation problems are detailed. This analysis builds on the work of Caney and Watson (2003), who described mental strategies for some fraction, decimal, and percent problems that transferred from the more familiar whole number domain. This chapter further examines the strategies, reporting the frequency of strategy use among the students. Additionally, students' mental computation responses are also considered in relation to procedural and conceptual thinking as described in Section 3.5.4.3. Working procedurally involves strategies that are learned by rote and have no accompanying explanation that displays conceptual understanding of the processes taking place. Working conceptually then involves strategies in which students do appear to connect their knowledge of part-whole quantities and operations to solve problems mentally.

Data are reported for students in the High group (n = 24) and Middle group (n = 22), as defined by mental computation performance in Chapter 5. Generally the two groups were relatively even in the numbers of primary and secondary school students. Although a small number of students (n = 9) from the Low mental computation performance group made up the sample of interview participants, responses of these students are not considered. Students in Group L were generally not very successful with the part-whole interview questions and answered fewer interview questions than those students in the other groups overall. Data collected from these students was therefore extremely limited.

Student quotes are used throughout this chapter, with students identified by an individual number. Each number is preceded by a *P* denoting a primary student or an *S* denoting a secondary student, for example, *P98* and *S130*. Associated appendices are detailed in Appendix D.

6.2 Mental Computation Strategies with Percents

The first problem in the percent section of the interview was 50% of 24; a smaller number of students were also asked 50% of 21 as a variation of the first problem using an odd number. For the secondary students, the problem 50% of 21 was reserved for students who displayed difficulties with any further problems presented. Problems involving 25% and 75%, commonly referred to as "benchmark" percents followed, with smaller numbers of students also solving problems that involved 10% and multiples of 10%. Responses to the four problems involving the benchmark percents were considered conceptual in nature. Conversely, in problems concerning 10%, students produced some responses that were considered procedural, particularly with the inclusion of rule-based strategies. These are highlighted with the discussion for each of the relevant problems.

The problem 50% of 24 (at Level 5) involves a halving concept; three strategies by which students solved the problem are presented in Table 6.1. The majority of the students changed the representation of 50% to ½ (n = 30), using the half in an equivalent manner to 50%, for example, "Twelve. Because fifty percent is just half, all the time, just half" (P19). Only one student referred to the alternative decimal representation point five in his explanation, "Fifty percent is half or point five of it so you just have to halve twenty-four" (P63), although this was not listed as a separate strategy.

Table 6.1

Mental Strategies Associated with 50% of 24

Strategy description	Number of responses	
	Group H	Group M
Changed representation (50% = ½)	17*	13
Number knowledge related to 50%	0	3
Split by place value (50% of 20 + 50% of 4)	1	1
Total number of responses	18	17

Students also used their number knowledge related to 50% (n = 3). One student, for example, referred to the link with the operation of division, "I just divided it by two because fifty percent is half of it" (S118). Addition and multiplication facts were also referred to: "Half of twenty-four is twelve – I worked it out with my times table and I knew that twelve twos are twenty-four" (P24). A slightly different response involved a student describing 100 as a whole, for example, "Well you just halve twenty-four which is twelve. I just knew that fifty is half of a hundred so you are halving twenty-four" (P98).

Place value was explicitly described in the explanations of 50% of 24 provided by two students although both approached the problem differently. The first student started working from the left (tens first): "Twelve. Just halve the twenty which is ten and then add the four – oh – add the two which is half of four" (S120). The second student worked from the right (units first):

"Twelve. Just halve it. Half the four and half the two – I basically knew but then did the maths thing, with the half four, half two" (P43).

Table 6.2 details three strategies used by students to solve 50% of 21 (at Level 5). In describing responses to 50% of 21, students predominantly halved using a strategy that involved a place value split ($n = 9$), for example, "That'd be ten point five. Well you couldn't half twenty-one, you had to do half of twenty, is ten, then you have to half one by a decimal or a fraction so it would be ten point five" (P63). Again some students reported working both from the left (tens first) and some from the right (units first). The representation of the answer shifted between 10½ and 10.5. One student described his response using the context of money:

> In dollars it would be fifty percent so, it would be ten dollars fifty or ten point five. I rounded it down to an even number, to the nearest number which was easiest, which was twenty, so fifty percent of twenty add a half. (P91)

Table 6.2

Mental Strategies Associated with 50% of 21

Strategy description	Number of responses	
	Group H	Group M
Split by place value (50% of 20 + 50% of 1)	6	3
Number knowledge related to 50%	0	3
Changed representation (50% = ½)	2	0
Total number of responses	8	6

Three students were observed using their knowledge of near numbers in relation to 50%. The first student used 11 to solve 50% of 21:

> Ten and a half, ten point five. Because odd numbers don't really have half, so you have to take it from the nearest number cause it can't be eleven because eleven plus eleven is twenty-two, fifty percent of twenty-one would be ten and a half. (P37)

Two students used their knowledge of multiplication; one student gave a lengthy description with mention of the times tables, an excerpt of which follows for 50% of 21 (P31):

Interviewer:	[*Very long response time prompts interviewer to encourage the student*] What are you thinking about?
Student:	I know that two numbers go into twenty-one; I was just thinking what would go into that. Fifty percent of twenty-one would be ten and a half. I was just thinking of the times tables and like, I was just thinking of the sevens and sixes, nines and eights, that go into that number.
Interviewer:	Right, so you were looking for something that went into it?
Student:	Yeah and then I found out that it had like two of the one number into that – ten and a half." (P31)

A further two students described thinking of 50% as a fraction which is a similar strategy reported in relation to 50% of 24.

Moving from 50% to 25%, students were asked to solve the problem 25% of 80 (at Level 6). Four strategies were observed and these are detailed in Table 6.3. The first strategy involved students drawing on their number knowledge related to 25% (n = 18). In some cases students gave extra information that involved describing the link between 25%, ¼, and 100, for example, "Twenty-five percent is a quarter of a hundred, so I did quarter of eighty is twenty" (S131). An extension of this strategy involved students further explaining the link with division, for example, "I think it's twenty because twenty-five percent is a quarter of a hundred and so I just divided eighty by four and I got twenty" (S157), or referring to a multiplicative relationship, "Twenty-five percent of eight is two because it's four twos makes eight, so four twenty's makes eighty" (P93). There was one case where repeated counting was observed:

> Is that a quarter of it? Cause I went twenty, forty, sixty, eighty and like that's four, cause how you have one whole in the fraction, there's like

four in it, cause there's a quarter, half, three quarters, and then a whole. (P6)

Table 6.3

Mental Strategies Associated with 25% of 80

Strategy description	Number of responses	
	Group H	Group M
Number knowledge related to 25%	12	6
Repeated halving	5	7
Changed representation (25% = ¼)	3	2
Split by place value (10% + 10% + 5%)	1	0
Total number of responses	21	15

A second strategy that students used was a repeated halving strategy (n = 12), for example, "I went fifty percent of eighty is forty and then I made it fifty percent of the fifty percent, half of the half, and then went half of forty is twenty" (P62) and "Twenty. Because twenty-five percent is the same as a quarter and to find out a quarter you just halve it, then halve the half" (S149).

The third strategy involved changing the representation of 25% to a ¼ (n = 5); this was similar to that described for 50% of 24 although in this case students simply reasoned, for example, that, "Twenty-five is a quarter and a quarter of eighty is twenty" (P91).

The fourth and final strategy, used by only one student, involved splitting the 25% into parts (by place value) rather than splitting the 80, which is what the other examples essentially involved. The student responded, "Ten percent of eighty is eight and times that by two you get sixteen and then the five percent is half of eight, so it's four and add that to sixteen and get twenty" (S152).

The problem 25% of 80 was followed with 75% of 200 (at Level 7): four strategies are summarised in Table 6.4. In using number knowledge related to quarters (n = 13), students described division with a link to 100%, for example, "One hundred and fifty. Basically I divided two hundred into four because you know that it goes like twenty-five, fifty, seventy-five, and that it takes four twenty-fives to get to one hundred so you divide it by four" (S147). The link with multiplication was also reported, for example, "I did a quarter of two hundred first and then – that would be fifty – and then I just timesed it by three to make one hundred and fifty" (P44) and "If you split them up into quarters, then each quarter is fifty, three fifty's together" (S120).

Table 6.4

Mental Strategies Associated with 75% of 200

Strategy description	Number of responses	
	Group H	Group M
Number knowledge related to 75%	8	5
Repeated halving	6	5
Split by other quantity (75% of 100)	6	2
Changed representation (75% = ¾)	1	0
Total number of responses	21	12

The repeated halving strategy was used by eleven students, for example, "Seventy-five percent of two hundred, if it was fifty percent it would become a hundred but there's another twenty-five there to make it seventy-five and so you take, you get half of a hundred and add it on to the hundred" (P95).

Students also reported another strategy for 75% of 200 that involved halving and doubling but in a different fashion to the repeated halving strategy (n = 8). This strategy, involved a first step of working out 75% of 100, for example, "Because I know that seventy-five percent of a hundred is seventy-five and two hundred is twice as much as a hundred, so I just double the seventy-five" (P157). There was only one instance of a strategy where the student explicitly changed the representation from 75% to ¾: "Because seventy-five percent is three quarters and three quarters of two hundred is one hundred and fifty" (P101).

The problem 10% of 45 (at Level 7) was the first problem that did not involve one of the benchmark percents (50%, 25%, or 75%), and it appeared to be a more difficult problem for students. Three strategies for solving 10% of 45 are detailed in Table 6.5. Ten students employed a rule they had learnt in association with 10%, for example, "Four point five. Because this is a trick that you can do if it's like ten percent or something – move the decimal point forward one" (P48). Some students were able to include within their explanations that this rule was associated with dividing by ten, for example, "Four point five. Just dividing by ten so I come back to my decimal table with decimal point to the ones, tens, hundreds, and it just moves along" (P47). These responses were categorised as procedural, given their rule-based nature.

Table 6.5

Mental Strategies Associated with 10% of 45

Strategy description	Number of responses	
	Group H	Group M
Used a rule	9	1
Number knowledge related to 10%	6	1
Split by place value (10% of 40 + 10% of 5)	1	0
Total number of responses	16	2

In using number knowledge related to 10%, students (n = 7) referred to the link with division, for example, "Four point five. Because I found out – I divided ten by forty-five because ten percent of a hundred is ten so I just divided ten by forty-five and I got four point five" (S157). This example is interesting because the description of the operation is incorrectly stated although correctly calculated. A few students also referred to "how many," for example, "Four point five. Ten percent of forty-five, that's how many tens goes into forty-five – that's four times and remainder five" (S70). One student reasoned using his knowledge of near numbers forty and fifty to work with forty-five.

Student: Four and a half. Well it couldn't be four because it is too small and it couldn't be five because that's too big.

Interviewer: *What do you mean too big?*

Student: Like if it was five it would be fifty. If it was four it would be forty. Put it in the middle of them two. (P93)

One student used a place value split, for example, "Four and a half. Ten percent of forty is four and then I was working out ten percent of five which is a half, so four and a half (P101). No students referred to 10% as one tenth.

Three strategies were observed for the problem, 20% of 15 (at Level 7), as summarised in Table 6.6. The first strategy reported by nine students involved starting with 10% (half of 20%) and doubling the answer, for example, "Well you'd do ten percent which is one point five and then times it by two, so it's three" (S158). Instances of the students using a money context were also observed in association with this strategy: "Three. A dollar and a half is ten percent, and just double it" (S144).

Table 6.6

Mental Strategies Associated with 20% of 15

Strategy description	Number of responses	
	Group H	Group M
Split by other quantity (10% of 15 + 10% of 15)	8	1
Changed representation (20% = $^1/_5$)	4	0
Related number knowledge	2	1
Total number of responses	14	2

Four students changed the 20% to a fraction representation of $^1/_5$, for example, "Twenty percent is the same as a fifth and a fifth of 15 is 3" (P47). A more detailed example of a student working with $^1/_5$ is the following.

Student: OK I'm going to do it this way now, so I'm going to do twenty over one hundred and then two over ten and then one over five and I think I'm going to say three.

Interviewer:	So how did that help you work out, what did you do next? That's such a big jump to then know that it's three!
Student:	I went it's a fifth and then the fifth goes into fifteen, like five goes into fifteen how many times and that was just basically it! (P43)

The third strategy that students used was their related number knowledge, in this case linking 20% to 100% as "the whole," and working with multiplication or division. An example of a student working in this way is as follows.

Student:	Three. Well with, you get fifteen and twenty percent you just need five of those numbers and three times five is fifteen so it works up to be three.
Interviewer:	When you say five of those numbers what do you mean by that?
Student:	Like twenty times five makes the one hundred, so that's a whole. (P93)

Ten out of 21 students were successful in solving 30% of 80 (at Level 8); the responses are summarised in Table 6.7. Four students split 30% working out 10% first, for example, "I just did ten percent of eighty is eight and then timesed it by three" (P44). Two students worked with fractions representations, for example: starting with tenths: "So that's three tenths, thirty percent of eighty, one tenth of eighty would be eight, times three, twenty-four." Another student also persisted with a fraction strategy, "Well thirty over one hundred, then three over ten then, that's all I can go to! So thirty percent of eighty, three over ten and ten goes into eighty eight times, eight times three is twenty-four? Twenty-four!" Just one student applied a rule related to decimals starting with, "I knew that if you moved the decimal point one it would be eight, and times eight times three" (S48); again the application of a rule was considered procedural.

Table 6.7

Mental Strategies Associated with 30% of 80

Strategy description	Number of responses	
	Group H	Group M
Split by other quantity (10% of 80)	4	0
Changed representation (30% to $^{3}/_{10}$)	3	1
Used a rule	1	0
Split by other quantity (20% of 80 + 10% of 80)	1	0
Total number of responses	9	1

Finally, one student was observed solving 30% of 80 in a different way. First, he worked out 20% of 80, "Twenty make it twenty, divide eighty by five which would give me ... that's sixteen, yeah sixteen." He then went on to work out ten percent of eighty, "Then divide it by ten that time, ten that's eight – so add that on, it's twenty-four" (S160).

Four students were also successful in solving 40% of 64. Again, these students worked with 10%, then two students "timesed it by four," whereas the other two students described using doubling, for example, "I did ten percent of sixty-four and then doubled it and then doubled it again" (P44).

6.3 Mental Computation Strategies with Fractions

Addition. The problem $^{2}/_{7}$ + $^{3}/_{7}$ (at Level 5) was for the most part given to the primary students. Of the 15 successful responses to $^{2}/_{7}$ + $^{3}/_{7}$, most students reported adding the "top numbers." Several students pointed out that, for example, "I left them because they weren't higher, they weren't different and the three and the two didn't go over seven" (P41) One of these students was also asked how it would be different if the problem was $^{2}/_{7}$ + $^{6}/_{7}$; she replied, "It would have to go into mixed numerals so one and one seventh" (P63). There was some variation in how students talked about the "bottom numbers" or the "sevens." Some students reported, "You don't need to add them because they just mean the same thing" (P21) or that "The bottom numbers are equal" (P49).

For the problem ½ + ¾ (at Level 6) most students employed a strategy that essentially involved bridging to 1 (or the whole), as reported in Table 6.8. The difference in approach was based on whether students bridged from the ½ or from the ¾. Twenty-four students used two halves to make a whole, for example, "One and a quarter. Basically two quarters is the same as half so if you've got two halves that equals a whole and then you just add one quarter to that" (S147). Eight students worked up to a whole from three quarters: "So half the half which gives you a quarter and that makes a whole if you add three quarters, and you've got a quarter left over" (S114). One student further described a mental picture that supported this strategy.

Interviewer:	Let's try something like ½ + ¾.
Student:	I'm doing another circle.
Interviewer:	OK talk me through it then.
Student:	One whole and one quarter. I used a clock again.
Interviewer:	That's fine!
Student:	And just imagined it, shading in the adds, like adding three and just came up to one whole...
Interviewer:	So what did you do when you got to your one whole?
Student:	I imagined another one, another clock and put on the quarter. (P41)

A second strategy reported by students involved performing addition after converting the ½ to quarters and arriving at $^5/_4$, for example, "So two quarters and three quarters would be five quarters or one and one quarter" (S77). Students demonstrated an implicit understanding of equivalence.

Table 6.8

Mental Strategies Associated with ½ + ¾

Strategy description	Number of responses	
	Group H	Group M
Bridging (using ½ or using ¾)	15	17
Number knowledge related to equivalence	5	2
Total number of responses	17	19

Subtraction. For the problem 1 – $^1/_3$ (at Level 6), all students explained the relationship of $^1/_3$ to a whole (n = 19), for example, "Because three thirds makes a whole and if you take one off then it would be two thirds" (P48). In describing their responses two students also used mental pictures including one student who reported using a clock with a "Y" shape to divide the pieces before taking them away. The second student talked about the using pizzas, "You just break the 1 down into the three parts – say if it was a pizza, cut it into three pieces and you take 1 piece, you've got two left, which means you've got two thirds" (S118).

The problem 1¼ – ½ (at Level 6) was asked of five students and four were successful in solving the problem. Two students made the one half into two quarters and was then able to take the quarters off individually: for example, "Two quarters is a half so if I've got 1 and then quarter left over; well, then I could take that away and then take another quarter off" (P62). Two students reported a slightly different approach, for example, "Well you just take one quarter off and then you take half and then you add the quarter back on" (P19).

Multiplication. Four successful strategies for the problem 4 x ¾ (at Level 7) are detailed in Table 6.9. First was a strategy that involved splitting the 4 and preserving the ¾, although some variations were observed. Some students used a multiplicative split involving the 4, (2 x ¾) x 2. One student working this way, for example, reported, "Two times three quarters which is one and a half and then just timesed that by two to make it four times" (P93). As well some students referred to "doubling" the 1½. Students also worked with the 4 in a more distributive fashion, (2 x ¾) + (2 x ¾), for example, "Three. I doubled three quarters - well I timesed three quarters by two and I got one and a half, and added one and a half and one and a half to get the answer" (S157). Students were also observed adding two three quarters and then recognising the link with multiplication (or doubling), for example, "Three. I just added three quarters, two of them to one and half and just doubled that" (S66).

Table 6.9

Mental Strategies Associated with 4 x ¾

Strategy description	Number of responses	
	Group H	Group M
Split by other quantity (4 into (2 x ¾) x 2 <u>or</u> 4 into (2 x ¾) + (2 x ¾))	7	8
Split by other quantity (4 x ½) + (4 x ¼)	5	0
Algorithm with quarters	3	1
Bridging	0	3
Total number of responses	15	12

A second strategy involved preserving the 4 and splitting the ¾ according to the distributive property (4 x ½) + (4 x ¼); for example, "I did four times half which is two and then four times a quarter which is another whole number, two plus one is three" (S140). The explicit use of addition was also described in this strategy, "You add half four times that's two and then add three quarters, I mean one quarter four times, that's one and add them together, that's three" (S117). Similarly, a secondary student explained the counting process involved in this strategy: "Three. I took the three quarters and made all of them into half and put those – cause that's half, half, half, half, whole, whole, and now I've got two and there's four quarters left over, so there's a whole" (S65).

A third strategy reported by four students involved working through an algorithm with quarters, for example, "Four times three quarters, so three quarters times four so that's six, nine, twelve over four as an improper fraction. So twelve into four is three so it would be three, three whole, something like that" (S77). These responses were considered to demonstrate procedural thinking.

Bridging was a final strategy reported by just three students. The descriptions, however, were very detailed and revealed three slightly different versions of a bridging strategy. The first involved a form of bridging that appeared to stem from counting. In a lengthy description the student counted quarters keeping track of the whole numbers on his fingers:

> Well what I thought was, you start off with three quarters and then you add another three to make six quarters but then you add one of those quarters back to three to make a whole and so now you've four quarters to make a whole and then you've got two left. Then you add another three onto that, you add the other two out of the three to make two wholes sort of thing and a quarter and that's two times I think, three times! Then you add another one there [counts on with fingers] got that four times, so one, two, three – three whole. (P31)

Another student reported a bridging strategy that was supported by a mental picture, "I can imagine all the three quarters, four of the three quarters and I take one – it would be three wholes because you take one of the three quarters and put quarter back into each of the other three quarters and that's three wholes" (P95). The last bridging approach, from a secondary student, involved making all the three quarters into wholes and then subtracting, "Three. I just made them all into one and then take away one, take away four quarters" (S131).

Division. Many students gave 1½ as the answer for 3 ÷ ½ (at Level 7), mistaking the "by half" with "in half." This distinction was pointed out by the researcher with the chance for students to try again, or in some cases the question was rephrased as "three how many halves?" Table 6.10 details the five successful strategies that students described.

The strategy that the largest number of students reported involved knowing that dividing by a half would double the whole number. The detail of the responses distinguished students working procedurally and working conceptually. Approximately half of the students who used a doubling rule simply imparted, for example, "If it's half on one side, you just double this number over here" (S70). Other students, however, revealed more of the thinking behind this process. An example of one student working this way was, "Because it is one whole and you take the halves, and there's two halves to each whole, so it doubles the amount of wholes" (P95).

Table 6.10

Mental Strategies Associated with 3 ÷ ½

Strategy description	Number of responses	
	Group H	Group M
Doubling (Procedural and Conceptual)	9	6
Split by other quantity (1 ÷ ½ x 3)	9	2
Repeated addition (with ½'s)	1	8
Number knowledge related to whole number referent	1	0
Used algorithm	0	1
Total number of responses	20	17

Split by other quantity was the second strategy observed and involved spitting the three in ones or wholes, identifying that each whole has two halves and then multiplying it by 3, for example, "So there's two halves in a whole, which would be two for each one, so I timesed the two, three times" (P48). Often students interchanged talking about "timesing" and "adding."

Students also reported a third strategy that involved repeatedly combining halves in an additive manner. A secondary student reported, for example, "Basically you just see how many halves make a whole so you've got like two halves make one and then you add another two halves to get two and then another two halves to get three" (S147), as well as counting, "To halve one whole is two, so you just two, four, and six" (P98).

One student demonstrated a unique approach working with a whole number referent of 300, also describing 30 in his thinking:

Interviewer:	Try this one, $3 \div \frac{1}{2}$.
Student:	Six. Because say it was three hundred, fifty into three hundred goes six, if it's like half.
Interviewer:	Is that what you were thinking of when you worked out the six there?
Student:	No, I just go five into thirty, six. I said three hundred ... just easier. (P49)

There was just a single example of a secondary student, who when prompted to give more detail, described a type of algorithm rule or shortcut:

Interviewer:	$3 \div \frac{1}{2}$.
Student:	That would be six.
Interviewer:	How does that work?
Student:	Because I just figured out how many times a half goes into three.
Interviewer:	Can you break that down even more?
Student:	I don't know if it would work with all of them, but to divide the whole number by – not divide, times the whole number by the umm, denominator, I think, the one on the bottom. I think that would work with those ones, yeah. (S157)

Again, these two final responses, although quite different to each other, were considered procedural.

Two problems that used "of" as the operator were also presented to students, $\frac{1}{2}$ of $\frac{1}{3}$ (at Level 7) and $\frac{1}{2}$ of $\frac{3}{4}$ (at Level 7). Those students who solved the problems successfully demonstrated an understanding of equivalence through doubling and halving. The problem $\frac{1}{2}$ of $\frac{1}{3}$, answered successfully by 16 students, most students simply reasoned, for example, "Well two sixths is the same as one third so I just halved it into one sixth" (S140); others reported doubling the 1 and the 3 from the third first. A few students described mental pictures in working through the problem, for example, "I imagined a third, because it is sort of a Y shape, I imagined a third filled in and I cut that in half because if it was a third, half of it would be one sixth (P95)." Some students appeared extremely uncertain about their answers (which were correct), but their explanations were often more detailed than those from students for whom the problem appeared easier. The following is an excerpt from a secondary student:

Interviewer: ½ of $^1/_3$?

Student: Would it be one sixth or something? Don't know!

Interviewer: Is there a way you can check that? How are you thinking about it?

Student: Well you couldn't really do half into a third.

Interviewer: So what do you have to do?

Student: So if I did the sixth it might to into it easier. That's an even number.

Interviewer: And does it work?

Student: I think so. (S131)

Fewer students were asked to solve ½ of ¾ (n = 13) and seven students were successful. The students reported "making it" or "converting it" (the ¾) into $^6/_8$. As was the case for ½ of $^1/_3$, some students were more explicit in their explanations as to how they did this, for example, "Three eighths. I double them and halved the fraction from there. [*So you doubled the…*] Both of the numbers and then halved one of them" (P48). One student described a mental picture that supported her thinking that added an interesting dimension to this problem.

Interviewer: What about ½ of ¾?

Student: I wouldn't have a clue!

Interviewer: How do you think you might be able to start with that one?

Student:	With a whole I guess, cut it into quarters...its three eights.
Interviewer:	How did you come up with that?
Student:	Well I had my picture of three quarters and then I halved.
Interviewer:	With the rectangle? [*Student referred to using a rectangle in a previous question*]
Student:	Yeah – and then I halved every quarter and then I got rid of one of the quarters and then saw what was in the middle, which was three.
Interview:	So you were actually turning it into...
Student:	Eighths. (P101)

6.4 Mental Computation Strategies with Decimals

Addition. Fifteen students solved the decimal problem 0.25 + 0.25 (at Level 5); see Table 6.11. The majority of students changed the representation of the problem to whole numbers, which involved, for example, explaining the problem: "Twenty-five plus twenty-five is fifty."

Table 6.11

Mental Strategies Associated with 0.25 + 0.25

Strategy description	Number of responses	
	Group H	Group M
Number knowledge related to whole number referent	5	8
Used a rule	0	1
Used written algorithm	1	0
Total number of responses	6	9

The answers that students gave indicated that language was important, as three types of answers were scored as successful, point five, point five zero and point fifty. Further questioning revealed some interesting aspects of knowledge regarding working with place value and zeros in the decimal domain.

Student:	Point five zero or point five because it is just doubling, twenty-five plus twenty-five basically. Sometimes I say point five sometimes I don't. Sometimes I just feel like it.
Interviewer:	Does that zero matter?
Student:	No because you can have it in anything, you can have zero, zero, zero it doesn't matter. (P49)

Other times further questioning revealed the student did not necessarily have a good understanding of place value with decimals. When asked about the additional zero in point five zero, one student replied, "It means it's in the tens and not a unit" (P13). Just one student had leant a rule from a teacher of a previous grade, "Point five. I just went like twenty-five plus twenty-five that's fifty so you take the zero off and put that in front and then it's zero point five. That's what we've learnt; I learnt that last year" (P6).

There was only one student (P63) who solved the problem 0.25 + 0.25 differently. Although this student followed an algorithmic procedure making use of place value, he changed the representation of the problem to fractions: "Point five. Because the zero point zero five, you add them together and that makes one tenth, so add it on to one of the twos which makes three and then you do the two plus the three tenths makes five tenths." This response was considered to show clear conceptual understanding of both place value and the connection to fractions. A procedural version of this strategy would have involved a more traditional use of a written algorithm based on the use of carrying and positioning of the decimal points. No students referred to the link between 0.25 and ¼.

The problem 0.5 + 0.75 (at Level 7) elicited a number of strategies that for most of the students involved splitting the 0.75 in one of two ways: by 0.7 and 0.05 or by 0.5 and 0.25. Strategies are described in Table 6.12. Splitting by place value (0.7 and 0.05) was more frequently used than splitting by other quantity (0.5 and 0.25). In using both strategies, however, students tended to emphasise either whole number knowledge or decimal place value knowledge, although both were considered procedural in nature.

Students who described working with 50 and 70 as 120 were considered to be splitting by place value based on connections to whole numbers. Conversely students who described working with 5 and 7 as 1.2 were considered to be splitting by place value based on place value knowledge. An example of each for 0.5 + 0.75 follows:

> Student: One point two five. Seventy and fifty makes one hundred and twenty plus five and it was the point before so now it is up to a certain point it goes over to one point two five. (S114)

> Student: Would be one point two five. Well I just added seven to the point, well point seven to the point five to get one point two and just add like point zero five. (S152)

Table 6.12

Mental Strategies Associated with 0.5 + 0.75

Strategy description	Number of responses	
	Group H	Group M
Split by place value (0.7 + 0.05)		
Whole number knowledge	4	2
Decimal place value knowledge	3	1
Split by other quantity (0.5 + 0.25)		
Decimal place value knowledge	2	2
Number knowledge related to whole number referent	3	3
Changed representation (0.5 = ½ and 0.75 = ¾)	2	1
Used written algorithm	1	1
Total number of responses	15	10

Four students demonstrated splitting the 0.75 into a different quantity, for example, "Cause point five and point five it gives you a total of one and the plus point five gives you one point two five" (S160).

The responses of six students were classified as using whole number knowledge. Like responses to the first problem 0.25 + 0.25, students simply reported adding "Fifty and seventy-five" for 0.5 + 0.75, with little indication of how they linked this interpretation to decimal place value.

Three students changed the representation of the problem from decimals to fractions: retaining a decimal answer:

> One point two five. I just did it as sort of like a fraction, as half, I mean three quarters add half and then so it's one and a quarter and then a quarter is equal to point two five, so it's one point two five. (P44)

A written algorithm procedure was the strategy behind the responses of two students. Both students, however, described the place value positions with reference to fractions. One of the students followed on from the first addition problem 0.25 + 0.25, again describing an algorithmic procedure with reference to fractions: "One point two five. Leave the hundredths five where it is and just add five and seven which is twelve and move the one into the units and the two stays in the tenths" (P63). Although both students lapsed into using the language of whole numbers, this response was considered conceptual with their apparent understanding of place value and fractions.

The problem 6.2 + 1.9 (at Level 7) required students to work with both the whole numbers that come before the decimal point and the following decimal digits. Although a number of different strategies were suggested by the students, and are described in Table 6.13, the majority of responses were focused around splitting the two numbers according to the decimal point, "I added one onto six, so that's seven and then I added the two and the nine and that's one point one and I added that onto the seven" (S158). In most cases the students started working from left with the whole numbers, although a few students started working from right with the decimals. One response stood out as being a little different and was considered conceptual as the student employed a strategy based on the principle of commutativity, "I went six point nine plus point two which gave you seven point one plus a whole, gave me eight point one" (S160). She rearranged the numbers in the problem so that the first number comprised the larger whole number and larger decimal value.

Table 6.13

Mental Strategies Associated with 6.2 + 1.9

Strategy description	Number of responses	
	Group H	Group M
Split by place value (6 + 1) + (0.2 + 0.9)	18	16
Split by other quantity (1 + 0.9)	0	3
Bridging (6.2 + 2) – 0.1	1	0
Total number of responses	19	19

Three secondary students used a strategy whereby one number in the problem is split and added cumulatively, for example:

> Eight point one. I added the nine, point nine first to get seven point one and then just added one to it, point nine to the six point two, get seven point one, and then added just one to seven. (S152)

Just one student reported a form of bridging: "Eight point one. I made one point nine, two and then six point two add two is eight point two and then just took one off to make it eight point one" (P44).

An additional problem, 0.19 + 0.1 (at Level 7) was given to just seven students, six of whom answered successfully. Although most students used their whole number knowledge, for example, "it's just like nineteen plus ten," one student reported, "Point two nine. Because point one is one tenth so it is just ten plus ten, add on the nine" (S63). This student is the same student who reported using fraction representations for the earlier addition problems.

Subtraction. Of the 19 successful responses to the problem 1 – 0.4 (at Level 6) there were two different whole number representations reported by students. Referring to the 1 as 10 was the main strategy, for example, "I just put the one as a ten so I put it as a whole decimal number, just moved it over from the other side of the thing which put it into a ten take four which equals point six" (S118). The second whole number variation involved thinking of the one as one hundred and was used by just one student: "I could think of it as one hundred and the point four as forty and so there's got to be sixty left" (P21).

To solve the problem 4.5 – 3 (at Level 6), just one student used a whole number representation: "If you made it forty-five and thirty you just do forty-five take thirty is fifteen and then put a point in between the one and the five" (P44). The majority of the students ($n = 22$) used a strategy that involved attending to the whole numbers first, 4 – 3, and then working with the 0.5, effectively splitting the 4.5 by place value: refer to Table 6.14.

Student:	One point five. Because it is three ones is a whole number and taking three off four, because four is a whole number which leaves you one point five.
Interviewer:	So what were you doing with the point five then?
Student:	Nothing, you don't need to do anything with it. (P49)

Table 6.14

Mental Strategies Associated with 4.5 – 3

Strategy description	Number of responses	
	Group H	Group M
Split by place value	11	11
Changed representation (4.5 = 45)	1	0
Total number of responses	12	11

All strategies recorded for the two decimal subtraction problems were considered procedural. Although three students referred to fractions during their explanations their responses were not considered conceptual in the same way that several of the addition problems were.

Multiplication. Twenty-seven students were successful in solving 3 x 0.6 (at Level 7). Most students used the number fact "3 x 6 is 18" as their immediate response. It was how the students justified where the decimal point should go that distinguished responses, although all responses were considered procedural (refer to Table 6.15). When questioned about the position of the decimal point, students gave many varied explanations. A number of students explained how it "becomes" a whole number, or why "it is more than 1." In doing this the whole was often referred to as being "a ten" or "in the tens," for example, "One point eight. So three sixes are eighteen, so it is over ten so every ten is another group of tens before the point" (P95). Similarly, several students used a representation of a hundred in the same way to mean a whole, for example:

> That would be one point eight. I thought of the point six as sixty and I timesed it by three to get eighteen and then I put the decimal point in the middle. Since I got one hundred and eighty if there's anything over one hundred that has to go before the decimal point. (S149)

Another student (S152) reasoned in the following way.

Student:	One point eight. I just times three by six and got the one point across from the six, the decimal point goes in between.
Interviewer:	Now that decimal point, how do you know that it goes in the middle there?
Student:	Cause once a number just after the decimal point, once it gets higher than nine, it becomes a whole point something.

Table 6.15

Mental Strategies Associated with 3 x 0.6

Strategy description	Number of responses	
	Group H	Group M
Used a rule	14	6
Number knowledge related to whole number referent	0	1
Used a rule (with reasoning)	2	2
Doubling/addition	1	1
Total number of responses	17	10

Six students were considered to be working more conceptually with the problem 3 x 0.6. Four students initially gave responses that were based on a decimal rule. These students were, however, able to reason further by eliminating alternative answers that were not appropriate, for example, "I just sort of knew it because it wouldn't be eighteen point zero or zero point one eight" (P44) and "Because if it didn't it would be zero point one eight which is less than what we started with, or eighteen which it just wouldn't be!" (S158).

Finally, two students reported a strategy that involved doubling and adding, also considered to demonstrate working conceptually: "One point eight. Just added six, doubled six – one point two – and then add another point six on that" (S66). Another student extended an initial response to reason: "You could just add up like the sixes, like six, twelve so you've still got the one and you've got the decimal point and then the six plus" (S153). Interestingly many students were observed making several attempts at answering this problem before settling on 1.8, these included, "point eighteen," "point one eight," and "one point eighty."

For the problem 0.6 x 10 (Level 7), the majority of students referred to a rule in much the same way as reported for the problem 3 x 0.6; refer to Table 6.16. Two rules, however, were observed as students reasoned about the appropriate place for the decimal point. The first rule emphasised a learnt procedure linking the decimal point to place value:

Student: It is like behind the decimal and then you just times it by ten and it goes above the decimal, like in front of it. (S117)

Or alternatively:

Student: When you are dividing by ten you move the number down a column so the six would go into the hundredths; but if you are timesing by ten, it goes up to the left in the units column so it would just be six point zero. (P63)

A different type of rule focused on "adding zeros," for example, "You times it by ten so you put a zero after it. So you replace where the six is with the zero and put the six before the decimal" (S153).

Table 6.16

Mental Strategies Associated with 0.6 x 10

Strategy description	Number of responses	
	Group H	Group M
Used a rule		
Decimal rule related to place value	9	3
Zero rule	5	3
Number knowledge related to a number fact	2	4
Used a rule (with reasoning)	0	2
Total number of responses	16	12

Six students used their whole number knowledge referring to the number fact 6 x 10 and were unable to offer any further explanation relating to the decimal point.

Two students were classed as working conceptually as they both went on to explain the use of a rule, for example, "Well point sixty is the same as point six so it can't be that and sixty would be too big cause point six times ten wouldn't be that much" (S140). One student (S118) worked in a similar way:

Interviewer:	What about this one, 0.6 x 10?
Student:	Which would be six. I've dropped the decimal point which would be six times ten which equals sixty and then you go back and put the decimal point in which would be after the six.
Interviewer:	Why does it have to go after the six?
Student:	It is a bit unrealistic saying when it was point six to start off, it would be sixty to finish sort of thing. You know it has got to be bigger than 1 but not all that much bigger.
Interviewer:	Right.
Student:	It is not into like the twenties and thirties it would be sort of in between ten and zero sort of thing.

Division. Although the problem 3 ÷ ½ (at Level 7) was given to most students in the fraction section, the problem 3 ÷ 0.5 (also at Level 7) was presented to some students as part of the decimal section; strategies are detailed in Table 6.17.

Table 6.17

Mental Strategies Associated with 3 ÷ 0.5

Strategy description	Number of responses	
	Group H	Group M
Changed operation (to multiplication)	8	3
Whole number knowledge	1	0
Total number of responses	9	3

Nine students were considered to be working conceptually in solving 3 ÷ 0.5 by changing the operation of division to multiplication referring to, for example, "doubling" or "how many halves." Additionally four of the nine students also changed the decimal representation to fractions: "That would be six I think. You've got the three and then you've got the point five, which is basically half, and it takes six halves to make three" (S63).

Just one student solved the problem differently using whole number knowledge in a procedural fashion: "Six. Because it is the same as using thirty, how many fives in thirty...I just go back to normal numbers instead of decimals because that's like complicating and then you know where the decimal point is" (P49).

6.5 Chapter Summary

The chapter focused on the successful strategies middle years students used to solve part-whole problems mentally. In Table 6.18 the prevalence of strategies across fractions, decimals, and percents is considered. In relation to the work of Watson and Caney (2004), some strategy descriptions have been refined. Two of the strategies described were common across each of the three domains (shown in bold text).

Table 6.18

Overview of Strategies Across Percents, Fractions, and Decimals

Strategy description	Percents	Fractions	Decimals
Changed representation	Yes	No	Yes
Related number knowledge	Yes	Yes	Yes
Split by place value	Yes	No	Yes
Split by other quantity	Yes	Yes	Yes
Doubling/halving	Yes	No	Yes
Bridging	No	Yes	Yes
Used a rule	Yes	Yes	Yes
Used written algorithm	No	Yes	Yes

Changed representation. In this study the students changed the representation of percent and decimal problems but not fraction problems. Fraction representations, were however, cited frequently by the students in relation to problems involving percents and decimals, particularly the intuitive use of halves and quarters. Decimal values were rarely referred to outside of the decimal problems. There was some use of "point five" in providing initials answers although this appeared to be as an alternate version of a half rather than as an individual strategy. For the percent problems, the use of this strategy decreased as the percent value moved away from the benchmark percents.

Related number knowledge. This strategy is perhaps less defined than some of the other strategies. At times "knowledge" was specific to one of the areas being considered, for example, equivalence with some of the fraction problems. At other times the three areas shared "themes," for example, in relation to the percent problems, students' related number knowledge often involved discussion of links with the concepts behind the four operations and the relationship of the whole.

Split by place value. This strategy was observed for problems involving percents and decimals which is not surprising as fraction representations do not explicitly denote place value relationships. For the percent problems students were observed splitting either the operand, for example, 50% of 20 and 50% of 4 or the percent operator, for example, 20% of 80 and 5% of 80. Similarly, for the problem 0.5 + 0.75 students were observed splitting the 0.75 into 0.7 and 0.05. For one of the decimal problems that elicited a place value split strategy, 6.2 + 1.9, it was the decimal point itself that encouraged the split and students were observed splitting both operators or preserving one and splitting the other.

Split by other quantity. In splitting by a quantity not specified by place value students used number relationships in percent, for example in 75% of 200, splitting the 200 into 100 and similar examples were observed for some of the fractions problems, for example in 4 x ¾, students split both the 4 or the ¾. An example of students splitting by other quantity for decimals, the problem 0.5 + 0.75 elicited the use of 0.5 and 0.25. The strategy was, however, similar to split by place value in that generally one operand in the problem was preserved.

Doubling and halving. For percent students were observed using repeated halving to solve problems comprising 25% and 75% in relation to 50% as a benchmark value. The strategy was also used implicitly to solve problems based on 10%, such as 20% of 15, which was classed as split by other quantity. In relation to decimals, doubling was observed for the problem 3 x 0.6 whereby students started working through the problem by doubling 0.6.

Bridging. Students were observed using bridging to solve fraction problems and essentially this involved bridging to one or a whole. Just a single example of bridging was observed for the decimal problem 6.2 + 1.9 where the student made the 1.9 a 2. This strategy was not observed for percent problems.

Used a rule. The application of rules for the percent problems generally involved "moving" the decimal point and was only observed for the higher level problem involving 10%. Similarly, there were only a few instances where students used a rule to solve a fraction problem. This strategy was more common with the decimals problems although interestingly the level of explanation generally determined whether students were working procedurally or conceptually.

Used written algorithm. Generally, this strategy did not feature greatly across the three areas: it was not observed at all with the percent problems.

Visualisation was one of the other strategies described by Watson and Caney (2003). Although a number of examples of students describing pictures to solve problems mentally were observed, this was not listed as an individual strategy as generally the use of mental pictures appeared to be supplementing other strategies. As well, working from the right (or from the left) does not appear to be as important when working with part-whole numbers as it does when working with whole numbers.

Overall, the strategies used by the students to solve percents and fractions were considered to be more conceptual in nature than those reported for solving the decimal problems; the number of responses in each category are reported in Table 6.19. Although few examples of rules or use of algorithm procedures were observed it was the procedural use of whole number representations that shaped this finding. This shows that for the areas of fractions and percents, mental computation is one good avenue to take in pursing assessment of conceptual understanding as the strategies themselves would seem good indicators of some key part-whole concepts. This does not seem as clear-cut in developing decimal understanding. Here, the strategy itself does often not reveal understanding of some the students' ideas behind concepts such as place value and manipulation of the decimal point. In many cases, it was only further questioning that prompted students to reveal a little more of their "number sense" thinking or not as the case may have been.

Table 6.19

Summary of Procedural and Conceptual Responses for Percents, Fractions, and Decimals

Strategy description	Group H		Group M		Total
	Procedural	Conceptual	Procedural	Conceptual	
Percents	10	97	1	54	162
Fractions	7	76	3	58	144
Decimals	91	19	67	11	188

Throughout this chapter, strategy use was reported separately for students identified as Group H and Group M. In general the numbers of students successful in solving individual problems was uneven making direct comparisons between the groups difficult. Overall there appeared to be very little difference between the two groups in regard to strategy use, which suggests that once students can solve problems involving part-whole numbers choosing to use a particular strategy is not linked to mental computation ability. For those problems where the number of responses across the two groups was similar, however, a trend in the results is worth reporting. Across the fraction and decimal problems, generally responses for the problems involving addition or subtraction were similar for both groups of students in that responses tended to cluster around the most common strategy. This included $\frac{1}{2} + \frac{3}{4}$, 0.25 + 0.25, 6.2 + 1.9, and 1 – 0.4; the exception was for 0.5 + 0.75 where both groups used a number of different strategies. For a number of the problems involving multiplication or division the groups did differ in their strategy use. For example, students in Group H used a doubling strategy or split by other quantity strategy to solve the problem $3 \div \frac{1}{2}$. Students in Group M predominantly used a repeated addition strategy. A similar pattern was observed for 4 x ¾ and 0.6 x 10. It could be that strategy use is related to the sophistication of the operation and is likely to be associated with the higher level items, particularly Level 7.

In the final results chapter – Chapter 7 – the spotlight returns to the teachers as data from the seven key teachers who participated in an interview session is presented (Phase 4). Although five of Shulman's teacher knowledge domains are considered, how teachers position fractions, decimals, and percents in relation to mental computation is a particular focus of the chapter.

Chapter 7

Results (Phase 4): Key Teacher Interviews

7.1 Introduction

This chapter reports on Phase 4 of the study, where key teachers participated in an interview session following the completion of the two student data collection phases. The data generated from the teacher interviews is presented in a similar fashion to the results of the teacher questionnaire (Chapter 4), with the interview questions organised by Shulman (1986; 1987). Five elements of the framework are addressed: knowledge of educational ends, purposes, and values, knowledge of contexts, curriculum knowledge, pedagogical content knowledge, and knowledge of learners' characteristics. The selection of the eight key teachers was described in Section 3.5.1.2. The sample comprised four primary teachers and four secondary teachers, with each teacher located at a different primary, secondary, or district school. As well the teachers represented three different levels of professional development: extensive, moderate, and limited. One teacher did not participate in the interview session; therefore the responses of seven key teachers are discussed in this chapter guided by the research question: how do teachers position the teaching and learning of fractions, decimals, and percent in relation to mental computation? The professional backgrounds of each of the key teachers are outlined in Appendix E.

This chapter refers to earlier teacher data reported in Chapter 4 and draws on student data reported in Chapter 6. Teachers were asked specifically to comment on mental computation in relation to fractions, decimals, and percents to reflect the emphasis of the student interviews (Chapter 6).

7.2 Knowledge of Educational Ends, Purposes, and Values

As part of the mental computation questionnaire, teachers were asked to provide reasons as to why they value mental computation. In the interview the key teachers were asked to consider the position of mental computation in relation to numeracy. Additionally, some of the teachers were specifically asked how they would like their students to leave their class at the end of the year, in terms of the students' numeracy experiences.

Overall, the primary and the secondary teachers expressed a different emphasis on mental computation in relation to numeracy. For the secondary teachers in particular, mental computation had a clear mathematical value for students in terms of having "a choice of methods" and as an avenue for "continual revision and strengthening of number facts" (Teacher C, secondary). Teacher D (secondary) emphasised the role mental computation can play in developing conceptual understanding in mathematics. Teacher B (secondary) listed a number of aspects of mental computation that he considered important. Like Teacher C, Teacher B started with a focus on mathematical understanding, for example, "understanding of the basic workings of numbers" and "thinking about the reasonableness of solutions and answers," and then moved on to the issue of mental computation as an avenue for communication: "I hope my students can communicate their maths to someone else fairly coherently, and talk about maths with other people." In relation to his Grade 7 class, Teacher E (secondary) said he specifically wanted his students to be "confident and competent and working with a range of whole number strategies," as well as understanding the links between fractions, decimals, and percents in terms of "equivalent representations."

Responses provided by the primary teachers emphasised mathematical understanding but also considered mental computation in relation to the broader territory of numeracy. Teacher A (primary), for example, considered mental computation as "an avenue for promoting success and enjoyment [in mathematics] based upon strategies that they [the students] have constructed their own way" and for Teacher G (primary) mental computation was simply about providing "the opportunity and the time to think [mathematically] for themselves." Teacher F (primary) focussed on the idea of developing confidence through mental computation. She also discussed the issue of efficiency and using mental computation to encourage students to be critical of the answers they get, and in choosing strategies.

7.3 Knowledge of Educational Contexts

The teachers were asked to provide further comment on the question: how do you think mental computation might change as students move from primary to secondary school? This question was originally part of the teacher questionnaire (refer to Section 4.4).

A theme shared by three teachers, two primary and one secondary, was that mental computation would never be given enough emphasis unless it was backed by a whole school approach. For example, "With a whole school approach there would be less resistance from students if it [mental computation] is just what they're used too" (Teacher G, primary). Teacher A (primary) expressed a similar sentiment: "It's a better result when it comes up from the primary and it's permissible to go down that track and continue. Developing a mental computation culture from scratch is hard." As a secondary teacher, Teacher E made several comments. First, he acknowledged the challenge of working mentally with secondary students:

> The older the kid is, the harder I've found it and then it's an absolute ongoing tension. For high school kids it seems like a real safety net, going back to the algorithms and it's because they see it as *proper* maths and what's regarded as the *really important stuff*. This perception is really challenging to break down.

Second, he reiterated his comments which he detailed at the end of his questionnaire:

> At a more systematic level we need to engage teachers in questions about what they believe to be the "big picture" concepts in mathematics.

> From here and this re-evaluation we need to focus on deeper understanding and teaching for this. I have no doubt that mental computation is a major *big picture* item. A major up-skilling of teachers – especially middle and upper primary to lower secondary – on mental computation is needed desperately. (Teacher E)

7.4 Curriculum Knowledge

Fractions, decimals, and percents. As part of the questionnaire, teachers were asked to consider mental computation across the four operations with basic number facts and multi-digit numbers, and also with fractions, decimals, and percents (refer to Section 4.5.2). During the interview session the teachers were asked to predict what they thought the collective pattern of teachers' responses might be for that question and provide further comment.

One of the primary teachers (Teacher A) speculated that teachers working to develop mental computation skills would "definitely be weighted towards the whole number domain." This comment reflected the comments of all the teachers interviewed, for example, "It wouldn't surprise me if the emphasis on mental computation is quite low across the decimals, fractions, and percents area" (Teacher C, secondary), adding that many people think of "number as whole number." Teacher D, a secondary teacher, expressed a similar view, although he emphasised the confidence of teachers in relation to this area:

> I would guess that mental computation might decrease for fractions, decimals, and percents. For me working mentally is a whole approach to number – it is not something I pick and choose to do as such. I guess this could be related to the confidence of the teachers with fractions, etc? I think it might be something some teachers avoid as they don't feel that confident working mentally in this area.

The same issue of teachers' confidence was also expressed by Teacher E (secondary). As a secondary teacher he speculated that teachers interpreting mental computation as automatic response may indicate frequently working with mental computation when really "they don't trust their own mental facility particularly with fractions and decimals." He went on to comment:

> I tend to go a lot more slowly with fractions and decimals than I would if you're doing work with whole numbers. I really like hearing kids say *I changed them to decimals* but a lot of the time the kids get to me in high school and generally they have to ask permission if they can work this way, asking *am I really allowed to do that?* That's an interesting sort of comment to hear.

One of the other secondary teachers (Teacher B) also suggested that although teachers at the primary level would work "a little" with whole numbers, they would "rarely" work mentally with those numbers that involve a part-whole relationships. Generally, however, the teachers did not direct comments specifically at the primary or secondary level.

Following discussion of the first question, some of the teachers were presented with a quote from the *A National Statement on Mathematics for Australian Schools* (Australian Education Council (AEC), 1991): "People need to carry out straightforward calculations mentally, and students should regard metal arithmetic as a first resort in many situations where a calculation is needed" (p. 104). They were then asked to respond to the quote, by reflecting on fractions, decimals, and percent.

The responses of two of the secondary teachers were interesting. First Teacher B emphasised the link between the whole and part-whole number domain:

> It shouldn't be different – there's no reason you can't extend basic skills of whole numbers and apply to new concepts – maybe we need to be more explicit about this? Some kids can't even see where concepts like fractions and decimals (other than money) fit in society let alone consider solving problems in their heads.

The comments of the second teacher, Teacher D (secondary), identified with the quote in a different way:

> I tell them all the time that *a real skill* is being able to work things out themselves and not rely on a calculator or mobile phone, but this is not a

> message that you just pass on, it needs to be a message they get all the time in maths, every year, from every teacher.

Developing written, calculator, and mental computation. As part of the curriculum knowledge section in the questionnaire, teachers were also asked to estimate the comparative amount of time they devoted to developing written, calculator, and mental computation skills. In the interview, however, the teachers were asked if working with specific types of number (e.g., fractions) would have an effect on their responses.

Four of the teachers responded that they would not change the time allocated to the three methods of computation for different types of numbers. The original responses of these teachers from the questionnaire were, however, quite diverse. Two teachers (one primary and one secondary) indicated the highest amount of time was devoted to written computation. Of the other two secondary teachers, the first preferenced mental computation although responses were virtually even for written and calculator computation. The other teacher reported a much higher percentage of time devoted to mental computation with almost no time allocated to written computation.

Three of the teachers acknowledged they would adjust the time allocated to the three methods of computation, particularly fractions. Teacher G (primary) remarked that she would actually "raise" the percent she had previously allocated to mental computation for fractions "because when you write fractions down too early, that's when they get muddled." Teacher F (primary) responded similarly with her response linking back to the first interview question:

> Actually I would probably even put mental computation a bit higher for fractions. I've found that if you do the pen and paper without the mental computation skills and without understanding what they're doing, they loose it [mental skills]. You don't quite know what they are understanding unless they're telling you and talking about it.

Both of these primary teachers had participated in extensive professional development involving mental computation.

Teacher C (secondary) was surprised by the high proportion of time some teachers allocated to working mentally. She commented that for fractions, decimals, and percents her mental computation work would probably decrease slightly from that which she allocated in her questionnaire. She reported increasing written computation from 60% to 70%, and decreasing calculator work from 25% to 20% along with mental computation from 15% to 10%.

7.5 Pedagogical Content Knowledge

The teachers were asked to report some of their teaching practices associated with mental computation in the questionnaire. This included mental computation teaching activities and assessment of mental computation. Both aspects were addressed in the interview although again teachers were asked whether working with fractions, decimals, and percents would influence their responses.

Mental computation activities. Teacher C (secondary) outlined what appeared to be a more traditional approach to mental computation compared to the other teachers, describing a session of questions and answers in which students worked individually. When asked about the level of discussion involved, she commented that the class really only discussed problems if a lot of students were making the same errors. In this scenario, students who got a correct answer would tell the class how they had worked it out.

Most of the teachers focussed on discussions with their students. One of the primary teachers (Teacher A) tied mental computation to problem solving, which he identified as "the crux of mathematical investigation." He also highlighted real life problems and open-ended questions with "multiple pathways" as the activities on which he founded mental computation. He also added that the verbal aspect of mental computation was important to develop a kind of "self talk."

The response of Teacher B (secondary) was similar in that he nominated open-ended questions with more than one answer and real life problems as the most common mental computation activities that he used. He added, however, that he probably did not do enough visualisation when it came to fractions, decimals, and percents that would support the computation aspect, particularly in working towards written computation.

In relation to games, Teacher G (primary) reflected that she "could do more with games," focussing more on activities involving open ended challenges and lots of class discussion of mental strategies. She commented, "when I first started teaching I had heaps of games that were great for mental but it was very much *that's not what we should be doing* – that was 18 years ago and now I wish I'd kept some!"

Assessment of mental computation. Like the previous question on mental computation activities, most of the teachers focussed on the use of observation and discussion with their students as the basis for assessment. The only teacher to convey a different and more traditional approach to assessment was again Teacher C (secondary). Following on from her explanation of a typical mental computation session involving testing, she reported using a class average on the tests, "they [the students] mark their own work, but then we put the results up in the classroom to share them." For several of the other teachers, however, the informal use of observation and discussion featured. For example:

> Assessment of mental computation is really informal in the sense that I look for clues in to how they're [the students] coping through their responses and also body language. Overall I take the attitude that an important element of assessment is feedback. Where will a wrong answer with no feedback get you? This can be a very lonely place to be. (Teacher A, primary)

> There's a lot of bravery involved in mental work. It's not hard to tell when kids have some mental computation strategies through informal discussion. (Teacher B, secondary)

> The key thing I'm looking for is their ability to justify so I look for assessment tasks that always, always ask the kids to explain and critique. (Teacher E, primary)

The comments of Teacher F, a primary teacher, placed the assessment of mental computation in the broader context of numeracy:

> For me, after years of teaching the higher grades in primary school, mathematics and numeracy is the area I still find the hardest to assess in the curriculum. Sometimes I really think the kids have got a concept and the next week it seems like we never even covered it. Working mentally has really opened the lines of communication for me.

7.6 Knowledge of Learners' Characteristics

Three number problems with examples of the students working mentally were shown to the teachers, one example each for fractions, decimals, and percents. The teachers were given the opportunity to provide comment on the examples, which were collated from the student interviews and involved the problems 75% of 200 and 4 x ¾. Overall, several of the teachers commented on the similarity of strategies with those that they were familiar with from the whole number domain. One secondary teacher added that she "hadn't considered working mentally with fractions, decimals, and percents in the same way as whole number" (Teacher C). She also added she would like to see "more of this made available for teachers." One of the primary teachers (Teacher G) acknowledged that she was "not as familiar with how well known mental computation strategies were applied to working with fractions, decimals, and percents."

A different response, from one of the primary teachers (Teacher A) was that generally he did not feel "the need to have the students working out complex fraction computations mentally," however he noted that all three problems involved a whole number (e.g. 4 x ¾) and that he really liked this link between the whole and part-whole number domains. A final comment by one of the secondary teachers:

> Students walking away with only a symbolic understanding of the number system that is supported by very little conceptual understanding really worries me. These are really good examples of students working with good conceptual understanding. (Teacher D, secondary)

7.7 Chapter Summary

As the final phase of the study, the interviews with the key teachers provided the opportunity to further explore some of the results generated through the questionnaire, and also encourage teachers to think about and discuss mental computation with part-whole numbers. A number of themes were extracted from the interview data. First, in relation to teachers' knowledge of the educational ends, purposes, and values associated with mental computation, the primary teachers generally considered mental computation within the broader context of numeracy, whereas the secondary teachers focussed almost solely on the mathematical value. Second, the need for educational systems to develop a whole school approach to mental computation was raised in relation to the teachers' knowledge of educational contexts. Third, as part of the teachers' curriculum knowledge, the teachers indicated that mental computation was largely associated with whole numbers. Several teachers suggested that teachers' confidence in working with part-whole numbers was a possible reason for this. The fourth theme that emerged was that generally the key teachers advised that their approach to and practice of mental computation would not change between whole and part-whole numbers, however, a few differences between the key teachers across the areas related to curriculum and pedagogical content knowledge were noted.

This chapter concludes the presentation of the results of the study. A discussion of the links between the teachers and the students, as well as recommendations for professional development and suggestions for future research based on the findings are considered in the next chapter.

Chapter 8

Discussion

8.1 Introduction

In addressing the essential needs of numeracy, opportunities in mathematics abound in the middle years of schooling. Investigating the potential role of mental computation in strengthening numeracy practices across the middle years was the motivation for the study. The study was conducted through four phases – each addressing a research question – and presented in four results chapters:

- Phase 1 – How is mental computation being addressed by teachers in middle years mathematics classrooms? (Chapter 4)
- Phase 2 – How is mental computation being experienced by middle years students? (Chapter 5)
- Phase 3 – What strategies do students use to solve mental computation problems involving fractions, decimals, and percents? (Chapter 6)
- Phase 4 – How do teachers position the teaching and learning of fractions, decimals, and percents in relation to mental computation? (Chapter 7)

The seminal work of Shulman (1986, 1987) provided the theoretical framework underpinning the design of the study, and the thread that connected each of the four phases. Shulman described seven domains of teacher knowledge: content knowledge; general pedagogical knowledge; curricular knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purposes and values. In the study these knowledge domains provided a framework for profiling the experiences of teachers in relation to mental computation. The framework has been used in three distinctive ways; in the first place, all of the teacher knowledge domains have been explored in a single study, although some domains have been considered in more detail than others. Watson (2001) also used all the domains to design a teacher profiling instrument for chance and data. Similarly, Watson, Beswick, Caney, & Skalicky (2006) designed a profiling instrument to assess teachers' knowledge in relation to middle years numeracy. The instrument was used to measure change in teachers' knowledge during a numeracy professional development program. Generally, other studies have focused on just one or two of the domains (e.g., Ball & Bass, 2000; Kanes & Nisbet, 1996; Mayer & Marland, 1997). Second, the application of the framework has focussed on mental computation as one aspect of the mathematics curriculum whereas the domains have been more frequently used to describe the pedagogy of teaching more generally. In this sense, using the work of Shulman, the focus of the study has been to use the knowledge domains in a practical application rather than to discuss their development and whether the description of each knowledge domain represents entirety. Third, by profiling the students' experiences, the study captured not only the teachers' knowledge of the students' as learners but importantly evidenced and captured some of the mental computation experiences teachers' *should* know about their students.

In this final chapter, the nature of each of Shulman's teacher knowledge domains is considered in order of importance as they relate to the study. In the first place teachers' general pedagogical knowledge is considered in relation to the total sample of teachers and the key teachers. Following this, curriculum knowledge, pedagogical content knowledge, and the characteristics of learners are considered in relation to mental computation from both the perspective of the teachers and their students. Content knowledge is briefly discussed as this domain overlaps with both general pedagogical knowledge and knowledge of learners' characteristics. Finally, knowledge of educational ends, purposes, and values, and educational contexts are considered within the larger picture of numeracy and mathematics.

8.2 General Pedagogical Knowledge

Teachers' general pedagogical knowledge is concerned broadly with teaching principles and concepts and for Shulman (1987) it was a type of knowledge that appeared to "transcend subject matter" (p. 8). In the study, the intention was not to investigate what general pedagogical knowledge might be needed to support mental computation nor how it could be developed, but rather as a descriptive base for positioning the results within the sample of teachers. General pedagogical knowledge was considered to be related to teachers' experience, and was captured in Phase 1, through data on the professional backgrounds of the middle years teachers.

In detailing their professional backgrounds, the teachers provided information of their current and previous teaching experience and mathematical expertise, as well as details of professional development related to mental computation that teachers had participated in during the last five years (Section 4.2). Responses to the questionnaire were analysed first, by the teachers' school level (either primary or secondary) and second, by the teachers' level of participation in professional development (extensive, moderate, limited, or none).

Overall, the sample of 34 teachers were relatively experienced teachers, with two thirds of the group reporting over ten years in the profession. Additionally their experiences were largely associated with the grades they were currently teaching. Of the secondary mathematics teachers, only one-third reported science or mathematics backgrounds that complemented their educational qualifications. The potential impact of the lack of specialist knowledge of the group is considered later in this chapter in relation to teachers' content knowledge.

The spread of professional development participation was relatively even across the four possible categories for the sample of teachers, however, primary teachers had participated in more professional development related to mental computation than the secondary teachers. With these details in mind, most of the characteristics of the main teacher sample were reflected across the seven key teachers interviewed in Phase 4. Key teachers had all participated to some extent in professional development. Teacher C was the only key teacher to be classified as having participated in *limited* professional development and her perspective on mental computation was generally different from that of the other key teachers as highlighted in Chapter 7.

8.3 Curriculum Knowledge

Curriculum knowledge was originally referred to as the "tools of the trade" by Shulman (1987, p. 9). It was interpreted in this study as the awareness of how mental computation relates to aspects of the curriculum including differences from other methods of computation (written and calculator) and between different types of numbers (whole and part-whole numbers).

8.3.1 Time spent developing part-whole numbers

Generally, the teachers in the study reported that they spent more time developing written computation over mental or calculator computation (Section 4.5.1). During discussions with pre-service teachers, McIntosh (1990) posed the question "What percentage of the time devoted to computation in primary schools is concerned with: a) written computation, b) calculator use, and c) mental computation?" He reported that the responses varied little across the sample of teachers and followed a pattern of approximately 90% devoted to written computation and 5% each of calculator and mental computation. Inspection of the actual percentages as distributions of time reported by the teachers in this study, however, revealed that in many cases the balance only just favoured written computation, with the average percentage of 56%. Teachers in this study did appear to be allocating a lot more time to developing mental computation skills; although the strong hold of the more familiar domain of written computation was still apparent. It seems some middle years teachers in this study are not yet ready to whole heartedly support the view that mental computation should be the main form of computation in schools (Willis, 1990, 1992).

When considering mental computation with fractions, decimals, and percents, the key teachers indicated that generally they would not change the time allocated to developing mental computation skills. Although, two key teachers indicated they would perhaps increase mental computation a little for fractions at the primary level. In comparison with the larger sample of teacher responses, however, working with decimals was reported more frequently than working with fractions with mental computation. Teacher C was the only key teacher to report that she would decrease working mentally across the part-whole domain and would increase the focus on written computation. She considered that the symbolic representation of part-whole numbers was particularly important for students in the early secondary years.

Interestingly, two of the key teachers commented that the competency of the teachers in relation to fractions, decimals, and percents would play a large part in determining how much time teachers might allocate to working mentally in this area. This assumption, however, is not being made in regards to Teacher C in the absence of an assessment of Teacher C's own understanding of fractions, decimals, and percents (content knowledge). The theme of teacher competence emerged in response to other questions on the questionnaire suggesting teachers' content knowledge itself is a key factor when considering the teaching and learning of mathematics (Ma, 1999).

Of the students who were asked a similar question as part of the questionnaire instrument (Phase 2), two thirds reported *using* written computation more than mental computation and calculators in their mathematics classroom (Section 5.4.4). Their responses reflect the prominence given to written computation. There was a smaller group of students (approximately one-quarter) who reported *using* mental computation more than the other two methods of computation. The majority of these students were from the primary level, consistent with the teacher reports of spending more time on mental computation also being from the primary level.

8.3.2 Developing mental computation with particular types of numbers

Developing mental computation strategies with basic whole number facts was reported by the majority of teachers (Section 4.5.2). It is possible that teachers attribute mental computation at the level of basic number facts as within the realm of recall for middle years students, particularly as responses across the four operations were alike. For some teachers, however, developing mental computation strategies with multi-digit whole numbers was not reported as frequently as for basic facts with whole numbers. Developing mental computation strategies for decimals was reported similarly to multi-digit numbers; perhaps the traditional link with whole number place value accounts for this. Reports of developing mental computation strategies with fractions and percents were fewer in number, possibly teachers were not as comfortable working in this area as with the more familiar whole number domain or did not view mental computation as being relevant to working with fractions or percents. Teachers certainly reported fewer strategies for these part-whole number domains when asked to detail the strategies students might use to solve problems of this type mentally. One of the primary key teachers initially expressed a view that he did not feel the need for his students (at Grade 6) to be performing "complex" computations with fractions that involved "adding different numerators and denominators." The type of computation he was referring to, however, is considerably more complex than the problems used in this study, many of which comprised both a whole number and a fraction for example. This is perhaps an area where more direction is needed for middle years teachers to ascertain what type of problems are reasonable for their students to be solving mentally. The work of Callingham and McIntosh (2001, 2002) and Callingham and Watson (2004) provide the foundation for this, particularly in linking types of problems to mental computation assessment.

Students were also asked a similar question as part of the questionnaire, although the item was framed in terms of *use* of mental computation skills

(Section 5.4.6). A high use of mental computation was reported by the students when adding and subtracting whole numbers. For the students, however, the items concerning whole numbers were not separated into basic number facts and multi-digit number as was the case in the questions for teachers. The distribution of students' responses agreed more closely with the teachers' reports on working to develop multi-digit whole numbers for addition and subtraction than with the teachers' reports on basic number facts. For the item involving multiplication and division with whole numbers, the distribution of student responses also agreed more closely with the teachers' reports on multi-digit numbers than with the teachers' reports on basic number facts but there was slightly less agreement overall by the students. It might be that students prefer other methods of calculating, such as written or using a calculator, especially if the operations of multiplication and division are perceived as "harder." Generally the four operations with whole numbers are more traditionally associated with mental computation and as such students might be more likely to consider working and calculating mentally with these.

In terms of using mental computation with fractions, decimals, and percents the modal response reported by the students was *sometimes*, although only for fractions did the teachers' patterns of responses mirror the students. For the question relating to decimals, the responses were more varied than for fractions, with the modal responses for the teachers being *frequently* followed by *sometimes*, whereas for the students it was *sometimes* followed by *rarely*. Similarly with percents, the students reported using mental computation less often than the teachers, although the majority of teachers indicated it was a topic where they *frequently* or a least *sometimes* developed mental computation skills. Possibly this discrepancy arises because students are not encouraged to use their mental computation skills for solving problems involving fractions, decimals, and percents, or do not feel as competent. It is also possible that what teachers regard as developing mental strategies with part-whole numbers is not influencing the students to work mentally with these types of numbers when faced with a problem.

8.4 Pedagogical Content Knowledge

The concept of pedagogical content knowledge was identified by Shulman (1986, 1987) as a "special amalgam of content and pedagogy that is uniquely the province of teachers" (p. 8). Essentially, pedagogical content knowledge is about knowing how to represent content to make it accessible for students. In this study pedagogical content knowledge was addressed in Chapter 4 through consideration of teachers' self-reported classroom practices in relation to mental computation, including teaching activities, assessment activities, general mathematical competencies, and classroom organisation. A number of these aspects were followed up through the teacher interviews, presented in Chapter 7 specifically in relation to part-whole numbers. With the exception of assessment activities, similar questions were asked of students as reported in Chapter 5.

8.4.1 Mental computation activities and assessment

In Chapter 4, it was reported that compared to the primary teachers, more secondary teachers recorded traditional types of mental computation activities when asked to describe a common mental computation session or activity (Section 4.6.1). Generally this was also the same group of teachers who had limited or no professional development related to mental computation. An "output" of professional development in relation to mathematics is often an opportunity for teachers to gain exposure to a range of teaching activities. It seems that in this study, the primary teachers who had participated in professional development were implementing activities aligned with a strategies approach to mental computation that many of the teachers in Tasmania were being exposed to at the time the data collection phase of the study took place. When provided with a list of specific teaching activities, slightly more primary teachers also reported using "strategy discussion" to develop mental computation skills than secondary teachers.

On the questionnaire, "strategy discussion" also appeared in relation to assessment of mental computation (Section 4.6.2). The patterns of response across the two items were similar with responses the same for many teachers suggesting that in discussing strategies mental computation is conducted in a formative manner. Similarly, the teachers' pattern of response for use of quick recall questions as a mental computation activity and testing (timed and un-timed) as an assessment activity was distributed evenly across the five Likert indicators. In this case, however, teachers were not generally providing the same response, suggesting some teachers would question the appropriateness of mental computation testing in relation to assessment but might still use it as a specific classroom activity, such as a "warm up" to start a lesson.

Responses of the students (Section 5.5.2) agreed closely with those of the teachers for memory activities, with both groups indicating this was an activity that was only *occasionally* or *rarely* used. For games, however, there was less agreement, with students reporting games less than that reported by the teachers. The teachers on the other hand indicated that this was one of the most common activities used to develop mental computation in the classroom. For the teachers, discussion of strategies was another common activity for developing mental computation and overall the students' responses supported this. Although the modal response was frequent for the teachers, the students' responses were distributed over frequent to occasional. For activities involving 20 quick questions, the students' perception of this activity with mental computation departed from that provided by the teachers. Teachers reported that they conducted 20 quick mental computation questions more frequently than was reported by the students, the modal response for students being *rare*.

8.4.2 Classroom organisation

Overall, mental computation was reported as a daily, whole class activity but more so for the primary teachers than the secondary teachers (Section 4.6.3). Alternatively, secondary teachers reported that mental computation was more of an independent task in their classrooms. Possibly, this is related to the finding that secondary teachers reported more traditional, testing-based mental computation activities which Morgan (2000) describes as "activities conducted in isolation" (p. 2).

It was also possible to compare how the teachers reported organising their classrooms for mental computation activities with the students' perceptions. The students' perceptions of how the class was organised differed from how the teachers reported organising it. For the students, mental computation was more frequently reported as an independent activity whereas more than half of the teachers reported working with the whole class for daily sessions as the most frequent way of organising mental computation. Students were only asked to report on mental computation as a whole class activity, whereas for the teachers it was separated into whole daily sessions and whole class weekly sessions. With this in mind the responses of the students agreed more closely with the teachers' reports of whole class weekly sessions than for whole class daily sessions. Being in a primary classroom or a secondary classroom did not affect student responses.

8.4.3 Associated mathematical competency

Generally, the teachers did not distinguish between the selection of nine mathematical competencies presented in the questionnaire (Section 4.6.4) in terms of supporting mental computation. Comparatively, responses to the item "using knowledge of written algorithms" received slightly less support than the other items and was it noted that all but one of the teachers with no professional development marked the item as *essential* or *important.*

For each of the competencies associated with mental computation the pattern of response reported by the students was very similar to that reported by the teachers. One difference between the two sets of responses was that teachers reported "using knowledge of written algorithms" as being slightly less important than students reported in the companion item, "being able to work things out on paper." This would seem to fit with the level of association that the students reported with written computation and mathematics class time as described in Section 5.4.4.

8.5 Knowledge of Learners' Characteristics

In the study, knowledge of learners' characteristics was addressed in two ways. First, from the perspective of the teachers, as they were asked to anticipate students' responses to a set of attitudes and detail the strategies they might expect their students to use in solving mental computation problems. Second, from the perspective of the students themselves, data were gathered to inform teachers of what might be important for them to know about their students.

8.5.1 Teachers' knowledge of their students' attitudes

Teachers were asked to consider how students would respond to a list of attitude statements (Section 4.7.2) and the same list was included in the student questionnaire (Section 5.4.3). In comparing the students' responses to the positive views associated with mental computation with the teachers' perception of the students, several similarities and differences were apparent. Generally both the teachers and the students responded positively to the view "It's fun." There was, however, an increase in the number of secondary students who did not associate "fun" with mental computation and this was recognised by some of the secondary teachers, five of whom reported that this attitude would be *rarely* or *never* heard. The pattern of responses for both the teacher the students to the view "I'd rather do it my head then write it down" were relatively similar. The distribution of primary responses (teachers and students) was slightly more positive than for the secondary groups. The students' responses also agreed closely with the teachers responses for the view "It's the quickest way to work things out."

The students were more positive in their responses to the view "It's really useful outside of school" than was indicated by the teachers: many of whom indicated this view would be *rare* amongst their students, the students do appear to see the value of mental computation in terms of its real life application. This was certainly a value that the teachers themselves associate with mental computation.

The two views where the responses of the teachers and the students did not closely concur were "It's hard because I can't remember everything" and "It's hard because I'm not very quick." For the former view, teachers reported this as relatively frequent amongst their students, although many of the students expressed disagreement with this view. For these students an association with memory did not appear to have a negative affect on the students' view of mental computation. The results show a similar picture for the latter view that emphasised speed, "It's hard because I'm not very quick." The teachers perceived this view as more common that the students reported it, although secondary teachers tended to support this more than the primary students did.

For the three more negative views associated with mental computation, the responses of the teachers and the students agreed more closely. In particular the distribution of the secondary students mimicked the distribution of the secondary teaches for the view "It's for the younger kids."

8.5.2 Mental computation strategies

Strategies for solving mental computation problems are very much the focal point of the mental computation literature and are positioned in this study as a key aspect of teachers' knowledge in relation to appreciating the ways students might solve part-whole problems mentally. In Phase 1 of the study, the teachers were asked to list the mental strategies they would expect from their students for six mental computation problems. Four problems are reported in this section and compared with the actual strategies that the students were observed using during the interviews. The students' strategies for solving the whole number problems were not described in Chapter 6 which focussed on mental computation strategies for part-whole numbers. Student data is, however, provided as part of Appendix D.

For the multi-digit addition problem, 58 + 34, the most common strategy reported by the teachers involved a place value split (e.g. 50 + 30 and 8 + 4). It was also the most common strategy used by the students. The second most common strategy that the teachers would expect students to use was that of levelling (e.g., 60 + 32), however, no students were observed using this strategy, preferring either a cumulative strategy or describing a version of a written algorithm. Again, for the multi-digit multiplication problem, 24 x 3, the most common strategy – split by other quantity (e.g. 20 x 3 and 4 x 3) – was reported by the teachers and used by the students. However, bridging (e.g., 25 x 3) the second most common strategy reported by the teachers was not used by any of the students in the interviews.

Two problems involving part-whole numbers were also given to the teachers and asked of students. In the first place, the most common strategy reported by the teachers for the problem 4 x ¾ involved a form of an algorithm. Students, however, were observed using a strategy that involved splitting by other quantity using either the 4 or the ¾. Again, for the problem 0.5 + 0.75, the teachers expected responses did not concur with those provided by the students. In this case, the most common strategy recorded by the teachers involved changing the representation of the problem from decimals to fractions. The most common strategy used by the stúdents, however, involved splitting by place value. Based on the four examples, it seems teachers knowledge of their students' strategies aligned more closely for the problems involving whole numbers than for problems involving part-whole numbers.

8.5.3 Working conceptually, working procedurally

The students' interview responses were classified as *working procedurally* or *working conceptually*, an approach used by Callingham (2004), Caney and Watson (2003), McIntosh (2002), and Weber (1999) in relation to mental computation. Generally, the students were classified as working procedurally if their mental computation strategy reflected a traditional written algorithm or a rule and was not accompanied by any further reasoning. For decimals in particular many more responses were classed as procedural and this was largely due to the procedural use of whole number language. A key finding of the student interviews (Phase 3) was that few examples of students working procedurally with part-whole numbers were observed for problems involving percents and fractions. There are several points to consider in relation to this finding. In the first place, at least half of the sample of middle years students were at the primary level (Grade 5 and Grade 6) and it was likely that students had limited exposure to the traditional written methods for solving part-whole problems, particularly for percents and fractions. A version of a traditional written algorithm with quarters was observed for the problem 4 x ¾, which indicated that some of the students were familiar with working through this type of problem in a written context. There were, however, no examples observed of students simply "cancelling," a rule demonstrated by several teachers in the questionnaire. A single example of a procedural response was recorded for the fraction problem 3 ÷ ½. Interestingly, however, just two examples of a student using a mental form of a written algorithm for a decimal problem were recorded for the problem, 0.5 + 0.75. This is perhaps surprising, as it is reasonable to expect more use of a traditional written algorithm given that decimal algorithms (for the four operations) are likely to be more familiar to students from the whole number domain. Percents and fractions, on the other hand, have written methods that are quite different.

The finding that examples of students working procedurally with part-whole numbers were generally limited may also be an artefact of the problems used in the interviews. The majority of the problems involved benchmark values, for example, 50%, 25%, ½, ¾, 0.5, and 0.25, and often in combination with a whole number. The use of these benchmarks values is discussed in the literature in detail in relation to number sense. McIntosh (1992), for example, includes working with benchmarks in his overview of a number sense framework. As well, Moss (2002) provides several scenarios in which students demonstrate their flexible approach in moving between alternate representations of 50%, 25%, and 75%. Being able to use benchmarks capably is viewed as an important aspect for students in developing an understanding of the links between the different rational number constructs. In relation to percent, Parker and Leinhardt (1995) stress that an over emphasis on benchmark values may hinder students in being able to deal with harder problems that they may encounter later in their schooling years. In this study, however, the students were generally quite competent in working with the benchmark values, suggesting that for mental computation the benchmark values may be a very good level of mental computation to cover in the middle years.

The procedural/conceptual distinction is a familiar dichotomy for teachers. The characteristics of each have an intuitive feel to them, which is perhaps why the terms have endured as contemporary researchers attempt to explore the complexities of each notion. Hiebert and LeFevre (1986) write, "The core of each is easy to describe, but the outside edges are hard to pin down" (p. 3). A recommendation for teachers in using the distinction for mental computation is to regard strategies that reflect written algorithms or rules as indicators that students might be working procedurally. This could be a cue for teachers to prompt students to expand on their thinking, for example, to explain why a rule might work and how it could be applied to a similar problem. Asking for alternate strategies would be another avenue for teachers to explore with their students – particularly if students appear to be relying on versions of written algorithms for mental computation (Caney, 2004). It should be noted that in the whole number domain, some counting strategies could also be considered procedural. They are used widely by students, often inappropriately, and are on the whole inefficient particularly when working with multi-digit numbers. Counting strategies, however, did not feature particularly in this study. Traditional written procedures have been shown to be very powerful (Weber, 1999; Reys et al. 1995) and once learned persist in students' thinking even in situations when a strategy is clearly inefficient and somewhat arduous to work through. Although current debate centres on the role of teacher-taught mental computation strategies versus student-invented strategies (e.g. Threlfall, 2000), perhaps discussion should also focus on the role of teachers in facilitating mental computation conversations with students with an emphasis on intervention being not so much about what strategies to use but how to delve deeper into their students' thinking. There are, however, many pedagogical challenges associated with listening to students that middle years teachers face. English and Doerr (2004) acknowledges that "Within a given classroom ... teachers are faced with the challenge of understanding the multiple ways that children might interpret a problem situation and the multiple pathways they might take for refining and revising their ideas" (p. 215). They also note that research concerning how teachers learn about their students (e.g., questioning techniques) is limited for the middle and secondary years.

Hiebert and LeFevre (1986) contend that although the procedural and conceptual distinction has a long history, "current discussions treat the two forms of knowledge as distinct but linked in critical mutually beneficial ways" (p. 2). In this study, the procedural/conceptual distinction was applied to student responses in their entirety, and not necessarily associated with individual strategies as such. It seems appropriate to keep the distinction just a shade blurred, some strategies are almost default examples of working conceptually. Through a description of bridging, for example, students are demonstrating their knowledge of near numbers, usually accompanied by an understanding of the operations involved as the problem is readjusted at the end to achieve an answer. Reys et al. (1995) write that "The acts of both generating and applying a strategy are significant" (p. 304).

8.6 Content Knowledge

Mathematical content knowledge has not been specifically addressed in this study, although two aspects of the questionnaire are worth discussing in relation to what is widely known as teachers knowledge of particular "subject matter." It was mentioned in Section 8.2 that of the 18 secondary teachers who participated in Phase 1, two-thirds had no particular mathematical educational qualifications to complement their teacher qualifications. Although it is widely acknowledged that a qualification per se may not be an adequate single measure of a teacher's knowledge of a content strand, the issue has concerned some in that lack of a particular level (of mathematics, for example) will impede or place limitations of how far the teacher can take their students with that subject. In this study, the complexity of mathematical knowledge expected by teachers was not great, and could be considered as part of the domain of "everyday" numeracy; hence it was not further considered.

Another aspect of the questionnaire that can be considered a secondary measure of teachers' content knowledge are the number of responses that teachers' provided in relation to their expected use of mental strategies in the part-whole domain by students. A higher proportion of teachers did not report any strategies across the part-whole domain problems in comparison to whole numbers. It is possible that teachers themselves were not familiar with strategies for solving problems involving these types of numbers.

8.7 Knowledge of Educational Ends, Purposes, and Values

Mental computation is embedded within the broader context of mathematics and numeracy. When asked to consider the value of mental computation, the teachers who completed the questionnaire identified the link to mathematical understanding, its real life applicability, and an affective value. Inherently these three values are perhaps long-term values in that they are the part of the collective role teachers' play in society in preparing students to become competent community members. A value of a different nature, identified by a smaller number of teachers, was that of mental computation in relation to teaching activities. This value is of short-term, individual interest for teachers in that it is likely to be related to a lesson or unit of work.

In Section 4.3 it was reported that there was the tendency for teachers at the secondary level to emphasise the value of mathematical understanding in relation to mental computation. This finding was reinforced through the interviews with the key teachers as discussions with the secondary key teachers were focused on mental computation in relation to its mathematical links with number. The primary teachers, however, linked mental computation with what could be considered the broader aspects of numeracy. The secondary teachers appeared to have a narrower view of the value of mental computation. Given that the secondary teachers were more likely to teach only mathematics or science, whereas the primary teachers taught mathematics as one part of a suite of curriculum areas, this is perhaps not surprising. It highlights, however, a potential obstacle for developing a consistent approach to mental computation that bridges both the primary and secondary levels.

8.8 Knowledge of Educational Contexts

Like Shulman's domain of knowledge of educational ends, purposes, and values, knowledge of educational contexts is fundamentally about the milieu in which mental computation is embedded. Shulman's interpretation of educational contexts was fairly broad encompassing, "workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures" (p. 8). In the current study it relates to the teachers' consideration of the primary and secondary school contexts and how they might impact on the teaching and learning of mental computation.

Regardless of the teachers' own beliefs about the importance of mental computation in both primary and secondary school, the teachers' overall perception was that mental computation declined as students moved into the secondary years. Reasons for the decline were mainly curriculum-related (emphasis on other methods of computation and aids, and teaching activities) and environment-related (constraints imposed by the curriculum and parental expectations) with fewer teachers reporting teacher- or student-related reasons. Generally this finding aligns with the well ingrained view of primary schooling as student-centred and secondary schooling as subject or discipline centred (Carrington, Pendergast, Bahr, Kapitzke, Mayer, & Mitchell, 2001).

In reviewing the results of this study, three of the seven key teachers raised the lack of a whole school approach to mental computation (and in relation to numeracy) as an important issue. This issue is topical within the realm of middle school generally (Carrington et al, 2001; Hill & Russell, 1999) and has surfaced specifically in relation to middle years numeracy (Luke et al, 2002). It was interesting that the three teachers were from different schooling environments, including one district teacher, one primary teacher and one secondary teacher. Comments seemed to be school specific. On the surface district schools would appear to have an advantage in that at the very least all grades and teachers are contained within one school site. Strengthening middle years numeracy in Tasmania, however, requires coordination between multiple school sites which will be an ongoing challenge for those involved in planning the directions education in the state will take.

8.9 Directions for Further Research

A number of potential directions for further research have been generated by this study. In relation to the students, interview data was collected across three areas – fractions, decimals, and percents – and therefore data on strategies relating to particular operations is, in one sense limited. For the purposes of the study, however, it seemed necessary to explore all three in one study given the links between the three domains. A different path that this study could have taken would have been to consider just one of the areas and expand on the problems relating to the four operations (in the case of fractions and decimals) or relating to the three types of percent application problems (Ashlock, Johnson, Wilson, & Jones, 1983). Additionally, the responses of the students from the Low Group, as defined by mental computation performance, were not included in this study. The teaching community would certainly benefit from a more detailed analysis of students working at the lower level of mental computation. This would be particularly useful in charting the development of mental computation involving part-whole numbers.

An observation from the literature reviewed for this study is that mental computation research involving students outweighs research involving teachers. Expanding the teachers' profiles as used in this study with observations of the teachers working in their classrooms would be one way of addressing this imbalance. Our understanding of middle years mathematics classrooms could also be enhanced through case studies involving middle years teachers' development of mental computation with their students. Both avenues align with Shulman's (1987a) view that multiple data sources are needed to assess teachers. Generally, further research on the teaching and learning of mental computation is likely to be influenced more broadly by the tenets of quantitative literacy (Steen, 2001), including: cross-curriculum learning opportunities and authentic assessment.

Another avenue for further research concerns the development of an evidence base to support informal written strategies or written ways of recording students' thinking that are not confined to standard written algorithms. Mackinlay (1996) writes "we tend not to see as much evidence of informal written methods in the classroom because there is usually firmer direction given about the way children should record their work on paper" (p. 2). It seems a logical progression from the current emphasis on mental computation to encourage students to document their thinking in ways that are meaningful to them (Australian Education Council, 1991; Campbell, Rowan, & Suarez, 1998). The focus of work in this area has again involved primarily whole numbers (e.g., Carroll & Porter, 1998; McIntosh, 2002, 2005), however, building a body of knowledge and research involving part-whole numbers could also be an essential factor in strengthening numeracy across the middle years.

8.10 Recommendations for the Department *of* Education Tasmania

In funding both this study and the project – Assessing and Improving the Mental Computation of School-Aged Students – the D*o*ET has created a body of information on mental computation collected from local teachers and their students. The information generated from this study will be useful to the D*o*ET in terms of supporting professional development activities for teachers, particularly as the study engaged both teachers and students. This section therefore addresses the first objective outlined for the study:

> To provide the D*o*ET with a set of recommendations to assist the on-going development and evaluation of numeracy targets for mental computation.

First, not only do teachers need to be encouraged to continue a strong mental computation focus in the middle years, but also this focus needs to extend further than whole number mental computation. In this study, there was a decline in the number of teachers reporting that they developed mental computation with part-whole numbers compared to the number developing whole number mental computation. This was largely attributed to the need to develop written computation and also to the pressure of a crowded mathematics curriculum. Mental computation with part-whole numbers need not be another "topic" to cover, a perception that can perhaps be attributed to the structure of the curriculum, but rather a way of teaching mathematical content. Certainly highlighting the importance of mental computation in relation to the part-whole number domain sends the message to middle years teachers that mental computation is relevant to the secondary classroom (Callingham & Watson, 2004). The D*o*ET's support for research in this very area adds weight to that message.

Second, in building teachers' capabilities to develop mental computation with middle years students, knowledge of learners' characteristics (Shulman, 1987) must be a key feature of any program or initiative supported by the department. This means that teachers need to be exposed to examples of students' thinking and students' working through problems. Using examples such as the ones presented in Chapter 6 can connect teachers with their students and encourage teachers to look further than their own methods of solution, particularly for part-whole numbers.

Third, in relation to teachers' pedagogical content knowledge, this study establishes that the benchmark values, e.g. 50%, ½, 0.5, 25%, ¼, and 0.25, are an appropriate level of mental computation for students in the middle years. At this level many of the strategies that students need are likely to be familiar to them from working with whole numbers, for example, doubling and halving strategies and also splitting numbers. Additionally problems comprising a combination of a whole and part-whole numbers, for example 4 x ¾, can potentially bridge the whole number, part-whole number divide.

Both aspects are starting points for teachers preparing students to work mentally with fractions, decimals and percents.

Fourth, teachers need to pay special attention to decimal mental computation. In this study it was found that students, although generally successful in solving decimal problems, were more often working procedurally rather than demonstrating their conceptual understanding. Although whole number strategies were seen to transfer to the decimal problems, the understandings behind the use of the strategies were often masked by the use of whole number language. In the interviews it was not until the students were asked "why" in relation to their answers, and not just "how," that their understandings were exposed. For this reason, teachers will need to create multiple opportunities for students to expand on their thinking with decimal mental computation.

Fifth, it is important to find ways to look at students' ability levels in relation to numeracy that go further than grade-based distinctions. For the middle years in particular, Hill and Russell (1999) identify that a "convergence in structures and approaches to teaching and learning between the final year of primary schooling and the first year of second schooling" (p. 9) is necessary. In this study, the use of the levels of mental computation competence (Callingham & McIntosh, 2002) showed that middle years students were spread primarily across Levels 5, 6, and 7, with no clear cut association of level with grade.

Sixth, if consistency across the primary/secondary divide is to be addressed in the future, the D*o*ET needs to consider targeted professional development programs for middle years teachers. Very few Tasmanian schools have separate middle years programs and most primary and secondary schools are situated on different sites with individual mathematics and numeracy programs. In this study more primary teachers had accessed professional development in relation to mental computation and overall than their secondary colleagues.

Finally, Shulman's work provided a useful and encompassing framework for this study. It also has the potential for providing a comprehensive framework from which to design professional development programs for teachers. Individually, Shulman's domains are familiar to teachers and are applicable to everyday life in a classroom. Together, however, they constitute a framework for capturing the breadth of teachers' experiences, an example of which is detailed by Watson, Beswick, Caney and Skalicky (2006). Collaborating with the numeracy team at the D*o*ET, the authors show how a professional development numeracy program used Shulman's work to underpin its design and implementation. In a similar fashion to this study, Shulman's domains were used to devise a teacher profiling instrument the results of which contributed to providing a program that would meet the specific needs of the participating middle years teachers.

8.11 Limitations

The limitations outlined in Chapter 3 concerned particular methodological choices. In this section several reflective limitations are briefly drawn to attention, representing lessons learned during the data collection phases of the study. They are largely drawn from the experience of interviewing students. In the first place, it was difficult on occasions to accommodate the time individual students needed to think and work mentally within an interview session. Unavoidably, within a set interview time, students who were slower at working through problems mentally were not able to be presented with the number of problems, as would have been ideal. Alternatively, it was possible to give students who worked at a quicker pace additional problems to solve. This was particularly an issue when working within the secondary environment, as the school timetable was generally tight and inflexible. At the primary level, however, interview sessions were able to take a little longer if necessary.

A second issue is that of the intervention of the researcher with some students during the interview. Again, on occasion, some students were prompted with a question categorised as a *guided* (refer to Section 3.5.2.3). From a research perspective the discussion that then takes place with the student following a guided question is not as valid as an initial response, particularly for mental computation. This said, however, in an interview situation it is important for the researcher to support students and keep the interview experience both positive and interesting. In a classroom setting the use of guided questions would be used by the teacher more freely to generate discussion and support a culture where students' thinking is at the forefront of mathematical activity.

A third point relates to the classification of mental strategies in general. Threlfall (2002) maintained that the array of the attempts to name and group strategies in the literature still does not capture the diversity of student thinking that would be found in any given classroom. Although there is perhaps a lack of agreement on the terminology used to describe strategies, fundamentally it is important that teachers understand the general ways in which students work with numbers. For whole numbers this aspect is well captured by researchers and available for teachers. Threlfall's concern is perhaps exacerbated by the tendency for researchers to capture "complete strategies" whereas in reality some strategies may be used in a singular sense or in a combined fashion. Students sometimes change their strategy part way through a discussion and research in this area does not necessarily capture these scenarios.

8.12 Conclusion

This thesis establishes that middle years numeracy practices could be strengthened through a greater emphasis on mental computation that extends further than working with whole numbers to embedding a strategies approach within the part-whole number domain. The research grows out of an interest in mental computation that spans some two decades and was largely inspired by the emergence of constructivist thought. In general, the research that has transpired showcases mental computation strategies with whole numbers, particularly addition and subtraction, and perhaps for that reason the focus of the research has been at the primary school level. It is not feasible, however, to consider numeracy in relation to the middle years mathematics curriculum without acknowledging the significance of the part-whole number domain, including fractions, decimals, and percents. As key mathematical ideas these three concepts have collectively been the subjects of a large body of research within the mathematics education community but not in relation to mental computation – one of the main contributions of this study.

In this study a profile of middle years students was constructed using multiple data sets. Three groups of students' mental computation competence – high, middle, and low – were established. These groups were then used to look at performance on two comparisons tests involving decimals and fractions, and also the students' experiences in relation to mental computation. For the students in the middle and high performing groups, the mental computation strategies that the students used were then documented for problems involving fractions, decimals, and percents. In relation to numeracy education, profiling students has proven to be a constructive way to look at both the multiplicity of students' abilities for a particular mathematical content area, as well as their perspectives, modelling an approach that transcends traditional grade-based classifications.

In keeping with goals of the project, Assessing and Improving the Mental Computation of School-Aged Students, this study has engaged both students and their teachers. A different approach to profiling the teachers was employed whereby a profiling instrument was constructed to capture the experiences of the middle years teachers. The work of Shulman (1986, 1987) in describing seven essential domains of teacher knowledge was used as the theoretical framework underpinning the profiling instrument. Additionally, differences between the teachers in terms of school level (primary and secondary) and level of professional development could be assessed. Previously, much of the advice afforded to teachers regarding mental computation has filtered through numeracy discourse with the exception of student thinking strategies which, for whole numbers, have been well documented. As part of this study, a set of teaching recommendations concerning the teaching and learning of mental computation has been provided. It is likely that professional development programs will be the platform from which the recommendations can start to be addressed. The recommendations are intended to complement the body of mental computation research that has already been conducted in Tasmania.

As an approach to the teaching and learning of number, the meaning of mental computation has moved away from the confines of mental arithmetic to encompass students' thinking strategies with the goal of developing conceptual understanding. Ultimately, the value of mental computation is that it brings life to the mathematics classroom through conversation and discussion. The tenets of mental computation therefore extend further than working with number, to model what should be embedded more broadly as the pedagogical approach to mathematics teaching. If indeed mental computation is the heart of numeracy, then the life blood of mental computation is surely the student voice.

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