INDUCED GRAVITY AND THE GAUGE TECHNIQUE

by

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DECLARATION

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by any person, except where due reference is made in the text of the thesis.

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ABSTRACT

The apparent incompatibility of the quantum theory with general relativity is well known. In this thesis we consider a possible solution to this problem, namely the program of induced gravity.

The problem of quantum gravity, namely its nonrenormalizability, is due to its scale non-invariance. The assumption of the induced gravity program is to begin with a fundamental scale invariant Lagrangian which is renormalizable. Quantum fluctuations can break scale invariance and thus it is possible that the Einstein-Hilbert Lagrangian will be induced, as first shown by Sakharov. This breaking of a classical symmetry by quantum fluctuations is called dynamical symmetry breaking.

It is possible to derive a relation between the induced Newtonian gravitational constant, G, and the stress-energy tensor of the matter fields. This formula, due to Adler and Zee, is derived. A review is given of all previous model calculations of G and their successes and failures noted. The extension to a quantized metric is considered and the properties of the scale invariant fundamental gravitational Lagrangian are studied.

Since the idea of inducing gravity as a quantum effect is essentially a non-perturbative effect, we require non-perturbative techniques to obtain useful information. One such technique is the Delbourgo-Salam Gauge Technique. A review of this technique is given, followed by its application to the program of induced gravity. The philosophy of this ansätze is used to calculate an

approximation to the contribution to G from a general fermion-graviton theory in terms of the spectral function of the fermion. The details of the Gauge Technique are then used to perform an actual calculation of the contribution to G from QED.

The result is quite small, signifying that the contribution to G from the electrodynamic interactions of the low mass fermions does not lead to any unexpected surprises.

CONTENTS

1. INTRODUCTION

1.1. OVERVIEW

Einstein's theory of gravitation has been with us for two thirds of a century and still it agrees perfectly with every experiment yet devised. Why is this? In a way, the answer is already known; general relativity is the simplest theory of gravitation consistent with the idea of general covariance. Yet physicists demand more. A prediction is required, a prediction of the strength of gravity. The magnitude of the interaction between space-time and matter is embodied in a constant which dates back to Newton. This is of course the gravitational constant, G.

Furthermore, physicists require a consistency for all interaction, and it is here that the theory of general relativity causes problems. It has recently been established that to quantize matter fields, but not gravity, is not only inconsistent but also inefficient. The field equations of general relativity can actually be predicted if and only if the metric itself is quantized. Furthermore, there is even tentative experimental evidence to support the necessity of quantization. These facts, and a faith in the consistency of nature, support the common belief for a quantized gravitational field.

However the consequences of this are disastrous, since to quantize gravity leaves us with a theory that at high energies has no predictive power whatsoever. The villain of this nonrenormalizability is well known; it is the non-dimensionless scale in the coupling, the gravitational constant.

Theories with scale invariance have many desirable features, not least being their apparent manifest renormalizability. A scale invariant theory with only one coupling constant would, in principle, be able to predict all dimensionless numbers, including mass ratios. Although a scale invariant theory contains no mass terms, quantum effects break scale invariance, and thus can dynamically induce a mass scale. The idea thus naturally arises; can one induce the scale of gravitation from a theory which has classical scale invariance?

This is the idea of induced gravity.

1.2 A HISTORICAL PERSPECTIVE

The idea of inducing Einstein's gravity from quantum fluctuations of the matter action is due to Sakharov (1967), who was motivated by a paper of Zel'dovich (1967) in which the idea of an induced cosmological constant was discussed. Besides a brief discussion in Misner, Thorne & Wheeler (1970) there was apparently virtually no published work directly in this area until 1980. However, a similar idea was being evolved, that of replacing the gravitational constant with a scalar field, which then acquires a non zero vacuum expectation value. This idea can be traced back to Fujii (1974) although the idea of replacing the gravitational constant by a scalar, and thus allowing the possibility of a scale invariant gravitational theory, dates back further, to the work of Gürsey (1963) and Brans & Dicke (1961). During the seventies the theory evolved to a more modern formalism (see for example Zee 1980) but always there was the scalar field. The presence of a scalar field is rather unsatisfying, due to the free parameters in the Lagrangian. These parameters imply a vacuum expectation value which is arbitrary since the effect of the scalar field is not manifested elsewhere. Consequently the induced gravitational constant is completely arbitrary.

Meanwhile, however, the modern methods of field theory were being advanced; gauge theories, renormalization group, dimensional regularization, dynamical symmetry breaking, etc. These are essential to the modern understanding of induced gravity, which began with the paper by Adler (1980a) who showed that with a renormalizable Lagrangian with only spin 1/2 and spin 1 fields, the induced gravitational constant must be finite. The existence of an induced term now requires dynamical symmetry breaking.

A few preliminary calculations were performed by Hasslacher & Mottola (1980) using an instanton gas approximation, and by Zee (1980) using a 1-loop calculation. However, the most important development was the derivation of a general formula for G, derived independently by Adler (1980b) and Zee (1981a). Since then, there has been further attempts at model calculations; for example for a asymptotically stable theory (Zee 1982a) and an outline of a lattice program to calculate G (Adler 1982). There has been extensions to the case when the metric is quantized (Adler 1982; Zee 1983a), and to the induced $O(R^2)$ terms (Zee 1982b; Brown & Zee 1983). There has also been the important work of Khuri, who finds some general theorems concerning the upper and lower bounds for the possible magnitude of the induced G, as well as the sign (Khuri 1982a, 1982b, 1982c).

By now there are quite a few review papers on induced gravity (Adler 1980c; Zee 1981b, 1981c, 1983b) but the most extensive review is by Adler (1982).

The central aim of induced gravity is of course, the calculation of G. This requires a technique for studying the non-perturbative behavior of a gauge theory; in fact we require a technique that takes into consideration an infinite number of Feynman diagrams. The Gauge Technique is one such method. This idea goes back to Salam (1963) and others (Delbourgo & Salam 1964; Strathdee 1964). However it was not until later (Delbourgo & West 1977a, 1977b) that the method was formulated into a productive technique, and by now it has had many applications. The gauge technique embodies the essence of dynamical symmetry breaking. It dynamically generates mass terms without any added fields, and so would appear to be a good candidate for a technique in the study of induced gravity.

I shall use the philosophy of the gauge technique to derive a general expression for G^{-1}/m^2, and then use the actual results of the QED gauge technique to calculate an approximation for the contribution to G from QED. We find this contribution to be very small, namely

$$G^{-1}/m^2 \approx (\frac{8}{9\alpha}) \approx 1.2 \times 10^2.$$

1.3. SUMMARY OF THESIS

I now give a short summary of each chapter.

Chapter two is an introduction to the philosophy and formalism of induced gravity. Here we explain the notion of dynamical symmetry breaking and derive the main result; namely the Adler-Zee expression for G.

We follow this with a review of all previous model calculations for G, and explain their successes and failures.

Chapter 4 discusses the effects of the induced R^2 terms, as well as the bare $O(R^2)$ Lagrangian whose presence is required for renormalization. The property of asymptotic freedom is discussed, as well as the notorious unitarity problem. These three chapters constitute the review section of this thesis; they have relied heavily on the review paper of Adler (1982), as well as the original papers.

In chapter 5 we consider the fermionic contribution to G, and derive the Feynman rules and Ward identity for a graviton-fermion theory. The chapter ends with an expression for G in terms of the spectral function of the fermion.

To calculate G we need to know the precise form of this spectral function. This is obtained from the gauge technique which is introduced in chapter 6, along with its successes and failures and its application to QED. Furthermore in this chapter we derive the necessary equations for the evaluation of the fermion spectral function.

In our second last chapter we calculate a definite result for the value of G^{-1}/m^2 purely in terms of the electromagnetic coupling constant.

We conclude with a discussion of the general implications of our result, of the future possibilities of this approach, and of induced gravity in general.

An appendix is included on the subject of quantizing the gravitational field. The necessity of this quantization is argued, followed by a discussion of the consequences of quantization to the program of induced gravity.

2. INDUCED GRAVITY

2.1. VIA SPONTANEOUS SYMMETRY BREAKING

The idea that gravity may be an induced quantum effect can be simply realized by the spontaneous symmetry breaking of an extra scalar field (Fujii 1974; Englert et al. 1975; Englert, Truffin & Gastmans 1976; Minkowski 1977; Chudnovskii 1978; Matsuki 1978; Smolin 1979; Linde 1979, 1980; Zee 1979, 1980). Note however that the scalar field is not necessarily elementary, thus this approach could be considered as starting from a Lagrangian which is not really fundamental. It does, however, illustrate the general idea of inducing gravity by a spontaneous symmetry breaking approach.

The gravity action is assumed to be

$$S = \int d^4x\sqrt{-g}\{\varepsilon\phi^2 R + \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - V(\phi)\} \tag{2.1}$$

where ε is a dimensionless coupling constant.

Let $\phi = \phi_0$ be the minimum of the potential $V(\phi)$. For ϕ at the minimum, we have

$$S = \int d^4x\sqrt{-g}\{(16\pi G)^{-1}R + \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - V(\phi_0)\} \tag{2.2}$$

where $16\pi G = \dfrac{1}{\varepsilon\langle\phi_0\rangle^2}$.

The Einstein-Hilbert Lagrangian has thus been induced, simply by requiring a non-zero vacuum expectation value for the field ϕ.

In general, G will be a function of both the temperature and of R, since $\langle\phi\rangle$ will depend on these. This will obviously have implications for cosmology (Linde 1980; Zee 1980).

Note that since ε is arbitrary, these models do not give a calculable G. Nevertheless, this model is still being studied

(Cerveró & Estévez 1982; Fujii 1982).

2.2. SCALE INVARIANCE

Scale invariance is a very useful symmetry to impose on the Lagrangian for a number of reasons. Firstly scale invariant theories appear to be renormalizable by power counting. Furthermore the reduction of freedom implies potentially greater calculability as we shall see. They thus have a great aesthetic appeal.

A scale transformation is as follows

$$\begin{aligned} x_\mu &\to a x_\mu \qquad \text{where a is a constant} \\ \Phi &\to a^{-d}\Phi \end{aligned} \tag{2.3}$$

constants are invariant.

Φ is any field that has canonical dimension $[\Phi] = d$, where the canonical dimension is defined as [mass] = 1, [length] = -1.

For a gauge theory in 2ℓ dimensions with

$$\mathcal{L} = \frac{-1}{4g^2} F^i_{\mu\nu} F^{i\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{2.4}$$

where $D_\mu = \partial_\mu + \frac{1}{2} i \lambda^j A^j_\mu$, we have $[A^\mu] = 1$,

$[F^i_{\mu\nu}] = 2$, $[\psi] = \frac{2\ell-1}{2}$, $[g] = 2 - \ell$.

Note that in $2\ell \neq 4$ dimensions the coupling constant has attained a canonical dimensionality. We can of course define a dimensionless coupling constant g_R by $g_R = g(\mu)^{\ell-2}$ where μ is an arbitrary mass parameter.

A scale transformation is a special kind of conformal transformation; the latter is a 15-parameter group which is defined by adding to the Poincaré group the scale transformations and the special conformal transformations $\delta x^\mu = (2x^\mu x^\rho - g^{\mu\rho}x_\tau x^\tau)C_\rho$. This is equivalent to a local scale transformation, the metric and fields thus transform as

$$\begin{aligned} g_{\mu\nu} &\to a^2(x)g_{\mu\nu} \\ \Phi &\to \{a(x)\}^{-d}\Phi \end{aligned} \tag{2.5}$$

Thus Yang-Mills theory, for instance, is conformally invariant in $2\ell = 4$ dimensions but not for $2\ell \neq 4$.

Note also that R^2 is scale invariant but not conformally invariant.

Although conformal invariance may be useful to impose, we shall until further notice only assume scale invariance.

Classically, massless gauge theories and the massless ϕ^4 theory are scale invariant. However, radiative corrections can break scale invariance, and this is one of the key principles behind dynamically induced gravity. To illustrate this dynamical symmetry breaking of classical scale invariance, we consider ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\lambda\phi^4}{4!} \tag{2.6}$$

If we consider only the 1-loop quantum effects then it can be shown by either a direct 1-loop calculation or by finding the 1-loop effective potential (Coleman & E. Weinberg 1973; see also Ramond 1981) that under a scale transformation $x'^\mu \to ax^\mu$,

$$\lambda \to \lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log a \tag{2.7}$$

The renormalized coupling constant λ is thus scale dependent, since it depends on the scale a. The simplest way to see the scale dependence is to consider the renormalization of λ via a 1-loop Feynman calculation. To keep λ dimensionless we must introduce an arbitrary mass scale. Since the Feynman diagram contains an infinite part (i.e., a pole at $2\ell = 4$) the finite part, and thus λ, will change if this mass parameter or the scale is altered.

The dependence of λ on the scale a is often expressed by the β function

$$\beta = \frac{d\ \lambda(a)}{d\ \log a^2} = b_0\lambda^2 + \ldots \tag{2.8}$$

For ϕ^4 theory,

$$\beta = \frac{3\lambda^2}{32\pi} + \ldots \tag{2.9}$$

Higher loop calculations give the further terms in β.

All this implies that the coupling has to be defined at some particular scale, usually called the renormalization point, μ. μ has the dimensions of mass and under the above scale change, $\mu \to \mu'$ where $a = \mu/\mu'$.

Similarily, for an SU(n) gauge theory with N_f flavours of massless fermions in the fundamental rep, we have

$$g^2(\mu'^2) = \frac{g^2(\mu^2)}{1 + \frac{1}{2}b_0 g^2(\mu^2)\log(\frac{\mu'^2}{\mu^2}) + \ldots} \tag{2.10}$$

Again, the further terms are given by higher loop calculations.

The function $\beta(g) = -\frac{1}{2}b_0 g^3 + O(g^5)$ is also used to describe the scale dependence.

$$b_0 = \frac{1}{8\pi^2}\left(\frac{11}{3}n - \frac{2}{3}N_f\right) \tag{2.11}$$

For a particle interaction, $-\mu'^2$ can be interpreted as the four momentum squared, q^2, a kinetic invariant that governs the energy scale of the process in question.

Note that if $b_0 > 0$, i.e., $N_f < \frac{11n}{2}$, we have $g^2(-q^2) \to 0$ as $-q^2 \to \infty$ in eqn. 2.10 which demonstrates the asymptotic freedom of nonabelian gauge theories.

The gauge theory at the tree level had one free parameter; and now, with the loop corrections, it has two nonindependent parameters, g and μ. We can however find a new free parameter $M(g,\mu)$ which is independent of the renormalization point.

From (2.10) we can see that the mass parameter

$$M(g,\mu) = \mu e^{-1/b_0 g^2(\mu^2)} \tag{2.12}$$

is independent of the scale, to one loop order. The expression incorporating all loop orders can also be found (Gross & Neveu 1974)

$$M(g,\mu) = \mu e^{-\int^{g(\mu)} \frac{dg'}{\beta(g')}}. \tag{2.13}$$

Alternative ways of expressing the above is to say that M is renormalization group invariant, or that M satisfies the Callan-Symanzik equation:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] M(g,\mu) = 0 \tag{2.14}$$

M is now the only free parameter in the theory and all quantities must be able to be expressed as a function of M. However M has the dimensions of mass, so any physical parameter $P(g,\mu)$ with canonical dimension d must be expressible in terms of a power $(M)^d$, simply by dimensional analysis. Furthermore, any dimensionless physical parameter (thus renormalization group invariant) must be calculable from the theory. For example, mass ratios. M itself cannot of course be calculated in isolation. It can however be measured. In QCD it is related by a calculable constant to the QCD parameter, which can be measured by the scaling violation in electroproduction.

This process, of transforming a theory with a dimensionless parameter to one with a dimensional parameter is called dimensional transmutation (Coleman & E. Weinberg 1973). This process is obviously of great importance since a realistic fundamental theory with only one free parameter would in principle give all mass ratios including, for instance, $\frac{M_{Planck}}{M_{electron}}$.

Note that the mass parameter M obviously breaks the scale invariance, just as expected. The above illustrates the mechanism of dynamical symmetry breaking via the renormalization process.

There are other ways of considering dynamical symmetry breaking, although none as clear as the above; (Nambu & Jona-Lasinio 1961; Johnson, Baker & Willey 1964; Jackiw & Johnson 1973; Cornwall & Norton 1973). These include considering analogies with the BCS theory of superconductivity or of trying to generate composite fields such as $\bar{\psi}\psi$; for example the Nambu-Jona-Lasinio type models, where there is also a nonvanishing fermion mass generated. For instance the Higgs mechanism requires a scalar field ϕ, however this could be a composite of the fundamental fermions, so that $\langle\phi\rangle = \langle\bar{\psi}\psi\rangle^{1/3} \neq 0$. The analogy with superconductivity will be elaborated upon later.

2.3 THE FUNDAMENTAL LAGRANGIAN

In flat space-time, there will be a purely matter Lagrangian, $\mathcal{L}_{matter}$, which will describe all the fundamental particles of physics. The exact form of this is of course not yet known, but it is expected to involve spin 1/2 fermions, spin 1 gauge particles, and possibly their supersymmetric partners. Since the theme that is being pursued here is that of dynamical symmetry breaking, fundamental scalars are viewed with reluctance. However, their optional appearance can be allowed under certain conditions stated shortly.

In a general space time, $\mathcal{L}_{matter}$ will be a function of the metric $g_{\mu\nu}$, and $\mathcal{L}_{matter}\,[g_{\mu\nu},\Phi]$ will be the generally covariant form of $\mathcal{L}_{matter}$; i.e., all derivatives ∂_μ will be replaced by the covariant derivatives ∇_μ. (Φ generically denotes all matter fields).

There must also be added a gravitational Lagrangian $\mathcal{L}_{grav}[g_{\mu\nu}]$ such that $\mathcal{L}_{grav}[\eta_{\mu\nu}] = 0$. The terms in $\mathcal{L}_{grav}$ must be a maximal set of generally covariant local composite operators constructed from the metric and the fields. Furthermore they must be of canonical dimension 4 and satisfy the symmetries of the theory (i.e. those of $\mathcal{L}_{matter}$). These conditions comprise the dimensional algorithm (Weinberg 1957). For instance, terms like $\nabla_\mu A^\mu R$ or $A^i_\mu A^{\mu i} R$ are not allowed, because they do not satisfy the required gauge invariance.

We now assume that the fundamental Lagrangian is scale invariant (see section 2.2). The resulting Lagrangian is much more aesthetically appealing than the arbitrariness implicit in a scale non-invariant fundamental Lagrangian.

Scale invariance implies that a term like $(1/16\pi G_0)R$ is not allowed; neither is a bare cosmological constant Λ_0.

We also assume either (a) that there are no elementary scalar fields or (b) that the Lagrangian is invariant under supersymmetry. Case (b) combined with the assumption that the Lagrangian is of polynomial form, implies that the term $\phi^2 R$ can not be allowed, since $\delta\phi \neq 0$.

With these conditions there are only three possible terms, those which are quadratic in the metric. These are

$$\begin{aligned} &(a) \quad R^2 \\ &(b) \quad \mathbb{G} = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ &(c) \quad C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} \end{aligned} \tag{2.15}$$

where $C_{\mu\nu\sigma\tau}$ is the conformally invariant Weyl tensor. In 2ℓ-dimensional space time, we have (Weinberg 1972)

$$\begin{aligned} C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} &- \frac{1}{2\ell-2}(g_{\mu\sigma}R_{\nu\tau} - g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau} + g_{\nu\tau}R_{\mu\sigma}) \\ &+ \frac{R}{(2\ell-1)(2\ell-2)}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma}) \end{aligned} \tag{2.16}$$

The fundamental gravitational Lagrangian is thus

$$L_{grav} = r_0 R^2 + s_0 \mathbb{G} + c_0 C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} = \frac{\mathcal{L}_{grav}}{\sqrt{-g}} \tag{2.17}$$

This has been proven to be renormalizable (Stelle 1977).

The coefficients r_0, s_0 and c_0 are allowed to be non-finite. This Lagrangian will be discussed in chapter 4, along with the possible resolution of the problem usually cited in connection with $O(R^2)$ theories; namely the nonunitarity of the S-matrix. We note however a trivial resolution of all such problems, simply take $r_0 = s_0 = c_0 = 0$. This however appears to be impossible to allow, since $O(R^2)$ terms are required as counterterms in the renormalization of $\mathcal{L}_{matter}[g_{\mu\nu},\Phi]$.

So far, no mention of the known low energy gravity Lagrangian $\mathcal{L} = \frac{\sqrt{-g}R}{16\pi G}$ has been made, other than its scale non-invariance. For such a term to appear, we require that it is only an effective Lagrangian, as explained in the next section.

2.4 EFFECTIVE ACTIONS

The idea of an effective action is simple. Consider a field theory with interacting quantum fields. To find the complete description of one of the fields, one simply functionally integrates over all the others in the expression for the partition function. This can be illustrated by the weak interaction.

Historically, the experimental data on weak interactions pointed to a current-current interaction:

$$\mathcal{L}_F = \frac{4G_F}{\sqrt{2}}(j_\mu^+ j^{\mu-} + j_\mu^n j^{\mu n}) + \text{hermitian conjugate} \qquad (2.18)$$

where for two generations, for instance,

$$j_\mu^- = (\bar{\nu}_e \;\; \bar{\nu}_\mu)\gamma_\mu \begin{pmatrix} e_L \\ \mu_L \end{pmatrix} + (\bar{u} \;\; \bar{c})\gamma_\mu U_c \begin{pmatrix} d_L \\ s_L \end{pmatrix} \qquad (2.19)$$

$$U_c = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \qquad (2.20)$$

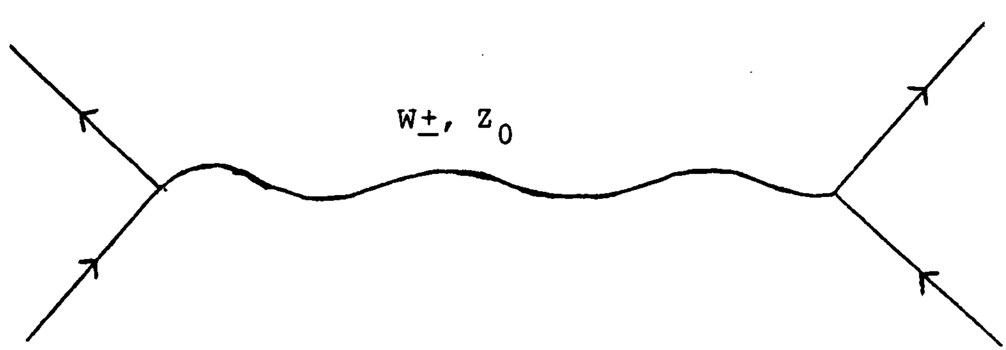
This theory describes the interactions of the fermions at low energies (i.e., less than 80 GeV), but fails at higher energies.

Furthermore it is not renormalizable; the reason for this is basically the presence of the dimensional coupling constant G_F = $(300 \text{ GeV})^{-2}$.

It is now known that the above is only an effective Lagrangian of the true Weinberg-Salam Lagrangian: (For a review, see Fritzsch & Minkowski 1981).

$$
\begin{aligned}
\mathcal{L}_{WS} = & -\tfrac{1}{4}F^{i}_{\mu\nu}F^{i\mu\nu} - \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} + \bar{e}_R(i\not\partial+g'\not B)e_R \\
& + (\bar{\nu}\ \bar{e}_L)(i\not\partial-\tfrac{g}{2}\tau_i\not A_i+\tfrac{g'}{2}\not B)\begin{pmatrix}\nu \\ e_L\end{pmatrix} \\
& - G_e[\bar{e}_R\phi^{\dagger}\begin{pmatrix}\nu \\ e_L\end{pmatrix} + (\bar{\nu}\ \bar{e}_L)\phi e_R] \\
& + [G_1\bar{d}_R\phi^{\dagger}\begin{pmatrix}u \\ d\end{pmatrix}_L + G_2\bar{u}_R\tilde{\phi}^{\dagger}\begin{pmatrix}u \\ d\end{pmatrix}_L + \text{h.c.}] \\
& + (\partial^{\mu}-\tfrac{i}{2}g'B^{\mu}-\tfrac{i}{2}g\tau_i A^{\mu}_i)\phi^{\dagger}(\partial_{\mu}+\tfrac{i}{2}g'B_{\mu}+\tfrac{i}{2}g\tau^i A^i_{\mu})\phi \\
& - \mu^2\phi^{\dagger}\phi - \lambda(\phi^{\dagger}\phi)^2 \\
& + \text{second generation and mixing terms.} \qquad (2.21)
\end{aligned}
$$

This is a renormalizable theory describing the behaviour of fermions and the gauge bosons at all energies. Furthermore, it is scale invariant. The usual correspondence between $\mathcal{L}_{WS}$ and $\mathcal{L}_F$ is obtained by considering the 4-fermion interaction at low energies.



However the full effective fermion action, S_{eff}, is obtained by functionally integrating out the gauge bosons and Higgs fields.

$$\exp(iS_{eff}(\psi)) = \int d[\text{All Bosons}]\ \exp(i\int d^4x\,\mathcal{L}_{WS}) \qquad (2.23)$$

ψ denotes all the fermion fields. Note that the partition function is then $Z = \int d[\psi] \exp(iS_{eff}(\psi))$. $S_{eff} = \int d^4x \mathcal{L}_{eff}$ describes, completely, all the interactions of the fermions when no external gauge or Higgs fields are involved, it is valid at all energies. It also contains a dimensional parameter, G_F, so the scale invariance has been broken.

If we now consider only low energies, then we find that $\mathcal{L}_{eff} = \mathcal{L}_F$ + small corrections.

The principle is the same for gravity. The full Lagrangian is the generally covariant matter Lagrangian, $\mathcal{L}_{matter}[g_{\mu\nu}, \Phi]$, plus the fundamental gravitational Lagrangian $\mathcal{L}_{grav}[g_{\mu\nu}]$.

The effective action $S_{eff}[g_{\mu\nu}]$ is obtained simply by

$$\exp(iS_{eff}[g_{\mu\nu}]) = \int d[\Phi] \exp(i \int d^4x \mathcal{L}_{matter}[g_{\mu\nu}, \Phi]) \qquad (2.24)$$

Let $S_{eff} = \int d^4x \mathcal{L}_{eff}$. $\mathcal{L}_{eff} + \mathcal{L}_{grav}$ describes the interactions of gravitons and takes into consideration all the effects of internal matter fields. It is valid at all energies and contains a dimensional scale $G \sim (10^{19} \text{GeV})^{-2}$, just like in the weak interactions. The analogy between weak interactions and gravity should not be taken too far, however, since in the Weinberg-Salam theory we integrate over the gauge fields, while in the gravity case we integrate over the matter fields and not the gravitational gauge field.

Note that as this effective Lagrangian is induced by the quantization of the matter fields, it is not a classical result. The metric $g_{\mu\nu}$ may or may not be quantized, the viewpoint is unaltered.

We now consider the situation of slowly varying metrics, i.e. $(\partial_\lambda g_{\mu\nu}) \times$ Planck length $<< 1$.

(By "$g_{\mu\nu}$" I mean the background metric, $\bar{g}_{\mu\nu}$, in the background field method of quantizing the gravitational field in which $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.)

Thus it should be possible to expand a function of $g_{\mu\nu}$ in powers of $\partial_\lambda g_{\mu\nu}$ with constant coefficients. $\mathcal{L}_{\text{eff}}[g_{\mu\nu}]$ is a scalar density, so all terms in the power expansion will also be scalars. There are no scalars with just odd powers of $\partial_\lambda g_{\mu\nu}$ so we can write

$$\mathcal{L}_{\text{eff}}[g_{\mu\nu}] = \sqrt{-g}\left(\frac{1}{16\pi G}(-2\Lambda) + \frac{1}{16\pi G}R\right) + O(\partial_\gamma g_{\mu\nu})^4 \qquad (2.25)$$

R is the only scalar of second order in $\partial_\lambda g_{\mu\nu}$. The constants G and Λ are defined to be the coefficients as above and are termed the induced quantities. This idea was first put forward by Sakharov (1967; see also 1982).

The $O(\partial_\gamma g_{\mu\nu})^4$ terms will contain R^2, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, and possibly logarithmic terms due to trace anomalies from massless fields.

$\mathcal{L}_{\text{matter}}[g_{\mu\nu}, \Phi] + \mathcal{L}_{\text{grav}}[g_{\mu\nu}]$ is renormalizable and scale invariant. Thus no scale symmetry violating counterterms are possible, so the coefficients G^{-1} and Λ must be finite. (However the coefficients of $O(\partial_\lambda g_{\mu\nu})^4$ need not be.) Thus the induced cosmological constant and Newton's constant should be calculable from the original Lagrangian $\mathcal{L}_{\text{matter}}$, with no possible ambiguity.

It is possible, in fact, to find a general formula for these, as is done in the next section:

As an aside, we note the analogy of induced gravity with that of superconductivity.

BCS model of superconductivity	Mass gap Δ	Weakly varying electro-magnetic field	Ginsburg-Landau theory
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Gauge theory	scale-mass term M of section 2.2	weakly varying metric	Einsteins gravity

2.5 FORMULAE FOR Λ_{ind} AND G^{-1}_{ind}

We shall derive expressions for $\frac{\Lambda}{G}$ and for G^{-1} in terms of the flat space-time Lagrangian, or actually the trace of the stress energy tensor. This derivation is due to Adler (1980b, 1980c).

For simplicity, we restrict $g_{\mu\nu}$ to be classical from this point on. Note, however that for $g_{\mu\nu}$ quantized, $\mathcal{L}_{grav}[g_{\mu\nu}]$ will make a contribution to the low energy $\mathcal{L}_{eff}$ via the high momentum gravitons. Now, the relationship between the effective Lagrangian $\mathcal{L}_{eff}$, and the matter Lagrangian $\mathcal{L}_m[\Phi]$ is

$$\exp i\int d^4x\,\mathcal{L}_{eff}[g_{\mu\nu}] = \int d[\Phi]\exp i\int d^4x\,\mathcal{L}_m[g_{\mu\nu},\Phi] \qquad (2.26)$$

where $\mathcal{L}_{eff}/\sqrt{-g} = \frac{1}{16\pi G}(R-2\Lambda)$ and remember that we neglect all terms of order $(\partial_\lambda g_{\mu\nu})^4$ and higher.

Operate on both sides with $2g_{\mu\nu}(y)\frac{\delta}{\delta g_{\mu\nu}(y)}$ where y represents an arbitrary point in an arbitrary region of space-time. So

$$\exp(i\int d^4x\,\mathcal{L}_{eff}[g_{\mu\nu}])2g_{\mu\nu}(y)\frac{\delta}{\delta g_{\mu\nu}(y)}\int d^4x\sqrt{-g}\frac{1}{16\pi G}(R-2\Lambda)$$

$$= \int d[\Phi]e^{iS_m[g_{\mu\nu},\Phi]}\,2g_{\mu\nu}(y)\frac{\delta}{\delta g_{\mu\nu}(y)}\int d^4x\,\mathcal{L}_m[g_{\mu\nu},\Phi] \qquad (2.27)$$

Now use the known results:

$$\delta\mathcal{L} = \frac{1}{2}\sqrt{-g}\;T_{\mu\nu}\,\delta g^{\mu\nu} \qquad (2.28)$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\;g_{\mu\nu}\delta g^{\mu\nu} \qquad (2.29)$$

$$\delta(\sqrt{-g}R) = \sqrt{-g}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \frac{\partial}{\partial x^{\ell}}(\sqrt{-g}\ w^{\ell}) \tag{2.30}$$

where $w^{\ell} = g^{ik}\delta\Gamma^{\ell}_{ik} - g^{i\ell}\delta\Gamma^{k}_{ik}$.

Divide by $\exp i\int d^4x\, \mathcal{L}_{eff}[g_{\mu\nu}] = \int d[\Phi]\exp iS_m[g_{\mu\nu},\Phi]$ to get:

$$\frac{-\sqrt{-g(0)}}{8\pi G}(R(y)-4\Lambda) = \frac{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi])\sqrt{-g(y)}\ T_{\mu}{}^{\mu}[g_{\mu\nu},y]}{\int d[\Phi]\exp iS_m[g_{\mu\nu},\Phi]} \tag{2.31}$$

$$= \langle\sqrt{-g}\ T_{\mu}{}^{\mu}[g_{\mu\nu},y]\rangle_0 \tag{2.32}$$

Take $g_{\mu\nu} = \eta_{\mu\nu}$ and thus

$$\frac{1}{2\pi}\frac{\Lambda}{G} = \langle T_{\mu}{}^{\mu}[\eta_{\mu\nu},y]\rangle_0 \tag{2.33}$$

This is the expression for the induced cosmological constant term for a classical metric. For a quantized metric, the expression is very similar (see appendix).

To obtain the expression for G^{-1}, we vary eqn. 2.31 around Minkowski space time so that $g^{\mu\nu} = \eta^{\mu\nu} + \delta g^{\mu\nu}$. (2.34)

Now, expand the metric using the general Riemann normal coordinates about the point y, but choose coordinates so that y=0.

i.e.
$$g^{\mu\nu}(x) = \eta^{\mu\nu} - \tfrac{1}{3}R^{\mu\alpha\nu\beta}x_{\alpha}x_{\beta} + \ldots \tag{2.35}$$

Thus
$$\delta g^{\mu\nu} = -\tfrac{1}{3}R^{\mu\alpha\nu\beta}x_{\alpha}x_{\beta} \tag{2.36}$$

and $\delta R(y) = R(y)$ and $g(y) = 1$.

So
$$\begin{aligned}\frac{-R(y)}{8\pi G} &= \frac{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi])\,\delta(T_{\mu}{}^{\mu}[g_{\mu\nu},y])}{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi])} \\ &+ \frac{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi])\sqrt{-g}(y)\ T_{\mu}{}^{\mu}[g_{\mu\nu},y]\, i\int d^4x\,\delta\mathcal{L}[g_{\mu\nu},\Phi]}{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi]} \\ &- \{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi]T_{\mu}{}^{\mu}[g_{\mu\nu},y]\} \\ &\times \frac{\{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi])\, i\int d^4x\,\delta\mathcal{L}[g_{\mu\nu},\Phi]\}}{\{\int d[\Phi]\exp(iS_m[g_{\mu\nu},\Phi])\}^2}\end{aligned} \tag{2.37}$$

$$= \langle \delta (T_\mu{}^\mu[g_{\mu\nu},y])\rangle_0$$
$$+ \langle T_\mu{}^\mu[g_{\mu\nu},y]\, i\int d^4x\, \delta\mathcal{L}[g_{\mu\nu},\Phi]\rangle_0^T$$
$$- \langle T_\mu{}^\mu[g_{\mu\nu},y]\rangle_0 \langle i\int d^4x\, \delta\mathcal{L}[g_{\mu\nu},\Phi]\rangle_0 \qquad (2.38)$$

where $\langle\ \ \rangle_0^T$ means the vacuum expectation value of the covariant time ordered product.

Now, from eqns. 2.36 and 2.28 we have

$$\int d^4x\, \delta\mathcal{L}[g_{\mu\nu},\Phi] = \frac{1}{2}\int d^4x \sqrt{-g(x)}\; T_{\mu\nu}[g_{\mu\nu},x]\, \delta g^{\mu\nu}$$
$$= -\frac{1}{6}R^{\mu\alpha\nu\beta}(y)\int d^4x\; x_\alpha x_\beta \sqrt{-g(x)}\; T_{\mu\nu}[g_{\mu\nu},x] \qquad (2.39)$$

Thus $-\frac{R(y)}{8\pi G} = \langle \delta T_\mu{}^\mu[g_{\mu\nu},y]\rangle_0$

$$-\frac{i}{6}R^{\mu\alpha\nu\beta}(y)\int d^4x\; x_\alpha x_\beta \{\langle T^\sigma_\sigma[g_{\mu\nu},y]\sqrt{-g(x)}\; T_{\mu\nu}[g_{\mu\nu},x]\rangle_0^T$$
$$-\langle T^\sigma_\sigma[g_{\mu\nu},y]\rangle_0 \langle\sqrt{-g(x)}\; T_{\mu\nu}[g_{\mu\nu},x]\rangle_0\} \qquad (2.40)$$

Since we are only working to $O(\partial_\lambda g_{\mu\nu})^2$, the first term gives zero. This can be seen by noticing that $\delta T_\mu{}^\mu$ is to be evaluated at $x = 0$ but $\delta g^{\mu\nu} \propto R^{\mu\alpha\nu\beta}x_\alpha x_\beta$ near $x = 0$. But in the expression for $\delta(\sqrt{-g}\; T_\mu{}^\mu)$,

$$\delta(g^{\mu\nu}\sqrt{-g}\; T_{\mu\nu}) = \sqrt{-g}\; T_{\mu\nu}\delta g^{\mu\nu} + 2g^{\mu\nu}\frac{\delta(\delta\mathcal{L})}{\delta g^{\mu\nu}} \qquad (2.41)$$

since $\delta\mathcal{L} = \frac{1}{2}\sqrt{-g}\; T_{\mu\nu}\delta g^{\mu\nu}$.

The first term vanishes as $O(x^2)$, and if the second term is not to vanish then the $\delta g^{\mu\nu}$ which is implicit in $\delta\mathcal{L}$ will have to be operated on by two derivatives. These derivatives can come only from either a $\partial^\lambda\partial^\theta\frac{\partial}{\partial(\partial^\lambda\partial^\theta g^{\mu\nu})}$ operator or from the square of a $\partial^\lambda\frac{\partial}{\partial\partial^\lambda g^{\mu\nu}}$ operator acting on $\mathcal{L}$. These operators appear in

$$-\sqrt{-g}\; T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \partial^\lambda\frac{\partial\mathcal{L}}{\partial(\partial^\lambda g^{\mu\nu})} + \partial^\lambda\partial^\theta\frac{\partial\mathcal{L}}{\partial(\partial^\lambda\partial^\theta g^{\mu\nu})}.$$

However these operators will give zero since does not depend on $\partial^\lambda\partial^\theta g^{\mu\nu}$ for a spin 1/2 or spin 1 field, and depends on $\partial^\lambda g^{\mu\nu}$ only once for a spin 1/2 field and not at all for a spin 1 field.

For a spin 0 field the only term that can contribute is the $R\phi^2$ term, but this was assumed (section 2.3) to be nonexistent in $\mathcal{L}$. Thus $\langle \delta T_\mu{}^\mu[g_{\mu\nu},y]\rangle_0$ is zero to order $(\partial^\lambda g^{\mu\nu})^2$. This would not be so if $g^{\mu\nu}$ were quantized.

With this in mind we can now partially take the Minkowski limit of eqn. 2.40 and write:

$$\frac{-R(y)}{8\pi G} = -\frac{i}{6}R^{\mu\alpha\nu\beta}(y)\int d^4x\; x_\alpha x_\beta\{\langle T(y)T_{\mu\nu}(x)\rangle_0^T - \langle T(y)\rangle_0\langle T_{\mu\nu}(x)\rangle_0\} \tag{2.42}$$

where for convenience the notation $T(x) = T^\sigma_\sigma[\eta_{\mu\nu},x]$ is used.

We now note that both these vacuum expectation values can be written as total divergences which then allows us to twice integrate by parts.

Note first that Lorentz covariance implies that $\langle T_{\mu\nu}(x)\rangle_0$ can be written as

$$\langle T_{\mu\nu}(x)\rangle_0 = \Lambda_1(x^2)\eta_{\mu\nu} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\Lambda_2(x^2) \tag{2.43}$$

[Perhaps more familiar as a Fourier transform in p_μ
$\langle \tilde{T}_{\mu\nu}(p)\rangle_0 = \tilde{\Lambda}_1(p^2)\eta_{\mu\nu} - p_\mu p_\nu\tilde{\Lambda}_2(p^2)$.]

Now the conservation law $\frac{\partial}{\partial x_\mu}\langle T_{\mu\nu}(x)\rangle_0 = 0$ (2.44)

implies, as a Fourier transform, $p^\mu\langle \tilde{T}_{\mu\nu}(p)\rangle_0 = 0$.

Thus $$p_\nu(\tilde{\Lambda}_1(p^2) - p^2\tilde{\Lambda}_2(p^2)) = 0 \tag{2.45}$$

so $$\Lambda_1(x^2) = \Box^2\Lambda_2(x^2) \tag{2.46}$$

Thus $$\langle T_{\mu\nu}(x)\rangle_0 = [\Box^2\eta_{\mu\nu} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}]\Lambda(x^2) \tag{2.47}$$

for some scalar function $\Lambda(x^2)$.

Substituting this into the integral, we have

$$R^{\mu\alpha\nu\beta}\int d^4x \; x_\alpha x_\beta \; \langle T_{\mu\nu}(x)\rangle_0$$

$$= R^{\mu\alpha\nu\beta}\int d^4x \; \Lambda(x^2) \; (\Box^2 \eta_{\mu\nu} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}) x_\alpha x_\beta$$

$$= 3R\int d^4x \; \Lambda(x^2)$$

$$= \frac{3}{8}R\int d^4x \; \Lambda(x^2)\Box x^2$$

$$= \frac{1}{8}R\int d^4x \; x^2\langle T^\mu_\mu(x)\rangle_0 \qquad (2.48)$$

This argument proceeds exactly the same for $\langle T(0)T_{\mu\nu}(x)\rangle_0^T$ since it obeys the same conservation law.

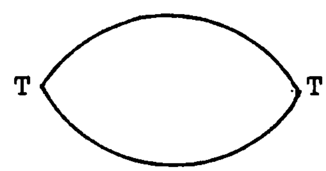
Thus, we can now divide by $R(y)$ to find

$$\frac{1}{8\pi G} = \frac{i}{48}\int d^4x \; x^2\{\langle T(y)T(x)\rangle_0^T - \langle T(y)_0\langle T(x)\rangle_0\} \qquad (2.49)$$

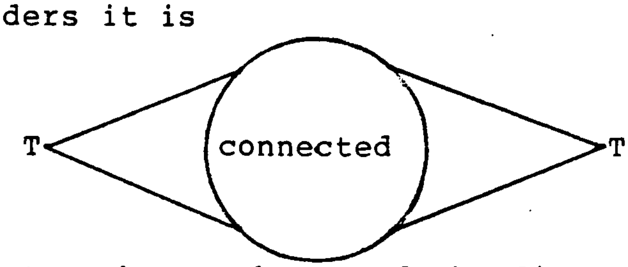
If we prefer, we can put y to zero and define $\hat{T}(x) = T(x) - \langle T(x)\rangle_0$, so that $\frac{1}{16\pi G} = \frac{i}{96}\int d^4x \; x^2 \langle\hat{T}(0)\hat{T}(x)\rangle_0^T$ (2.50)

This is the usual expression for the induced gravitational constant.

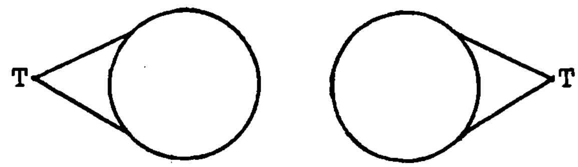
It is interesting to consider the diagram for $\langle\hat{T}(0)\hat{T}(x)\rangle_0^T$. To lowest order it is



To all orders it is



Recall that we have subtracted the disconnected part, which is



Now let $\Psi(k)$ be the Fourier transform of $\langle \hat{T}(0)\hat{T}(x)\rangle_0^T$, so

$$\frac{1}{16\pi G} = \frac{i}{96}\int d^4x \frac{d^4k}{(2\pi)^4}\, x^2 e^{-ikx}\Psi(k)$$

$$= \frac{-i}{96}\int d^4k \frac{d^4x}{(2\pi)^4} e^{-ikx} \Box_k^2 \Psi(k)$$

$$= \frac{-i}{96}\Box_k^2 \Psi(k)\Big|_{k=0} \tag{2.51}$$

But $\Psi(k)$ is a scalar function so is only a function of k^2, and

$$\Box_k^2 \Psi(k^2) = 8\Psi'(k^2) + 4k^2\Psi''(k^2)$$

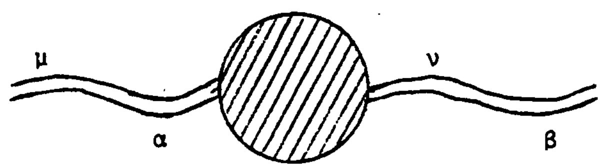
therefore

$$\frac{1}{16\pi G} = \frac{-i}{12}\Psi'(k^2)\Big|_{k=0} \tag{2.52}$$

Note the simplicity of our final expression.

The diagram for $\Psi(k^2)$, with the contraction of indices as $\Psi^{\mu}{}_{\mu}{}^{\nu}{}_{\nu}(k^2)$, is the amputated version of the following:

$\Psi^{\mu\alpha\nu\beta}$ =



(The double wavy lines represent gravitons.)

Observe that since $g_{\mu\nu}$ is classical there are no graviton loops in this diagram. If $g_{\mu\nu}$ were to be quantized then there would be graviton loops since the virtual graviton fluctuations would have to be included into the energy-momentum tenson $T_{\mu\nu}$.

The next chapter deals with attempted calculations of G^{-1}.

The Feynman diagram will in general give a divergent result and this must be removed, usually by analytic continuation of the dimension of the integrals, although other regularization methods are possible.

3. PREVIOUS ATTEMPTS AT MODEL CALCULATIONS FOR G^{-1}

In this chapter we consider all previously published work on the calculation of G^{-1}, both model calculations and more general considerations. These calculations are in approximate chronological order.

3.1 INSTANTON GAS APPROXIMATION IN A PURE SU(2) GAUGE THEORY

It is possible to evaluate an approximation to the full non-perturbative generating functional Z, by considering only certain non-perturbative effects, the instantons. An instanton is a finite action solution to the classical equations of motion obeying certain boundary conditions (localization in space-time). It is thus a stationary point of the Euclidean action. The generating functional $Z = \int d[\phi] e^{-S[\phi]}$ is dominated by stationary points, so to approximate the functional integral we consider 1-loop quantum fluctuations about the instanton. We then integrate over all positions and sizes of the instantons, and sum over all possible multi-instanton contributions. However when instantons overlap calculations become complicated and other non-pertubative effects may become important (e.g. merons), so we only consider non overlapping instantons. Thus we integrate over sizes of instantons, ρ, up to a maximum ρ_{max}. This is the "dilute instanton gas approximation". For a good review see Coleman (1977). The ρ_{max} is an artificial cut-off.

The 1-loop correction to the pure SU(2) gauge theory generating functional with a general conformally flat Euclidean metric $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$ was considered by Hasslacher & Mottola (1980), using the dilute instanton gas approximation. The resultant 1-loop action $S_{1\text{-loop}} = -\ln Z$ is an approximation to $S_{eff}[g_\mu]$ (eqn. 2.24), and thus enables G^{-1} and Λ to be evaluated without the use of the general formula $\frac{3}{4}iG^{-1} = \pi\psi'(k^2)\big|_{k=0}$ (eqn. 2.52).

The actual quantity calculated was

$$T^{\mu}_{\mu\ 1\text{-loop}} = \frac{2g^{\mu\nu}}{g^{1/2}} \frac{\delta S_{1\text{-loop}}}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}(R-2\Lambda) + O(\partial_\lambda g_{\mu\nu})^4 \tag{3.1}$$

The restriction to slowly varying metrics, so as to drop $O(\partial^\lambda g^{\mu\nu})^4$ terms, ensures a simple solution:

$$\frac{1}{8\pi G}(R-2\Lambda) = \int_0^{\rho_{max}(R)} \frac{d\rho}{\rho^5}\left[\frac{22}{3} + \frac{5}{6}\left(\frac{19}{9} - \log\frac{48}{\rho^2 R}\right)\rho^2 R\right] D(\rho) \tag{3.2}$$

$$D(\rho) = bx^4 e^{-x} \quad \text{(Bernard 1979)}$$

where $x = \dfrac{8\pi^2}{g^2(\rho)}$ and $b \approx 0.016$.

We know how $g^2(\rho)$ varies with ρ for small g (the ultraviolet region) from the scale dependence given in eqn. 2.10

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g^2(\mu)} - \frac{22}{3}\ln(\mu\rho) \tag{3.3}$$

where μ is an arbitrary renormalization point.

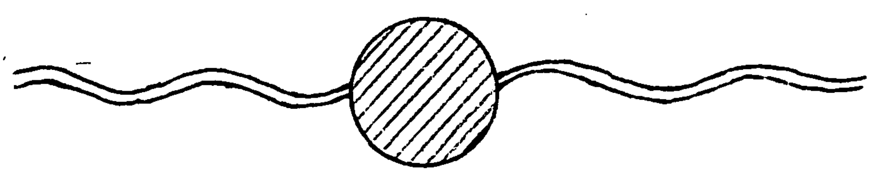
Thus $e^{-x} = (\mu\rho)^{22/3} e^{-x(\mu)}$ and consequently the above integral converges in the ultraviolet region.

The first thing to note from the above is that the existence of an induced G^{-1} is indicated. Unfortunately $\rho_{max}(R)$ is not known, it requires a more detailed analysis of the infrared region, and so G^{-1} can not be calculated reliably. Also, it is possible that the $(\log \rho^2 R)\rho^2 R$ term is merely an artifact of the dilute instanton gas approximation, as argued by Adler (1982), and that this will be compensated by either other non-perturbative effects or by the dependence of ρ_{max} on R. Thus the significance of this term is unclear, which is unfortunate since the sign of G^{-1} depends on this term and ρ_{max}!

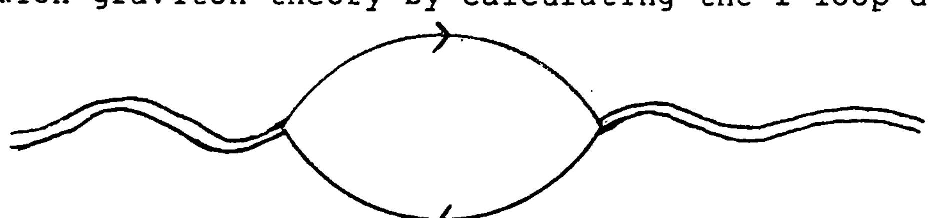
Observe, however, two things. The first is the presence of a term $e^{-8\pi^2/g^2}$ which is associated with non-perturbative effects and which can be very small for g small. The other thing to notice is that, in the UV region at least, the minus sign in e^{-x} ensures that with a low energy μ, there cannot be a large contribution to G^{-1} from the gauge fields of an SU(2) gauge theory, i.e. $G^{-1} = O(\mu^2)$. Of course, this need not be valid for other fields.

3.2 A ONE LOOP CALCULATION WITH MASSIVE FERMIONS

Zee (1981a) has calculated an approximation to



for a fermion-graviton theory by calculating the 1-loop diagram



The fermion is given a mass m, and two Pauli-Villars regulators are used. (This calculation has been done before. See

Sakharov 1975; Akama, Chikashige, Matsuki & Terazawa 1978.) However the purpose of these regulators is not just to control the infinities, it is to simulate dynamical symmetry breaking. i.e. the two masses of the regulators, m_1 and m_2, are to be considered as perfectly physical. The momentum dependence of the mass m(q) is then approximated by a cutoff at the regulator mass. Only p^2 terms are considered, so only two regulator masses are required. The result Zee obtains is

$$G^{-1} = \frac{2}{3\pi}\{m_2{}^2(\frac{m_1{}^2-m^2}{m_1{}^2-m_2{}^2})\log\frac{m_1{}^2}{m_2{}^2} - m^2\log\frac{m_1{}^2}{m_2{}^2}\} \tag{3.4}$$

Although this does not give a definite value for G^{-1}, it is still possible to draw some conclusions. We see that the sign of G^{-1} depends on the mass ratios between m_1, m_2 and m; G^{-1} is positive for $m^2 < m_1{}^2, m_2{}^2$. Since m_1 and m_2 are supposed to simulate the dynamical symmetry breaking, we can conclude that the sign of G^{-1}, and not just the value, depends on the detailed mechanism of this breaking. (This dependence on the detailed dynamics was seen before in the instanton gas approximation.)

A second conclusion is that to obtain a realistic value for G^{-1}, the masses of the regulators must be of the order of the Planck mass. This conclusion is not very strong, however, since a 1-loop calculation can hardly simulate non-perturbative effects.

The actual calculation by Zee is interesting in that it introduces a number of tricks to simplify the algebra. These tricks comprise replacing the fermion loop by a 1-loop scalar-graviton vertex, and then letting one of the scalar momenta go to zero. This is only useful in obtaining a value of G^{-1}, and one loses the tensor information of the original diagram.

3.3 SPECTRAL ANALYSIS OF $\langle \hat{T}(x)\hat{T}(0)\rangle_0^T$

If we consider $\langle \hat{T}(x)\hat{T}(0)\rangle_0^T$ as a scalar function of x, then we can formally write down its spectral representation.

$$\langle \hat{T}(x)\hat{T}(0)\rangle_0^T = \int_0^\infty d\sigma^2 \; \rho(\sigma^2)\, i\Delta_F(x,\sigma) \tag{3.5}$$

$$\text{i.e. } \Psi(k) = \int_0^\infty d\sigma^2 \, \frac{i\rho(\sigma^2)}{k^2-\sigma^2+i\varepsilon} \tag{3.6}$$

where
$$\rho(\sigma^2) = (2\pi)^3 \sum_n \delta^4(p_n-\sigma)\,|\langle 0|T(0)|n\rangle|^2 \tag{3.7}$$

Thus
$$G^{-1} = -\frac{4}{3\pi}\int_0^\infty \frac{\rho(\sigma^2)}{\sigma^4} d\sigma^2 \tag{3.8}$$

Since however $\rho(\sigma^2)$ is gauge invariant, it can be evaluated in a gauge where there are no ghosts (such as the axial gauge) and consequently the Hilbert space metric is positive definite. Thus $\rho(\sigma^2) \geq 0$ and G^{-1} is always negative.

As pointed out by Adler (1980b), this argument is flawed. This is because $\frac{\rho(\sigma^2)}{\sigma^4}$ is divergent as $\sigma \to \infty$. This is because T contains a trace anomaly which in gauge theories is (Collins, Duncan & Joglekar 1977)

$$T = \frac{2\beta(g)}{g}\,\frac{1}{4}(F^a_{\mu\nu}F^{a\mu\nu})^{\text{renormalized}} + (1+\delta(g))(\bar{\psi}m_0\psi)^{\text{renormalized}} \tag{3.9}$$

This is true to all orders in perturbation theory.

Since we have an asymptotically free theory, the $\sigma \to \infty$ is just the free field limit, thus

$$\langle\hat{T}(x)\hat{T}(0)\rangle_0^T \quad \text{behaves as} \quad \left(\frac{\partial}{\partial x^\mu}\right)^4 (\Delta_1(x))^2$$

where $\Delta_1(x) \sim \frac{m}{x}K_1(mx)$ is the free field space-like propagator for a spin-1 particle with mass m, where K_1 is the modified Bessel function of the third kind. (Bjorken & Drell 1965, Appendix C)

Thus, since $\Delta_1(x) \sim \frac{1}{x^2}$, we have

$$\langle\hat{T}(x)\hat{T}(0)\rangle_0^T \sim \frac{1}{(x^2)^4} . \tag{3.10}$$

Actually, the true behavior is the above modified by logarithms, as can be seen from the Wilson operator product expansion (Wilson 1968; Zimmerman 1970; see also Itzykson & Zuber 1980, p. 672).

$$\langle\hat{T}(x)\hat{T}(0)\rangle_0^T \sim \sum_N C_N(x)\; O_N\left(\frac{x}{2}\right) \tag{3.11}$$

where O_N are a sequence of operators, and C_N are C-number coefficients. The behavior, perturbatively, is

$$x^\gamma \times \text{(possible power series in log x)}$$

where γ is the canonical dimensionality of the operator O_N.

Thus $$\langle\hat{T}(x)\hat{T}(0)\rangle_0^T \sim \frac{1}{(x^2)^4} \times \text{logarithms} + \ldots \tag{3.12}$$

so $$\rho(\sigma^2) \sim \sigma^4 \times \text{logarithms}(\sigma^2) \tag{3.13}$$

Regardless of logarithms, the integral $\int^\infty \frac{\rho(\sigma^2)}{\sigma^4}$ is divergent. In the language of dispersion relations, we can say that $\Psi(k)$ does not obey an unsubtracted dispersion relation. Alternatively, we could hope to handle the diverges using a dimensional regularization approach. However an apparently positive integral can turn out negative once the divergences have been subtracted off.

3.4 AN INFRARED-STABLE YANG-MILLS THEORY

The problems in the previous model calculations have been in the infrared. In the theories considered so far the problem has been the lack of knowledge of the IR behavior. However, it is possible to obtain the IR behavior of a Yang-Mills theory in a certain restrictive case, and that is an infrared stable theory with a small coupling constant. This is, of course, only a toy model. With such a model, G^{-1} is calculable (Zee 1982a).

The infrared-stable fixed point is $g_*{}^2 = \frac{-b_0}{2b_1}$

where $$\beta(g) = -\tfrac{1}{2}b_0 g^3 - b_1 g^5 - b_2 g^7 + \ldots \tag{3.14}$$
we thus require b_0 to be small and positive (so as to retain asymptotic freedom), and b_1 to be negative. An example of such a theory is QCD with 16 flavours; however, these calculations are valid for any IR-stable Yang-Mills theory.

With g_*^2 sufficiently small, all terms of $O(g_*{}^2)^2$ are ignored. The exact behavior of $g(-q^2)$ can then be found. Note that once the nature of the theory has been chosen (i.e. the gauge group and the representations) g_*^2 becomes a fixed quantity.

The UV behavior is the usual asymptotic freedom as expected and the IR behavior is

$$\frac{g^2(x)}{g_*^2} \to 1 - e(\mu x)^{-b_0 g_*^2} \tag{3.15}$$

Since g^2 is a function of q^2 and μ^2, and since g_*^2 is fixed, the renormalization point, μ, is fixed by choosing $g^2(-q^2) = \frac{1}{2}g_*^2$ at $q^2 = \mu^2$. (Note that the factor $\frac{1}{2}$ is chosen arbitrarily, but it must be a positive number less than 1.) Thus μ^2 is now a measurable quantity.

The function $\Psi(x) = \langle\hat{T}(x)\hat{T}(0)\rangle_0^T$ is now approximated by C_Ψ/x^8, i.e. its free field value, see eqn. 3.10. Adler (1982) has evaluated the constant C_Ψ for an SU(n) gauge theory to be

$$C_\Psi = \frac{3\times 2^6}{(2\pi)^4}(n^2-1) \tag{3.16}$$

From all this, Zee obtains the formula

$$\frac{G^{-1}}{\mu^2} = \frac{-\pi^3}{3}\,\frac{C_\Psi b_0}{16}\,g_*^2 \lim_{\eta\to\gamma} e^{-\eta}J(\eta) \tag{3.17}$$

where $\quad \gamma = \dfrac{2}{b_0 g_*^2} > 0.$

The function $J(\eta)$ is a divergent integral for $\eta > 0$.

$$J(\eta) = \int_0^\infty dx\,\frac{x^{1+\eta}}{(1+x)^3}e^{\eta x} \tag{3.18}$$

Thus to evaluate it we require a regularization process for instance a dimensional one. The integral is evaluated for $\eta < 0$, where it is defined, and the resulting expression is then analytically continued to $\eta > 0$. The resulting expression for $J(\eta)$ still has problems, since it has a cut along the positive real axis, which is where γ lies. Thus $J(\gamma)$ is not yet defined.

We define $J(\gamma)$ by symmetrizing from just above and just below the cut; i.e.

$$J(\gamma) = \frac{1}{2}[J(\gamma+i\varepsilon)+J(\gamma-i\varepsilon)] \tag{3.19}$$

This process of defining a divergent integral by analytically continuing a parameter which it contains, gives a unique and unambiguous answer. The essence lies in the property of analytic continuation, which is known to always give a unique process no matter how it is carried out. For further discussion, see Adler (1982) or Zee (1982a).

The resultant expression for $J(\gamma)$ can be evaluated by numerical integration, or by the method of steepest descent, which is what Zee does.

The result is

$$\frac{G^{-1}}{\mu^2} = \frac{\pi^4 C_\Psi}{48\ g_*^4} \exp(-4/b_0 g_*^2)\ \sin(\frac{2\pi}{b_0 g_*^2}) \tag{3.20}$$

Thus, the mass ratio G^{-1}/μ^2 is in terms of entirely known quantities. Furthermore μ is measurable, so the induced Newtonian constant for this toy model is completely predicted. Note however, that the sign of G is very sensitive to the position of the fixed point, i.e. to the details of the infrared region. Zee (1981b) has suggested that this is a universal feature, and that a general argument for the sign of G may not exist.

This toy model is useful in that it illustrates the principle of induced gravity, and the problems with which one must come to grips.

3.5 A LATTICE CALCULATION FOR A PURE SU(n) GAUGE THEORY

A procedure was outlined by Adler (1982) to calculate G^{-1} using the technique of Monte Carlo simulation on a lattice. The idea of using a lattice for gauge theories was originated by

Wilson (1974). The Monte Carlo method is one of the more successful methods employed for lattice calculations, and was introduced by Wilson and extended by Creutz (1980). This method is non-perturbative and is thus used to examine the IR behaviour of a gauge theory. The successes of lattice gauge theories have been to both pure Yang-Mills theory and also Yang-Mills theory with fermions. However, we shall only consider the former.

Pure Yang-Mills gauge theory has no classical scale, but it has a non-perturbative dynamically induced scale. This scale can be written in a number of forms and related to the scale parameters used in QCD, such as Λ_{MS}, Λ_{mom}. We also have the lattice scale Λ_L, the string tension $\sqrt{K}$, and the scale introduced in eqn. 2.3, M. (These are all formally renormalization point independent, though the perturbative QCD scale parameters do depend on the renormalization scheme.) The most accurately measured scale is the string tension $\sqrt{K}$, defined as the coefficient in the heavy quark-antiquark potential

$$V_{static}(x) \rightarrow Kx \quad \text{as} \quad x \rightarrow \infty.$$

Its value can be obtained either from the Regge slope or from the phenomenology of heavy quark bound states, and it is found that $\sqrt{K}$ lies between 400 and 500 MeV (Eichten, Gottfried, Kinoshita, Lane & Yan 1980). Using Monte Carlo techniques, this can be related to the lattice scale Λ_L via $\Lambda_L = (6 \pm 1) \times 10^{-3}\sqrt{K}$ (Creutz & Moriaty 1982). This parameter can be related to the Λ parameter in continuum QCD, (Hasenfratz & Hasenfratz 1980, Billoire 1981), which can be measured from deep inelastic scattering experiments.

There is, for us, a more relevent parameter, namely the vacuum energy density, or gluon condensate

$$\frac{-2\beta(g)}{b_0 g^3}\langle 0|\frac{g^2}{4\pi^2}F_{\mu\nu}^{i}F^{i\mu\nu}|0\rangle.$$

This parameter can be measured by using the sum rules for charm production in e^+e^- annihilation (Shifman, Vainshtein & Zakharov 1979) or it can be related to the string tension via Monte Carlo techniques (see, for example, Di Giacomo & Paffuti 1982). Its value is approximately 0.012 GeV^4.

This quantity is of interest since it is related to the trace anomaly $\langle T_\mu{}^\mu\rangle_0$, and this is related to the induced cosmological constant (eqn. 2.5), so

$$\frac{1}{2\pi}\frac{\Lambda_{ind}}{G_{ind}} = -\pi^2 b_0 \times (\text{gluon condensate}) \tag{3.21}$$

$$= \frac{\beta(g)}{2g}\langle F_{\mu\nu}^{i}F^{i\mu\nu}\rangle_0$$

Compared with the experimental value, which is $\frac{1}{2\pi}\frac{\Lambda}{G} \sim 10^{-44}$ GeV^4, we see we have a discrepancy. This is the famous cosmological constant problem. It is usually said that a resolution of this problem will come from a new symmetry which will imply that the induced $\frac{\Lambda}{G}$ from other particles will exactly cancel out the above so the total $\frac{\Lambda}{G}$ will be zero. This, of course, is speculation.

This calculation of G^{-1} from a lattice has not yet been performed, for the method requires knowledge of the IR behavior of the function $\Psi(x) = \langle \hat{T}(x)\hat{T}(0)\rangle_0^T$. The approximate behavior is known from general considerations, since $\Psi(x)$ is a correlation function, and has to be exponentially finite, i.e.

$$\Psi(x) \to \exp(-m_g x) \quad \text{for} \quad x \to \infty. \tag{3.22}$$

The mass parameter m_g, called the glueball mass or mass gap, is the mass of the lowest lying glueball states; suspected to have $J^{PC} = 0^{++}$.

This parameter can be calculated using Monte Carlo techniques, see example Berg & Billoire (1983), who give $m_g = (280 \pm 50)\Lambda_L \approx 750$ MeV.

The exact nature of $\Psi(x)$ in the IR region, however, has yet to be computed. It is however non-divergent and so can be integrated.

On the other hand the UV part of the $\Psi(x)$ is known from asymptotic freedom and the Wilson operator product expansion, (see eqn. 3.11). Consequently we know that $\Psi(x)$ behaves like $\frac{1}{(x^2)^4}\times$logs for $x \to 0$, and thus the integral $\int d^4x\, x^2\Psi(x)$ is divergent. To evaluate this integral we subtract off the divergent piece $\Psi_d(x)$ leaving a convergent integral which can be evaluated by numerical integration. The divergent piece $\Psi_d(x)$ must then be evaluated separately by a process of analytic continuation.

We can easily evaluate the most divergent part of $\Psi_d(x)$ by asymptotic freedom, since this implies we consider the lowest loop contribution.

$$\text{Thus } g^2(-x^2) = \frac{g^2(\mu^2)}{1-\frac{1}{2}b_0 g^2(\mu^2)\log(-x^2\mu^2)} \tag{3.23}$$

$$\sim (\log(-x^2\mu^2))^{-1} \quad \text{as} \quad x^2 \to 0 \tag{3.24}$$

Furthermore $\frac{\beta(g)}{g} \sim g^2$, so

$$T \sim \frac{1}{x^4}(\log(-x^2))^{-1} \tag{3.25}$$

Thus $\Psi_d(x) \sim \frac{1}{x^8(\log(-x^2))^2}$ as $x^2 \to 0$ (3.26)

We have thus evaluated the logarithms in eqn. 3.12.

Consideration of all loops leads to the general expression

$$\Psi_d(x) = \frac{1}{4} b_0{}^2 C_\Psi \frac{g^4}{x^8}(1+\sum_{n=1}^{\infty} C_n g^{2n}) \tag{3.27}$$

where C_Ψ is given in eqn. 3.16. Note that evaluation of C_n requires consideration of all n+1 loop contributions. At present, only two loop calculations have been performed (Kataev, Krasnikov & Pivovarov 1982). Note also that there cannot be a $1/x^6$ term in Ψ_d, since by the operator product expansion this would imply an internal symmetry (including scale) invariant operator of mass dimension 2, and no such operator exists in YM gauge theories.

Since we are evaluating the UV and IR pieces by different techniques, we must define a crossover point, x_0 say. This would presumably be a small fixed number, but would depend on the accuracy of our other calculations, namely the coefficients C_n and the Monte Carlo simulation of the IR region of $\Psi(x)$.

The integration of the above divergent integral, namely $\int_0^{x_0} d^4x\ x^2\Psi_d(x)$ has been evaluated by Adler to the extent that, by a process of analytic continuation, it has been re-expressed in a form which no longer has divergences. The actual analytic continuation is interesting in that the integral is reformulated into an expression which has a branch cut in the space-time dimension, 2ℓ, from $2\ell = 2$ to $2\ell = \infty$. Consequently the true value at $2\ell = 4$ must be found by approaching the point $2\ell = 4$ from just below and just above the branch cut.

Adler has thus reduced the calculation of G for a pure SU(n) gauge theory in terms of Monte Carlo simulations, numerical integrals and evaluation of perturbation theory coefficients. This result is interesting in that it should be possible within the near future to obtain some information from this program. Furthermore, although present efforts are concentrated on SU(2) or SU(3), it is not too difficult to extend these to SU(5) and thus relate G^{-1} to the various mass scales of the grand unified theories. Indications are that no great surprises are expected in the results from SU(2), SU(3) or SU(5), except, of course, for the actual values of the scale parameters.

3.6 BOUNDS ON G^{-1} IN AN ASYMPTOTICALLY FREE GAUGE THEORY

(In this section only, we shall adopt Khuri's notation $\Psi(q^2) = -i\int d^4x\, e^{iqx}\langle\hat{T}(x)\hat{T}(0)\rangle_0^T$ and so $(16\pi G)^{-1} = \frac{1}{12}\Psi'(0)$, instead of eqn. 2.52.)

Khuri (1982a, 1982b, 1982c) has established some general results for an asymptotically free gauge theory with massless fermions. These conditions imply (see eqn. 3.26 and Khuri 1982b)

$$\Psi(q^2) \to -C_A q^4 \left(\log\frac{-q^2}{\Lambda^2}\right)^{-1} \qquad \text{as } -q^2 \to \infty \tag{3.28}$$

where $\qquad C_A = C_\Psi(\pi^2/2^6 \times 3) \qquad$ (see 3.16)

Khuri's results originate from a study of the zeros of $\Psi(q^2)$, the techniques used are an adaptation of the work by Jin & Martin (1964). The main ingredients are consequences of analyticity.

As an example of the power but simplicity of this approach, we assume $\Psi(q^2)$ has precisely one zero, at $-\mu^2 < 0$. We define a function $H(q^2)$ by

$$H(q^2) = \frac{-\Psi(q^2)}{q^2+\mu^2} \tag{3.29}$$

By eqn. 3.28 $H(q^2)/q^2 \to 0$ as $|q^2| \to \infty$ and so we can write down a once-subtracted dispersion relation for $H(q^2)$

$$H(q^2) = H(0) + \frac{q^2}{\pi}\int_0^\infty \frac{\mathrm{Im}\, H(\sigma^2)}{\sigma^2(\sigma^2-q^2)}\, d\sigma^2 \tag{3.30}$$

Thus

$$\Psi(q^2) = \frac{\Psi(0)}{\mu^2}(q^2+\mu^2) - \frac{q^2(q^2+\mu^2)}{\pi}\int_0^\infty \frac{\mathrm{Im}\, H(\sigma^2)}{\sigma^2(\sigma^2-q^2)}\, d\sigma^2 \tag{3.31}$$

Differentiating and putting $q^2 = 0$, we have

$$(16\pi G)^{-1} = \frac{\Psi(0)}{12\mu^2} + \frac{\mu^2}{12\pi}\int_0^\infty \frac{\mathrm{Im}\ \Psi(\sigma^2)}{\sigma^4(\sigma^2+\mu^2)}\, d\sigma^2 \tag{3.32}$$

But $$\Psi(k^2) = \int_0^\infty d\sigma^2\ \frac{\rho(\sigma^2)}{k^2-\sigma^2+i\varepsilon} \tag{3.33}$$

(see eqns. 3.6, 3.7)

and so $$\mathrm{Im}\ \Psi(\sigma^2) = -\pi\rho(\sigma^2) < 0 \text{ for } \sigma^2 > 0. \tag{3.34}$$

Thus $$(16\pi G)^{-1} = \frac{\Psi(0)}{12\mu^2} - \frac{\mu^2}{12}\int_0^\infty \frac{\rho(\sigma^2)}{\sigma^4(\sigma^2+\mu^2)}\, d\sigma^2 \tag{3.35}$$

Thus, in one case, we have determined an upper bound on G^{-1},

$$(16\pi G)^{-1} < \frac{\Psi(0)}{12\mu^2}\ . \tag{3.36}$$

The properties of $\Psi(q^2)$ (analyticity on the plane with the exception of $0 \leq q^2 < \infty$; and eqn. 3.34) imply that $H(q^2)$ defined in eqn. 3.29 is a particular type of function called a Herglotz function. (Khuri 1982a; Jin & Martin 1964; Khuri 1969; Shohat & Tamarkin 1943)

If $\Psi(Z)$ has n real zeros (not necessarily distinct) at $Z = -\mu_j^2$ $j=1,\ldots,n$, and N complex-conjugate zeros at m_i^2 $i=1,\ldots,N$; then

$$H(Z) = \frac{-\Psi(Z)}{\prod_{i=1}^{N}(Z_i-m_i^2)(Z_i-m_i^{*2})\prod_{j=1}^{n}(Z+\mu_j^2)} \tag{3.37}$$

is a Herglotz function.

A key property of Herglotz functions is their boundedness;

$$C|Z|^{-1} \leq |H(Z)| \leq C'|Z| \quad \text{as } |Z| \to \infty. \tag{3.38}$$

This result, and the restriction of eqn. 3.28 combine to limit the number and type of zeros that $\Psi(q^2)$ can have. It is easy to see that there must be at most two zeros. Khuri analyzes all possible cases and derives sum rules analogous to eqn. 3.35 for each case. Assuming that the induced G^{-1} is positive, then Khuri shows that if $\Psi(q^2)$ has one real zero, then $\Psi(0)$ must be positive (see eqn. 3.35), if $\Psi(q^2)$ has one real zero at $q^2 = 0$ then $\Psi(0) = 0$, and if $\Psi(q^2)$ has a pair of complex-conjugate zeros, then $\Psi(0)$ must be less than zero.

To derive an upper bound in terms of the mass scale of the theory we again use dispersion relations. In the case of one real zero, for instance, let

$$h(q^2) = \frac{(q^2+\mu^2)}{\Psi(q^2)} \tag{3.39}$$

Now $$\operatorname{Im} h(\sigma^2) = -(\sigma^2+\mu^2)\,\frac{\operatorname{Im}\Psi(\sigma^2)}{|\Psi(\sigma^2)|^2}$$

$$= \pi(\sigma^2+\mu^2)\ \frac{\rho(\sigma^2)}{|\Psi(\sigma^2)|^2} \tag{3.40}$$

So $\text{Im}\ h(\sigma^2) > 0$ and $|h(\sigma^2)| \to 0$ as $|q^2| \to \infty$ (3.41)

Thus $h(q^2)$ obeys an unsubtracted dispersion relation

$$h(q^2) = \int_0^\infty \frac{\rho(\sigma^2)(\sigma^2+\mu^2)}{|\Psi(\sigma^2)|^2(\sigma^2-q^2)}\, d\sigma^2 \tag{3.42}$$

Thus $h(0) > 0$ and $h'(0) > 0$.

Now, from eqn. 3.39,

$$\Psi'(0) = (h(0) - \mu^2 h'(0))/[h(0)]^2.$$

so $\quad \Psi'(0) \leq 1/h(0)$

From 3.42, we can write

$$16\pi G \geq 12 \int_{L^2}^\infty \frac{\rho(\sigma^2)}{|\Psi(\sigma^2)|^2}\, d\sigma^2 \tag{3.43}$$

Choose L^2 large enough so that $\Psi(\sigma^2)$ can be approximated by eqn. 3.28. (e.g. $L^2 = 100\Lambda^2$)

Now since

$$\rho(\sigma^2) \to C_A\ \sigma^4 (\ln\frac{\sigma^2}{\Lambda^2})^{-2} \tag{3.44}$$

we have

$$(16\pi G)^{-1} \leq \frac{100}{12}\, C_A\ \Lambda^2 \tag{3.45}$$

Khuri calculates the other two cases in a similar manner. The final result is the greatest of the three upper bounds and gives for a pure SU(N) gauge theory.

$$(16\pi G)^{-1} \leq (\frac{25}{12\pi^2})(N^2-1)\Lambda_N^{\ 2} \tag{3.46}$$

(The number $\frac{25}{12\pi^2}$ would vary if we had a different prescription for the onset of the validity of asymptotic freedom.)

This result is the most important result since it implies that the origin of the induced gravitational constant must come from either a non-asymptotically free theory or a theory with a very large mass scale.

If we assume that the scale Λ is very large, then the question arises as to how close G^{-1} comes to the upper bound. Khuri (1982c) argues that, if we assume G^{-1} to be positive then $G^{-1} = O(M_{zero}^2)$, where M_{zero} is the mass of the zero of $\Psi(q^2)$. Furthermore Khuri shows that one would expect to obtain a small value of M_{zero} and thus not obtain a realistic value of G^{-1}. The only way to obtain a realistic value of G^{-1} is if the zeros of $\Psi(q^2)$ are a pair of complex conjugates at

$$q^2 = m_0^2 \pm i\gamma M_0 \qquad (3.47)$$

where $M_0 \sim O(\Lambda)$ and $\gamma \ll M_0$.

Whether or not $\Psi(q^2)$ does have this type of zeros is not known, but Khuri shows that it is not impossible.

Assuming the above zeros, it is possible by similar techniques to the derivation of the upper bound to find a lower bound for G^{-1}. This is

$$\frac{\pi^2}{16(\log 10)144} C_\Psi M_0^2 \leq (16\pi G)^{-1} \qquad (3.48)$$

These results give tight restrictions on the properties that $\Psi(q^2)$ must have to give a realistic G^{-1}. Furthermore, the lower and upper bounds on G^{-1} are very restrictive. If we assume asymptopia starts at $q^2 \sim 10\Lambda^2$, then the ratio of the upper to the lower bound is approximately 50.

4. THE FUNDAMENTAL GRAVITATIONAL LAGRANGIAN

We now consider one further formal development, namely that of a Lagrangian which is of fourth order in the metric derivatives. Such terms must be considered since a general covariant matter Lagrangian will require such terms as counter terms upon quantization of the gravitational field.

4.1 OVERVIEW OF THE QUANTIZED LAGRANGIAN

The general scale invariant polynomial Lagrangian of the quantized metric (or equivalently the vierbein) on a four-dimensional manifold without boundaries is

$$L_{grav} = aR^2+bR_{\mu\nu}R^{\mu\nu}+dR_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} \tag{4.1}$$

This can obviously also be written as

$$L = r_0R^2+s_0\mathbb{G}+c_0C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} \tag{4.2}$$

where $C_{\mu\nu\sigma\tau}$ is the conformally invariant Weyl tensor. In 2ℓ-dimensions we have

$$\begin{aligned} C_{\mu\nu\sigma\tau} &= R_{\mu\nu\sigma\tau} - \frac{1}{2\ell-2}(g_{\mu\sigma}R_{\nu\tau}-g_{\mu\tau}R_{\nu\sigma}-g_{\nu\sigma}R_{\mu\tau}+g_{\nu\tau}R_{\mu\sigma}) \\ &+ \frac{R}{(2\ell-1)(2\ell-2)}(g_{\mu\sigma}R_{\nu\tau}-g_{\mu\tau}R_{\nu\sigma}) \end{aligned} \tag{4.3}$$

The tensor $\mathbb{G} = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}-4R_{\mu\nu}R^{\mu\nu}+R^2$ is the Gauss Bonnet density

so that $$\chi = \frac{1}{32\pi^2}\int\sqrt{-g}\ \mathbb{G}\ d^4x \tag{4.4}$$

is a topological invariant (the Euler number). i.e. $\frac{\delta\chi}{\delta g_{\mu\nu}} = 0.$

In 4 dimensions χ will not contribute to the field equations which for the Lagrangian

$$L = aR^2 + bR_{\mu\nu}R^{\mu\nu} \tag{4.5}$$

are $\frac{1}{2}aR^2g^{\mu\nu}+\frac{1}{2}bR_{\sigma\tau}R^{\sigma\tau}g^{\mu\nu}-2aR\,R^{\mu\nu}-2bR_{\sigma\tau}R^{\mu\sigma\nu\tau}-(2a+\frac{1}{2}b)\Box Rg^{\mu\nu}$ (4.6)

$$-b\Box R^{\mu\nu}+(2a+b)R^{;\mu\nu} = 0$$

As an aside note that any vacuum solution of the Einstein field equations is also a solution of the above equations.

Although χ does not appear in the field equations, it is necessary for it to be retained since $\chi \neq 0$ (the general case) will imply that the coefficient of $\mathbb{G}$ will have to be renormalized. In fact, even if $\chi = 0$ we still must retain this term since it is only a topological invariant in $2\ell = 4$ dimensions. Consequently when we perform a dimensional regularization of the theory, we must treat this term just like any other. Furthermore, dimensional regularization is required to maintain gauge invariance and also to maintain the general coordinate invariance of the integration measure $d[g_{\mu\nu}]$, which is otherwise destroyed (Fradkin & Vilkovisky 1975).

We consider now the issue of renormalizability of L_{grav}. Being a fourth derivative theory, the graviton propagator at high energies will behave as $1/k^4$. So loops containing gravitons will converge much faster than with $L = (16\pi G)^{-1}R$. In fact we see by power counting that no infinite series of counterterms should be required and thus the theory should be renormalizable (as expected by DeWitt & Utiyama 1962; Deser, van Nieuwenhuizen & Tsao 1974).

The rigorous proof of renormalizability (Stelle 1977) had to wait until the machinery of BRS identities was discovered. Stelle proved that the general $O(R^2)$ Lagrangian (without torsion) was renormalizable and that this is maintained when the gravity is minimally coupled to a renormalizable matter field.

Nonabelian gauge theories are renormalizable and also asymptotically free. It is natural to ask what is the ultraviolet behaviour of the $O(R^2)$ theories, i.e. the momentum dependence of the coupling constants r^{-1}, s^{-1} and c^{-1}. It has in fact been proved that these Lagrangians are asymptotically free (Fradkin & Tseytlin 1981, which followed from the work of Julve & Tonin 1978, Salam & Strathdee 1978). This was done by quantizing the theory using the background field method and taking the 1-loop approximation, to give the 1-loop UV coupling constant behaviour via the renormalization group equations; using, of course, dimensional regularization.

$$\alpha^2(k^2) = \frac{\alpha^2(\mu^2)}{1+\frac{1}{2}b\alpha^2(\mu^2)\log(k^2/\mu^2)} \quad \text{where } b>0 \text{ always} \qquad (4.7)$$

and where we have $c = \frac{-1}{\alpha^2}$, which must be taken to be negative to ensure that we can take a meaningful Euclidean continuation of the generating function;

$$Z = \int d[g_{\mu\nu}]\, e^{+\int_E d^4x\ g^{1/2} cC_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} + \ldots} \qquad (4.8)$$

The above result was also obtained (Tomboulis 1980) by taking a $1/N$ expansion, where N is the number of massless non-interacting fields. The limit $N \to \infty$ provides an exactly solvable theory which is asymptotically free. This is essentially a non-perturbative result.

To summarize, work on the quantization of $O(R^2)$ Lagrangians, and the consequences thereof, is only just beginning. One must mention the recent work of Christensen (1982), Barth & Christensen (1983) and Boulware, Horowitz & Strominger (1983).

4.2 THE DYNAMICALLY INDUCED LAGRANGIAN

The quantum fluctuations of the matter fields will not only induce a $\frac{1}{16\pi G}R$ term, but also induces $O(R^2)$ terms. The three $O(R^2)$ terms are all scale-invariant dimension 4 operators, so the renormalization process will induce all of them as necessary counterterms, unless there exists a symmetry imposed on the entire Lagrangian. This symmetry will forbid induced terms from appearing with infinite coefficients. Indications are that $C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}$ and $\mathbb{G}$ are always required in the bare gravity Lagrangian. However for a conformally invariant matter Lagrangian, the R^2 term appears not to be needed as a counterterm. The evidence for this (Tsao 1977) comes from a calculation of the 1-loop counterterms for a general conformally invariant Lagrangian with unquantized metric, using a generalization of an algorithm ('t Hooft 1973) to calculate explicitly the coefficients. The result is

$$L_{\text{counterterm}} = \frac{1}{2\ell-4}\,\frac{1}{16\pi^2}(k_1 C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} + k_2\mathbb{G}) \qquad (4.9)$$

where $k_1 = \frac{1}{40}$, $k_2 = \frac{-11}{720}$ for a real spin 1/2 field.

The coefficient of R^2 in the gravity Lagrangian, r_0, is thus arbitrary, and can be put to zero.

Because of higher loops, however, this result is not conclusive. (However Englert, Truffin & Gastmans 1976 have given an all-order proof that only conformally-invariant counterterms are required. However they assume that the Lagrangian is conformally invariant in all dimensions.)

Another indication that the induced R^2 term is finite is due to Zee (1982b). The result is for scale invariant asymptotically free Yang-Mills theories and is a consequence of a general formula for the induced coefficient, r, in terms of $T_\mu{}^\mu$. This is a non-perturbative exact result:

$$r = \frac{-i}{13824} \int d^4x\ x^4 \langle \hat{T}(x)\hat{T}(0)\rangle_0^T \tag{4.10}$$

Zee finds this by expanding the Lagrangian to 1st order in $h_{\mu\nu}$ as $\mathcal{L} = \frac{1}{2}\int d^4x\ T^{\mu\nu}h_{\mu\nu} + \ldots$, and writing

$$e^{iS_{eff}} = \langle \exp(i\int d^4x\mathcal{L})\rangle_0^T \tag{4.11}$$

Each of these exponentials is expanded, the right hand side to $O(h_{\mu\nu})^2$. The perturbation $h_{\mu\nu}$ is then specialized to $\frac{1}{4}\eta_{\mu\nu}h$ and h is Taylor expanded about some point. The term quartic in derivatives gives the above formula. (See also Zee 1981a, 1983a.)

This formula is equivalent, by a Fourier transform, to

$$\begin{aligned} r &= \frac{-i}{13824}\Box_k^4 \Psi(k)\,|_{k=0} \\ &= \frac{-i}{144}\ \Psi''(k^2)\,|_{k^2=0} \end{aligned} \tag{4.12}$$

To decide if r is finite or not, we need to consider the behaviour of $\Psi(x^2) = \langle\hat{T}(x)\hat{T}(0)\rangle_0^T$. We know that the IR region converges since for Yang-Mills fields we have the characteristic exponential decay; eqn. 3.22, but in the UV region we have

$\Psi(x^2) \sim \frac{1}{(x^2)^4} \times$logs, by the Wilson operator product expansion, (eqn. 3.11). Thus the logarithm terms are the decisive factor.

These logarithm factors are easily evaluated for an asymptotically free theory, (eqns. 3.23 to 3.26) and give

$$\Psi(x) \sim \frac{1}{x^8(\log(-x^2))^2} \quad \text{as } x^2 \to 0.$$

Thus the integral converges and r is finite.

The problem with this is that it leaves the metric unquantized. However, since $O(R^2)$ theories are also asymptotically free, we suspect that the contribution to r will be finite when gravity is quantized as well. This belief is reinforced by the trace anomaly for $\mathcal{L} = -\sqrt{-g}\, c\, C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}$ given by Zee (1983a) as

$$\frac{2\beta(c)}{c} \sqrt{-g}\, C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}. \tag{4.13}$$

To summarize, r is probably finite, but this is not yet conclusive.

Using the spectral function $\sigma(m^2)$ of the scalar operator $\Psi(k^2)$ we can actually find out the sign of the induced r. For;

$$-i\Psi(k^2) = \int_0^\infty dm^2 \sigma(m^2) \frac{1}{k^2 - m^2} \tag{4.14}$$

and $$r = \frac{-i}{144} \Psi''(k^2)\big|_{k=0}, \text{ so}$$

$$r = \frac{1}{72}\int_0^\infty \frac{dm^2}{m^6} \sigma(m^2) \tag{4.15}$$

Since $T^2 \sim \frac{1}{x^8(\log(-x^2))^2}$, eqn. 3.26, we have that (Khuri 1982b) $\sigma(m^2) \sim m^4 \times (\log m^2)^{-2}$ and thus the integral converges, and so r is positive.

Note that it is important to have r positive, as this ensures that the rR^2 term does not produce a tachyon.

It should be possible to calculate r just as easily as G^{-1}. For Zee's infrared stable gauge theory considered in section 3.4, the result is

$$r = (221184)^{-1}\pi^2 C_\Psi b_0 g_*^2 \tag{4.16}$$

Concerning the induced coefficient of $C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}$ we suspect that it is not finite; one reason being that it appears as a counterterm in conformal matter theories; and another reason is that the expression for the induced c involves the full $T_{\mu\nu}$ tensor and so the ultraviolet softening of T via the asymptotic freedom property of $\beta(g)$ probably does not occur.

4.3 THE UNITARITY PROBLEM

This is the main unsolved problem of all $O(R^2)$ gravity Lagrangians and is the main criticism levelled at the induced gravity program.

The unitarity problem is the non-realistic behaviour of the tree level propagator, namely the appearance of a ghost.

This is guaranteed to appear from the 4th derivative terms which imply that the propagator will have the form

$$\frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{k^2+m^2}\right). \tag{4.17}$$

The negative residue of the 2nd term signifies a ghost.

If the theory has a mass scale (or if it is dynamically induced) then the mass of the ghost is related to it. Otherwise the limit $m \to 0$ is taken.

To be more specific, the Lagrangian $L = \frac{-1}{4\alpha^2}C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}$ will have the tree level propagator of the form

$$\frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{k^2+\alpha^2 m^2}\right) \tag{4.18}$$

However, as was pointed out by a number of authors (Julve & Tonin 1978; Salam & Strathdee 1978) radiative corrections may alter the behaviour; for example push the mass up to infinity.

However there is another way radiative corrections enter and that is through the momentum dependence of the coupling constant:

$$\frac{1}{\alpha^2(k^2)} = \frac{1}{\alpha^2(\mu^2)} + \frac{b}{2}\log(k^2/\mu^2) \tag{4.19}$$

Thus the pole will shift with the momentum. At the pole itself $k^2 = -\alpha^2 m^2$, but for $k^2 < 0$, $\alpha^2(k^2)$ has an imaginary part. The pole is thus actually not on the real axis and so is not a physical particle, but is two unstable ghost particles. (See Hasslacher & Mottola 1981; Fradkin & Tseytlin 1981; Tomboulis 1977, 1980.)

Consequently it should not enter into the asymptotic states, and it is thus possible that the S-matrix remains unitary. (Lee & Wick 1969a, 1969b, 1970; Cutkosky, Landshoff, Olive & Polkinghorne 1969.)

Whether this mechanism is in fact operative or not is not clear. Whether higher loop effects and non-perturbative effects destroy the mechanism is also not clear. What is clear, however, is that unitarity is a dynamical question and that there are definite mechnaisms by which it is possible to maintain unitarity.

More work is obviously required in this area, but the unthinking rejection of $O(R^2)$ Lagrangians due to unitarity is invalid.

It is possible that the solution to the combined problem of renormalizability and unitarity lies in a completely different approach to gravity. Two of the more important ideas are supersymmetry and the incorporation of torsion. Recent work on torsion has been done by Neville (1978, 1980, 1981, 1982) and Sezgin & van Nieuwenhuizen (1980). The idea is usually to make the spin connection an independent propagating field, along with the graviton. So far, no realistic unitary theory of gravity which is either renormalizable or finite has yet been constructed.

5. CALCULATION OF G^{-1} FROM THE FERMION PROPAGATOR

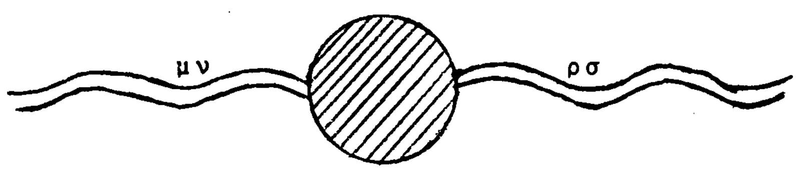
The exact calculation of G^{-1} would require consideration of all fundamental particles present, and all the allowed interactions. In this chapter we consider how to obtain an approximate expression for G^{-1} in terms of the spectral function of the fermion, we do this by considering the contribution from only certain interactions.

5.1 THE CONTRIBUTING FEYNMAN DIAGRAM

Recall the formula for G^{-1}; (eqn. 2.51)

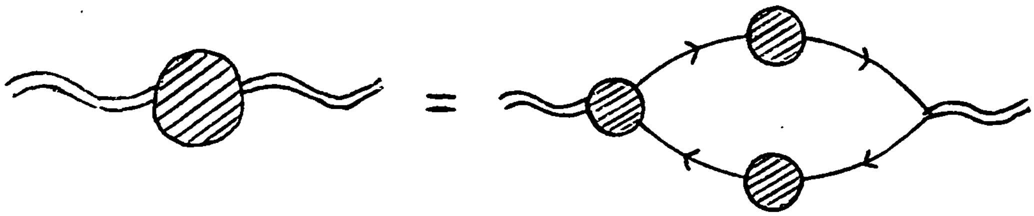
$$G^{-1} = -i\frac{\pi}{6}\Box_k^2 \ \Psi(k)\Big|_{k=0} \tag{5.1}$$

where $\Psi(k) = \Psi^{\mu}{}_{\mu}{}^{\rho}{}_{\rho}(k)$ and $\Psi^{\mu\nu\rho\sigma}(k)$ is the amputated version of the diagram:



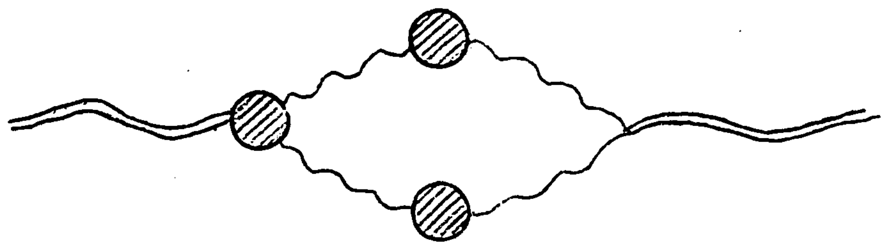
The first restriction on the possible interactions is obtained by expanding the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This is substituted into the general covariant matter Lagrangian density and the only interactions retained are those of first order in $h_{\mu\nu}$.

The above diagram can then be expanded as

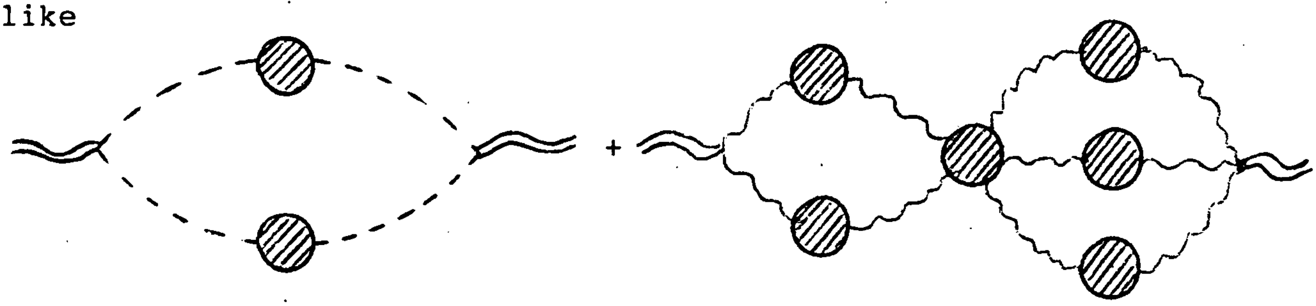


+ others.

For QED, say, the "others" will simply be



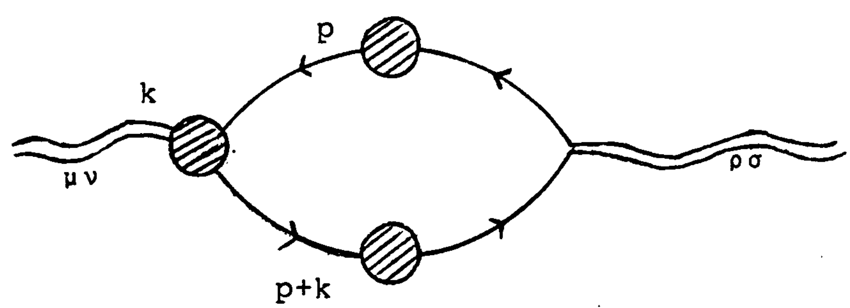
For a non-abelian gauge theory there will be additional terms like



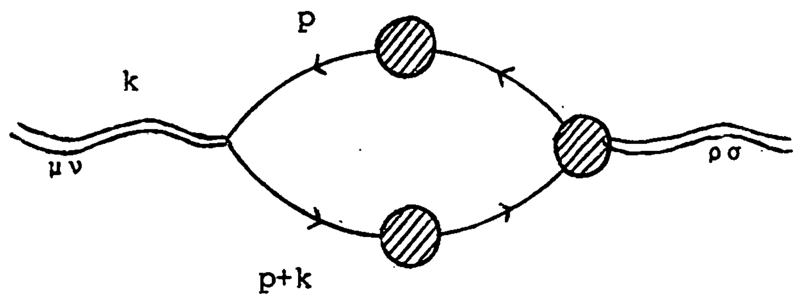
+ many more.

For calculational purposes, we consider only the contributions to G^{-1} from the first diagram, and so we could define

$\Psi_{\mu\nu\rho\sigma}(k)$ =



Note that we also could choose the definition

$\Psi_{\mu\nu\rho\sigma}(k) =$ 

These two diagrams are not the same, and so the definition of $\Psi_{\mu\nu\rho\sigma}(k)$ that is chosen is the average of the two. This is equivalent to pair symmetrization in the indices, i.e. $(\mu\nu)\leftrightarrow(\rho\sigma)$. This symmetrization will be denoted by "sym".

Letting $iS(p)$ be the full Dirac fermion propagator, $-i\Lambda_{\rho\sigma}(k)$ be the undressed vertex and $-i\Gamma_{\mu\nu}(k)$ be the full vertex, we have

$$\Psi_{\mu\nu\rho\sigma}(k) = -\text{sym}\int \bar{d}^4p \; \mathrm{Tr}[iS(p+k)(-i)\Gamma_{\mu\nu}(k)iS(p)(-i)\Lambda_{\rho\sigma}(k)] \qquad (5.2)$$

where $\bar{d}^4p = d^4p/(2\pi)^4$. Although we can write this either in terms of renormalized or bare quantities, we choose renormalized quantities. Thus any coupling constants and masses are the measurable renormalized ones.

5.2 THE FERMION-GRAVITON INTERACTION TERM

The Lagrangian for a spin-1/2 particle with mass m is

$$\mathcal{L} = \bar{\psi}(\tfrac{1}{2}i\overleftrightarrow{\not{\partial}}-m)\psi \quad \text{where} \quad \bar{\psi}\overleftrightarrow{\not{\partial}}\psi = \bar{\psi}\not{\partial}\psi - (\not{\partial}\bar{\psi})\psi.$$

So the general covariant Lagrangian density is

$$\mathcal{L} = \sqrt{-g}\,\{\tfrac{1}{2}i(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi-(\nabla_{\mu}\bar{\psi})\gamma^{\mu}\psi)-\bar{\psi}m\psi\} \qquad (5.3)$$

where $\nabla_{\mu}\psi = \partial_{\mu}\psi + \frac{1}{2}\Sigma^{\alpha\beta}V_{\alpha}{}^{\nu}(\partial_{\mu}V_{\beta\nu})\psi$

and $\nabla_{\mu}\bar{\psi} = \partial_{\mu}\bar{\psi} + \frac{1}{2}\,\bar{\psi}\Sigma^{\alpha\beta}V_{\alpha}{}^{\nu}(\partial_{\mu}V_{\beta\nu})$

where $V_{\beta\nu}$ is the vierbein.

$$g_{\mu\nu}(x) = V^{\alpha}{}_{\mu}(x)V^{\beta}{}_{\nu}(x)\,\eta_{\alpha\beta} \qquad (5.4)$$

The α, β, γ indices are the Lorentz indices of the vierbein and are raised and lowered simply with $\eta_{\alpha\beta}$.

Tensors with Lorentz indices, eg γ_α, are defined via the formula $\gamma_\alpha = V_{\alpha\mu}\gamma^\mu$, etc. (For a review of the vierbein formalism, see Weinberg 1972, or Birrell & Davies 1982.)

We now expand the metric as a perturbation about the covariant Minkowski metric. This can be done either with the metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ or via the vierbein $V_{\alpha\mu} = \eta_{\alpha\mu} + \varepsilon_{\alpha\mu}$. The final result $\Psi^{\mu}{}_{\mu}{}^{\rho}{}_{\rho}(k)$ will not depend on which method is employed, as we will see.

Note that now we have $\eta_{\alpha\mu}\gamma^\alpha = \gamma_\mu$, etc. and $V^{\alpha\mu} = \eta^{\alpha\mu} - \varepsilon^{\alpha\mu}$ where $\eta^{\mu\nu} = \eta^{\alpha\nu}\delta_\alpha{}^\mu$ is the contravariant Minkowski metric.

Now the result we require is the vertex $-i\Lambda_{\mu\nu}$, so we need only consider the linear term in $\varepsilon^{\alpha\mu}$.

$$(-g)^{1/2} = (-\det(g_{\mu\nu}))^{1/2} = \det(V^\alpha{}_\mu)$$
$$= 1 + \varepsilon_{\alpha\mu}\eta^{\alpha\mu} = 1 + \varepsilon^{\alpha\mu}\eta_{\alpha\mu}.$$

Thus we have

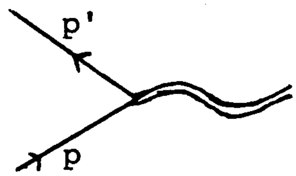
$$\begin{aligned} &= (1+\varepsilon^{\alpha\mu}\eta_{\alpha\mu})\{\tfrac{1}{2}i\bar\psi[V^{\alpha\nu}\gamma_\alpha\partial_\nu\psi - (\partial_\nu\bar\psi)\gamma_\alpha V^{\alpha\nu}\psi] - \bar\psi m\psi\} \\ &+ (1+\varepsilon^{\delta\tau}\eta_{\delta\tau})\tfrac{1}{4}i\bar\psi\, V^{\beta\mu}\gamma_\beta\Sigma^\gamma{}_\alpha V_{\gamma\nu}(\partial_\mu V^{\alpha\nu})\psi \\ &+ (1+\varepsilon^{\delta\tau}\eta_{\delta\tau})\tfrac{1}{4}i\bar\psi\Sigma^\gamma{}_\alpha V_{\gamma\nu}(\partial_\mu V^{\alpha\nu})V^{\beta\mu}\gamma_\beta\psi \end{aligned} \tag{5.5}$$

Since $\partial_\mu V^{\alpha\nu} = -\partial_\mu\varepsilon^{\alpha\nu}$, we have to first order in $\varepsilon_{\alpha\mu}$:

$$\begin{aligned} &= \bar\psi(\tfrac{1}{2}i\overleftrightarrow{\partial\!\!\!/} - m)\psi + \varepsilon^{\alpha\nu}\bar\psi(\eta_{\alpha\nu}(\tfrac{1}{2}i\overleftrightarrow{\partial\!\!\!/} - m) - \tfrac{1}{2}i\gamma_\alpha\overleftrightarrow{\partial}_\nu)\psi \\ &- \tfrac{1}{4}i\bar\psi(\partial^\mu\Sigma_{\nu\alpha} + \Sigma_{\nu\alpha}\gamma^\mu)(\partial_\mu\varepsilon^{\alpha\nu})\psi \end{aligned} \tag{5.6}$$

We then integrate the last term by parts.

The Feynman rule for the vertex



$$= -i\Lambda_{\mu\nu}(k)$$

is thus

$$\begin{aligned}\Lambda_{\mu\nu} &= \eta^{\alpha}{}_{\mu}\Lambda_{\alpha\nu}\\ &= -\eta_{\mu\nu}(\tfrac{1}{2}(\not p+\not p')-m) + \tfrac{1}{2}((p+p')_{\nu}\gamma_{\mu})\\ &\quad + \tfrac{1}{4}(p_{\lambda}-p'_{\lambda})\{\gamma^{\lambda},\Sigma_{\nu\mu}\}\end{aligned} \tag{5.7}$$

A comment on the non-symmetric part will be given at the end of section 5.3 (see also van Nieuwenhuizen 1974).

Note however the sign convention that has been choosen; this is due to the positive T_{00} requirement.

For, $\Lambda_{00} = m + \frac{1}{2}(p_i+p_i')\cdot\gamma^{i}$, and if we sandwich Λ_{00} between two eigenstates of positive energy we get $\bar\psi\Lambda_{00}\psi = m \geq 0$ as it should be.

5.3 THE WARD IDENTITY

It is easy to verify the following identity:

$$\Lambda_{\lambda\nu}k^{\nu} = (\not p+\not k-m)(p_{\mu}-\tfrac{1}{2}\Sigma_{\mu\nu}k^{\nu})-(p_{\mu}+k_{\mu}-\tfrac{1}{2}\Sigma_{\mu\nu}k^{\nu})(\not p-m)$$

where $p_{\mu}+k_{\mu}=p'_{\mu}$.

The analogy with the QED Ward identity

$$\Gamma_{\mu}k^{\mu} = S^{-1}(p+k)-S^{-1}(p),$$

suggests that the above identity may be the lowest order expression of a fermion-graviton Ward identity, which would read

$$\Gamma_{\mu\nu}k^{\nu} = S^{-1}(p+k)(p_{\mu}-\tfrac{1}{2}\Sigma_{\mu\nu}k^{\nu})-(p_{\mu}+k_{\mu}-\tfrac{1}{2}\Sigma_{\mu\nu}k^{\nu})S^{-1}(p) \tag{5.8}$$

This is in fact an identity, as has been shown by Just &

Rossberg (1965) (see also DeWitt 1967; Brout & Englert 1966).

The above identity shall now be proved using the generating functional

$$Z[J] = \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\mathcal{D}[A_\mu] \exp i\int d^4x \mathcal{L}[J]$$

$$\text{where } \mathcal{L}[J] = \mathcal{L}_m + \bar{\psi}J + \bar{J}\psi + J^{\alpha\mu}\varepsilon_{\alpha\mu} + \mathcal{L}_B + \mathcal{L}_G + \mathcal{L}_{other} \tag{5.9}$$

$\mathcal{L}_m$ is the matter Lagrangian of all the fields,

$\mathcal{L}_B$ is the general coordinate gauge breaking term

$\mathcal{L}_G$ is the gravitational ghost term

$\mathcal{L}_{other}$ contains source terms, gauge breaking terms and ghost terms for the other fields (the A_μ). These will have no effect on the discussion as will be seen.

We shall now perform an infinitesimal coordinate transformation at the point x.

$$x^\mu \to x'^\mu = x^\mu - \Lambda^\mu(x)$$

$$\text{we have } g'_{\mu\nu}(x) = g'_{\mu\nu}(x') + (\partial_\lambda g_{\mu\nu})\Lambda^\lambda \tag{5.10}$$

$$\text{Thus } \delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = (\partial_\lambda g_{\mu\nu})\Lambda^\lambda + g_{\lambda\nu}\partial_\mu\Lambda^\lambda + g_{\lambda\mu}\partial_\nu\Lambda^\lambda \tag{5.11}$$

$$\text{and } \quad \delta\varepsilon^\alpha{}_\mu = \delta V^\alpha{}_\mu = (\partial_\lambda V^\alpha_\mu)\Lambda^\lambda + V^\alpha{}_\lambda\partial_\mu\Lambda^\lambda \tag{5.12}$$

The infinitesimal coordinate transformation of ψ and $\bar{\psi}$ are

$$\delta\psi(x) = \psi'(x) - \psi(x) = (\partial_\lambda\psi)\Lambda^\lambda - \frac{1}{2}\Sigma_{\mu\nu}\psi\Lambda^{\mu,\nu} \tag{5.13}$$

$$\delta\bar{\psi}(x) = (\partial_\lambda\bar{\psi})\Lambda^\lambda + \frac{1}{2}\bar{\psi}\Sigma_{\mu\nu}\Lambda^{\mu,\nu} \tag{5.14}$$

By general covariance, $\delta\mathcal{L}_m = 0$, so

$$\delta\mathcal{L} = \bar{J}\delta\psi + \delta\bar{\psi}J + J^{\alpha\mu}\delta\varepsilon_{\alpha\mu} + \delta(\mathcal{L}_B + \mathcal{L}_G + \mathcal{L}_{other}) \tag{5.15}$$

We now substitute the above transformations, integrate by parts to remove the derivatives from the Λ^λ and expand the exponential $\exp i\int d^4x\delta\mathcal{L}[J]$ to first order in Λ^λ. Note that since $Z[J]$ is a scalar quantity it is unchanged by the coordinate transformation, so the $\int d^4x\delta\mathcal{L}$ term will functionally integrate to give zero for any Λ^λ. So we can factor Λ^λ out to get

$$\int \mathcal{D}[A_\mu]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ \exp(i\int d^4x\mathcal{L}[J])\ \{\bar{J}\partial_\lambda\psi+\tfrac{1}{2}\partial^\nu(\bar{J}\Sigma_{\lambda\nu}\psi)$$
$$+(\partial_\lambda\bar{\psi})J-\tfrac{1}{2}\partial^\nu(\bar{\psi}\Sigma_{\lambda\nu}J)+J^{\alpha\mu}\partial_\lambda V_{\alpha\mu}-\partial_\mu(J^{\alpha\mu}V_{\alpha\lambda})$$
$$+\delta(\mathcal{L}_B+\mathcal{L}_G+\mathcal{L}_{other})/\Lambda^\lambda\} = 0 \tag{5.16}$$

Now use the functional derivatives of Z[J] to express this as a functional differential equation.

eg $\int \mathcal{D}[A_\mu]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ \exp(i\int d^4x\mathcal{L}[J](\partial_\lambda\bar{\psi})J = \frac{1}{i}J\partial_\lambda\frac{\delta}{\delta J}Z[J]$

Note that $\frac{\delta Z[J]}{\delta J^{\alpha\mu}} = \epsilon_{\alpha\mu} = V_{\alpha\mu} - \eta_{\alpha\mu}$.

We then convert this to an equation in the generating functional of connected graphs, W[J], defined by Z = exp(iW[J]). We thus obtain

$$[\bar{J}\partial_\lambda\frac{\delta}{\delta\bar{J}}+\frac{1}{2}\partial^\nu(\bar{J}\Sigma_{\lambda\nu}\frac{\delta}{\delta\bar{J}})+J\partial_\lambda\frac{\delta}{\delta J}-\frac{1}{2}\partial^\nu(\frac{\delta}{\delta J}\Sigma_{\lambda\nu}J)+J^{\alpha\mu}\partial_\lambda\frac{\delta}{\delta J_{\alpha\mu}}$$
$$-\partial_\mu(J^{\alpha\mu}\frac{\delta}{\delta J^{\alpha\lambda}})-\frac{\partial_\mu J^{\alpha\mu}\eta_{\alpha\lambda}+\text{"other"}}{W}]\ W[J] = 0 \tag{5.17}$$

We now transform this to an equation in the generating functional of one particle irreducible amputated graphs,

$$\Gamma(\psi,\bar{\psi},\epsilon_{\alpha\mu},A_\mu) = W[J] - \int dx(\bar{J}\psi+\bar{\psi}J+J^{\alpha\mu}\epsilon_{\alpha\mu}+J^\mu A_\mu)$$

We thus obtain the relevent part of the gravitational gauge identity, namely

$$\frac{\partial\Gamma}{\delta\psi}\partial_\lambda\psi+\frac{1}{2}\partial^\nu(\frac{\partial\Gamma}{\delta\psi}\Sigma_{\lambda\nu}\psi)+\frac{\delta\Gamma}{\delta\bar{\psi}}\partial_\lambda\bar{\psi}-\frac{1}{2}\partial^\nu(\bar{\psi}\Sigma_{\lambda\nu}\frac{\delta\Gamma}{\delta\bar{\psi}})+\frac{\delta\Gamma}{\delta\epsilon_{\alpha\mu}}\partial_\lambda\epsilon_{\alpha\mu}$$
$$-\partial_\mu(\frac{\delta\Gamma}{\delta\epsilon_{\alpha\mu}}V_{\alpha\lambda})+\text{"other"} = 0 \tag{5.18}$$

Recall that the "other" terms contain only the other fields, no ψ or $\bar{\psi}$.

The above equation is defined at a certain point, call it x. To obtain the Ward identity required, we take $\frac{\delta}{\delta\psi(y)}\frac{\delta}{\delta\bar{\psi}(z)}$ and then set all fields (ψ, $\bar{\psi}$, $\varepsilon_{\alpha\mu}$, A_μ, ghosts) to zero. We then have:

$$\frac{\delta^2\Gamma}{\delta\psi(x)\,\delta\bar{\psi}(z)}\partial_\lambda\delta(x-y) + \frac{1}{2}\partial^\nu\left(\frac{\delta^2\Gamma}{\delta\psi(x)\,\delta\bar{\psi}(z)}\Sigma_{\lambda\nu}\delta(x-y)\right)$$

$$+ \frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\,\delta\psi(y)}\partial_\lambda\delta(x-z) - \frac{1}{2}\partial^\nu\left(\delta(x-z)\,\Sigma_{\lambda\nu}\frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\,\delta\psi(y)}\right)$$

$$- \partial_\mu\left(\frac{\delta^3\Gamma}{\delta\psi(y)\,\delta\varepsilon_{\alpha\mu}(x)\,\delta\bar{\psi}(z)}V_{\alpha\lambda}\right) = 0 \tag{5.19}$$

All derivatives are with respect to x.

We now Fourier transform this equation using $\int(\text{above}) \times \exp(ikx+ipy-ip'z)\, d^4x\, d^4y\, d^4z$ and integrate by parts once or twice, to obtain:

$$- \int\frac{\delta^3\Gamma\, e^{ikx+ipy-ip'z}}{\delta\psi(y)\,\delta\varepsilon_{\alpha\mu}(x)\,\delta\bar{\psi}(z)}V_{\alpha\lambda}k_\mu\; d^4x\, d^4y\, d^4z$$

$$= \int\frac{\delta^2\Gamma}{\delta\psi(x)\,\delta\bar{\psi}(z)}e^{i(p+k)x}e^{-p'z}d^4x d^4z(p_\lambda-\tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)$$

$$- (p_\lambda+k_\lambda-\tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)\int\frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\,\delta\psi(y)}e^{ipy}e^{i(p-p')x}\; d^4x\, d^4y \tag{5.20}$$

and so;

$$\Gamma_{\lambda\mu}k^\mu = S^{-1}(p+k)(p_\lambda-\tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)-(p_\lambda+k_\lambda-\tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)S^{-1}(p)$$

which is the previously stated Ward identity (eqn. 5.8).

Note that this is for the complete vertex, i.e. symmetric and antisymmetric parts. It is also true just for the symmetric part, as is easily seen by noting the changes in the above derivation for a symmetric source and metric perturbation, i.e. $J^{\mu\nu}h_{\mu\nu}$ where

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The transformation $\delta h_{\mu\nu}(x)$ is given by eqn. 5.11 which is symmetric in μ and ν. So the Ward identity is unaltered. Note then that $k^{\mu}\Gamma^{\text{antisymmetric}}_{\lambda\mu} = 0$.

However, we shall only be interested in the symmetric part of $\Gamma_{\lambda\mu}$ for a simple reason. The aim is to find

$$G^{-1} = \frac{-i\pi}{6}\Box_k^2 (g_{\rho\sigma} g_{\mu\nu} \Psi^{\mu\nu\rho\sigma})\Big|_{k=0} \tag{5.21}$$

and so by eqn. 5.1 none of the antisymmetric parts of $\Gamma_{\mu\nu}$ or $\Lambda_{\rho\sigma}$ will contribute at all.

So from now on we shall simply symmetrize $\Lambda_{\rho\sigma}$ and $\Gamma_{\mu\nu}$.

It is interesting to note some properties of the antisymmetric part of $\Gamma_{\mu\nu}$ (and thus of $T_{\mu\nu}$); it does not enter into the value of Λ or G^{-1}, or the Ward identity above, or Einstein's field equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$. Furthermore if $\Gamma_{\mu\nu}$ is sandwiched between Dirac spinors then the antisymmetric part cancels on application of the equation of motion. It is no wonder that $\Gamma^{\text{antisymmetric}}$ can usually be ignored. (Note however that one cannot neglect $\Gamma^{\text{antisymmetric}}$ if the torsion is chosen to also be a propagating field.)

5.4 THE FERMION SPECTRAL FUNCTION

To calculate the function $\Psi_{\mu\nu\rho\sigma}(k)$ we need to have some access to the full vertices and propagators. The Ward identity relates the full vertex to the full propagator and we use the Lehmann spectral function, $\rho(\omega)$, to relate the full and bare propagators. With suitable approximations, it will then be possible to write G^{-1} in terms of $\rho(\omega)$.

We know $S(p) = \int_m^\infty \frac{\not{p}\rho_1(s) + m\rho_2(s)}{p^2 - s + i\epsilon} ds$

$$= \left(\int_{-\infty}^{-m} + \int_m^\infty\right) \frac{\rho(\omega)\, d\omega}{\not{p} - \omega + i\epsilon\ \epsilon(\omega)} \tag{5.22}$$

where $\epsilon(\omega) = \mathrm{sign}(\omega)$ and m is the dynamically induced fermion mass.

Now, by the Ward identity;

$$S(p+k)\Gamma_{\lambda\mu}(k)S(p)k^\mu = (p_\lambda - \tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)S(p) - S(p+k)(p_\lambda + k_\lambda - \tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)$$
$$= \int\rho(\omega)\,d\omega\ \left[(p_\lambda - \tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)\frac{1}{\not{p}-\omega} - \frac{1}{\not{p}+\not{k}-\omega}(p_\lambda + k_\lambda - \tfrac{1}{2}\Sigma_{\lambda\nu}k^\nu)\right]$$
$$= \int\rho(\omega)\,d\omega\ \frac{1}{\not{p}+\not{k}-\omega}\Lambda_{\lambda\mu}(k)\frac{1}{\not{p}-\omega}k^\mu \tag{5.23}$$

Note that the undressed vertex has mass parameter ω.

We consider only the longitudinal solution of the above equation; i.e.

$$S(p+k)\Gamma_{\lambda\mu}(k)S(p) = \int\rho(\omega)\,d\omega\ \frac{1}{\not{p}+\not{k}-\omega}\Lambda_{\lambda\mu}(k)\frac{1}{\not{p}-\omega} \tag{5.24}$$

The neglecting of the transverse contribution may or may not be serious.

We now substitute this solution into equation 5.2 to obtain

$$\Psi_{\mu\nu\rho\sigma}(k) = -\mathrm{sym}\int\bar{d}^4p\,d\omega\ \rho(\omega)\ \mathrm{Tr}\ \frac{1}{\not{p}+\not{k}-\omega}\Lambda_{\mu\nu}(k,\omega)\frac{1}{\not{p}-\omega}\Lambda_{\rho\sigma}(k,m) \tag{5.25}$$

To do this calculation we need to know the condition on $\rho(\omega)$ such that the result is finite. We do this by using dimensional regularization of the space-time dimension 2ℓ. (For a review, see Delbourgo 1976.)

We thus define the spectral function in 2ℓ dimensions $\rho(\omega,\ell)$, the propagator $S(p,\ell)$ and also define

$$\Pi_{\mu\nu\rho\sigma}(k,\omega,\ell) = \mathrm{sym}\int\bar{d}^{2\ell}p\ \mathrm{Tr}\ \frac{1}{\not{p}+\not{k}-\omega}\Lambda_{\mu\nu}(k,\omega)\frac{1}{\not{p}-\omega}\Lambda_{\rho\sigma}(k,m) \tag{5.26}$$

Thus

$$G^{-1} = \frac{i\pi}{6} \int d\omega \rho(\omega,\ell) \Box_k^2 \Pi_\mu{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega,\ell) \Big|_{k=0} \tag{5.27}$$

5.5 SPECTRAL RULES FOR $\rho(\omega)$

In calculating $\Box_k^2 \Pi_\mu{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega,\ell)\Big|_{k=0}$ we shall be as general as possible and examine some structure of the entire $\Pi_{\mu\nu\rho\sigma}(k,\omega,\ell)$.

To calculate $\Pi_{\mu\nu\rho\sigma}$, we proceed in the usual way:

We introduce the Feynman parameter α, transform $p \to p + \alpha k$ and take the trace over spinor indices. We use $Tr(\gamma_\mu \gamma_\nu) = g_{\mu\nu} 2^\ell$. Only terms which are even in powers of p need be kept, by symmetric integration.

We then transform the indices on the p_μ via

$$\int \bar{d}^{2\ell}p \ f(p) p_\mu p_\nu = \frac{1}{2\ell} g_{\mu\nu} \int \bar{d}^{2\ell}p \ f(p) p^2 \tag{5.28}$$

and

$$\int \bar{d}^{2\ell}p f(p) p_\mu p_\nu p_\rho p_\sigma = \frac{1}{4\ell(\ell+1)} (\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}) \int \bar{d}^{2\ell}p \ f(p) p^4 \tag{5.29}$$

This enables the result to be written as simply an integral over the Feynman parameter α.

To express the result so far, we define five convenient tensors:

$$K_1 = \eta_{\mu\nu}\eta_{\rho\sigma} \tag{5.30a}$$

$$K_2 = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} \tag{5.30b}$$

$$K_3 = k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu} \tag{5.30c}$$

$$K_4 = k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} \tag{5.30d}$$

$$K_5 = k_\mu k_\nu k_\rho k_\sigma \tag{5.30e}$$

We find, after some calculation,

$$\Pi_{\mu\nu\rho\sigma}(k,\omega,\ell) = 2^{\ell-2}\int_0^1 d\alpha \ (T_1I_1+T_2I_2+T_3I_3) \tag{5.31}$$

where
$$T_1 = \frac{1}{\ell(\ell+1)}((4\ell^2-2)K_1-(\ell-1)K_2) \tag{5.32}$$

$$\begin{aligned} T_2 = {} & 4\omega m(\omega^2-\alpha(1-\alpha)k^2)K_1 \\ & +(1-2\alpha)^2\{\tfrac{1}{4}(\alpha(1-\alpha)k^2+\omega^2)K_4-2\alpha(1-\alpha)K_5 \\ & +(\omega m+\alpha(1-\alpha)k^2)K_3 \\ & -(\omega^2+2\omega m+\alpha(1-\alpha)k^2)k^2K_1\} \end{aligned} \tag{5.33}$$

$$\begin{aligned} \ell T_3 = {} & (\tfrac{1}{4}(1-\ell)(1-2\alpha)^2-\alpha(1-\alpha))K_4 \\ & +(4\alpha(1-\alpha)-\ell(1-2\alpha)^2)K_3 \\ & +[(3\ell-1)(1-2\alpha)^2+4(\ell-2)\alpha(1-\alpha)]k^2K_1 \\ & +4\omega[(1-\ell)m-\ell\omega]K_1+(\omega^2+\alpha(1-\alpha)k^2)K_2 \end{aligned} \tag{5.34}$$

where the integrals I_1, I_2, I_3 are defined by

$$I_1 = \int \bar{d}^{2\ell}p \ f(p^2)p^4 = \frac{i}{(4\pi)^\ell}\frac{\ell+1}{\ell-1}\Gamma(2-\ell)(\omega^2-\alpha(1-\alpha)k^2)^\ell \tag{5.35}$$

$$I_2 = \int \bar{d}^{2\ell}p \ f(p^2) = \frac{i}{(4\pi)^\ell}\Gamma(2-\ell)(\omega^2-\alpha(1-\alpha)k^2)^{\ell-2} \tag{5.36}$$

$$I_3 = \int \bar{d}^{2\ell}p \ f(p^2)p^2 = \frac{i}{(4\pi)^\ell}\frac{\ell}{\ell-1}\Gamma(2-\ell)(\omega^2-\alpha(1-\alpha)k^2)^{\ell-1} \tag{5.37}$$

where
$$f(p^2) = (p^2+\alpha(1-\alpha)k^2-\omega^2)^{-2} \tag{5.38}$$

The three momentum integrals were evaluated by using:

$$\int \bar{d}^{2\ell}p \ \frac{(p^2)^T}{(p^2-m^2)^\Sigma} = \frac{i(-1)^{T-\Sigma}\Gamma(\ell+T)\Gamma(\Sigma-\ell-T)}{(4\pi)^\ell\Gamma(\ell)\Gamma(\Sigma)(m^2)^{\Sigma-\ell-T}} \ . \tag{5.39}$$

At this point we could take the limit $\ell \to 2$ and separate the pole at $\ell = 2$ from the finite part. First however, we proceed via the shorter route and take $\Box_k^2\big|_{k=0}$.

We obtain

$$\Box_k^2 \Pi_{\mu\nu\rho\sigma}(0,\omega,\ell) = \frac{i\Gamma(2-\ell)(\omega^2)^{\ell-1}}{2^{2-\ell}(4\pi)^{\ell}} \int_0^1 d\alpha \; \{\frac{4\alpha(1-\alpha)}{\ell-1}(\eta_{\mu\nu}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho})$$

$$+ [12(1-2\alpha)^2(1-\frac{m}{\omega}) + \frac{1}{\ell-1}(12(1-2\alpha)^2-32\alpha(1-\alpha))]\eta_{\mu\nu}\eta_{\rho\sigma}\} \quad (5.40)$$

Integrating over α, we find

$$\Box_k^2 \Pi_{\mu\nu\rho\sigma}(0,\omega,\ell) = \frac{i\Gamma(2-\ell)(\omega^2)^{\ell-1} 2^{\ell-2}}{3(4\pi)^{\ell}} \times$$

$$\{\frac{2}{\ell-1}(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}) + (12(1-\frac{m}{\omega})-\frac{4}{\ell-1})\eta_{\mu\nu}\eta_{\rho\sigma}\} \quad (5.41)$$

Thus

$$\Box_k^2 \Pi_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(0,\omega,\ell) = \frac{16i\Gamma(2-\ell)(\omega^2)^{\ell-1}2^{\ell-2}}{(4\pi)^{\ell}}(4(1-\frac{m}{\omega}) - \frac{1}{\ell-1}) \quad (5.42)$$

and by equation 5.27, we have G^{-1} as

$$G^{-1} = \frac{-1}{6\pi}\int \frac{(\omega^2)^{\ell-1}}{(2\pi)^{\ell-2}}\Gamma(2-\ell)(4(1-\frac{m}{\omega}) - \frac{1}{\ell-1})\rho(\omega,\ell)\,d\omega \quad (5.43)$$

Notice that there is a pole in ℓ for all even dimensional spaces. We expand the terms and take the limit $\ell \to 2$, so

$$G^{-1} = \frac{-1}{2\pi}\int(\omega^2-\frac{4}{3}m\omega)\frac{\rho(\omega)}{2-\ell}d\omega$$

$$+ \frac{1}{2\pi}\int(\omega^2-\frac{4}{3}m\omega)[\rho(\omega)(\ln\frac{\omega^2}{2\pi}+\frac{1}{3}+\gamma)+\frac{\partial}{\partial\ell}\rho(\omega,2)]d\omega \quad (5.44)$$

Since we do not want G^{-1} infinite we require the spectral integral conditions;

$$\int\omega^2\rho(\omega)d\omega = 0 \quad \text{and} \quad \int\omega\rho(\omega)d\omega = 0 \quad (5.45)$$

Note that $\int\omega\rho(\omega)d\omega = m_0 Z_\psi{}^{-1}$, where $Z_\psi{}^{-1} = \int\rho(\omega)d\omega \neq 0$. Thus m_0, the bare fermion mass, is required to be zero. We see once again the necessity of scale invariance.

If these spectral integral conditions are satisfied then we have $G^{-1} = \frac{1}{2\pi}\int[(\omega^2-\frac{4}{3}m\omega)\ln\omega^2 \; \rho(\omega)+(\omega^2-\frac{4}{3}m\omega)\frac{\partial}{\partial\ell}\rho(\omega,\ell)\big|_{\ell=2}]d\omega$ (5.46)

Notice that the ω^2 in $\ln\omega^2$ can be scaled if desired to

$\ln(\omega^2/\mu^2)$ and the result will not be changed. This can also be expressed by noting that the replacement $\int \bar{d}^4p \to \int \bar{d}^{2\ell}p$ is not unique since we could have $\int \bar{d}^4p \to \frac{1}{(\mu^2)^{\ell-2}}\int \bar{d}^{2\ell}p$.

The parameter μ is usually referred to as the renormalization point. The neglect of such a term is called "minimal renormalization."

An obviously equivalent method of finding the above result is to take the limit $\ell \to 2$ in equation 5.31 and to separate the pole at $\ell = 2$ from the finite part. Since we have need of this result, we write down the answer using the tensors defined previously in eqn. 5.30.

$$
\begin{aligned}
\Pi_{\mu\nu\rho\sigma}(k,\omega) = \frac{i}{4\pi^2}\int_0^1 d\alpha\ \{ & 3f^2[(C_1-\ln f)\tfrac{1}{24}(14K_1-K_2)-\tfrac{13}{72}K_1+\tfrac{1}{144}K_2] \\
& +2f[(C_3-\ln f)\{(\tfrac{1}{2}m\omega-\omega^2)K_1+\tfrac{5}{8}k^2(1-2\alpha)^2K_1+\frac{\omega^2+\alpha(1-\alpha)k^2}{8})K_2 \\
& -\tfrac{1}{4}(1-6\alpha(1-\alpha))K_3-\tfrac{1}{32}K_4\} \\
& +\tfrac{1}{16}(\omega^2+\alpha(1-\alpha)k^2)K_2+\tfrac{1}{4}\alpha(1-\alpha)K_3+\tfrac{1}{4}m\omega K_1 \\
& -(\tfrac{1}{16}(1-2\alpha)^2+\frac{\alpha(1-\alpha)}{2})k^2K_1+\frac{8\alpha(\alpha-1)+1}{64}K_4] \\
& +(C_2-\ln f)\{-\tfrac{1}{2}\alpha(1-\alpha)(1-2\alpha)^2K_5+\tfrac{1}{4}(\omega m+\alpha(1-\alpha)k^2)(1-2\alpha)^2K_3 \\
& -\tfrac{1}{4}(1-2\alpha)^2(\omega^2+2\omega m+\alpha(1-\alpha)k^2)k^2K_1 \\
& +\tfrac{1}{16}(1-2\alpha)^2(\omega^2+\alpha(1-\alpha)k^2)K_4 \\
& +\omega m(\omega^2-\alpha(1-\alpha)k^2)K_1\}\ \}
\end{aligned}
\tag{5.47}
$$

$f = \omega^2-\alpha(1-\alpha)k^2$

where $C_1 = \ln 4\pi - \gamma - \ln 2 + \frac{2}{3}$

$C_2 = \ln 4\pi - \gamma - \ln 2$

$$C_3 = \ln 4\pi - \gamma - \ln 2 + \frac{1}{2}$$

The infinite part (i.e. factor of $\frac{1}{2-\ell}$) can be simply read off from the above expression as the terms which are multiplied by $-\ln f$.

We shall, however, only be interested in the trace, i.e. $\Pi_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega)$.

This simplifies the expression to:

$$\Pi_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega) = \frac{i}{4\pi^2}\int_0^1 d\alpha \; \{f^2[(C_1-\ln f)27-\tfrac{51}{6}]$$
$$+2f\{(C_3-\ln f)[-15\omega^2+8m\omega+(10(1-2\alpha)^2+13\alpha(1-\alpha)-\tfrac{17}{8})k^2]$$
$$+\omega^2+4m\omega-[(1-2\alpha)^2+6\alpha(1-\alpha)-\tfrac{1}{16}]k^2\}$$
$$+(C_2-\ln f)[-\tfrac{15}{4}(1-2\alpha)^2\omega^2k^2-\tfrac{9}{4}\alpha(1-\alpha)(1-2\alpha)^2k^4]$$
$$+16\omega m(\omega^2-\alpha(1-\alpha)k^2)-6\omega m(1-2\alpha)^2k^2\} \qquad (5.48)$$

Taking $\Box_k^2\big|_{k=0}$ of the above equation we get the finite part

$$\Box_k^2\Pi_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega)\big|_{k=0} = \frac{-i}{\pi^2}(3\omega^2-4m\omega)\ln\omega^2+C\omega^2 \qquad (5.49)$$

which gives the sames G^{-1} as before, namely

$$G^{-1} = \frac{1}{2\pi}\int[(\omega^2-\tfrac{4}{3}m\omega)(\ln\omega^2)\rho(\omega)+(\omega^2-\tfrac{4}{3}m\omega)\frac{\partial}{\partial\ell}\rho(\omega,\ell)\big|_{\ell=2}]\,d\omega \qquad (5.50)$$

In the next chapter we find a spectral function $\rho(\omega,\ell)$, which will enable us to calculate G^{-1}. However before we do this, we consider one further aspect of induced gravity, the induced rR^2 term, which was given in eqn. 4.12 as

$$r = \frac{-i}{144}\,\Psi''(k^2)\big|_{k^2=0}$$

In general, both the finite part and infinite part of $\Pi_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega)$ would be expected to contain k^4 terms, thus r would contain a contribution $\frac{1}{2-\ell}\int\rho(\omega)d\omega$. But $\int\rho(\omega)d\omega = Z_{\psi}^{-1} \neq 0$ and thus r would be infinite. However an actual calculation of the k^4

term in the infinite part of $\Pi_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(k,\omega)$ determines its coefficient to be zero. Thus r is finite, which agrees with the tentative conclusions of section 4.2.

Calculating the induced R^2 coefficient gives a positive result, and thus the induced rR^2 term will not produce a tachyon.

6. THE GAUGE TECHNIQUE

To calculate G^{-1} for a realistic theory, such as QED, we require $\rho(\omega,\ell)$. In this chapter we shall show how to find an approximation to this function, using the method known as the gauge technique.

6.1 THE GAUGE TECHNIQUE ANSATZE

The essential theme of the gauge technique is that it is a non-perturbative method of solving the Schwinger-Dyson equations and, furthermore, guaranteeing that the gauge identities are automatically respected. These Ward identities relate the (n+1)-point function in terms of lower point functions and thus, with the Schwinger-Dyson equations which also couple the n-point functions to a (n+1)-point function, we can combine and solve these identities. Of course, since the Ward identity only used the longitudinal part of the (n+1)-point function, some information will be lacking. The true solution would require consideration of all the Green functions.

The initial idea of such a non-perturbative method (Salam 1963; Delbourgo & Salam 1964; Strathdee 1964) used as a starting point an approximation known as two-particle unitarity. This approximation was not all that productive, it was not for some time that a much better method was found (Delbourgo & West 1977a, 1977b. For a review, see Delbourgo 1979a, or Parker 1983, or Atkinson & Slim 1979). The technique is to write out the Dyson-

Schwinger equation of the electron propagator;

$$S(p)(\not{p}-m_0) = Z_1^{-1} + ie^2\int \bar{d}^4k\ S(p)\Gamma_\mu(p,p-k)S(p-k)\gamma_\nu D^{\mu\nu}(k) \quad (6.1)$$

and replace the $S(p)\Gamma_\mu(p,p-k)S(p-k)$ by an expression obtained from the Ward identity, which is

$$(p-p')^\mu S(p)\Gamma_\mu(p,p')S(p') = S(p')-S(p)$$

$$= \int d\omega\ \rho(\omega)\frac{1}{\not{p}-\omega}\gamma_\mu\frac{1}{\not{p}'-\omega}(p-p')^\mu \quad (6.2)$$

The ansatz is to consider only the longitudinal contribution of this to the full non-perturbative behavior and, as a first approximation, to neglect photon dressing for the propagator $D_{\mu\nu}(k)$.

We shall not, however, completely ignore the transverse corrections, as we will see. Basically, the above ansatz leads to a pair of coupled Volterra integral equations which are solvable in any gauge for $\rho(\omega)$, and thus $S(p)$. This resulting solution can, in principle, then be substituted back into the Schwinger-Dyson equations and improved versions of $S(p)$, $D_{\mu\nu}(k)$ and $\Gamma_\mu(p,p-k)$ obtained in an iterative fashion.

If we substitute the above ansatz into the Schwinger-Dyson equation (eqn. 6.1) we find that

$$Z_1^{-1} = \int \frac{\rho(\omega)}{\not{p}-\omega + i\varepsilon\ \varepsilon(\omega)}(\not{p}-m_0-\Sigma(p,\omega))\,d\omega \quad (6.3)$$

where $-i\Sigma(p,\omega)$ is the self energy;

$$\Sigma(p,\omega) = ie^2\int \bar{d}^4k\ \frac{1}{\not{p}-\omega+i\varepsilon}\gamma_\mu\frac{1}{\not{p}-\not{k}-\omega+i\varepsilon}\gamma_\nu\left(-\eta^{\mu\nu}+\frac{(1-a)k_\mu k_\nu}{k^2+i\varepsilon}\right)\frac{1}{k^2+i\varepsilon} \quad (6.4)$$

We now take the imaginary part of eqn. 6.3 and use the fact that $\rho(\omega)$ is real to obtain

$$\int d\omega\ \rho(\omega)(\omega-m_0-\mathrm{Re}(p,\omega))\,\mathrm{Im}\frac{1}{p-\omega+i\varepsilon\ \varepsilon(\omega)} = \int d\omega\ \rho(\omega)\frac{\mathrm{Im}\Sigma(p,\omega)}{\omega-p} \tag{6.5}$$

where we have taken $p = (p^2)^{1/2}$ as the positive eigenvalue of the matrix $\not{p}$.

Using $$\mathrm{Im}\,\frac{1}{p-\omega+i\varepsilon\ \varepsilon(\omega)} = \lim_{\varepsilon\to 0^+}\frac{-\varepsilon\ \varepsilon(\omega)}{(p-\omega)^2+\varepsilon^2} = -\pi\delta(p-\omega)\varepsilon(\omega) \tag{6.6}$$

we have $$\varepsilon(\omega)(\omega-m_0-\mathrm{Re}\Sigma(\omega,\omega))\rho(\omega) = \int d\omega'\rho(\omega')\frac{\mathrm{Im}\Sigma(\omega,\omega')}{\pi(\omega'-\omega)} \tag{6.7}$$

We would like, however, to have an equation fully in terms of renormalized quantities, so we need to replace m_0 by $m - \mathrm{Re}\Sigma(m,m) + O(e^4)$.

The left hand side will then become

$$\varepsilon(\omega)(\omega-m+\mathrm{Re}\Sigma(m,m)-\mathrm{Re}\Sigma(\omega,\omega)+O(e^4))\rho(\omega) \tag{6.8}$$

It would appear at first sight that the real part contains a divergence in the space-time dimension as $\ell\to 2$. This apparent problem is due to the fact that we have neglected the transverse corrections which will appear on the right hand side. These will cancel off the divergence since we know that $\rho(\omega)$ is a renormalized function with no divergences at $\ell\to 2$. Furthermore, as well as the $O(e^4)$ terms on the left hand side there will be $O(e^4)$, $O(e^6)$, terms in the unknown transverse part on the right hand side, and so it would be an inconsistent approximation to retain these perturbative terms, especially since we are interested in the non-perturbative behavior. Consequently, we have

$$\varepsilon(\omega)(\omega-m)\rho(\omega) = \int d\omega'\rho(\omega')\ \frac{\mathrm{Im}\Sigma(\omega,\omega')}{\pi(\omega'-\omega)} \tag{6.9}$$

The imaginary part of $\Sigma(\omega,\omega')$ can be easily calculated and thus this equation can be solved for $\rho(\omega)$.

Before we go on to calculate $\rho(\omega)$, it should be noted that the gauge technique is not confined to QED. It has been applied to scalar electrodynamics (Delbourgo 1977), vector electrodynamics (Delbourgo 1978), two dimensional field theories such as the Schwinger model (Delbourgo & Shepherd 1978), the Thirring models (Delbourgo & Thompson, 1982; Thompson 1983), the Rothe-Stamatescu axial model (Thompson 1983), and also QED with massive photon (Delbourgo, Kenny & Parker 1982), Chiral QED (Delbourgo & Keck 1980a), the Bloch-Nordsieck model in QED (Alekseev & Rodionov 1980), Flavordynamics (Delbourgo & Kenny 1981), electrodynamics in the axial gauge (Delbourgo 1978; Delbourgo & Phocas-Cosmetatos 1979) and also QCD (Delbourgo 1979b; Ball & Zachariasen 1978; Anishetty et. al. 1979; Baker 1979; Baker, Ball, Lucht & Zachariasen 1980; Khelashvili 1981; Cornwall 1980, 1983).

6.2 THE IMAGINARY PART OF THE SELF ENERGY

Since we require $\rho(\omega,\ell)$, and not just $\rho(\omega,2)$, we shall calculate $\mathrm{Im}\Sigma(p,\omega)$ in arbitrary 2ℓ dimensions. We have:

$$\Sigma(p,\omega) = ie^2\int\frac{\bar{d}^{2\ell}k}{k^2+i\varepsilon}\,\gamma^\mu\,\frac{1}{\not p+\not k-\omega+i\varepsilon}\,\gamma^\nu\left(-g_{\mu\nu}+(1-a)\frac{k_\mu k_\nu}{k^2+i\varepsilon}\right) \tag{6.10}$$

we define Σ by $\Sigma(p,\omega) = \Sigma^1 + \Sigma^a$

$$\text{where } \Sigma^1(p,\omega) = -ie^2\int\frac{\bar{d}^{2\ell}k}{k^2+i\varepsilon}\,\gamma^\mu\,\frac{\not p-\not k+\omega}{(p-k)^2-\omega^2+i\varepsilon}\,\gamma_\mu \tag{6.11}$$

By the Cutkosky rules (Cutkosky 1960; see also Itzykson & Zuber 1980, pp. 315-316), we have

$$2\mathrm{Im}\Sigma^1(p,\omega) = (2\pi)^2 e^2 \int \bar{d}^{2\ell}k \ (\gamma^\mu(\not p-\not k+\omega)\gamma_\mu)\,\delta_+(k^2)\,\delta_+((p-k)^2-\omega^2)$$

where $\qquad \delta_+(p^2-m^2) = \delta(p^2-m^2)\,\theta(p_0)$ (6.12)

So $2\mathrm{Im}\Sigma^1(p,\omega) = 2(2\pi)^2 e^2 \int \bar{d}^{2\ell}k \ [(\not k-\not p)(\ell-1)+\ell\omega]\,\delta_+(k^2)\,\delta_+((p-k)^2-\omega^2)$

It is trivial to show that

$$\mathrm{Im}F(p,\omega) = \not p\,\mathrm{Im}F_1(p^2,\omega^2)+\omega\,\mathrm{Im}F_2(p^2,\omega^2) \tag{6.13}$$

where $\qquad \mathrm{Im}F_1 = \dfrac{1}{2^\ell p^2}\,\mathrm{Tr}(\not p\,\mathrm{Im}F)$ (6.14)

$$\mathrm{Im}F_2 = \frac{1}{2^\ell \omega}\,\mathrm{Tr}(\mathrm{Im}F) \tag{6.15}$$

Thus $\mathrm{Im}\Sigma_1^1(p^2,\omega) = \dfrac{4\pi^2e^2}{2^\ell p^2}\displaystyle\int \bar{d}^{2\ell}k \ \mathrm{Tr}[\not p(\not k-\not p)(\ell-1)+\ell\not p\omega]$

$$\times\ \delta_+(k^2)\,\delta_+((p-k)^2-\omega^2)$$

$$= (\ell-1)\,\frac{4\pi^2e^2}{p^2}\int \bar{d}^{2\ell}k\,(p\cdot k-p^2)\,\delta_+(k^2)\,\delta_+((p-k)^2-\omega^2)$$

Using the δ functions, we have

$$\mathrm{Im}\Sigma_1^1(p^2,\omega) = -(\ell-1)2\pi^2e^2\,\frac{(p^2+\omega^2)}{p^2}\int \bar{d}^{2\ell}k \ \ \delta_+(k^2)\,\delta_+(p^2-2pk-\omega^2) \tag{6.16}$$

Similarily, we have

$$\mathrm{Im}\Sigma_2^1(p^2,\omega) = 4\ell\pi^2e^2\int \bar{d}^{2\ell}k \ \ \delta_+(k^2)\,\delta_+((p-k)^2-\omega^2) \tag{6.17}$$

We now consider the second half of $\Sigma(p,\omega)$, namely

$$\Sigma^a(p,\omega) = -i(a-1)e^2\int \frac{\bar{d}^{2\ell}k}{(k^2+i\varepsilon)^2}\,\not k\,\frac{1}{\not p-\not k-\omega+i\varepsilon}\,\not k \tag{6.18}$$

we use the trick $\dfrac{1}{k^4} = \lim\limits_{\mu\to 0}\left(\dfrac{1}{k^2-\mu^2}-\dfrac{1}{k^2}\right)\dfrac{1}{\mu^2}$ (6.19)

to handle the quadratic singularity, and so define

$$\Sigma^a(p,\omega,\mu^2) = -i(a-1)e^2\int \frac{\bar{d}^4k}{(k^2-\mu^2+i\varepsilon}\,\not k\,\frac{1}{\not p-\not k-\omega+i\varepsilon}\,\not k \tag{6.20}$$

Proceed as before to obtain

$$\mathrm{Im}\Sigma_1^a(p,\omega,\mu^2) = \frac{2(a-1)e^2\pi^2}{p^2}\int \bar{d}^{2\ell}k\ [2(p\cdot k)^2-p^2k^2-k^2p\cdot k]$$

$$\times\ \delta_+(k^2-\mu^2)\,\delta_+(p-k)^2-\omega^2)$$

$$= (a-1)\frac{\pi^2e^2}{p^2}[(p^2-\omega^2)^2-\mu^2(p^2+\omega^2)]\int \bar{d}^{2\ell}p\ \delta_+(k^2-\mu^2)\delta_+(p-k)^2-\omega^2) \quad (6.21)$$

and $\mathrm{Im}\Sigma_2^a(p^2,\omega^2,\mu^2) = 2(a-1)e^2\pi^2\mu^2\int \bar{d}^{2\ell}k\ \delta_+(k^2-\mu^2)\delta_+(p-k)^2-\omega^2)$ (6.22)

To evaluate the integral, we use the volume element formula

$$d^{2\ell}k = \frac{\pi^{\ell-1/2}}{\Gamma(\ell-1/2)}(k^2)^{\ell-1}dk^2(\sinh\xi)^{2\ell-2}d\xi \qquad (6.23)$$

$$0 \leq i\xi \leq \pi$$

where ξ is the parameter defined by the time component of k; $k_0 = (k^2)^{1/2}\cosh\xi$, which is allowable, since k must be timelike because of $\delta(k^2-\mu^2)$.

We now choose a Lorentz frame where $p = ((p^2)^{1/2},0,0,\ldots,0)$. Thus $\int \bar{d}^{2\ell}k\ \delta_+(k^2-\mu^2)\delta_+((p-k)^2-\omega^2)$

$$= \frac{\pi^{\ell-1/2}}{\Gamma(\ell-1/2)(2\pi)^{2\ell}}\int dk^2d\xi(k^2)^{\ell-1}(\sinh\xi)^{2\ell-2}\delta_+(k^2-\mu^2)\delta_+(p^2-\omega^2+\mu^2 -2\mu(p^2)^{1/2}\cosh\xi)$$

$$= \frac{\pi^{\ell-1/2}}{\Gamma(\ell-1/2)}\ \frac{(\mu^2)^{\ell-1}}{(2\pi)^{2\ell}}\ \frac{(\sinh\xi)^{2\ell-3}}{2\mu(p^2)^{1/2}}\ \theta(p^2-(\omega+\mu)^2)$$

(The θ-fn comes from the conditions $\cosh\xi \geq 1$ and $(p^2)^{1/2} \geq \mu\cosh\xi$.)

$$= \frac{\pi^{\ell-1/2}}{(2\pi)^{2\ell}\Gamma(\ell-1/2)}\ \frac{1}{(4p^2)^{\ell-1}}\ \Delta^{2\ell-3}(p^2,\omega^2,\mu^2)\,\theta(p^2-(\omega+\mu)^2) \qquad (6.24)$$

where $\Delta^2 = (p^2-\omega^2+\mu^2)^2 - 4\mu^2p^2$

$= (p^2+\omega^2-\mu^2)^2 - 4\omega^2p^2$

Thus $$\mathrm{Im}\Sigma_1^a(p,\omega) = \frac{(a-1)}{4^{\ell-1}(2\pi)^{2\ell}} \frac{e^2\pi^2}{p^{2\ell}} \frac{\pi^{\ell-1/2}}{\Gamma(\ell-1/2)} \times$$

$$\lim_{\mu\to 0} \frac{1}{\mu^2}[\Delta^{2\ell-3}(p^2,\omega^2,\mu^2)((p^2-\omega^2)^2-\mu^2(p^2+\omega^2))-\Delta^{2\ell-3}(p^2,\omega^2)(p^2-\omega^2)^2]$$

$$= \frac{-4(\ell-1)(a-1)e^2\pi^2\pi^{\ell-1/2}}{4^{\ell-1}(2\pi)^{2\ell}(p^2)^\ell\Gamma(\ell-1/2)} \frac{(p^2+\omega^2)(p^2-\omega^2)^{4\ell-4}}{2(p^2-\omega^2)^{2\ell-1}}$$

where $p^{2\ell} = (p^2)^\ell$ and we also obtain $\mathrm{Im}\Sigma_2^a$

$$\mathrm{Im}\Sigma_2^a(p,\omega) = \frac{2(a-1)e^2\pi^2}{4^{\ell-1}(2\pi)^{2\ell}} \frac{\pi^{\ell-1/2}}{\Gamma(\ell-1/2)(p^2)^{\ell-1}} (p^2-\omega^2)^{2\ell-3} \qquad (6.25)$$

Collecting terms we have $\mathrm{Im}\Sigma_1 = \mathrm{Im}\Sigma_1^1 + \mathrm{Im}\Sigma_1^a$

$$= (1-\ell)C_\ell \frac{a(p^2+\omega^2)(p^2-\omega^2)^{2\ell-3}}{p^{2\ell}} \cdot \frac{e^2}{16\pi} \qquad (6.26)$$

where $$C_\ell = \frac{\pi^{5/2-\ell}}{2^{4\ell-7}\Gamma(\ell-1/2)} \qquad \text{(so that } C_2 = 1)$$

$$\mathrm{Im}\Sigma_2 = \mathrm{Im}\Sigma_2^1 + \mathrm{Im}\Sigma_2^a$$

$$= C_\ell \frac{(a+2\ell-1)(p^2-\omega^2)^{2\ell-3}}{(p^2)^{\ell-1}} \cdot \frac{e^2}{16\pi} \qquad (6.27)$$

Thus, replacing the θ-fn and combining, we have

$$\mathrm{Im}\Sigma(p,\omega) = \frac{e^2C_\ell}{16\pi}(p^2-\omega^2)^{2\ell-3}\left((1-\ell)a\frac{(p^2+\omega^2)}{(p^2)^\ell}\cancel{p} + \frac{(a+2\ell-1)\omega}{(p^2)^{\ell-1}}\right)\theta(p^2-\omega^2) \qquad (6.28)$$

So:

$$\mathrm{Im}\Sigma(\omega,\omega') = \frac{-e^2C_\ell}{16\pi} \frac{(\omega^2-\omega'^2)^{2\ell-3}}{(\omega^2)^\ell} [(\ell-1)a(\omega^2+\omega'^2)\omega + (a+2\ell-3)\omega'\omega^2] \times \theta(\omega^2-\omega'^2) \qquad (6.29)$$

Taking the limit $\ell\to 2$ gives the usual result.

6.3 THE SOLUTION TO THE ANSATZE

We now seek to solve the equation

$$\varepsilon(\omega)(\omega-m)\rho(\omega,\ell) = \int d\omega'\rho(\omega')\frac{\mathrm{Im}\Sigma(\omega,\omega',\ell)}{\pi(\omega'-\omega)}$$

for $\rho(\omega,\ell)$

For the time being we shall retain the general covariant gauge a, but later we shall see that it is necessary to choose the Landau gauge, a=0.

We shall convert this equation into two coupled integral equations, which are amenable to solution.

First of all convert to dimensionless variables, namely $\beta = \omega/m$ and define $\rho(\beta)$ as $m\rho(\beta m)$.

Thus the above equation becomes

$$\varepsilon(\beta)(\beta-1)\rho(\beta) = \frac{e^2}{16\pi^2}C_\ell\int^\beta d\beta'\rho(\beta')\frac{(\beta^2-\beta'^2)^{2\ell-3}}{(\beta-\beta')(\beta^2)^{2\ell-1}\beta}[(\ell-1)a(\beta^2+\beta'^2) - (a+2\ell-1)\beta\beta'] \qquad (6.30)$$

where the symbol $\int^\beta d\beta'$ means $(\int_{-|\beta|}^{-1} + \int_1^{|\beta|})d\beta'$.

Change to a new choice of dependent variables;

$$s(\beta) = \varepsilon(\beta)\beta^2\rho(\beta)$$

and define $s_1(\beta^2)$ and $s_2(\beta^2)$ by

$$s(\beta) = \beta s_1(\beta^2)+s_2(\beta^2) \qquad (6.31)$$

We now use the observation that

$$\int^\beta d\beta' f(\beta')\varepsilon(\beta') = \int_1^\beta [f(\beta')-f(-\beta')]d\beta'$$

and rewrite the above expression as

$$(\beta-1)(\beta s_1(\beta^2)+s_2(\beta^2)) = \xi^2\int_1^{\beta^2} d\beta'^2 \frac{(\beta^2-\beta'^2)^{2\ell-4}}{(\beta^2)^{\ell-1}\beta'^2}\{[a(\ell-1)(\beta^2+\beta'^2)$$

$$-(a+2\ell-1)\beta'^2]\beta^2 s_1(\beta'^2) + [a(\ell-1)(\beta^2+\beta'^2)-(a+2\ell-1)\beta^2]\beta s_2(\beta'^2)\}$$

where $\xi^2 = \dfrac{e^2}{16\pi^2}C_\ell$.

Examining the above, we compare the even functions of β in both sides and equate; to obtain

$$Zs_1(Z)-s_2(Z) = \xi^2\int_1^{Z} \frac{(Z-Z')^{2\ell-4}}{Z^{\ell-2}}\{a(\ell-1)\frac{Z}{Z'}+a(\ell-2)+1-2\ell\}s_1(Z') \quad (6.32)$$

where $Z = \beta^2$ and $Z' = \beta'^2$.

Similarily, comparing the odd functions of β, we obtain

$$s_2(Z)-s_1(Z) = \xi^2\int_1^{Z} \frac{(Z-Z')^{2\ell-4}}{Z^{\ell-2}}\{\frac{a(\ell-1)}{Z} + \frac{a(\ell-2)}{Z'} + \frac{1-2\ell}{Z'}\}s_2(Z') \quad (6.33)$$

These equations, if solved, would give s_1 and s_2, and thus $\rho(\omega,\ell,a)$ for any gauge and for any spacetime dimension.

Unfortunately, these equations cannot be solved in general. They are two coupled linear Voltera integral equations of the 2nd kind, with non-degenerate kernel. Furthermore, the kernel $K(Z,Z')$ is not even of the Faltung type $K(Z-Z')$. These statements are true for any value of a.

In principle, of course, one could solve these equations numerically and/or using Picard's process of successive approximations (the Neumann series) obtained by an iterative process. See, for example, Tricomi (1955). It is possible to find a solution, however, for particular values of ℓ, namely $2\ell-4$ = non-negative integer.

The simplest solution is obtained for $\ell=2$, which is the situation encountered in the normal QED gauge technique problem. To find the solution, we differentiate each equation four times to arrive at two 4th order linear differential equations, whose solutions are ${}_4F_3$ hypergeometric functions (Delbourgo, Keck & Parker 1981).

Unfortunately the theory of ${}_4F_3$ functions is insufficiently developed for our purposes. Consequently we choose a particular value of a, namely a=0, which simplifies the problem to two 2nd order linear differential equations:

$$\left(Z(1-Z)\frac{d^2}{dZ^2} + (1-3Z(1+2\xi^2))\frac{d}{dZ} - (1+3\xi^2)^2\right)s_1(Z) = 0 \tag{6.34}$$

$$\left(Z(1-Z)\frac{d^2}{dZ^2} - 2(1+3\xi^2)Z\frac{d}{dZ} - 3\xi^2(1+3\xi^2)\right)s_2(Z) = 0 \tag{6.35}$$

The general solutions of these equations are

$$s_1(Z) = C\ {}_2F_1(1-\zeta,1-\zeta;2-2\zeta;1-Z)+D_1(Z-1)^{2\zeta-1}{}_2F_1(\zeta,\zeta;2\zeta;1-Z) \tag{6.36}$$

$$s_2(Z) = K\ {}_2F_1(-\zeta,1-\zeta;2-2\zeta;1-Z)+D_2 Z(Z-1)^{2\zeta-1}{}_2F_1(\zeta,1+\zeta;2\zeta;1-Z)$$

$$\text{where} \qquad \zeta = -3\xi^2 \tag{6.37}$$

However, we have lost information in differentiating and separating. If we substitute these back into the two coupled integral equations and consider the behavior around Z=1 (i.e. do an expansion in powers of Z-1) and compare coefficients we find $D_1 = D_2$ and $C = K = 0$.

We thus have (Delbourgo & West 1977b; Delbourgo & Keck 1980b)

$$\rho(\omega) = \frac{\varepsilon(\omega)(\omega^2/m^2-1)^{2\zeta-1}}{m\ 4^{\zeta}\Gamma(2\zeta)}\left[\frac{m}{\omega}F(\zeta,\zeta;2\zeta;1-\frac{\omega^2}{m^2})+F(\zeta,1+\zeta;2\zeta;1-\frac{\omega^2}{m^2})\right] \tag{6.38}$$

where the constant D_1 has been chosen via appropriate normalization;

$$D_1 = \frac{1}{4^{\zeta}\Gamma(2\zeta)} \tag{6.39}$$

6.4 THE SPECTRAL FUNCTION IN FOUR DIMENSIONS

The infrared limit of $\rho(\omega)$ as $\omega \to m$ is easy to see:

$$\rho(\omega) \sim \varepsilon(\omega)\left(\frac{\omega^2}{m^2}-1\right)^{2\zeta-1} . \tag{6.40}$$

This however can be generalized to an arbitrary gauge from calculation directly from the integral, to derive

$$\rho(\omega) \sim \varepsilon(\omega)\left(\frac{\omega^2}{m^2}-1\right)^{2(1-\frac{a}{3})\zeta - 1} \tag{6.41}$$

or
$$S(p) = \text{constant} \times \frac{\not{p}+m}{p^2-m^2}\left(\frac{m^2}{p^2-m^2}\right)^{e^2(a-3)/8\pi^2} \tag{6.42}$$

which reproduces the known result for all gauges (Abrikosov 1956; see also Ball, Horn & Zachariasen 1978).

The ultraviolet behavior $\frac{\omega}{m}\to\infty$ is not so simple (Delbourgo, Keck & Parker 1981; Atkinson & Slim 1979; Slim 1981). It is easy in the Landau gauge; we obtain

$$\rho(\omega) \sim (\omega^2/m^2)^{\zeta-1}\left(1 + 2\frac{m}{\omega}\ln\frac{\omega}{m}\right) \tag{6.43}$$

which leads to $S(p) \sim \frac{\not{p}}{p^2} - \frac{1}{\zeta m}(-p^2/m^2)^{\zeta-1}$

which agrees with Baker, Johnson & Willey (1964).

In an arbitrary gauge, the situation is more complicated since we need to consider the full ${}_4F_3$ solution of $\rho(\omega)$ and take $\omega\to m$. It would appear however that the solution is of the form $\rho(\omega) = \varepsilon(\omega)\omega^{-2}(\omega\sigma_1+m\sigma_2)$

where $\sigma_1 \sim (\frac{\omega^2}{m^2})^{-a_2}$ $\quad a_2 = \frac{1}{2}(1+\frac{a\zeta}{3}-\zeta)+\frac{1}{2}(1-\frac{2a\zeta}{3}-2\zeta+\frac{(a-\zeta)^2\zeta^2}{9})^{\frac{1}{2}}$ (6.44)

$$\sigma_2 \sim (\frac{\omega^2}{m^2})^{1-a_3} \quad a_3 = \frac{1}{2}(3+\frac{a\zeta}{3}-\zeta)-\frac{1}{2}(1+\frac{2a\zeta}{3}+2\zeta+\frac{(a-3)^2\zeta^2}{9})^{\frac{1}{2}} \qquad (6.45)$$

It should be noted that in both these limits it is known that the apparent manifest gauge covariance of the gauge technique is actually true gauge covariance, i.e. satisfies

$S(x;a) = (-m^2x^2)^{-ae^2/16\pi^2}S(x;0)$ (Delbourgo & Keck 1980b). However, in intermediate regions this is not so, it would appear that one must incorporate at least part of the transverse vertices to achieve full gauge covariance. This fact can lead to ambiguous conclusions in particular applications (Atkinson & Slim 1979; Delbourgo, Keck & Parker 1981; Slim 1981; Gardner 1981).

Before we move on to consideration of $\rho(\omega,\ell)$ we note a number of further properties of $\rho(\omega,2)$.

Using the general integral:

$$\int_0^\infty x^{c-1}(x+y)^{-d}F(a,b;c;-x)\,dx = \frac{\Gamma(a-c+d)\,\Gamma(b-c+d)\,\Gamma(c)}{\Gamma(a+b-c+d)\,\Gamma(d)}$$

$$\times\ F(a-c+d,b-c+d;a+b-c+d;1-y) \qquad (6.46)$$

we see that $Z_\psi^{-1} = \int\rho(\omega)\,d\omega = 2^{-2\zeta}(\Gamma(1-\zeta))^2.$ (6.47)

and also $\int\rho(\omega)(\frac{\omega^2}{m^2})^{\ell-2}\omega\,d\omega = \frac{\Gamma(2-\ell-\zeta)\,\Gamma(3-\ell-\zeta)}{4^\zeta\ \ \Gamma(3-\ell)}\ \frac{1}{\Gamma(2-\ell)}\ m$ (6.48)

Thus as $\ell\to 2$, $m_0 = \int\rho(\omega)\omega\,d\omega \sim (\ell-2)m \to 0.$

This observation was used by Khare & Kumar (1978) to conjecture that QED is actually a finite theory in the sense of the JBW program (Baker, Johnson & Willey 1964; Adler 1972). However Slim

(1979) showed that with the approximations made, the gauge technique does not imply a finite theory.

Note however that we do have dynamical symmetry breakdown. The bare QED Lagrangian has no bare mass terms and thus is scale invariant.

We also have

$$\int \rho(\omega)\left(\frac{\omega^2}{m^2}\right)^{\ell-1} d\omega = \frac{(\Gamma(2-\ell-\zeta))^2}{4^\zeta\,\Gamma(2-\ell)\,\Gamma(2-\ell)} \tag{6.49}$$

Thus as $\ell \to 2$

$$\int \rho(\omega)\,\omega^2 d\omega \to O(\ell-2)^2 + O(\ell-2)^3 \tag{6.50}$$

The vanishing of these integrals is very important for the elimination of ultraviolet divergences, both in some other theories (e.g. Chiral QED; Delbourgo and Keck 1980a) and in induced gravity.

6.5 THE APPLICABILITY OF THE GAUGE TECHNIQUE TO INDUCED GRAVITY

We summarize the results of the previous chapter:

$$G^{-1} = \lim_{\ell \to 2} \frac{1}{2\pi(\ell-2)} \int (\omega^2)^{\ell-2} (\omega^2 - \tfrac{4}{3} m\omega)\,\rho(\omega,\ell)\,d\omega \tag{6.51}$$

For this to be finite, we require the spectral rules

$$\int \rho(\omega)\,\omega^2 d\omega = 0 \tag{6.52}$$

$$\int \rho(\omega)\,\omega\, d\omega = 0 \tag{6.53}$$

The gauge technique provides a spectral function $\rho(\omega)$ for QED that automatically satisfies these identities. Furthermore the gauge technique forces us to take the bare fermion mass as zero, so the dynamically induced fermion mass m signifies the very essence of induced gravity. The fundamental Lagrangian is scale invariant and all masses are produced through a non-perturbative mechanism.

Thus the philosophies of the gauge technique and induced gravity are not only compatible, but essentially the same.

We thus proceed to calculate $\rho(\omega,\ell)$ which is provided by the differential equations 6.32 and 6.33. Since we only need $\rho(\omega,\ell)$ for ℓ near 2, we can write

$$\rho(\omega,\ell) = \rho(\omega) + (\ell-2)\eta(\omega) \tag{6.54}$$

or equivalently

$$s_i(Z,\ell) = s_i(Z) + (\ell-2)h_i(Z) \quad i = 1,2 \tag{6.55}$$

where s_1 and s_2 are defined in eqn. 6.36 and 6.37.

These will be substituted back into the differential equations, but only terms of order $(\ell-2)$ will be kept. This will enable us to calculate $h_1(Z)$ and $h_2(Z)$, and thus G^{-1}.

7. A CALCULATION OF G^{-1}/m^2 for QED

Using the results of the last chapter, we perform an actual calculation of G^{-1}/m^2, giving the result purely in terms of the coupling constant for QED.

7.1 SOLVING THE INTEGRAL EQUATION

Recall the results of Chapter 5, (eqn. 5.46), namely

$$G^{-1} = \frac{1}{2\pi}\int \omega^2 \ln\frac{\omega^2}{m^2}\rho(\omega)\,d\omega - \frac{2}{3\pi}m\int \omega \ \ln\frac{\omega^2}{m^2}\rho(\omega)\,d\omega$$

$$+ \frac{1}{2\pi}\int \omega^2\,\eta(\omega)\,d\omega - \frac{2}{3\pi}m\int \omega \ \eta(\omega)\,d\omega \tag{7.1}$$

where $\rho(\omega,\ell) = \rho(\omega) + (\ell-2)\eta(\omega) + O(\ell-2)^2$. Now, by equation 6.48 and 6.49 we have

$$\int \omega \ \ln\frac{\omega^2}{m^2}\rho(\omega)\,d\omega = -\Gamma(-\zeta)\,\Gamma(1-\zeta)4^{-\zeta}m \tag{7.2}$$

$$\int \omega^2 \ \ln\frac{\omega^2}{m^2}\rho(\omega)\,d\omega = 0 \tag{7.3}$$

Equations 6.55 and 6.31 define h_1 and h_2. These are equivalent to

$$\eta(\omega) = \frac{m\ \varepsilon(\omega)}{\omega^2}\ \left(\frac{\omega}{m}\ h_1\left(\frac{\omega^2}{m^2}\right) + h_2\left(\frac{\omega^2}{m^2}\right)\right) \tag{7.4}$$

It is thus immediate that

$$\int \omega \ \eta(\omega)\,d\omega = m\int_1^\infty \frac{h_2(Z)}{Z}dZ \tag{7.5}$$

and $$\int \omega^2 \, \eta(\omega) d\omega = m^2 \int_1^\infty h_1(Z) \, dZ \tag{7.6}$$

where $Z = \omega^2/m^2$

Thus $$G^{-1}/m^2 = \frac{2}{3\pi}\Gamma(-\zeta)\Gamma(1-\zeta)4^{-\zeta} - \frac{2}{3\pi}\int_1^\infty \frac{h_2(Z)}{Z} dZ + \frac{1}{2\pi}\int_1^\infty h_1(Z) dZ \tag{7.7}$$

What remains to be done is the finding of the functions $h_1(Z)$ and $h_2(Z)$ and the calculations of the integrals.

Now, from the differential equations (eqns 6.32 and 6.33) we choose the Landau gauge a = 0 and expand to first order in $\ell-2$, to obtain

$$Zh_1(Z) - h_2(Z) = \zeta\int_1^Z \{[2\ln(Z-Z')-\ln Z+d]\, s_1(Z') + h_1(Z')\}dZ' \tag{7.8}$$

$$h_2(Z) - h_1(Z) = \zeta\int_1^Z \{[2\ln(Z-Z')-\ln Z+d]\frac{s_2(Z')}{Z} + \frac{h_2(Z')}{Z}\}dZ' \tag{7.9}$$

where d = constant and $s_1(Z)$, $s_2(Z)$ are given by eqns 6.36 and 6.37 with C = K = 0.

This is of the form

$$Zh_1 - h_2 = \zeta\int_1^Z h_1(Z')dZ' + f_1(Z) \tag{7.10}$$

$$h_2 - h_1 = \zeta\int_1^Z \frac{h_2(Z')}{Z'}dZ' + f_2(Z) \tag{7.11}$$

Now, $f_1(Z)$ and $f_2(Z)$ can be calculated, so these coupled integral equations are solvable by differentiation.

Twice differentiating, we find

$$Z(1-Z)h_1'' + [1+Z(2\zeta-3)]h_1' - (1-\zeta)^2 h_1 = -(Zf_2''+Zf_1''+f_2'+(1-\zeta)f_1') \equiv R_1(Z) \tag{7.12}$$

$$Z(1-Z)h_2'' + 2Z(\zeta-1)h_2' - \zeta(\zeta-1)h_2 = -(Z^2 f_2''+Zf_1''+(2-\zeta)Zf_2') \equiv R_2(Z) \tag{7.13}$$

We write

$$h_1 = A_1h_{11} + B_1h_{12} + h_{1p} \tag{7.14}$$

$$h_2 = A_2h_{21} + B_2h_{22} + h_{2p} \tag{7.15}$$

A_1, A_2, B_1 and B_2 are constants.

The solutions to the homogeneous part are

$$h_{11}(Z) = {}_2F_1(1-\zeta,1-\zeta;2-2\zeta;1-Z) \tag{7.16}$$

$$h_{12}(Z) = (Z-1)^{2\zeta-1}\,{}_2F_1(\zeta,\zeta;2\zeta;1-Z) \tag{7.17}$$

$$h_{21}(Z) = Z\,{}_2F_1(1-\zeta,2-\zeta;2-2\zeta;1-Z) \tag{7.18}$$

$$h_{22}(Z) = Z(Z-1)^{2\zeta-1}\,{}_2F_1(\zeta,1+\zeta;2\zeta;1-Z) \tag{7.19}$$

From the general theory of the method of variation of parameters, we have

$$h_{1p}(Z) = -h_{11}(Z)\int_1^Z \frac{h_{12}(Z')R_1(Z')}{Z'(1-Z')W_1(Z')}dZ' + h_{12}(Z)\int_1^Z \frac{h_{11}(Z')R_1(Z')}{Z'(1-Z')W_1(Z')}dZ' \tag{7.20}$$

where $W_1 = h_{11}h_{12}' - h_{11}'h_{12}$ is the Wronskian for the differential equation 7.12.

We have of course a similar equation for $h_{2p}(Z)$.

These Wronskians are easily evaluated by setting up a 1st order differential equation for them.

$$\frac{dW_1}{dZ} = \frac{Z(2\zeta-3)+1}{Z(Z-1)}\,W_1 \tag{7.21}$$

so

$$W_1 = (2\zeta-1)\frac{(Z-1)^{2\zeta-3}}{Z} \tag{7.22}$$

This is all the information needed to calculate G^{-1}, as shown in section 7.2. However, for completeness, some discussion on h_1 and h_2 will now be given.

The coefficients A_1 and A_2 are not arbitrary. To verify this, in fact to show that $A_1 = A_2 = 0$, one proceeds to find the infrared behaviour ($Z \backsim 1$) of h_{11}, h_{12}, h_{21}, h_{22}, and of the actual solution $\rho(\omega)$ of the integral equation 6.30. This is done by taking the limit $Z \backsim 1$ and $\ell \to 2$ (retaining $O(\ell-2)$ terms) of eqn. 6.30 which converts it to a Volterra integral equation with a kernel of the Faltung, or convolution, type. This can then be solved via Laplace transforms to obtain

$$\rho(x) \backsim x^{k-1}\{1+k(\ell-2)[(\ln x)^2+k_1\ln x+k_2]+O(\ell-2)^2\} \qquad (7.23)$$

where $\quad k = 2\zeta(1-\frac{a}{3})$ for the gauge a, and $x = Z-1$.

7.2 A VALUE FOR G^{-1}/m^2

There are a number of ways one might try to evaluate the integrals in eqn. 7.7. The easiest way however is to note that the integrals to be evaluated are the same integrals which appear in the integral equations 7.10 and 7.11. Thus all that needs to be done is the evaluation of the ultraviolet ($Z \to \infty$) limit of the functions h_{11}, h_{12}, h_{21}, h_{22}, h_{1p}, h_{2p}, f_1, and f_2. These are all of the form

$$Z^{\zeta} + O(Z^{\zeta-1}) \quad \text{as} \quad Z \to \infty.$$

$$\text{Since } \zeta = -3\xi^2 = \frac{-3e^2}{16\pi^2} < 0 \ ,$$

the limit of all these functions is zero!

$$\text{Thus} \qquad \int_1^{\infty} h_1(Z)dZ = \int_1^{Z} \frac{h_2(Z)}{Z}\, dZ = 0$$

Substituting this into eqn. 7.7 gives

$$\frac{G^{-1}}{m^2} = \frac{2}{3\pi}\,\Gamma(-\zeta)\,\Gamma(1-\zeta)\,4^{-\zeta}$$

Substituting in the value $\xi^2 = \frac{\alpha}{4\pi}$ for QED we find

$$\frac{G^{-1}}{m^2} \simeq \frac{8}{9\alpha} \simeq 120$$

Note that this is very small but positive. It thus clearly does not give a realistic G^{-1}. Nevertheless, it is the contribution to G^{-1} from QED as given by our approximations. Thus we can conclude that the contribution to G^{-1} from QED is negligibly small.

8. CONCLUSIONS

8.1 RESULTS

The results obtained are important in a number of ways. Firstly, we have established the existence of a non-zero contribution to G^{-1} from QED. Thus the basic principle of inducing the Einstein-Hilbert Lagrangian is exhibited, and the dynamical symmetry breaking of scale invariance is established.

Secondly, we note that the calculated contribution to G^{-1} from the fermionic sector of QED is very small, namely

$$\frac{G^{-1}}{m^2} \approx \frac{8}{9\alpha} \approx 1.2 \times 10^2$$

Thus no suggestion is being made that the force of gravity is induced from the quantization of the electron field. However, note that the induced G^{-1} is positive, and so the third point to note is that a realistic G^{-1} could be induced from the charged sector of a theory which has a fermion of mass 10^{18} GeV, (or a charge $q = 1/3$ fermion of mass 10^{17} GeV), or less if there are many such fermions.

This conclusion, namely the necessity of a large mass scale, is also indicated by all previous model calculations of G^{-1}. It is expected that the future lattice calculations will support this. Thus, the necessity for a GUT is clearly indicated; the contributions to G^{-1} from the various particles in the GUT are clearly many, but the results of this work has been to show that a significant contribution may come from the charged fermions of the theory.

8.2 EXTENSIONS

A number of extensions come immediately to mind. One could repeat the above calculations for a variety of other models, for example scalar electrodynamics, or vector electrodynamics. One could easily calculate a general formula for the contribution of a spin 1 particle to G^{-1}, in analogy to the calculations of chapter 5. It may then be possible to find an approximation to the contribution of the dressed photon loop to G^{-1}.

More realistically, one could try and calculate a model spectral function for a quark, taking into account the gluon dressing. This could be approached by using the known limits of QCD and then guessing an interpolating model function for the spectral function. Unfortunately the interpolation is too ambiguous to give useful results, especially since this must all be done in 2ℓ dimensions. A better approach to QCD would be to use the gauge technique; a success in this area may soon be forthcoming but no truly realistic spectral function has yet been calculated for all values of the argument.

We can indicate extensions in a more spectulative area, by incorporating the idea of dimensional reduction, since the idea of induced gravity is not restricted to 4 dimensions. However, since we demand scale invariance, we cannot have a fundamental $\frac{1}{g^2} F^2$ term. Higher-dimensional theories already exist where this term is absent, namely Kaluza-Klein theories, where the $\frac{1}{g^2} F^2$ term is a result of dimensional reduction for the original Lagrangian which contains a $\frac{1}{16\pi G}R$ term. This curvature term need not be fundamental, but could be induced from, say, the fermionic sector of the theory. Unfortunately it is probable that this scenario will produce non-renormalizable infinities.

A further extension is to note that the Lagrangian of supergravity also contains a mass scale, and so it would be possible to incorporate the idea of induced gravity into supergravity. Furthermore this may solve the problem with the counterterms in the dimensional reduction approach.

To summarize, the concept of induced gravity is an idea of profound importance, not necessarily tied to the conventional low energy theories, but as a key concept in the physics at the higher energies.

APPENDIX

Up until now, we have almost entirely considered the metric $g_{\mu\nu}$ as being a classical background field (with the exception of chapter 4). Given the problems posed by quantizing $O(R^2)$ theories, we may wish to circumvent the issue by not quantizing the metric at all. A theory with a classical metric does not require $O(R^2)$ terms in the fundamental gravity Lagrangian. The idea of induced gravity is still useful since the value of G^{-1} could still in principle be predicted. Clearly the metric appears to us in a rather different way than the other fields, so the idea of leaving $g_{\mu\nu}$ classical is not without its motivation or merits.

There are, however, a number of arguments against this idea. The first argument is that leaving $g_{\mu\nu}$ classical implies that the Einstein field equations must be introduced as a postulate, while a quantized metric enables a derivation to be given. (Fradkin & Vilkovisky 1976, 1977. See also Adler 1982.) In the context of induced gravity, the "heavy" matter fields (the unobservable ones) induce the Einstein-Hilbert Lagrangian, but the "light" fields can be assumed, as an approximation, to not contribute to G^{-1}; the resulting approximate Lagrangian is then a sum of $\frac{R}{16\pi G}$ and $\mathcal{L}_{matter}(\phi_{light}, g_{\mu\nu})$.

If the metric is quantized, then the effective action can be shown to be stationary with respect to the metric and we thus have the semi-classical Einstein field equations

$$\bar{G}^{\mu\nu} + \Lambda\bar{g}^{\mu\nu} = 8\pi G\langle 0^+|T^{\mu\nu}_{matter}|0^-\rangle$$

(The metric $\bar{g}^{\mu\nu}$ is the classical background field metric, as used in the background field formalism, where the quantized metric $g^{\mu\nu}$ is split up into $\bar{g}^{\mu\nu}$ and a quantum fluctuation so that $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.)

A second argument against taking the semi-classical approach to gravity is that of inconsistency. Duff (1981) has shown that if one quantizes two classically equivalent systems which differ only by field redefinitions, one can obtain two non-equivalent quantum systems. Since our theories must be invariant under field redefinitions, the semi-classical quantum theory is inconsistent and so must be excluded. This conclusion is valid for any finite number of loops considered and it is only when infinite number of loops are considered, i.e. the full quantum gravity theory, that consistency is maintained.

The third argument is the existence of tentative experimental evidence against the semi-classical theory (Geilker & Page 1981). The possibility of this has been discussed before (see, for example, Kibble 1981). The argument rests upon the idea of the total wave function of experiment plus observer which thus does not collapse. This is the familiar Everett formulation of quantum mechanics (Everett 1957 and references in Geilker & Page 1981). This is shown to be necessary since a semi-classical theory of gravity is inconsistent with the idea of the collapse of the wave function. For instance, one can show (Eppley & Hannah 1977) that it is possible to collapse the wave function outside the light cone, and thus violate causality. The Everett formulation is thus assumed.

The argument against the semi-classical theory stems from the fact that it relates a definite classical metric with all of the components of the matter wave function. If one assumes that the observer is part of the system under consideration, and arranges masses according to a quantum-mechanical decision process (eg. radioactive decay), the "observed" mass configuration (along with the "observer") is just one component of the total wave function of the system. But the metric responds to all components of the wave function, and so one would expect a low correlation (can be made zero) of the metric with the mass configuration. This result is not only intuitively distasteful but also experimentally incorrect, as Geilker & Page have clearly shown.

The full quantum theory has no definite metric and thus each component of the wave function is related to its own distinct metric, which would have a high correlation with the mass distribution in that component. This is of course the intuitively more attractive idea.

We thus conclude that the metric must almost certainly be quantized. Assuming this the general principles of induced gravity (sections 2.1-2.4) remains unaltered but the details change, namely the resulting value of the induced G^{-1}. This is because the quantization of $g_{\mu\nu}$ implies the existence of virtual graviton loops and so these must be included in $\Psi_{\mu\alpha\nu\beta}$ (eqn. 2.53). The induced G^{-1} and Λ will still however be finite by the arguments of scale invariance and renormalizability of section 2.4. The resulting expressions for G^{-1} and Λ have been given by Adler (1982) and Zee (1983a). Zee also gives the expression for r, the induced coefficient of R^2. The method used is the background field formalism,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

The field $h_{\mu\nu}(x)$ is then treated no differently from the other fields and the quantization proceeds in the usual way. A gauge fixing term is introduced as well as the corresponding Faddeev-Popov ghosts. We define the total action by

$$S = S_{matter} + S_{gravity} + S_{ghost} + S_{gauge\ breaking}$$

where each of the terms depends on both $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$. We also introduce the external current $J_{\mu\nu}$, which is determined formally by the restriction $\langle h_{\mu\nu}\rangle_J = 0$ and so we can write $J_{\mu\nu}(\bar{g}_{\alpha\beta})$.

Following Zee, we define

$$T_1(x) = 2g^{\mu\nu} \left.\frac{\delta S}{\delta g^{\mu\nu}}\right|_{g_{\sigma\tau} = \eta_{\sigma\tau}}$$

$$T_2(x) = g_{\mu\nu}(x) \int d^4y\ h_{\alpha\beta} \left.\frac{\delta J^{\alpha\beta}}{\delta g_{\mu\nu}(x)}\right|_{g_{\sigma\tau} = \eta_{\sigma\tau}}$$

$$W(x) = 4g_{\mu\nu}(x)g_{\alpha\beta}(0)\left.\frac{\delta^2 S}{\delta g_{\mu\nu}(x)\,\delta g_{\alpha\beta}(0)}\right|_{g_{\sigma\tau} = \eta_{\sigma\tau}}$$

The results are

$$\frac{\Lambda}{2\pi G} = \langle T_1(x)\rangle_J$$

$$(16\pi G)^{-1} = \frac{i}{96}\int d^4x\ x^2(\langle\hat{T}_1(x)\hat{T}_1(0)\rangle_J^T - \langle T_2(x)T_2(0)\rangle_J^T - \langle W(x)\rangle_J^T)$$

$$r = \frac{-i}{13824}\int d^4x\ x^4(\langle\hat{T}_1(x)\hat{T}_1(0)\rangle_J^T - \langle T_2(x)T_2(0)\rangle_J^T - \langle W(x)\rangle_J^T)$$

Clearly the results are similar to those obtained for a classical metric (eqns. 2.33, 2.50 and 4.10). No calculations have yet been performed from these formula but clearly some calculations must eventually be done to obtain a proper picture of the importance of induced gravity in nature.

REFERENCES

Abrikosov A. A., 1956, J. Exper. Theoret. Phys. USSR 30, 96, (Soviet Physics JETP 3, 71).[6.4]

Adler S. L., 1972, Phys. Rev. D5, 3021.[6.4]

Adler S. L., 1980a, Phys. Rev. Lett. 44, 1567.[1.2]

Adler S. L., 1980b, Phys. Lett. 95B, 241.[1.2,2.5,3.3]

Adler S. L., 1980c, "The High Energy Limit," Proceedings of the 1980 Erice Summer School, ed. A. Zichichi, (Plenum 1983). [1.2,2.5]

Adler S. L., 1982, Rev. Mod. Phys. 54, 729.[1.2,1.3,3.1,3.4,3.5, App.]

Akama K., Chikashige Y., Matsuki T. & Terazawa H., 1978, Progr. Theoret. Phys. 60, 868.[3.2]

Alekseev A. I. & Rodionov A. Ya., 1980, Theor. and Math. Phys. 43, 503.[6.1]

Anishetty R., Baker M., Kim S. K., Ball J. S. & Zachariasen F., 1979, Phys. Lett. 86B, 52.[6.1]

Atkinson D. & Slim H. A., 1979, Il Nuovo Cimento 50A, 555.[6.1, 6.4]

Baker M., 1979, "High-Energy Physics in the Einstein Centennial Year," Proceedings of Orbis Scientiae, Coral Gables, Miami, eds. A. Perlmutter, F. Krausz & F. Scott, (Plenum).[6.1]

Baker M., Ball J. S., Lucht P. & Zachariasen F., 1980, Phys. Lett. 89B, 211.[6.1]

Baker M., Johnson K. & Willey R., 1964, Phys. Rev. 136B, 1111. [6.4]

Ball J. S., Horn D. & Zachariasen F., 1978, Nucl. Phys. B132, 509. [6.4]

Ball J. S. & Zachariasen F., 1978, Nucl. Phys. B143, 148.[6.1]

Barth N. H. & Christensen S. M., 1983, Phys. Rev. D28, 1876.[4.1]

Berg B. & Billoire A., 1983, Nucl. Phys. B221, 109.[3.5]

Bernard C., 1979, Phys. Rev. D19, 3013.[3.1]

Billoire A., 1981, Phys. Lett. 104B, 472.[3.5]

Birrell N. D. & Davies P. C. W., 1982, "Quantum Fields in Curved Space," (Cambridge).[5.2]

Bjorken J. D. & Drell S. D., 1965, "Relativistic Quantum Fields," (McGraw Hill).[3.3]

Boulware D. G., Horowitz, G. T. & Strominger A., 1983, Phys. Rev. Lett. 50, 1726.[4.1]

Brans C. & Dicke R., 1961, Phys. Rev. 124, 925.[1.2]

Brout R. & Englert F., 1966, Phys. Rev. 141, 1231.[5.3]

Brown L. S. & Zee A., 1983, J. Math. Phys. 24, 1821.[1.2]

Cerverò J. M. & Estèvez P. G., 1982, Ann. Phys. 142, 64.[2.1]

Christensen S. M., 1982, "Quantum Structure of Space and Time," Proceedings of the Nuffield Workshop, Imperial College, eds. M. J. Duff & C. J. Isham, (Cambridge Univ. Press).[4.1]

Chudnovskii E., 1978, Theor. and Math. Phys. 35, 538.[2.1]

Coleman S., 1977, in "The Whys of Subnuclear Physics," ed. A. Zichichi, (Plenum Press, 1979).[3.1]

Coleman S. & Weinberg E., 1973, Phys. Rev. D7, 1888.[2.2]

Collins J. C., Duncan A. & Joglekar S. D., 1977, Phys. Rev. D16, 438.[3.3]

Cornwall J. M., 1980, Phys. Rev. D22, 1452.[6.1]

Cornwall J. M., 1983, Preprint UCLA/83/TEP/4, Univ. of California at Los Angeles.[6.1]

Cornwall J. M. & Norton R. E., 1973, Phys. Rev. D8, 3338.[2.2]

Creutz M., 1980, Phys. Rev. D21, 2308.[3.5]

Creutz M. & Moriaty K. J. M., 1982, Phys. Rev. D26, 2166.[3.5]

Cutkosky, R. E., 1960, J. Math. Phys. 1, 429.[6.2]

Cutkosky, R. E., Landshoff P. V., Olive D. I. & Polkinghorne J. C., 1969, Nucl. Phys. B12, 281.[4.3]

Delbourgo R., 1976, Rep. Prog. Phys. 39, 345.[5.4]

Delbourgo R., 1977, J. Phys. A10, 1369.[6.1]

Delbourgo R., 1978, J. Phys. A11, 2057.[6.1]

Delbourgo R., 1979a, Il Nuovo Cimento 49A, 484.[6.1]

Delbourgo R., 1979b, J. Phys. G5, 603.[6.1]

Delbourgo R. & Keck, B. W., 1980a, J. Phys. G6, 275.[6.1,6.4]

Delbourgo R. & Keck, B. W., 1980b, J. Phys. A13, 701.[6.3,6.4]

Delbourgo R., Keck B. W. & Parker C. N., 1981, J. Phys. A14, 921. [6.3,6.4]

Delbourgo R. & Kenny B. G., 1981, J. Phys. G7, 417.[6.1]

Delbourgo R., Kenny B. G. & Parker C. N., 1982, J. Phys. G8, 1173. [6.1]

Delbourgo R. & Phocas-Cosmetatos P., 1979, J. Phys. A12, 191.[6.1]

Delbourgo R. & Salam A., 1964, Phys. Rev. 135, 1398.[1.2,6.1]

Delbourgo R. & Shepherd T., 1978, J. Phys. G4, L197.[6.1]

Delbourgo R. & Thompson G., 1982, J. Phys. G8, L185.[6.1]

Delbourgo R. & West P., 1977a, J. Phys. A10, 1049.[1.2,6.1]

Delbourgo R. & West P., 1977b, Phys. Lett. 72B, 96.[1.2,6.1,6.3]

Di Giacomo A. & Paffuti G., 1982, Phys. Lett. 108B, 327.[3.5]

Deser S., Tsao H. S., & van Nieuwenhuizen P., 1974, Phys. Rev. D10, 3337.[4.1]

DeWitt B. S., 1967, Phys. Rev. 162, 1239.[5.3]

DeWitt B. S. & Utiyama R., 1962, J. Math. Phys. 3, 608.[4.1]

Duff M., 1981, in "Quantum Gravity 2," eds. C. J. Isham, R. Penrose & D. W. Sciama, (Oxford Univ. Press).[App.]

Eichten E., Gottfried K., Kinoshita T., Lane K. D. & Yan T. M., 1980, Phys. Rev. D21, 203.[3.5]

Englert F., Gastmans R. & Truffin C., 1976, Nucl. Phys. B117, 407. [2.1,4.2]

Englert F., Gunzig E., Truffin C. & Windley P., 1975, Phys. Lett. 57B, 73; (and references therein).[2.1]

Eppley K. & Hannah E., 1977, Found. Phys. 7, 51.[App.]

Everett H., 1957, Rev. Mod. Phys. 29, 454.[App.]

Fradkin E. S. & Tseytlin A. A., 1981, Phys. Lett. 104B, 377.[4.1, 4.3]
and Nucl. Phys. B201, 469.(1982).

Fradkin E. S. & Vilkovisky G. A., 1975, Phys. Lett. 55B, 224.[4.1, App.]

Fradkin E. S. & Vilkovisky G. A., 1976, in "Proceedings of the 18th International Conference on High Energy Physics," Tbilisi, USSR, vol. 2, Sec. C, p. 28.[App.]

Fritzsch H. & Minkowski P., 1981, Phys. Rep. 73, 67.[2.4]

Fujii, Y., 1974, Phys. Rev. D9, 874.[1.2,2.1]

Fujii, Y., 1982, Phys. Rev. D26, 2580.[2.1]

Geilker C. D. & Page D. N., 1981, Phys. Rev. Lett. 47, 979.[App.]

Gardner E., 1981, J. Phys. G7, L269.[6.4]

Gross D. & Neveu A., 1974, Phys. Rev. D10, 3235.[2.2]

Gürsey F., 1963, Ann. Phys. 24, 211.[1.2]

Hasenfratz A. & Hasenfratz P., 1980, Phys. Lett. 93B, 165; (and references therein).[3.5]

Hasslacher B. & Mottola E., 1980, Phys. Lett. 95B, 237.[1.2,3.1]

Hasslacher B. & Mottola E., 1981, Phys. Lett. 99B, 221.[4.3]

't Hooft G., 1973, Nucl. Phys. B62, 444.[4.2]

Itzykson C. & Zuber J., 1980, "Quantum Field Theory," (McGraw-Hill).[3.3,6.2]

Jackiw R., & Johnson K., 1973, Phys. Rev. D8, 2386.[2.2]

Julve J. & Tonin M., 1978, Il Nuovo Cimento 46B, 137.[4.1,4.3]

Jin Y. S. & Martin A., 1964, Phys. Rev. 135, B1369.[3.6]

Just K. & Rossberg K., 1965, Il Nuovo Cimento 40, 1077.[5.3]

Kataev A. L., Krasnikov N. V. & Pivouarov A. A., 1982, Nucl. Phys. B198, 508.[3.5]

Khare A. & Kuman S., 1978, Phys. Lett. 78B, 94.[6.4]

Khelashvili A. A., 1981, Theor. and Math. Phys. 46, 149.[6.1]

Khuri N. N., 1969, in "Theories of Strong Interactions at High Energies," (BNL, Upton, New York), pp. 75-120.[3.6]

Khuri N. N., 1982a, Phys. Rev. D26, 2644.[1.2,3.6]

Khuri N. N., 1982b, Phys. Rev. Lett. 49, 513.[1.2,3.6,4.2]

Khuri N. N., 1982c, Phys. Rev. D26, 2671.[1.2,3.6]

Kibble T. W. B., 1981, in "Quantum Gravity 2," eds. C. J. Isham, R. Penrose & D. W. Sciama, (Oxford Univ. Press).[App.]

Lee T. D. & Wick G. C., 1969a, Nucl. Phys. B9, 209.[4.3]

Lee T. D. & Wick G. C., 1969b, Nucl. Phys. B10, 1.[4.3]

Lee T. D. & Wick G. C., 1970, Phys. Rev. D2, 1033.[4.3]

Linde A. D., 1979, Pišma Zh. Eksp. Teor. Fiz. 30, 479, (JETP Lett. 30, 447).[2.1]

Linde A. D., 1980, Phys. Lett. 93B, 394.[2.1]

Matsuki T., 1978, Prog. Theor. Phys. 59, 235.[2.1]

Minkowski P., 1977, Phys. Lett. 71B, 419.[2.1]

Misner C. W., Thorne K. S. & Wheeler J. A., 1970, "Gravitation," (W. H. Freedman & Co.).[1.2]

Neville D. E., 1978, Phys. Rev. D18, 3535.[4.3]

Neville D. E., 1980, Phys. Rev. D21, 867.[4.3]

Neville D. E., 1981, Phys. Rev. D23, 1244.[4.3]

Neville D. E., 1982, Phys. Rev. D26, 2638.[4.3]

Nambu Y. & Jona-Lasino G., 1961, Phys. Rev. 122, 345.[2.2]

Parker C. N., 1983, Ph.D. Thesis, University of Tasmania, (unpublished).[6.1]

Ramond P., 1981, "Field Theory, a modern primer," (Benjamin/Cummings).[2.2]

Sakharov A. D., 1967, Doklady Akad. Nauk. SSSR 177, 70, (Sov. Phys. Doklady 12, 1040, (1968)).[1.2,2.4]

Sakharov A. D., 1975, Theor. and Math. Phys. 23, 435.[3.2]

Sakharov A. D., 1982, "Collected Scientific Works," (Marcel Dekker).[1.2,2.4]

Salam, A., 1963, Phys. Rev. 130, 1287.[1.2,6.1]

Salam A. & Strathdee J., 1978, Phys. Rev. D18, 4480.[4.1,4.3]

Sezgin E. & van Nieuwenhuizen P., 1980, Phys. Rev. D21, 3269.[4.3]

Shifman M. A., Vainshtein A. I. & Zakharov V. I., 1979, Phys. Rev. Lett. 42, 297.[3.5]

Shohat J. A. & Tamarkin J. D., 1943, "The Problem of Moments," (American Mathematical Society).[3.6]

Slim H. A., 1979, Phys. Lett. 81B, 19.[6.4]

Slim H. A., 1981, Nucl. Phys. B177, 172.[6.4]

Smolin L. 1979, Nucl. Phys. B160, 253.[2.1]

Stelle K. S., 1977, Phys. Rev. D16, 953.[2.3,4.1]

Strathdee J., 1964, Phys. Rev. 135, 1428.[1.2,6.1]

Thompson G., 1983, Preprint SHEP 82/83-6, University of Southhampton.[6.1]

Tomboulis E., 1977, Phys. Lett. 70B, 361.[4.3]

Tomboulis E., 1980, Phys. Lett. 97B, 77.[4.1,4.3]

Tricomi, F. G., 1955, "Integral Equations," (John Wiley & Sons). [6.3]

Tsao H. S., 1977, Phys. Lett. 68B, 79.[4.2]

van Nieuwenhuizen P., 1974, Phys. Rev. D10, 411.[5.2]

Weinberg S., 1957, Phys. Rev. 106, 1301.[2.3]

Weinberg S., 1972, "Gravitation and Cosmology," (John Wiley & Sons).[2.3,5.2]

Wilson K., 1968, Phys. Rev. 179, 1499.[3.3]

Wilson K., 1974, Phys. Rev. D14, 2455.[3.5]

Zee A., 1979, Phys. Rev. Lett. 42, 417.[2.1]

Zee A., 1980, Phys. Rev. Lett. 44, 703.[1.2,2.1]

Zee A., 1981a, Phys. Rev. D23, 858.[1.2,3.2,4.2]

Zee A., 1981b, in "Proceedings of the 1981 Erice School," ed. A. Zichici, (Plenum).[1.2,3.4]

Zee A., 1981c, in "Proceedings of the Fourth Kyoto Summer School on Grand Unified Theories and Related Topics," eds. M. Konuma & T. Maskawa, (World Science Publishing Co.).[1.2]

Zee A., 1982a, Phys. Lett. 48, 295.[1.2,3.4]

Zee A., 1982b, Phys. Lett. 109B, 183.[1.2,4.2]

Zee A., 1983a, Preprint 40048-32 P2, University of Washington. [1.2,4.2,App.]

Zee A., 1983b, in "The Proceedings of the 20th Annual Orbis Scientiae Dedicated to P. A. M. Dirac's 80th Year," to appear.[1.2]

Zel'dovich Ya. B., 1967, Zh. Eksp. Teor. Fiz. Pis'ma 6, 883, (JETP Lett. 6, 316).[1.2]

Zimmerman W., 1970, in "Lectures on Elementary Particles and Quantum Field Theory," vol. I, eds. S. Deser, M. Grisaru & H. Pendelton, (MIT Press).[3.3]