ORTHOSYMPLECTIC SUPERALGEBRAS IN MATHEMATICS AND PHYSICS

by

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DECLARATION

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by any person, except where due reference is made in the text of the thesis.

R Farmer.

(R.J. Farmer)

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DEDICATED TO

MY WIFE, VICKI

ABSTRACT

This thesis is devoted to the study of the representation theory of orthosymplectic superalgebras and their applications to physical theories. Techniques are developed to educe typical and atypical finite-dimensional, irreducible representations of orthosymplectic superalgebras. These include superfield and weight space procedures which are illustrated for several low-rank orthosymplectic superalgebras. Young supertableaux are used to enumerate finite-dimensional typical, tensor representations and spinor representations of OSp(M/N), and atypical, tensor representations of OSp(2/2), OSp(3/2) and OSp(4/2). Relations between Kac-Dynkin and supertableau labels are obtained and used to present conditions on diagram shape, necessary and sufficient for atypicality. Modification rules for typical supertableaux of OSp(M/N), and for atypical supertableaux of OSp(2/2), OSp(3/2) and OSp(4/2) are presented. Dimension formulae, in diagram notation, are discussed for typical, representations of OSp(M/N).

New superfield realisations are presented for the determination of infinite-dimensional irreducible representations of N-extended super-Poincaré algebras with central charges. These are illustrated for the $N=2$ extended super-Poincaré algebra with one central charge. Finally, a discussion of the roles played by orthosymplectic supergroups in some physical theories is presented.

CONTENTS

1. INTRODUCTION

The purpose of this chapter is to place the subject matter of this thesis in an historical perspective so that the significance of the results reported here may be appreciated in their proper mathematical and physical context.

1.1 AN HISTORICAL PRELUDE

Symmetry principles play an important role in physics, lending simplicity and elegance to physical laws and physical systems amid the complexity which so often accompanies them. In particular, global and local symmetries have become established as a fundamental feature of modern particle physics. In the early 1970's a new symmetry principle was introduced to physics which involved transformations relating states of different quantum spin-statistics [1-8]. It has become known as supersymmetry. The algebraic structure of supersymmetry is that of a graded Lie algebra, which is an extension of an ordinary Lie algebra to include anticommutators. Graded Lie algebras first appeared in the mathematical literature with the work of Nijenhuis and Fröhlicher [9,10] and later in connection with cohomology and deformation theories [11,12]. The sequel contains a brief historical review of the mathematical development of the theory of Lie superalgebras followed by a discussion of the applications they have found in physics.

Prior to embarking on this, a few sentences will be devoted to establishing precisely what characterises a Lie superalgebra.

Superalgebra is the term which has been adopted for Z_2-graded algebras, $A = A_{\bar{0}} + A_{\bar{1}}$, which are algebras which satisfy the following: if $a \in A_\alpha$, $b \in A_\beta$ and $\alpha, \beta \in Z_2 = \{\bar{0}, \bar{1}\}$, then $ab \in A_{\alpha+\beta}$. A Lie superalgebra is a superalgebra $G = G_{\bar{0}} + G_{\bar{1}}$ with an operation $[\ ,\ ]$ which satisfies

$$[a,b] = -(-1)^{\alpha\beta}[b,a] \qquad \text{for } a \in G_\alpha,\ b \in G_\beta$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]] \quad \text{for } a \in G_\alpha,\ b \in G_\beta.$$

A theory establishing the connection between Lie superalgebras and Lie supergroups has been developed by Berezin and Kats [13] and Berezin and Leites [14]. An extensive discussion of supergroups and supermanifolds has been given by Kostant [48].

The program of the classification of Lie superalgebras was begun by Pais and Rittenberg [17]. Under some rather strong restrictions, including that the Killing form be nonsingular and the bosonic part be simple, they find the only algebras to be OSp(1/2n). This was followed by the work of Kaplansky and Freund [15,16] who exhibited two infinite families of simple Lie superalgebras, the special linear, SU(m/n), and orthosymplectic, OSp(m/2n), algebras and postulated the existence of the exceptional Lie superalgebras, F(4), G(3) and D(2,1;α). The classification of all simple Lie superalgebras whose Lie algebra is reductive was given by Scheunert, Nahm and Rittenberg [18,19]. This work provided a complete classification for all the 'classical' Lie superalgebras which, in addition to the 'basic classical' Lie superalgebras mentioned above, includes the two 'strange' series, P(n) and Q(n). The complete classification of all finite-dimensional, simple Lie superalgebras has been obtained by Kac [20]. He has shown that in addition to the classical Lie superalgebras there exist four series of the 'Cartan' type, W(n), S(n), H(n), $\tilde{S}(n)$. Filtrations and Z-gradations of Lie superalgebras play an important role in this classification. Parker [21] has given a classification for the real forms of all the classical Lie superalgebras.

The representation theory of Lie superalgebras has seen the following developments. Kac [22] has obtained character and supercharacter formulae for 'typical' finite-dimensional, irreducible representations of the basic classical Lie superalgebras. For these algebras he has derived [23] necessary and sufficient conditions for a finite-dimensional, irreducible representation to be 'typical' and has obtained dimension formulae for these representations. Djokovic and Hochschild [24,25] showed that the only Lie superalgebras for which all the finite-dimensional representations are completely reducible are those which are isomorphic to a direct product of a semi-simple Lie algebra with finitely many Lie superalgebras of the type OSp(1/2n). With the work of Corwin [26] and Djokovic [27] a quite detailed representation theory for the Lie superalgebras OSp(1/2n) was developed.

The concept of Hermitian representations of simple Lie algebras was generalized to classical Lie superalgebras by Scheunert, Nahm and Rittenberg [28]. They demonstrated the existence of two classes of such representations defined on a graded Hilbert space. These are called star and grade star representations and are defined through adjoint and grade adjoint operations. The finite-dimensional, star and grade star representations of OSp(1/2) and SU(2/1) were subsequently obtained by these authors [29].

The Casimir invariants for the general linear, special linear and orthosymplectic Lie superalgebras and for the strange Lie superalgebras have been constructed by Jarvis and Green [30] and Jarvis and Murray [31] respectively. Following the classical approach of Perelomov and Popov, Scheunert has constructed generating functions to obtain the eigenvalues of the Casimir elements for the general linear, special linear and orthosymplectic Lie superalgebras [32] while Balantekin and Bars [36] have used characters to obtain formulae for the eigenvalues of all Casimir operators of SU(m/n), OSp(m/2n) and P(n).

Diagram techniques were introduced to the study of representations of Lie superalgebras by Dondi and Jarvis [33,34] and Bars and Balantekin [35,36]. These authors have studied the Lie superalgebras U(m/n), SU(m/n) OSp(m/2n) and P(n), developing branching rules, character formulae and dimension formulae for some representations. Later work by Balantekin and Bars [37] and Bars, Morel and Ruegg [38] saw Young supertableaux techniques applied to contravariant, covariant and mixed representations of SU(m/n), while Delduc and Gourdin [39] have investigated SU(n/1) to establish which supertableaux correspond to irreducible representations. Morel, Sciarrino and Sorba [40] have recently developed new diagram techniques for the study of OSp(m/2n). They have been successful in obtaining branching rules, in closed form, for all typical tensor and spinor representations of these algebras. King [41] has used standard schur function operations to derive simple Kronecker product rules and branching rules for all representations of SU(m/n) and for tensor representations of OSp(m/2n). He has also given dimension formulae in terms of partition labels for these representations, provided they are typical.

Superfield techniques were first applied to the study of finite-dimensional representations of Lie superalgebras by Dondi and Jarvis [33], who studied U(m/n) and SU(m/n). These techniques have been further developed and applied to the orthosymplectic Lie superalgebras by Farmer and Jarvis [42].

Following the initial work of Kac [23], weight space techniques were further developed by Hurni and Morel [43,44] where applications to the basic classical Lie superalgebras were considered. Further developments were made by Farmer and Jarvis [45], applying these techniques to the orthosymplectic superalgebras and explicitly constructing all finite-dimensional, irreducible, star and grade star representations of OSp(1/2), OSp(2/2, OSp(3/2) and D(2,1;α). Some very interesting developments have been made recently by Thierry-Mieg [46], who has obtained theorems which allow the explicit construction of the irreducible representations of the basic classical Lie superalgebras.

Lie superalgebras have become an important influence in the physics world and in particular in theoretical particle physics, where a very significant fraction of the literature is currently devoted to theories which are based on these algebras in some form. The first applications of Lie superalgebras came with the work of Neveu and Schwarz [1] and Ramond [2] on string models. Independently, Gol'fand and Likhtman [5] and Volkov and Akulov [6] showed how to generalize the Poincaré group to include fermionic charges. With the construction of an interacting field theory, invariant under this graded Poincaré group, by Wess and Zumino [7,8], a way was opened to circumvent the 'no-go' theorems of O'Raifearteagh [49] and Coleman and Mandula [50] and unify in a non-trivial way internal with space-time symmetries. These supersymmetric field theories [51] have turned out to have a less divergent ultraviolet behaviour than non-supersymmetric field theories and it is even hoped that some theories, such as the $N=4$ super Yang-Mills theory may even be finite. Despite this it is still far from certain that these theories describe the real physical world. With the work of Ferrara, Freedman and van Nieuwenhuizen [52,53] and Deser and Zumino [54], supersymmetry became a *local* gauge symmetry and supergravity was born. From its inception this theory generated much interest with the possibility of unifying gravity with the other forces of nature in a finite field theory. It has now become a vast subject (see [55,56] for reviews), though the technical complexities have, to date, thwarted the complete construction of what is hoped will be this unifying theory, the N=8- extended supergravity. The majority of these global and local supersymmetric field theories are based on OSp(N/4) or SU(N/4) either directly or via Inonu-Wigner contraction.

Although the greatest efforts in applying Lie superalgebras to physical problems have been in the above areas a number of other applications have also been found in recent years. One of the most useful is the BRS invariance of quantum gauge theories [57] where the symmetries are generated by translations in a superspace [58,59]. Another interesting application is in relation to composite models of quarks and leptons. In these models SU(M/N) plays the role of a classification group which helps solve 'anomaly matching' and 'decoupling' constraints [60,61,62]. These are necessary for the dynamical survival of chiral symmetries, which are needed to explain the small masses of the quarks and leptons relative to their physical size, or to the binding energy of their composite structure.

There have been other applications for instance to internal symmetries [63] and supersymmetric grand unification [64,65,66,67,68], however, I would like to close this discussion with the one application of superalgebras in Nature which has experimental support. This is in the area of nuclear physics and is a model based on the algebra SU(6/M) [69,70]. It provides a classification scheme for many low lying nuclear states of muclei in the Platinum-Gold region and predicts energy levels, relations among decay rates and relations between nucleon transfer reactions with accuracies of 10-20%.

1.2 THE THESIS STRUCTURE

Throughout this thesis it is assumed that the reader if familiar with the representation theory of Lie algebras. Although some knowledge of Lie superalgebras and their representations would be useful, chapter two should serve as a brief introduction for those unfamiliar with this subject. It should also serve to establish the notation used here and the terminology necessary for communication. For more comprehensive treatments of this subject the reader is referred to the works of Kac [20,23] and Scheunert [47] on which the material of chapter two is based.

As mentioned, chapter two serves as an introduction to the theory of Lie superalgebras. The first section introduces the concepts of graded vector spaces and graded algebras, from which are defined the special class of Z_2-graded algebras called Lie superalgebras. Many of the formal definitions associated with these structures are given here. The second section provides the classification of all finite-dimensional simple Lie superalgebras which has been given by Kac [20]. The structures of the classical Lie superalgebras and their root systems are discussed in some detail. In particular, the origin of the orthosymplectic superalgebras becomes apparent here. The third section provides an introduction to the representation theory of basic classical Lie superalgebras.

The study of orthosymplectic superalgebras is begun in earnest in chapter three. The general structure of the algebra is discussed in §3.2, incorporating an explicit choice of simple roots and presenting the general form of the Cartan matrix. This allows the complete algebra to be constructed for any orthosymplectic superalgebra. Weight space techniques are then developed for educing finite-dimensional, typical and atypical, irreducible representations of these algebras.

These techniques are then used to determine *all* finite-dimensional, irreducible representations of the superalgebras B(1,1), C(2) and D(2,1;α). For $\alpha = 1$ these are the lowest rank algebras from each of the three orthosymplectic classes. The star and grade star representations are determined for each of these algebras.

Chapter four develops superfield techniques for the determination of irreducible, typical and atypical, representations of orthosymplectic superalgebras. These methods are based on the theory of induced representations. Using these techniques, all irreducible representations of the superalgebras B(0,1), B(1,1), C(2) and D(2,1) are found. These are in agreement with the results of chapter three.

Young supertableaux are introduced into the study of the representations of orthosymplectic superalgebras in chapter five. The relation between the Kac-Dynkin labels and the supertableau labels is first established and used to express the conditions for atypical representations in diagram notation. Modification rules are found for the typical supertableaux of all orthosymplectic superalgebras and the atypical supertableaux of B(1,1), C(2) and D(2,1). Dimension formulae for typical representations are presented here in diagram notation and branching rules to the underlying Lie algebra are given for spinor representations of all orthosymplectic superalgebras and atypical representations of B(1,1), C(2) and D(2,1).

New superfield techniques are introduced in chapter six for the study of irreducible realisations of the N-extended supersymmetry algebra in the presence of central charges. After a general discussion of the procedure, which is based on an induced representation construction, the $N = 2$ case is considered in detail. The results are found to be in agreement with those obtained via the conventional methods, with the 'spin reducing' cases arising analogously to atypical representations.

A review of the roles which orthosymplectic supergroups have found in physical theories is presented in chapter seven. Perhaps the most useful application currently known is the elegant formulation it lends to the extended BRS symmetries of quantum gauge theories. Discussed here are applications to non-abelian gauge theories, Kaluza-Klein theories and gravity. Orthosymplectic supergroups play a quite fundamental role in supersymmetric Yang-Mills and supergravity theories, since the N-extended super Poincaré algebras, on which these theories are based, can be obtained by Inönü-Wigner contraction of OSp(N/4). This contraction procedure is presented in §7.2. This chapter concludes with a discussion of Kaluza-Klein supergravity theories, wherein orthosymplectic supergroups play the role of the ground state symmetry of some compactifying solutions of these theories.

The thesis concludes in chapter eight with a summary, reiterating the main new results obtained, and indicating avenues for future research.

The appendices contain details of notation and some techniques which have been employed in the course of this work. Also presented are two proofs pertaining to the work of chapters three and five and some useful identities for handling the θ-calculus of chapters four and six.

Each chapter contains its own set of references, which though leading to some duplication has the advantage of making each chapter self-contained.

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2. AN INTRODUCTION TO LIE SUPERALGEBRAS

The primary function of this chapter is to provide the reader with the notation, definitions and basic mathematical theory of Lie superalgebras, necessary to make the remainder of this thesis intelligible and to provide the appropriate context for the representation theory of orthosymplectic Lie superalgebras. Consequently this chapter is basically a review of the mathematical theory of Lie superalgebras which relies heavily on the works of Corwin, Ne'eman and Sternberg [1], Pais and Rittenberg [2]. Freund and Kaplansky [3], Nahm, Rittenberg and Scheunert [4,5,6,7], Rittenberg and Scheunert [8] and particularly the comprehensive treatments by Kac [9,10] and Scheunert [11]. To contain the length of this chapter it has been thought expedient to only state results and refer the reader to the literature for the relevant proofs.

In §2.1 the necessary basic definitions and concepts pertaining to graded algebraic structures and Lie superalgebras are introduced. Since the primary concern of this thesis is with Lie superalgebras these refer substantially to Z_2 - graded structures. The extensions to more general gradings are discussed by Scheunert [12].

The classification of Lie superalgebras is discussed in §2.2. This deals mainly with the complete classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero which has been provided by Kac [9].

Finally §2.3 provides a short review of the work of Kac [10] on finite-dimensional irreducible representations of simple Lie superalgebras.

2.1 INTRODUCTION TO GRADED ALGEBRAIC STRUCTURES AND LIE SUPERALGEBRAS

Let Γ be the ring of integers, Z, or the residue class ring of Z modulo 2Z, $Z_2 = Z/2Z$ [13]. The two elements of Z_2 will be denoted by $\bar{0}$ and $\bar{1}$. All spaces and algebras are regarded over a ground field, K, which is algebraically closed and of characteristic zero.

A Γ-*graded vector space*, V, over the field K contains a family of subspaces, V_γ, where $\gamma\in\Gamma$, such that

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma .$$

An element of V is said to be homogeneous of degree $\gamma\in\Gamma$ if it is an element of V_γ. If $\Gamma = Z_2$ the element of $V_{\bar{0}}(V_{\bar{1}})$ is called even (odd). On any Z-graded vector space $V = \bigoplus_{j\in Z} V_j$ there exists a natural Z_2-grading, induced by the Z-grading and defined by

$$V_{\bar{0}} = \bigoplus_{j\in Z} V_{2j} \quad ; \qquad V_{\bar{1}} = \bigoplus_{j\in Z} V_{2j+1} .$$

A subspace U of V is called a Γ-*graded subspace* if $U = \bigoplus_{\gamma\in\Gamma} (U \cap V_\gamma)$.

Let V and W be two Γ-graded vector spaces. A linear mapping $g\colon V \to W$ is said to be *homogeneous* of degree V, $V\in\Gamma$, if $g(V_\alpha) \subset W_{\alpha+\gamma}$ $\forall\alpha\in\Gamma$. The mapping g is called a *homomorphism* of V into W if it is homogeneous of degree 0.

An algebra A, over the field K, is a Γ-*graded algebra* if its underlying vector space is Γ-graded, $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, and if $A_\alpha A_\beta \subset A_{\alpha+\beta}$ $\forall\alpha_1\ \beta\in\Gamma$. If A has a unit element, e, it follows that $e\in A_0$.

A *homomorphism of* Γ-*graded algebras* is a homomorphism of the underlying algebras and of the underlying Γ-graded vector spaces. It is homogeneous of degree 0.

A graded subalgebra (or ideal) of a Γ-graded algebra, A, is a subalgebra (or ideal) of the algebra A which is, in addition, a graded subspace of the underlying Γ-graded vector space A.

A *superalgebra* is a Z_2-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$. The elements of $A_{\bar{0}}(A_{\bar{1}})$ are called even (odd). If $a \in A_\alpha$ $(\alpha = \bar{0},\bar{1})$, then a is called homogeneous of degree α.

The *graded tensor product*, $A \otimes B$, of two associative subalgebras A and B is the tensor product of the underlying vector spaces with multiplication defined by the requirement that

$$(a \otimes b)(a' \otimes b') = (-1)^{\beta\alpha'} (aa') \otimes (bb')$$

$$\forall a \in A,\ a' \in A_{\alpha'},\ b \in B_\beta,\ b' \in B\ ;\ \ \alpha', \beta \in Z_2.$$

With this multiplication $A \otimes B$ is an associative superalgebra.

A *Lie superalgebra* is a superalgebra, $G = G_{\bar{0}} \oplus G_{\bar{1}}$, with an operation [,] satisfying the following:

$$[a,b] = -(-1)^{\alpha\beta} [b,a] \qquad \text{(graded skew-symmetry)}$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta} [b,[a,c]] \quad \text{(graded Jacobi identity)}$$

$$\forall a \in G_\alpha,\ b \in G_\beta,\ c \in G\ ;\quad \alpha,\beta \in Z_2.$$

If A is an associative superalgebra then defining [,] by

$$[a,b] = ab -(-1)^{\alpha\beta} ba \qquad \forall a \in A_\alpha,\ b \in A_\beta\ \ ;\ \alpha,\beta \in Z_2$$

turns A into a Lie superalgebra denoted A_L.

The *universal enveloping algebra* of a Lie superalgebra is constructed in the following way [9,11]. Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra and T(G) the tensor algebra over the vector space G. The Z_2-grading of G induces a Z_2-grading of T(G). Let R be the two-sided ideal of T(G) generated by elements of the form:

$$R = [a,b] - a \otimes b + (-1)^{\alpha\beta}\, b \otimes a.$$

The factor algebra U(G) defined by U(G) = T(G)/R is an associative superalgebra.

The canonical mapping $G \to U(G)$ induces a homomorphism $i: G \to U(G)_L$ of Lie superalgebras. The pair $(U(G),i)$ is the universal enveloping algebra of G.

The *Poincaré-Birkhoff-Witt theorem* can be used to construct a canonical basis for $U(G)$ as follows. Let $a_1,\ldots,a_m$ be a basis of $G_{\bar{0}}$ and $b_1,\ldots,b_n$ be a basis of $G_{\bar{1}}$, then the elements of the form

$$a_1^{k_1}\ldots a_m^{k_m}\ b_{i_1}\ldots b_{i_n}, \quad \text{where } k_i \geq 0 \text{ and } 1 \leq i_1 < \ldots < i_s \leq n,$$

form a basis of $U(G)$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a Z_2-graded vector space, with $V_{\bar{0}}$ and $V_{\bar{1}}$ of dimension m and n respectively, and let

$$\mathrm{End}_\alpha(V) = \{ A \in \mathrm{End}(V): \ A\, V_\beta = V_{\beta+\alpha}\} .$$

$\mathrm{End}(V) = \oplus\, \mathrm{End}_\alpha(V)$ is endowed with a Z_2-grading and a Lie superalgebra structure, denoted by $\ell(V) \equiv \ell(m,n)$, can be defined on it by setting

$$[A,B] = AB - (-1)^{\alpha\beta}\, BA \quad , \qquad A,B \in \mathrm{End}(V).$$

A *linear representation*, ρ, of a Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ in V is a homomorphism

$$\rho: \ G \to \ell(V).$$

The map ad: $G \to \ell(G)$ for which

$$(\mathrm{ad}\ g)(a) = [g,a] \quad , \qquad a,g \in G$$

is a linear representation of the Lie superalgebra, G, called the *adjoint representation.*

The adjoint representation of a Lie superalgebra, G, induces a representation of the Lie algebra $G_{\bar{0}}$ in the odd subspace $G_{\bar{1}}$ and is denoted by $\mathrm{ad}|_{G_{\bar{1}}}$ or $G_{\bar{0}}|G_{\bar{1}}$.

It is now possible to introduce a generalized adjoint operation for a Lie superalgebra, G, and the concepts of star and grade star representations of G in a graded vector space, V, [7].

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional Z_2-graded vector space. Assume that on V there exists a non-degenerate hermitian form denoted by (,) such that $V_{\bar{0}}$ and $V_{\bar{1}}$ are orthogonal with respect to this form, i.e., $(V_{\bar{0}},V_{\bar{1}}) = \{0\}$. If (,) is positive definite then V is called a graded Hilbert space.

For any linear operator A in V the adjoint operator, A^+, with respect to (,) is defined by

$$(A^+x,y) = (x,Ay) \qquad \forall x,y \in V.$$

For the Lie superalgebra $\ell(V)$ consider the following rules:

(i) The adjoint of an even (odd) element is even (odd).

(ii) $(aA + bB)^+ = a^*A^+ + b^*B^+$.

(iii) $[A,B]^+ = [B^+,A^+]$.

(iv) $(A^+)^+ = A$.

$$\forall A,B \in \ell(V) \quad \text{and} \quad a,b \in C.$$

An adjoint operation in a Lie superalgebra, G, is a mapping $A \to A^+$ of G into itself which satisfies the conditions (i) - (iv) above.

Let A be a homogeneous linear operator in V of degree α. The grade adjoint operator, $A^‡$, with respect to (,) is defined by

$$(A^‡x,y) = (-1)^{\alpha\xi}(x,Ay) \qquad \forall x \in V_\xi, \quad y \in V_\eta .$$

For the Lie superalgebra $\ell(V)$ consider the following rules:

(i') The grade adjoint of an even (odd) element is even (odd).

(ii') $(aA + bB)^‡ = a^*A^‡ + b^*B^‡$

$$\text{(iii')}\quad [A,B]^{\ddagger} = (-1)^{\alpha\beta}\,[B^{\ddagger},A^{\ddagger}]$$
$$\text{(iv')}\quad (A^{\ddagger})^{\ddagger} = (-1)^{\alpha}\,A$$
$$\forall A \in \ell(V)_{\alpha}\,,\quad B \in \ell(V)_{\beta} \quad \text{and} \quad a,b \in C\,.$$

A grade adjoint operation in a Lie superalgebra, G, is a mapping $A \to A^{\ddagger}$ of G into itself which satisfies the conditions (i') - (iv') above.

Let G be a Lie superalgebra equipped with an adjoint (grade adjoint) operation, and let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a graded vector space. A star representation (grade star representation) of G in V is a graded representation ρ of G in V which satisfies

$$\rho(A^{+}) = \rho(A)^{+} \qquad (\rho(A^{\ddagger}) = \rho(A)^{\ddagger})\,.$$

Let V be a finite-dimensional Z_2-graded vector space and let $\gamma : V \to V$ be the linear mapping which satisfies

$$\gamma\,(v) = (-1)^{\alpha}v \qquad \text{if } v \in V_{\alpha}\;;\;\; \alpha \in Z_2\,.$$

The *supertrace,* str, is a linear form on $\ell(V)$ defined by

$$\mathrm{str}(A) = \mathrm{Tr}(\gamma A) \qquad \forall A \in \ell(V)\,.$$

From this definition it follows that

$$\mathrm{str}([A,B]) = 0 \qquad \forall A,B \in \ell(V)\,.$$

Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Z_2-graded space and let f be a bilinear form on G. Then f is called

consistent if $f(a,b) = 0$ for $a \in G_{\bar{0}}$, $b \in G_{\bar{1}}$,

and *supersymmetric* if $f(a,b) = (-1)^{\alpha\beta}f(b,a)$ for $a \in G_{\alpha}$, $b \in G_{\beta};\alpha,\beta \in Z_2$.

If G is a Lie superalgebra, f is called

invariant if $f([a,b],c) = f(a,[b,c])$.

The bilinear form $(a,b) = \mathrm{str}(ab)$ on $\ell(V)$ is consistent, supersymmetric and invariant. The *killing form* on a Lie superalgebra, G, is the bilinear form

$$(a,b) = \mathrm{str}((\mathrm{ad}\, a)(\mathrm{ad}\, b)) .$$

A superalgebra, G, is said to be *Z-graded* if we are given a family $(G_j)_{j \in Z}$ of Z_2-graded subspaces of G such that

(i) $G = \bigoplus_{j \in Z} G_j$,

(ii) $G_i G_j \subset G_{i+j}$ $\quad \forall i,j \in Z$.

The Z-grading is said to be *consistent* with the Z_2-grading of G if

$$G_{\bar{0}} = \bigoplus_{j \in Z} G_{2j} \quad , \quad G_{\bar{1}} = \bigoplus_{j \in Z} G_{2j+1} \; .$$

If G is a Z-graded Lie superalgebra, then G_0 is a subalgebra and $[G_0, G_i] \subseteq G_i$. Thus the adjoint representation, restricted to G_0, induces linear representations of G_0 in the subspaces G_i, denoted by $G_0|G_i$. G is called irreducible if the representation of G_0 in G_{-1} is irreducible.

A Lie superalgebra, $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is *solvable* if and only if its Lie algebra $G_{\bar{0}}$ is solvable. $G_{\bar{0}}$ is solvable if $G_{\bar{0}}^{(n)} = 0$ for some n, where $G_{\bar{0}}^{(i)}$ is defined by

$$G_{\bar{0}}^{(0)} = G_{\bar{0}} \; , \quad G_{\bar{0}}^{(1)} = [G_{\bar{0}}^{(0)}, G_{\bar{0}}^{(0)}], \quad G_{\bar{0}}^{(2)} = [G_{\bar{0}}^{(1)}, G_{\bar{0}}^{(1)}],$$

$$\ldots, G_{\bar{0}}^{(i)} = [G_{\bar{0}}^{(i-1)}, G_{\bar{0}}^{(i-1)}].$$

A Lie superalgebra, G, is called *semisimple* if it contains no solvable ideals.

A Lie superalgebra, G, is called *simple* if it does not have any graded ideals which are different from $\{0\}$ and G and if $[G,G] \neq 0$.

2.2 CLASSIFICATION OF SIMPLE LIE SUPERALGEBRAS

The following discussion will be restricted to finite-dimensional, simple Lie superalgebras, $G = G_{\bar{0}} \oplus G_{\bar{1}}$, over an algebraically closed field, K, of characteristic zero. A classification for all simple Lie superalgebras has been obtained by Kac [9] although partial results, particularly for the classical superalgebras, have also been obtained by others [2,3,4,5,6,14]. The two main categories are called *Cartan* and *classical* superalgebras. The classification of the Cartan superalgebras relies on the concept of a filtration of G [9] which will not be discussed here. A Lie superalgebra, G, is called classical if it is simple and the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$, $G_{\bar{0}}|G_{\bar{1}}$, is completely reducible. These can be subdivided into two categories depending on whether $G_{\bar{0}}$ in $G_{\bar{1}}$ is reducible (*type I*) or irreducible (*type II*). The type I and type II classical superalgebras can be further subdivided into those with non-degenerate killing form and those with zero killing form. In Figure 2.1 the classification scheme is sketched.

2.2a CLASSIFICATION OF CLASSICAL LIE SUPERALGEBRAS*

1. A(m,n):

Let $S\ell(m,n) = \{ A \in \ell(m,n) | str(A) = 0 \}$.

Then from the property $str([A,B]) = 0$ it can be seen that $S\ell(m,n)$ is an ideal in $\ell(m,n)$ of dimension one less than the dimension of $\ell(m,n)$. Z_2- and Z-gradings of $\ell(m,n)$ induce the same gradings on $S\ell(m,n)$.

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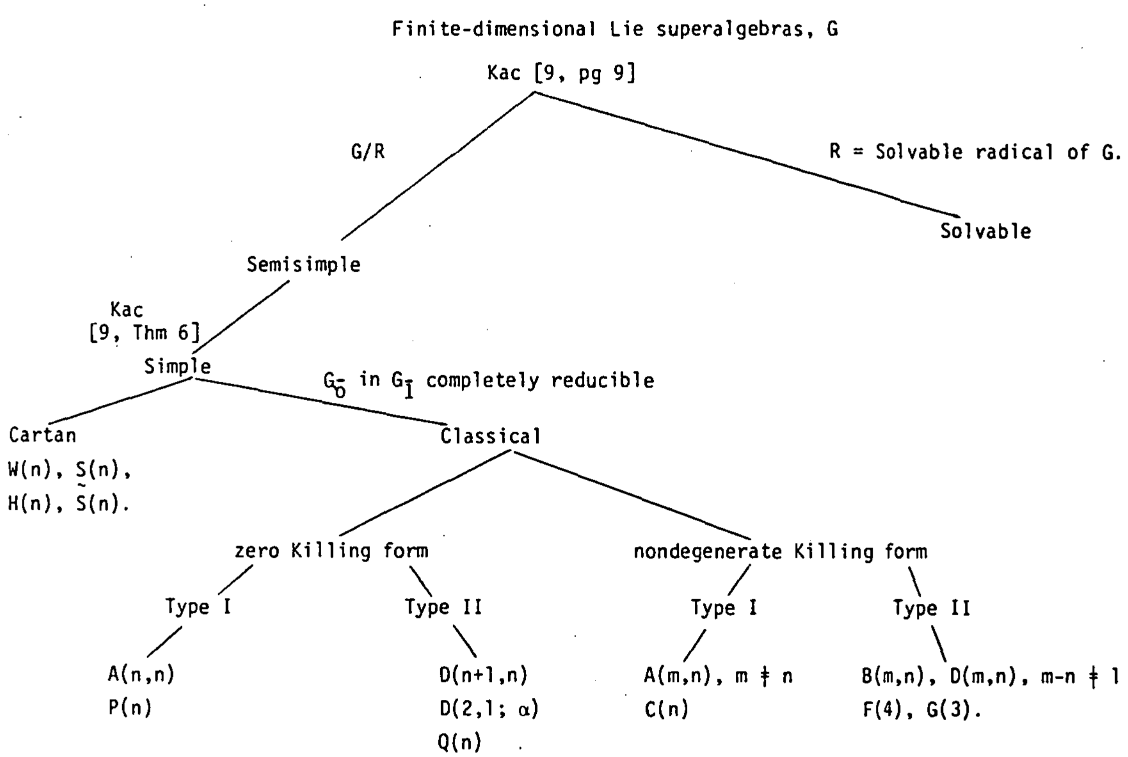
* The following notation is used in this section: $S\ell_n$, Sp_n, SO_n stand for the fundamental representations of these Lie algebras, $spin_k$ stands for the irreducible spinor representation of SO_k, g_2 denotes the simplest representation of the Lie algebra G_2, CSp is Sp plus the 1-dimensional centre, and $S\ell_n$ stands for the adjoint representation of $S\ell_n$, *denotes the dual module and S^2 and Λ^2 denote symmetrical and exterior products.

Figure 2.1 Classification of finite-dimensional Lie superalgebras over an algebraically closed field of characteristic zero.



$S\ell(n,n)$ contains the one-dimensional ideal consisting of scalar matrices λI_{2n}.

We set

$$A(m,n) = S\ell(m+1,n+1) \quad \text{for} \quad m \neq n \quad , \qquad m,n \geq 0.$$

$$A(n,n) = S\ell(n+1,n+1)/\lambda I_{2n+2} \quad , \qquad n > 0.$$

The Killing form of $S\ell(m,n)$ is given by

$$(A,B) = 2(m-n)\ \mathrm{str}(AB) \qquad A,B \in S\ell(m,n)$$

From this it is found that for $A(m,n)$ the Killing form is non-degenerate while for $A(n,n)$ the Killing form is zero.

These are also known as *unitary* superalgebras.

2. $B(m,n)$, $D(m,n)$, $C(n)$:

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a Z_2-graded vector space with $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$. Let F be a nondegenerate, consistent, supersymmetric bilinear form on V.

We define $OSp(m/n) = OSp(m/n)_{\bar{0}} + OSp(m/n)_{\bar{1}}$ by

$$OSp(m/n)_s = \{ A \in \ell(m,n)_s \,|\, F(A(x),y) = -(-1)^{s(\deg x)} F(x,A(y))\}$$

where $s \in Z_2$; $x,y \in V$.

(i) If $m = 2\ell + 1$, $n = 2r$:

In some homogeneous basis of V the matrix of the form F can be written as

$$\left[\begin{array}{ccc|cc} 0 & I_\ell & 0 & & \\ I_\ell & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & I_r \\ & & & -I_r & 0 \end{array}\right]$$

and $OSp(2\ell+1,2r)$ consists in this basis of the matrices of the form

$$\left[\begin{array}{ccc|cc} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ \hline y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{array}\right]$$

where a is any $(\ell \times \ell)$-matrix, b and c are skew-symmetric $(\ell \times \ell)$-matrices, d is any $(r \times r)$-matrix, e and f are symmetric $(r \times r)$-matrices, u and v are $(\ell \times 1)$-matrices, x and y are $(\ell \times r)$-matrices and z is an $(r \times 1)$-matrix.

Two important properties of this are

(a) $OSp(2\ell+1,2r)_{\bar{0}}$ is a Lie algebra of type $B_\ell \oplus C_r$,

(b) the representation of $OSp(2\ell+1,2r)_{\bar{0}}$ in $OSp(2\ell+1,2r)_{\bar{1}}$ is isomorphic to $SO_{2\ell+1} \otimes Sp_{2r}$.

(ii) If $m = 2\ell$, $n = 2r$:

For this case the matrix of the form F and the matrices of $OSp(2\ell,2r)$ are the same as for (i) with the middle row and column deleted.

The properties analogous to (i) are

(a) $OSp(2\ell,2r)_{\bar{0}}$ for $\ell \geq 2$ is a Lie algebra of type $D_\ell \oplus C_r$;

(b) the representation of $OSp(2\ell,2r)_{\bar{0}}$ in $OSp(2\ell,2r)_{\bar{1}}$ is isomorphic to $SO_{2\ell} \otimes Sp_{2r}$.

The case $\ell = 1$ admits the consistent Z-grading $G_{-1} \oplus G_0 \oplus G_1$ where G_0, G_{-1} and G_1 consist, respectively, of matrices of the form:

$$\left[\begin{array}{cc|cc} a & 0 & & \\ 0 & -a & & \\ \hline & & d & e \\ & & f & -d^T \end{array}\right] \qquad \left[\begin{array}{cc|cc} & & 0 & 0 \\ & & y & y_1 \\ \hline y_1^T & 0 & & \\ -y^T & 0 & & \end{array}\right] \qquad \left[\begin{array}{cc|cc} & & x & x_1 \\ & & 0 & 0 \\ \hline 0 & x_1^T & & \\ 0 & -x^T & & \end{array}\right]$$

where the various elements are matrices of the form as discussed in (i) with $\ell = 1$.

We set

$$B(m,n) = OSp(2m+1/2n) , \qquad m \geq 0,\ n > 0$$

$$D(m,n) = OSp(2m/2n) , \qquad m \geq 2,\ n > 0$$

$$C(n) = OSp(2/2n-2) , \qquad n \geq 2.$$

These are also known as *orthosymplectic* superalgebras.

3. $P(n)$, $n \geq 2$:

This is a subalgebra of $S\ell(n+1,n+1)$ consisting of matrices of the form $\left[\begin{array}{c|c} a & b \\ \hline c & -a^T \end{array}\right]$

where tr $a = 0$, b is a symmetric matrix and c is a skew-symmetric matrix.

$P(n)$ admits the Z-graded structure $P(n) = G_{-1} \oplus G_0 \oplus G_1$ where G_0, G_{-1} and G_1 consist, respectively of matrices of the form:

$$\left[\begin{array}{c|c} a & \\ \hline & -a^T \end{array}\right] \qquad \left[\begin{array}{c|c} 0 & b \\ \hline 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|c} 0 & 0 \\ \hline c & 0 \end{array}\right]$$

4. $Q(n)$, $n \geq 2$:

This the subalgebra of $S\ell(n+1,n+1)$, $Q(n) = \tilde{Q}(n)/I_{2n+2}$ where $\tilde{Q}(n)$ consists of matrices of the form $\left(\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right)$ where tr $b = 0$ and

I_{2n+2} is the one-dimensional centre of $\tilde{Q}(n)$.

5. F(4):

This is a 40-dimensional Lie superalgebra for which $F(4)_{\bar{0}}$ is a Lie algebra of type $B_3 \oplus A_1$ and the representation of $F(4)_{\bar{0}}$ in $F(4)_{\bar{1}}$ is $spin_7 \otimes s\ell_2$.

6. G(3):

This is a 31-dimensional Lie superalgebra for which $G(3)_{\bar{0}}$ is a Lie algebra of type $G_2 \oplus A_1$ and the representation of $G(3)_{\bar{0}}$ in $G(3)_{\bar{1}}$ is $g_2 \otimes s\ell_2$.

7. $D(2,1;\alpha)$, $\alpha \in K\backslash\{0,-1\}$:

This is a one-parameter family of 17-dimensional Lie superalgebras consisting of all simple Lie superalgebras for which $D(2,1;\alpha)_{\bar{0}}$ is a Lie algebra of type $A_1 \oplus A_1 \oplus A_1$ and the representation of $D(2,1;\alpha)_{\bar{0}}$ in $D(2,1;\alpha)_{\bar{1}}$ is $s\ell_2 \otimes s\ell_2 \otimes s\ell_2$.

In Table 2.1 are listed all the classical Lie superalgebras for which the representations of $G_{\bar{0}}$ in $G_{\bar{1}}$ is irreducible. Also presented are the corresponding Lie algebra $G_{\bar{0}}$ and the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$.

Table 2.1

G	$G_{\bar{0}}$	$G_{\bar{0}} \vert G_{\bar{1}}$	G	$G_{\bar{0}}$	$G_{\bar{0}} \vert G_{\bar{1}}$
$B(m,n)$	$B_m \oplus C_n$	$SO_{2m+1} \otimes Sp_{2n}$	$F(4)$	$B_3 \oplus A_1$	$Spin_7 \otimes S\ell_2$
$D(m,n)$	$D_m \oplus C_n$	$SO_{2m} \otimes Sp_{2n}$	$G(3)$	$G_2 \oplus A_1$	$g_2 \otimes S\ell_2$
$D(2,1;\alpha)$	$A_1 \oplus A_1 \oplus A_1$	$S\ell_2 \otimes S\ell_2 \otimes S\ell_2$	$Q(n)$	A_n	ad $S\ell_{n+1}$

In Table 2.2 are listed all the classical Lie superalgebras for which the representation of $G_{\bar{0}}$ in $G_{\bar{1}}$ is reducible. These admit a unique consistent Z-grading of the form $G_{-1} \oplus G_0 \oplus G_1$ and the representations of G_0 in G_{-1} and G_1 are irreducible and for A(m,n) and C(n) contragredient. Also presented are the corresponding Lie algebra $G_{\bar{0}}$ and the representations of G_0 in G_{-1} and G_1.

Table 2.2

G	G_0	$G_0 \mid G_{-1}$	$G_0 \mid G_1$
A(m,n), m ≠ n	$A_m \oplus A_n \oplus K$	$S\ell_{m+1} \otimes S\ell_{n+1} \otimes K$	$S\ell_{m+1} \otimes S\ell_{n+1} \otimes K$
A(n,n)	$A_n \oplus A_n$	$S\ell_{n+1} \otimes S\ell_{n+1}$	$S\ell_{n+1} \otimes S\ell_{n+1}$
C(n)	$C_{n-1} \oplus K$	CSp_{2n-2}	CSp^*_{2n-2}
P(n)	A_n	$\Lambda^2\, S\ell^*_{n+1}$	$S^2\, S\ell_{n+1}$

A(m,n) and C(n) are called *basic classical Lie superalgebras of type I* and B(m,n), D(m,n), D(2,1;α), F(4) and G(3) *basic classical Lie superalgebras of type II*. The remainder of this chapter will concentrate on enumerating and discussing various properties of the basic classical Lie superalgebras.

2.2b ROOT SYSTEMS

Before discussing the properties of the root systems for the basic classical superalgebras the notation used here for weights, weight vectors, roots and root vectors is introduced.

Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a basic classical Lie superalgebra and let H be a Cartan subalgebra of $G_{\bar{0}}$. Let ρ be a representation of G in a vector space V. For $\lambda \in H^*$ we set

$$V_\lambda = \{v \in V \mid \rho(h)v = \lambda(h)v \;, \quad h \in H\}.$$

If $V_\lambda \neq 0$ then λ is called a *weight* of ρ and a nonzero vector $v_\lambda \in V_\lambda$ is called a *weight vector*.

A weight of the adjoint representation of G is called a *root* of G. For $\alpha \in H^*$ we set

$$G_\alpha = \{ e \in G | [h,e] = \alpha(h)e, \quad h \in H \} .$$

If $G_\alpha \neq 0$ then α is called a root of G and $e_\alpha \in G_\alpha$ is called a *root vector*.

A root α is called even if $G_\alpha \cap G_{\bar{0}} \neq 0$ and odd if $G_\alpha \cap G_{\bar{1}} \neq 0$. Let Δ, Δ_0 and Δ_1 denote the sets of all roots, even roots and odd roots respectively. We also introduce the following sets

$$\bar{\Delta}_0 = \{ \alpha \in \Delta_0 | \alpha/2 \notin \Delta_1 \}$$
$$\bar{\Delta}_1 \quad \{ \alpha \in \Delta_1 | 2\alpha \notin \Delta_0 \}$$

The cartan subalgebra, H, can be considered as a subspace of the space of diagonal matrices D. Consequently the roots are expressed in terms of the standard basis ε_i of D^*. The systems of non-zero even roots Δ'_0 and odd roots Δ_1 for all the basic classical Lie superalgebras have been given by Kac [10] and since we will need to refer to them later they are reproduced here in Table 2.3.

<u>Table 2.3</u>

A(m,n). The roots are expressed in terms of linear functions

$\varepsilon_1, \ldots, \varepsilon_{m+1}, \delta_1 = \varepsilon_{m+2}, \ldots, \delta_{n+2} = \varepsilon_{m+n+2}$.

$\Delta'_0 = \{\varepsilon_i - \varepsilon_j; \ \delta_i - \delta_j\}, \ i \neq j; \quad \Delta_1 = \{\pm(\varepsilon_i - \delta_j)\}$.

B(m,n). The roots are expressed in terms of linear functions

$\varepsilon_1, \ldots, \varepsilon_m, \ \delta_1 = \varepsilon_{2m+1}, \ldots, \delta_n = \varepsilon_{2m+n}$.

$\Delta'_0 = \{\pm \varepsilon_i \pm \varepsilon_j; \ \pm 2\delta_i; \ \pm\varepsilon_i; \ \pm\delta_i \pm \delta_j\}$, $i \neq j$;

$\Delta_1 = \{\pm \delta_i; \ \pm\varepsilon_i \pm\delta_j\}$.

C(n). The roots are expressed in terms of linear functions

ε_1, $\delta_1 = \varepsilon_3, \ldots, \delta_{n-1} = \varepsilon_{n+1}$.

$\Delta'_0 = \{\pm 2\delta_i;\ \pm\delta_i \pm \delta_j\}$; $\Delta_1 = \{\pm\varepsilon_1 \pm \delta_i\}$.

D(m,n). The roots are expressed in terms of linear functions

$\varepsilon_1, \ldots, \varepsilon_m$, $\delta_1 = \varepsilon_{2m+1}, \ldots, \delta_n = \varepsilon_{2m+n}$.

$\Delta'_0 = \{\pm\varepsilon_i \pm\varepsilon_j;\ \pm 2\delta_i;\ \pm\delta_i \pm\delta_j\}$, $i \neq j$;

$\Delta_1 = \{\pm\varepsilon_i \pm\delta_j\}$.

D(2,1; α). The roots are expressed in terms of linear functions

ε_1, ε_2, ε_3 .

$\Delta'_0 = \{\pm 2\varepsilon_i\}$; $\Delta_1 = \{\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3\}$

F(4). The roots are expressed in terms of linear functions

ε_1, ε_2, ε_3, δ_1 .

$\Delta'_0 = \{\pm\varepsilon_i \pm\varepsilon_j;\ \pm\varepsilon_i;\ \pm\delta_1\}$, $i \neq j$;

$\Delta_1 = \{\frac{1}{2}(\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3 \pm\delta_1)\}$.

G(3). The roots are expressed in terms of linear functions

ε_1, ε_2, ε_3, δ_1 with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$.

$\Delta'_0 = \{\varepsilon_i - \varepsilon_j;\ \pm\varepsilon_i;\ \pm 2\delta_1\}$; $\Delta_1 = \{\pm\varepsilon_i \pm\delta_1;\ \pm\delta_1\}$.

Some general properties of basic classical Lie superalgebras, which are relevant for the explicit construction of the algebra and the discussion of representation theory later in the chapter, will now be presented. In all future work, unless explicitly stated otherwise, G will refer to a basic classical Lie superalgebra.

It is first noted that if H is the Cartan subalgebra of G then $G = \bigoplus_{\alpha\in H^*} G_\alpha$ and $G_0 = H$. Furthermore, dim $G_\alpha = 1$, for $\alpha \neq 0$ except for A(1,1) and $[G_\alpha, G_\beta] \neq 0$ if and only if α, β and $\alpha + \beta$ are all elements of Δ.

An invariant, non-degenerate, supersymmetric bilinear form, $(\, , \,)$, may be fixed on G. This form is unique, up to a constant factor, and such that $(G_\alpha, G_\beta) = 0$ for $\alpha \neq -\beta$. If now a bilinear form is defined on H^* by $(\alpha,\beta) = (h_\alpha, h_\beta)$ then $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$ where h_α is a non-zero vector determined by $(h_\alpha, h) = \alpha(h)$ where $h \in H$.

Finally it is noted that (i) if $\alpha \in \Delta$ (respectively $(\Delta_0, \Delta_1, \overline{\Delta}_0, \overline{\Delta}_1)$ then $-\alpha \in \Delta$ (respectively $\Delta_0, \Delta_1, \overline{\Delta}_0, \overline{\Delta}_1$) and (ii) $k\alpha \in \Delta$, for $\alpha \neq 0$ and $k \neq \pm 1$, if and only if $\alpha \in \Delta_1$ and $(\alpha,\alpha) \neq 0$ in which case $k = \pm 2$.

Let $B_{\bar{0}}$ be a Borel subalgebra of $G_{\bar{0}}$ (i.e. a maximal solvable subalgebra of $G_{\bar{0}}$), containing H. Having fixed a Borel subalgebra $B = B_{\bar{0}} + B_{\bar{1}}$ of G then, since the adjoint representation of H in G is diagonalizable, G may be decomposed as follows:

$$G = N^- \oplus H \oplus N^+ \quad \text{and} \quad B = H \oplus N^+$$

where N^- and N^+ are subalgebras with the properties that $[H, N^+] \subset N^+$ and $[H, N^-] \subset N^-$.

A root α is called *positive* if $G_\alpha \cap N^+ \neq 0$ and *negative* if $G_\alpha \cap N^- \neq 0$. Let ρ_0 (resp ρ_1) denote half the sum of all the even (resp odd) positive roots and let $\rho = \rho_0 - \rho_1$. A positive root α is called simple if it cannot be decomposed into a sum of two positive roots. Let $\Pi = \{\alpha_1,\ldots,\alpha_r\}$, where r is the rank of G, be the set of all simple roots.

With the introduction of the above structures further useful properties of the basic classical Lie superalgebras can be enumerated. It is first noted that all the subspaces $G_\alpha \cap N^\pm$ are one dimensional. Thus, non-zero elements $e_i^+ \in G_{\alpha_i} \cap N^+$, $e_i^- \in G_{-\alpha_i} \cap N^-$ and $h_i \in H$, $i = 1,\ldots,r$ may be chosen such that e_i^+, e_i^- and h_i is the system of generators of G which satisfies the following relations:

$$[e_i^+, e_j^-] = \delta_{ij}\, h_i \quad ; \quad [h_i, h_j] = 0$$

$$[h_i, e_j^+] = a_{ij}\, e_j^+ \quad ; \quad [h_i, e_j^-] = -a_{ij} e_j^-$$

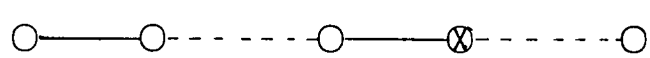
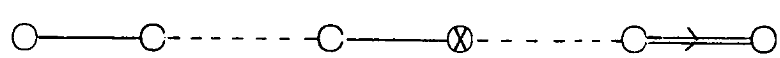
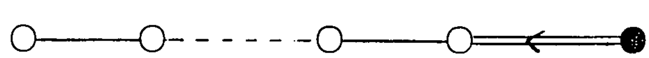
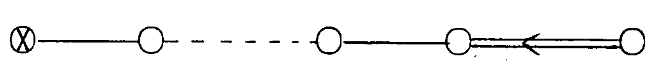
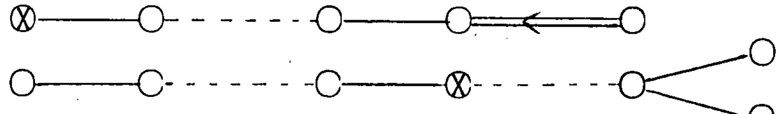
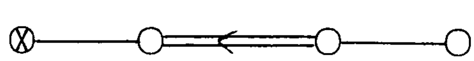
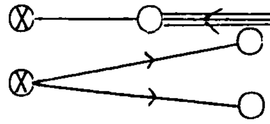
where (a_{ij}) is the *Cartan matrix* which will be chosen to satisfy the following normalizing conditions:
(i) $a_{ii} = +2$ or 0; (ii) if $a_{ii} = 0$ then the first non-zero element among a_{ii+k} is $+1$. The Cartan matrix will depend on the choice of B. The above elements e_i^+, e_i^- and h_i generate G. The elements $h_1, \ldots, h_r$ span H and are linearly independent for all G except for G = A(n,n) for which case there is a unique linear dependence:

$$(h_1 + h_{2n+1}) + 2(h_2 + h_{2n}) + \ldots + (n-1)(h_{n-1} + h_{n+1}) + nh_n = 0.$$

Having defined the Cartan matrix, G can be uniquely determined, up to an isomorphism, by the pair $((a_{ij}), \tau)$ where τ is a subset of $\{1, \ldots, r\}$ consisting of those i for which α_i is an odd root.

Basic classical Lie superalgebras admit a Borel subalgebra, B, for which the corresponding Dynkin diagram has the form represented in Table 2.4.

Table 2.4

G	Dynkin diagram
A(m,n)	
B(m,n), m > 0	
B(0,n)	
C(n), n > 2	
D(m,n)	
F(4)	
G(3)	
D(2,1; α)	

These diagrams consist of r-nodes of the form o, ⊗ and • which are called white, grey and black respectively. The i-th node is white if $i \notin \tau$ and grey or black if $i \in \tau$ and $a_{ii} = 0$ or $+2$ respectively. The i-th and j-th nodes are joined by $|a_{ij}\, a_{ji}|$ lines except in the case $D(2,1;\alpha)$. If $a_{ij}\, a_{ji} = 0$ then $a_{ij} = a_{ji} = 0$ and if $a_{ii} = +2$ then all the entries in the i-th row are non-positive integers.

The pair $((a_{ij}), \tau)$ is uniquely determined by the Dynkin diagram except for $D(2,1;\alpha)$ and $D(2,n)$. The Cartan matrix of $D(2,1;\alpha)$ is

$$D_\alpha = \begin{bmatrix} 0 & +1 & \alpha \\ -1 & +2 & 0 \\ -1 & 0 & +2 \end{bmatrix}$$

and the 3×3 - submatrix corresponding to the last 3 nodes of the Dynkin diagram of $D(2,n)$ is D_1.

The remaining classical Lie superalgebras, $P(n)$ and $Q(n)$, have special properties which would necessitate a separate treatment to that given here. Since the body of this thesis is concerned with orthosymplectic superalgebras, which belong to the class of basic classical Lie superalgebras it was thought to be inexpedient to discuss these algebras in detail. Rather some general properties of $P(n)$ and $Q(n)$ which differ from the basic classical Lie superalgebras will be noted.
Let G be a classical Lie superalgebra with $G = \bigoplus_{\alpha \in H^*} G_\alpha$ its root decomposition with respect to the Cartan subalgebra H. Then if $G = Q(n)$, $G_0 \neq H$ and if G is any of $P(2)$, $P(3)$ or $Q(n)$ then the property $\dim G_\alpha = 1$ for $\alpha \neq 0$ does not generally hold. Furthermore, for G any of $P(n)$ or $Q(n)$, there does not generally exist on G a unique,

non-degenerate, invariant, supersymmetric bilinear form.

Finally, the properties (i) $[G_\alpha, G_\beta] \neq 0$ if and only if $\alpha,\beta,\alpha+\beta \in \Delta$, (ii) $(G_\alpha, G_\beta) = 0$ for $\alpha \neq -\beta$, (iii) $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, presented earlier as being valid for the basic classical Lie superalgebras, are no longer generally valid for P(n) and Q(n). These differing features are a consequence of the fact that the basic classical Lie superalgebras belong to the class of contragredient Lie superalgebras [9] where as P(n) and Q(n) do not.

2.3 REPRESENTATIONS OF BASIC CLASSICAL LIE SUPERALGEBRAS

This section contains a short review of the work of Kac [10] on finite-dimensional representations of simple Lie superalgebras. This work is based on the theory of induced representations which are now defined.

Let G be a Lie superalgebra with universal enveloping superalgebra U(G). Let H be a subalgebra of G and V be a H-module. Since V is equally well a U(H)-module it makes sense to form the tensor product $U(G) \otimes_{U(H)} V$ where $U(G) \otimes_{U(H)} V$ is a Z_2-graded space defined as the factor space of $U(G) \otimes V$ by the Z_2-graded subspace, I, spanned by elements of the form $gh \otimes v - g \otimes h(v)$, $g \in U(G)$, $h \in U(H)$, $v \in V$. The space $U(G) \otimes_{U(H)} V$ can be endowed with the structure of a G-module by defining the left action of g as $g(u \otimes v) = gu \otimes v$, $g \in G$, $u \in U(G)$, $v \in V$. That $U(G) \otimes_{U(H)} V$ has the structure of a G-module follows from the observation that I is invariant under the action of G. Thus, if we consider $x \in U(G) \otimes_{U(H)} V$ say

$$x = (u \otimes v) + (g_1 h \otimes w - g_1 \otimes h(w)) = (u \times v) + i, \quad \text{where } i \in I$$

then $gx = g(u \otimes v) + g(g_1 h \otimes w - g_1 \otimes h(w))$

$$= (g(u) \otimes v) + (g(g_1 h) \otimes w - gg_1 \otimes h(w))$$

$$= (g(u) \otimes v) + ((gg_1)h \otimes w - (gg_1) \otimes h(w))$$

$$= (g(u) \otimes v) + (g_2 h \otimes w - g_2 \otimes h(w)) \ , \quad \text{where } g_2 = gg_1 \in G$$

$$= (g(u) \otimes v) + i' \qquad , \quad \text{where } i' \in I$$

$$= x' \qquad , \text{ where } x' \in U(G) \otimes_{U(H)} V \ .$$

Therefore the action of G on $U(G) \otimes_{U(H)} V$ is well defined by $g\colon\ u \otimes v + I \to g(u) \otimes v + I \quad \forall g \in G,\ u \in U(G),\ v \in V$. This G-module, constructed as above, is said to be induced from the H-module V and is denoted by $\mathrm{Ind}_H^G\, V$ [9,10]. This construction is now used to develop the representation theory of simple Lie superalgebras.

If $G = G_{\bar{0}} + G_{\bar{1}}$ is a basic classical Lie superalgebra, excluding $A(n,n)$, and H is a Cartan subalgebra of $G_{\bar{0}}$ then we can fix a Borel subalgebra, B, of G containing H as $B = H \oplus N^+$. Let $\Lambda \in H^*$ be a linear function on H and define a one-dimensional B-module v_Λ by $h(v_\Lambda) = \Lambda(h) v_\Lambda$ for $h \in H$ and $N^+(v_\Lambda) = 0$. Setting $\tilde{V}(\Lambda) = \mathrm{Ind}_B^G\, v_\Lambda$, $\tilde{V}(\Lambda)$ is a G-module which contains a unique maximal submodule $I(\Lambda)$. Setting $V(\Lambda) = \tilde{V}(\Lambda)/I(\Lambda)$, the G-module $V(\Lambda)$ is an irreducible representation and is called an *irreducible representation with highest weight* Λ.

Let e_i^+, e_i^-, h_i, $i = 1,\ldots,r$ be the generators of G described in §2.2b and set $a_i = \Lambda(h_i)$ where $\Lambda \in H^*$. The representation $V(\Lambda)$ is finite-dimensional if and only if the following conditions are satisfied [10].

1) $a_i \in Z_+$ for $i \neq s$, where s is the number of the non-white node in the Dynkin diagram.

2) for type II superalgebras $b \in Z_+$, where b is given in Table 2.5.

3) for $b < \ell$ in Table 2.5 the following supplementary conditions must also be satisfied:

$B(m,n)$: $a_{n+k+1} = \dots = a_{m+n} = 0.$

$D(m,n)$: $a_{n+k+1} = \dots = a_{m+n} = 0$, $b \leq m-2$;

$a_{m+n-1} = a_{m+n}$, $b = m-1.$

$D(2,1;\alpha)$: all $a_i = 0$ if $b = 0$;

$(a_3+1)\alpha = \pm(a_2+1)$ if $b = 1.$

$F(4)$: all $a_i = 0$ if $b = 0$; $b \neq 1$;

$a_2 = a_4 = 0$ if $b = 2$; $a_2 = 2a_4+1$ if $b = 3.$

$G(3)$: all $a_i = 0$ if $b = 0$; $b \neq 1$;

$a_2 = 0$ if $b = 2.$

Table 2.5

G	b	ℓ
$B(0,n)$	$\frac{1}{2} a_n$	0
$B(m,n)$, $m > 0$	$a_n - a_{n+1} - \dots - a_{m+n-1} - \frac{1}{2} a_{m+n}$	m
$D(m,n)$	$a_n - a_{n+1} - \dots - a_{m+n-2} - \frac{1}{2} (a_{m+n-1} + a_{m+n})$	m
$D(2,1;\alpha)$	$\frac{1}{1+\alpha} (2a_1 - a_2 - \alpha a_3)$	2
$F(4)$	$\frac{1}{3} (2a_1 - 3a_2 - 4a_3 - 2a_4)$	4
$G(3)$	$\frac{1}{2} (a_1 - 2a_2 - 3a_3)$	3

A general property of simple Lie superalgebras is that they contain finite-dimensional representations which are not completely reducible. In fact it has been shown by Djokovic and Hochschild [15,16] that if G is a Lie superalgebra then all the finite dimensional representations of G are completely reducible if and only if G is isomorphic to the direct product of a semi-simple Lie algebra with finitely many Lie superalgebras of the type B(0,n), $n > 0$. Finite-dimensional representations of a Lie superalgebra, G, which are completely reducible are called *typical*.

Kac [10] has derived necessary and sufficient conditions for a finite-dimensional, irreducible G-module, $V(\Lambda)$, with highest weight Λ, to be typical. For example a sufficient condition is that $V(\Lambda)$ is typical if $(\Lambda+\rho,\alpha) \neq 0$ for any $\alpha \in \bar{\Delta}_1^+$. A necessary condition for $V(\Lambda)$ to be typical is that $\dim V_{\bar{0}} = \dim V_{\bar{1}}$ provided G is not isomorphic to one of the algebras B(0,n).

In Table 2.6 the conditions for $V(\Lambda)$ to be typical are presented where G is a basic classical Lie superalgebra [10].

<u>Table 2.6</u>

A(m,n) : $$a_{m+1} \neq \sum_{t=m+2}^{j} a_t - \sum_{t=1}^{m} a_t - 2m - 2 + i + j$$

$$\text{for } 1 \leq i \leq m+1 \leq j \leq m+n+1$$

B(m,n) : $$\sum_{t=i}^{n} a_t - \sum_{t=n+1}^{j} a_t + 2n - i - j \neq 0$$

$$\sum_{t=i}^{n} a_t - \sum_{t=n+1}^{j} a_t - 2\sum_{t=j+1}^{m+n-1} a_t - a_{m+n} - i + j - 2m + 1 \neq 0$$

$$\text{for } 1 \leq i \leq n \leq j \leq m+n-1 .$$

B(0,n) : All finite-dimensional representations $V(\Lambda)$ are typical.

C(n) : $a_1 \neq \sum_{t=2}^{i} a_t + i - 1$

$$a_1 \neq \sum_{t=2}^{i} a_t + 2 \sum_{t=i+1}^{i} a_t + 2n - i - 1$$

for $1 \leq i \leq n - 1$.

D(m,n) : $\sum_{t=i}^{n} a_t - \sum_{t=n+1}^{j} a_t + 2n - i - j \neq 0$

for $1 \leq i \leq n \leq j \leq m + n - 1$.

$$\sum_{t=i}^{n} a_t - \sum_{t=n+1}^{m+n-2} a_t - a_{m+n} + n - m - i + 1 \neq 0$$

for $1 \leq i \leq n$.

$$\sum_{t=i}^{n} a_t - \sum_{t=n+1}^{j} a_t - 2 \sum_{t=j+1}^{m+n-2} a_t - a_{m+n-1} - a_{m+n} - i + j - 2m+2 \neq 0$$

for $1 \leq i \leq n \leq j \leq m + n - 2$. *

D(2,1;α) : $a_1 \neq 0$; $a_1 \neq a_2 + \alpha a_3 + 1 + \alpha$;

$a_1 \neq a_2+1$; $a_1 \neq \alpha a_3 + \alpha$.

G(3) : $a_1 \neq 0$; $a_1 \neq a_2 + 1$;

$a_1 \neq a_2 + 3a_3 + 4$; $a_1 \neq 3a_2 + 3a_3 + 6$;

$a_1 \neq 3a_2 + 6a_3 + 9$; $a_1 \neq 4a_2 + 6a_3 + 10$;

F(4) : $a_1 \neq 0$; $a_1 \neq a_2 + 1$;

$a_1 \neq a_2 + 2a_3 + 3$; $a_1 \neq 2a_2 + 2a_3 + 4$;

$a_1 \neq a_2 + 2a_3 + 2a_4 + 5$; $a_1 \neq 2a_2 + 2a_3 + 2a_4 + 6$;

$a_1 \neq 2a_2 + 4a_3 + 2a_4 + 8$; $a_1 \neq 3a_2 + 4a_3 + 2a_4 + 9$.

---

* There is an error in this expression in ref.[10] which is corrected here.

Kac [10] has also derived the expression for the dimension of a typical G-module $V(\Lambda)$, with highest weight Λ, where G is a basic classical Lie superalgebra. If $d = \dim \Delta_1^+$ then

$$\dim V = 2^d \prod_{\alpha\in\Delta_0^+} \frac{(\Lambda+\rho,\alpha)}{(\rho_0,\alpha)}$$

These dimension formulae are given explicitly in terms of the Kac-Dynkin labels, a_i, defined earlier in Table 2.7.

Table 2.7

$A(m,n)$: $$\dim V(\Lambda) = 2^{(m+1)(n+1)} \prod_{1\le i\le j\le m} \frac{a_i+a_{i+1}+\ldots+a_j+j-i+1}{j-i+1} \times \prod_{m+2\le i\le j\le m+n+1} \frac{a_i+\ldots+a_j+j-i+1}{j-i+1}$$

$C(n)$: $$\dim V(\Lambda) = 2^{2n-2} \prod_{2\le i\le j\le n-1} \frac{a_i+\ldots+a_j+j-i+1}{j-i+1} \times \prod_{2\le i\le j\le n} \frac{a_i+\ldots+a_{j-1}+2a_j+\ldots+2a_n}{2n-i-j+2}$$

$B(m,n)$: $$\dim V(\Lambda) = 2^{(2m+1)n} \prod_{1\le i\le j\le n-1} \frac{a_i+\ldots+a_j+j-i+1}{j-i+1} \prod_{n+1\le i\le j\le m+n-1} \frac{a_i+\ldots+a_j+j-i+1}{j-i+1}$$
$$\times \prod_{1\le i\le j\le n} \frac{(a_i+\ldots+a_{j-1})+2(a_j+\ldots+a_n-a_{n+1}-\ldots-a_{n+m-1})-a_{n+m}+2n-2m+1-i-j}{2n+2-i-j}$$
$$\times \prod_{n+1\le i\le j\le m+n} \frac{(a_i+\ldots+a_{j-1}+2(a_j+\ldots+a_{m+n-1})+a_{m+n}+2m-i-j+1}{2m-i-j+1}$$

$$D(m,n) \quad : \quad \dim V(\Lambda) = 2^{2mn} \prod_{1\le i\le j\le n-1} \frac{a_i+\ldots+a_j+j-i+1}{j-i+1} \prod_{n+1\le i\le j\le m+n-1} \frac{a_i+\ldots+a_j+j-i+1}{j-i+1}$$

$$\times \prod_{1\le i\le j\le n} \frac{(a_i+\ldots+a_{j-1})+2(a_j+\ldots+a_n-a_{n+1}-\ldots-a_{m+n-1})-a_{m+n}+2n-2m+1-i-j}{2n+2-i-j}$$

$$\times \prod_{n+1\le i\le j\le m+n-1} \frac{(a_i+\ldots+a_{j-1})+2(a_j+\ldots+a_{m+n-1})+a_{m+n}+2m-i-j+1}{2m-i-j+1}$$

$$B(0,n) \quad : \quad \dim V(\Lambda) = \prod_{1\le i\le j\le n} \frac{(a_i+\ldots+a_j)+2(a_{j+1}+\ldots+a_{n-1})+a_n+2n-i-j}{2n-i-j}$$

$$\times \prod_{1\le i\le n} \frac{2(a_i+\ldots+a_{n-1})+a_n+2n-2i+1}{2n-2i+1}$$

$$D(2,1;\alpha) \quad : \quad \dim V(\Lambda) = 16(a_2+1)(a_3+1)[(2a_1-a_2-\alpha a_3)(1+\alpha)^{-1}-1]$$

$$G(3) \quad : \quad \dim V(\Lambda) = \frac{8}{15}(a_2+1)(a_3+1)(a_2+a_3+2)(a_2+3a_3+4)$$

$$\times(a_2+2a_3+3)(2a_2+3a_3+5)(a_1-2a_2-3a_3-5)$$

$$F(4) \quad : \quad \dim V(\Lambda) = \frac{32}{45}(a_2+1)(a_3+1)(a_4+1)(a_2+a_3+2)(a_3+a_4+2)$$

$$\times(a_2+2a_3+3)(a_2+a_3+a_4+3)(a_2+2a_3+2a_4+5)$$

$$\times(a_2+2a_3+a_4+4)(2a_1-3a_2-4a_3-2a_4-9)\ .$$

CHAPTER 2 - REFERENCES

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3. REPRESENTATIONS OF ORTHOSYMPLECTIC SUPERALGEBRAS: WEIGHT SPACE TECHNIQUES

3.1 INTRODUCTION

In this chapter weight space techniques are used to explicity construct irreducible representations of orthosympletic superalgebras. The general method follows on the work of Kac [1,2]; explicit results have been obtained by Hurni and Morel [3] for several particular representations of various orthosymplectic superalgebras and also by Thiery-Mieg and Morel [4] and Hurni and Morel [5] for various special linear superalgebras.

The general construction of the algebra is first presented followed by the procedure for obtaining irreducible representations. This is illustrated by a complete analysis of the finite-dimensional irreducible representations of the lowest rank superalgebras from each orthosymplectic class, namely C(2), B(1,1) and D(2,1;α) (from which D(2,1) is obtained by setting $\alpha = +1$).

3.2 STRUCTURE OF THE ALGEBRA

The notation of chapter 2 is modified slightly to make the relation between the simple roots and their corresponding generators more apparent. Let h_i ($i = 1,2,\ldots r$; r = rank of the superalgebra) be the generators of the Cartan subalgebra and let α_i^+ (α_i^-) be the generator corresponding to th i th positive (negative) simple root. As discussed in [1] and ch.2 the algebra in this basis can be written in the following form

$$
\begin{aligned}
[\alpha_i^+, \alpha_j^-] &= \delta_{ij}\, h_i \\
[h_i, h_j] &= 0 \\
[h_i, \alpha_j^\pm] &= \pm a_{ij}\, \alpha_j^\pm
\end{aligned}
$$

where the a_{ij} are the elements of the Cartan matrix. The remaining generators may be defined from those corresponding to simple roots by (anti-) commutation [3,5,6].

The weight space decomposition of a representation is given by the eigenvalues a_i' and b' of a vector with respect to h_i and k respectively. The odd simple root 'hides' an even simple root of the even subalgebra. Consequently there exists a hidden Cartan generator, k, which is defined by equations (3.2, 3.5, 3.7, 3.9) for $B(o,n)$, $B(m,n)$, $D(m,n)$ and $C(n)$ respectively.

<u>B(o,n)</u>

The Dynkin diagram with the set of simple positive roots chosen and their associated generators is

$$\begin{array}{cccccc} \delta_1-\delta_2 & \delta_2-\delta_3 & \delta_3-\delta_4 & \delta_{n-2}-\delta_{n-1} & \delta_{n-1}-\delta_n & \delta_n \\ \overset{+}{\alpha_1} & \overset{+}{\alpha_2} & \overset{+}{\alpha_3} & \overset{+}{\alpha}_{n-2} & \overset{+}{\alpha}_{n-1} & \overset{+}{\alpha}_n \equiv \beta^{n+} \end{array}$$

The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix}$$

The remaining odd generators are constructed in the following way:

$$\beta^{i\pm} = [\,[\ldots [\,[\beta^{n\pm}, \alpha^{\pm}_{n-1}], \alpha^{\pm}_{n-2}], \ldots], \alpha^{\pm}_{i}] \qquad 3.1$$

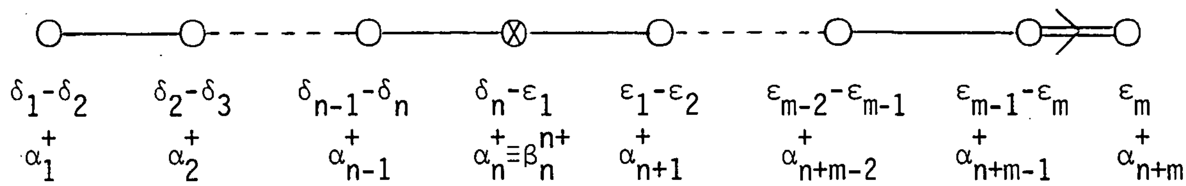
where $1 \leq i \leq n-1$. The generator, k, in the Cartan subalgebra of $Sp(2n)$ is

$$k = \tfrac{1}{2}\, h_n. \tag{3.2}$$

The "hidden" Sp(2n) generator associated with the n th node of the Dynkin diagram is taken as $\{\beta^{n\pm},\ \beta^{n\pm}\}$.

B(m,n) m > 0

The Dynkin diagram with the set of simple positive roots chosen and their associated generators is



The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix}
2 & -1 & & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & -1 & 2 & -1 & & & & & & \\
 & & & & & -1 & 0 & +1 & & & & & \\
 & & & & & & -1 & 2 & +1 & & & & \\
 & & & & & & & -1 & 2 & -1 & & & \\
 & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & -1 & 2 & -1 \\
 & & & & & & & & & & & -2 & 2
\end{bmatrix}$$

The remaining odd generators are constructed in the following way:

$$\begin{aligned}
\beta_n^{i\pm} &= [\,[\ldots [\,[\beta_n^{n\pm},\ \alpha_{n-1}^{\pm}],\ \alpha_{n-2}^{\pm}],\ \ldots],\ \alpha_i^{\pm}] \\
\beta_j^{i\pm} &= [\,[\ldots [\,[\beta_n^{i\pm},\ \alpha_{n+1}^{\pm}],\ \alpha_{n+2}^{\pm}],\ \ldots],\ \alpha_j^{\pm}] \\
\tilde{\beta}_j^{i\pm} &= [\,[\ldots [\,[\beta_{n+m}^{i\pm},\ \alpha_{n+m}^{\pm}],\ \alpha_{n+m-1}^{\pm}],\ \ldots],\ \alpha_j^{\pm}]
\end{aligned} \tag{3.3}$$

where $1 \le i \le n$; $n+1 \le j \le n+m$.

The 'hidden' generator, k, in the Cartan subalgebra of Sp(2n) will be some linear combination of the h_i's which satisfies the requirements

$$[k, \alpha_j^{\pm}] = 0 \quad , \qquad n+1 \le j \le n+m$$
$$[k, \alpha_{n-1}^{\pm}] = \mp \alpha_{n-1}^{\pm} \qquad\qquad 3.4$$
$$[k, \{\beta,\beta\}^{\pm}] = \pm\, 2\{\beta,\beta\}^{\pm}$$

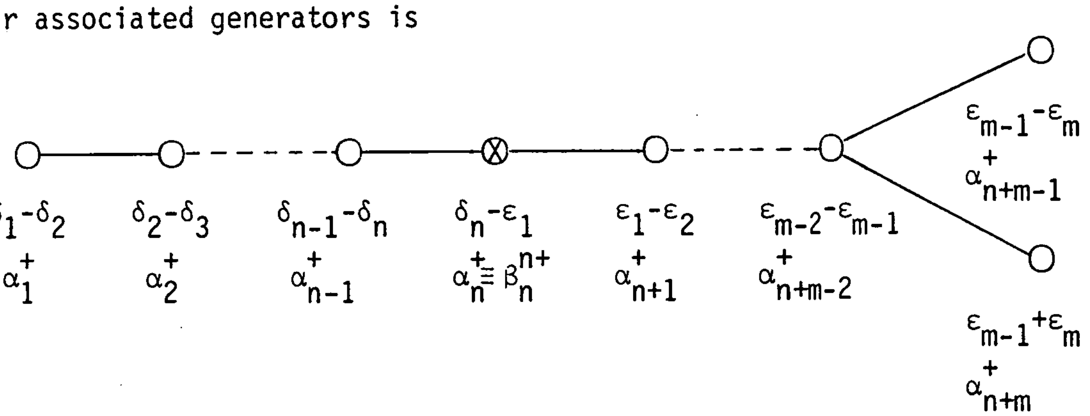
where $\{\beta,\beta\}$ refers to one of the 'hidden' generators given below. We find

$$k = h_n - h_{n+1} - h_{n+2} - \dots - h_{n+m-1} - \tfrac{1}{2}h_{n+m} \qquad 3.5$$

Associated with the n th node of the Dynkin diagram there exists a 'hidden' Sp(2n) generator which in the basis chosen can be taken as one of $\{\beta_j^{n\pm}, \tilde{\beta}_{j+1}^{n\pm}\}$ where $n \le j \le n+m-1$ or as $\{\beta_{n+m}^{n\pm}, \beta_{n+m}^{n\pm}\}$.

<u>D(m,n)</u>

The Dynkin diagram with the set of simple positive roots chosen and their associated generators is



The Cartan matrix is

$$
[a_{ij}] = \begin{bmatrix}
2 & -1 & & & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & & & & \\
 & & & & & & & & & & & & & \\
 & & & & -1 & 2 & -1 & & & & & & & \\
 & & & & & -1 & 0 & +1 & & & & & & \\
 & & & & & & -1 & 2 & -1 & & & & & \\
 & & & & & & & & & & & & & \\
 & & & & & & & & & -1 & 2 & -1 & & \\
 & & & & & & & & & & -1 & 2 & -1 & -1 \\
 & & & & & & & & & & & -1 & 2 & 0 \\
 & & & & & & & & & & & -1 & 0 & 2
\end{bmatrix}
$$

The remaining odd generators are constructed in the following way:

$$
\begin{aligned}
\beta_n^{i\pm} &= [\,[\ldots\,[\,[\beta_n^{n\pm}, \alpha_{n-1}^{\pm}], \alpha_{n-2}^{\pm}], \ldots], \alpha_i^{\pm}] \\
\beta_j^{i\pm} &= [\,[\ldots\,[\,[\beta_n^{i\pm}, \alpha_{n+1}^{\pm}], \alpha_{n+2}^{\pm}], \ldots], \alpha_j^{\pm}] \\
\beta_{n+m}^{i\pm} &= [\beta_{n+m-2}^{j\pm}, \alpha_{n+m}^{\pm}]. \\
\tilde{\beta}_j^{i\pm} &= [\,[\ldots\,[\,[\beta_{n+m}^{i\pm}, \alpha_{n+m-1}^{\pm}], \alpha_{n+m-2}^{\pm}], \ldots], \alpha_j^{\pm}] \\
&\quad n+1 \le j \le n+m-1 \quad ; \quad 1 \le i \le n.
\end{aligned}
\qquad 3.6
$$

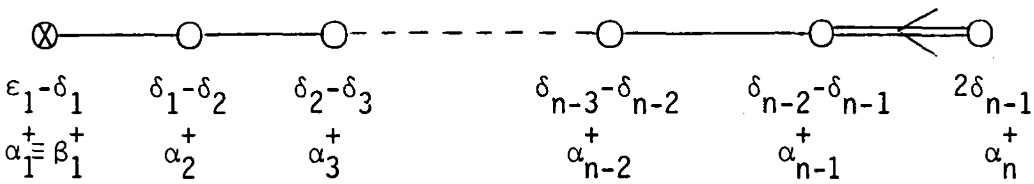
The 'hidden' generator in the Cartan subalgebra of Sp(2n) is

$$
k = h_n - h_{n+1} - h_{n+2} - \ldots - h_{n+m-2} - \tfrac{1}{2}(h_{n+m-1} + h_{n+m}) \qquad 3.7
$$

The 'hidden' Sp(2n) generator associated with the n th Dynkin node can be taken as one of $\{\beta_j^{n\pm}, \tilde{\beta}_{j+1}^{n\pm}\}$, where $n \le j \le n+m-2$ or as $\{\beta_{n+m-1}^{n\pm}, \beta_{n+m}^{n\pm}\}$, in the basis of simple roots chosen.

$\underline{C(n),\ n > 2^*}$

The Dynkin diagram with the set of simple positive roots chosen and their associated generators is



The Cartan matrix is

$$[a_{ij}] = \begin{bmatrix} 0 & +1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & & & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -2 \\ & & & & & & -1 & 2 \end{bmatrix}$$

The remaining odd generators are constructed in the following way:

$$\begin{aligned} \beta_i^{\pm} &= [\,[\ldots [\,[\beta_1^{\pm}, \alpha_2^{\pm}], \alpha_3^{\pm}], \ldots], \alpha_i^{\pm}] && 2 \le i \le n \\ \tilde{\beta}_j^{\pm} &= [\,[\ldots [\,[\beta_n^{\pm}, \alpha_{n-1}^{\pm}], \alpha_{n-2}^{\pm}], \ldots], \alpha_j^{\pm}] && 2 \le j \le n-1 \end{aligned} \qquad 3.8$$

The 'hidden' O(2) generator is

$$k = h_1 - h_2 - h_3 - \ldots - h_n \qquad 3.9$$

---

* For the $C(2) \simeq A(1,0)$ case, see §3.4.

3.3 FORMALISM FOR THE CONSTRUCTION OF REPRESENTATIONS

Consider a representation possessing a highest weight vector, Λ, of weight, λ, such that $h_i\Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$. Let $g_i^+ \in G_{\alpha_i}$ and $g_i^- \in G_{-\alpha_i}$ such that $\alpha_i \in \Delta_0^+$ and let $\beta_i^+ \in G_{\alpha_i}$ and $\beta_i^- \in G_{-\alpha_i}$ such that $\alpha_i \in \Delta_1^+$ then $g_i^+\Lambda = 0$ and $\beta_i^+\Lambda = 0$ for all positive root vectors g_i^+ and β_i^+ and the representation with highest weight vector Λ is spanned by the vectors

$$(g_1^-)^{i_1}(g_2^-)^{i_2}\dots(g_m^-)^{i_m}(\beta_1^-)^{k_1}(\beta_2^-)^{k_2}\dots(\beta_{\frac{1}{2}MN}^-)^{k_{\frac{1}{2}MN}}\Lambda \qquad 3.10$$

where $i_j \in \{0,Z^+\}$, $k_j \in \{0,1\}$ and m is the dimension of the even subalgebra $O(M) \times Sp(N)$. This is a consequence of the Poincaré-Birkhoff-Witt theorem. The distinct multiplets of the even subalgebra are generated from the $2^{\frac{1}{2}MN}$ states

$$\psi_j = (\beta_1^-)^{k_1}(\beta_2^-)^{k_2}\dots(\beta_{\frac{1}{2}MN}^-)^{k_{\frac{1}{2}MN}}\Lambda \qquad 3.11$$

by application of even generators.

Kac [1,2] has given conditions on the a_i under which the representation is finite-dimensional and irreducible (or typical). If the conditions for irreducibility are not satisfied, the representation is indecomposable, and the $O(M) \times Sp(N)$ structure of the irreducible composition factors (atypical representations) may be explicitly determined. In certain cases some of the χ_j belong to infinite dimensional subspaces and it is necessary to revert to the induced module construction as discussed in chapter 2 (see also [2,9]).

For the construction presented below it is useful to introduce an 'inner product' on the representation space. Scheunert et al. [7] have shown that for a Lie superalgebra there are two ways to do this depending on the choice of conjugation operation on the algebra.

As discussed in chapter 2 this can be either an adjoint (†) or a superadjoint (‡) to which correspond star and grade star representations respectively. Given that either exists, we have two different inner products $(\; , \;)_A$ or $(\; , \;)_S$ defined with respect to a fixed basis of the superalgebra by

$$(g_1^- g_2^- \ldots g_p^- \Lambda , f_1^- f_2^- \ldots f_q^- \Lambda)_A = x \qquad 3.12$$

if $\quad (g_p^-)^+ \ldots (g_2^-)^+ (g_1^-)^+ f_1^- f_2^- \ldots f_q^- \Lambda = x\Lambda$

and $\quad (g_1^- g_2^- \ldots g_p^- \Lambda , f_1^- f_2^- \ldots f_q^- \Lambda)_S = (-1)^{\gamma_1+\gamma_2+\ldots+\gamma_p} y \qquad 3.13$

if $\quad (g_1^- g_2^- \ldots g_p^-)^{\ddagger} f_1^- f_2^- \ldots f_q^- \Lambda = y\Lambda$

and zero otherwise (i.e. if the vectors have different weights). Here g_i^-, f_j^- are negative root vectors of degrees γ_i and η_j respectively and $(g_i g_j)^{\ddagger} = (-1)^{\gamma_i \gamma_j} g_j^{\ddagger} g_i^{\ddagger}$, and adjoints and superadjoints are given in Appendix A. A characterisation of a vector v which belongs to an invariant subspace is that its length (v,v) should vanish [9, exercise 20.9]; this criterion is applied to 'highest weight' vectors χ_j of the even subalgebra $O(M) \times Sp(N)$.

Given the ψ_j and the inner product, the first stage is to write down the χ_j by Schmidt orthogonalisation,

$$\chi_j = \psi_j - \sum_{\ell} C_\ell \, \phi_\ell \qquad 3.14$$

where the set ϕ_ℓ consists of all states of the form $\phi_\ell = (g_1^-)^{i_1} (g_2^-)^{i_2} \ldots (g_m^-)^{i_m} \chi_i$ such that the weight of ϕ_n equals the weight of ψ_j and not all of the i_j are zero. The coefficients C_ℓ can be determined by imposing the conditions $(\phi_\ell, \chi_j) = 0$ for all ϕ_ℓ. This gives

$$(\phi_m,\chi_j) = (\phi_m,\psi_j) - \sum_\ell (\phi_m,\phi_\ell)\, C_\ell$$

$$= y_m \quad - \quad (\Phi)_{m\ell}\, C_\ell \quad = \quad 0 \qquad 3.15$$

$$\Rightarrow \; y \quad = \quad \Phi C$$

$$\Rightarrow \; C \quad = \quad \Phi^{-1}\, y$$

or in components $C_\ell = (\Phi^{-1})_{\ell m}\, y_m$. 3.16

That this procedure ensures χ_j is a highest weight of the even subalgebra is proved in Appendix B. These coefficients are not dependent on whether the inner product is defined using an adjoint or a superadjoint operation. In practice since Φ will in general be block diagonal its inversion will not be as difficult as first appears. Despite this it is often easier to determine these coefficients by requiring χ_j to be a highest weight of the even subalgebra, i.e. requiring $\alpha_i^+\, \chi_j = 0$ for all positive, even, simple root vectors, α_i^+, leads to a set of simultaneous equations which can be solved for the C_ℓ.

The second stage is to evaluate the lengths (χ_j,χ_j) and identify atypicality conditions and invariant subspaces. If a degeneracy exists, in the sense that there is more than one χ_j of a given weight, then to determine whether the states of this weight belong to an invariant subspace mappings of the following form must be considered

$$(\beta_1^{\pm})^{k_i}\, (\beta_2^{\pm})^{k_2}\, ,\ldots\, (\beta_{\frac{1}{2}MN}^{\pm})^{k_{\frac{1}{2}MN}}\, \chi_k = \tilde{\chi}_j + \sum_\ell b_\ell\, \phi_\ell \qquad 3.17$$

where χ_k belongs to an invariant subspace. The $\tilde{\chi}_j$ will be some linear combination of the degenerate states and the b_ℓ are some coefficients. The linear dependence of the $\tilde{\chi}_j$'s under these mappings will tell us how many of the degenerate states belong to the invariant subspace.

The above construction shows that the whole representation can be made star or grade star. Indeed since the individual $(\chi_j,\chi_j)_A$ and $(\chi_j,\chi_j)_S$ differ at most by a sign, the crucial question is whether the representation is on a graded Hilbert space. In fact, we find no such finite-dimensional star representations for B(m,n) and D(m,n), but two classes for C(2), depending on how the adjoint is defined, in agreement with Scheunert et al. [7,8]. In the grade star case there exist two classes of finite-dimensional representations on a graded Hilbert space depending on how the adjoint is defined, as discussed in Appendix B. These representations are given for the cases studied in the following sections.

The result, that no finite-dimensional star representations exist for B(m,n), D(m,n) and D(2,1;α) can be easily demonstrated as follows. If $E^{\pm}$ designate the 'hidden' Sp(2n) generators defined in §3.2 and §3.6 then $[E^+,E^-] = -ak$, where a = -16 for B(m,n) and a = -4 for D(m,n) and D(2,1;α). For a given representation with highest weight vector Λ, let $k\Lambda = b\Lambda$. A finite-dimensional representation requires $b \geq 0$. Therefore if

$$(\Lambda,\ \Lambda)_A = (\Lambda\ \ \Lambda)_S = +1$$

then for star representations:

$$(E^-\Lambda,E^-\Lambda)_A = ((E^-)^+E^-\Lambda,\Lambda)_A = -ab$$

$$\Rightarrow \qquad (E^-\Lambda,E^-\Lambda)_A \leq 0 \quad \text{if} \quad b \geq 0.$$

However for grade star representations:

$$(E^-\Lambda,E^-\Lambda)_S = ((E^-)^{\neq}E^-\Lambda,\Lambda)_S = (-E^+E\Lambda,\Lambda) = ab$$

$$\Rightarrow \qquad (E^-\Lambda,E^-\Lambda)_S \geq 0 \quad \text{if} \quad b \geq 0.$$

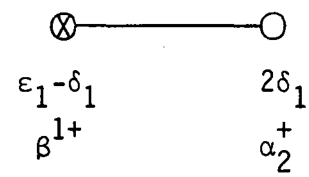
In the examples considered in the following sections we find that if in (3.14) $C_k^{-1} = 0$, then for the procedure to be consistent (3.14) must be written as

$$\chi_j' = \psi_j - \sum_{\substack{\ell \\ \ell\neq k}} C_\ell\ \phi_\ell\ . \qquad 3.18$$

It is found that although χ_j' is not a highest weight of the even subalgebra it is part of the infinite-dimensional invariant subspace and therefore does not appear in the finite dimensional factor space.
If the Kac-Dynkin labels have been chosen appropriately [2] so that Λ is the highest weight vector of a finite-dimensional factor space (so that supplementary conditions may apply), then $(\chi_j', \chi_j') = 0$. To determine the irreducible representations for these 'special' cases, it is necessary to examine explicity mappings from states in the invariant subspace to states χ_j for which $(\chi_j, \chi_j) \neq 0$.

3.4 <u>$C(2) \equiv OSp(2/2) \simeq A(1,0)$</u>

Dynkin diagram:



Cartan matrix: $(a_{ij}) = \begin{bmatrix} 0 & +1 \\ -1 & +2 \end{bmatrix}$

As discussed in §3.2 the odd generators are $\beta^{1\pm}$ and $\beta_2^{1\pm} = [\beta^{1\pm}, \alpha_2^{\pm}]$. The even generators corresponding to the even positive and negative simple roots are $\alpha_2^{\pm}$. The generators of the Cartan subalgebra are h_1 and h_2. The hidden O(2) generator is

$$k = 2h_1 - h_2 \; . \tag{3.19}$$

The complete algebra is given in Appendix A.

The highest weight vector of an OSp(2/2) representation will be designated by Λ, with weight components $(a_1, a_2;\ b = 2a_1 - a_2)$ where $h_i\Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$ and $k\Lambda = \lambda(k)\Lambda = b\Lambda$. Any OSp(2/2) representation can be uniquely decomposed in terms of $O(2) \times Sp(2)$ irreducible representations. In general we have four of these (see §3.3). The weight components of the $O(2) \times Sp(2)$ highest weight vectors are given below.

$$\begin{aligned}
\psi_1 &= \Lambda & &: (a_1, a_2;\ b) \\
\psi_2 &= \beta^{1-}\Lambda & &: (a_1, a_2+1;\ b-1) \\
\psi_3 &= \beta_2^{1-}\Lambda & &: (a_1-1, a_2-1;\ b-1) \\
\psi_4 &= \beta^{1-}\beta_2^{1-}\Lambda & &: (a_1-1, a_2;\ b-2)
\end{aligned} \tag{3.20}$$

Applying the procedure discussed in §3.3, we find the corresponding $O(2) \times Sp(2)$ highest weight vectors are given by the following:

$$\begin{aligned}
\chi_1 &= \psi_1 \\
\chi_2 &= \psi_2 \\
\chi_3 &= \psi_3 + \frac{1}{\alpha_2+1}\,\alpha_2^-\,\chi_2 \\
\chi_4 &= \psi_4
\end{aligned} \tag{3.21}$$

As discussed in §3.3, to find the conditions under which a state χ_i decouples from the highest weight we look for those conditions under which $(\chi_i, \chi_i) = 0$. The inner products of the above states are given by the following:

$$(\chi_1,\chi_1)_{A1,2} = (\chi_1,\chi_1)_{S1,2} = 1$$

$$(\chi_2,\chi_2)_{A1} = -(\chi_2,\chi_2)_{A2} = -(\chi_2,\chi_2)_{S1} = (\chi_2,\chi_2)_{S2} = +a_1$$

$$(\chi_3,\chi_3)_{A1} = -(\chi_3,\chi_3)_{A2} = -(\chi_3,\chi_3)_{S1} = (\chi_3,\chi_3)_{S2} = -a_2(a_2-a_1+1)/(a_2+1)$$

$$(\chi_4,\chi_4)_{A1,2} = -(\chi_4,\chi_4)_{S1,2} = -a_1(a_2-a_1+1) \tag{3.22}$$

It can be seen that under the conditions (i) $a_2 = 0$, and (ii) $a_2-a_1+1 = 0$ the $OSp(2/2)$ representation specified by the highest weight vector, Λ, is not irreducible and can be decomposed as shown in Table 3.1.

We require a_2 to be a non-negative integer for the representation to be finite dimensional.

Table 3.1

Atypicality condition	Factor space	Invariant subspace
$a_1 = 0$	χ_1, χ_3	χ_2, χ_4
$a_2-a_1+1 = 0$	χ_1, χ_2	χ_3, χ_4

From (3.22) it can be seen that the only finite-dimensional irreducible representations defined on a graded Hilbert space are the following.

Star representations:

A1: $\{\chi_1,\chi_2\ \chi_3,\chi_4\}$ if $b > a_2 + 2$, $\{\chi_1,\chi_2\}$ if $b = a_2 + 2$,

$\{\chi_1\}$ if $a_2 = b = 0$;

A2: $\{\chi_1,\chi_2,\chi_3,\chi_4\}$ if $b + a_2 < 0$, $\{\chi_1,\chi_3\}$ if $a_2 + b = 0$.

Grade star representations:*

S1: $\{\chi_1,\chi_3\}$ if $a_2 + b = 0$;

S2: $\{\chi_1,\chi_2,\chi_4\}$ if $a_2 = 0$ and $0 < b < 2$,

$\{\chi_1,\chi_2\}$ if $\frac{1}{2}a_2 - \frac{1}{2}b + 1 = 0$, $\{\chi_1\}$ if $a_2 = b = 0$.

These results are in agreement with those of Scheunert et al. [8] where the representation labels (b,q) correspond to $(\frac{1}{2}b-\frac{1}{2},\frac{1}{2}a_2+\frac{1}{2})$ in the present notation.

---

* The sets of grade star representations designated here as S1 and S2 have been determined using the convention that the grading of Λ is of degree zero. If the grading of Λ is chosen to be of degree 1, then S1 and S2 will simply interchange. This is also the case for the remaining results of this chapter.

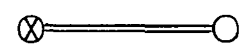
Taking C(2) simply as the n = 2 case of the general treatment of C(n) as given in Chapters 2 and 5 and [2] corresponds to taking the Cartan matrix as

$$(a_{ij}) = \begin{bmatrix} 0 & +2 \\ -1 & +2 \end{bmatrix}$$

With this, the value of the a_1 label in Chapters 2 and 5 and [2] will be twice the value of the a_1 label in this section.

3.5 B(1,1) ≡ OSp(3/2)

Dynkin diagram:



$\delta_1 - \varepsilon_1$ $\quad$ ε_2

β^{1+} $\quad$ α_2^+

Cartan matrix: $\quad (a_{ij}) = \begin{bmatrix} 0 & +1 \\ -2 & +2 \end{bmatrix}$

As discussed in §3.2 the odd generators are $\beta^{1\pm}$, $\beta_2^{1\pm} = [\beta^{1\pm}, \alpha_2^{\pm}]$ and $\tilde{\beta}_2^{1\pm} = [\beta_2^{1\pm}, \alpha_2^{\pm}]$. The even generators are $\alpha_2^{\pm}$ corresponding to the even positive and negative simple roots. The 'hidden' Sp(2) generators are given by $\{\beta_2^{1\pm}, \beta_2^{1\pm}\}$. The generators of the Cartan subalgebra are h_1 and h_2. The 'hidden' Cartan generator corresponding to the Sp(2) sector is given by

$$k = h_1 - \tfrac{1}{2}h_2 . \qquad 3.23$$

The complete algebra is given in Appendix A.

The highest weight vector of an OSp(3/2) representation will be designated by Λ, with weight components $(a_1, a_2;\ b = a_1 - \tfrac{1}{2}a_2)$, where $h_i \Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$, $k\Lambda = \lambda(k)\Lambda \equiv b\Lambda$. Any OSp(3/2) representation can be uniquely decomposed in terms of O(3) × Sp(2) irreducible representations. In general there will be eight of these (see §3.3) The weight components of the $O(3) \times Sp(2)$ highest weight vectors are given below:

$$
\begin{array}{lll}
\psi_1 = \Lambda & : & (a_1,a_2;\ b) \\
\psi_2 = \beta^{1-}\Lambda & : & (a_1,a_2+2;\ b-1) \\
\psi_3 = \beta_2^{1-}\Lambda & : & (a_1-1,a_2;\ b-1) \\
\psi_4 = \tilde{\beta}_2^{1-}\Lambda & : & (a_1-2,a_2-2;\ b-1) \\
\psi_5 = \beta^{1-}\beta_2^{1-}\Lambda & : & (a_1-1,a_2+2;\ b-2) \\
\psi_6 = \beta^{1-}\tilde{\beta}_2^{1-}\Lambda & : & (a_1-2,a_2;\ b-2) \\
\psi_7 = \beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda & : & (a_1-3,a_2-2;\ b-2) \\
\psi_8 = \beta^{1-}\beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda & : & (a_1-3,a_2;\ b-3)
\end{array}
\qquad 3.24
$$

Applying the procedure discussed in §3.3, we find the corresponding $O(3) \times Sp(2)$ highest weight vectors are given by the following:

$$\chi_1 = \psi_1 \qquad , \qquad \chi_2 = \psi_2$$

$$\chi_3 = \psi_3 + \frac{2}{a_2+2}\alpha_2^- \quad \chi_2$$

$$\chi_4 = \psi_4 + \frac{2}{a_2}\alpha_2^- \quad \chi_3 - \frac{2}{(a_2+1)(a_2+2)}\alpha_2^- \quad \alpha_2^- \quad \chi_2$$

$$\chi_5 = \psi_5$$

$$\chi_6 = \psi_6 + \frac{2}{(a_2+2)} \quad \alpha_2^- \quad \chi_5 + \frac{(a_2-a_1)}{(a_2-2a_1)}\{\beta_2^{1-},\beta_2^{1-}\}\ \chi_1$$

$$\chi_7 = \psi_7 + \frac{2}{a_2}\alpha_2^- \quad \chi_6 - \frac{2}{(a_2+1)(a_2+2)}\alpha_2^- \quad \alpha_2^- \quad \chi_5 - \frac{1}{(a_2-2a_1)}\alpha_2^- \quad \{\beta_2^{1-},\beta_2^{1-}\}\ \chi_1$$

$$\chi_8 = \psi_8 - \frac{(a_2-a_1+2)}{(a_2-2a_1+2)}\{\beta_2^{1-},\beta_2^{1-}\}\ \chi_3 + \frac{1}{(a_2+2)}\alpha_2^-\{\beta_2^{1-},\beta_2^{1-}\}\ \chi_2 \ . \qquad 3.25$$

As discussed in §3.3 the conditions for which $(\chi_i,\chi_i) = 0$ are the conditions for which χ_i decouples from the highest weight. The inner products of the above states are given below:

$$(\chi_1,\chi_1)_{S1} = (\chi_1,\chi_1)_{S2} = 1$$

$$(\chi_2,\chi_2)_{S1} = -(\chi_2,\chi_2)_{S2} = -a_1$$

$$(\chi_3,\chi_3)_{S1} = -(\chi_3,\chi_3)_{S2} = +a_2(a_2-2a_1+2)/(a_2+2)$$

$$(\chi_4,\chi_4)_{S1} = -(\chi_4,\chi_4)_{S2} = +4(a_2-1)(a_2-a_1+1)/(a_2+1) \ ; \quad a_2 \neq 0$$

$$(\chi_5,\chi_5)_{S1} = (\chi_5,\chi_5)_{S2} = +a_1(a_2-2a_1+2)$$

$$(\chi_6,\chi_6)_{S1} = (\chi_6,\chi_6)_{S2} = +4a_1a_2(a_2-a_1+1)(a_2-2a_1+2) \\ /[(a_2+2)(a_2-2a_1)] \ ; \quad b \neq 0$$

$$(\chi_7,\chi_7)_{S1} = (\chi_7,\chi_7)_{S2} = -4(a_2-1)(a_2-a_1+1)(a_2-2a_1+2) \\ /(a_2+1); \ a_2 \neq 0, \ b \neq 0.$$

$$(\chi_8,\chi_8)_{S1} = -(\chi_8,\chi_8)_{S2} = 4a_1(a_2-a_1+1)(a_2-2a_1+4) \ ; \ b \neq 1. \tag{3.26}$$

It can be seen that under the condition $(a_2-a_1+1) = 0$, the OSp(3/2) representation specified by the highest weight vector, Λ, is not irreducible and can be decomposed as shown in Table 3.2. As discussed in §3.3, if $b = 0,1$ or $a_2 = 0$, then (3.25) must be modified as per (3.18). If $b = 0$, then to obtain a finite-dimensional representation we must also impose the supplementary condition $a_2 = 0$ [2] and the representation is atypical. This gives the singlet, χ_1, as the only finite-dimensional irreducible representation. For the 'special' cases $a_2 = 0$ or $b = 1$, the only finite-dimensional irreducible representations occur as factor spaces. These are: $a_2 = 0$, $\{\chi_1,\chi_2,\chi_5,\chi_8\}$, the adjoint is obtained from this by setting $b = 2$; $b = 1$, $\{\chi_1,\chi_2,\chi_4\}$. If $a_2 = 0$ and $b = 1$, we obtain the fundamental $\{\chi_1,\chi_2\}$. The decompositions for all atypical, irreducible, finite-dimensional representations are given in Table 3.2. For the existence of a finite-dimensional representation, we require a_2 and b to be non-negative integers.

Table 3.2

Atypicality condition	Factor space	Invariant subspace
$a_1 = 0$	χ_1	
$a_2 - a_1 + 1 = 0$	$\chi_1, \chi_2, \chi_3, \chi_5$	$\chi_4, \chi_6, \chi_7, \chi_8$

From (3.26) and the above discussion we see that the only finite-dimensional, irreducible representations defined on a graded Hilbert space are the following grade star representations:

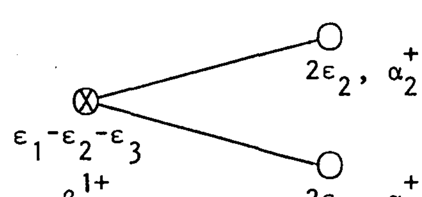
S1: $\{\chi_1\}$ if $a_2 = b = 0$

S2: $\{\chi_1\}$ if $a_2 = b = 0$

$\{\chi_1 \chi_2\}$ if $b = 1$, $a_2 = 0,1$.

3.6 D(2,1; α)

Dynkin diagram:



Cartan matrix: $$(a_{ij}) = \begin{bmatrix} 0 & +1 & \alpha \\ -1 & +2 & 0 \\ -1 & 0 & +2 \end{bmatrix}$$

As discussed in §3.2 the odd generators are $\beta^{1\pm}$, $\beta_2^{1\pm} = [\beta^{1\pm},\alpha_2^{\pm}]$, $\beta_3^{1\pm} = [\beta^{1\pm},\alpha_3^{\pm}]$ and $\tilde{\beta}_2^{1\pm} = [\beta_2^{1\pm},\alpha_3^{\pm}] = [\beta_3^{1\pm},\alpha_2^{\pm}]$. The even generators corresponding to the even positive and negative simple roots are $\alpha_2^{\pm}$, $\alpha_3^{\pm}$. The 'hidden' Sp(2) generators are given by $\{\beta_2^{1\pm},\beta_3^{1\pm}\}$. The generators of the Cartan subalgebra are h_1, h_2 and h_3. The 'hidden' Cartan generator is given by $k = \frac{1}{1+\alpha}(2h_1 - h_2 - \alpha h_3)$.

$$k = (2h_1 - h_2 - \alpha h_3)/(1+\alpha) \ . \qquad 3.27$$

The complete algebra is given in Appendix A.

The highest weight vector of a $D(2,1;\,\alpha)$ representation will be designated by Λ, with weight components $(a_1, a_2, a_3;\ b = \frac{1}{1+\alpha}(2a_1 - a_2 - \alpha a_3))$, where $h_i\Lambda = \lambda(h_i)\Lambda \equiv a_i\Lambda$ and $k\Lambda = \lambda(k)\Lambda \equiv b\Lambda$. Any $D(2,1;\,\alpha)$ representation can be uniquely decomposed in terms of $SU(2) \times SU(2) \times SU(2)$ irreducible representations. In general there will be sixteen of these (see §3.3). The weight components of the $SU(2) \times SU(2) \times SU(2)$ highest weight vectors are given below:

$$\begin{array}{lll}
\psi_1 = \Lambda & : (a_1, a_2, a_3;\ b) & \\
\psi_2 = \beta^{1-}\Lambda & : (a_1, a_2+1, a_3+1;\ b-1) & \\
\psi_3 = \beta_2^{1-}\Lambda & : (a_1-1, a_2-1, a_3+1;\ b-1) & \\
\psi_4 = \beta_3^{1-}\Lambda & : (a_1-\alpha, a_2+1, a_3-1;\ b-1) & \\
\psi_5 = \tilde{\beta}_2^{1-}\Lambda & : (a_1-1-\alpha, a_2-1, a_3-1;\ b-1) & \\
\psi_6 = \beta^{1-}\beta_2^{1-}\Lambda & : (a_1-1, a_2, a_3+2;\ b-2) & \\
\psi_7 = \beta^{1-}\beta_3^{1-}\Lambda & : (a_1-\alpha, a_2+2, a_3;\ b-2) & \\
\psi_8 = \beta^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-1-\alpha, a_2, a_3;\ b-2) & \\
\psi_9 = \beta_2^{1-}\beta_3^{1-}\Lambda & : (a_1-1-\alpha, a_2, a_3;\ b-2) & \\
\psi_{10} = \beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-2-\alpha, a_2-2, a_3;\ b-2) & \\
\psi_{11} = \beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-1-2\alpha, a_2, a_3-2;\ b-2) & \\
\psi_{12} = \beta^{1-}\beta_2^{1-}\beta_3^{1-}\Lambda & : (a_1-1-\alpha, a_2+1, a_3+1;\ b-3) & \\
\psi_{13} = \beta^{1-}\beta_2^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-2-\alpha, a_2-1, a_3+1;\ b-3) & \\
\psi_{14} = \beta^{1-}\beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-1-2\alpha,\ a_2+1, a_3-1;\ b-3) & \\
\psi_{15}\ \ \beta_2^{1-}\beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-2-2\alpha, a_2-1, a_3-1;\ b-3) & \\
\psi_{16}\ \ \beta^{1-}\beta_2^{1-}\beta_3^{1-}\tilde{\beta}_2^{1-}\Lambda & : (a_1-2-2\alpha, a_2, a_3;\ b-4) & 3.28
\end{array}$$

Applying the procedure discussed in §3.3, we find the corresponding SU(2) × SU(2) × SU(2) highest weight vectors are given by the following:

$$\chi_1 = \psi_1 \qquad , \chi_2 = \psi_2$$

$$\chi_3 = \psi_3 + \frac{1}{a_2+1}\alpha_2^- \chi_2$$

$$\chi_4 = \psi_4 + \frac{1}{a_3+1}\alpha_3^- \chi_2$$

$$\chi_5 = \psi_5 + \frac{1}{a_2+1}\alpha_2^- \chi_4 + \frac{1}{a_3+1}\alpha_3^- \chi_3 - \frac{1}{(a_2+1)(a_3+1)}\alpha_2^- \alpha_3^- \chi_2$$

$$\chi_6 = \psi_6$$

$$\chi_7 = \psi_7$$

$$\tilde{\chi}_8 = \psi_8 + \frac{1}{a_2+2}\alpha_2^- \chi_7 + \frac{1}{a_3+2}\alpha_3^- \chi_6 + \frac{a_2+\alpha a_3-a_1}{a_2+\alpha a_3-2a_1}\{\beta_2^{1-},\beta_3^{1-}\}\chi_1$$

$$\tilde{\chi}_9 = \psi_9 + \frac{1}{a_2+2}\alpha_2^- \chi_7 - \frac{1}{a_3+2}\alpha_3^- \chi_6 - \frac{\alpha a_3-a_1}{a_2+\alpha a_3-2a_1}\{\beta_2^{1-},\beta_3^{1-}\}\chi_1$$

$$\chi_{10} = \psi_{10} + \frac{1}{a_2}\alpha_2^- \tilde{\chi}_9 + \frac{1}{a_2}\alpha_2^- \tilde{\chi}_8 - \frac{1}{(a_2+1)(a_2+2)}\alpha_2^- \alpha_2^- \chi_7$$
$$- \frac{1}{a_2+\alpha a_3-2a_1}\alpha_2^- \{\beta_2^{1-},\beta_3^{1-}\}\chi_1$$

$$\chi_{11} = \psi_{11} - \frac{1}{a_3}\alpha_3^- \tilde{\chi}_9 + \frac{1}{a_3}\alpha_3^- \tilde{\chi}_8 - \frac{1}{(a_3+1)(a_3+2)}\alpha_3^- \alpha_3^- \chi_6$$
$$- \frac{\alpha}{a_2+\alpha a_3-2a_1}\alpha_3^- \{\beta_2^{1-},\beta_3^{1-}\}\chi_1$$

$$\chi_{12} = \psi_{12} - \frac{\alpha a_3-a_1+\alpha}{a_2+\alpha a_3-2a_1+1+\alpha}\{\beta_2^{1-},\beta_3^{1-}\}\chi_2$$

$$\chi_{13} = \psi_{13} + \frac{1}{a_2+1}\alpha_2^- \chi_{12} - \frac{a_2+\alpha a_3-a_1+1+\alpha}{a_2+\alpha a_3-2a_1+1+\alpha}\{\beta_2^{1-},\beta_3^{1-}\}\chi_3$$
$$+ \frac{\alpha a_3-a_1+\alpha}{(a_2+1)(a_2+\alpha a_3-2a_1+1+\alpha)}\alpha_2^- \{\beta_2^{1-},\beta_3^{1-}\}\chi_2$$

$$\chi_{14} = \psi_{14} - \frac{1}{a_3+1}\,\alpha_3^-\ \chi_{12} - \frac{a_2+\alpha a_3-a_1+1+\alpha}{a_2+\alpha a_3-2a_1+1+\alpha}\,\{\beta_2^{1-},\beta_3^{1-}\}\ \chi_4$$

$$+ \frac{a_2-a_1+1}{(a_3+1)(a_2+\alpha a_3-2a_1+1+\alpha)}\,\alpha_3^-\ \{\beta_2^{1-},\beta_3^{1-}\}\ \chi_2$$

$$\chi_{15} = \psi_{15} + \frac{1}{a_2+1}\,\alpha_2^-\ \chi_{14} - \frac{1}{a_3+1}\,\alpha_3^-\ \chi_{13} + \frac{1}{(a_2+1)(a_3+1)}\,\alpha_2^-\,\alpha_3^-\ \chi_{12}$$

$$- \frac{\alpha a_3-a_1+\alpha}{a_2+\alpha a_3-2a_1+1+\alpha}\,\{\beta_2^{1-},\beta_3^{1-}\}\ \chi_5 + \frac{a_2+\alpha a_3-a_1+1+\alpha}{(a_2+1)(a_2+\alpha a_3-2a_1+1+\alpha)}\,\alpha^-\ \{\beta_2^{1-},\beta_3^{1-}\}\ \chi_4$$

$$- \frac{a_1}{(a_3+1)(a_2+\alpha a_3-2a_1+1+\alpha)}\,\alpha_3^-\ \{\beta_2^{1-},\beta_3^{1-}\}\ \chi_3 - \frac{a_2-a_1+1}{(a_2+1)(a_3+1)(a_2+\alpha a_3-2a_1+1+\alpha)}$$

$$\alpha_2^-\,\alpha_3^-\ \{\beta_2^{1-},\beta_2^{1-}\}\ \chi_2$$

$$\chi_{16} = \psi_{16} + \frac{a_2+\alpha a_3-a_1+2+2\alpha}{a_2+\alpha a_3-2a_1+2+2\alpha}\,\{\beta_2^{1-},\beta_3^{1-}\}\ \tilde{\chi}_9$$

$$- \frac{\alpha a_3-a_1+2\alpha}{a_2+\alpha a_3-2a_1+2+2\alpha}\,\{\beta_2^{1-},\beta_3^{1-}\}\ \tilde{\chi}_8 + \frac{1}{a_3+2}\,\alpha_3^-\ \{\beta_2^{1-},\beta_3^{1-}\}\ \chi_6$$

$$+ \frac{(a_1^2+\alpha^2 a_3^2-a_1a_2-2\alpha a_1a_3+\alpha a_2a_3-a_1+\alpha(1+\alpha)a_3)}{(a_2+\alpha a_3-2a_1)(a_2+\alpha a_3-2a_1+1+\alpha)}\,\{\beta_2^{1-},\beta_3^{1-}\}\{\beta_2^{1-},\beta_3^{1-}\}\ \chi_1 \qquad 3.29$$

Examination of the above states reveals a degeneracy in the sense that ψ_8 and ψ_9 possess the same weight and the same eigenvalues with respect to the even subalgebra Casimin operators. Since the orthogonalization procedure we have used does not allow us to overcome this multiplicity problem, we have been obliged to determine the irreducible spaces to which the corresponding SU(2) × SU(2) × SU(2) highest weight vectors, χ_8 and χ_9, belong by mapping from states in the invariant subspace to linear combinations of $\tilde{\chi}_8$ and $\tilde{\chi}_9$. We can then determine from the nature of these linear combinations whether both, none or only one of χ_8 and χ_9 belong to the invariant subspace. The inner products, (χ_i,χ_i), of the remaining states are given below:

$$(\chi_1,\chi_1)_{S1} = (\chi_1,\chi_1)_{S2} = +1$$

$$(\chi_2,\chi_2)_{S1} = -(\chi_2,\chi_2)_{S2} = -a_1$$

$$(\chi_3,\chi_3)_{S1} = -(\chi_3,\chi_3)_{S2} = +a_2(a_2-a_1+1)/(a_2+1)$$

$$(\chi_4,\chi_4)_{S1} = -(\chi_4,\chi_4)_{S2} = +a_3(\alpha a_3-a_1+\alpha)/(a_3+1)$$

$$(\chi_5,\chi_5)_{S1} = -(\chi_5,\chi_5)_{S2} = +a_2a_3(a_2+\alpha a_3-a_1+1+\alpha)/[a_2+1)(a_3+1)]$$

$$(\chi_6,\chi_6)_{S1} = (\chi_6,\chi_6)_{S2} = +a_1(a_2-a_1+1)$$

$$(\chi_7,\chi_7)_{S1} = (\chi_7,\chi_7)_{S2} = +a_1(\alpha a_3-a_1+\alpha)$$

$$(\chi_{10},\chi_{10})_{S1} = (\chi_{10},\chi_{10})_{S2} = -(a_2-a_1+1)(a_2+\alpha a_3-a_1+1+\alpha)(a_2-1) \\ /(a_2+1) \ ; \quad a_2 \neq 0 \ , \ b \neq 0.$$

$$(\chi_{11},\chi_{11})_{S1} = (\chi_{11},\chi_{11})_{S2} = -(\alpha a_3-a_1+\alpha)(a_2+\alpha a_3-a_1+1+\alpha)(a_3-1) \\ /(a_3+1) \ ; \quad a_3 \neq 0 \ , \ b \neq 0 \ .$$

$$(\chi_{12},\chi_{12})_{S1} = -(\chi_{12},\chi_{12})_{S2} = a_1(a_2-a_1+1)(\alpha a_3-a_1+\alpha) \\ (a_2+\alpha a_3-2a_1+2+2\alpha)/(a_2+\alpha a_3-2a_1+1+\alpha) \ ; \ b \neq 1.$$

$$(\chi_{13},\chi_{13})_{S1} = -(\chi_{13},\chi_{13})_{S2} = a_1a_2(a_2-a_1+1)(a_2+\alpha a_3-2a_1+2+2\alpha) \\ (a_2+\alpha a_3-a_1+1+\alpha)/(a_2+1)(a_2+\alpha a_3-2a_1+1+\alpha) \ ; \ b \neq 1.$$

$$(\chi_{14},\chi_{14})_{S1} = -(\chi_{14},\chi_{14})_{S2} = a_1a_3(\alpha a_3-a_1+1)(a_2+\alpha a_3-2a_1+2+2\alpha) \\ (a_2+\alpha a_3-a_1+1+\alpha)/(a_3+1)(a_2+\alpha a_3-2a_1+1+\alpha) \ ; \ b \neq 1.$$

$$(\chi_{15},\chi_{15})_{S1} = -(\chi_{15},\chi_{15})_{S2} = -a_2a_3(a_2-a_1+1)(\alpha a_3-a_1+\alpha)$$
$$(a_2+\alpha a_3-a_1+1+\alpha)(a_2+\alpha a_3-2a_1+2+2\alpha)/$$
$$(a_2+1)(a_3+1)(a_2+\alpha a_3-2a_1+1+\alpha) \;;\quad b \neq 1.$$

$$(\chi_{16},\chi_{16})_{S1} = (\chi_{16},\chi_{16})_{S2} = -a_1(a_2-a_1+1)(\alpha a_3-a_1+\alpha)$$
$$(a_2+\alpha a_3-a_1+1+\alpha)(a_2+\alpha a_3-2a_1+3+3\alpha)/(a_2+\alpha a_3-2a_1+1+\alpha) \;;\quad b \neq 0,2.$$

3.30

It can be seen that under the conditions (i) $a_1 = 0$, (ii) $a_2-a_1+1 = 0$, (iii) $\alpha a_3-a_1+\alpha = 0$ and (iv) $a_2+\alpha a_3-a_1+1+\alpha = 0$, the OSp(4/2) representation specified by the highest weight vector, Λ, is not irreducible and can be decomposed as shown in Table 3.3. As discussed in §3.3, if $b = 0,1,2$ or $a_2 = 0$ or $a_3 = 0$, then (3.29) must be modified as per (3.18). If $b = 0$, then to obtain a finite-dimensional representation the supplementary conditions $a_2 = a_3 = 0$ must be imposed [2]. This gives the singlet, χ_1, as the only finite-dimensional irreducible representation. Similarly, if $b = 1$, then either of the supplementary conditions C_+: $(a_2+1) = \alpha(a_3+1)$ or C_-: $(a_2+1) = -\alpha(a_3+1)$ must be imposed. If C_+ is taken the only finite-dimensional, irreducible representation consists of $\{\chi_1,\chi_2,\chi_5\}$. If C_- is imposed the only finite-dimensional, irreducible representation consists of $\{\chi_1,\chi_3,\chi_4\}$. Other 'special' cases are: if $b = 2$ or $a_2 = 0$ or $a_3 = 0$, then one of χ_8 or χ_9 is part of the infinite-dimensional subspace; if $a_2 = a_3 = 0$ or $a_2 = 0$ and $b = 2$ or $a_3 = 0$ and $b = 2$, then *both* χ_8 and χ_9 belong to the infinite-dimensional subspace. For the following atypical representations Table 3.3 must be modified to include *both* χ_8 and χ_9 in the invariant subspace: if condition (ii) above and $a_3 = 0$ or condition (iii) and $a_2 = 0$ or condition (iv) and $a_2 = a_3$ are imposed.

If $b = 1$ and $a_2 = a_3 = 0$ the fundamental $\{\chi_1, \chi_2\}$ is obtained. If $b = 2$ and $a_2 = a_3 = 0$ the adjoint $\{\chi_1, \chi_2, \chi_6, \chi_7\}$ is obtained. Table 3.3 contains the decompositions for all atypical, finite-dimensional, irreducible representations. For the existence of a finite-dimensional representation a_2, a_3 and b are required to be non-negative integers.

Table 3.3

Atypicality condition	Factor space	Invariant subspace
$a_1 = 0$	$\chi_1, \chi_3, \chi_4, \chi_5, \chi_{10}, \chi_{11}, \chi_{15}$	$\chi_2, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{16}$
$a_2 - a_1 + 1 = 0$	$\chi_1, \chi_2, \chi_4, \chi_5, \chi_7, \chi_8, \chi_{11}, \chi_{14}$	$\chi_3, \chi_6, \chi_9, \chi_{10}, \chi_{12}, \chi_{13}, \chi_{15}, \chi_{16}$
$\alpha a_3 - a_1 + \alpha = 0$	$\chi_1, \chi_2, \chi_3, \chi_5, \chi_6, \chi_8, \chi_{10}, \chi_{13}$	$\chi_4, \chi_7, \chi_9, \chi_{11}, \chi_{12}, \chi_{14}, \chi_{15}, \chi_{16}$
$a_2 + \alpha a_3 - a_1 + 1 + \alpha = 0$	$\chi_1, \chi_2, \chi_3, \chi_4, \chi_6, \chi_7, \chi_8, \chi_{12}$	$\chi_5, \chi_9, \chi_{10}, \chi_{11}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}$

From an analysis of (3.30) and considering the above discussion it is observed that the only finite-dimensional, irreducible representations defined on a graded Hilbert space are the following grade star representations with highest weight vectors of the even subalgebra written $\chi_i(a_2, a_3, b)$

S1: $\{\chi_1(0,0,0)\}$

$\{\chi_1(1,0,-\frac{1}{\alpha+1}),\ \chi_3(0,1,-\frac{\alpha+2}{\alpha+1})\}$

$\{\chi_1(-\alpha-1,0,1),\ \chi_3(-\alpha-2,1,0)\}$

S2: $\{\chi_1(0,0,0)\}$

$\{\chi_1(0,1,-\frac{\alpha}{\alpha+1}),\ \chi_4(1,0,-\frac{2\alpha+1}{\alpha+1})\}$

$\{\chi_1(0,\ -\frac{\alpha+1}{\alpha},\ 1),\ \ \chi_4(1,\ -\frac{2\alpha+1}{\alpha}\ ,\ 0)\}$

$\{\chi_1(0,\ \frac{1-\alpha}{\alpha},\ 1),\ \ \chi_2(1,\ \frac{1}{\alpha}\ ,\ 0)\}$

$\{\chi_1(\alpha-1,0,1),\ \ \chi_2(\alpha,1,0)\}$

For the above representations to be finite-dimensional α must be chosen such that for $\chi_1(a_2,a_3,b)$ each of a_2,a_3 and b must be a non-negative integer.

CHAPTER 3 - REFERENCES

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4. REPRESENTATIONS OF ORTHOSYMPLECTIC SUPERALGEBRAS: SUPERFIELD TECHNIQUES

4.1 CONSTRUCTION OF INDUCED REPRESENTATIONS

The technique of induced representations for finding irreducible representations of a group is a well established procedure in group theory [1,2]. There is a large class of groups which has irreducible representations which can be written as induced representations. For example Mackey [1] has shown that for the class of groups having invariant subgroups all unitary irreducible representations can be written as induced representations. The application of induced representations to supergroups, G, involves the construction of functions (superfields), Φ, defined on graded coset spaces, G/H, and taking values in a representation space, V, of the subgroup H of G. Application of these techniques to supergroups was first made by Salam and Strathdee [3] who considered the graded Poincare group. Subsequently much work has been done on superfield formulations of supersymmetry and supergravity (see van Nieuwenhuizen [4] for a review). The use of induced representations to determine finite-dimensional irreducible representations of simple graded Lie algebras was begun by Dondi and Jarvis [5,6] who considered SU(m/1). Applications to orthosymplectic superalgebras have been made by Farmer and Jarvis [7] and it is principally these results which are reported here.

The procedure elucidated here was proposed by P.D.Jarvis and is, conceptually, a graded extension of a technique pioneered by Bargmann [11]. Bargmann considered the application of function spaces R^2, being homogeneous polynomials in two complex variables, to the study of the rotation group. This is a realization of a more abstract work by

Schwinger [12] in which he introduces certain operators a_ζ, a_ζ^+ which act as creation and annihilation operators of boson fields. The orthonormal vector basis of the Hilbert space on which the operators a_ζ act is then defined in terms of the a_ζ themselves. In Bargmann's approach the Hilbert space is given a priori as a function space, R^2, while the creation and annihilation operators are realized as operators in R^2 and consequently the representations are directly defined on the function space. These boson operator techniques were used to construct explicit states of irreducible representations of the unitary groups by Baird and Biedenharn [13] and later extended to the orthogonal and symplectic groups by Lohe and Hurst [14,15] and Zhelobenko [16]. This brings us to expound the method used here as applied to supergroups.

Consider a supergroup G and subgroup H, with corresponding superalgebras $\mathcal{G}$ and $\mathcal{H}$. Representations of $\mathcal{G}$ are afforded by functions Φ on coset spaces G/H and taking their values in a representation space V of $\mathcal{H}$. If x and y are coset representatives of G/H, then for $g \in G$ the group action in an appropriate basis for V is

$$(g\ \Phi)_a\ (x) = \hat{h}_a^b\ \Phi_b(y) \qquad 4.1$$

where y is such that $g \cdot x = yh^{-1}$, $h \in H$ and $\hat{h}_a^b$ is the matrix representing h in the chosen basis for V.

The coset space G/H is the space of orbits that the subgroup H sweeps out in G. One can choose an origin in this space and coordinatize its neighbourhood by exponentiating the coordinates in the tangent space at that point; i.e. a point in the coset space can be written as $\exp\Sigma(xX + \theta Q)$, where X and Q are generic even and odd elements of $\mathcal{G}/\mathcal{H}$ and x and θ are c-number and a-number parameters respectively.

If now $S(R)$ is an odd (even) element of $\mathcal{H}$ and $\eta(y)$ is an a-(c-) number parameter then the group action on G/H is infinitessimally

$$\exp(\eta s)\ \exp\Sigma(xX + \theta Q) = \exp\Sigma[(x + \eta\ \theta\ f_1(x,\theta^2))X + (\theta + \eta\ g_1(x,\theta^2))Q]\ \exp\Sigma(\eta\ k_1(x,\theta)K) \qquad 4.2$$

$$\exp(yR)\ \exp\Sigma(xX + \theta Q) = \exp\Sigma[(x + y\ f_2(x,\theta^2))X + (\theta + y\ \theta\ g_2(x,\theta^2))Q]\ \exp\Sigma(y\ k_2(x,\theta)K) \qquad 4.3$$

where $K \in \mathcal{H}$. The particular basis chosen will determine the precise form of the functions f,g and k and for an appropriate $\mathcal{H}$ they may be restricted to polynomials of low degree which can be obtained directly via BCH formula. From (4.1), (4.2) and (4.3) it can be seen that the group action induces a motion in the parameter space. This motion may be generated by differential operators

$$S \to \Sigma[f_1(x,\theta^2)\ \theta\partial/\partial x + g_1(x,\theta^2)\ \partial/\partial\theta - k_1(x,\theta)\hat{K} \qquad 4.4$$

$$R \to \Sigma[f_2(x,\theta)^2\ \partial/\partial x + g_2(x,\theta^2)\ \theta\partial/\partial\theta - k_2(x,\theta)\hat{K} \qquad 4.5$$

where $\hat{K}$ is the matrix of the infinitessimal generator K in the representation space V. Often it will be possible to decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_+$, where $\mathcal{H}_+$ is an ideal ($[\mathcal{H}, \mathcal{H}_+] \subset \mathcal{H}_+$). Representations of $\mathcal{H}_0$ are then easily extended to $\mathcal{H}$ by taking them to be zero on $\mathcal{H}_+$.

The action on superfields corresponding to (4.2) and (4.3) is given by

$$\delta_S\ \Phi(x,\theta) = S\ \Phi(x,\theta) \qquad \text{and} \qquad \delta_R\ \Phi(x,\theta) = R\ \Phi(x,\theta)$$

respectively. The representation obtained by expanding as power series in x and polynomially in θ is in general infinite dimensional, but possessing a finite-dimensional factor related to the choice of V.

As discussed in chapter 2, Kac [8] has argued that all irreducible representations can be obtained by choosing $\mathcal{H}$ as a Borel subalgebra, $\mathcal{H}_0$ the Cartan subalgebra and V one-dimensional. However, in general, this leads to large dimensional coset spaces making the algebra prohibitively complex. Consequently, in general, $\mathcal{H}$ will be chosen larger than the Borel subalgebra and thus V greater than one-dimensional.

The form of the algebra used in this chapter will not be that of chapters 2 and 3 but rather the covariant form given by Jarvis and Green [9]. Here the OSp(m/n) generators are $M_{AB} = -[AB]\, M_{BA}$, $1 \le A,B \le m+n$. These consist of the O(m) generators $M_{ab} = M_{ba}$, $1 \le a,b \le m$, the Sp(n) generators $M_{\alpha\beta} = M_{\beta\alpha}$, $1 \le \alpha,\beta \le n$, and the odd generators $M_{a\alpha} = M_{\alpha a}$. The generators satisfy the superalgebra

$$[M_{AB},M_{CD}] = g_{BC}\, M_{AD} - [AB]\, g_{AC}\, M_{BD} - [CD]\, g_{BD}\, M_{AC} + [AB][CD]\, g_{AD}\, M_{BC} \tag{4.6}$$

where $g_{AB} = [AB]\, g_{BA}$ is the orthosymplectic metric and the sign factors [AB] are +1 if $1 \le A,B \le m$ or $1 \le A \le m$, $m+1 \le B \le m+n$ (or vice versa) and -1 if $m+1 \le A,B \le m+n$. The metric is taken as follows

$$g_{ab} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & m \text{ even} \\[2ex] \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & m \text{ odd} \end{cases} \qquad g_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.7}$$

$$g_{\alpha a} = g_{a\alpha} = 0.$$

In the following sections Sp(1/2), OSp(2/2), OSp(3/2) and OSp(4/2) are examined. In each case the superfield transforms as an *arbitrary, irreducible* representation of the chosen little group. Full decompositions with respect to the even subalgebra for typical and atypical representations are derived.

4.2 OSp(1/2):

In the notation of (4.6) the OSp(1/2) superalgebra consists of the even Sp(2) generators $M_{\alpha\beta}$ and the odd generators $M_{1\alpha}$ where $1 \le \alpha,\beta \le 2$. The odd generators will be writeen $M_{1\alpha} \equiv Q_\alpha$ and the $M_{\alpha\beta}$ will be transformed to the spherical basis M_+, M_- and M_3 via $M_{\alpha\beta} = 2\underset{\sim}{M}\cdot(\underset{\sim}{\sigma}\varepsilon)_{\alpha\beta}$ where $\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices and $(\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or $M_+ = \frac{1}{2}M_{22}$, $M_- = -\frac{1}{2}M_{22}$ and $M_3 = \frac{1}{2}M_{12}$. With the generators in this form the superalgebra becomes:

$$[M_3,Q_\alpha] = -\tfrac{1}{2}(\sigma_3)_\alpha{}^\beta\, Q_\beta \qquad [M_\pm,Q_\alpha] = -(\sigma_\pm)_\alpha{}^\beta\, Q_\beta$$

$$[M_+,M_-] = 2M_3 \qquad [M_3,M_\pm] = \pm M_\pm$$

$$\{Q_\alpha,Q_\beta\} = -2(\sigma_+\varepsilon)_{\alpha\beta}\, M_- - 2(\sigma_-\varepsilon)_{\alpha\beta}\, M_+ - 2(\sigma_3\varepsilon)_{\alpha\beta}\, M_3 \qquad 4.8$$

with all other (anti-) commutators zero.

The subalgebra $\mathcal{H}$ will be taken as $\mathcal{H} = \{M_3,M_+,Q_2\}$ with $\mathcal{H}_0 = \{M_3\} \simeq U(1)$. The cosets are labelled by the elements $\exp(xM_- + \theta Q_1)$ and the superfields are functions $\Phi(x,\theta)$ carrying a charge $\hat{M} \equiv -M$. Expanding the superfield in θ gives simply

$$\Phi(x,\theta) = A(x) + \theta\, \psi(x) \qquad 4.9$$

The differential representation of these generators (see 4.4 and 4.5) is

$$\begin{aligned}
M_- &= \partial/\partial x \\
M_+ &= -x^2\,\partial/\partial x - x\theta\,\partial/\partial\theta + 2xM \\
M_3 &= -x\,\partial/\partial x - \tfrac{1}{2}\theta\,\partial/\partial\theta + M \\
Q_1 &= -\theta\,\partial/\partial x + \partial/\partial\theta \\
Q_2 &= -\theta x\,\partial/\partial x + x\,\partial/\partial\theta + 2\theta M
\end{aligned} \qquad 4.10$$

Acting on the superfield with the above set of generators yields the following variations for the component fields, writing $A' \equiv \partial A/\partial x$, etc.,

$$
\begin{array}{llll}
M_- : & \delta A = A' & \delta\psi = \psi' & \\
M_+ : & \delta A = -x^2A' - 2MxA & \delta\psi = -x^2\psi' - x\psi - 2Mx\psi & \\
M_3 : & \delta A = -xA' - MA & \delta\psi = -x\psi' - \frac{1}{2}\psi - M\psi & \\
Q_1 : & \delta A = \psi & \delta\psi = -A' & \\
Q_2 : & \delta A = x\psi & \delta\psi = -xA' - 2MA & 4.11
\end{array}
$$

Now expand A(x) and ψ(x) as power series in x:

$$
A(x) = \sum_{n=0}^{\infty} A^n x^n \qquad \text{and} \qquad \psi(x) = \sum_{n=0}^{\infty} \psi^n x^n . \qquad 4.12
$$

Substituting these into (4.11) and equating like powers of x gives the following results

$$
\begin{array}{llll}
M_- : & \delta A^n = (n+1)A^{n+1} & \delta\psi^n = (n+1)\psi^{n+1} & \\
M_+ : & \delta A^n = -(n-1-2M)A^{n-1},\ n \geq 1 & \delta\psi^n = -(n-2M)\psi^{n-1},\ n \geq 1 & \\
M_3 : & \delta A^n = -(n-M)A^n & \delta\psi^n = -(n+\frac{1}{2}-M)\psi^n & \\
Q_1 : & \delta A^n = \psi^n,\ n \leq 2M-1 & \delta\psi^n = -(n+1)A^{n+1} & \\
Q_2 : & \delta A^n = \psi^{n-1},\ n \geq 1 & \delta\psi^n = -(n-2M)A^n,\quad n \geq 1 & \\
 & & & 4.13
\end{array}
$$

with all other variations zero.

If M is taken as half-integral, then it is clear from the explicit component form of the variations (4.13), especially M_+ and Q_α, that the infinite set $\{A^0,A^1,\ldots;\ \psi^0,\psi^1,\ldots\}$ has an infinite invariant subset $\{A^{2M+1},A^{2M+2},\ldots;\ \psi^{2M}\ \psi^{2M+1},\ldots\}$. If these components are set to zero, then the remaining finite subset $\{A^0,A^1,\ldots,A^{2M};\ \psi^0,\psi^1,\ldots,\psi^{2M-1}\}$ is invariant (i.e. as a factor space).

Thus an arbitrary finite-dimensional irreducible representation of OSp(1/2) has dimension 4M+1 and 'superspin' M[10]. The superspin is with respect to the second order Casimir invariant, C_2, (which for OSp(1/2) is the only independent Casimir invariant) acting on the superfield $\Phi(x,\theta)$ with eigenvalue $M(M+\frac{1}{2})$, where

$$C_2 = M_-M_+ + M_3^2 + M_3 - \frac{1}{4}\varepsilon_{\alpha\beta}\, Q_\alpha\, Q_\beta .$$

The component fields A and ψ have spins M and $M-\frac{1}{2}$, respectively under Sp(2). The matrix elements acquire a more symmetrical form in the basis defined by $B^\mu = A^{\mu+M}, \chi^\nu = \psi^{\nu+M-\frac{1}{2}}$ where $\mu = -M,-M+1,\ldots,M$ and $\nu = -M+\frac{1}{2}$, $-M+\frac{3}{2}$, ..., $M-\frac{1}{2}$:

$$\begin{array}{llll} M_3 & : \ \delta B^\mu = -\mu B^\mu & \delta\chi^\nu = -\nu\chi^\nu & \\ M_+,M_- & : \ \delta B^\mu = (M+1\mp\mu)B^{\mu\mp 1} & \delta\chi^\nu = (M+\frac{1}{2}\mp\nu)\chi^{\nu\mp 1} & \\ Q_1,Q_2 & : \ \delta B^\mu = \chi^{\mu\mp\frac{1}{2}} & \delta\chi^\nu = -(\nu\pm M\pm\frac{1}{2})B^{\nu-\frac{1}{2}\pm 1} & 4.14 \end{array}$$

where $B^{\pm(M+1)} \equiv \chi^{\pm(M+\frac{1}{2})} \equiv 0$. An alternative form for these matrices is given in Appendix E in terms of spin projection operators (see, for example, E1, E7, E8); it is in this form that they are required for OSp(3/2) as treated in §4.4.

4.3 OSp(2/2):

In the notation of (4.6) the OSp(2/2) superalgebra consists of the odd generators $Q_{a\alpha} \equiv M_{a\alpha}$, the O(2) generator $L_{ab} \equiv M_{ab}$ and the Sp(2) generators $M_{\alpha\beta}$. Here $1 \le a,b \le 2$ refers to O(2) and $1 \le \alpha,\beta \le 2$ refers to Sp(2). The Sp(2) generators are again written in the spherical M_+,M_-,M_3 basis as in §4.2. These generators satisfy the superalgebra

$$\begin{array}{llll} [L_{ab}, Q_{c\alpha}] = -\delta_{ac}\, Q_{b\alpha} + \delta_{bc}\, Q_{a\alpha} & & & \\ [M_3\, , Q_{a\alpha}] = (-1)^\alpha \frac{1}{2} Q_{a\alpha} & & & \\ [M_+\, , Q_{a1}] = -Q_{a2} & [M_-, Q_{a2}] = -Q_{a1} & & \\ [M_+\, , M_-] = 2M_3 & [M_3, M_\pm] = \pm M_\pm & & \\ \{Q_{a1}, Q_{b1}\} = 2\delta_{ab}M_- & \{Q_{a2}, Q_{b2}\} = -2\delta_{ab}M_+ & & \\ \{Q_{a1}, Q_{b2}\} = -2\delta_{ab}M_3 - L_{ab} & & & 4.15 \end{array}$$

with all other (anti) commutators zero.

The subalgebra $\mathcal{H}$ will be taken as $\mathcal{H} = \{L_{ab}, M_+, M_3, Q_{a2}\}$ with $\mathcal{H}_0 = \{L_{ab}, M_3\} \simeq U(1) \times U(1)$. The cosets are labelled by the elements $\exp(xM_- + \theta^a Q_{a1})$. Superfields are functions $\Phi(x, \theta^a)$ which form a representation of the $U(1) \times U(1)$ little group carrying changes $\hat{M} = -M$ and $\hat{L}_{ab} = -\varepsilon_{ab}\, iL$. Expanding the superfield in θ^a yields

$$\Phi(x, \theta^a) = A(x) + \theta^a\, \psi_a(x) + \tfrac{1}{2}\, \theta^a\, \theta^b\, \varepsilon_{ab}\, H(x). \qquad 4.16$$

Note that indices can be lowered or raised using the $O(2)$ metric δ_{ab} or inverse metric δ^{ab} respectively and $\varepsilon_{12} = \varepsilon^{12} = +1$.

Following (4.4, 4.5) the differential form of the generators, writing $\partial_a \equiv \partial/\partial\theta^a$, is

$$\begin{aligned}
L_{ab} &= \theta^b \partial_a - \theta^a \partial_b + \varepsilon_{ab}\, iL \\
M_- &= \partial/\partial x \\
M_+ &= -x^2\, \partial/\partial x - x\theta^a \partial_a + 2xM + \tfrac{1}{2}\, \theta^a\, \theta^b\, \varepsilon_{ab}\, iL \\
M_3 &= -x\, \partial/\partial x - \tfrac{1}{2}\, \theta^a \partial_a + M \\
Q_{a1} &= -\theta^a\, \partial/\partial x + \partial_a \\
Q_{a2} &= -\theta^a\, x\partial/\partial x + x\partial_a - \theta^a\, \theta^b\, \partial_b + 2\theta^a M - \theta^b\, \varepsilon_{ab}\, iL \qquad 4.17
\end{aligned}$$

The following field redefinitions are introduced so that the components of the superfield transform as eigenvectors of the $O(2) \times Sp(2)$ even subalgebra.

$$\tilde{H} = H - \frac{iL}{2M} A', \quad M \neq 0; \qquad \psi_+ = \psi_1 + i\, \psi_2; \quad \psi_- = \psi_1 - i\, \psi_2 .$$

Following the procedure of §4.1, $A(x)$, $\psi_\pm(x)$ and $\tilde{H}(x)$ are expanded as power series in x. Examining the transformations of the components A^n, $\psi_\pm{}^n$ and $\tilde{H}^n$ under the above generators shows that the infinite set $\{A^0, A^1, \ldots;\ \psi_\pm{}^0\ \ \psi_\pm{}^1, \ldots;\ \tilde{H}^0, \tilde{H}^1, \ldots\}$ decomposes into an infinite dimensional invariant subset $\{A^{2M+1},\ A^{2M+2}, \ldots;\ \psi_\pm{}^{2M}, \psi^{2M+1}, \ldots;$

$\tilde{H}^{2M-1}$, $\tilde{H}^{2M}$,...} and a finite dimensional factor space $\{A^0, A^1, \ldots A^{2M};$ $\psi_\pm{}^0, \psi_\pm{}^1, \ldots, \psi_\pm{}^{2M-1};\ \tilde{H}^0, \tilde{H}^0, \ldots\ \tilde{H}^{2M-2}\}$.

Thus in general an arbitrary finite-dimensional representation of OSp(2/2) has dimension 8M and the component fields A, ψ_+, ψ_- and $\tilde{H}$ have spins M, M + $\frac{1}{2}$, M - $\frac{1}{2}$ and M - 1 respectively under Sp(2) and charges iL, i(L+1), i(L-1) and il respectively under O(2).

These representations may be atypical (see Chapters 2 and 3) and thus reducible. To determine the conditions under which this may occur the transformations of the component fields under the odd generators are examined. This yields the following results

$$
\begin{aligned}
Q_{11} :\ \delta A &= \frac{1}{2}\,\psi_+' + \frac{1}{2}\,\psi_-' \\
\delta\psi_\pm &= \pm i\tilde{H} - \left(1 \pm \frac{L}{2M}\right)A' \\
\delta\tilde{H} &= \frac{1}{2}\,i\,\left(1 - \frac{L}{2M}\right)\psi_+' - \frac{1}{2}\,i\,\left(1 + \frac{L}{2M}\right)\psi_-'
\end{aligned}
\qquad 4.18
$$

$$
\begin{aligned}
Q_{21} :\ \delta A &= -\frac{1}{2}\,i\,\psi_+' + \frac{1}{2}\,i\,\psi_-' \\
\delta\psi_\pm &= -\tilde{H} \mp \left(1 \pm \frac{L}{2M}\right)A' \\
\delta\tilde{H} &= \frac{1}{2}\left(1 - \frac{L}{2M}\right)\psi_+' + \frac{1}{2}\left(1 + \frac{L}{2M}\right)\psi_-'
\end{aligned}
\qquad 4.19
$$

$$
\begin{aligned}
Q_{12} :\ \delta A &= \frac{1}{2}\,x\,\psi_+ + \frac{1}{2}\,x\,\psi_- \\
\delta\psi_\pm &= \pm\,i\,x\,\tilde{H} - \left(1 \pm \frac{L}{2M}\right)(xA' - 2MA) \\
\delta\tilde{H} &= \frac{1}{2}\,i\,\left(1 - \frac{L}{2M}\right)(x\,\psi_+' - 2M\,\psi_+ + \psi_+) \\
&\quad - \frac{1}{2}\,i\,\left(1 + \frac{L}{2M}\right)(x\,\psi_-' - 2M\,\psi_- - \psi_-)
\end{aligned}
\qquad 4.20
$$

$$Q_{22} : \quad \delta A = -\frac{1}{2} x \psi_+ + \frac{1}{2} x \psi_-$$

$$\delta\psi_\pm = - x \tilde{H} \mp i \left(1 \pm \frac{L}{2M}\right)(x A' \mp 2MA)$$

$$\delta\tilde{H} = \frac{1}{2}\left(1 - \frac{L}{2M}\right)(x \psi_+' - 2M \psi_+ + \psi_+)$$

$$+ \frac{1}{2}\left(1 + \frac{L}{2M}\right)(x \psi_-' - 2M \psi_- + \psi_-) \qquad 4.21$$

It becomes apparent from these results that if $L = \pm 2M$ the set $\{\psi_\mp, \tilde{H}\}$ form an invariant subspace of dimension (4M-1) with the set $\{A, \psi_\pm\}$ invariant as a factor space of dimension (4M+1).

The irreducible representations obtained here are in agreement with those of §3.4 where the label correspondence is $a_2 = 2M - 1$ and $b = L + 1$.

4.4 OSp(3/2):

The OSp(3/2) superalgebra consists of the odd generators $Q_{a\alpha} \equiv M_{a\alpha}$, the O(3) generators $L_{ab} = M_{ab}$ and the Sp(2) generators $M_{\alpha\beta}$. Here $1 \le a, b \le 3$ refer to O(3) and $1 \le \alpha, \beta \le 2$ refer to Sp(2). These generators can be recast in the form

$$L_+ = L_{31} + iL_{32} \qquad L_- = -L_{31} + iL_{32} \qquad L_3 = iL_{21}$$

$$Q_{+\alpha} = Q_{1\alpha} + iQ_{2\alpha} \qquad Q_{-\alpha} = Q_{1\alpha} - iQ_{2\alpha} \qquad Q_{3\alpha} = Q_{3\alpha} \ .$$

In this form the generators satisfy the following superalgebra

$$[L_+, L_-] = 2L_3 \qquad [L_3, L_\pm] = \pm L_\pm$$

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \varepsilon_{\beta\gamma} M_{\alpha\delta} + \varepsilon_{\alpha\delta} M_{\beta\gamma} + \varepsilon_{\alpha\gamma} M_{\beta\delta} + \varepsilon_{\beta\delta} M_{\alpha\gamma}$$

$$[L_\pm, Q_{\mp\alpha}] = \pm 2Q_{3\alpha} \qquad [L_\pm, Q_{3\alpha}] = \mp Q_{\pm\alpha}$$

$$[L_3, Q_{\pm\alpha}] = \pm Q_{\pm\alpha}$$

$$[M_{\alpha\beta}, Q_{\pm\gamma}] = \varepsilon_{\alpha\gamma} Q_{\pm\beta} + \varepsilon_{\beta\gamma} Q_{\pm\alpha}$$

$$[M_{\alpha\beta}, Q_{3\gamma}] = \varepsilon_{\alpha\gamma} Q_{3\beta} + \varepsilon_{\beta\gamma} Q_{3\alpha}$$

$$\{Q_{3\alpha}, Q_{\pm\beta}\} = \mp \varepsilon_{\alpha\beta} L_{\pm} \qquad \{Q_{3\alpha}, Q_{3\beta}\} = - M_{\alpha\beta}$$

$$\{Q_{+\alpha}, Q_{-\beta}\} = -2M_{\alpha\beta} - 2\varepsilon_{\alpha\beta} L_3 \tag{4.22}$$

with all other (anti) commutators zero.

The subalgebra $\mathcal{H}$ will be taken as $\mathcal{H} = \{L_+, L_3, Q_{+\alpha}, Q_{3\alpha}, M_{\alpha\beta}\}$ with $\mathcal{H}_0 = \{L_3, Q_{3\alpha}, M_{\alpha\beta}\} \simeq U(1) \times OSp(1/2)$. Cosets are labelled by the elements $\exp(xL_- + \theta^\alpha Q_{-\alpha})$. Superfields are functions $\Phi_A(x, \theta_\alpha)$ carrying change $\hat{L} \equiv -L$, and a 'superspin' M representation of the $U(1) \times OSp(1/2)$ little group (see also E.1).

$$\Phi_A (x, \theta_\beta) = \begin{pmatrix} \phi_a(x, \theta_\beta) \\ \phi_{a\alpha}(x, \theta_\beta) \end{pmatrix} \tag{4.23}$$

In the following the spin-M indices will be suppressed. Expanding the superfield in θ^α gives

$$\begin{pmatrix} \phi(x, \theta_\beta) \\ \\ \phi_\alpha(x, \theta_\beta) \end{pmatrix} = \begin{pmatrix} A(x) \\ \\ a_\alpha(x) \end{pmatrix} + \theta^\beta \begin{pmatrix} \psi^+_\beta(x) + \psi^-_\beta(x) \\ \\ P_{\alpha\beta}(x) \end{pmatrix} + \frac{1}{2}\theta^2 \begin{pmatrix} H(x) \\ \\ h_\alpha(x) \end{pmatrix} \tag{4.24}$$

where $\theta^2 \equiv \varepsilon_{\alpha\beta}\, \theta^\alpha \theta^\beta$. The Sp(2) indices can be lowered or raised using the Sp(2) metric $\varepsilon_{\alpha\beta}$ or inverse metric $\varepsilon^{\alpha\beta}$ respectively where $\varepsilon^{\alpha\beta}$ respectively where $\varepsilon_{12} = \varepsilon^{21} = +1$ and $\theta_\beta = \varepsilon_{\beta\gamma}\, \theta^\gamma$, $\theta^\beta = \varepsilon^{\beta\gamma}\, \theta_\gamma$.

The components have the following spins under Sp(2): A and H, M; ψ^+_α, $M + \frac{1}{2}$; ψ^-_α, a_α and h_α, $M - \frac{1}{2}$. $P_{\alpha\beta}$ may be decomposed into fields of definite spin under Sp(2) by the following procedure:

$$P_{\alpha\beta} = \frac{1}{2}(P_{\alpha\beta} - P_{\beta\alpha}) + \frac{1}{2}(P_{\alpha\beta} + P_{\alpha\beta})$$

$$= \frac{1}{2}\varepsilon_{\alpha\beta}\,\varepsilon^{\gamma\delta}\,P_{\delta\gamma} + P^{0}{}_{\alpha\beta} + P^{-1}{}_{\alpha\beta} \qquad 4.25$$

where $P^{0,-1}_{\alpha\beta} = \Pi^{0,-1}_{\alpha\beta}{}^{\gamma\delta}\,\frac{1}{2}(P_{\delta\gamma} + P_{\delta\gamma})$

are the spin M and spin (M-1) projections defined in Appendix C. Since $\theta^{\beta}\,P_{\beta\alpha}$ has spin $(M - \frac{1}{2})$ but $\Pi_{\alpha}^{-\frac{1}{2}\,\beta}\,\Pi^{+1}_{\beta\gamma}{}^{\delta\varepsilon} = 0$, there is no spin (M+1) projection. Furthermore using (D5,D7) and (D.20)

$$\varepsilon^{\gamma\delta}\,P_{\delta\gamma} = \hat{M}^{\gamma\delta}\,P_{\gamma\delta}/2(M+1) \equiv \hat{M}^{\gamma\delta}\,P^{0}_{\gamma\delta}/2(M+1). \qquad 4.26$$

It should be noted that $P^{0}_{\alpha\beta}$ is not an eigenvector of $\Pi^{\pm\frac{1}{2}\,\delta}_{\gamma}$. However, it can be rewritten as

$$(2M+1)\,P^{0}_{\alpha\beta} = M(P^{0}_{+})_{\alpha\beta} + (M+1)(P^{0}_{-})_{\alpha\beta} \qquad 4.27$$

where $(P^{0}_{\pm})_{\alpha\beta} = (P^{0}_{\alpha\beta} - \varepsilon_{\alpha\beta}\,\hat{M}^{\gamma\delta}\,P^{0}_{\gamma\delta}/2M^{\pm})$

$$M^{+} \equiv M \qquad\qquad M^{-} \equiv -M - 1$$

such that $\Pi_{\alpha}^{\pm\frac{1}{2}\,\beta}(P^{0}_{\pm})_{\gamma\beta} = (P^{0}_{\pm})_{\gamma\alpha}$ and $\Pi_{\alpha}^{\pm\frac{1}{2}\,\beta}(P^{0}_{\mp})_{\gamma\beta} = 0.$

Thus we finally have in (4.24)

$$P_{\alpha\beta} = (P^{0}_{-})_{\alpha\beta} + P^{-1}_{\alpha\beta} \qquad 4.28$$

The differential representation of the generators is (see (4.4,4.5) and Appendix E)

$$M_{\alpha\beta} = \theta_{\alpha}\,\partial_{\beta} + \theta_{\beta}\,\partial_{\alpha} - \hat{\mathcal{M}}_{\alpha\beta}$$

$$L_{-} = \partial/\partial x$$

$$L_{+} = -x^{2}\,\partial/\partial x - \theta^{2}\,\partial/\partial x \;\; -2x\,\theta^{\alpha}\,\partial_{\alpha} + 2xL - 2\theta^{\alpha}\,\hat{\mathcal{Q}}_{3\alpha}$$

$$L_3 = -x\partial/\partial x - \theta^\alpha \partial_\alpha + L$$

$$Q_{-\alpha} = \partial_\alpha$$

$$Q_{+\alpha} = -2\theta_\alpha x\partial/\partial x - x^2\partial_\alpha + 2\theta^2\partial_\alpha + 2\theta_\alpha L - 2\theta^\beta \hat{\mathcal{M}}_{\beta\alpha} - 2x\,\hat{\mathcal{Q}}_{3a}$$

$$Q_{3\alpha} = -\theta_\alpha\,\partial/\partial x - x\partial_\alpha - \hat{\mathcal{Q}}_{3\alpha}. \qquad 4.29$$

Upon examination of the action of $L_\pm$ on the superfield a modified basis for the component fields is obtained in which they transform as eigenvectors under the $O(3) \times Sp(2)$ even subalgebra. The necessary field redefinitions are (where $[M] = (2M+1)^{\frac{1}{2}}$)

$$\tilde{\psi}_\alpha = L[M]^{-1}\,\psi^-_\alpha + a'_\alpha$$

$$\tilde{P}^0_{\alpha\beta} = -L[M]\,P^0_{\alpha\beta} + \frac{1}{2}\,\hat{M}_{\alpha\beta}\,A'$$

$$\tilde{H} = (M+1)[M]^{-1}(L-1)H - \frac{1}{2}\,L^{-1}[M]^{-1}\,M^{\alpha\beta}\,\tilde{P}'^{0}_{\alpha\beta}$$

$$+ [M]^{-1}L^{-1}(2L-1)^{-1}(L-1)(M+1)(L+2M)A''$$

$$\tilde{h}_\alpha = (L-1)[M]^{-1}h_\alpha + [M]\,L^{-1}\,\tilde{\psi}^{1-}_\alpha$$

$$+ [M]^{-1}\,L^{-1}(2L-1)^{-1}(L-1)(L-2M-1)a''_\alpha \qquad 4.30$$

and A, ψ^+_α, a_α and $P^{-1}_{\alpha\beta}$ are unchanged.

Following the procedure of the previous sections and expanding the component fields as power series in x reveals a finite dimensional factor space in which their degrees (highest power of x in the finite factor) are A and a_α, $2L$; ψ^+_α, $\tilde{\psi}^-_\alpha$, $\tilde{P}^0_{\alpha\beta}$ and $P^{-1}_{\alpha\beta}$, $(2L-2)$; $\tilde{H}$ and $\tilde{h}_\alpha$, $(2L-4)$.

From (4.29), taking into account (4.30), the $O(3) \times Sp(2) \simeq SU(2) \times SU(2)$ decompositions obtained for arbitrary induced representations with the chosen little group (corresponding to superfields of arbitrary half-integer change L and 'superspin' M) are given in Table 4.1.

In general this class of representations is typical and thus irreducible with total dimension $4(2L-1)(4M+1)$ for $L \geq \frac{3}{2}$ and $M \geq 0$. For $M = 0$ the superfield Φ_A is a singlet under the little group and (4.23) and (4.24) reduce to $\Phi_A(x,\theta_\beta) = \phi(x,\theta_\beta) = A(x) + \theta^\beta\psi^+_\beta(x) + \frac{1}{2}\theta^2 H(x)$ 4.31

and consequently A, ψ^+_β and H form an invariant set. Examining the transformations of the component fields under the odd generators reveals that for certain (L,M) the above set is reducible, corresponding to atypical representations. Table 4.2 demonstrates this for Q^+_γ.
With $(L = 2M + 1, M \geq \frac{1}{2})$ the set $P^{-1}_{\alpha\beta}$, $\tilde{H}$, $\tilde{\psi}^-_\alpha$ and $\tilde{h}_\alpha$ form an invariant subspace of dimension $(32M^2 - 2) = (16M^2 - 2|16M^2)$, with the set $(A, \tilde{P}^0_{\alpha\beta}, a_\alpha, \psi^+_\alpha)$ invariant as a factor space. For $(L = 2, M = \frac{1}{2})$ $\tilde{H}$ and $\tilde{\psi}_\alpha$ form a further invariant subspace equivalent to the fundamental 5. From (4.24) it is evident that $L = 0, \frac{1}{2}$ and 1 must be treated as special cases; if $L = 0$ the only finite-dimensional representation occurs for $M = 0$, corresponding to the singlet A; no finite-dimensional $(L = \frac{1}{2}, M \geq 0)$ superfield can be constructed; the sequence $(L = 1, M \geq 0)$ has an invariant set $(A, P^-_{\alpha\beta}, \psi^+_\alpha, a^-_\alpha)$ which includes the fundamental $5 = (3 \times 1)/(1 \times 2)$ for $M = 0$ and the adjoint $12 = (3 \times 1 + 1 \times 3)/(3 \times 2)$ for $M = \frac{1}{2}$. The $O(3) \times Sp(2) \simeq SU(2) \times SU(2)$ decompositions obtained for these cases are summarised in Table 4.2.

The irreducible representations presented here are in agreement with those of §3.5 where the label correspondence is $a_2 = 2L - 2$ and $b = 2M + 1$.

Table 4.1 $O(3) \times Sp(2) \simeq SU(2) \times SU(2)$ decomposition of *typical* $OSp(3/2)$ induced representations from little group $U(1) \times OSp(1/2)$ for $L \geq 3/2$, $M \geq 0$.

'Even'	Dimension	'Odd'	Dimension
$A(L,M)$	$(2L+1)(2M+1)$	$a(L,M-\frac{1}{2})$	$(2L+1)(2M)$
$\tilde{H}(L-2,M)$	$(2L-3)(2M+1)$	$\tilde{h}(L-2,M-\frac{1}{2})$	$(2L-3)(2M)$
$\tilde{p}^{0}(L-1,M)$	$(2L-1)(2M+1)$	$\tilde{\psi}^{-}(L-1,M-\frac{1}{2})$	$(2L-1)(2M)$
$p^{-1}(L-1,M-1)$	$(2L-1)(2M-1)$	$\psi^{+}(L-1,M+\frac{1}{2})$	$(2L-1)(2M+2)$
Total	$2(2L-1)(4M+1)$	Total	$2(2L-1)(4M+1)$

Table 4.2 $OSp(3/2)$ component field variations under Q^{+}_{γ} .

$$\delta A = -x^2\psi^{+}_{\gamma} - L^{-1}[M]x^2\tilde{\psi}^{-}_{\gamma} + L^{-1}[M]x^2a'_{\gamma} - 2[M]xa_{\gamma}$$

$$\begin{aligned}\delta\psi^{+}_{\alpha} = {} & (\pi^{+}\varepsilon)_{\alpha\gamma}[M](M+1)^{-1}(L-1)^{-1}x^2\tilde{H} \\ & + (\pi^{+}\varepsilon)_{\alpha\gamma}(L+2M)\{-L^{-1}(2L-1)^{-1}x^2A'' + 2L^{-1}xA' - 2A\} \\ & + 2L^{-1}[M]^{-2}\{(L-1)^{-1}x^2\tilde{p}'^{0}_{+\gamma\alpha} - 2x\tilde{p}^{0}_{+\gamma\alpha}\}\end{aligned}$$

$$\begin{aligned}\delta\tilde{\psi}^{-}_{\alpha} = {} & (\pi^{-}\varepsilon)_{\alpha\gamma}L(M+1)^{-1}(L-1)^{-1}x^2\tilde{H} \\ & - (\pi^{-}\varepsilon)_{\alpha\gamma}(L-2M-1)(L-1)[M]^{-1}\{L^{-1}(2L-1)^{-1}x^2A'' - 2L^{-1}xA' - 2A\} \\ & + (L-2M-1)L^{-1}[M]^{-3}\{(L-1)^{-1}x^2\tilde{p}'^{0}_{-\gamma\alpha} - 2x\tilde{p}^{0}_{-\gamma\alpha}\} - x^2p'^{-1}_{\gamma\alpha} + 2(L-1)xp^{-1}_{\gamma\alpha}\end{aligned}$$

$$\begin{aligned}\delta\tilde{H} = {} & (M+1)[M](L-1)^{-1}\{x^2\tilde{h}'_{\gamma} - 2Lx\tilde{h}_{\gamma}\} \\ & - (L+2M)(M+1)[M](2L-1)^{-1}\{(L-1)^{-1}x^2\tilde{\psi}''^{-}_{\gamma} - 2(2L-3)(L-1)^{-1}x\tilde{\psi}'^{-}_{\gamma} + 2(2L-3)\tilde{\psi}^{-}_{\gamma}\} \\ & - (L-2M-1)(M+1)(2L-1)^{-1}\{x^2\psi''^{+}_{\gamma} - 2(2L-3)x\psi'^{+}_{\gamma} + 2(L-1)(2L-3)L^{-1}\psi^{+}_{\gamma}\}\end{aligned}$$

$$\delta a_{\alpha} = L^{-1}[M]^{-1}x^2\tilde{p}^{0}_{-\gamma\alpha} - x^2p^{-1}_{\gamma\alpha} + (\pi^{-}\varepsilon)_{\alpha\gamma}[M]\{L^{-1}x^2A' = 2xA\}$$

$$\delta\tilde{P}^{0}_{\alpha\beta} = -\hat{M}_{\alpha\beta}M^{-1}[M]^2L(L-1)^{-1}x^2\tilde{h}_\gamma - 2\hat{M}_{\alpha\beta}\{x^2\psi'^{+}_{\gamma} + 2(L-1)x\psi^{+}_{\gamma}\}$$
$$+ \hat{M}_{\alpha\beta}[M](L+2M)L^{-1}M^{-1}\{(L-1)^{-1}x^2\tilde{\psi}'^{-}_{\gamma} - 2x\tilde{\psi}^{-}_{\gamma}\}$$
$$+ \hat{M}_{\alpha\beta}[M](L+2M)(L-1)M^{-1}\{L^{-1}(2L-1)^{-1}x^2a''_\gamma - 2L^{-1}xa'_\gamma + 2a_\gamma\}$$

$$\delta P^{-1}_{\alpha\beta} = (M^{-1}\hat{M}_{\alpha\beta}\delta_\gamma{}^\rho + \tfrac{1}{2}\varepsilon_{\gamma\alpha}\delta_\beta{}^\rho + \tfrac{1}{2}\varepsilon_{\alpha\beta}\delta_\alpha{}^\rho)$$
$$\times(-[M](L-1)^{-1}x^2\tilde{h}_\rho + [M]^2L^{-1}\{(L-1)^{-1}x^2\tilde{\psi}'^{-}_{\rho} - 2x\tilde{\psi}_\rho\}$$
$$+ (L-2M-1)\{L^{-1}(2L-1)^{-1}x^2a''_\rho - 2L^{-1}xa'_\rho + 2a_\rho\})$$

$$\delta\tilde{h}_\alpha = (L-2M-1)[M]^{-2}\{(L-1)^{-1}(2L-1)^{-1}x^2\tilde{P}''^{0}_{-\gamma\alpha} - 2(2L-3)(L-1)^{-1}(2L-1)^{-1}x\tilde{P}'^{0}_{-\gamma\alpha}$$
$$+ 2(2L-3)\tilde{P}^{0}_{-\gamma\alpha}\}$$
$$- (L+2M)[M]^{-1}(2L-1)^{-1}\{x^2P''^{-1}_{\gamma\alpha} - 2(2L-3)xP'^{-1}_{\gamma\alpha} - 2(2L-3)(L-1)P^{-1}_{\gamma\alpha}\}$$
$$+ (\pi^{-}\varepsilon)_{\alpha\beta}[M](L-1)^{-1}(M+1)^{-1}\{x^2\tilde{H}' - 2(L-2)x\tilde{H}\}$$

Table 4.3 $O(3)\times Sp(2) \simeq SU(2)\times SU(2)$ decompositions of $OSp(3/2)$ induced representations from little group $U(1)\times OSp(1/2)$ for *atypical* representations and 'special' cases (see text).

$(L = 2M+1,\ M \geq 1)$ invariant space			
$P^{-1}(2M,M-1)$	$(4M+1)(2M-1)$	$\tilde{\psi}^{-}(2M,M-\frac{1}{2})$	$(4M+1)(2M)$
$\tilde{H}\ (2M-1,M)$	$(4M-1)(2M+1)$	$\tilde{h}\ (2M-1,M-\frac{1}{2})$	$(4M-1)(2M)$
Total	$16M^2-2$	Total	$16M^2$

$(L = 2M+1,\ M \geq \frac{1}{2})$ factor space			
$A(2M+1,\ M)$	$(2M+1)(4M+3)$	$a(2M+1,\ M-\frac{1}{2})$	$(4M+3)(2M)$
$\tilde{P}^0(2M,M)$	$(2M+1)(4M+1)$	$\psi^{+}(2M,\ M+\frac{1}{2})$	$(4M+1)(2M+2)$
Total	$16(M+\frac{1}{2})^2$	Total	$16(M+\frac{1}{2})^2-2$

$(L = 1,\ M \geq 0)$ invariant space			
$A(1,M)$	$3(2M+1)$	$\psi^+(0, M+\frac{1}{2})$	$1(2M+2)$
$P^{-1}(0,M-1)$	$1(2M-1)$	$a^-(1, M-\frac{1}{2})$	$3(2M)$
Total	$8M+2$	Total	$8M+2$

$(L \geq 1,\ M = 0)$ invariant space			
$A(L,0)$	$2L+1$	$\psi^+(L-1,\frac{1}{2})$	$2(2L-1)$
$\tilde{H}(L-2,0)$	$2L-3$		
Total	$4L-2$		$4L-2$

4.5 <u>OSp(4/2):</u>

The generators of the OSp(4/2) superalgebra can be written in the following way. Let $0 \leq \mu, \nu \leq 3$ refer to the O(4) indices and $1 \leq \alpha, \beta \leq$ refer to the Sp(2) indices, then from (4.6) define

$$L_{\dot{a}\dot{b}} = -\frac{1}{2}(\bar{\sigma}^{\mu\nu})_{\dot{a}\dot{b}}\, M_{\mu\nu}$$

$$M_{ab} = \frac{1}{2}(\sigma^{\mu\nu})_{ab}\, M_{\mu\nu}$$

$$N_{\alpha\beta} \equiv M_{\alpha\beta}$$

$$Q_{\dot{a}a\alpha} = \sigma^{\mu}_{a\dot{a}}\, M_{\mu\alpha} \qquad 4.32$$

where $\sigma_\mu = (1, \underset{\sim}{\sigma})$ $\quad \bar{\sigma}_\mu = (1, -\underset{\sim}{\sigma})$ $\quad \sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\eta_{\mu\nu}$

$$\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu) \qquad \bar{\sigma}_{\mu\nu} = \frac{1}{2}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)\ .$$

These generators satisfy the superalgebra

$$\{Q_{\dot{a}a\alpha},\ Q_{\dot{b}b\beta}\} = -2\varepsilon_{ab}\ \varepsilon_{\dot{a}\dot{b}}\ N_{\alpha\beta} + \varepsilon_{ab}\ \varepsilon_{\alpha\beta}\ L_{\dot{a}\dot{b}} + \varepsilon_{\alpha\beta}\ \varepsilon_{\dot{a}\dot{b}}\ M_{ab}$$

$$[M_{ab}, M_{cd}] = \varepsilon_{bc}\ M_{ad} + \varepsilon_{ac}\ M_{bd} + \varepsilon_{bd}\ M_{ac} + \varepsilon_{ad}\ M_{bc}$$

$$[M_{ab}, Q_{\dot{c}c\gamma}] = \varepsilon_{bc}\ Q_{\dot{c}a\gamma} \quad \varepsilon_{ac}\ Q_{\dot{c}b\gamma} \tag{4.33}$$

and similarly for $L_{\dot{a}\dot{b}}$ and $N_{\alpha\beta}$. The generators $L_{\dot{a}\dot{b}}$ can be written in a spherical basis L_+, L_-, L_3 as in §4.2

The subalgebra $\mathcal{H}$ will be taken as

$\mathcal{H} = \{L_3, M_{ab}, N_{\alpha\beta}, L_+, Q_{\dot{2}a\alpha}\}$ with

$\mathcal{H}_0 = \{L_3, M_{ab}, N_{\alpha\beta}\} \simeq U(1) \times SU(2) \times SU(2)$. Cosets are labelled by the elements $\exp(xL_- + \theta^{a\alpha}\ Q_{\dot{1}a\alpha})$ and superfields are functions $\Phi(x,\theta_{a\alpha})$ carrying a charge $\hat{L} \equiv -L$ and spins $M \times N$ under the little group $U(1) \times SU(2) \times SU(2)$:

$$\Phi(x,\theta_{a\alpha}) = A(x) + \theta^{a\alpha}\ \Sigma(\psi^{\pm\pm}_{a\alpha}(x) + \psi^{\pm\mp}_{a\alpha}(x))$$
$$+ (\theta\theta)^{ab}\ \Sigma(F^{0,\pm}_{ab}(x)) + (\theta\theta)^{\alpha\beta}\ \Sigma(G^{0,\pm}_{\alpha\beta}(x))$$
$$+ (\theta^3)^{a\alpha}\ \Sigma(\chi^{\pm\pm}_{a\alpha}(x) + \chi^{\pm\mp}_{a\alpha}(x)) + \theta^4\ D(x). \tag{4.34}$$

where $(\theta\theta)^{ab} = \theta^{a\alpha}\ \theta^{b}_{\alpha}$ $\qquad (\theta\theta)^{\alpha\beta} = \theta^{a\alpha}\ \theta^{\beta}_{a}$

$(\theta^3)^{a\alpha} = (\theta\theta)^{ab}\ \theta^{\alpha}_{b}$ $\qquad \theta^4 = (\theta^3)^{a}\ \theta_{a\alpha}$

and the summations are over all possible projections onto total spin $(M \pm \frac{1}{2}, N \pm \frac{1}{2})$ for $\psi_{a\alpha}(x)$ and $\chi_{a\alpha}(x)$ $(M, M\pm1)$ for $F_{ab}(x)$ and $(N, N\pm1)$ for $G_{\alpha\beta}(x)$. The relevant projection operators are given in Appendix D.

The differential representation for the generators, writing $\partial_{a\alpha} \equiv \partial/\partial\theta^{a\alpha}$, is (see (4.4,4.5) and Appendix D)

$$L_- = \partial/\partial_x$$

$$L_+ = -(x^2 - \tfrac{1}{2}\theta^4)\partial/\partial x \;\; -(x\theta+\theta^3)^{a\alpha}\;\partial_{a\alpha} + 2xL + \tfrac{1}{2}(\theta\theta)^{ab}\,\hat{M}_{ab} - (\theta\theta)^{\alpha\beta}\;\hat{N}_{\alpha\beta}$$

$$L_3 = -x\,\partial/\partial x - \tfrac{1}{2}\theta^{a\alpha}\;\partial_{a\alpha} + L$$

$$M_{ab} = \theta^{\alpha}_{a}\;\partial_{b\alpha} + \theta^{\alpha}_{b}\partial_{a\alpha} - \hat{M}_{ab}$$

$$N_{\alpha\beta} = \theta^{a}_{\alpha}\;\partial_{a\beta} + \theta^{a}_{\beta}\;\partial_{a\alpha} - \hat{N}_{\alpha\beta}$$

$$Q_{ia\alpha} = \theta_{a\alpha}\;\partial/\partial x + \partial_{a\alpha}$$

$$Q_{2a\alpha} = (x\theta_{a\alpha} + \theta^3_{a\alpha})\,\partial/\partial x + x\,\partial_{a\alpha} - 2(\theta\theta)^{b}_{a}\,\partial_{b\alpha} + (\theta\theta)^{\beta}_{\alpha}\,\partial_{a\beta} + \theta_{b\alpha}\,\hat{M}^{b}_{a} - 2\theta_{a\beta}\,\hat{N}^{\beta}_{\alpha} \qquad 4.35$$

As for the previous cases the following field redefinitions are introduced so that all component fields will transform as eigenvectors under the even subalgebra $O(4) \times Sp(2)$:

$$\tilde{F}^0_{ab} = F^0_{ab} - \hat{M}_{ab}\,A'/4L$$

$$\tilde{G}^0_{\alpha\beta} = G^0_{\alpha\beta} + \hat{N}_{\alpha\beta}\,A'/2L$$

$$\tilde{X}^{mn} = X^{mn} + (2M^m + 4N^n + 3)\;\psi^{mn}/6(L - \tfrac{1}{2})$$

$$\tilde{D} = D - \hat{M}^{ab}\;F^{0'}_{ab}/12(L-1) - \hat{N}^{\alpha\beta}\;G^{0'}_{\alpha\beta}/6(L-1) - [3L+2M(M+1) - 8N(N+1) - 3]A''/24(L - \tfrac{1}{2})(L-1) \qquad 4.36$$

with $m,n = \pm$ and $2M^{\pm} + 1 = \pm(2M+1)$.

Expanding the component fields as power series in x yields a finite-dimensional factor space in which their degrees (highest power of x in the finite factor) are:

$A, 2L$; $\psi^{mn}, (2L-1)$; $F^m, \tilde{F}^0, G^n$ and $\tilde{G}^0, (2L-2)$;

$\tilde{\chi}^{mn}, (2L-3)$; $\tilde{D}, (2L-4)$.

From (4.35), taking into account the definitions (4.36), the $O(4) \times Sp(2) \simeq SU(2) \times SU(2) \times SU(2)$ decompositions obtained for arbitrary induced representations with the chosen little group (corresponding to superfields of arbitrary half-integer change L and spins M and N) are given in Table 4.4. This class of irreducible representations is in general typical (with even and odd dimensions the same), and total dimension $16(2L-1)(2M+1)(2N+1)$, with $L \geq \frac{3}{2}$ and $M, N \geq 0$. In the basis (4.36) it is found that superfields which cannot be decomposed arise for certain (L,M,N) values, corresponding to atypical representations. This is demonstrated for the component field variations under $Q_{\dot{2}a\alpha}$ in Table 4.5. It is apparent from this table that when a particular atypicality condition is imposed only a certain linear combination of $G^0_{\alpha\beta}$ and F^0_{ab} appears in the invariant subspace. This indicates that, in general, for atypical representations, of the two fields with weights (L-1,M,N) one belongs to the invariant subspace and the other to the factor space. The general atypicality conditions are $L = M^m - 2N^n$, $m,n = +,-$. These are in agreement with the results of Kac [8] and of §3.6 for $D(2,1;\alpha = 1)$ where the label correspondence is $a_2 = 2M$, $a_3 = 2(L-1)$, $b = 2(N+1)$. The condition $L = M^- - 2N^+$ is however never realized in this approach. It corresponds to the condition $a_1 = 0$ of §3.6 for which only the trivial representation ($a_2 = a_3 = b = 0$) occurs. The complete $O(4) \times Sp(2) \simeq SU(2) \times SU(2) \times SU(2)$ decompositions for the remaining atypicality conditions are presented in Table 4.6. As a specific example it can be noted that the adjoint $17 = (3\times1\times1 + 1\times3\times1 + 1\times1\times3)/(2\times2\times2)$ is derived from the invariant set $(A, F^+, G^+)/(\psi^{++})$ for $(L = 1, M = N = 0)$.

The atypical representations found here and listed in Table 4.6 are identical to those found for D(2,1; α=1) in §3.6 and listed in Table 3.3.

Table 4.4 $O(4) \times Sp(2) \simeq SU(2) \times SU(2) \times SU(2)$ decomposition of typical OSp(4/2) irreducible induced representations from little group $U(1) \times SU(2) \times SU(2)$ for $L \geq \frac{3}{2}$, $M, N \geq 0$.

'Even'	Dimension	'Odd'	Dimension
$A, \tilde{D}(L-1\pm1,M,N)$	$2(2L-1)(2M+1)(2N+1)$	$\psi^{+\pm}(L-\frac{1}{2},M+\frac{1}{2},N\pm\frac{1}{2})$	$2(2L)(2M+2)(2N+1)$
$\tilde{F}^0(L-1,M,N)$	$(2L-1)(2M+1)(2N+1)$	$\psi^{-\pm}(L-\frac{1}{2},M-\frac{1}{2},N\pm\frac{1}{2})$	$2(2L)(2M)(2N+1)$
$F^{\pm}(L-1,M\pm1,N)$	$2(2L-1)(2M+1)(2N+1)$	$\tilde{\chi}^{+\pm}(L-\frac{3}{2},M+\frac{1}{2},N\pm\frac{1}{2})$	$2(2L-2)(2M+2)(2N+1)$
$\tilde{G}^0(L-1,M,N)$	$(2L-1)(2M+1)(2N+1)$	$\tilde{\chi}^{-\pm}(L-\frac{3}{2},M-\frac{1}{2},N\pm\frac{1}{2})$	$2(2L-2)(2M)(2N+1)$
$G^{\pm}(L-1,M,N\pm1)$	$2(2L-1)(2M+1)(2N+1)$		
Total	$8(2L-1)(2M+1)(2N+1)$	Total	$8(2L-1)(2M+1)(2N+1)$

Table 4.5 OSp(4/2) component field variations under $Q_{2c\gamma}$.

$$\delta A = \sum_{m,n} x\, \psi^{mn}_{c\gamma}$$

$$\begin{aligned}\delta\psi^{mn}_{a\alpha} = {} & 2x(\pi^n\varepsilon)_{\alpha\gamma}\, F^m_{ac} + 2x(\pi^m\varepsilon)_{ac}\, F^n_{\alpha\gamma} \\ & - \frac{i}{2}(\pi^m\varepsilon)_{ac}\,(\pi^n\varepsilon)_{\alpha\gamma}\left[\frac{1}{(N^n+1)}\, x\hat{N}^{\delta\varepsilon}\tilde{G}^0_{\delta\varepsilon} + \frac{1}{(M^m+1)}\, x\hat{M}^{de}\tilde{F}^0_{de} \right. \\ & \left. \qquad - \frac{2(L-M^m+2N^n)}{L}\,\{2MA - xA'\}\right]\end{aligned}$$

$$\delta F^m_{ab} = \sum_n P^{m\;de}_{ab}\;\varepsilon_{dc}\left[\frac{3}{2}\, x\tilde{\chi}^{mn}_{e\gamma} + \frac{(-L+M^m+2N^n+2)}{2L-1}\,\{\psi^{mn}_{e\gamma},\psi'^{mn}_{e\gamma}\}\right]$$

$$\delta G^n_{\alpha\beta} = \sum_m P^{n\;\delta\varepsilon}_{\alpha\beta}\;\varepsilon_{\delta\gamma}\left[-\frac{3}{2}\, x\tilde{\chi}^{mn}_{c\varepsilon} - \frac{(L+M^m+2N^n+1)}{2L-1}\{\psi^{mn}_{c\varepsilon},\psi'^{mn}_{c\varepsilon}\}\right]$$

$$\delta\tilde{F}^0_{ab} = \sum_{m,n} \frac{1}{4M(M+1)} \hat{M}_{ab} M^m \left[-\frac{3}{2} x\tilde{\chi}^{mm}_{c\gamma} + \{\frac{M^m+1}{L} - \frac{(-L+M^m+2N^n+2)}{2L-1}\} \{\psi^{mn}_{c\gamma}, \psi'^{mn}_{c\gamma}\} \right]$$

$$\delta\tilde{G}^0_{\alpha\beta} = \sum_{m,n} \frac{1}{4N(N+1)} \hat{N}_{\alpha\beta} N^n \left[\frac{3}{2} x\tilde{\chi}^{mn}_{c\gamma} + \{-\frac{2(N^n+1)}{L} + \frac{(L+M^m+2N^n+1)}{2L-1}\} \{\psi^{mn}_{c\gamma}, \psi'^{mn}_{c\gamma}\} \right]$$

$$\delta\chi^{mn}_{a\alpha} = -\frac{4}{3}(\pi^n\varepsilon)_{\alpha\gamma} \frac{(L+M^m+2N^n+1)}{2L-1} \{F^m_{ac}, F'^m_{ac}\}$$

$$-\frac{4}{3}(\pi^m\varepsilon)_{ac} \frac{(-L+M^m+2N^n+2)}{2L-1} \{G^n_{\alpha\gamma}, G'^n_{\alpha\gamma}\}$$

$$-\frac{1}{3}(\pi^m\varepsilon)_{ac}(\pi^n\varepsilon)_{\alpha\gamma} \Big[\{\frac{1}{L-1} - \frac{(L+M^m+2N^n+1)}{(M^m+1)(2L-1)}\} \hat{M}^{de} \{\tilde{F}^0_{de}, \tilde{F}'^0_{de}\}$$

$$+ \{\frac{2}{L-1} - \frac{(-L+M^m+2N^n+2)}{(N^n+1)(2L-1)}\} \hat{N}^{\delta\varepsilon} \{\tilde{G}^0_{\delta\varepsilon}, \tilde{G}'^0_{\delta\varepsilon}\} - 12\, x^2\tilde{D} \Big]$$

$$\delta D = \sum_{m,n} \frac{1}{4} \frac{(L+M^n-2N^n-1)}{L-1} \{(2m-3)\, \tilde{\chi}^{mn}_{c\gamma} - x\, \tilde{\chi}'^{mn}_{c\gamma}\}$$

Where ${\pi^m}_a{}^b$ and ${\pi^n}_\alpha{}^\beta$ are spin $M \pm \frac{1}{2}$ and $N \pm \frac{1}{2}$ projectors respectively and ${P^m}_{ab}{}^{cd}$ and ${P^n}_{\alpha\beta}{}^{\gamma\delta}$ are spin $M \pm 1$ and spin $N \pm 1$ projectors respectively. Also if B is any of $\tilde{F}^0, \tilde{G}^0$, F^m or G^n

then $\qquad \{B, B'\} = (2M-2)B - xB'$

and $\qquad \{\psi^{mn}_{a\alpha}, \psi'^{mn}_{a\alpha}\} = (2M-1)\, \psi^{mn}_{a\alpha} - x\psi'^{mn}_{a\alpha}$.

Table 4.6 $O(4) \times Sp(2) \simeq SU(2) \times SU(2) \times SU(2)$ decompositions of atypical $OSp(4/2)$ irreducible, induced representations from little group $U(1) \times SU(2) \times SU(2)$ for $L \geq 3/2$, $M,N \geq 0$.

Atypicality Condition	Irreducible Representations			Dimensions
$L = M - 2N$	Factor Space	'Even'	$A, F^-, G^-, \tilde{F}^0$	$2(16M^2N + 4M^2 - 32MN^2 - 8MN - 8N^2 + 1)$
		'Odd'	$\psi^{+-}, \psi^{-+}, \psi^{--}, \tilde{\chi}^{--}$	$8(4M^2N + M^2 - 8MN^2 - 2MN - 2N^2)$
	Invariant Space	'Even'	$G^+, F^+, \tilde{D}, \tilde{G}^0$	$2(16M^2N + 12M^2 - 32MN^2 - 24MN - 24N^2 - 24N - 5)$
		'Odd'	$\psi^{++}, \tilde{\chi}^{++}, \tilde{\chi}^{-+}, \tilde{\chi}^{+-}$	$8(4M^2N + 3M^2 - 8MN^2 - 6MN - 6N^2 - 6N - 1)$
$L = M + 2N + 2$	Factor Space	'Even'	$A, G^+, F^-, \tilde{F}^0$	$2(16M^2N + 12M^2 + 32MN^2 + 56MN + 24M + 8N^2 + 16N + 7)$
		'Odd'	$\psi^{++}, \psi^{-+}, \psi^{--}, \tilde{\chi}^{-+}$	$8(4M^2N + 3M^2 + 8MN^2 + 14MN + 6M + 2N^2 + 4N + 2)$
	Invariant Space	'Even'	$\tilde{D}, F^+, G^-, \tilde{G}^0$	$2(16M^2N + 4M^2 + 32MN^2 + 40MN + 8M + 24N^2 + 24N + 5)$
		'Odd'	$\psi^{+-}, \tilde{\chi}^{++}, \tilde{\chi}^{+-}, \tilde{\chi}^{--}$	$8(4M^2N + M^2 + 8MN^2 + 10MN + 2M + 6N^2 + 6N + 1)$
$L = -M + 2N + 1$	Factor Space	'Even'	$A, G^+, F^+, \tilde{F}^0$	$2(-16M^2N - 12M^2 + 32MN^2 + 24MN + 24N^2 + 24N + 5)$
		'Odd'	$\psi^{++}, \psi^{+-}, \psi^{-+}, \tilde{\chi}^{++}$	$8(-4M^2N - 3M^2 + 8MN^2 + 6MN + 6N^2 + 6N + 1)$
	Invariant Space	'Even'	$F^-, G^-, \tilde{D}, \tilde{G}^0$	$2(-16M^2N - 4M^2 + 32MN^2 + 8MN + 8N^2 - 1)$
		'Odd'	$\psi^{--}, \tilde{\chi}^{+-}, \tilde{\chi}^{-+}, \tilde{\chi}^{--}$	$8(-4M^2N - M^2 + 8MN^2 + 2MN + 2N^2)$

The above dimensions are, in general, only applicable for $L \geq 2$, $M, N \geq 1$.

See text for the discussion regarding $\tilde{F}^0$ and $\tilde{G}^0$.

CHAPTER 4 - REFERENCES

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5. REPRESENTATIONS OF ORTHOSYMPLECTIC SUPERALGEBRAS: YOUNG SUPERTABLEAUX

5.1 INTRODUCTION

Young tableau and Schur function techniques provide a useful and elegant description of many properties related to irreducible representations of semi-simple Lie algebras. The extension of these techniques to Lie superalgebras was first made by Dondi and Jarvis [1,8] and Bars and Balantekin [6,7]. Dondi and Jarvis [1] presented the following branching rules for purely covariant or purely contravariant representations of U(m/n) and SU(m/n)

$$
\begin{array}{lll}
U(m/n) \downarrow U(m) \otimes U(n) & \{\lambda\} \downarrow \sum_{\xi} \{\lambda/\xi\} \otimes \{\tilde{\xi}\} & \\
U(m+\nu/\mu+n) \downarrow U(m/\mu) \otimes U(\nu/n) & \{\lambda\} \downarrow \sum_{\xi} \{\lambda/\xi\} \otimes \{\xi\} & \\
U(m\mu+n\nu)m\nu+n\mu) \downarrow U(m/n) \otimes U(\mu/\nu) & \{\lambda\} \downarrow \sum_{\sigma\in Sr} \{\lambda o \sigma\} \otimes \{\sigma\} & 5.1
\end{array}
$$

where ξ is summed over all possible partitions and the operation (o) is that of the inner Kronecker product of representations of the symmetric group Sr, r being the rank of $\{\lambda\}$. The above branching rules, with the inclusion of a U(1) label, also apply for SU(m/n) $\downarrow$ SU(m) x SU(n) x U(1). They have also given rules for Kronecker products, and dimensions of some representations, for U(m/n) and SU(m/n).

Bars and Balantekin [6,7] have used Young tableau techniques generalized to supergroups to derive character and dimension formulae for representations of SU(m/n) and OSp(m/2n) which may be derived from direct products of (covariant and contravariant) fundamental representations. In particular they have noted that the characters for the orthosymplectic and superunitary groups formally look the same as the characters for the orthogonal and unitary groups respectively, except for the replacement of supertraces by traces and superdeterminants by determinants. These works have also drawn attention to the result that for SU(m/n) and OSp(m/n) the fundamental representation acts in a graded (m+n) dimensional space $V = V_{\bar{0}} + V_{\bar{1}}$, where $V_{\bar{0}}$ is of degree zero and $V_{\bar{1}}$ is of degree one. There are two possible gradings of V and consequently, by taking tensor products, two classes of representations. These are designated, class I for which dim $V_{\bar{0}} = m$ and dim $V_{\bar{1}} = n$ and class II for which dim $V_{\bar{0}} = n$ and dim $V_{\bar{1}} = m$. For OSp(m/n) King [2] has referred to these classes by OSp(m/n) and SpO(m/n) respectively. Later work [10] saw the derivation of a generating function to obtain the eigenvalues of all Casimir operators of SU(m/n), while the branching rules $SU(m/n) \downarrow SU(m) \otimes SU(n) \times U(1)$, $SU(m+\nu/\mu+n) \downarrow SU(m/\mu) \times SU(\nu/n) \times U(1)$, and $SU(m\mu+n\nu/m\nu+n\mu) \downarrow SU(m/n) \otimes SU(\mu/\nu)$ have been extended to contravariant, covariant and mixed supertableaux [9,11], though the mixed supertableaux may be reducible but indecomposable, i.e. atypical.

Bars, Morel and Ruegg [11] have established the relation between Young supertableaux and the Kac-Dynkin diagrams for SU(m/n). The connection is made by realising that the highest weight of the representation corresponds to that state in the decomposition $SU(m/n) \downarrow SU(m) \otimes SU(n) \otimes U(1)$ for which the U(1) charge is maximum if $m < n$ or minimum if $m > n$. They have thus established that purely covariant or purely contravariant tableaux correspond to irreducible representations while mixed supertableaux may also correspond to indecomposable representations. An investigation of the mixed supertableaux of SU(n/1) has been carried out by Delduc and Gourdin [16] for typical and atypical representations. They have established the cases for which the supertableau corresponds to an irreducible representation of SU(n/1). These are typical or atypical covariant, contravariant or mixed supertableaux for which $c_1+\bar{c}_1 \leq n$ or typical covariant, contravariant and mixed supertableux for which $c_1+\bar{c}_1 > n$, where $c_1+\bar{c}_1$ is the sum of covariant and contravariant boxes of the first columns. A mixed atypical supertableaux, for which $c_1+\bar{c}_1 > n$, is only a part of an indecomposable representation of SU(n/1) and the indecomposable representation is a sum of four atypical components.

Wybourne [17] has used the theory of symmetric functions to provide concise expressions for characters, dimensions and branching rules for representations of U(m/n).

Some recent work of Morel, Sciarrino and Sorba [13] has considered the development of Young supertableaux for the study of representations of OSp(M/N). They develop branching rules for the decomposition of a supertableaux, corresponding to an irreducible representation of OSp(m/N), into the irreducible representations of O(M) x Sp(N) which compose it. For this purpose extensive use is made of generalized Young tableaux. These are diagrammatic techniques which have been developed by Girardi, Sciarrino and Sorba for the study of Kronecker products of representations of SO(2m) [14] and Sp(2n) [15]. In the interests of space it will not be possible to develop these here, however the reader is encouraged to examine these works for the authors have been successful in obtaining a closed form of the branching rules for all typical, tensor and spinor representations of OSp(M/N). They have also given rules for obtaining the decomposition of atypical representations, however this requires a knowledge of all irreducible, atypical representations of lower dimension which satisfy the same atypicality condition as the representation under investigation. If these should exist within the decomposition obtained by applying the rules used for typical diagrams then they are simply deleted to yield an irreducible, atypical representation.

King [2] has developed Kronecker product rules and branching rules, for tensor representations of OSp(M/N), in terms of standard Schur function operations. He has also given dimension formulae for these representations in terms of partition labels. The Kronecker product of two representations $[\lambda]$ and $[\mu]$ of OSp(M/N) is given by

$$[\lambda] \times [\mu] = \sum_{\rho} [(\lambda/\rho) \cdot (\mu/\rho)] \quad .$$

Branching rules for tensor representations [2] are given in (5.2) and (5.3) while branching rules for spinor representations follow from the character formulae [12] (5.16) and (5.17) for OSp(2m/2n) and (5.20) and (5.17) for OSp(2m+1/2n). The dimension formulae are presented in §5.5.

5.2 ATYPICALITY CONDITIONS AND RELATIONS BETWEEN KAC-DYNKIN AND SUPERTABLEAU LABELS

This section examines finite-dimensional, tensor representations of OSp(M/N) via standard Young diagrams. These can be realised by graded symmetrised, supertraceless tensors [1] and can be decomposed in terms of irreducible representations of O(M) x Sp(N), with branching rule [2],

$$[\lambda] \downarrow \sum_{\xi} [\lambda/\xi] \langle\tilde{\xi}/\beta\rangle \quad \sum_{\xi} \sum_{\beta} \; \lambda/\xi \; \langle\tilde{\xi}/\beta\rangle \qquad 5.2$$

or

$$[\lambda] \downarrow \sum_{\xi} [\xi/D] \widetilde{\langle\lambda/\xi\rangle} = \sum_{\xi} \sum_{\delta} [\xi/\delta] \widetilde{\langle\lambda/\xi\rangle} \qquad 5.3$$

where ξ runs over all divisors of λ, β corresponds to partitions with even column lengths and δ to partitions with even row lengths.

In order to present necessary and sufficient conditions on the diagram shape for the representation to be atypical, each of the algebras B(m,n), C(n) and D(m,n) are examined to establish the correspondence between the Kac-Dynkin labels which label the highest weight of an OSp(M/N) representation (as discussed in chs. 2 & 3) and the diagram labels.

In this section we consider only 'standard' supertableaux in the following sense: for B(m,n) and D(m,n) the diagrams are such that if c_i is the length of the i^{th} column then $c_i \leq m$ for $i > n$. For C(n) $\cong$ OSp(2/2n-2) we require $c_i \leq 1$ for $i > n-1$. Of course all diagrams must be regular in the sense that for all column lengths c_k and row lengths r_k, $c_k \geq c_{k+1}$ and $r_k \geq r_{k+1}$.

The conditions for atypicality are given in table 5.1. These conditions are general for OSp(M/N) when diagram labels are used.

Table 5.1 Atypicality conditions for OSp(M/N)

(i) $\mu_i + \lambda_j + \frac{1}{2} N = i + j - 1$

(ii) $\mu_i + \frac{1}{2} N + j + 1 = \lambda_j + M + i$

where $1 \leq i \leq \frac{1}{2} N$; $1 \leq j \leq [\frac{1}{2} M]$

($[\frac{1}{2} M]$ is the largest integer less than or equal to $\frac{1}{2}$ M.)

The diagram labelling is as given in (5.4).

For $C(n) \equiv OSp(2/2n-2)$ the correlation between the above diagram labelling and that of §5.2c is $k_1 = \lambda_1 + n-1$, and $\nu_i = \mu_i - 1$. If $k_1 < n-1$ then $\mu_i = 1$ for $k_1 + 1 \leq i \leq n-1$.

It has been noted by King [12] that these atypicality conditions may be interpreted as conditions on the ($[\frac{1}{2} M] \times \frac{1}{2} N$) box positions in the upper-left corner of the tableaux. The above conditions may be written as

(i') $h_{ij} = 0$

(ii') $h_{ij} = h_j$

where $h_{ij} = (\mu_i - j) + (\lambda_j + \frac{1}{2} N - i) + 1$

and $h_j = 2(\mu_i - i + 1) - (M - N)$.

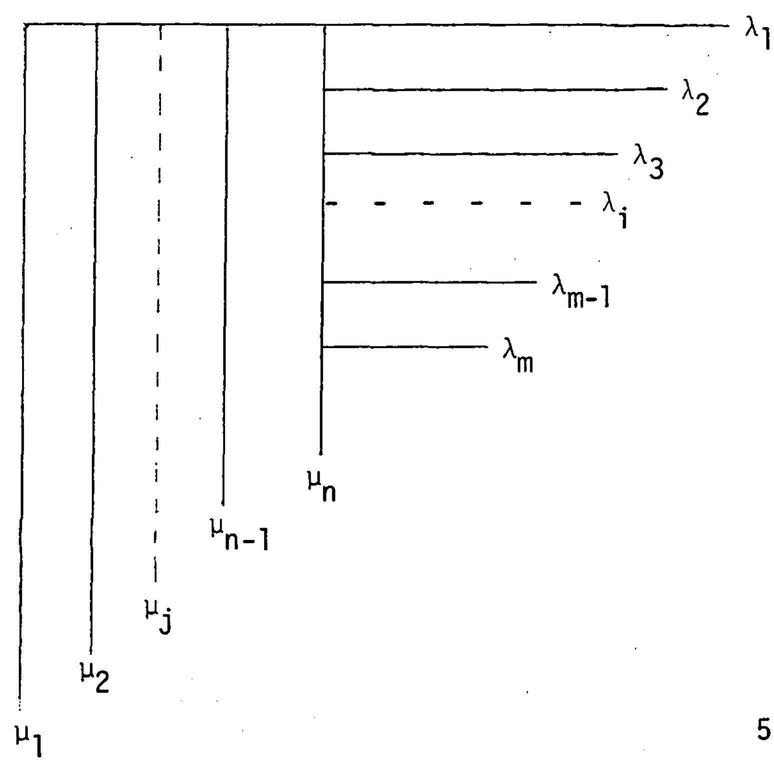
Thus (i') can be interpreted as the condition for the 'hook length' of the (i,j) position to be zero while (ii') can be interpreted as the condition necessary for a modification of the tableau in O(M-N) to yield a regular tableau when starting the hook removal in the i^{th} column. In view of the close connection between the orthosymplectic and orthogonal characters, as discussed earlier, and particularly the equivalence between the super-character of OSp(M/N) and the

O(M-N) character this seems an interesting observation though its implications remain to be explored.

5.2.a $B(m,n)$, $m \geq 0$

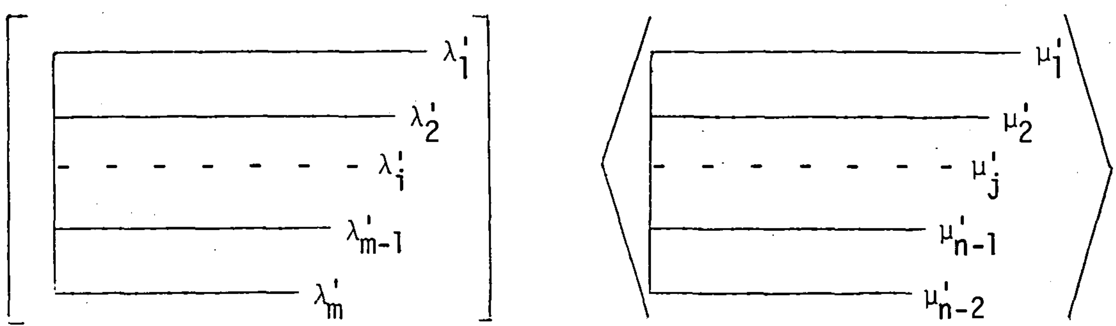
Consider the supertableau

$[\lambda] =$



5.4

where λ_i is the number of boxes beyond the n^{th} column in the i^{th} row, with $i \leq m$ and μ_j is the number of boxes in the j^{th} column, with $j \leq n$. This diagram will be designated as $[\lambda_1,\lambda_2,\ldots,\lambda_m;\ \mu_1,\mu_2,\ldots,\mu_n]$. A general diagram in the decomposition (5.2), after the appropriate modification, will have the form



5.5

The relationships, for (5.5), between the O(2M+1) x Sp(2n) Dynkin labels and the diagram labels are given by [3]:

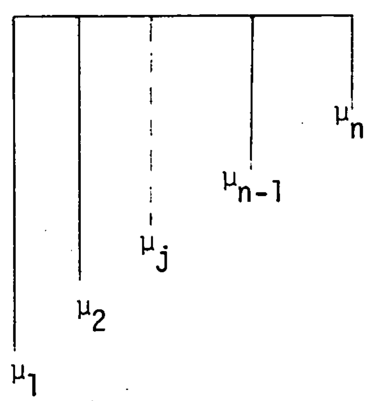
$$a'_1 = \mu'_1 - \mu'_2 \ , \ a'_2 = \mu'_2 - \mu'_3 \ , \ \dots \ , \ a'_{n-1} = \mu'_{n-1} - \mu'_n \ , \ b' = \mu'_n \ ,$$

$$a'_{n+1} = \lambda'_1 - \lambda'_2 \ , \ a'_{n+2} = \lambda'_2 - \lambda'_3 \ , \ \dots \ , \ a'_{n+m-1} = \lambda'_{m-1} - \lambda'_m \ , \ a'_{n+m} = 2\lambda'_m \qquad 5.6$$

where a'_1, a'_2 , ... , a'_{n-1} and b' refer to the Sp(2n) labels and a'_{n+1}, a'_{n+2} , ... , a'_{n+m} refer to the O(2m+1) labels.

To determine which diagram in (5.2) corresponds to the highest weight vector, Λ, with weight components $\lambda(h_i) = a_i$, $\lambda(k) = b$, of a B(m,n) representation we first consider the action of the odd negative generators on Λ. The weights of these are presented in tables 5.2 and 5.3. The action of all the odd negative generators can be obtained by considering each of those in table 5.2 by themselves and in conjunction with each of the even supplementary operators in table 5.3. Examination of these reveals that Λ can be uniquely determined by application of the following sequence of selection criteria: (i) select those states of maximum b', (ii) within this subset select those states of maximum a'_{n-1}, (iii) from these select those states of maximum a'_{n-2}, etc., until we finally select the state of maximum a'_1. This state will be Λ. Expressed in diagram notation these criteria are: (i) select those diagrams of maximum μ'_n, (ii) of these select those diagrams of maximum μ'_{n-1}, etc., until we finally select the diagram with maximum μ'_1. The diagram which corresponds to Λ is obtained by taking $\beta = \{0\}$ and

$\xi = \tilde{\mu} =$



5.7

This diagram will be given by taking $\lambda'_i = \lambda_i$ and $\mu'_j = \mu_j$ in (5.5). Therefore the Kac-Dynkin labels a_k and b in terms of the supertableau labels λ_i and μ_j are:

$$a_1 = \mu_1 - \mu_2\ ,\ a_2 = \mu_2 - \mu_3\ ,\ \cdots\ ,\ a_{n-1} = \mu_{n-1} - \mu_n\ ,$$

$$a_n = \mu_n + \lambda_1\ ,\ a_{n+1} = \lambda_1 - \lambda_2\ ,\ a_{n+2} = \lambda_2 - \lambda_3\ ,\ \cdots\ ,$$

$$a_{n+m-1} = \lambda_{m-1} - \lambda_m\ ,\ a_{n+m} = 2\lambda_m\ ,\ b = \mu_n\ . \qquad 5.8$$

Using (5.8) we can now rewrite the conditions for atypicality [4] in diagram notation. These results are given in table 5.1. The proof that the above choice (5.7) for ξ, uniquely determines Λ, is presented in Appendix F.

Note that for the B(0,n) algebra, as defined in ch 2., there is a direct correspondence with the above by setting $\lambda_i = 0 \; \forall \; i$. There are no atypical representations for B(0,n) [4].

5.2.b D(m,n)

Consider again the supertableau (5.4) for which (5.2) and (5.5) are still applicable. The relationships for (5.5) between the O(2m) x Sp(2n) Dynkin labels and the diagram labels are given by [3]:

$$a_1' = \mu_1' - \mu_2' \; , \; a_2' = \mu_2' - \mu_3' \; , \; \ldots \; , \; a_{n-1}' = \mu_{n-1}' - \mu_n' \; , \; b' = \mu_n' \; ,$$

$$a_{n+1}' = \lambda_1' - \lambda_2' \; , \; a_{n+2}' = \lambda_2' - \lambda_3' \; , \; \ldots \; , \; a_{n+m-2}' = \lambda_{m-2}' - \lambda_{m-1}' \; ,$$

$$a_{n+m-1}^{+\prime} = \lambda_{m-1}' - \lambda_m' \; , \; a_{n+m}^{+\prime} = \lambda_{m-1}' + \lambda_m' \; ,$$

$$a_{n+m-1}^{-\prime} = \lambda_{m-1}' + \lambda_m' \; , \; a_{n+m}^{-\prime} = \lambda_{m-1}' - \lambda_m' \qquad 5.9$$

where a_1' , a_2' , ..., a_{n-1}' and b' refer to the Sp(2n) labels and a_{n+1}' , a_{n+2}' , ... , $a_{n+m-1}^{\pm\prime}$, $a_{n+m}^{\pm\prime}$ refer to the O(2m) labels. If $\lambda_m' \neq 0$, both signs arise for $a_{n+m-1}^{\pm\prime}$ and $a_{n+m}^{\pm\prime}$. This corresponds to the fact that the O(2m) representation is self-associated and reduces to a sum of two inequivalent irreducible representations of SO(2m) under this restriction, so that under O(2m) $\downarrow$ SO(2m) we have $[\lambda] \downarrow [\lambda]_+ + [\lambda]_-$. $[\lambda]_+$ and $[\lambda]_-$ are conjugate to one another under an involutary outer-automorphism of SO(2m) involving a matrix of determinant -1.

To determine which diagram in (5.2) corresponds to Λ, we again consider the action of the odd negative generators on Λ. These weights are presented in tables 5.2 and 5.4. The action of all the odd negative generators can be obtained by considering each of those in table 5.2 by themselves and in conjunction with each of the even supplementary operators in table 5.4. Examination of these reveals that Λ can be uniquely determined by application of the same sequence of selection criteria as for B(m,n). Consequently the diagram which corresponds to Λ is again obtained by taking $\beta = \{0\}$ and ξ as in (5.7). If $\lambda_m \neq 0$ the sign ambiguity corresponds to the decomposition of the graded tensor $[\lambda]$ into a sum of two conjugate representations of D(m,n) with distinct Kac-Dynkin labels:

$$a_1 = \mu_1 - \mu_2 \ , \ a_2 = \mu_2 - \mu_3 \ , \ \cdots \ , \ a_{n-1} = \mu_{n-1} - \mu_n \ ,$$

$$a_n = \mu_n + \lambda_1 \ , \ a_{n+1} = \lambda_1 - \lambda_2 \ , \ a_{n+2} = \lambda_2 - \lambda_3 \ , \ \cdots \ ,$$

$$a^+_{n+m-1} = \lambda_{m-1} - \lambda_m \ , \ a^+_{n+m} = \lambda_{m-1} + \lambda_m \ ;$$

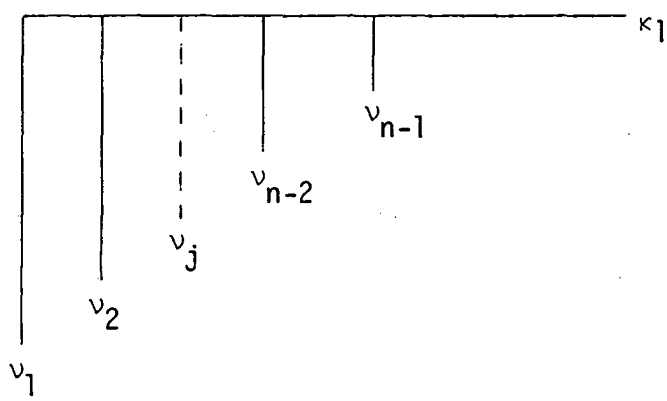
$$a^-_{n+m-1} = \lambda_{m-1} + \lambda_m \ , \ a^-_{n+m} = \lambda_{m-1} - \lambda_m \ , \ b = \mu_n \ . \qquad 5.10$$

This decomposition is the super-analogue of the D(m) tensor reduction described above and is related to the outer-automorphism of D(m,n) generated by $\alpha^+_{n+m} \leftrightarrow \alpha^+_{n+m-1}$ for the simple roots. It is clear from table 5.4 that this corresponds to the usual automorphism of D(m) on each irreducible representation of O(2m) x Sp(2n).

Using (5.10), the conditions for atypicality [4] are presented in diagram notation in table 5.1. Note that the conditions are independent of the sign choice for $\lambda_m \neq 0$. The proof that the choice (5.7) for ξ uniquely determines Λ is given in Appendix F.

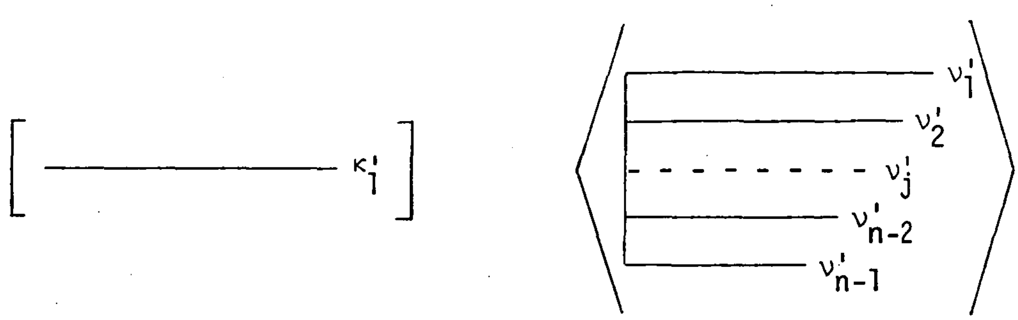
5.2.c $C(n)$

Consider the supertableau

$[\lambda]$ = 

5.11

where κ_1 is the number of boxes in the first row and ν_{j+1} is the number of boxes in the j^{th} column, with $j \leq n-1$. In the decomposition (5.3) a general diagram will take the following form after modification,



5.12

The relationships, for (5.12), between the $O(2) \times Sp(2n-2)$ Dynkin labels and the diagram labels are given by [3]:

$$b' = \kappa_1' \; , \; a_2' = \nu_1' - \nu_2' \; , \; a_3' = \nu_2' - \nu_3' \; , \; \ldots \; ,$$

$$a_{n-1}' = \nu_{n-2}' - \nu_{n-1}' \; , \; a_n' = \nu_{n-1}' \qquad 5.13$$

where b' is the $O(2)$ label and $a_2' , \ldots , a_n'$ are the $Sp(2n-2)$ labels. Since the branching rule for $O(2) \downarrow U(1) \cong SO(2)$ is

$[b'] \downarrow \{b'\} + \{\bar{b}'\}$ [5], then an O(2) tensor $[\lambda]$ with Dynkin label b' will decompose into a direct sum $[\lambda]_+ + [\lambda]_-$ with Dynkin labels $b'^+ = +b'$ and $b'^- = -b'$ respectively.

To determine which diagram in (5.3) corresponds to Λ, the action of the odd, negative generators on Λ is again considered. These results are presented in table 5.5. Examination of these reveals that the O(2) x Sp(2n-2) highest weight state of maximum b' must be Λ. The diagram (5.12) which corresponds to Λ must, therefore, have $\kappa_1' = \kappa_1$. This state is unique and is obtained by taking $\xi = \{\kappa_1\}$ and $\delta = \{0\}$. For this situation (5.12) becomes $[\kappa_1] < \nu_1, \nu_2, \ldots, \nu_{n-1} >$. To show that this is indeed the only diagram in (5.3) containing $[\kappa_1]$ we need only show that if ξ contains more than one row, then $[\xi/D]$ contains only diagrams $[\kappa_1']$ with $\kappa_1' < \kappa_1$. This is achieved by consideration of the chain $O(2) \uparrow r\ U(2) \downarrow O(2)$ which diagrammatically can be expressed as $[\xi/D) \uparrow r \{\xi\} \downarrow [\xi/D]$ [5]. In U(2) we need consider only $\{\xi\}$. If it has more than two rows it will be zero and if it has two rows, i.e. if $\{\xi\} = \{\xi_1, \xi_2\}$, then it will have the same O(2) content as $\{\xi_1 - \xi_2\}$. Thus when we consider the branching $\{\xi\} \downarrow [\xi/D]$ there will be no diagram consisting of just $[\kappa_1]$ if $\xi_2 \neq 0$.

If $\kappa_1 > n-1$ the graded tensor $[\lambda]$ decomposes in a sum of two conjugate representations of C(n) with distinct Kac-Dynkin labels:

$$b^+ = +\kappa_1 \ , \ b^- = 2n - \kappa_1 - 2 \ , \ a_1^+ = \kappa_1 + \nu_1 \ , \ a_1^- = 2n - \kappa_1 - 2 + \nu_1 \ ,$$

$$a_2 = \nu_1 - \nu_2 \ , \ a_3 = \nu_2 - \nu_3 \ , \ \ldots \ , \ a_{n-1} = \nu_{n-2} - \nu_{n-1} \ , \ a_n = \nu_{n-1} \ . \tag{5.14}$$

Using these we present the atypicality conditions in table 5.1. Note that the conditions are independent of which of these two conjugate representations they are written for.

TABLE 5.2 Weight components for some odd negative generators of B(m,n) and D(m,n).

	β_n^{n-}	β_n^{n-1-}	β_n^{n-2-}	β_n^{n-3-}	...	β_n^{3-}	β_n^{2-}	β_n^{1-}
h_1						0	+1	-1
h_2						+1	-1	0
h_3						-1	0	
0								
h_{n-4}				+1				
h_{n-3}			+1	-1				
h_{n-2}		+1	-1	0				
h_{n-1}	+1	-1	0	0				
h_n	0	+1	+1	+1		+1	+1	+1
(h_n)	(-2)	(0)	(0)	(0)		(0)	(0)	(0)
h_{n+1}	+1	+1	+1	+1		+1	+1	+1
h_{n+2}								
h_{n+m}								
k	-1	0	0	0		0	0	0

The terms in brackets indicate the appropriate values for consideration of B(0,n).

TABLE 5.3 Weight components for even negative 'supplementary' generators of B(m,n).

	e_1^-	e_2^-	e_3^-	e_4^-	e_{m-1}^-	e_m^-	$\tilde{e}_m^-$	$\tilde{e}_{m-1}^-$	$\tilde{e}_{m-2}^-$	$\tilde{e}_3^-$	$\tilde{e}_2^-$	$\tilde{e}_1^-$
h_1												
h_2												
h_{n-1}												
h_n	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2
h_{n+1}	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	-2
h_{n+2}	+1	-1	0	0	0	0	0	0	0	+1	-1	0
h_{n+3}		+1	-1	0						-1	0	
h_{n+4}			+1	-1						0		
h_{n+5}				+1								
h_{n+m-3}									+1			
h_{n+m-2}								+1	-1			
h_{n+m-1}					-1	0	+1	-1	0			
h_{n+m}					+2	0	-2	0	0			
k	0	0	0	0	0	0	0	0	0	0	0	0

where $e_i^- = [\ldots[[\alpha_{n+1}^-, \alpha_{n+2}^-], \alpha_{n+3}^-], \ldots \alpha_{n+i}^-]$

$\tilde{e}_i^- = [\ldots[[e_m^-, \alpha_{n+m}^-], \alpha_{n+m-1}^-], \ldots \alpha_{n+i}^-]$

and $1 \leq i \leq m$.

TABLE 5.4 Weight components for even negative 'supplementary' generators for $D(m,n)$.

	f_1^-	f_2^-	f_3^-	f_4^-	f_{m-2}^-	f_{m-1}^-	f_m^-	$\tilde{f}_{m-1}^-$	$\tilde{f}_{m-2}^-$	$\tilde{f}_3^-$	$\tilde{f}_2^-$	$\tilde{f}_1^-$
h_1												
h_2												
h_{n-1}												
h_n	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2
h_{n+1}	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	-2
h_{n+2}	+1	-1	0	0	0	0	0	0	0	+1	-1	0
h_{n+3}		+1	-1	0						-1	0	
h_{n+4}			+1	-1						0		
h_{n+5}				+1								
h_{n+m-3}					0	0	0	0	+1			
h_{n+m-2}					-1	0	0	+1	-1			
h_{n+m-1}					+1	-1	+1	-1	0			
h_{n+m}					+1	+1	-1	-1	0			
k	0	0	0	0	0	0	0	0	0	0	0	0

where $f_i^- = [\ldots[[\alpha_{n+1}^-, \alpha_{n+2}^-, \alpha_{n+3}^-], \ldots \alpha_{n+i}^-]$

$f_m^- = [f_{m-2}^-, \alpha_{n+m}^-]$

$\tilde{f}_i^- = [\ldots[[f_m^-, \alpha_{n+m-1}^-], \alpha_{n+m-2}^-], \ldots \alpha_{n+i}^-]$

and $1 \leq i \leq m-1$.

TABLE 5.5 Weight components for all negative generators of C(n).

	β_1^-	β_2^-	β_3^-	β_4^-	β_{n-1}^-	β_n^-	$\tilde{\beta}_{n-1}^-$	$\tilde{\beta}_{n-2}^-$	$\tilde{\beta}_3^-$	$\tilde{\beta}_2^-$
h_1	0	-1	-1	-1	-1	-1	-1	-1	-1	-2
h_2	+1	-1	0	0	0	0	0	0	+1	-1
h_3	0	+1	-1	0	0	0	0	0	-1	0
h_4		0	+1	-1	0	0	0	0	0	0
h_5			0	+1	0	0	0	0	0	
h_{n-3}							0	+1		
h_{n-2}					0	0	+1	-1		
h_{n-1}					-1	+1	-1	0		
h_n					+1	-1	0	0		
k	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

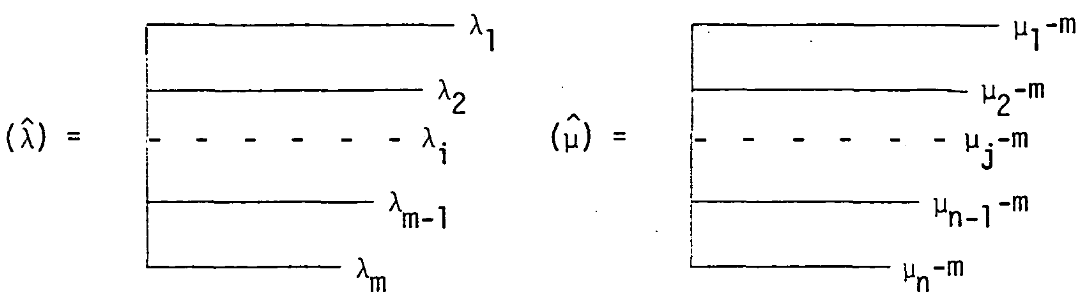
The above tables show the weight components $\alpha(h_i)$ and $\alpha(k)$ where α are the roots associated with the indicated root vectors.

5.3 BRANCHING RULES

5.3.a SPINOR REPRESENTATIONS

In this section branching rules are obtained which decompose finite-dimensional, irreducible, spinor representations of OSp(M/N) in terms of irreducible representations of O(M) x Sp(N). The spinor representations of OSp(M/N) are characterised by a_{n+m} being an odd positive integer for OSp(2m+1/2n) m > 0, and by a_{n+m-1} being an odd positive integer with a_{n+m} an even positive integer (or vice-versa) for OSp(2m/2n) m > 0. They can be represented in terms of Young supertableau by defining a standard, spinor supertableau, $[\Delta; \lambda]$, where (λ) refers to the partition defined in (5.4) and $[\Delta; \lambda]$ is the partition of (5.4) with an additional m 'spinorial' boxes in the (n+1)th column. This spinor supertableau is labelled analogously with (5.4) by $\mu_1, \ldots, \mu_n$ as in (5.4) and by $\lambda_1^s = \lambda_1 + \frac{1}{2}, \ldots, \lambda_m^s = \lambda_m + \frac{1}{2}$ where the λ_i are the labels of (5.4). The relations between the Kac-Dynkin labels (a_i, b) and the spinor supertableau labels are taken to be (5.8) and (5.10) for OSp(2m+1/2n) and OSp(2m/2n) respectively with λ_i replaced by $\lambda_i^s = \lambda_i + \frac{1}{2}$. Two notable features are, there are no spinor representations for $b = \mu_n < m$ and there are no atypical spinor representations.

In the following $\chi_{2m/2n}[\kappa]$, $\chi_{2m}[\kappa]$ and $\chi_{2n}\langle\kappa\rangle$ refer respectively to the OSp(2m/2n), O(2m) and Sp(2n) characters of a partition (κ). Similar notation is used for OSp(2m+1/2n) and O(2m+1). If (λ) is taken as defined in (5.4) then $(\hat{\lambda})$ and $(\hat{\mu})$ are partitions of the form



5.15

Osp(2m/2n) m > 0:

The branching rules expressed here for spinor representations of OSp(2m/2n) are based on two results obtained by King [12]. Firstly, a consequence of Kac's character formulae [4] is that

$$\chi_{2m/2n}[\Delta;\,\lambda] = \chi_{4mn}[\Delta] \cdot \chi_{2m}[\Delta;\,\hat{\lambda}] \cdot \chi_{2n}\,\langle\hat{\mu}\rangle \qquad 5.16$$

Secondly,

$$\chi_{4mn}[\Delta] = \chi_{2m/2n}[n^m/A] \quad . \qquad 5.17$$

Consequently,

$$\chi_{2m/2n}[\Delta;\,\lambda] = \chi_{2m/2n}[n^m/A] \cdot \chi_{2m}[\Delta;\,\hat{\lambda}] \cdot \chi_{2n}\,\langle\hat{\mu}\rangle \qquad 5.18$$

The following sequence of Schur function operations can now be performed.

$$\begin{aligned}[\Delta;\,\lambda]_{2m/2n} &= [n^m/A]_{2m/2n}\,[\Delta;\,\hat{\lambda}]_{2m}\,\langle\hat{\mu}\rangle_{2n} \\ &= \sum_{\xi} [n^m/A\xi]_{2m}\,\langle\tilde{\xi}/B\rangle_{2n}\,[\Delta;\,\hat{\lambda}]_{2m}\,\langle\hat{\mu}\rangle_{2n} \\ &= \sum_{\xi,\tau,\eta} [\Delta;\,(n^m/G\xi\tau)(\hat{\lambda}/\tau)]_{2m}\,\langle(\tilde{\xi}/B\eta)(\hat{\mu}/\eta)\rangle_{2n} \qquad 5.19\end{aligned}$$

<u>OSp(2m+1/2n) m > 0</u>:

For this case King [12] has obtained the following result

$$\chi_{2m+1/2n}\,[\Delta;\,\lambda] = \chi_{2m/2n}\,[n^m/A]\cdot\chi_{2m+1}\,[\Delta;\,\hat{\lambda}]\cdot\chi_{2n+1}[\hat{\mu}] \quad 5.20$$

This can be expressed in a more edifying form through the following sequence of Schur function operations:

$$\begin{aligned}
[\Delta;\,\lambda]_{2m+1/2n} &= [n^m/A]_{2m/2n}\,[\Delta;\,\hat{\lambda}]_{2m+1}\,[\hat{\mu}]_{2n+1} \\
&= \sum_{\xi}\,[n^m/A\xi]_{2m}\,\langle\tilde{\xi}/B\rangle_{2n}\,[\Delta;\,\hat{\lambda}]_{2m+1}\,[\hat{\mu}/M]_{2n} \\
&= \sum_{\xi}\,[n^m/AL\xi]_{2m+1}\,\langle\tilde{\xi}/B\rangle_{2n}\,[\Delta;\,\hat{\lambda}]_{2m+1}\,\langle\hat{\mu}/MBC\rangle_{2n} \\
&= \sum_{\xi}\,[n^m/E\xi]_{2m+1}\,\langle\tilde{\xi}/B\rangle_{2n}\,[\Delta;\,\hat{\lambda}]_{2m+1}\,\langle\hat{\mu}/Q\rangle_{2n} \qquad 5.21 \\
&= [n^m/E]_{2m+1/2n}\,[\Delta;\,\hat{\lambda}]_{2m+1}\,\langle\hat{\mu}/Q\rangle_{2n} \qquad 5.22
\end{aligned}$$

This gives the result

$$\chi_{2m+1/2n}\,[\Delta;\,\lambda] = \chi_{2m+1/2n}\,[n^m/E]\cdot\chi_{2m+1}\,[\Delta;\,\hat{\lambda}]\,\chi_{2n}\,\langle\hat{\mu}/Q\rangle \quad 5.23$$

From (5.21) the following expressions can now be obtained

$$\begin{aligned}
[\Delta;\,\lambda]_{2m+1/2n} &= \sum_{\xi}\,[n^m/E\xi]_{2m+1}\,[\Delta;\,\hat{\lambda}]_{2m+1}\,\langle\tilde{\xi}/B\rangle_{2n}\,\langle\hat{\mu}/Q\rangle_{2n} \\
&= \sum_{\xi,\tau,\eta}\,[\Delta;\,(n^m/C\xi\tau)(\hat{\lambda}/\tau)]_{2m+1}\,\langle(\tilde{\xi}/B\eta)(\hat{\mu}/Q\eta)\rangle_{2n} \qquad 5.24
\end{aligned}$$

The branching rules (5.19) and (5.24) are valid for all spinor representations of OSp(2m/2n) and OSp(2m+1/2n), respectively. Unfortunately these expressions are very complex involving negative terms through the Schur function series G and C. It would be hoped that future work could provide more compact forms for these branching rules.

5.3.b ATYPICAL REPRESENTATIONS

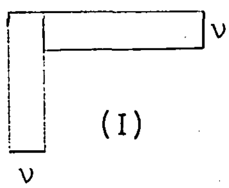
From an explicit knowledge of the atypical representations for OSp(2/2), OSp(3.2) and OSp(4/2) as derived in chapters 3 and 4 it has been possible to determine branching rules for these cases which decompose irreducible, atypical representations of OSp(M/2) M = 2,3,4 in terms of irreducible representations of O(M) x Sp(2). These results as disclosed in (5.25), (5.26) and (5.27) highlight the phenomenon that the indecomposable, atypical representations contain an irreducible, atypical representation as a factor and an irreducible, atypical representation of the same atypicality type, but of lower dimension, in an invariant subpsace. This is just the phenomenon which is apparent in tables 3.1, 3.2 and 3.3 and has been exploited by Morel, Sciarrino and Sorba [13] in their branching rules for atypical diagrams and has been noted by Delduc and Gourdin [16] in their work on indecomposable representations of SU(n/1) as discussed in §5.1. Thierry-Mieg [17] has in fact proved this result quite generally for the basic classical Lie algebras and used it to aid in compiling tables of irreducible representations for a number of these algebras. The irreducible representations obtained in this thesis for OSp(2/2), OSp(3/2) and OSp(4/2) agree with Thierry-Mieg's work.

OSp(2/2):

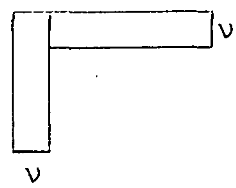
Standard Young supertableaux for tensor representations of OSp(2/2) are characterised by

λ_1 = (number of boxes in the first row -1) and

μ_1 = (number of boxes in the first column). From table 5.1, atypical representations are characterised by $\lambda_1 = \mu_1$. If the supertableau corresponding to the irreducible character of the atypical representation with $\lambda_1 = \nu$ and $\mu_1 = \nu$ is denoted by the supertableau with the subscript (I) then for $\nu > 1$ (if $\nu = 1$ the final term on the R.H.S. of (5.25) disappears)



$=$



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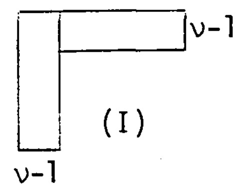


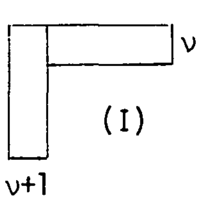
$+ (-1)^{\nu-1} \bullet$

5.25

Using (5.2) or (5.3) the right hand side of (5.25) may be decomposed into irreducible representations of $O(2) \times Sp(2)$ to give the complete branching.

<u>OSp(3/2)</u>:

As described in §5.2.a supertableaux are labelled by λ_1 and μ_1 (see 5.4) and from table 5.1 atypical representations are characterised by $\mu_1 = \lambda_1+1$. Again denoting the supertableau corresponding to the irreducible character of the atypical representation with $\lambda_1 = \nu$ and $\mu_1 = \nu+1$ by the supertableau with the subscript (I) then for $\nu > 1$ (for $\nu = 1$ (5.26) has no singlet on the R.H.S.)



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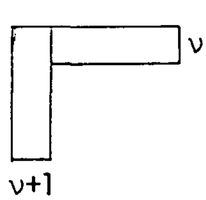

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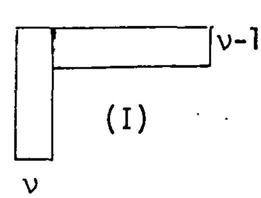


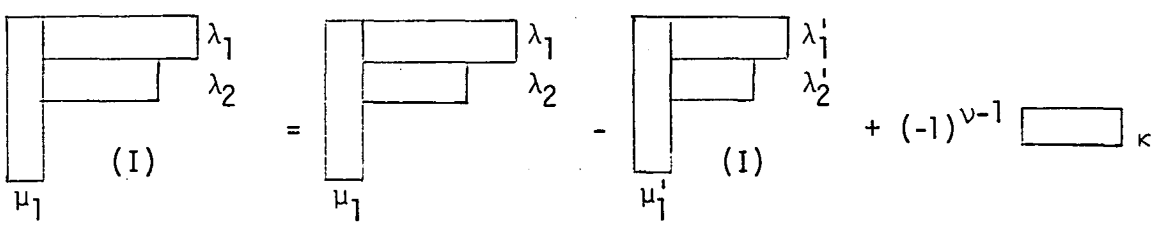
$+ (-1)^{\nu-1} \bullet$

5.26

The right hand side of (5.26) can now be decomposed into irreducible representations of O(3) x Sp(2) through the use of (5.2) or (5.3) to obtain the complete branching.

<u>OSp(4/2)</u>:

As described in §5.2.b supertableaux are labelled by λ_1, λ_2 and μ_1 (see 5.4). The atypical representations of interest here are characterised by (see table 5.1) (1) $\mu_1 = \lambda_1+2$ and (2) $\mu_1 = \lambda_2+1$. Again the irreducible supertableau of an atypical representation is denoted by the subscript (I). Consider the following expression



5.27

This expression is valid when interpreted in the following way.

(1) $\mu_1 = \lambda_1 + 2$:

(a) If $\lambda_1 > \lambda_2$ then set $\mu_1 = \nu+2$, $\lambda_1 = \nu$, $\lambda_2 = \kappa$

$\mu_1' = \nu+1$, $\lambda_1' = \nu-1$, $\lambda_2' = \kappa$.

(b) If $\lambda_1 = \lambda_2$ then set $\mu_1 = \nu+2$, $\lambda_1 = \nu$, $\lambda_2 = \kappa = \nu$

$\mu_1' = \nu$, $\lambda_1' = \nu-1$, $\lambda_2' = \nu-1$.

(2) $\mu_1 = \lambda_2 + 1$:

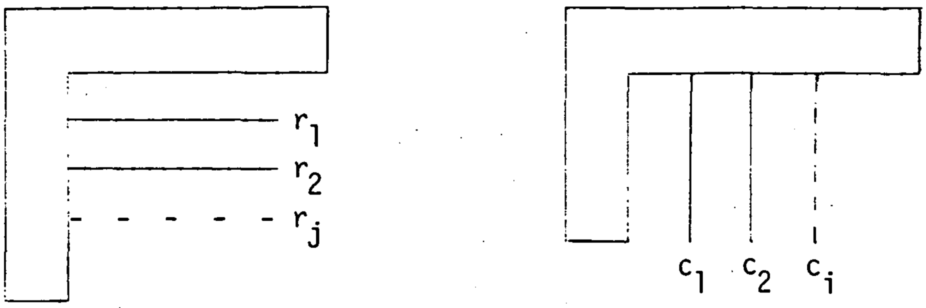
Let $\mu_1 = \nu+1$, $\lambda_2 = \nu$, $\lambda_1 = \kappa$, $\mu_1' = \nu$, $\lambda_2' = \nu-1$, $\lambda_1' = \kappa$.

For each of these cases the final term on the R.H.S. of (5.27) is to be dropped for $\nu = 1$. Using (5.2) or (5.3) the R.H.S. of (5.27) can now be decomposed into irreducible representations of O(4) x Sp(2).

5.4 MODIFICATION RULES FOR YOUNG SUPERTABLEAUX

In this section consideration will be given to the treatment of non-standard supertableaux. In particular the modification rules will be presented through which non-standard supertableaux may be expressed in terms of standard supertableaux. In this section we maintain the same relations between the Kac-Dynkin labels and the diagram labels as given in §5.2 and call those diagrams typical which do not satisfy either of the conditions of table 5.1 and those diagrams atypical which satisfy either of these conditions. Typical and atypical diagrams modify in significantly different ways. In §5.4.a general modification rules are presented for all typical, tensor supertableaux of OSp(M/N). In §5.4.b the modification rules for atypical supertableaux are discussed and explicit results given for OSp(2/2), OSp(3/2) and OSp(4/2). The results presented here have been obtained by explicitly decomposing numerous supertableaux using (5.2) or (5.3), with the aid of the group theory computer package SCHUR. General proofs for these results remain the work of future investigations, but for the typical case a proof would presumably follow directly from the results expressed in (5.35) and (5.36).

As described earlier, standard tableaux for OSp(M/N) lie within the envelope shown in (5.4) and (5.11). Non-standard tableaux will include boxes outside this envelope. These 'extra' boxes we label by row lengths, r_j, or column lengths, c_i as shown below.



5.28

5.4.a TYPICAL SUPERTABLEAUX

<u>OSp(2m+1/2n)</u>:

(1) If $r_1 \geq c_1$ the modification rule is

$$[\lambda] \to [\lambda]_M = (-1)^{r-1} \quad [\lambda-h] \quad , \quad h = 2r_1-1 \qquad 5.29$$

where h is the hook boundary length to be removed from $[\lambda]$ starting from the end box in r_1 and working to the left and down with r being the row in which the removal ends.

(2) If $c_1 \geq r_1$ the modification rule is

$$[\lambda] \to [\lambda]_M = (-1)^{c-1} \quad [\lambda-h] \quad , \quad h = 2c_1-1 \qquad 5.30$$

where h is again the hook boundary length to be removed from $[\lambda]$ starting from the end box in c_1 and working to the right and up with c being the column in which the removal ends.

If any modification results in an irregular diagram this diagram is set to zero.

OSp(2m/2n):

(1) If $r_1 > c_1$ the modification rule is

$$[\lambda] \rightarrow [\lambda]_M = (-1)^r \quad [\lambda-h] \quad , \quad h = 2r_1-2 \qquad 5.31$$

and proceed as for OSp(2m+1/2n) case (1).

(2) If $c_1 \geq r_1$ the modification rule is

$$[\lambda] \rightarrow [\lambda]_M = (-1)^{c-1} \quad [\lambda-h] \quad , \quad h = 2c_1 \qquad 5.32$$

and proceed as for OSp(2m+1/2n) case (2).

These results have a natural interpretation in terms of the character formulae of King [12]. For a typical, tensor representation with corresponding standard Young supertableau $[\lambda]$, as defined by (5.4), he has noted that for

$$\text{OSp}(2m+1/2n) : \chi_{2m+1/2n}[\lambda] = \chi_{2m+1/2n}[n^m/E] \cdot \chi_{2m+1}[\hat{\lambda}] \cdot \chi_{2n+1}[\hat{\mu}] \qquad 5.33$$

and for $\text{OSp}(2m/2n) : \chi_{2m/2n}[\lambda] = \chi_{2m/2n}[n^m/A] \cdot \chi_{2m}[\hat{\lambda}] \cdot \chi_{2n} <\hat{\mu}>$ 5.34

where $(\hat{\lambda})$ and $(\hat{\mu})$ are defined in (5.15).

If $[\lambda']$, as defined in (5.28), is a non-standard, typical supertableau then the modification rules (5.29)-(5.32) tell us, in the light of (5.33) and (5.34), that

$$[\lambda']_{2m+1/2n} = [n^m/E]_{2m+1/2n} \, [\hat{\lambda}']_{2m+1} \, [\hat{\mu}']_{2n+1} \qquad 5.35$$

and

$$[\lambda']_{2m/2n} = [n^m/A]_{2m/2n} \, [\hat{\lambda}']_{2m} \, <\hat{\mu}'>_{2n} \qquad 5.36$$

where, if $r_1 > c_1$:

$$(\hat{\lambda}') = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \lambda_m)$$

$$(\hat{\mu}') = \mu_1-m, \mu_2-m, \ldots, \mu_n-m, c_1, c_2, \ldots, c_i)$$

and if $c_1 \geq r_1$:

$$(\hat{\lambda}') = (\lambda_1, \lambda_2, \ldots, \lambda_m, r_1, r_2, \ldots, r_j)$$

$$(\mu') = (\mu_1-m, \mu_2-m, \ldots, \mu_n-m).$$

Thus the modification of a supertableau is essentially a modification of $(\hat{\lambda}')$ in O(2m+1) or O(2m) or of $(\hat{\mu}')$ in O(2n+1) or Sp(2n).

5.4.b ATYPICAL SUPERTABLEAUX

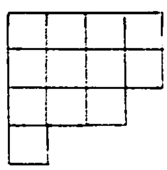
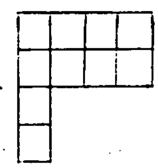
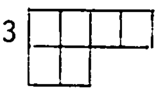
For a supertableaux to be atypical one or more of the conditions of table 5.1 must be fulfilled. An analysis of these conditions reveals that for regular, non-standard supertableaux, none of the conditions (i) of this table can be fulfilled. However, for regular, non-standard supertableaux in OSp(M/N) the maximum number of the conditions (ii), of table 5.1, which may be simultaneously realized is the lesser of (N/2) and [M/2]. In the cases examined below it is possible to realize only one atypicality condition for a given regular, non-standard supertableaux.

In the notation of (5.4) non-standard, atypical diagrams for OSp(2/2) are characterised by $\lambda_1 = \mu_1 = \nu$. Non-standard, atypical diagrams for OSp(3.2) are characterised by $\lambda_1 = \mu_1-1 = \nu$. Non-standard, atypical diagrams for OSp(4/2) are characterised by either (i) $\lambda_1 = \mu_1-2 = \nu$ or (ii) $\lambda_2 = \mu_1-1 = \nu$. In the above ν is any positive integer. For each of these cases the appropriate modification rule is

$$[\lambda] = [\lambda]_M + (-1)^{\nu}\,[\lambda]_A + (-1)^{\nu-1}\,[\lambda]_B \qquad 5.37$$

where $[\lambda]_M$ is diagram obtained from $[\lambda]$ by application of the rules given in §5.4.a and $[\lambda]_A$ and $[\lambda]_B$ are obtained as follows. If $[\lambda]$ is an atypical diagram by virtue of it satisfying one of the conditions (ii) of table 5.1 which relates say μ_i and λ_j then $[\lambda]_A$ is obtained from $[\lambda]$ by removal of a boundary which starts from the final box of row j and ends in the final box of column i. $[\lambda]_B$ is obtained by removal of a similar boundary from $[\lambda]_M$ remembering to carry any sign factors $[\lambda]_M$ possesses onto $[\lambda]_B$. The branching rules developed in §5.3 can now be used to yield irreducible representations for any of these algebras. As a demonstration of the working of (5.37) an example is given, for OSp(4/2), in table 5.6.

TABLE 5.6 Modification in OSp(4/2) of the supertableau $[\lambda] \equiv [\lambda_1=3,\ \lambda_2=3,\ \mu_1=4,\ r_1=2]$ and its decomposition into irreducible representations of O(4) x Sp(2). This tableau satisfies the atypicality condition $\mu_1 = \lambda_2+1$ and $\nu = 3$.

Modification :  = -  + $(-1)^3$  + $(-1)^2$ x - 

$[\lambda]$ $[\lambda]_M$ $[\lambda]_A$ $[\lambda]_B$

o(4) x Sp(2) decomposition:

$[\lambda] \downarrow$ - [44] <2> - [43] <3> - [43] <1> - [42] <2>

- [42] <0> - [33] <4> - [33] <2> - [33] <0>

- [41] <1> - [32] <3> - [32] <1> - [32] <1>

- [4] <0> - [31] <2> - [31] <0> - [22] <2>

- [22] <0> - [3] <1> - [21] <1> - [2] <0>

$[\lambda]_M \downarrow$ - [44] <2> - [43] <3> - [43] <1> - [42] <2>

- [33] <4> - [33] <2> - [33] <0> - [32] <3>

- [32] <1> + [4] <0> - [22] <2> + [3] <1> + [2] <0>

$[\lambda]_A \downarrow$ + [42] <0> + [41] <1> + [32] <1> + [4] <0> + [31] <2>

+ [31] <0> + [22] <0> + [3] <1> + [21] <1> + [2] <0>

$[\lambda]_B \downarrow$ - [4] <0> - [3] <1> - [2] <0>

5.5 DIMENSION FORMULAE

The dimensions, in supertableaux notation, of all irreducible, typical representations for U(M/N) and OSp(M/N) have been given by King [2,12]. For completeness these results for OSp(M/N) are reproduced here in the present notation. If (λ) is a standard, typical partition as given by (5.4) then for

$$\text{OSp}(2m/2n) : D_{(2m,2n)}\,[\lambda] = 2^{2mn}\, D_{2m}[\hat{\lambda}] \cdot D_{2n}\,\langle\hat{\mu}\rangle \qquad 5.38$$

$$D_{(2m,2n)}\,[\Delta;\,\lambda] = 2^{2mn}\, D_{2m}[\Delta;\,\hat{\lambda}] \cdot D_{2n}\,\langle\hat{\mu}\rangle \qquad 5.39$$

$$\text{for OSp}(2m+1/2n) : D_{(2m+1/2n)}\,[\lambda] = 2^{2mn}\, D_{2m+1}[\hat{\lambda}] \cdot D_{2n+1}[\hat{\mu}] \qquad 5.40$$

$$D_{(2m+1/2n)}\,[\Delta;\,\lambda] = 2^{2mn}\, D_{2m+1}[\Delta;\,\hat{\lambda}] \cdot D_{2n+1}[\hat{\mu}] \qquad 5.41$$

where $D_{(M/N)}[\ ]$, $D_M[\ ]$ and $D_N\langle\ \rangle$ refer to the dimension of the relevant representation in OSp(M/N), O(M) and Sp(N) respectively and $(\hat{\lambda})$ and $(\hat{\mu})$ are defined in (5.15).

The above is always sensible for $\langle\hat{\mu}\rangle$ since, as pointed out by King [2] and can easily be derived from (i) of table 5.1, if $\mu_n < m$ then $[\lambda]$ is atypical. El Samra and King [19] have given compact dimension formulae for the classical Lie groups which for a partition, (κ), with row lengths κ_i and column lengths $\tilde{\kappa}_i$ are for O(M) and Sp(N) respectively

$$D_M[\kappa] = \prod_{(i\leq j)}^{\kappa} (M+\kappa_i+\kappa_j-i-j) \prod_{(i>j)}^{\kappa} (M-\tilde{\kappa}_i-\tilde{\kappa}_j+i+j-2)/H(\kappa) \qquad 5.42$$

$$D_N\,\kappa = \prod_{(i<j)}^{\kappa} (N+\kappa_i+\kappa_j-i-j+2) \prod_{(i\geq j)}^{\kappa} (N-\tilde{\kappa}_i-\tilde{\kappa}_j+i+j)/H(\kappa) \qquad 5.43$$

where the products are taken over all pairs (i,j) specifying positions of boxes of the Young diagrams with i specifying the column number and j the row number. The denominators, $H(\kappa)$, in (5.42 & 5.43) refer to Robinson's hook length formula [20] which is a product of hook lengths given by

$$H(\kappa) = \prod_{(i,j)}^{\kappa} (\kappa_i + \tilde{\kappa}_j - i - j + 1) \ . \qquad 5.44$$

For spinor representations of O(M) the dimension is obtained from

$$D_M[\Delta;\ \kappa] = 2^{[M/2]}\ D_{M-1} <\kappa> \ . \qquad 5.45$$

CHAPTER 5 - REFERENCES

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6. NEW SUPERFIELDS FOR N-EXTENDED SUPERSYMMETRY WITH CENTRAL CHARGES

6.1 INTRODUCTION

In this chapter new methods are developed for the study of N-extended supersymmetry in superspace. Although the ultimate goals of the superfield programme, as applied to the study of super space-time algebras, are the construction of realistic interacting models with a view to their quantum behaviour, the work here remains at the linearized level. Specifically, new representations are introduced which generalize, to the case of N-extended supersymmetry with unrestricted central charges, the notion of chiral superfields; a step which general arguments from the usual superfield framework would indicate as problematical. To the extent that a pluralistic attack is needed on unresolved questions of maximally extended $N=4$ super Yang-Mills and $N=8$ supergravity models [1,2], the present work and extensions of it may find application alongside other approaches. Thus, although rapid progress has been made recently in component formalisms at the classical level [e.g. 3], comprehensive results with the quantized models will require full local and covariant techniques. The complexities of the latter have engendered such modifications as N-supersymmetry in an $N=1$-'component superfield' basis [4,5,6,7,8] and light-cone formalisms [9,10,11,12] which necessitate sacrifices such as auxiliary field content, manifest Lorentz invariance or locality. There are indications based on counting arguments that beyond $N=2$ the full N-superspace is intrinsically inadequate to represent physical multiplets [13,14] unless particular 'spin-reducing' representations are used [4,5,15,16]. These emerge naturally in the present work.

The conventional method of nonlinear realisations on coset spaces, as applied to N-extended supersymmetry, considers functions (superfields), Φ, on the coset spaces $((Z \times O(3,1) \times G) \wedge T_{4/4N})/O(3,1) \times G$, where $O(3,1)$ is the Lorentz group, G is an internal symmetry group (a subgroup of Sp(2N)), $T_{4/4N}$ is the nilpotent algebra of translations and Z denotes the abelian central charges [17,18,19]. As discussed in Chapter 4, induced representations of the algebra are afforded by these superfields, $\Phi^{\{\lambda\}}_{(p,q)}(x^\mu,\bar{\theta}^{ai},\theta^{ai},\ldots)$, which are functions of spinor parameters $(\bar{\theta}^{ai},\theta^{ai})$ transforming as $(\frac{1}{2},0) \times \{1\} + (0,\frac{1}{2}) \times \{\bar{1}\}$ under $O(3,1) \times G$, plus the usual Minkowski space coordinates, x^μ, and some additional bosonic central charge coordinates. The superfields take their values in a representation space of the little group, $O(3,1) \times G$, labelled by $(p,q) \times \{\lambda\}$. The representations of the N-extended superalgebra, realised by these superfields, are highly reducible and it is a nontrivial exercise to extract the irreducible content of a given superfield.

These superfields and the physical multiplets contained within them have been analyzed extensively [20-29]. The superfield is a function of 4N Grassmann coordinates, and consequently when expanded in these coordinates, contains 2^{4N} component fields. A satisfactory analysis of the representation content of these fields, requires the use of the maximal automorphism symmetry of the algebra, this being Sp(2N) in the absence of central charges or a subgroup of Sp(2N) if central charges are present. The irreducible representations are obtained by realising that the superfields are in fact irreducible under an enlarged algebra containing covariant derivatives which anticommute with the supertranslations. These can then be used to provide labelling operators, including Casimir invariants, from which projectors can be constructed to provide differential constraints on the superfields. These suitably constrained superfields provide irreducible representations of the original superalgebra

A useful set of projectors corresponds to the 'chiral' case where a superfield is constrained to have vanishing covariant derivative and consequently can be solved in terms of a function only of $x^\mu + \bar{\theta}\sigma^\mu\theta$ and say θ^{ai}, thereby having only 2^{2N} components. However, since central charges arise from the anticommutation of covariant derivatives, care must be exercised, lest on-shell conditions (e.g. $P^2 = 0 = |Z|^2$) be applied already as constraints [30,31].

The approach expounded here differs in two fundamental respects to the conventional procedure. First, the central charges are realised as multiplicative, complex parameters rather than extra coordinates. Second, the superfields are functions of Grassmann parameters of only a particular chirality but take their values in a graded representation space of a superalgebra. Thus superfields are functions on the coset space $((Z \times O(3,1) \times G) \wedge T_{4/4N})/((Z \times O(3,1) \times G) \wedge T_{0/2N})$, where $T_{0/2N}$ is the superalgebra of supertranslations of a particular chirality. These superfields are functions of only 2N Grassmann coordinates but possess 'external' representations of $(Z \times O(3,1) \times G) \wedge T_{0/2N}$. As will be seen in the next section, these include $2^{2[\frac{1}{2}N]}$ irreducible representations of the Lorentz group giving a total of $2^{2N+2[\frac{1}{2}N]}$ component fields.

It has been observed [4,5,15,16] that for $P^2 + |Z|^2 = 0$ where $|Z|^2 = |\bar{Z}_{ij}Z_{ij}|$, a constraint is imposed on the supertranslation generators effecting a drastic reduction in the number of component fields contained in an irreducible representation. In fact, in the presence of the maximal number of central charges, all of which fulfil the above condition, the number of component fields reduces from 2^{2N} to 2^N.

In the present treatment $P^2 + |Z|^2 = 0$ is an atypicality condition (see chapters 2, 3 and 4) under which otherwise irreducible superfields become indecomposable;on each factor space the constraint is implemented modulo coset elements.

It is via these so-called 'spin-reducing' cases (which will become $P^2 = |Z|^2 = 0$ on shell) that one hopes to avoid the 'component explosion', and give a full off shell formalism for $N \geq 3$ supersymmetry (for results of a different implementation of this approach see [32]). As far as the present work is concerned we observe that bilinear invariants may always be written down (at least in component form) which in fact serve as definitions of the contragrediently-transforming superfield; presumably a corresponding projector formalism could be found [26,27,29,33]. However in practice such projections are implemented via gauge freedoms and other constraints, so there is little to be gained in the absence of these and without interactions. In this connection the possibility of a geometrical framework for the present superfield realizations also raises interesting questions.

6.2 CONSTRUCTION OF INDUCED REPRESENTATIONS

The SO(N)-extended super-Poincaré algebra, $\mathcal{G}$ consists of the generators $(P_\mu, J_{\mu\nu})$ of the Poincaré algebra, spinorial generators $(Q_{\alpha i}, \bar{Q}_{\alpha i})$, $\frac{1}{2}N(N-1)$ SO(N) generators $T_{ij} = -T_{ji}$ and at most $\frac{1}{2}N(N-1)$ complex central charges $Z_{ij} = -Z_{ij}$ where $i,j = 1,\ldots,N$. These generators satisfy the following superalgebra

$$[J_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu)$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho})$$

$$\{Q_{\alpha i}, \bar{Q}_{\dot{\alpha} j}\} = -2\delta_{ij}(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu$$

$$\{Q_{\alpha i},Q_{\beta j}\} = 2\varepsilon_{\alpha\beta}Z_{ij}$$

$$\{\bar{Q}_{\dot{\alpha} i},\bar{Q}_{\dot{\beta} j}\} = 2\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{Z}_{ij}$$

$$[J_{\mu\nu},Q_{\alpha i}] = -i(\sigma_{\mu\nu})_{\alpha}{}^{\beta}Q_{\beta i}$$

$$[J_{\mu\nu},\bar{Q}_{\dot{\alpha} i}] = i(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta} i}$$

$$[Q_{\alpha i},T_{jk}] = (t_{jk})_{i}{}^{\ell}Q_{\alpha\ell}$$

$$[\bar{Q}_{\alpha i},Tjk] = (t_{jk})_{i}{}^{\ell}\bar{Q}_{\dot{\alpha}\ell}$$

$$[T_{ij},T_{k\ell}] = C^{mn}_{ij,k\ell}T_{mn} \qquad 6.1$$

where $(t_{jk})_{i}{}^{\ell}$ is an hermitian representation of T_{jk} and $C^{mn}_{ij,k\ell}$ are the structure constants of SO(N). All other (anti-)commutators are zero.

The technique of constructing induced representations of $\mathcal{G}$ is analogous to the procedure used in chapter 4 to which the reader is referred for a more formal discussion of the inducing construction. This technique was first applied to supersymmetry by Salam and Strathdee [34]. In the present work superspace is taken as the coset space, G/H, where G is the SO(N)-extended super-Poincaré group whose corresponding superalgebra is $\mathcal{G}$ and H is a subgroup of G with corresponding superalgebra $\mathcal{H} = \{J_{\mu\nu},Q_{\alpha i},T_{ij},Z_{ij}\}$. This coset space can be parametrized as $\exp i(x^{\mu}P_{\mu} + \bar{\theta}^{\dot{\alpha} i}\bar{Q}_{\dot{\alpha} i})$ with coordinates $(x^{\mu},\bar{\theta}^{\dot{\alpha} i})$, where $x^{\mu}(\bar{\theta}^{\dot{\alpha} i})$ is a c-(a-) number parameter. Representations of $\mathcal{G}$ are afforded by superfields $\Phi_A(x^{\mu},\bar{\theta}^{\dot{\alpha} i})$ which are functions on G/H taking their values in a representation space V, of $\mathcal{H}$.

The group action on G/H is infinitesimally

$$\exp i\ (sS) \cdot \exp i\ (X^{\mu}\ P_{\mu} + \bar{\theta}^{\dot{\alpha} i}\ \bar{Q}_{\dot{\alpha} i}\ )$$

$$= \exp i[(x^{\mu} + sf^{\mu}(x,\bar{\theta}))P_{\mu} + (\bar{\theta}^{\dot{\alpha} i} + sg^{\dot{\alpha} i}(x,\bar{\theta}))\bar{Q}_{\dot{\alpha} i}]$$

$$\cdot \exp i\ \Sigma(sk(x,\bar{\theta})K) \qquad 6.2$$

where $S \in \mathcal{G}$ and $K \in \mathcal{H}$. The precise form of $f^{\mu}(x,\bar{\theta})$, $g^{\dot{\alpha}i}(x,\bar{\theta})$ and $k(x,\bar{\theta})$ can be obtained via the BCH formula. From (6.2) the differential representation of the generators will be

$$S \to -if^{\mu}(x,\bar{\theta})\partial/\partial x^{\mu} - ig^{\dot{\alpha}i}(x,\bar{\theta})\partial/\partial\bar{\theta}^{\dot{\alpha}i} - \Sigma\ k(x,\bar{\theta})K^{0} \qquad 6.3$$

where K^0 is the matrix of the infinitesimal generator K in the representation carried by V. The action on superfields is given simply by

$$\delta\Phi_A(x^{\mu},\bar{\theta}^{\dot{\alpha}i}) = S\,\Phi_A(x^{\mu},\bar{\theta}^{\dot{\alpha}i}). \qquad 6.4$$

Obtaining irreducible representations of $\mathcal{G}$ in the above manner presumes the irreducible representations of $\mathcal{H}$ are known. Since $\mathcal{H}$ is also a superalgebra an analogous procedure to the above is followed. Firstly, it is noted that with respect to the positive, negative and zero roots of SO(N) the generators T_{ij} and $Q_{\alpha i}$ may be written in bases $T_{ij} = \{T_a^+, T_a^-, T_n^0\}$ and $Q_{\alpha i} = \{Q_{\alpha n}^+, Q_{\alpha n}^-, Q_{\alpha}^0\}$ respectively, where $n = 1,\ldots,[\frac{1}{2}N]$, $a = 1,\ldots,[\frac{1}{2}N][\frac{1}{2}(N-1)$ and Q_{α}^0 only exists for N odd, for which $[T^{\pm},T^0] \subset T^{\pm}$, $[T^+,T^-] \subset T^0$, $\{Q^+,Q^-\} \subset Z$, $[T^{\pm},Q^{\mp}] \subset Q^{\pm}$ and all other (anti-) commutators involving these generators are zero. To implement the inducing construction on $\mathcal{H}$, a subgroup, H', of H is chosen with corresponding superalgebra $\mathcal{H}' = \{J_{\mu\nu},T_a^+,T_n^0,Q_{\alpha n}^+,Q_{\alpha}^0,Z_{ij}\}$. It is possible to decompose $\mathcal{H}'$ as $= \mathcal{H}'_0 + \mathcal{H}'_+$, where $\mathcal{H}'_+ = \{T_a^+,Q_{\alpha n}^+,Q_{\alpha}^0\}$ is an ideal. Representations of $\mathcal{H}'_0$ are then extended to $\mathcal{H}$ by taking them to be zero on $\mathcal{H}'_+$. Representations of $\mathcal{H}$ are afforded by superfields $\Psi_B(y^a,\theta^{\alpha n}\quad)$ which are functions on the coset space H/H' and taking their values in a representation space of $\mathcal{H}'$. This coset space is parametrized as $\exp i(y^aT_a^- + \theta^{\alpha n}Q_{\alpha n}^-)$ with coordinates $(y^a,\theta^{\alpha n})$. In the manner described above, the generators of $\mathcal{H}$ can be realized as differential operators on the coset space and the action on the superfields examined to determine the finite-dimensional, irreducible representations of $\mathcal{H}$ by the method expounded in chapter 4.

Since an expansion of $\Psi_B(y^a,\theta^{\alpha n}\quad)$ in $\theta^{\alpha n}$ yields $2^{2[\frac{1}{2}N]}$ component fields and an expansion of $\Phi_A(x^\mu,\bar{\theta}^{\dot{\alpha}i})$ in $\bar{\theta}^{\dot{\alpha}i}$ yields 2^{2N} component fields, each of which carries a representation of $\mathcal{H}$, there are a total of $4^{N+[\frac{1}{2}N]}$ component fields. There may however be fewer than this if the representation of $\mathcal{H}$ carried by Ψ_B is reducible.

The problem of determining irreducible representations of $\mathcal{G}$ must now be addressed. In the conventional procedure, discussed in §6.1, the algebra $\mathcal{G}$ is extended to include covariant derivatives which, together with the generators of $\mathcal{G}$, provide a basis in superspace for the enveloping algebra and under which the superfields are still invariant. Since the superfields provide a representation space for the extended algebra they are expected to be reducible under $\mathcal{G}$. A similar situation exists in the present case.

The differential form of the spinorial generators is

$$\bar{Q}_{\dot{\alpha}i} = -i\ \partial/\partial\bar{\theta}^{\dot{\alpha}i} + i\ \bar{\theta}^{\dot{\beta}j}\ \varepsilon_{\dot{\alpha}\dot{\beta}}\ Z^0_{ij} \qquad 6.5$$

$$Q_{\alpha i} = 4\bar{\theta}^{\dot{\alpha}i}(\sigma^\mu)_{\alpha\dot{\alpha}}\ P_\mu - Q^0_{\alpha i} \qquad 6.6$$

where Z^0_{ij} and $Q^0_{\alpha i}$ are matrix representations of the corresponding generator. Remembering that Z_{ij} is totally antisymmetric, (6.5) and (6.6) tell us that a basis for the enveloping algebra in superspace is provided by extending the superalgebra to include a new set of generators, $\bar{S}_{\dot{\alpha}i} = \partial/\partial\bar{\theta}^{\dot{\alpha}i}$. It is noted however that the complete set of differential operators, $\partial/\partial\bar{\theta}^{\dot{\alpha}i}$, is not required for this basis if N is odd. This becomes apparent if one regards the second term on the right of (6.5) as a set of N linear equations in the variables $\bar{\theta}^j$ with coefficients $Z^0{}_{ij}$. Since $Z^0{}_{ij}$ is a totally antisymmetric N × N matrix it will have zero determinant, for odd N, and consequently the equations will be linearly dependent. Thus, for N odd, at least one of the generators, $\bar{S}_{\dot{\alpha}i}$, can be regarded as being constructed from linear combinations of the other generators.

This extended algebra is denoted by $\bar{\mathcal{G}}$. The generators $\bar{S}_{\dot{\alpha}e}$ are, however, significantly different to the covariant derivatives of the conventional procedure in that they do not anticommute with $\bar{Q}_{\dot{\alpha}i}$ and $Q_{\alpha i}$ and thus cannot be used to generate irreducible representations of $\bar{\mathcal{G}}$ from irreducible representations of $\mathcal{G}$. Rather than adopting the treatment based upon the construction of Casimir invariants and associated projection operators [20,21,22,23, 26,27,29] the procedure here is based upon recognition of highest (and lowest) weight components, and explicit construction of the invariant subspaces therefrom. First, it is noted that $\bar{S}_{\dot{\alpha}i}$ and $\bar{Q}_{\dot{\alpha}i}$ may be cast in bases $\bar{S}_{\dot{\alpha}i} = \{\bar{S}^+_{\dot{\alpha}n}, \bar{S}^-_{\dot{\alpha}n}, \bar{S}^0_{\dot{\alpha}}\}$ and $\bar{Q}_{\dot{\alpha}i} = \{\bar{Q}^+_{\dot{\alpha}n}, \bar{Q}^-_{\dot{\alpha}n}, \bar{Q}^0_{\dot{\alpha}}\}$ with similar properties to $Q^{\pm}_{\alpha n}$ and $Q^0{}_{\alpha}$. From the discussion of the previous paragraph we note that it is possible to regard $\bar{S}^0_{\dot{\alpha}}$ as a linear combination of the other generators of $\bar{\mathcal{G}}$ and thus it is not an independent generator. Consequently in the following work it will not be counted in the explicit construction of states.

Irreducible representations of $\bar{\mathcal{G}}$ (and hence superfields) are obtained by an inducing construction from irreducible representations of $\mathcal{G}$. The irreducible representations of $\mathcal{G}$ may also be obtained from an inducing construction by choosing a subalgebra $\mathcal{H}$, of $\bar{\mathcal{G}}$ where $\mathcal{H} = \{\bar{S}^+_{\dot{\alpha}n}, \bar{Q}^+_{\dot{\alpha}n}, Q^+_{\alpha n}, Q^0_{\alpha}, T^+_a, T^0_n, Z_{ij}, J_{\mu\nu}\}$ and states, Λ, which are irreducible representations of the little algebra $\mathcal{H}_0 = \{T^0_n, J_{\mu\nu}, Z_{ij}\}$

and which satisfy $\bar{S}^{+}_{\dot{\alpha}n}\Lambda = \bar{Q}^{+}_{\dot{\alpha}n}\Lambda = Q^{+}_{\alpha n}\Lambda = Q^{0}_{\alpha}\Lambda = T^{+}_{a}\Lambda = 0$. This last requirement is justified from the fact that $\mathcal{H}^{+} = \{\bar{S}^{+}_{\dot{\alpha}n}, \bar{Q}^{+}_{\dot{\alpha}n}, Q^{+}_{\alpha n}, Q^{0}_{\alpha}, T^{+}_{a}\}$ is an ideal of $\mathcal{H}$. A basis for an irreducible representation of $\mathcal{G}$, of states which are representations of $\mathcal{H}$, is obtained by acting with monomials of $\bar{Q}^{-}_{\dot{\alpha}n}$, $\bar{Q}^{0}_{\dot{\alpha}}$ and $Q^{-}_{\alpha n}$ on Λ. A similar basis, for irreducible representations of $\bar{\mathcal{G}}$, is obtained by acting with monomials of $\bar{Q}^{-}_{\dot{\alpha}n}$, $\bar{Q}^{0}_{\dot{\alpha}}$, $Q^{-}_{\alpha n}$ and $\bar{S}^{-}_{\dot{\alpha}n}$ on Λ. Thus, a superfield will possess $2^{2[\frac{1}{2}N]}$ irreducible multiplets of $\mathcal{G}$ each of which contains 2^{2N} irreducible multiplets of $\mathcal{H}_{0}$, giving a total of $4^{N+[\frac{1}{2}N]}$ component fields as required by the superfield analysis. Unlike the conventional case, where the irreducible multiplets of $\mathcal{G}$ are invariant under the covariant derivatives, the $\bar{S}_{\dot{\alpha}e}$ will mix these representations.

The 'spin-reducing' cases can be obtained by introducing further field redefinitions for which the constraint, $P^2 + |Z|^2 = 0$, is an atypicality condition under which otherwise irreducible multiplets become indecomposable. This programme is carried out in detail in §6.3 for SO(2)-extended supersymmetry.

6.3 N = 2 EXTENDED SUPERSYMMETRY WITH CENTRAL CHARGE

6.3a Algebra

The SO(2) graded extension of the Poincaré algebra, $\mathcal{G}$, is obtained by taking, in addition to the generators of the Poincaré algebra, P_{μ} and $J_{\mu\nu}$, the generator for SO(2) transformations, T, and the Majorana spinor charges $Q_{\alpha a}$ and $\bar{Q}_{\dot{\alpha}a}$, where $1 \leq \alpha, \dot{\alpha} \leq 2$ and $a = +,-$. In its most general form the algebra may also include a central charge, Z. In the Weyl representation these generators satisfy the following graded Lie algebra

$$[J_{\mu\nu},P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu)$$

$$[J_{\mu\nu},J_{\rho\sigma}] = i(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho})$$

$$\{Q_{\alpha\pm},\bar{Q}_{\dot{\alpha}\mp}\} = -4(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu$$

$$\{Q_{\alpha a},Q_{\beta b}\} = 4i\ \varepsilon_{\alpha\beta}\ \varepsilon_{ab}Z$$

$$\{\bar{Q}_{\dot{\alpha}a},\bar{Q}_{\dot{\beta}b}\} = 4i\ \varepsilon_{\dot{\alpha}\dot{\beta}}\ \varepsilon_{ab}\bar{Z}$$

$$[J_{\mu\nu},Q_{\alpha a}\} = -i(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta a}$$

$$[J_{\mu\nu},\bar{Q}_{\dot{\alpha}a}\} = i(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}a}$$

$$[T,Q_{\alpha\pm}] = \pm Q_{\alpha\pm}$$

$$[T,\bar{Q}_{\dot{\alpha}\pm}] = \pm\bar{Q}_{\dot{\alpha}\pm} \qquad 6.7$$

where $\varepsilon_{-+} = -\varepsilon_{+-} = +1$ and all other (anti-) commutators are zero. The metric is taken as $\eta_{\mu\nu} = (-,+,+,+)$.

Following the procedure discussed in §6.2 the subalgebra, $\mathcal{H}$, is taken to be $\mathcal{H} = \{J_{\mu\nu},T,Z,Q_{\alpha a}\}$ with little group $\mathcal{H}_0 = \mathcal{H}$. The cosets $\mathcal{G}/\mathcal{H}$ are labelled by the elements $\exp i(x^\mu P_\mu + \bar{\theta}^{\dot{\alpha}a}\bar{Q}_{\dot{\alpha}a})$ and the superfields are defined as functions, $\Phi_A(x^\mu,\bar{\theta}^{\dot{\alpha}a})$, taking their values in a representation space of $\mathcal{H}_0$.

Since $\mathcal{H}$ still defines a superalgebra the first task is to determine the irreducible representations of $\mathcal{H}$. To do this the above procedure is repeated with the subalgebra $\mathcal{H}'$ of $\mathcal{H}$ taken as $\mathcal{H}' = \{J_{\mu\nu},T,Z,Q_{\alpha+}\}$ and little group $\mathcal{H}_0' = \{J_{\mu\nu},T,Z\}$. The cosets $\mathcal{H}/\mathcal{H}'$ are labelled by the elements $\exp i(\theta^{\alpha-}Q_{\alpha-})$ and the superfields are defined as functions $\psi(\theta^{\alpha-})$ taking their values in a representation space of $\mathcal{H}_0'$. In §6.3b the irreducible representations of $\mathcal{H}$ are determined and subsequently used to deduce the irreducible representations of $\mathcal{G}$ in §6.3c.

6.3b Irreducible Representations of $\mathcal{H}$

The generators of $\mathcal{H}$ can be realized as differential operators in the coset space $\mathcal{H}/\mathcal{H}'$ and as matrix representatives in the representation space of $\mathcal{H}_0'$. Explicitly they take the form:

$$
\begin{aligned}
Q_{\alpha-} &= -i\partial_{\alpha-} \\
Q_{\alpha+} &= 4\theta^{\beta-}\varepsilon_{\beta\alpha}Z^0 \\
T &= -\theta^{\alpha-}\partial_{\alpha-} - T^0 \\
Z &= -Z^0 \\
J_{\alpha\beta} &= i\theta^{\gamma-}(\varepsilon_{\gamma\alpha}\partial_{\beta-} + \varepsilon_{\gamma\beta}\partial_{\alpha-}) - J^0{}_{ab}{}^{*} \\
J_{\dot\alpha\dot\beta} &= -J^0{}_{\dot\alpha\dot\beta}{}^{*}
\end{aligned}
\qquad 6.8
$$

where $\partial_{\alpha-} = \partial/\partial\theta^{\alpha-}$ and $T^0, Z^0, J^0{}_{\alpha\beta}$ and $J^0{}_{\dot\alpha\dot\beta}$ are matrix representations of the 'little superalgebra'. T^0 and Z^0 may be represented simply as charges $T^0 = -T$ and $Z^0 = -Z$ while $J^0{}_{\dot\alpha\dot\beta}$ and $J^0{}_{\alpha\beta}$ may be represented in terms of spin $p \times \frac{1}{2}$ and spin $q \times \frac{1}{2}$ projectors, $\Pi_{\dot\alpha}^{\pm\dot\beta}$ and $\Pi_{\alpha}^{\pm\beta}$, respectively (see Appendix D) as

$$J^0{}_{\dot\alpha\dot\beta} = 2p(\Pi^+\varepsilon)_{\dot\alpha\dot\beta} - 2(P+1)(\Pi^-\varepsilon)_{\dot\alpha\dot\beta} \qquad 6.9$$

$$J^0{}_{\alpha\beta} = 2q(\Pi^+\varepsilon)_{\alpha\beta} - 2(q+1)(\Pi^-\varepsilon)_{\alpha\beta} \qquad 6.10$$

where the spin p and q indices have been suppressed.

---

* $J_{\mu\nu}$ has been expressed in bispinor form in the following way

$$
\begin{aligned}
J_{\mu\nu} &= \frac{1}{8}(\bar\sigma_\mu{}^{\dot\alpha\alpha}\bar\sigma_\nu{}^{\dot\beta\beta} - \bar\sigma_\nu{}^{\dot\alpha\alpha}\bar\sigma_\mu{}^{\dot\beta\beta})J_{\alpha\dot\alpha\beta\dot\beta} \\
&= \frac{1}{8}(\bar\sigma_\mu{}^{\dot\alpha\alpha}\bar\sigma_\nu{}^{\dot\beta\beta} - \bar\sigma_\nu{}^{\dot\alpha\alpha}\bar\sigma_\mu{}^{\dot\beta\beta})(\varepsilon_{\alpha\beta}J_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}J_{\alpha\beta})
\end{aligned}
$$

where $J_{\alpha\beta} = -\frac{1}{2}J_{\alpha\dot\gamma\beta}{}^{\dot\gamma}$ and $J_{\dot\alpha\dot\beta} = -\frac{1}{2}J_{\gamma\dot\alpha}{}^{\gamma}{}_{\dot\beta}$

Expanding the superfield $\psi(\theta^{\alpha-})$ in $\theta^{\alpha-}$ gives

$$\Psi(\theta^{\alpha-}) = V + \theta^{\alpha-}(\Sigma^+_\alpha + \Sigma^-_\alpha) + \frac{1}{2}\,\theta^2 W \qquad 6.11$$

where all components possess spin p under $J_{\dot\alpha\dot\beta}$ and charge Z under Z. V and W possess spin q under $J_{\dot\alpha\dot\beta}$ and charges T and T-2 respectively under T, while Σ^+_α and Σ^-_α possess spin $q + \frac{1}{2}$ and $q - \frac{1}{2}$ respectively under $J_{\alpha\beta}$ and charge T-1 under T.

All component fields are eigenfunctions of the even generators and have the following variations under the odd generators

$$Q_{\alpha-} : \quad \delta_{\alpha-}V = -i\,\Sigma^+_\alpha - i\,\Sigma^-_\alpha \qquad \delta_{\alpha-}\Sigma^\pm_\beta = i(\Pi^\pm\varepsilon)_{\beta\alpha}W$$

$$\delta_{\alpha-}W = 0 \qquad 6.12$$

$$Q_{\alpha+} : \quad \delta_{\alpha+}V = 0 \qquad \delta_{\alpha+}\Sigma^\pm_\beta = -4(\Pi^\pm\varepsilon)_{\beta\alpha}ZV$$

$$\delta_{\alpha+}W : -4Z\Sigma^+_\alpha - 4Z\Sigma^-_\alpha \qquad 6.13$$

From this explicit component form of the variations it is clear that $\psi(\theta^{\alpha-})$ is irreducible for non-zero central charge. Having found the irreducible representations of $\mathcal{H}$ the principal task of determining the irreducible representations of $\mathcal{G}$ can now be broached.

6.3c Irreducible Representations of $\mathcal{G}$

The generators of $\mathcal{G}$ can be realized as differential operators in the coset space $\mathcal{G}\,/\,\mathcal{H}$ and as matrix representatives in the representation space of $\mathcal{H}_0$. Their explicit form is

$$P_\mu = -i\partial_\mu$$

$$J_{\mu\nu} = i(\eta_{\nu\rho}x^\rho\partial_\mu - \eta_{\mu\rho}x^\rho\partial_\nu) + i\,\bar\theta^{\dot\beta a}(\bar\sigma_{\mu\nu})_{\dot\beta}{}^{\dot\alpha}\partial_{\dot\alpha a} - J^0{}_{\mu\nu}$$

$$\bar Q_{\dot\alpha\pm} = -i\partial_{\dot\alpha\pm} \pm 2\bar\theta^{\dot\beta\pm}\varepsilon_{\dot\alpha\dot\beta}\,\bar Z^0$$

$$Q_{\alpha\pm} = 4\bar{\theta}^{\dot{\alpha}\mp} (\sigma^{\mu})_{\alpha\dot{\alpha}} \partial_{\mu} - Q^{0}{}_{\alpha\pm}$$

$$Z = - Z^{0}$$

$$T = \bar{\theta}^{\dot{\alpha}+} \partial_{\dot{\alpha}+} - \bar{\theta}^{\dot{\alpha}-} \partial_{\dot{\alpha}-} - T^{0} \qquad 6.14$$

where $\partial_{\dot{\alpha}a} = \partial/\partial\bar{\theta}^{\dot{\alpha}a}$, $\partial_{\mu} = \partial/\partial x^{\mu}$ and $J^{0}{}_{\mu\nu}$, Z^{0}, T^{0} and $Q^{0}{}_{\alpha a}$ are the matrix representations of the 'little superalgebra' which determines the external transformation rules of the superfield. Suitable forms for Z^{0} and T^{0} are

$$Z^{0} = -Z \begin{bmatrix} \delta_{a}{}^{c} & & & 0 \\ & \Pi^{+\beta c}_{\alpha a} & & \\ 0 & & \Pi^{-\beta c}_{\alpha a} & \\ & & & \delta_{b}{}^{d} \end{bmatrix} \qquad 6.15$$

$$T^{0} = \begin{bmatrix} -T\delta_{a}{}^{c} & & & 0 \\ & -(T-1)\Pi^{+\beta c}_{\alpha a} & & \\ 0 & & -(T-1)\Pi^{-\beta c}_{\alpha a} & \\ & & & -(T-2)\delta_{b}{}^{d} \end{bmatrix} \qquad 6.16$$

The algebra satisfied by $\mathcal{G}$ requires matrices $Q^{0}{}_{\alpha a}$ which satisfy $\{Q^{0}{}_{\alpha a}, Q^{0}{}_{\beta b}\} = 4i\, \varepsilon_{\alpha\beta}\, \varepsilon_{ab}\, Z^{0}$.

These are found to be of the form

$$Q^{0}_{\gamma+} = +y\mu \begin{bmatrix} 0 & 0 & 0 & 0 \\ (\Pi^{+}\varepsilon)_{\alpha\gamma a}{}^{c} & 0 & 0 & 0 \\ (\Pi^{-}\varepsilon)_{\alpha\gamma a}{}^{c} & 0 & 0 & 0 \\ 0 & \Pi^{+\beta c}_{\gamma b} & \Pi^{-\beta c}_{\gamma b} & 0 \end{bmatrix} \qquad 6.17$$

$$Q^{0}_{\gamma_-} = +y\mu \begin{bmatrix} 0 & -\Pi^{+\beta c}_{\gamma a} & -\Pi^{-\beta c}_{\gamma a} & 0 \\ 0 & 0 & 0 & (\Pi^{+}\varepsilon)_{\alpha\gamma a}{}^{d} \\ 0 & 0 & 0 & (\Pi^{-}\varepsilon)_{\alpha\gamma a}{}^{d} \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 6.18$$

where $\mu = \sqrt{z}$ and $y = (1+i)\sqrt{2}$ has been chosen simply to render the most symmetrical form for $Q^{0}_{\gamma\pm}$. The only essential requirement for these coefficients is that their product is $4iz$.

The superfields $\Phi_A(x^\mu, \bar{\theta}^{\dot{\alpha}a})$ form a representation of $\mathcal{H}$ - labelled by {(p,q), T, Z} as described in §6.2:

$$\Phi_A(x^\mu, \bar{\theta}^{\dot{\alpha}a}) = \begin{pmatrix} V_a(x^\mu, \bar{\theta}^{\dot{\alpha}a}) \\ \Sigma^{+}_{a\alpha}(x^\mu, \bar{\theta}^{\dot{\alpha}a}) \\ \Sigma^{-}_{a\alpha}(x^\mu, \bar{\theta}^{\dot{\alpha}a}) \\ W_a(x^\mu, \bar{\theta}^{\dot{\alpha}a}) \end{pmatrix} \qquad 6.19$$

The general form of the superfield when expanded in $\bar{\theta}^{\dot{\alpha}a}$ is (spin-q indices will be suppressed in the following work; $\bar{\theta}$-monomials are defined in Appendix G together with some useful identities):

$$\Phi_A(x^\mu, \bar\theta^{\dot\alpha a}) = \begin{pmatrix} A \\ \alpha^+_\alpha \\ \alpha^-_\alpha \\ a \end{pmatrix} + \bar\theta^{\dot\alpha a} \sum_m \begin{pmatrix} \Psi^m_{\dot\alpha a} \\ P^{m+}_{\dot\alpha\alpha a} \\ P^{m-}_{\dot\alpha\alpha a} \\ \psi^m_{\dot\alpha a} \end{pmatrix} + (\bar\theta\ \bar\theta)^{ab} \sum_k \begin{pmatrix} F_k \\ \phi^+_{\alpha k} \\ \phi^-_{\alpha k} \\ f_k \end{pmatrix}^{*}$$

$$+ (\bar\theta\ \bar\theta)^{\dot\alpha\dot\beta} \sum_\ell \begin{pmatrix} G^\ell_{\dot\alpha\dot\beta} \\ \Upsilon^{\ell+}_{\dot\alpha\dot\beta\alpha} \\ \Upsilon^{\ell-}_{\dot\alpha\dot\beta\alpha} \\ g^\ell_{\dot\alpha\dot\beta} \end{pmatrix} + (\bar\theta^3)^{\dot\alpha a} \sum_m \begin{pmatrix} \Omega^m_{\dot\alpha a} \\ W^{m+}_{\dot\alpha\alpha a} \\ W^{m-}_{\dot\alpha\alpha a} \\ \omega^m_{\dot\alpha a} \end{pmatrix} + (\bar\theta^4) \begin{pmatrix} D \\ \delta^+_\alpha \\ \delta^-_\alpha \\ d \end{pmatrix} \qquad 6.20$$

where $m = +,-$ refer to spin $P+\frac{1}{2}$ and $P-\frac{1}{2}$ projections, $\ell = 0,+,-$ refer to spin P, P+1 and P-1 projections, $a = +,-$ refer to T+1 and T-1 projections and $k = 0,+,-$ refer to T,T+2,T-2 projections. All component fields are functions of x^μ.

To determine the irreducibility or indecomposability of $\Phi_A(x^\mu, \bar\theta^{\dot\alpha a})$ it is necessary to introduce appropriate field redefinitions for the component fields and examine their variations under the odd generators. To aid in this we recall that the algebra realized by 6.14 may be extended to include the generators $\bar S_{\dot\alpha a} = \partial_{\dot\alpha a}$ yielding the extended algebra $\bar{\mathcal{G}}$. As we have noted, since

---

* To be precise this term should read, considering for example the top component only,

$$(\bar\theta\ \bar\theta)^{ab} F_{ab} = (\bar\theta\ \bar\theta)^{++} F_{++} + (\bar\theta\ \bar\theta)^{--} F_{--} + (\bar\theta\ \bar\theta)^{+-} F_{+-} + (\bar\theta\ \bar\theta)^{-+} F_{-+} .$$

Thus we define $F_+ = F_{++}$, $F_- = F_{--}$ and $F_0 = F_{+-} + F_{-+}$.

the general superfield $\Phi_A(x^\mu, \bar\theta^{\dot\alpha a})$ is still a representation of $\bar{\mathcal{G}}$ it is expected to be reducible under $\mathcal{G}$. To find the irreducible representations of $\mathcal{G}$ contained in Φ we proceed as follows. Given an irreducible representation of $\mathcal{G}$ with highest weight vector $\Lambda = |(p,q), T, Z>$, such that

$$Q_{\alpha+}\Lambda = \bar Q_{\dot\alpha+}\Lambda = \bar S_{\dot\alpha+}\Lambda = 0 \quad ,$$

a basis for $\bar{\mathcal{G}}$ may be obtained from the four vectors Λ, $\Pi^{+}{}_{\dot\alpha}{}^{\dot\beta}\bar S_{\dot\beta-}\Lambda$, $\Pi^{-}{}_{\dot\alpha}{}^{\dot\beta}\bar S_{\dot\beta-}\Lambda$, $(\bar S_{\dot\alpha-})^2\Lambda$ by acting with monomials of $\bar Q_{\dot\alpha-}$ and $Q_{\alpha-}$. This suggests that a superfield $\Phi_A(x^\mu, \bar\theta^{\dot\alpha a})$, which carries a representation $((p,q), T, Z)$ of $\mathcal{H}$, contains four irreducible representations of $\mathcal{G}$. This is indeed found to be the case, with $\Lambda = |(p,q), T+2, Z>$ and consequently, $\Pi^{+}{}_{\dot\alpha}{}^{\dot\beta}\bar S_{\dot\beta-}\Lambda = |(p+\tfrac{1}{2},q), T+1, Z>$, $\Pi^{-}{}_{\dot\alpha}{}^{\dot\beta}\bar S_{\dot\beta-}\Lambda = |(p-\tfrac{1}{2},q), T+1, Z>$ and $(\bar S_{\dot\alpha-})^2\Lambda = |(p,q), T, Z>$. Each of these multiplets contains sixteen fields with weights as shown in table 6.1.

To obtain the basis which renders the irreducible multiplets of $\mathcal{G}$ evident we proceed as follows. From the superfield it is apparent that the highest weight vector, Λ, is F_+ since $\delta_{Q_{\gamma+}} F_+ = \delta_{\bar Q_{\dot\gamma+}} F_+ = \delta_{\bar S_{\dot\gamma+}} F_+ = 0$. The variations of, F_+, under $Q_{\gamma-}$ and $\bar Q_{\dot\gamma-}$ are:

$$\delta_{\gamma-}F_+ = -\,y\mu(\phi^{+}_{\gamma+} + \phi^{-}_{\gamma+}) + 2(\sigma^\mu)_\gamma{}^{\dot\alpha}\,(\partial_\mu\,\psi^{+}_{\dot\alpha+} + \partial_\mu\,\psi^{-}_{\dot\alpha+}) \qquad 6.21$$

$$\delta_{\dot\gamma-}F_+ = \frac{3}{2}\,i\,(\Omega^{+}_{\dot\gamma+} + \Omega^{-}_{\dot\gamma+}) - \bar\mu^2\,(\psi^{+}_{\dot\gamma+} + \psi^{-}_{\dot\gamma+}) \qquad 6.22$$

From (6.21) and (6.22) we project spin $q\pm\tfrac{1}{2}$ and spin $p\pm\tfrac{1}{2}$ states respectively, and define new fields proportional to these states. Thus, explicitly, we have

$$\Pi^{\pm}{}_{\alpha}{}^{\gamma}\ \delta_{\gamma-} F_{+} = -\ y\mu\ \tilde{\phi}^{\pm}_{\alpha+} \qquad 6.23$$

$$\Pi^{\pm}{}_{\dot\alpha}{}^{\dot\gamma}\ \delta_{\dot\gamma-} F_{+} = \tilde{\Omega}^{\pm}_{\dot\alpha+} \qquad 6.24$$

where $\tilde{\phi}^{\pm}_{\alpha+} = \phi^{\pm}_{\alpha+} - (2/y\mu)(\sigma_{\pm}{}^{\mu})_{\alpha}{}^{\dot\alpha}\ (\partial_{\mu}\Psi^{+}_{\dot\alpha+} + \partial_{\mu}\Psi^{-}_{\dot\alpha+})$

and $\tilde{\Omega}^{\pm}_{\dot\alpha+} = \frac{3}{2}\ i\ \Omega^{\pm}_{\dot\alpha+} - \bar{\mu}^{2}\ \Psi^{\pm}_{\dot\alpha+}$ (see end of Table 6.2 for notation).

We now consider the variation of each of these ~ fields under $Q_{\gamma-}$ and $\bar{Q}_{\dot\gamma-}$ and define new ~ fields by projecting Lorentz eigenstates from the field variations as in (6.23) and (6.24). This procedure is simply repeated until a basis for the sixteen states of this multiplet has been generated. This basis is given in Table 6.2a.

For the multiplet characterized by $(\bar{S}_{\dot\alpha-})^{2}\Lambda$ the judicious choice is to consider a lowest weight vector $\bar{\Lambda} = |(p,q),\ T\text{-}4,\ Z>$ such that $Q_{\alpha-}\bar{\Lambda} = \bar{Q}_{\dot\alpha-}\bar{\Lambda} = \bar{S}_{\dot\alpha-}\bar{\Lambda} = 0$ and obtain the remaining states of the multiplet by acting with monimials of $Q_{\alpha+}$ and $\bar{Q}_{\dot\alpha+}$ on $\bar{\Lambda}$. From the superfield we find that f_{-} is the field corresponding to $\bar{\Lambda}$, since $\delta_{Q_{\alpha-}} f_{-} = \delta_{\bar{Q}_{\dot\alpha-}} f_{-} = \delta_{\bar{S}_{\dot\alpha-}} f_{-} = 0$. By analogy with the F_{+} multiplet we now determine the variations of f_{-} under $Q_{\alpha+}$ and $\bar{Q}_{\dot\alpha+}$ and define new fields as proportional to the Lorentz eigenstates projected from these variations. Again by repeated application of this procedure we obtain a basis for the sixteen states of this multiplet. This basis is given in Table 6.2b.

For the remaining two multiplets characterized by $\Pi^{\pm}{}_{\dot\alpha}{}^{\dot\beta}\bar{S}_{\dot\beta-}\Lambda$, we see from the superfield that the highest weight states will be some linear combination of $\Omega^{\pm}_{\dot\alpha+}$ and $\Psi^{\pm}_{\dot\alpha+}$ which is linearly independent to $\tilde{\Omega}^{\pm}_{\dot\alpha+}$. For simplicity we choose $\tilde{\Psi}^{\pm}_{\dot\alpha+} = \Psi^{\pm}_{\dot\alpha+}$ for which

$\delta_{Q_{\gamma+}} \tilde{\Psi}^{\pm}{}_{\dot{\alpha}+} = 0$ and $\delta_{\bar{Q}_{\dot{\gamma}+}} \tilde{\Psi}^{\pm}{}_{\dot{\alpha}+} = \delta_{\bar{S}_{\dot{\gamma}+}} \tilde{\Psi}^{\pm}{}_{\dot{\alpha}+} = 2i(\Pi^{\pm}\varepsilon)_{\dot{\alpha}\dot{\lambda}} F_{+}$. Thus as will be seen presently, $\tilde{\Psi}^{\pm}{}_{\dot{\alpha}+}$, are highest weight vectors, modulo coset elements, F_{+}. Again by analogy with the F_{+} multiplet, the bases for the $\Psi^{\pm}{}_{\dot{\alpha}+}$ multiplets are obtained by acting with mon mials of $Q_{\alpha+}$ and $\bar{Q}_{\dot{\alpha}+}$ on $\tilde{\Psi}^{\pm}{}_{\dot{\alpha}+}$ and defining new fields as proportional to the Lorentz eigenstates projected from these variations. These bases are given in Tables 6.2c and 6.2d.

This procedure effectively provides a basis transformation of the superfield components into irreducible multiplets of $\mathcal{G}$. Such basis transformations may also be effected by constructing Casimirs of $\mathcal{G}$ which label different multiplets of $\mathcal{G}$ and finding functions of $\bar{\theta}^{\dot{\alpha}a}$ which form a complete set of eigenfunctions of this Casimir. Expanding the superfield in terms of these functions yields the appropriate basis directly as the component fields. Jarvis [35] has used this technique for the study of unitary, irreducible representations of the N=1 super Poincaré algebra. Bufton and Taylor [36] define similar basis functions for the N-extended supersymmetry algebras.

Given this new basis for the components of the superfield, the irreducibility of the multiplets we have generated can now be examined. It is found that the F_{+} and f_{-} multiplets are invariant subspaces while the $\Psi^{+}{}_{\dot{\alpha}+}$ and $\Psi^{-}{}_{\dot{\alpha}+}$ multiplets are invariant as factor spaces. This behaviour is typified by the following examples:

$$\delta_{Q_{\gamma+}} \bar{\omega}_{\dot{\alpha}+} = -\, y\mu\, \tilde{W}^{+-}_{\gamma\dot{\alpha}+} - y\mu\, \tilde{W}^{--}_{\gamma\dot{\alpha}+} - 4i(\sigma^{\mu})^{-}_{\gamma\dot{\alpha}}\, \partial_{\mu}\, \tilde{f}_{+}$$

$$\delta_{\bar{Q}_{\dot{\gamma}-}} \tilde{a} = -\, 2i\, \bar{\mu}^{-2}\, \tilde{\omega}^{+}_{\dot{\gamma}-} - 2i\, \bar{\mu}^{-2}\, \tilde{\omega}^{-}_{\dot{\gamma}-}$$

$$\delta_{\bar{Q}_{\dot{\gamma}+}} \tilde{F}_0 = \frac{2i}{(p+1)} \mu^{-2} \tilde{\Psi}^{+}_{\dot{\gamma}+} - \frac{i}{(p+1)(2p+1)} \tilde{\Omega}^{+}_{\dot{\gamma}+} - \frac{2i}{(2p+1)} \tilde{\Omega}^{-}_{\dot{\gamma}+}$$

$$\delta_{\bar{Q}_{\dot{\gamma}+}} \tilde{\psi}_{\dot{\alpha}} = 2i(\Pi^{-}\varepsilon)_{\dot{\alpha}\dot{\gamma}} \tilde{f}_{+} - \frac{4i}{y\mu} (\sigma^{\mu})_{\dot{\gamma}}{}^{\alpha} (\partial_{\mu} \tilde{P}^{-+}_{\alpha\dot{\alpha}+} + \partial_{\mu} \tilde{P}^{--}_{\alpha\dot{\alpha}+})$$

As discussed in §6.2, it has been pointed out [4,5,15,16] that if the constraint

$$\bar{Z} Q_{\alpha\pm} = \pm\, i(\sigma^{\mu})_{\alpha}{}^{\dot{\alpha}} P_{\mu} \bar{Q}_{\dot{\alpha}\pm} \qquad 6.25$$

is imposed, the number of fields in an irreducible representation is reduced from 2^4 to 2^2. This constraint implies that

$$P^2 + Z\bar{Z} = 0 \ . \qquad 6.26$$

In the present approach this reduction takes place via the imposition of only the weaker constraint (6.26) as described below.

To observe this phenomenon we note the intimate connection between (6.25) and (6.26) and use this to introduce further field redefinitions, for the fields in each multiplet which are obtained from acting with $Q_{\alpha-}$, $Q_{\alpha-} Q_{\beta-}$ and $Q_{\alpha-} \bar{Q}_{\dot{\alpha}-}$ on the highest weight state of the multiplet or with $Q_{\alpha+}$, $Q_{\alpha+} Q_{\beta+}$, $Q_{\alpha+} \bar{Q}_{\dot{\alpha}+}$ on the lowest weight state of the multiplet. These fields are constructed, up to a proportionality, from the ~ basis of table 6.2, by projecting Lorentz eigenstates either from

$$(Q_{\alpha-} + \frac{i}{\bar{Z}} (\sigma^{\mu})_{\alpha}{}^{\dot{\alpha}} P_{\mu} \bar{Q}_{\dot{\alpha}-})\tilde{B} \qquad 6.27$$

where $\tilde{B}$ is the generic title given to the fields obtained from Λ_i, $Q_{\alpha-}\Lambda_i$ and $\bar{Q}_{\dot{\alpha}-}\Lambda_i$ with Λ_i the highest weight state of a multiplet or from

$$(Q_{\alpha +} - \frac{i}{Z} (\sigma^{\mu})_{\alpha}{}^{\dot{\alpha}} P_{\mu} \bar{Q}_{\dot{\alpha}+})\tilde{B} \qquad 6.28$$

where $\tilde{B}$ refers here to the fields obtained from $\bar{\Lambda}$, $Q_{\alpha+}\bar{\Lambda}$ and $\bar{Q}_{\dot{\alpha}+}\bar{\Lambda}$ with $\bar{\Lambda}$ the lowest weight state of a multiplet. These field redefinitions are given in Table 6.3 and in this basis it is observed that the fields Λ_i, $\bar{Q}_{\dot{\alpha}\pm}\Lambda_i$ and $\bar{Q}_{\dot{\alpha}\pm}\ \bar{Q}_{\dot{\beta}\pm}\Lambda_i$ (taking upper (lower) signs if Λ_i is a lowest (highest) weight vector) are invariant as a factor space with the remaining fields of each multiplet decoupling when $P^2 + Z\bar{Z} = 0$. This is demonstrated for the F_+ multiplet in table 6.4, which clearly shows that when condition (6.26) is imposed, an irreducible realisation of the SO(2)-extended super Poincaré algebra consists of four fields with O(3,1) x U(1) labels

$$\{(p,q,T),\ (p+\tfrac{1}{2},q,T-1),\ (p-\tfrac{1}{2},q,T-1),\ (p,q,T-2)\}\ .$$

TABLE 6.1 Weights and defining fields of the four irreducible multiplets of $\mathcal{G}$ contained in the superfield.

	p, q, T	Λ_1	p, q, T	Λ_2	p, q, T	Λ_3	p, q, T	Λ_4
Λ_i	0, 0, +2	$\tilde{F}_+$	0, 0, -4	$\tilde{f}_-$	½, 0, +1	$\tilde{\Psi}^{+}{}_{\dot\alpha+}$	-½, 0, +1	$\tilde{\Psi}^{-}{}_{\dot\alpha+}$
$Q_{\alpha\pm}\ \Lambda_i$	0, ½, +1	$\tilde{\phi}^{+}{}_{\alpha+}$	0, ½, -3	$\tilde{\phi}^{+}{}_{\alpha-}$	½, ½, 0	$\tilde{P}^{++}_{\dot\alpha\alpha+}$	-½, ½, 0	$\tilde{P}^{-+}_{\dot\alpha\alpha+}$
	0, -½, +1	$\tilde{\phi}^{-}{}_{\alpha+}$	0, -½, -3	$\tilde{\phi}^{-}{}_{\alpha-}$	½, -½, 0	$\tilde{P}^{+-}_{\dot\alpha\alpha+}$	-½, -½, 0	$\tilde{P}^{--}_{\dot\alpha\alpha+}$
$\bar{Q}_{\dot\alpha\pm}\ \Lambda_i$	½, 0, +1	$\tilde{\Omega}^{+}{}_{\dot\alpha+}$	½, 0, -3	$\tilde{\omega}^{+}{}_{\dot\alpha-}$	1, 0, 0	$\tilde{G}^{+}{}_{\dot\alpha\dot\beta}$	-1, 0, 0	$\tilde{G}^{-}_{\dot\alpha\dot\beta}$
	-½, 0, +1	$\tilde{\Omega}^{-}{}_{\dot\alpha+}$	-½, 0, -3	$\tilde{\omega}^{-}{}_{\dot\alpha-}$	0, 0, 0	$\tilde{F}_0$	0, 0, 0	$\tilde{G}^{0}_{\dot\alpha\dot\beta}$
$Q_{\alpha\pm}Q_{\beta\pm}\Lambda_i$	0, 0, 0	$\tilde{f}_+$	0, 0, -2	$\tilde{F}_-$	½, 0, -1	$\tilde{\psi}^{+}{}_{\dot\alpha+}$	-½, 0, -1	$\tilde{\psi}^{-}{}_{\dot\alpha+}$
$\bar{Q}_{\dot\alpha\pm}\bar{Q}_{\dot\beta\pm}\Lambda_i$	0, 0, 0	$\tilde{D}$	0, 0, -2	$\tilde{a}$	½, 0, -1	$\tilde{\Psi}^{+}{}_{\dot\alpha-}$	-½, 0, -1	$\tilde{\Psi}^{-}{}_{\dot\alpha-}$
$Q_{\alpha\pm}\bar{Q}_{\dot\alpha\pm}\Lambda_i$	½, ½, 0	$\tilde{W}^{++}_{\dot\alpha\alpha+}$	½, ½, -2	$\tilde{W}^{++}_{\dot\alpha\alpha-}$	1, ½, -1	$\tilde{Y}^{++}_{\dot\alpha\dot\beta\alpha}$	-1, ½, -1	$\tilde{Y}^{-+}_{\dot\alpha\dot\beta\alpha}$
	½, -½, 0	$\tilde{W}^{+-}_{\dot\alpha\alpha+}$	½, -½, -2	$\tilde{W}^{+-}_{\dot\alpha\alpha-}$	1, -½, -1	$\tilde{Y}^{+-}_{\dot\alpha\dot\beta\alpha}$	-1, -½, -1	$\tilde{Y}^{--}_{\dot\alpha\dot\beta\alpha}$
	-½, ½, 0	$\tilde{W}^{-+}_{\dot\alpha\alpha+}$	-½, ½, -2	$\tilde{W}^{-+}_{\dot\alpha\alpha-}$	0, ½, -1	$\tilde{\phi}^{+}{}_{\alpha0}$	0, ½, -1	$\tilde{Y}^{0+}_{\dot\alpha\dot\beta\alpha}$
	-½, -½, 0	$\tilde{W}^{--}_{\dot\alpha\alpha+}$	-½, -½, -2	$\tilde{W}^{--}_{\dot\alpha\alpha-}$	0, -½, -1	$\tilde{\phi}^{-}{}_{\alpha0}$	0, -½, -1	$\tilde{Y}^{0-}_{\dot\alpha\dot\beta\alpha}$
$Q_{\alpha\pm}\bar{Q}_{\dot\alpha\pm}\bar{Q}_{\dot\beta\pm}\Lambda_i$	0, ½, -1	$\tilde{\delta}^{+}{}_{\alpha}$	0, ½, -1	$\tilde{\alpha}^{+}{}_{\alpha}$	½, ½, -2	$\tilde{P}^{++}_{\dot\alpha\alpha-}$	-½, ½, -2	$\tilde{P}^{-+}_{\dot\alpha\alpha-}$
	0, -½, -1	$\tilde{\delta}^{-}{}_{\alpha}$	0, -½, -1	$\tilde{\alpha}^{-}{}_{\alpha}$	½, -½, -2	$\tilde{P}^{+-}_{\dot\alpha\alpha-}$	-½, -½, -2	$\tilde{P}^{--}_{\dot\alpha\alpha-}$
$\bar{Q}_{\dot\alpha\pm}Q_{\alpha\pm}Q_{\beta\pm}\Lambda_i$	½, 0, -1	$\tilde{\omega}^{+}{}_{\dot\alpha+}$	½, 0, -1	$\tilde{\Omega}^{+}{}_{\dot\alpha-}$	1, 0, -2	$\tilde{g}^{+}{}_{\dot\alpha\dot\beta}$	-1, 0, -2	$\tilde{g}^{-}{}_{\dot\alpha\dot\beta}$
	-½, 0, -1	$\tilde{\omega}^{-}{}_{\dot\alpha+}$	-½, 0, -1	$\tilde{\Omega}^{-}{}_{\dot\alpha-}$	0, 0, -2	$\tilde{f}_0$	0, 0, -2	$\tilde{g}^{0}{}_{\dot\alpha\dot\beta}$
$Q_{\alpha\pm}Q_{\beta\pm}\bar{Q}_{\dot\alpha\pm}\bar{Q}_{\dot\beta\pm}\Lambda_i$	0, 0, -2	$\tilde{d}$	0, 0, 0	$\tilde{A}$	½, 0, -3	$\tilde{\psi}^{+}{}_{\dot\alpha-}$	-½, 0, -3	$\tilde{\psi}^{-}{}_{\dot\alpha-}$

The fields of the Λ_1, Λ_3 and Λ_4 multiplets are defined as proportional to $\Pi Q_{\alpha-}\Pi\bar{Q}_{\dot\alpha-}$ acting on the highest weight vectors $\tilde{F}_+$, $\tilde{\Psi}^{+}{}_{\dot\alpha+}$ and $\tilde{\Psi}^{-}{}_{\dot\alpha-}$ respectively. The fields of the Λ_2 multiplet are defined as proportional to $\Pi Q_{\alpha+}\Pi\bar{Q}_{\dot\alpha+}$ acting on the lowest weight vector $\tilde{f}_-$.

TABLE 6.2a Basis for the Λ multiplet

$$\tilde{F}_{+} = F_{+}$$

$$\tilde{\phi}^{\pm}{}_{\alpha+} = \phi^{\pm}{}_{\alpha+} - \frac{2}{y\mu} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\beta}} [\partial_{\mu} \Psi^{+}{}_{\dot{\beta}+} + \partial_{\mu} \Psi^{-}{}_{\dot{\beta}+}]$$

$$\tilde{\Omega}^{\pm}{}_{\dot{\alpha}+} = \frac{3}{2} i \, \Omega^{\pm}{}_{\dot{\alpha}+} - \bar{\mu}^{2} \, \Psi^{\pm}{}_{\dot{\alpha}+}$$

$$\tilde{f}_{+} = f_{+} + \frac{8}{y^{2}\mu^{2}} \partial^{2} A - \frac{2}{y\mu} (\sigma^{\mu})^{\dot{\alpha}\alpha} [\partial_{\mu} P^{++}_{\dot{\alpha}\alpha+} + \partial_{\mu} P^{+-}_{\dot{\alpha}\alpha+} + \partial_{\mu} P^{-+}_{\dot{\alpha}\alpha+} + \partial_{\mu} P^{--}_{\dot{\alpha}\alpha+}]$$

$$\tilde{D} = 3D + i\bar{\mu}^{2} F_{0} - \bar{\mu}^{4} A$$

$$\tilde{W}^{\pm\pm}_{\dot{\alpha}\alpha+} = \frac{3}{2} i \, W^{\pm\pm}_{\dot{\alpha}\alpha+} - \bar{\mu}^{2} P^{\pm\pm}_{\dot{\alpha}\alpha+} + \frac{2i}{y\mu} (\sigma^{\mu}_{\pm})^{\pm}{}_{\dot{\alpha}\alpha} \partial_{\mu} F_{0}$$

$$- \frac{4\bar{\mu}^{2}}{y\mu} (\sigma^{\mu}_{\pm})^{\pm}{}_{\dot{\alpha}\alpha} \partial_{\mu} A - \frac{4i}{y\mu} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\beta}} [\partial_{\mu} G^{\pm}{}_{\dot{\beta}\dot{\alpha}} + \partial_{\mu} G^{0}{}_{\pm\dot{\beta}\dot{\alpha}}]$$

$$\tilde{\delta}^{\pm}{}_{\alpha} = 3\delta^{\pm}{}_{\alpha} + i\bar{\mu}^{2} \phi^{\pm}{}_{\alpha 0} - \bar{\mu}^{4} \alpha^{\pm}{}_{\alpha}$$

$$- \frac{2i\bar{\mu}^{2}}{y\mu} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}} [\partial_{\mu} \Psi^{+}{}_{\dot{\alpha}-} + \partial_{\mu} \Psi^{-}{}_{\dot{\alpha}-}] + \frac{3}{y\mu} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}} [\partial_{\mu} \Omega^{+}{}_{\dot{\alpha}-} + \partial_{\mu} \Omega^{-}{}_{\dot{\alpha}-}]$$

$$\tilde{\omega}^{\pm}{}_{\dot{\alpha}+} = \frac{3}{2} i \, \omega^{\pm}{}_{\dot{\alpha}+} - \bar{\mu}^{2} \psi^{\pm}{}_{\dot{\alpha}+} - \frac{8i}{y^{2}\mu^{2}} \partial^{2} \Psi^{\pm}{}_{\dot{\alpha}-}$$

$$+ \frac{2i}{y\mu} (\sigma^{\mu})^{\pm}{}_{\dot{\alpha}}{}^{\alpha} [\partial_{\mu} \phi^{+}{}_{\alpha 0} + \partial_{\mu} \phi^{-}{}_{\alpha 0}] - \frac{4\bar{\mu}^{2}}{y\mu} (\sigma^{\mu})^{\pm}{}_{\dot{\alpha}}{}^{\alpha} [\partial_{\mu} \alpha^{+}{}_{\alpha} + \partial_{\mu} \alpha^{-}{}_{\alpha}]$$

$$- \frac{4i}{y\mu} (\sigma^{\mu})^{\dot{\beta}\alpha} [\partial_{\mu} \gamma^{\pm +}_{\dot{\beta}\dot{\alpha}\alpha} + \partial_{\mu} \gamma^{\pm -}_{\dot{\beta}\dot{\alpha}\alpha} + \partial_{\mu} \gamma^{0+}_{\pm\dot{\beta}\dot{\alpha}\alpha} + \partial_{\mu} \gamma^{0-}_{\pm\dot{\beta}\dot{\alpha}\alpha}]$$

$$\tilde{d} = 3d + i\bar{\mu}^{2} f_{0} - \bar{\mu}^{4} a - \frac{8}{y^{2}u^{2}} \partial^{2} F_{-}$$

$$- \frac{2i\bar{\mu}^{2}}{y\mu} (\sigma^{\mu})^{\dot{\alpha}\alpha} [\partial_{\mu} P^{++}_{\dot{\alpha}\alpha-} + \partial_{\mu} P^{+-}_{\dot{\alpha}\alpha-} + \partial_{\mu} P^{-+}_{\dot{\alpha}\alpha-} + \partial_{\mu} P^{--}_{\dot{\alpha}\alpha-}]$$

$$+ \frac{3}{y\mu} (\sigma^{\mu})^{\dot{\alpha}\alpha} [\partial_{\mu} W^{++}_{\dot{\alpha}\alpha-} + \partial_{\mu} W^{+-}_{\dot{\alpha}\alpha-} + \partial_{\mu} W^{-+}_{\dot{\alpha}\alpha-} + \partial_{\mu} W^{--}_{\dot{\alpha}\alpha-}]$$

TABLE 6.2b Basis for the $(\bar{S}_{\dot{\alpha}-})^2\Lambda$ multiplet

$$\tilde{f}_- = f_-$$

$$\tilde{\phi}^{\pm}{}_{\alpha-} = \phi^{\pm}{}_{\alpha-} + \frac{2}{y\mu}(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}}[\partial_{\mu}\psi^{+}{}_{\dot{\alpha}-} + \partial_{\mu}\psi^{-}{}_{\dot{\alpha}-}]$$

$$\tilde{\omega}^{\pm}{}_{\dot{\alpha}-} = \frac{3}{2}\, i\, \omega^{\pm}{}_{\dot{\alpha}-} - \bar{\mu}^{2}\,\psi^{\pm}{}_{\dot{\alpha}-}$$

$$\tilde{F}_- = F_- - \frac{8}{y^2\mu^2}\partial^2 a - \frac{2}{y\mu}(\sigma^{\mu})^{\dot{\alpha}\alpha}[\partial_{\mu}P^{++}_{\dot{\alpha}\alpha-} + \partial_{\mu}P^{+-}_{\dot{\alpha}\alpha-} + \partial_{\mu}P^{-+}_{\dot{\alpha}\alpha-} + \partial_{\mu}P^{--}_{\dot{\alpha}\alpha-}]$$

$$\tilde{a} = 3d - i\bar{\mu}^{2} f_0 - \bar{\mu}^{4} a$$

$$\tilde{W}^{\pm\pm}_{\dot{\alpha}\alpha-} = \frac{3}{2}\, i\, W^{\pm\pm}_{\dot{\alpha}\alpha-} - \bar{\mu}^{2} P^{\pm\pm}_{\dot{\alpha}\alpha-} + \frac{2i}{y\mu}(\sigma^{\mu}_{\pm})^{\pm}{}_{\dot{\alpha}\alpha}\,\partial_{\mu}f_0$$

$$+ \frac{4\bar{\mu}^{2}}{y\mu}(\sigma^{\mu}_{\pm})^{\pm}{}_{\dot{\alpha}\alpha}\,\partial_{\mu}a + \frac{4i}{y\mu}(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\beta}}[\partial_{\mu}g^{\pm}{}_{\dot{\beta}\dot{\alpha}} + \partial_{\mu}g^{0}{}_{\pm\dot{\beta}\dot{\alpha}}]$$

$$\tilde{\alpha}^{\pm}_{\alpha} = 3\delta^{\pm}{}_{\alpha} - i\bar{\mu}^{2}\,\phi^{\pm}{}_{\alpha 0} - \bar{\mu}^{4}\,\alpha^{\pm}{}_{\alpha}$$

$$- \frac{2i\bar{\mu}^{2}}{y\mu}(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}}[\partial_{\mu}\psi^{+}{}_{\dot{\alpha}+} + \partial_{\mu}\psi^{-}{}_{\dot{\alpha}+}] + \frac{3}{y\mu}(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}}[\partial_{\mu}\omega^{+}{}_{\dot{\alpha}+} + \partial_{\mu}\omega^{-}{}_{\dot{\alpha}+}]$$

$$\tilde{\Omega}^{\pm}{}_{\dot{\alpha}-} = \frac{3}{2}\, i\Omega^{\pm}{}_{\dot{\alpha}-} - \bar{\mu}^{2}\,\Psi^{\pm}{}_{\dot{\alpha}-} - \frac{8i}{y^2\mu^2}\partial^2\,\psi^{\pm}{}_{\dot{\alpha}+}$$

$$- \frac{2i}{y\mu}(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}}{}^{\alpha}[\partial_{\mu}\phi^{+}{}_{\alpha 0} + \partial_{\mu}\phi^{-}{}_{\alpha 0}] - \frac{4\bar{\mu}^{2}}{y\mu}(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}}{}^{\alpha}[\partial_{\mu}\alpha^{+}{}_{\alpha} + \partial_{\mu}\alpha^{-}{}_{\alpha}]$$

$$- \frac{4i}{y\mu}(\sigma^{\mu})^{\dot{\beta}\alpha}[\partial_{\mu}\gamma^{\pm +}_{\dot{\beta}\dot{\alpha}\alpha} + \partial_{\mu}\gamma^{\pm -}_{\dot{\beta}\dot{\beta}\alpha} + \partial_{\mu}\gamma^{0+}_{\pm\dot{\beta}\dot{\alpha}\alpha} + \partial_{\mu}\gamma^{0-}_{\pm\dot{\beta}\dot{\alpha}\alpha}]$$

$$\tilde{A} = 3D - i\bar{\mu}^{2} F_0 - \bar{\mu}^{4} A + \frac{8}{y^2\mu^2}\partial^2 f_+$$

$$- \frac{2i\bar{\mu}^{2}}{y\mu}(\sigma^{\mu})^{\dot{\alpha}\alpha}[\partial_{\mu}P^{++}_{\dot{\alpha}\alpha+} + \partial_{\mu}P^{+-}_{\dot{\alpha}\alpha+} + \partial_{\mu}P^{-+}_{\dot{\alpha}\alpha+} + \partial_{\mu}P^{--}_{\dot{\alpha}\alpha+}]$$

$$+ \frac{3}{y\mu}(\sigma^{\mu})^{\dot{\alpha}\alpha}[\partial_{\mu}W^{++}_{\dot{\alpha}\alpha+} + \partial_{\mu}W^{+-}_{\dot{\alpha}\alpha+} + \partial_{\mu}W^{-+}_{\dot{\alpha}\alpha+} + \partial_{\mu}W^{--}_{\dot{\alpha}\alpha+}]$$

TABLE 6.2c Basis for the $\Pi^{+}{}_{\dot\alpha}{}^{\dot\beta}\,\bar S_{\dot\beta -}\Lambda$ multiplet

$$\tilde\Psi^{+}{}_{\dot\alpha +} = \Psi^{+}{}_{\dot\alpha +}$$

$$\tilde P^{+\pm}_{\dot\alpha\alpha +} = P^{+\pm}_{\dot\alpha\alpha +} + \frac{4}{y\mu}\,(\sigma^{\mu}_{\pm})^{+}{}_{\dot\alpha\alpha}\,\partial_{\mu}A$$

$$\tilde G^{+}{}_{\dot\alpha\dot\beta} = G^{+}{}_{\dot\alpha\dot\beta}$$

$$\tilde F_{0} = \frac{i}{2P+1}\,F_{0} - \frac{2\bar\mu^{2}}{2P+1}\,A + \frac{i}{2(P+1)(2P+1)}\,\hat M^{\dot\alpha\dot\beta}\,G^{0}{}_{\dot\alpha\dot\beta}$$

$$\tilde\psi^{+}{}_{\dot\alpha +} = \psi^{+}{}_{\dot\alpha +} + \frac{4}{y\mu}\,(\sigma^{\mu})^{+}{}_{\dot\alpha}{}^{\alpha}\,[\partial_{\mu}\alpha^{+}{}_{\alpha} + \partial_{\mu}\alpha^{-}{}_{\alpha}]$$

$$\tilde\Psi^{+}{}_{\dot\alpha -} = \frac{3}{2}\,i\,\Omega^{+}{}_{\dot\alpha -} + \bar\mu^{2}\,\Psi^{+}{}_{\dot\alpha -}$$

$$\tilde\gamma^{+\pm}_{\dot\alpha\dot\beta\alpha} = \gamma^{+\pm}_{\dot\alpha\dot\beta\alpha} + \frac{1}{y\mu}\,(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot\lambda}\,M^{+}_{+\dot\alpha\dot\beta\dot\lambda}{}^{\dot\delta}\,\partial_{\mu}\Psi^{+}{}_{\dot\delta -}$$

$$\tilde\phi^{\pm}{}_{\alpha 0} = \phi^{\pm}{}_{\alpha 0} + 2i\bar\mu^{2}\,\alpha^{\pm}{}_{\alpha} + \frac{1}{2(P+1)}\,\hat M^{\dot\alpha\dot\beta}\,\gamma^{0\pm}_{\dot\alpha\dot\beta\alpha}$$

$$- \frac{2}{y\mu(P+1)}\,(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot\alpha}\,\partial_{\mu}\Psi^{+}{}_{\dot\alpha -} - \frac{4}{y\mu}\,(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot\alpha}\,\partial_{\mu}\Psi^{-}{}_{\dot\alpha -}$$

$$\tilde P^{+\pm}_{\dot\alpha\alpha -} = \frac{3}{2}\,i\,W^{+\pm}_{\dot\alpha\alpha -} + \bar\mu^{2}\,P^{+\pm}_{\dot\alpha\alpha -} + \frac{4i}{y\mu}\,(\sigma^{\mu}_{\pm})^{+}{}_{\dot\alpha\alpha}\,\partial_{\mu}F_{-}$$

$$\tilde g^{+}_{\dot\alpha\dot\beta} = g^{+}{}_{\dot\alpha\dot\beta} + \frac{1}{y\mu}\,(\sigma^{\mu})^{\dot\lambda\alpha}\,M^{+}_{+\dot\alpha\dot\beta\dot\lambda}{}^{\dot\delta}\,[\partial_{\mu}P^{++}_{\dot\delta\alpha -} + \partial_{\mu}P^{+-}_{\dot\delta\alpha -}]$$

$$\tilde f_{0} = f_{0} + 2i\bar\mu^{2}a + \frac{1}{2(P+1)}\,\hat M^{\dot\alpha\dot\beta}\,g^{0}{}_{\dot\alpha\dot\beta}$$

$$+ \frac{2}{y\mu}\,(\sigma^{\mu})^{\dot\alpha\alpha}\,\left[\frac{1}{P+1}\,\partial_{\mu}P^{++}_{\dot\alpha\alpha -} + \frac{1}{P+1}\,\partial_{\mu}P^{+-}_{\dot\alpha\alpha -} - \frac{2(P+1)}{P}\,\partial_{\mu}P^{-+}_{\dot\alpha\alpha -} - \frac{2(P+1)}{P}\,\partial_{\mu}P^{--}_{\dot\alpha\alpha -}\right]$$

$$\tilde\psi^{+}{}_{\dot\alpha -} = \frac{3}{2}\,i\,\hat\omega^{+}{}_{\dot\alpha -} + \bar\mu^{2}\psi^{+}{}_{\dot\alpha -} + \frac{4i}{y\mu}\,(\sigma^{\mu})^{+}{}_{\dot\alpha}{}^{\alpha}\,[\partial_{\mu}\phi^{+}{}_{\alpha -} + \partial_{\mu}\phi^{-}{}_{\alpha -}]$$

TABLE 6.2d Basis for the $\Pi^{-}{}_{\alpha}{}^{\dot\beta}\bar{S}_{\dot\beta-}$ multiplet

$$\tilde{\Psi}^{-}{}_{\dot\alpha+} = \Psi^{-}{}_{\dot\alpha+}$$

$$\tilde{P}_{\dot\alpha\alpha+}^{-\pm} = P_{\dot\alpha\alpha+}^{-\pm} + \frac{4}{y\mu}(\sigma^{\mu}_{\pm})^{-}{}_{\dot\alpha\alpha}\partial_{\mu} A$$

$$\tilde{G}^{-}{}_{\dot\alpha\dot\beta} = G^{-}{}_{\dot\alpha\dot\beta}$$

$$\tilde{G}^{0}{}_{\dot\alpha\dot\beta} = \frac{2i(P+1)}{(2P+1)} G^{0}{}_{\dot\alpha\dot\beta} - \frac{i}{2(2P+1)} \hat{M}_{\dot\alpha\dot\beta} F_{0} + \frac{\bar\mu^{2}}{(2P+1)} \hat{M}_{\dot\alpha\dot\beta} A$$

$$\tilde{\psi}^{-}{}_{\dot\alpha+} = \psi^{-}{}_{\dot\alpha+} + \frac{4}{y\mu}(\sigma^{\mu})^{-}{}_{\dot\alpha}{}^{\alpha}[\partial_{\mu}\alpha^{+}{}_{\alpha} + \partial_{\mu}\alpha^{-}{}_{\alpha}]$$

$$\tilde{\Psi}^{-}{}_{\dot\alpha-} = \frac{3}{2} i\, \Omega^{-}{}_{\dot\alpha-} + \bar\mu^{2} \Psi^{-}{}_{\dot\alpha-}$$

$$\tilde{\gamma}^{-\pm}_{\dot\alpha\dot\beta\alpha} = \gamma^{-\pm}_{\dot\alpha\dot\beta\alpha} + \frac{1}{y\mu}(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot\lambda} M^{-}{}_{-\dot\alpha\dot\beta\dot\lambda}{}^{\dot\delta}\, \partial_{\mu}\Psi^{-}{}_{\dot\delta-}$$

$$\tilde{\gamma}^{0\pm}_{\dot\alpha\dot\beta\alpha} = 2(P+1)\gamma^{0\pm}_{\dot\alpha\dot\beta\alpha} - \frac{1}{2}\hat{M}_{\dot\alpha\dot\beta}\, \phi^{\pm}{}_{\alpha 0} - i\bar\mu^{2}\hat{M}_{\dot\alpha\dot\beta}\, \alpha^{\pm}{}_{\alpha}$$
$$+ \frac{1}{y\mu}\hat{M}_{\dot\alpha\dot\beta}(\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot\gamma}[2\partial_{\mu}\Psi^{+}{}_{\dot\gamma-} - \frac{1}{P}\partial_{\mu}\Psi^{-}{}_{\dot\gamma-}]$$

$$\tilde{P}_{\dot\alpha\alpha-}^{-\pm} = \frac{3}{2} iW_{\dot\alpha\alpha-}^{-\pm} + \bar\mu^{2} P_{\dot\alpha\alpha-}^{-\pm} + \frac{4i}{y\mu}(\sigma^{\mu}_{\pm})^{-}{}_{\dot\alpha\alpha}\partial_{\mu}F_{-}$$

$$\tilde{g}^{-}{}_{\dot\alpha\dot\beta} = g^{-}{}_{\dot\alpha\dot\beta} + \frac{1}{y\mu}(\sigma^{\mu})^{\dot\lambda\alpha} M^{-}{}_{-\dot\alpha\dot\beta\dot\lambda}{}^{\dot\delta}[\partial_{\mu} P_{\dot\delta\alpha-}^{-+} + \partial_{\mu}P_{\dot\delta\alpha-}^{--}]$$

$$g^{0}{}_{\dot\alpha\dot\beta} = 2(P+1)g^{0}{}_{\dot\alpha\dot\beta} - \frac{1}{2}\hat{M}_{\dot\alpha\dot\beta} f_{0} - i\bar\mu^{2}\hat{M}_{\dot\alpha\dot\beta}\, a$$
$$+ \frac{1}{y\mu}(\sigma^{\mu})^{\dot\gamma\alpha}\hat{M}_{\dot\alpha\dot\beta}[\frac{2P}{P+1}\partial_{\mu}P_{\dot\gamma\alpha-}^{++} + \frac{2P}{P+1}\partial_{\mu}P_{\dot\gamma\alpha-}^{+-} + \frac{1}{P}\partial_{\mu}P_{\dot\gamma\alpha-}^{-+} + \frac{1}{P}\partial_{\mu}P_{\dot\gamma\alpha-}^{--}]$$

$$\tilde{\psi}^{-}{}_{\dot\alpha-} = \frac{3}{2} i\, \omega^{-}{}_{\dot\alpha-} + \bar\mu^{2}\, \psi^{-}{}_{\dot\alpha-} + \frac{4i}{y\mu}(\sigma^{\mu})^{-}{}_{\dot\alpha}{}^{\alpha}[\partial_{\mu}\phi^{+}{}_{\alpha-} + \partial_{\mu}\phi^{-}{}_{\alpha-}]$$

In these tables the following notation has been adopted.

$(\sigma^{\mu}_{\pm})_{\dot{\alpha}\alpha} = \Pi^{\pm}{}_{\alpha}{}^{\beta}\,(\sigma^{\mu})_{\dot{\alpha}\beta}$, $(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}\alpha} = \Pi^{\pm}{}_{\dot{\alpha}}{}^{\dot{\beta}}\,(\sigma^{\mu})_{\dot{\beta}\alpha}$,

$$M^{\pm}{}_{\pm\dot{\alpha}\dot{\beta}\dot{\lambda}}{}^{\dot{\gamma}} = \frac{1}{P^{\pm}}\left(\hat{M}_{\dot{\alpha}\dot{\beta}}\,\delta_{\dot{\lambda}}{}^{\dot{\gamma}} + \varepsilon_{\dot{\lambda}\dot{\alpha}}\,\delta_{\dot{\beta}}{}^{\dot{\gamma}} + \varepsilon_{\dot{\lambda}\dot{\beta}}\,\delta_{\dot{\alpha}}{}^{\dot{\gamma}}\right),$$

where $P^{+} = P$ and $P^{-} = -P-1$, and

$G^{0}_{\pm\dot{\alpha}\dot{\beta}} = \Pi^{\pm}{}_{\dot{\alpha}}{}^{\dot{\gamma}}\,G^{0}_{\dot{\gamma}\dot{\beta}}$ and $\gamma^{0\pm}_{\pm\dot{\alpha}\dot{\beta}\alpha}$ and $g^{0}_{\pm\dot{\alpha}\dot{\beta}}$ are

similarly defined. See Appendix D for further discussion of the properties of the spin $p\times\frac{1}{2}$ projectors, $\Pi^{\pm}{}_{\dot{\alpha}}{}^{\dot{\beta}}$, and the spin $q\times\frac{1}{2}$ projectors, $\Pi^{\pm}{}_{\alpha}{}^{\beta}$.

TABLE 6.3 Basis required to obtain the irreducible 'spin-reducing' multiplets

$$\hat{\phi}^{\pm}{}_{\alpha+} = \tilde{\phi}^{\pm}{}_{\alpha+} - \frac{1}{y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}} [\partial_{\mu} \tilde{\Omega}^{+}{}_{\dot{\alpha}+} + \partial_{\mu} \tilde{\Omega}^{-}{}_{\dot{\alpha}+}]$$

$$\hat{f}_{+} = \tilde{f}_{+} - \frac{1}{2y\mu\mu^{-2}} (\sigma^{\mu})^{\dot{\alpha}\alpha} [\partial_{\mu} \tilde{W}^{++}_{\dot{\alpha}\alpha+} + \partial_{\mu} \tilde{W}^{+-}_{\dot{\alpha}\alpha+} + \partial_{\mu} W^{-+}_{\dot{\alpha}\alpha+} + \partial_{\mu} \tilde{W}^{--}_{\dot{\alpha}\alpha+}]$$

$$\hat{W}^{\pm\pm}_{\dot{\alpha}\alpha+} = \tilde{W}^{\pm\pm}_{\dot{\alpha}\alpha+} - \frac{2}{y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})^{\pm}{}_{\dot{\alpha}\alpha}\, \partial_{\mu} \tilde{D}$$

$$\hat{\phi}^{\pm}{}_{\alpha-} = \tilde{\phi}^{\pm}{}_{\alpha-} + \frac{1}{y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}} [\partial_{\mu} \tilde{\omega}^{+}{}_{\dot{\alpha}-} + \partial_{\mu} \tilde{\omega}^{-}{}_{\dot{\alpha}-}]$$

$$\hat{F}_{-} = \tilde{F}_{-} - \frac{1}{2y\mu\mu^{-2}} (\sigma^{\mu})^{\dot{\alpha}\alpha} [\partial_{\mu} \tilde{W}^{++}_{\dot{\alpha}\alpha-} + \partial_{\mu} \tilde{W}^{+-}_{\dot{\alpha}\alpha-} + \partial_{\mu} \tilde{W}^{-+}_{\dot{\alpha}\alpha-} + \partial_{\mu} \tilde{W}^{--}_{\dot{\alpha}\alpha-}]$$

$$\hat{W}^{\pm\pm}_{\dot{\alpha}\alpha-} = \tilde{W}^{\pm\pm}_{\dot{\alpha}\alpha-} + \frac{2}{y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})^{\pm}{}_{\dot{\alpha}\alpha}\, \partial_{\mu} \tilde{d}$$

$$\hat{P}^{+\pm}_{\dot{\alpha}\alpha+} = \tilde{P}^{+\pm}_{\dot{\alpha}\alpha+} + \frac{1}{y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\beta}} [2i\partial_{\mu} \tilde{G}^{+}_{\dot{\beta}\dot{\alpha}} + \tfrac{1}{2} \hat{M}_{\dot{\beta}\dot{\alpha}}\, \partial_{\mu} \tilde{F}_{0}]$$

$$\hat{\psi}^{+}{}_{\dot{\alpha}+} = \tilde{\psi}^{+}{}_{\dot{\alpha}+} + \frac{i}{2y\mu\mu^{-2}} (\sigma^{\mu})^{+}{}_{\dot{\alpha}}{}^{\alpha} [\partial_{\mu} \tilde{\phi}^{+}{}_{\alpha 0} + \partial_{\mu} \tilde{\phi}^{-}{}_{\alpha 0}]$$

$$+ \frac{i}{y\mu\mu^{-2}} (\sigma^{\mu})^{\dot{\beta}\alpha} [\partial_{\mu} \tilde{Y}^{++}_{\dot{\beta}\dot{\alpha}\alpha} + \partial_{\mu} \tilde{Y}^{+-}_{\dot{\beta}\dot{\alpha}\alpha}]$$

$$\hat{Y}^{+\pm}_{\dot{\alpha}\dot{\beta}\alpha} = \tilde{Y}^{+\pm}_{\dot{\alpha}\dot{\beta}\alpha} - \frac{1}{2y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\lambda}}\, M^{+}_{+\dot{\alpha}\dot{\beta}\dot{\lambda}}{}^{\dot{\delta}}\, \partial_{\mu} \tilde{\psi}^{+}{}_{\dot{\delta}-}$$

$$\hat{\phi}^{\pm}{}_{\alpha 0} = \tilde{\phi}^{\pm}{}_{\alpha 0} - \frac{(2P+1)}{(P+1)} \frac{1}{y\mu\mu^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\alpha}}\, \partial_{\mu} \tilde{\psi}^{+}{}_{\dot{\alpha}-}$$

$$\hat{P}^{-\pm}_{\alpha\dot{\alpha}+} = \tilde{P}^{-\pm}_{\dot{\alpha}\alpha+} + \frac{1}{y\mu\bar{\mu}^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\beta}} \left[2i\ \partial_{\mu}\ \tilde{G}^{-}{}_{\dot{\beta}\dot{\alpha}} + \partial_{\mu}\ \tilde{G}^{0}{}_{\dot{\beta}\dot{\alpha}}\right]$$

$$\hat{\psi}^{-}_{\dot{\alpha}+} = \tilde{\psi}^{-}{}_{\dot{\alpha}+} + \frac{i}{2y\mu\bar{\mu}^{-2}} (\sigma^{\mu})^{\dot{\beta}\alpha} \left[\frac{1}{P+1}\ \partial_{\mu}\ \tilde{\gamma}^{0+}_{-\dot{\beta}\dot{\alpha}\alpha} + \frac{1}{P+1}\ \partial_{\mu}\ \tilde{\gamma}^{0-}_{-\dot{\beta}\dot{\alpha}\alpha}\right.$$

$$\left. + 2\ \partial_{\mu}\ \tilde{\gamma}^{-+}_{\dot{\beta}\dot{\alpha}\alpha} + 2\ \partial_{\mu}\ \tilde{\gamma}^{--}_{\dot{\beta}\dot{\alpha}\alpha}\right]$$

$$\hat{\gamma}^{-\pm}_{\dot{\alpha}\dot{\beta}\alpha} = \tilde{\gamma}^{-\pm}_{\dot{\alpha}\dot{\beta}\alpha} - \frac{1}{2y\mu\bar{\mu}^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\lambda}}\ M^{-}_{-\dot{\alpha}\dot{\beta}\dot{\lambda}}{}^{\dot{\delta}}\ \partial_{\mu}\ \tilde{\Psi}^{-}{}_{\dot{\delta}-}$$

$$\hat{\gamma}^{0\pm}_{\dot{\alpha}\dot{\beta}\alpha} = \tilde{\gamma}^{0\pm}_{\dot{\alpha}\dot{\beta}\alpha} + \frac{(2P+1)}{2P}\ \frac{1}{y\mu\bar{\mu}^{-2}} (\sigma^{\mu}_{\pm})_{\alpha}{}^{\dot{\gamma}}\ \hat{M}_{\dot{\alpha}\dot{\beta}}\ \partial_{\mu}\ \tilde{\Psi}^{-}{}_{\dot{\gamma}-}$$

TABLE 6.4 Variations of the fields of the $\Lambda \equiv F_+$ multiplet under $Q_{\gamma+}$ and $\bar{Q}_{\dot{\lambda}+}$. These demonstrate the irreducibility, as a factor space, of the fields F_+, $\tilde{\Omega}^+{}_{\dot{\alpha}+}$, $\tilde{\Omega}^-{}_{\dot{\alpha}+}$, $\tilde{D}$ under the constraint $P^2+\mu^2\bar{\mu}^{-2} = 0$.

$$\delta_{\gamma+} F_+ = \delta_{\dot{\lambda}+} F_+ = 0$$

$$\delta_{\gamma+} \hat{\phi}^{\pm}{}_{\alpha+} = \frac{y}{\mu\bar{\mu}^{-2}} (\Pi^{\pm}\varepsilon)_{\alpha\gamma} (P^2+\mu^2\bar{\mu}^{-2}) F_+$$

$$\delta_{\dot{\lambda}+} \hat{\phi}^{\pm}{}_{\alpha+} = 0$$

$$\delta_{\gamma+} \tilde{\Omega}^{\pm}{}_{\dot{\alpha}+} = - 4i(\sigma^{\mu})^{\pm}{}_{\gamma\dot{\alpha}} \partial_{\mu} F_+$$

$$\delta_{\dot{\lambda}+} \tilde{\Omega}^{\pm}{}_{\dot{\alpha}+} = - 4i\, \bar{\mu}^{-2} (\Pi^{\pm}\varepsilon)_{\dot{\alpha}\dot{\lambda}} F_+$$

$$\delta_{\gamma+} \hat{f}_+ = \frac{y}{\mu\bar{\mu}^{-2}} (\tfrac{1}{2} P^2+\mu^2\bar{\mu}^{-2})(\phi^+{}_{\gamma+} + \phi^-{}_{\gamma+})$$
$$+ \frac{2i}{y^2\mu^2\bar{\mu}^{-4}} (\sigma^{\mu})^{\dot{\alpha}}{}_{\gamma} (P^2+\mu^2\bar{\mu}^{-2})(\partial_{\mu} \tilde{\Psi}^+{}_{\dot{\alpha}+} + \partial_{\mu} \tilde{\Psi}^-{}_{\dot{\alpha}+})$$

$$\delta_{\dot{\lambda}+} \hat{f}_+ = \frac{2i}{y\mu} (\sigma^{\mu})_{\dot{\lambda}}{}^{\alpha} (\partial_{\mu} \hat{\phi}^+{}_{\alpha+} + \partial_{\mu} \hat{\phi}^-{}_{\alpha+})$$

$$\delta_{\gamma+} \tilde{D} = - 2i(\sigma^{\mu})^{\dot{\alpha}}{}_{\gamma} (\partial_{\mu} \tilde{\Omega}^+{}_{\dot{\alpha}+} + \partial_{\mu} \tilde{\Omega}^-{}_{\dot{\alpha}+})$$

$$\delta_{\dot{\lambda}+} \tilde{D} = -2i\bar{\mu}^{-2} (\tilde{\Omega}^+{}_{\dot{\lambda}+} + \tilde{\Omega}^-{}_{\dot{\lambda}+})$$

$$\delta_{\gamma+} \hat{W}^{\pm\pm}_{\dot{\alpha}\alpha+} = - 4i(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}\gamma} \partial_{\mu} \hat{\phi}^{\pm}{}_{\alpha+}$$
$$- \frac{y}{\mu\bar{\mu}^{-2}} (\Pi^{\pm}\varepsilon)_{\alpha\gamma} (P^2+\mu^2\bar{\mu}^{-2}) \tilde{\Omega}^{\pm}{}_{\alpha+}$$

$$\delta_{\dot{\lambda}+} \hat{W}^{\pm\pm}_{\dot{\alpha}\alpha+} = - 4i\bar{\mu}^{-2} (\Pi^{\pm}\varepsilon)_{\dot{\alpha}\dot{\lambda}} \hat{\phi}^{\pm}{}_{\alpha+}$$

$$\delta_{\gamma+}\,\tilde{\delta}^{\pm}{}_{\alpha} = \frac{y}{\mu\bar{\mu}^{2}}\,(\Pi^{\pm}\varepsilon)_{\alpha\gamma}\,(P^{2}+\mu^{2}\bar{\mu}^{2})\,\tilde{D}$$

$$-\,2i(\sigma^{\mu})^{\dot{\alpha}}{}_{\gamma}\,(\partial_{\mu}\,\hat{W}^{+\pm}_{\dot{\alpha}\alpha+} + \partial_{\mu}\,\hat{W}^{-\pm}_{\dot{\alpha}\alpha+})$$

$$\delta_{\dot{\lambda}+}\tilde{\delta}^{\pm}{}_{\alpha} = -\,2i\,\bar{\mu}^{2}\,(\hat{W}^{+\pm}_{\dot{\lambda}\alpha+} + \hat{W}^{-\pm}_{\dot{\lambda}\alpha+})$$

$$\delta_{\gamma+}\,\tilde{\omega}^{\pm}{}_{\dot{\alpha}+} = -\,4i(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}\gamma}\,\partial_{\mu}\,\hat{f}_{+} - y\mu(\hat{W}^{\pm+}_{\dot{\alpha}\gamma+} + W^{\pm-}_{\dot{\alpha}\gamma+})$$

$$-\,\frac{2i}{y\mu\bar{\mu}^{2}}\,(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}\gamma}\,(\sigma^{\nu})^{\dot{\beta}\beta}\,\partial_{\mu}\partial_{\nu}(\hat{W}^{++}_{\dot{\beta}\beta+} + \hat{W}^{+-}_{\dot{\beta}\beta+} + \hat{W}^{-+}_{\dot{\beta}\beta+} + \hat{W}^{--}_{\dot{\beta}\beta+})$$

$$-\,\frac{2}{\mu^{2}\bar{\mu}^{4}}\,(\sigma^{\mu})^{\pm}{}_{\dot{\alpha}\gamma}\,(P^{2}+\mu^{2}\bar{\mu}^{2})\,\partial_{\mu}\,\tilde{D}$$

$$\delta_{\dot{\lambda}+}\,\tilde{\omega}^{\pm}{}_{\dot{\alpha}+} = -\,4i\bar{\mu}^{2}(\Pi^{\pm}\varepsilon)_{\dot{\alpha}\dot{\lambda}}\,\hat{f}_{+} - \frac{2i}{y\mu}\,(\sigma^{\mu})_{\dot{\lambda}}{}^{\alpha}\,\partial_{\mu}(\hat{W}^{\pm+}_{\dot{\alpha}\alpha+} + \hat{W}^{\pm-}_{\dot{\alpha}\alpha+})$$

$$-\,\frac{2i}{y\mu}\,(\sigma^{\mu})^{\pm}_{\dot{\alpha}}{}^{\alpha}\,\partial_{\mu}(\hat{W}^{++}_{\dot{\lambda}\alpha+} + \hat{W}^{+-}_{\dot{\lambda}\alpha+} + \hat{W}^{-+}_{\dot{\lambda}\alpha+} + \hat{W}^{--}_{\dot{\lambda}\alpha+})$$

$$\delta_{\dot{\lambda}+}\,\tilde{d} = -\,2i\bar{\mu}^{2}(\tilde{\omega}^{+}{}_{\dot{\lambda}+} + \tilde{\omega}^{-}{}_{\dot{\lambda}+}) + \frac{y}{\mu}\,(\sigma^{\mu})_{\dot{\lambda}}{}^{\alpha}\,(\partial_{\mu}\,\tilde{\delta}^{+}{}_{\alpha} + \partial_{\mu}\,\tilde{\delta}^{-}{}_{\alpha})$$

$$\delta_{\gamma+}\,\tilde{d} = -\,2i(\sigma^{\mu})^{\dot{\alpha}}{}_{\gamma}\,(\partial_{\mu}\,\tilde{\omega}^{+}{}_{\dot{\alpha}+} + \partial_{\mu}\,\tilde{\omega}^{-}{}_{\dot{\alpha}+}) + y\mu(\tilde{\delta}^{+}{}_{\gamma} + \tilde{\delta}^{-}{}_{\gamma})$$

In the above $\delta_{\gamma+} \equiv \delta_{Q_{\gamma+}}$ and $\delta_{\dot{\lambda}+} \equiv \delta_{\bar{Q}_{\dot{\lambda}+}}$, and we recall $P^{2} = -\,\partial^{2}$, $\mu^{2} \equiv Z$ and $\bar{\mu}^{2} \equiv \bar{Z}$.

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7. ORTHOSYMPLECTIC SUPERGROUPS IN PHYSICAL THEORIES

7.1 EXTENDED BRS INVARIANCE

Perhaps the most *useful* application that orthosymplectic supergroups have found (to date) in the world of physics, is to the elegant formulation they provide of the extended BRS symmetries [1,2,3,4] of quantised gauge theories. The BRS symmetries [1,2] mix the gauge and ghost fields of non-abelian gauge theories in such a way as to leave the action invariant. They have powerful implications for the quantisation and renormalisation of these theories, and in particular Zinn-Justin [5] and Kluberg-Stern and Zuber [6,7] have been able to prove the renormalisability of Yang-Mills theories based on this invariance. Subsequent investigations [3,4] revealed that an 'extended' BRS set can be constructed, involving a two-parameter 'BRS group' where the roles of 'ghost' and 'antighost' can essentially be interchanged. Following earlier work on the unextended case [8,9], Bonora and Tonin [10] developed a superfield formulation of the extended BRS symmetry based on a six dimensional superspace in which the BRS group consists of supertranslations in two a-number superspace coordinates $(\theta,\bar{\theta})$. An alternative formulation of BRS supersymmetry has been proposed by Delbourgo and Jarvis [11]. This formulation is based upon a real form of the inhomogeneous OSp(4/2) supergroup [12] consisting of the usual transformations of the Poincaré group and, in addition, symplectic transformations in $(\theta,\bar{\theta})$ space as well as 'supertranslations' and 'super-Lorentz' transformations. This goes beyond the work of Bonora and Tonin [10] in the sense that the group of supertranslations is enlarged to include transformations mixing x^{μ} and $(\theta,\bar{\theta})$. The supertranslations again give rise to extended

BRS transformations amongst the superfield component fields.
The essence of this formulation by Delbourgo and Jarvis is sketched below and its extensions discussed briefly.

As mentioned above the space-time supersymmetry imposed is a real form [13] of the six-dimensional, inhomogeneous, orthosymplectic supergroup $OSp(4/2) \wedge T_{4/2}$, which is the group of all superlinear transformations preserving the distance [12]

$$(X-Y)^2 \equiv (X-Y)_M \, g^{MN} \, (X-Y)_N \qquad 7.1$$

between points in superpace, $X_M = (x_\mu, \theta_\alpha)$ where $\mu = 0,1,2,3$ and $\alpha = 1,2$. Here the orthosymplectic metric is

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \varepsilon^{\alpha\beta} \end{pmatrix}$$

where $\eta^{\mu\nu}$ is the usual diagonal Lorentz metric and $\varepsilon^{\alpha\beta}$ is the 2×2 antisymmetric matrix with $\varepsilon^{12} = +1$.

This space-time supergroup admits, in addition to the usual Poincaré transformations, supertranslations

$$(x_\mu, \theta_\alpha) \rightarrow (x_\mu, \theta_\alpha + \varepsilon_\alpha) \qquad 7.2$$

symplectic rotations on θ_α,

$$(x_\mu, \theta_\alpha) \rightarrow (x_\mu, \tau_\alpha^\beta \theta_\beta) \qquad 7.3$$

and super-Lorentz transformations,

$$(x_\mu, \theta_\alpha) \rightarrow (x_\mu + \lambda_\mu^\beta \theta_\beta, \;\; \theta_\alpha - \lambda_\alpha^\nu x_\nu) \; . \qquad 7.4$$

The conventions here are those of [11] in which $\theta_2 = \theta_1^*$ and $(\theta_1, \theta_2) = (\theta, \bar{\theta})$ (see [11] for further discussion of hermiticity questions).

In constructing local gauge theories over superspace, a gauge potential superfield is introduced

$$\Phi_M(x,\theta) = A_M(x) + (\text{higher-order terms in } \theta) \qquad 7.5$$

where $A_\mu(x)$ is a c-number field, $A_\alpha(x)$ is an a-number field and $A_M(x)$ transforms as the fundamental, six-dimensional, vector representation of OSp(4/2) and takes its values in the compact Lie algebra of the gauge group, $\Phi_M = \Phi_M^a T^a$, where the T^a are the generators of the Lie algebra.

The gauge field strength, $\Phi_{MN}(x,\theta)$, is a superfield transforming in the 17-dimensional graded-antisymmetrical tensor representation (i.e. the adjoint) of OSp(4/2) and is constructed as

$$\Phi_{MN} = \partial_M \Phi_N - [MN]\partial_N \Phi_M - ig[\Phi_M, \Phi_N]_\pm \qquad 7.6$$

where g is the gauge coupling constant and [MN] is a signature factor with $[\mu\nu] = [\mu\alpha] = +1$, $[\alpha,\beta] = -1$.

Gauge transformations for Φ_M and Φ_{MN} are given as usual by

$$\Phi'_M(x,\theta) = U^{-1}\Phi_M(x,\theta)U - i/g(\partial_M U^{-1})U \qquad 7.7$$

$$\Phi'_{MN}(x,\theta) = U^{-1}\Phi_{MN}(x,\theta)U \qquad 7.8$$

and, as shown by Bonora et al [14], $U(x,\theta)$ may be uniquely decomposed as

$$U(x,\theta) = \exp[-ig\Lambda(x)]\,\exp[-ig(\theta^\alpha\omega_\alpha(x) - \theta^\alpha\theta_\alpha B(x))] \equiv U_0 U_1 \qquad 7.9$$

To obtain a model in the six-dimensional space, which yields a Yang-Mills action upon dimensional reduction to four-dimensional Minkowski space-time, only those gauge potential superfields are admitted which satisfy

$$\Phi_M(x,\theta) = U_1^{-1}\begin{pmatrix} A_\mu(x) \\ 0 \end{pmatrix}U_1 - i/g(\partial_M U_1^{-1})U_1 \qquad 7.10$$

and

$$\Phi_{MN}(x,\theta) = U_1^{-1}\begin{pmatrix} F_{\mu\nu}(x) \\ 0 \\ 0 \end{pmatrix} U_1 \qquad 7.11$$

where $A_\mu(x)$ is the ordinary four-vector potential and $F_{\mu\nu}(x)$ is the usual Yang-Mills field strength. Expanding the exponential of (7.9) the components of $\Phi_M(x,\theta)$ may be written as

$$\Phi_\mu = A_\mu + \theta^\beta D_\mu \omega_\beta - \frac{1}{2}\theta^\beta\theta_\beta[D_\mu B + \frac{1}{2}g\,(D_\mu\omega^\alpha) \times \omega_\alpha] \qquad 7.12$$

$$\Phi_\alpha = \omega_\alpha + \theta^\beta[B\varepsilon_{\beta\alpha} - \frac{1}{2}g\,\omega_\beta \times \omega_\alpha]$$

$$- \frac{1}{2}\theta^\beta\theta_\beta[-gB \times \omega_\alpha + \frac{1}{6}g^2\,(\omega_\alpha \times \omega^\gamma) \times \omega_\gamma] \qquad 7.13$$

where D_μ is the covariant derivative $D_\mu B = \partial_\mu B + gA_\mu \times B$, etc.

The six-dimensional action, S, is taken to be the sum of a gauge-independent piece, S_0, and a gauge-dependent piece, S_1. S_0 is constructed to be gauge invariant, $OSp(4/2)\Lambda T_{4/2}$ invariant and to reduce to the usual Yang-Mills action in four-dimensional space-time. It is given by

$$S_0 = \int d^6X\,\frac{1}{4}X^2\,\Phi^{aMN}\Phi^a_{MN} \quad . \qquad 7.14$$

The choice of the gauge breaking term, S_1, is not unique but is required to break superlocal gauge invariance, to be both Sp(2) and supertranslation invariant, and to have canonical dimension 2 [15] (for an extended discussion of gauge breaking see Thompson [15]). A suitable candidate is

$$S_1 = \int d^6X \cdot 2\Phi^M\Phi_M/\xi \qquad 7.15$$

where ξ is a real constant. This leads to the action [11]

$$S = \int d^4x[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \partial^\mu A_\mu \cdot B + \frac{1}{2}\xi B^2 - \partial^\mu\bar{\omega}\cdot\partial_\mu\omega$$

$$- \frac{1}{2}gA^\mu \cdot \bar{\omega} \times \overleftrightarrow{\partial}_\mu\omega + \frac{1}{8}g^2\xi\,(\bar{\omega} \times \omega)^2 \qquad 7.16$$

after appropriate rescaling of the fields and taking $(\omega_1,\omega_2) = (\omega,\bar{\omega})$. This action incorporates the conventional Yang-Mills action; gauge fixing terms involving the gauge potential, A_μ, and multiplier field, B; and Faddeev-Popov ghost terms involving ghost fields, ω and $\bar{\omega}$, in addition to A_μ. It differs from the conventional action by the inclusion of a quartic ghost-term.

Supertranslations leave invariant the action (7.14) plus (7.15) and also respect the condition (7.10). From (7.12) and (7.13) the component field variations under supertranslations are found to be

$$\begin{aligned}
\delta A_\mu &= \bar{\varepsilon} D_\mu \omega - \varepsilon D_\mu \bar{\omega} , \\
\delta\omega &= -\tfrac{1}{2} g\, \bar{\varepsilon}\bar{\omega} \times \omega - \varepsilon B_- \\
\delta\bar{\omega} &= +\tfrac{1}{2} g\, \varepsilon\bar{\omega} \times \bar{\omega} - \bar{\varepsilon} B_+ \\
\delta B_+ &= - g\varepsilon B_+ \times \bar{\omega} , \qquad \delta B_- = g\bar{\varepsilon} B_-
\end{aligned} \qquad 7.17$$

where $B_\pm = B \pm \frac{1}{2} g\, \bar{\omega} \times \omega$.

These are the extended BRS symmetries, providing a set of transformations which leave the total action invariant. Further, it was shown [11] that this model is renormalisable in standard fashion and yields the same on-shell S-matrix as that obtained via the conventional approach [15].

It was later shown [16] that a satisfactory geometrical setting for the above scheme could be achieved based upon a coset space dimensional reduction procedure [17]. With this treatment, the ansatz (7.10), for the form of $\Phi_M(x,\theta)$, follows from superfield constraints, obtained from the requirement that the gauge potential superfield be invariant under the action of the OSp(4/2) supergroup, up to a gauge transformation.

This formalism also allows the introduction of matter fields, including fermions, which appear as the solutions of analogous constraint equations applied to an appropriate representation of the tangent space supergroup OSp(4/2). One such constraint requires fields which are Sp(2) singlets [16]. Thus, for example, to incorporate spinors into the theory we examine representations of OSp(4/2), labelled by $\{L,M,N\}$, and decompose them with respect to

$$O(4) \times Sp(2) \simeq SU(2) \times SU(2) \times SU(2) \quad \text{as} \quad (\text{see } \S 4.5)$$

$$\begin{aligned} \{L,M,N\} \downarrow\ & (L-1,M,N)^2 + (L-1\pm1,M,N) + (L-1,M\pm1,N) \\ & (L-1,M,N\pm1) + (L-1\pm\tfrac{1}{2},M\pm\tfrac{1}{2},N\pm\tfrac{1}{2}) \end{aligned} \qquad 7.18$$

From this, it is evident that there is a unique, typical, irreducible representation whose only Sp(2) singlet is $(\frac{1}{2},0)$, corresponding to a left handed spinor: namely the $\{3/2,0,1\}$, possessing dimension 96. Similarly the right handed spinor $(0,\frac{1}{2})$ occurs as the unique Sp(2) singlet in the $\{1,\frac{1}{2},1\}$ representation of dimension 96. Thus a Dirac Spinor, $\Psi_\alpha(x,\theta)$ would correspond to the reducible representation $\{3/2,0,1\} + \{1,\frac{1}{2},1\}$, of OSp(4/2). Using the supertableau techniques of Chapter 5, it is possible to show that the Kronecker product of $\{3/2,0,1\}$ with the fundamental representation of OSp(4/2) contains $\{1,\frac{1}{2},1\}$, the parity conjugate. Thus it is possible to construct a bilinear kinetic term and subsequently an OSp(4/2) invariant lagrangian $(\overline{\Psi}_\alpha\ C^M_{\alpha\beta}\ D_M\ \overline{\Psi}_\beta + m\ \overline{\Psi}_\alpha\ \overline{\Psi}_\alpha)$, where the $C^M_{\alpha\beta}$ are coupling coefficients, which becomes $(\overline{\Psi}\ i\gamma^\mu\ D_\mu\ \Psi + m\ \overline{\Psi}\Psi)$ upon dimensional reduction [16].

This formalism of dimensional reduction via supercoset space has been applied to the quantisation of an OSp(n/2) gauge theory over a six-dimensional superspace [18]. This results, after reduction to four dimensions, in a gauged O(n) model containing massless scalar Higgs fields in the fundamental representation of O(n) with quartic selfinteractions and gauge fixing and ghost terms which admit an extended BRS invariance.

It has also been shown [15,19,20] that the derivation of *extended* BRS symmetries for quantum gravity is also amenable to the orthosymplectic BRS supersymmetry formalism; suitably modified of course but retaining a six-dimensional superspace which admits an OSp(4/2) supergroup. This was a particularly satisfying result in view of the fact that the standard gauge fixing procedure for gravity, while possessing an invariance under a set of BRS transformations [21,22], does not allow a corresponding set of dual BRS transformations [23]. The actions obtained from the OSp(4/2) formalism and the standard formalism differ only by a BRS (or dual BRS) transformation and hence their on-shell S matrices agree.

Kaluza-Klein theories [24,25,26,27] involve the construction of a gravitational action in (4+N)-dimensions and the subsequent compactification of N-dimensions to yield an action which formally incorporates a four-dimensional gravitational action, a Yang-Mills action (if $N > 1$) and a cosmological term. By extending the above ideas to consider a $4 + N + 2$ -dimensional manifold admitting an OSp(4+N/2) symmetry, combined with a (4+N)-dimensional Kaluza-Klein theory, an action emerges, after an appropriate dimensional reduction, which contains not only the correct gauge-fixing and Faddeev-Popov terms for both the gravitational and non-abelian gauge theories, but also leads to a complete set of extended BRS transformations [15,28]. This result was first achieved by Hosoya, Ohkuwa and Omote [29] quite independently of the OSp(4+N/2) formalism, however the latter version is aesthetically more pleasing in that the ghost and antighost fields enter, quite generally, in a symmetric manner.

7.2 ORTHOSYMPLECTIC SUPERGROUPS IN SUPERSYMMETRY

Over the past ten years supersymmetric Yang-Mills and supergravity theories have become a dominant force in the physics literature (see [30,31) for reviews). The basic algebra underlying the majority of these theories is the N-extended super Poincaré algebra, SP_4^N, discussed in Chapter 6. The role of orthosymplectic superalgebras becomes apparent when it is realised that OSp(N/4) is the algebra of the N-extended graded de Sitter group and is related by contraction to SP_4^N. The gauging of a number of orthosymplectic supergroups has bestowed upon them a dynamical role in some theories. For example, MacDowell and Mansouri [32] showed that N = 1 supergravity with a cosmalogical constant follows from gauging OSp(1/4), while Townsend and van Nieuwenhuizen [59] arrived at a similar result for N = 2 supergravity from the gauging of OSp(2/4). Nath and Arrowitt [33] have constructed geometrical models of supergravity in superspace where the tangent space group is OSp(3,1/4N). Extensive use has been made of orthosymplectic groups in the group manifold approach to supergravity theories. D'Adda et al [34] have classified a large range of orthosymplectic groups which are suitable for the construction of supergravity theories in various dimensions. They have developed a comprehensive procedure by which such theories may be formulated on orthosymplectic supergroup manifolds.

A detailed discussion of these models is eschewed in favour of demonstrating the contraction procedure for OSp(N/4), based upon the work of Green and Jarvis [35] and Butchart [36], which leads to the N-extended super Poincaré algebra, and a discussion of the work of Lukierski and Rytel [37] wherein a contraction of OSp(2N/4) leads to the N-extended super Poincaré algebra with a complete set of N(N-1) real central charges.

The OSp(N/4) superalgebra consists of even generators, $L_{ab} = -L_{ab}$, of O(N) and, $M_{\alpha\beta} = M_{\beta\alpha}$, of Sp(4) and odd generators $S_{a\alpha} = S_{\alpha a}$, where $a = 1,\ldots,N$; $\alpha = 1,\ldots,4$. These generators satisfy the following relations

$$
\begin{aligned}
[L_{ab},L_{cd}] &= \eta_{cb}L_{ad} - \eta_{ac}L_{bd} + \eta_{nd}L_{ca} - \eta_{da}L_{cb} \\
[M_{\alpha\beta},M_{\gamma\delta}] &= C_{\gamma\beta}M_{\alpha\delta} + C_{\gamma\alpha}M_{\beta\delta} + C_{\delta\beta}M_{\gamma\alpha} + C_{\delta\alpha}M_{\gamma\beta} \\
[L_{ab},S_{c\alpha}] &= \eta_{cb}S_{a\alpha} - \eta_{ca}\,S_{b\alpha} \\
[M_{\alpha\beta},S_{a\gamma}] &= C_{\gamma\beta}S_{a\alpha} + C_{\gamma\alpha}S_{a\beta} \\
\{S_{a\alpha},S_{b\beta}\} &= \eta_{ab}M_{\alpha\beta} + C_{\alpha\beta}L_{ba}
\end{aligned}
\tag{7.19}
$$

where η_{ab} is the diagonal O(N) metric with signature $(+,+,\ldots,+)$ and $C_{\alpha\beta}$ is the Sp(4) metric. The isomorphism between the Sp(4) algebra and the algebra, SO(3,2) of the de Sitter group is established by identifying $C_{\alpha\beta}$ with the charge conjugation matrix of the Dirac spinor representation and expanding $M_{\alpha\beta}$ in terms of the symmetric matrices $(\gamma_\mu C)_{\alpha\beta}$ and $(\sigma_{\mu\nu}C)_{\alpha\beta}$

$$
M_{\alpha\beta} = -(\gamma_\mu C)_{\alpha\beta}M^\mu - \frac{1}{2}(\sigma_{\mu\nu}C)_{\alpha\beta}M^{\mu\nu} \tag{7.20}
$$

where $\mu = 0,1,2,3$. The de Sitter algebra is

$$
\begin{aligned}
[M^{\mu\nu},M^{\rho\sigma}] &= i(\eta^{\nu\rho}M^{\mu\sigma}-\eta^{\mu\rho}M^{\nu\sigma}-\eta^{\nu\sigma}M^{\mu\rho}+\eta^{\sigma\mu}M^{\nu\rho}) \\
[M^{\mu\nu},M^{\rho}] &= i(\eta^{\rho\nu}M^{\mu}-\eta^{\rho\mu}M^{\nu}) \\
[M^{\mu},M^{\nu}] &= -iM^{\mu\nu}
\end{aligned}
\tag{7.21}
$$

where $\eta_{\mu\nu} = (+\,---)$.

Defining new generators

$$\bar{Q}_{a\alpha} = \frac{1}{R} S_{a\alpha} , \qquad \bar{P}^{\mu} = \frac{1}{R^2} M^{\mu} , \qquad \bar{J}^{\mu\nu} = M^{\mu\nu} \qquad 7.22$$

and taking the limit, $R \to \infty$, in which the barred generators of (7.22) tend to smooth limits $Q_{a\alpha}$, P^{μ} and $J^{\mu\nu}$, (7.19) and (7.21) become

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\rho\nu} J_{\mu\sigma} - \eta_{\rho\mu} J_{\nu\sigma} - \eta_{\sigma\nu} J_{\mu\rho} + \eta_{\sigma\mu} J_{\nu\rho}) \\
[J_{\mu\nu}, P_{\rho}] &= i(\eta_{\nu\rho} P_{\mu} - \eta_{\mu\rho} P_{\nu}) \\
[J_{\mu\nu}, Q_{a\alpha}] &= \frac{1}{2} (\sigma_{\mu\nu})_{\alpha}{}^{\beta} Q_{a\beta} \\
\{Q_{a\alpha}, Q_{b\beta}\} &= -\eta_{ab} (\gamma_{\mu} C)_{\alpha\beta} P^{\mu} \qquad\qquad 7.23 \\
[P_{\mu}, P_{\nu}] &= [P_{\mu}, Q_{a\alpha}] = 0 \\
[L_{ab}, L_{cd}] &= \eta_{cb} L_{ad} - \eta_{ac} L_{bd} + \eta_{bd} L_{ca} - \eta_{da} L_{cb} \\
[L_{ab}, Q_{c\alpha}] &= \eta_{cb} Q_{a\alpha} - \eta_{ca} Q_{b\alpha} .
\end{aligned}$$

This is the algebra of O(N)-extended supersymmetry and has been obtained by a straightforward contraction of OSp(N/4). The Casimir invariants of OSp(N/4) have been shown [35,36] to contract in the required manner to yield the Casimir invariants of the extended supersymmetry.

As shown in chapter 6 the central charges constitute an important enlargement of SP_4^N . Indeed as noted there (see e.g. [38,39]) in some cases it is only the presence of central charges which allows for the existence of the appropriate field representations necessary for the construction of interacting theories. They also appear to be necessary for the formulation of $N \geq 3$-extended supergravities. Lukierski and Rytel [37] have shown that by a suitable contraction procedure, similar to that described above, though somewhat more complex, applied to OSp(2N/4) one can obtain SP_4^N with a complete set of N(N-1) real central charges. The O(2N) algebra may be decomposed into a direct sum U(N) $\oplus$ O(2N)/U(N). It is the O(2N)/U(N) generators which in the contraction limit yield the central charges, Z_{ij}, upon reduction of the internal symmetry group, U(N), to a subgroup which commutes the Z_{ij}. Such a reduction has been discussed by Ferrara et al [40] and Lopuszanski and Wolf [41]. It is this type of geometric understanding of the origin of central charges which may provide the key to their potential role in the extended supergravity theories.

7.3 KALUZA-KLEIN SUPERGRAVITY

Kaluza-Klein supergravity is a term which refers to the construction of a supergravity theory in a space of dimension, d = 4 + N, and the subsequent compactification of N of these dimensions to yield an effective theory of supergravity in four dimensions. These theories, and in particular the d = 11 version, have offered the exciting prospect of unifying gravity with the standard model, SU(3) × SU(2) × U(1), of elementary particles. Although this offer has not been fulfilled, despite an enthusiastic following, these are still early days in the investigation and much remains to be learned about the structure of these theories. Orthosymplectic supergroups arise as the ground state symmetry of some of the solutions which have been exhibited. The eleven dimensional supergravity models which compactify a seven dimensional internal manifold have received the closest scrutiny, since they appear at present to offer the best prospect for achieving the unification mentioned above. In these theories it is OSp(M/4) which arises as the ground state symmetry for some solutions and consequently it is the infinite dimensional representations of OSp(M/4) which must be ascertained if the full implications of these solutions are to be known. This is of particular consequence for those solutions for which the effective four dimensional theory is in a de Sitter space where the simple idea of keeping massless modes and discarding massive ones has been shown to be incorrect [42]. In the sequel the solutions of simple supergravity in $d=11$ will be presented for which the compact manifold, M^7, is the 'round' seven sphere, S^7, and the seven torus, T^7, both of which possess a ground state symmetry of OSp(8/4). This will be followed by a brief discussion of the possibilities for obtaining solutions which have $SU(3) \times SU(2) \times U(1)$ as the isometry group of M^7. Finally some recent work by Freedman and Nicolai [43] on unitary, irreducible representations of OSp(N/4) and their applications will be mentioned.

The fields of simple supergravity in $d=11$ [44] are the elfbein, $e_M{}^A$, a 32-component Majorana spinor ψ_M, and an antisymmetric 3-index tensor A_{MNP} where the world indices, M,N,..., and tangent space indices, A,B,..., all take values 1-11. To obtain vacuum solutions, the usual procedure [45,46] to require that the vacuum expectation value of the spinor field, $<\psi_M>$, be zero and to look for solutions of the bosonic field equations. These are

$$R_{MN} - \frac{1}{2} g_{MN} R = - \frac{1}{3} [F_{MPQR} F_N{}^{PQR} - \frac{1}{8} g_{MN} F_{PQRS} F^{PQRS}] \qquad 7.24$$

$$D_M F^{MNPQ} = - \frac{1}{576} \varepsilon^{M_1 \ldots M_8 NPQ} F_{M_1 \ldots M_4} F_{M_5 \ldots M_8} \qquad 7.25$$

where $F_{MNPQ} = 4\, \partial_{[M} A_{NPQ]}$, $D_M = (\partial_M + \frac{1}{4} \omega_M{}^{AB} \Gamma_{AB})$ is the covariant derivative, with $\Gamma^{AB} = \Gamma^{[A}\Gamma^{B]}$ and Γ^A the eleven dimensional Dirac matrices, $\varepsilon^{\cdots}$ is the totally antisymmetric tensor with $\varepsilon^{1 \ldots 11} = +1$, and g_{MN}, R_{MN} and R are the eleven dimensional metric tensor, Ricci curvature tensor and Ricci scalar, respectively. A solution to this system of equations is provided by the Freund-Rubin mechanism [47] which takes $F_{\mu\nu\rho\sigma} = 3M\ \varepsilon_{\mu\nu\rho\sigma}$, $F_{MNPQ} = 0$ and

$$g_{MN}(x^{\mu},y^{m}) = \begin{pmatrix} g_{\mu\nu}(x^{\mu}) & 0 \\ 0 & g_{mn}(y^{m}) \end{pmatrix} \quad \text{where } (\alpha,\beta,\ldots;\mu,\nu,\ldots = 1,\ldots,4)$$

and $(a,b,\ldots;m,n,\ldots = 1,\ldots 7)$. Some straightforward manipulation of (7.24) and (7.25), adopting these forms for F_{MNPQ} and g_{MN}, yields

$$R_{\mu\nu} = 12m^2 g_{\mu\nu} \ , \ R_{mn} = -6m^2 g_{mn} \ , \ R_{\mu m} = 0 \ . \qquad 7.26$$

Thus we have a solution of the field equations for which the ground state is a direct product of a non-compact four dimensional manifold and a compact seven dimensional manifold. There are, however, still infinitely many solutions of (7.26) and some criterion is necessary to distinguish the 'false' ground states from the 'true' ground state. Such a criterion can be provided by the requirement that the ground state be stable, a reason for which may be an unbroken supersymmetry. Such a supersymmetric vacuum would require that, $< \psi_M >$, stay zero under the local supersymmetry transformations, $\delta_\xi \psi_M$, which leave the action invariant. Assuming that the local spinor parameter $\xi(x,y)$ of these transformations factorizes as $\xi(x,y) = \xi(x)\eta(y)$ and recalling the above restrictions on F_{MNPQ} and g_{MN} then $\delta_\xi \psi_M = 0$ implies

$$\overline{D}_\mu \xi = D_\mu \xi + m\gamma_\mu \gamma_5 \xi = 0 \qquad 7.27$$

$$\overline{D}_m \eta = D_m \eta - \frac{1}{2} m\Gamma_m \eta = 0 \qquad 7.28$$

where the Γ matrices have decomposed as

$$\Gamma_A = (\gamma_\alpha \otimes I \ , \ \gamma_5 \otimes \Gamma_a) \ .$$

The integrability conditions of (7.27) and (7.28) are

$$[\bar{D}_\mu,\bar{D}_\nu]\xi = [\, \frac{1}{8} R_{\mu\nu\rho\sigma}\, \gamma^{\rho\sigma} - m^2 \gamma_{\mu\nu}]\xi = 0 \qquad 7.29$$

$$[\bar{D}_m,\bar{D}_n]\eta = [\, \frac{1}{8} R_{mnrs}\, \Gamma^{rs} + \frac{1}{4} m^2\, \Gamma_{mn}]\eta = 0 \qquad 7.30$$

If these conditions are satisfied then

$$R_{\mu\nu\rho\sigma} = 4m^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \qquad 7.31$$

$$R_{mnrs} = -m^2(g_{mr}g_{ns} - g_{ms}g_{nr}) \qquad 7.32$$

There remain only two possibilities for the complete solution: (i) if $m^2 = 0$, (7.31) and (7.32) are the standard Riemann curvature tensors for a four dimensional Minkowski space, M_4, and the seven-Torus, T^7, respectively; (ii) if $m^2 > 0$ (7.31) and (7.32) are the standard Riemann curvature tensors for anit- de Sitter space, AdS_4, and the seven sphere, S^7, respectively. The $m^2 = 0$ case (when only zero modes are retained) was the first solution found for the compactification of eleven dimensional supergravity to four dimensions. It was obtained by Cremmer and Julia [48] via a dimensional reduction procedure in which all fields were simply assumed to be independent of the 'extra' seven dimensions. The $m^2 > 0$ solution was first exhibited by Duff and Pope [46,49] by taking the results of Freund and Rubin [47], who obtained (7.26), and requiring the existence of eight unbroken supersymmetries in four dimensions. These are provided by the eight linearly independent solutions to (7.28), since η is an eight component spinor. Since S^7 is the coset space SO(8)/SO(7) which admits an SO(8) isometry group, this solution describes a theory with local SO(8) invariance. The full symmetry group of both these solutions has been found to be OSp(8/4) [50,51].

Variations of the above solutions have also been obtained for which the compact seven dimensional manifold is the 'squashed' [52] or parallelized [53] seven spheres or a product of spheres, $S^5 \otimes S^2$, $S^4 \otimes S^3$, $S^3 \otimes S^2 \otimes S^2$ [54]. Of these the only solution involving an orthosymplectic supergroup is the 'squashed' S^7 of Awada, Duff and Pope [52]. These authors observed that S^7 admits another Einstein metric besides the maximally symmetric 'round' one for which the isometry group is S^7. This is the 'squashed' S^7 which has $SO(5) \otimes SU(2)$ as its isometry group. The full symmetry group of this solution is $OSp(1/4) \otimes SO(5) \otimes SU(2)$.

Of the solutions mentioned above only, $S^5 \times S^2$, possesses an isometry group of the compact internal manifold large enough to contain the phenomenological $SU(3) \otimes SU(2) \otimes U(1)$ gauge group. A class of seven dimensional manifolds which do contain $SU(3) \otimes SU(2) \otimes U(1)$ are the coset spaces

$$M^{pqr} = \frac{SU(3) \otimes SU(2) \otimes U(1)}{SU(2) \otimes U(1) \otimes U(1)} \quad .$$

This classification has been given by Witten [55] where p, q and r are integers with no common divisor and $r \neq 0$. These integers parametrize the embedding of $SU(2) \otimes U(1) \otimes U(1)$ in $SU(3) \otimes SU(2) \otimes U(1)$. Castellani, D'Auria and Fré [56] have investigated compactifying solutions of $d = 11$ simple supergravity where the compact manifolds are M^{pqr} spaces. They have been able to classify all such solutions by the ratio p/q, there being no solution only for $p = q = 0$, while for all other values of p and q there is a unique invariant Einstein metric on the corresponding topological space M^{pqr}. No supersymmetry survives except in the case of $p/q = 1$ for which the full symmetry group is $OSp(2/4) \otimes SU(3) \otimes SU(2)$. There is obviously a long way to go to make $d = 11$ supergravity a realistic theory of nature but the fact that it does compactify onto manifolds which possess an SU(3) $\otimes$ SU(2) $\otimes$ U(1) symmetry is an encouraging result.

In order to obtain a complete classification of the various states present in the supermultiplets, the full invariance of the ground state must be considered. Thus for the 'round' S^7, which possesses an OSp(8/4) invariance of the ground state, the excitations corresponding to fluctuations about this ground state should form irreducible representations of OSp(8/4). For other solutions we have seen that OSp(M/4) is the relevant invariance and consequently a knowledge of the unitary, irreducible representations of OSp(M/4) is important for the construction of supersymmetric field theories in anti- de Sitter space. A study of such representations has begun with the work of Gunaydin and Bars [57,58] and Freedman and Nicolai [43]. Of particular interest in the latter work is a phenomenon which these authors have called 'multiplet shortening'. This phenomenon arises for certain restrictions on the vacuum quantum numbers, and effects a reduction of the maximal spin of a representation. Consequently it may have an important role to play in the construction of supersymmetric field theories. In this way it closely resembles the phenomenon of 'spin reduction' in the presence of central charges which has been discussed in chapter 6. As a final point it is worth stressing that both these phenomena are closely related to the existence of atypical representations, which as we have seen throughout chapters 2 to 5 play a fundamental role in the finite-dimensional, representation theory of the Lie superalgebras.

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8. CONCLUSION

We conclude by summarizing the work and results presented in this thesis and discussing avenues for future research.

8.1 SUMMARY

In chapter two a brief review of the theory of Lie superalgebras was presented . This served to introduce this subject to readers unfamiliar with it, and to introduce the notation and terminology used in the thesis. This work was based on the comprehensive treatises by Kac [1,2] and Scheunert [3].

The main work of the thesis was commenced in chapter three, wherein weight space techniques were developed to educe finite-dimensional, irreducible, typical and atypical, star and grade star representations of the orthosymplectic superalgebras. These techniques were then used to determine all such representations for the superalgebras $B(1,1)$, $C(2)$ and $D(2,1;\alpha)$. These representations for $C(2) \simeq A(1,0)$ have been reported by Scheunert, Nahm and Rittenberg [7]. The results for $B(1,1)$ and $D(2,1;\alpha)$ are new to the literature.

Chapter four saw the development of superfield techniques for the determination of all finite-dimensional, irreducible representations of the orthosymplectic superalgebras. These techniques are based on an induced representation construction and were used here to find all such representations for the superalgebras $B(0,1)$, $B(1,1)$, $C(2)$ and $D(2,1)$. The results for the latter three cases were found to be in agreement with those of chapter three. Finite dimensional, irreducible representations for $B(0,1)$ and $C(2)$ have been constructed using weight space techniques by Scheunert, Nahm and Rittenberg [7] while Dondi and Jarvis [8] have used superfield techniques to construct such representations for $C(2)$. The results for $B(1,1)$ and $D(2,1)$ are new to the literature.

Young supertableaux were investigated in chapter five. This chapter included a fairly comprehensive review of the development of young supertableau techniques for the study of representations of SU(M/N) and OSp(M/N). New results obtained here were the relations between the Kac-Dynkin labels and the supertableau labels for OSp(M/N) and the subsequent expression of Kac's atypicality conditions as conditions on the diagram shape. Modification rules were also obtained for all typical representations of OSp(M/N) and, in addition, for the atypical representations of OSp(2/2), OSp(3/2) and OSp(4/2). Branching rules for spinor representations of OSp(M/N) and for atypical representations of OSp(2/2),OSp(3/2) and OSp(4/2) were also presented.

Chapter six saw new 'chiral-like' superfield techniques developed for the study of irreducible realisations of the N-extended supersymmetry algebra in the presence of central charges. These techniques are based on the theory of induced representations. The $N = 2$-extended algebra was considered in full detail and all irreducible realisations, including the 'spin-reducing' cases, were exhibited.

Chapter seven was a review of some of the applications orthosymplectic supergroups have found in physical theories. The relationship between these supergroups and the extended BRS symmetries of quantum gauge theories was first discussed, based on the work of Delbourgo and Jarvis [4]. It was then demonstrated how the N-extended super Poincaré algebra could be obtained from a Inonu-Wigner contraction of OSp(N/4). This chapter concluded with a discussion of the role played by OSp(N/4) as the ground state symmetry of some compactifying solutions in Kaluza-Klein supergravity theories [5].

8.2 FUTURE RESEARCH

Although the techniques developed in chapters three and four are applicable to any orthosymplectic superalgebra they become rapidly more complex to work with as the rank of the algebra increases. This is principally due to the rapid increase in the number of irreducible representations of O(M) x Sp(N) contained generally in an irreducible representation of OSp(M/N). This goes as $2^{\frac{1}{2}MN}$ and consequently without computer assistance would soon become unmanageable. Thus, a high priority for the approaches of chapters three and four would be the simplification of these procedures to make the higher rank algebras more accessible. Indeed the recent work of Thierry-Mieg [6] takes a very significant step in this direction though it appears some degree of computer assistance may still be necessary.

There are a number of possible areas, related to Young supertableaux, which are open for development. Nearly all of these pertain to atypical representations. In particular the development of modification rules, branching rules and rules for Kronecker products, for atypical representations of OSp(M/N) and from which one could determine the irreducible representations would be most welcome. A simpler form of the branching rule for spinor representations than that given in §5.3 would also be useful.

The techniques of chapters three to five have been developed with the application to the orthsymplectic superalgebras as the immediate motivation. However, hybrid forms of these procedures should also be applicable to the study of representations of any of the classical Lie superalgebras. This is obviously a major task and it may be many years before the representation theory of Lie superalgebras has evolved to the level currently acquired by Lie algebras.

The work presented in chapter six is only the beginning of a program for which the next phase is the use of these superfields in the construction of models of supersymmetric field theories. It is hoped that these superfields will be more amenable to a fully covariant treatement with the minimal sets of auxiliary fields arising in a more transparent manner.

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APPENDIX A: THE ALGEBRAS: C(2), B(1,1), D(2,1;α)

The explicit form of the algebras for C(2), B(1,1) and D(2,1;α) as used in chapter three are presented here. The notation is as given in §3.2.

C(2) :

$$[h_1, \beta^{1\pm}] = 0 \qquad [h_1, \alpha_2^{\pm}] = \pm\alpha_2^{\pm}$$

$$[h_2, \beta^{1\pm}] = \mp\beta^{1\pm} \qquad [h_2, \alpha_2^{\pm}] = \pm 2\alpha_2^{\pm}$$

$$\{\beta^{1+}, \beta^{1-}\} = h_1 \qquad [\alpha_2^{+}, \alpha_2^{-}] = h_2$$

$$[\beta^{1\pm}, \alpha_2^{\mp}] = 0 \qquad [h_1, \beta_2^{1\pm}] = \pm\beta_2^{1\pm}$$

$$[h_2, \beta_2^{1\pm}] = \pm\beta_2^{1\pm} \qquad [\alpha_2^{\pm}, \beta_2^{1\mp}] = -\beta^{1\mp}$$

$$\{\beta^{1\pm}, \beta_2^{1\mp}\} = \mp\alpha_2^{\mp} \qquad \{\beta_2^{1+}, \beta_2^{1-}\} = h_2 - h_1$$

B(1,1) :

$$[h_1, \beta^{1\pm}] = 0 \qquad [h_1, \alpha_2^{\pm}] = \pm\alpha_2^{\pm}$$

$$[h_2, \beta^{1\pm}] = \mp 2\beta^{1\pm} \qquad [h_2, \alpha_2^{\pm}] = \pm 2\alpha_2^{\pm}$$

$$\{\beta^{1+}, \beta^{1-}\} = h_1 \qquad [\alpha_2^{+}, \alpha_2^{-}] = h_2$$

$$[\beta^{1\pm}, \alpha_2^{\mp}] = 0 \qquad \{\beta_2^{1\pm}, \beta^{1\mp}\} = \pm\alpha_2^{\pm}$$

$$[\beta_2^{1\pm}, \alpha_2^{\mp}] = 2\beta^{1\pm} \qquad \{\beta_2^{1+}, \beta_2^{1-}\} = h_2 - 2h_1$$

$$[\beta_2^{1\pm}, h_2] = 0 \qquad [\beta_2^{1\pm}, h_1] = \mp\beta_2^{1\pm}$$

$$[\tilde{\beta}_2^{1\pm}, h_2] = \mp 2\tilde{\beta}_2^{1\pm} \qquad [\tilde{\beta}_2^{1\pm}, h_1] = \mp 2\tilde{\beta}_2^{1\pm}$$

$$
\begin{aligned}
&[\tilde{\beta}_2^{1\pm}, \alpha_2^{\mp}] = 2\beta_2^{1\pm} && \{\tilde{\beta}_2^{1\pm}, \beta_2^{1\mp}\} = \mp 2\alpha_2^{\pm} \\
&\{\tilde{\beta}_2^{1\pm}, \beta^{1\mp}\} = 0 && \{\tilde{\beta}_2^{1+}, \tilde{\beta}_2^{1-}\} = 4(h_1 - h_2)
\end{aligned}
$$

$$[\{\beta_2^{1+}, \beta_2^{1+}\}, \{\beta_2^{1-}, \beta_2^{1-}\}] = 8(h_2 - 2h_1) = -16k$$

<u>$D(2,1;\alpha)$:</u>

$$
\begin{aligned}
&[\alpha_2^{+}, \alpha_2^{-}] = h_2 && \{\beta^{1+}, \beta^{1-}\} = h_1 \\
&[\alpha_3^{+}, \alpha_3^{-}] = h_3 && [\beta^{1\pm}, \alpha_2^{\mp}] = 0 \\
&[\beta^{1\pm}, \alpha_3^{\mp}] = 0 && [\alpha_2^{\pm}, \alpha_3^{\mp}] = 0 \\
&[h_1, \beta^{1\pm}] = 0 && [h_1, \alpha_2^{\pm}] = \pm\alpha_2^{\pm} \\
&[h_1, \alpha_3^{\pm}] = \pm\alpha\,\alpha_3^{\pm} && [h_2, \beta^{1\pm}] = \mp\beta^{1\pm} \\
&[h_2, \alpha_2^{\pm}] = \pm 2\alpha_2^{\pm} && [h_2, \alpha_3^{\pm}] = 0 \\
&[h_3, \alpha_2^{\pm}] = 0 && [h_3, \beta^{1\pm}] = \mp\beta^{1\pm} \\
&[h_3, \alpha_3^{\pm}] = \pm 2\alpha_3^{\pm} && \{\beta^{1\mp}, \beta_2^{1\pm}\} = \pm\alpha_2^{\pm} \\
&\{\beta^{1\mp}, \beta_3^{1\pm}\} = \pm\alpha\,\alpha_3^{\pm} && \{\beta^{1\mp}, \tilde{\beta}_2^{1\pm}\} = 0 \\
&\{\beta_2^{1+}, \beta_2^{1-}\} = h_2 - h_1 && \{\beta_3^{1+}, \beta_3^{1-}\} = \alpha h_3 - h_1 \\
&\{\tilde{\beta}_2^{1+}, \tilde{\beta}_2^{1-}\} = h_1 - h_2 - \alpha h_3 && [h_1, \beta_2^{1\pm}] = \pm\beta_2^{1\pm} \\
&[h_2, \beta_2^{1\pm}] = \pm\beta_2^{1\pm} && [h_3, \beta_2^{1\pm}] = \mp\beta_2^{1\pm} \\
&[h_1, \beta_3^{1\pm}] = \pm\alpha\,\beta_3^{1\pm} && [h_2, \beta_3^{1\pm}] = \mp\beta_3^{1\pm} \\
&[h_3, \beta_3^{1\pm}] = \pm\beta_3^{1\pm} && [h_1, \tilde{\beta}_2^{1\pm}] = \pm(1+\alpha)\tilde{\beta}_2^{1\pm} \\
&[h_2, \tilde{\beta}_2^{1\pm}] = \pm\tilde{\beta}_2^{1\pm} && [h_3, \tilde{\beta}_2^{1\pm}] = \pm\tilde{\beta}_2^{1\pm}
\end{aligned}
$$

$$[\alpha_2^{\pm}, \beta_2^{1\mp}] = -\beta^{1\mp} \qquad [\alpha_3^{\pm}, \beta_2^{1\mp}] = 0$$

$$[\alpha_2^{\pm}, \beta_3^{1\mp}] = 0 \qquad [\alpha_3^{\pm}, \beta_3^{1\mp}] = -\beta^{1\mp}$$

$$[\alpha_2^{\pm}, \tilde{\beta}_2^{1\mp}] = -\beta_3^{1\mp} \qquad [\alpha_3^{\pm}, \tilde{\beta}_2^{1\mp}] = -\beta_2^{1\mp}$$

$$\{\beta_2^{1\pm}, \beta_3^{1\mp}\} = 0 \qquad \{\beta_2^{1\mp}, \tilde{\beta}_2^{1\pm}\} = \mp\alpha\,\alpha_3^{\pm}$$

$$\{\beta_3^{1\mp}, \tilde{\beta}_2^{1\pm}\} = \mp\alpha_2^{\pm} \qquad [\beta^{1\pm}, \{\beta_2^{1\mp}, \beta_3^{1\mp}\}] = \pm 2\,\tilde{\beta}_2^{1\mp}$$

$$[\{\beta_2^{1+}, \beta_3^{1+}\}, \{\beta_2^{1-}, \beta_3^{1-}\}] = (1+\alpha)[h_2 + \alpha h_3 - 2h_1] = -(1+\alpha)^2 k$$

APPENDIX B: DEFINITION OF ADJOINT AND SUPERADJOINT OPERATIONS

The adjoints and superadjoints [1] of all even root vectors corresponding to simple roots and of all generators in the Cartan subalgebra are defined as follows: $(\alpha_j^{\pm})^{\dagger}=\alpha_j^{\mp}$, $(\alpha_j^{\pm})^{\ddagger}=\alpha_j^{\mp}$, $(h_i)^{\dagger}=h_i$, $(h_i)^{\ddagger}=h_i$.

The adjoints of the odd root vectors can be defined in two ways which we designate as Aσ, where σ=1 or 2:

B(m,n): $$(\beta^{n-j\,\pm}_{n+k})^{\dagger} = (-1)^{j+k+\sigma+1}\,\beta^{n-j\,\mp}_{n+k}$$

$$(\tilde{\beta}^{n-j\,\pm}_{n+m-l})^{\dagger} = (-1)^{j+m+l+\sigma}\,\tilde{\beta}^{n-j\,\mp}_{n+m-l}$$

where $0\le j\le n-1$, $0\le k\le m$, $0\le l\le m-1$.

D(m,n): $$(\beta^{n-j\,\pm}_{n+k})^{\dagger} = (-1)^{j+k+\sigma+1}\,\beta^{n-j\,\mp}_{n+k}$$

$$(\beta^{n-j\,\pm}_{n+m})^{\dagger} = (-1)^{j+m+\sigma}\,\beta^{n-j\,\mp}_{n+m}$$

$$(\tilde{\beta}^{n-j\,\pm}_{n+m-l})^{\dagger} = (-1)^{j+m+l+\sigma}\,\tilde{\beta}^{n-j\,\mp}_{n+m-l}$$

where $0\le j\le n-1$, $0\le k\le m-1$, $1\le l\le m-1$.

C(n): $$(\beta_j^{\pm})^{\dagger} = (-1)^{j+\sigma}\,\beta_j^{\mp}$$

$$(\tilde{\beta}^{\pm}_{n-k})^{\dagger} = (-1)^{n+k+\sigma}\,\beta^{\mp}_{n-k}$$

where $1\le j\le n$, $1\le k\le n-2$.

The superadjoint of the odd root vectors can be defined in two ways which we designate as Sσ, where σ=1 or 2:

B(m,n): $$(\beta^{n-j\,\pm}_{n+k})^{\ddagger} = \pm(-1)^{j+k+\sigma}\,\beta^{n-j\,\mp}_{n+k}$$

$$(\tilde{\beta}^{n-j\,\pm}_{n+m-l})^{\ddagger} = \pm(-1)^{j+m+l+\sigma+1}\,\tilde{\beta}^{n-j\,\mp}_{n+m-l}$$

where $0\le j\le n-1$, $0\le k\le m$, $0\le l\le m-1$.

D(m,n): $$(\beta^{n-j\,\pm}_{n+k})^{\ddagger} = \pm(-1)^{j+k+\sigma}\,\beta^{n-j\,\mp}_{n+k}$$

$$(\beta^{n-j\,\pm}_{n+m})^{\ddagger} = \pm(-1)^{j+m+\sigma+1}\,\beta^{n-j\,\mp}_{n+m}$$

$$(\tilde{\beta}^{n-j\,\pm}_{n+m-l})^{\ddagger} = \pm(-1)^{j+m+l+\sigma+1}\,\tilde{\beta}^{n-j\,\mp}_{n+m-l}$$

where $0\le j\le n-1$, $0\le k\le m-1$, $1\le l\le m-1$.

C(n): $$(\beta_j^{\pm})^{\ddagger} = \mp(-1)^{j+\sigma}\,\beta_j^{\mp}$$

$$(\tilde{\beta}^{\pm}_{n-k})^{\ddagger} = \mp(-1)^{n+k+\sigma+1}\,\tilde{\beta}^{\mp}_{n-k}$$

where $1\le j\le n$, $1\le k\le n-2$.

We note that for B(m,n) and D(m,n), but not C(n), the 'hidden' even Sp(2n) generator $\{\beta_a^{\pm}, \beta_b^{\pm}\}$ defined in §3.2 transforms as

$\{\beta_a^{\pm}, \beta_b^{\pm}\}^{\ddagger}=-\{\beta_a^{\mp}, \beta_b^{\mp}\}$ and $\{\beta_a^{\pm}, \beta_b^{\pm}\}^{\dagger}=+\{\beta_a^{\mp}, \beta_b^{\mp}\}$

corresponding to compact and non-compact real forms of Sp(2n) in the superadjoint and adjoint cases, respectively.

APPENDIX C: ORTHOGONALISED STATES, χ_j, AS HIGHEST WEIGHT STATES OF THE EVEN SUBALGEBRA.

In this appendix it is demonstrated, in a quite general manner that the states,

$$\chi_j = \psi_j - \sum_{k<j} \frac{(\psi_j,\phi_k)}{(\phi_k,\phi_k)} \phi_k \qquad \text{C.1}$$

as constructed in §3.3, are in fact highest weight states of the even subalgebra. The proof presented here is quite simple and rests only on the assumption that, given a particular basis in the enveloping algebra, the states constructed in (C1) by Schmidt orthogonalisation are unique.

It will be recalled from §3.3 that the ϕ_k are states of the same weight as ψ_j and such that $\phi_k = E_k^- \ \chi_k$, where E_k^- is a monomial of even,negative root vectors. The adjoint and superadjoint of E_k^- will be a monomial of even, positive root vectors, which will be designated F_k^+ and G_k^+ respectively (see Appendix B for a discussion of these operations): i.e. $(E_k^-)^\dagger = F_k^+$ and $(E_k^-)^\ddagger = G_k^+$.

If we now construct a highest weight state of the even subalgebra χ_j', from ψ_j and some subset of the set of (ϕ_k) from (C1) then

$$(\chi_j',\phi_k) = (\chi_j', E_k^-\chi_k) = ((E_k^-)^\dagger \chi_j', \chi_k) = (F_k^+ \chi_j', \chi_k) = 0 \qquad \text{C.2}$$

since $F_k^+ \chi_j' = 0$. Thus χ_j' is orthogonal to all the ϕ_k. If, however such a state can only be constructed uniquely then χ_j as constructed in (C1) must be a highest weight of the even subalgebra. An identical argument follows if the superadjoint is taken in (C2).

APPENDIX D: PROJECTION OPERATORS FOR SPIN M×½ AND SPIN M×1

Chapters four and six require the use of spin M×½ and spin M×1, with respect to SU(2), projection operators. As explained in §4.2 the two-index basis for SU(2) is related to the spherical basis via

$$\hat{M}_{\alpha\beta} = 2(\hat{\underset{\sim}{M}}\cdot\underset{\sim}{\sigma}\varepsilon)_{\alpha\beta} \tag{D.1}$$

where the generators are in a spin M matrix representation of SU(2). Where these act on superfield components such as ψ_α or $P_{\alpha\beta}$, the question arises of projections onto total spins (M±½) or (M, M±1), respectively. These are derived using the characteristic identity (quadratic or cubic, respectively) satisfied by the generators in the reducible M×½ and M×1 representations.

The general construction of projection operators proceeds as follows. Consider some reducible representation of an algebra with Casimir operator, C, and eigenvalues c_1, c_2, ..., c_n. Then there exists a complete set of projection operators

$$\Pi_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(C-c_j)}{(c_i-c_j)} \tag{D.2}$$

such that $\Pi_i\cdot\Pi_j = \delta_{ij}\Pi_i$ and $\sum_{i=1}^{n}\Pi_i = 1$. Each of the Π_i will extract a subspace with eigenvalue c_i with respect to C.

For M×½ we have for the Casimir (spin M indices are suppressed and indices α, β, ... are raised using the inverse metric $\varepsilon^{\alpha\beta}$)

$$(\hat{\underset{\sim}{M}}\cdot\underset{\sim}{\sigma})_\alpha{}^\beta = (\hat{\underset{\sim}{M}} + \tfrac{1}{2}\underset{\sim}{\sigma})^2{}_\alpha{}^\beta - (\hat{\underset{\sim}{M}})^2\delta_\alpha^\beta - (\tfrac{1}{2}\underset{\sim}{\sigma})^2{}_\alpha{}^\beta \tag{D.3}$$

where $\hat{\underset{\sim}{M}}$ and $\tfrac{1}{2}\underset{\sim}{\sigma}$ are spin M and spin ½ matrix representations respectively. The eigenvalues of $(\hat{\underset{\sim}{M}}\cdot\underset{\sim}{\sigma})_\alpha{}^\beta$ on the reducible M×½ space are given by

$$(M\pm\tfrac{1}{2}) \text{ subspace:} \quad (M\pm\tfrac{1}{2})(M\pm\tfrac{1}{2}+1) - M(M+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) = M^{\pm} \tag{D.4}$$

where $M^+ = M$ and $M^- = -M-1$. The projection operators are therefore given by

$$\Pi^{\pm\frac{1}{2}}{}_\alpha{}^\beta = (\hat{M}_\alpha{}^\beta - 2M^{\mp}\delta_\alpha^\beta)/2(2M^{\pm}+1) \tag{D.5}$$

where (D.1) has been used. The following expressions can easily be derived from (D.5) and are frequently used

$$\delta_\alpha^{\beta} = \Pi_\alpha^{+\frac{1}{2}\beta} + \Pi_\alpha^{-\frac{1}{2}\beta} \qquad \text{D.6}$$

$$\hat{M}_\alpha^{\ \beta} = 2M^{+}\Pi_\alpha^{+\frac{1}{2}\beta} + 2M^{-}\Pi_\alpha^{-\frac{1}{2}\beta} \qquad \text{D.7}$$

$$\hat{M}_\alpha^{\ \beta}\hat{M}_\beta^{\ \gamma} = 4M(M+1)\delta_\alpha^{\gamma} - 2\hat{M}_\alpha^{\ \gamma} \qquad \text{D.8}$$

For Mx1 we have the Casimir

$$(\hat{\underset{\sim}{M}}\cdot\underset{\sim}{\Sigma})^{\gamma\delta}_{\alpha\beta} = (\hat{\underset{\sim}{M}} + \tfrac{1}{2}\underset{\sim}{\Sigma})^{2\gamma\delta}_{\alpha\beta} - (\hat{\underset{\sim}{M}})^{2}1^{\gamma\delta}_{\alpha\beta} - (\tfrac{1}{2}\underset{\sim}{\Sigma})^{2\gamma\delta}_{\alpha\beta} \qquad \text{D.9}$$

where

$$\tfrac{1}{2}\underset{\sim}{\Sigma}^{\gamma\delta}_{\alpha\beta} = \tfrac{1}{2}(\underset{\sim}{\sigma}_\alpha^{\ \gamma}\delta_\beta^{\ \delta} + \delta_\alpha^{\ \gamma}\underset{\sim}{\sigma}_\beta^{\ \delta} + \underset{\sim}{\sigma}_\beta^{\ \gamma}\delta_\alpha^{\ \delta} + \delta_\beta^{\ \gamma}\underset{\sim}{\sigma}_\alpha^{\ \delta}) \qquad \text{D.10}$$

is the spin 1 matrix representation and

$$1^{\gamma\delta}_{\alpha\beta} = \tfrac{1}{2}(\delta_\alpha^{\ \gamma}\delta_\beta^{\ \delta} + \delta_\alpha^{\ \delta}\delta_\beta^{\ \gamma}). \qquad \text{D.11}$$

The eigenvalues of $(\hat{\underset{\sim}{M}}\cdot\underset{\sim}{\Sigma})^{\gamma\delta}_{\alpha\beta}$ on the reducible Mx1 space are given by

(M±1) subspace: $\quad (M\pm1)(M\pm1+1) - M(M+1) - 1(1+1) = 2M^{\pm}$ D.12

(M) subspace : $\quad M(M+1) - M(M+1) - 1(1+1) = -2$ D.13

Thus the projection operators are

$$\Pi^{\pm1\gamma\delta}_{\alpha\beta} = \left[\frac{\hat{N} + (2M^{\pm}+3)\hat{L} + 4(M^{\pm}+1)(M^{\pm}+2)1}{8(M^{\pm}+1)(2M^{\pm}+1)}\right]^{\gamma\delta}_{\alpha\beta} \qquad \text{D.14}$$

$$\Pi^{0\gamma\delta}_{\alpha\beta} = \left[\frac{\hat{N} + \hat{L} + 4M^{+}M^{-}1}{8M^{+}M^{-}}\right]^{\gamma\delta}_{\alpha\beta} \qquad \text{D.15}$$

where we have used (D.8) and the following definitions

$$\hat{L}^{\gamma\delta}_{\alpha\beta} = \tfrac{1}{2}(\hat{M}_\alpha^{\gamma}\delta_\beta^{\delta} + \delta_\alpha^{\gamma}\hat{M}_\beta^{\delta} + \hat{M}_\beta^{\gamma}\delta_\alpha^{\delta} + \delta_\beta^{\gamma}\hat{M}_\alpha^{\delta}) \qquad \text{D.16}$$

$$\hat{N}^{\gamma\delta}_{\alpha\beta} = \tfrac{1}{4}(\hat{M}_\alpha^{\gamma}\hat{M}_\beta^{\delta} + \hat{M}_\alpha^{\delta}\hat{M}_\beta^{\gamma} + \hat{M}_\beta^{\gamma}\hat{M}_\alpha^{\delta} + \hat{M}_\beta^{\delta}\hat{M}_\alpha^{\gamma}) \qquad \text{D.17}$$

From these definitions several useful identities can be derived which are necessary for the extraction of component field variations. Examples are

$$\Pi_\alpha^{\pm\frac{1}{2}\gamma}\,\delta_\beta^{\ \delta}\,P^{\pm1}_{\ \gamma\delta} = P^{\pm1}_{\ \alpha\beta}$$

$$\Pi_\alpha^{\pm\frac{1}{2}\gamma}\,\hat{M}_\beta^{\ \delta}\,P^{\pm1}_{\ \gamma\delta} = 2M^{\pm}\,P^{\pm1}_{\ \alpha\beta}$$

$$\Pi^{\pm\frac{1}{2}\gamma}_{\alpha}\ \delta_{\beta}^{\ \delta}\ P^{\mp 1}_{\ \gamma\delta} = 0 = \Pi^{\pm\frac{1}{2}\gamma}_{\alpha}\ \hat{M}_{\beta}^{\ \delta}\ P^{\mp 1}_{\ \gamma\delta}$$

$$\Pi^{\pm\frac{1}{2}\gamma}_{\alpha}\ \delta_{\beta}^{\ \delta}\ P^{0}_{\gamma\delta} = (M^{\pm}\ P^{0}_{\alpha\beta} + \tfrac{1}{4}\varepsilon_{\alpha\beta}\hat{M}^{\gamma\delta}P^{0}_{\gamma\delta})/(2M^{\pm} + 1)$$

$$\Pi^{\pm\frac{1}{2}\gamma}_{\alpha}\ \hat{M}_{\beta}^{\ \delta}\ P^{0}_{\gamma\delta} = -2(M^{\pm}\ P^{0}_{\alpha\beta} + \tfrac{1}{4}\varepsilon_{\alpha\beta}\hat{M}^{\gamma\delta}P^{0}_{\gamma\delta})(M^{\pm}+2)/(2M^{\pm}+1) \qquad \text{D.18}$$

where $P^{\pm 1}$ and P^{0} satisfy $\Pi^{\pm 1\gamma\delta}_{\alpha\beta}\ P^{\pm 1}_{(\gamma\delta)} = P^{\pm 1}_{(\alpha\beta)}$ and $\Pi^{0\gamma\delta}_{\alpha\beta}\ P^{0}_{(\gamma\delta)} = P^{0}_{(\alpha\beta)}$.

Other useful examples are

$$\tfrac{1}{2}(\delta_{\alpha}^{\ \gamma}\delta_{\beta}^{\ \delta} + \delta_{\beta}^{\ \gamma}\delta_{\alpha}^{\ \delta})\eta_{\gamma}\psi^{\pm\frac{1}{2}}_{\delta} = (\Pi^{\pm 1\gamma\delta}_{\alpha\beta} + \Pi^{0\gamma\delta}_{\alpha\beta})\eta_{\gamma}\psi^{\pm\frac{1}{2}}_{\delta}$$

$$\tfrac{1}{2}(\hat{M}_{\alpha}^{\ \gamma}\delta_{\beta}^{\ \delta} + \hat{M}_{\beta}^{\ \gamma}\delta_{\alpha}^{\ \delta})\eta_{\gamma}\psi^{\pm\frac{1}{2}}_{\delta} = 2M^{\pm}\ \Pi^{\pm 1\gamma\delta}_{\alpha\beta} - 2(M^{\pm}+2)\Pi^{0\gamma\delta}_{\alpha\beta}\eta_{\gamma}\psi^{\pm\frac{1}{2}}_{\delta} \qquad \text{D.19}$$

where $\Pi^{\pm\frac{1}{2}\beta}_{\alpha}\psi^{\pm\frac{1}{2}}_{\beta} = \psi^{\pm\frac{1}{2}}_{\alpha}$ and η_{γ} is a spinor parameter.

Finally we have

$$\Pi^{\pm 1\gamma\delta}_{\alpha\beta}\ \hat{M}_{\gamma\delta} = 0 = \hat{M}^{\alpha\beta}\ \Pi^{\pm 1\gamma\delta}_{\alpha\beta}$$

$$\Pi^{0\gamma\delta}_{\alpha\beta}\ \hat{M}_{\gamma\delta} = \hat{M}_{\alpha\beta}\ , \quad \hat{M}^{\alpha\beta}\ \Pi^{0\gamma\delta}_{\alpha\beta} = \hat{M}^{\gamma\delta}. \qquad \text{D.20}$$

APPENDIX E: MATRIX REPRESENTATIONS OF OSp(1/2)

In §4.4 the little group chosen involves the supergroup OSp(1/2) with generators $(M_{\alpha\beta}, Q_\alpha)$. The superfield technique requires explicit matrix representations $(\hat{\mathcal{M}}_{\alpha\beta}, \hat{\mathcal{Q}}_\alpha)$ of these generators acting on fields of arbitrary 'superspin' M (§4.2), i.e. two component superfields

$$\Phi_A = \begin{pmatrix} \phi_a \\ \phi_{a\alpha} \end{pmatrix} \qquad \text{E.1}$$

where ϕ_a has spin M and $\phi_{a\alpha}$ has spin M-½ or $\Pi^{-\frac{1}{2}b\beta}_{a\alpha}\ \phi_{b\beta} = \phi_{a\alpha}$.

Using the results of appendix D the matrices for the Sp(2) generators can be written as

$$(\hat{\mathcal{M}}_{\alpha\beta})^D_C = \begin{pmatrix} (\hat{M}_{\alpha\beta})^d_c & 0 \\ 0 & (\hat{M}^x_{\alpha\beta})^{d\delta}_{c\gamma} \end{pmatrix} \qquad \text{E.2}$$

where $(\hat{M}_{\alpha\beta})^d_c$ are spin M matrix representations and $(\hat{M}^x_{\alpha\beta})^{d\delta}_{c\gamma}$ correspond to the reducible Mx½ representation

$$(\hat{M}^x_{\alpha\beta})^{d\delta}_{c\gamma} = (\hat{M}_{\alpha\beta})^d_c\ \delta^\delta_\gamma + \varepsilon_{\gamma\alpha}\ \delta^\delta_\beta + \varepsilon_{\gamma\beta}\ \delta^\delta_\alpha. \qquad \text{E.3}$$

We actually want to project the spin M-½ component from $\hat{M}^x_{\alpha\beta}$, however, since the spin M±½ projectors commute with it the form presented in (E.2) is appropriate, i.e.

$$\Pi^{-\frac{1}{2}d\delta}_{c\gamma}\ (\hat{M}^x_{\alpha\beta})^{e\varepsilon}_{d\delta}\ \phi_{e\varepsilon} = (\hat{M}^x_{\alpha\beta})^{d\delta}_{c\gamma}\ \Pi^{-\frac{1}{2}e\varepsilon}_{d\delta}\ \phi_{e\varepsilon} = (\hat{M}^x_{\alpha\beta})^{d\delta}_{c\gamma}\ \phi_{d\delta} \qquad \text{E.4}$$

since $\phi_{a\alpha}$ has spin M-½.

The matrices $(\hat{\mathcal{Q}}_\alpha)^D_C$ must be chosen to satisfy the anticommutation relations

$$(\hat{\mathcal{Q}}_\alpha)^D_C\ (\hat{\mathcal{Q}}_\beta)^E_D + (\hat{\mathcal{Q}}_\beta)^D_C\ (\hat{\mathcal{Q}}_\alpha)^E_D = -(\hat{\mathcal{M}}_{\alpha\beta})^E_C \qquad \text{E.5}$$

as required by the OSp(1/2) algebra. The appropriate choice is found to be

$$(\hat{\mathcal{Q}}_\alpha)^C_B = (2M+1)^{\frac{1}{2}} \begin{pmatrix} 0 & \Pi^{-\frac{1}{2}c\gamma}_{b\alpha} \\ (\Pi^{-\frac{1}{2}}\varepsilon)^c_{b\beta\alpha} & 0 \end{pmatrix} \qquad \text{E.6}$$

When working with $\hat{\mathcal{Q}}_\alpha$ care must be taken to ensure that it anticommutes with a-numbers, though from the form (E.6) this property is not explicit

Finally, the action of $(\hat{\mathcal{M}}_{\alpha\beta})_C^D$ and $(\hat{\mathcal{Q}}_\alpha)_C^D$ on the superfield Φ_D is

$$(\hat{\mathcal{M}}_{\alpha\beta})_C^D\ \Phi_D = \begin{pmatrix} (\hat{M}_{\alpha\beta})_c^d\ \phi_d \\ (\hat{M}^x_{\alpha\beta})_{c\gamma}^{d\delta}\ \phi_{d\delta} \end{pmatrix} \qquad \text{E.7}$$

$$(\hat{\mathcal{Q}}_\alpha)_C^D\ \Phi_D = \begin{pmatrix} \phi_{\alpha c} \\ (\Pi^{-\frac{1}{2}}\varepsilon)_{\gamma\alpha c}^{d}\phi_d \end{pmatrix} \qquad \text{E.8}$$

APPENDIX F. A PROOF WHICH DETERMINES THE YOUNG TABLEAU CORRESPONDING TO THE HIGHEST WEIGHT VECTOR, Λ.

In the following we present a diagrammatic proof that the choice (5.7) $\tilde{\mu}$, for ξ and $\beta = (0)$ in (5.2) uniquely determines the highest weight vector, Λ, for B(m,n) and D(m,n). Given the selection criteria, for the diagram corresponding to Λ, which are presented in §§5.2.a, 5.2.b, this proof amounts to showing that if $\xi = \widetilde{(\mu+\chi)}$ has (n+x) columns where χ corresponds to the final x rows in $\tilde{\xi}$, then all the partitions in the series $< \tilde{\xi}/B >$ modify in Sp(2n) to a partition of rank $< |\mu|$. The rank of a partition (ρ) we designate as $|\rho|$.

We first note that if (χ) is a partition of the form

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & a_r \\ a_{1+1} & a_{2+1} \cdot\cdot & a_{r+1} \end{pmatrix} ,$$

in Frobenius notation [2], then $< \mu+\chi >$ modifies to $(-1)^{|\chi|/2} < \mu >$; otherwise it modifies to partitions of rank $< |\mu|$. Our proof is by induction in which we show that if $< \frac{\mu+\chi}{B} >$ contains no diagram of rank $|\mu|$ then $< \frac{\mu+(\chi+2)}{B} >$ contains no diagram of rank $|\mu|$, where $(\chi+2)$ is any partition for which $|(\chi+2)| = |\chi|+2$ and $\mu+(\chi+2)$ is a regular diagram. Since, in Sp(2n), modification involves removing a hook of length $h = 2(P-n-1) \geq 0$ [3], then unless $|\chi|$ is even $< \frac{\mu+\chi}{B} >$ will modify to partitions of rank $< |\mu|$. Consider now

$$\{\frac{\mu + (\chi+2)}{A}\} = \{\mu + (\chi+2)\} \quad \Sigma\{\mu + (\chi+2)''\}$$

$$+ \Sigma\{\mu'' +(\chi+2)\} + \Sigma\{\mu' + (\chi+2)'\} \qquad \text{F.1}$$

where $|\mu'| \leq |\mu|-1$, $|\mu''| \leq |\mu|-2$, $|(\chi+2)'| \leq |\chi|+1$, $0 \leq |(\chi+2)''| \leq |\chi|$, and A is the s-function series $A = \sum_{\alpha}(-1)^{|\alpha|/2}\{\alpha\}$. We now divide both sides by B and use $AB = 1 = \{0\}$ to give

$$\{\frac{\mu + (\chi+2)}{B}\} = \{\mu + (\chi+2)\} - \Sigma\{\frac{\mu + (\chi+2)''}{B}\}$$

$$- \Sigma\{\frac{\mu'' + (\chi+2)}{B}\} - \Sigma\{\frac{\mu' + (\chi+2)'}{B}\} \qquad \text{F.2}$$

Examining (F.2) we see that the final two terms explicitly modify to partitions of rank $< |\mu|$. Considering the first and second terms, we note two possibilities:

(i) $(\chi+2)$ is not a form (α). In this case, $2 \leq |(\chi+2)''| \leq |\chi|$ and by our assertion $< \frac{\mu+(\chi+2)''}{B} >$ will modify only to partitions of rank $< |\mu|$. Also as noted earlier the first term will modify to a partition of rank $< |\mu|$.

(ii) $(\chi+2)$ is of form (α). For this case in (F.1) we have

$$\Sigma\{\mu + (\chi+2)''\} = (-1)^{|\alpha|/2}\{\mu\} + \Sigma\{\mu + (\chi+2)'''\}$$

where now $2 \leq |(\chi+2)'''| \leq |\chi|$. The second term in (F.2) therefore takes the form

$$(-1)^{|\alpha|/2}\{\frac{\mu}{B}\} + \Sigma\{\frac{\mu + (\chi+2)'''}{B}\} .$$

By our assertion the last term here modifies only to partitions of rank $< |\mu|$ and the first term is explicitly of rank $< |\mu|$ except for $\beta = \{0\}$. The first term in (F.2) modifies however to $(-1)^{|\alpha|/2}<\mu>$. Thus the only terms contributing to $<\mu>$ in (F.2) will cancel.

To complete the proof we need only show that for $|\chi| = 2$, $< \frac{\mu+\chi}{B} >$ modifies only to partitions of rank $< |\mu|$. There are only two possibilities:

(i) $\chi = (1^2)$: $<\frac{\mu+(1^2)}{B}> = <\mu+(1^2)> + <\mu> + \Sigma<\mu'+(1)> + \Sigma<\mu''+(1^2)> + \Sigma<\mu''>$

$$\rightarrow -<\mu> + <\mu> - \Sigma<\mu''> + \Sigma<\mu'''>$$

$$= \Sigma<\mu'''> - \Sigma<\mu''> \quad \text{where } |\mu''|, \quad |\mu'''| < |\mu| .$$

(ii) $\chi = (2)$: $<\frac{\mu+(2)}{B}> = <\mu+(2)> + \Sigma<\mu' + (1)>$

$$+ \Sigma<\mu''+(2)> + \Sigma<\mu'''>$$

$$\rightarrow \Sigma<\mu'''> \quad \text{where } |\mu'''| < |\mu| .$$

APPENDIX G: θ - CONVENTIONS AND SOME USEFUL IDENTITIES

Conventions for chapter four:

$$\varepsilon^{ab} = \varepsilon^{\alpha\beta} = -\varepsilon_{ab} = -\varepsilon_{\alpha\beta}$$

where a,b = 1,2 , α,β = 1,2 and $\varepsilon^{12} = +1$.

$$\varepsilon^{ab}\,\varepsilon_{bc} = \delta^{a}_{c} \quad , \quad \varepsilon^{\alpha\beta}\,\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

$$\theta_{a\alpha} = \varepsilon_{\alpha\beta}\,\varepsilon_{ab}\,\theta^{b\beta} \quad , \quad \theta^{a\alpha} = \varepsilon^{\alpha\beta}\,\varepsilon^{ab}\,\theta_{b\beta}$$

Conventions for chapter six:

$$\varepsilon^{\dot\alpha\dot\beta} = \varepsilon^{ab} = -\varepsilon_{\dot\alpha\dot\beta} = -\varepsilon_{ab}$$

where $\dot\alpha,\dot\beta$ = 1,2 , a,b = +,- and $\varepsilon^{12} = \varepsilon^{+-} = +1$.

$$\varepsilon^{\dot\alpha\dot\beta}\,\varepsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}_{\dot\gamma} \quad , \quad \varepsilon^{ab}\,\varepsilon_{bc} = \delta^{a}_{c}$$

$$\bar\theta_{\dot\alpha a} = \varepsilon_{\dot\alpha\dot\beta}\,\varepsilon_{ba}\,\bar\theta^{\dot\beta b} \quad , \quad \bar\theta^{\dot\alpha a} = \varepsilon^{\dot\alpha\dot\beta}\,\varepsilon^{ba}\,\bar\theta_{\dot\beta b}$$

Metric $\eta_{\mu\nu} = (-,+,+,+)$

$$\sigma^{\mu} = (1,\sigma^{i}) \quad , \quad \bar\sigma^{\mu} = (1,-\sigma^{i}) \quad , \quad \mathrm{tr}\,\sigma^{\mu}\,\bar\sigma^{\nu} = -2\eta^{\mu\nu}$$

The monomial bases for $\theta^{\alpha a}$ and $\bar\theta^{\dot\alpha a}$ expansions in §4.5 and §6.3 respectively, is given below together with some useful identities associated with taking products and derivatives. These are given in the notation of §6.3, the corresponding relations for §4.5 are obtained by simply replacing $\bar\theta^{\dot\alpha a}$ by $\theta^{a\alpha}$, i.e. $\bar\theta \to \theta$, $\dot\alpha \to a$, $a \to \alpha$.

Calculus:

$$\partial_{\dot\alpha a}\,\bar\theta^{\dot\beta b} = \delta_{\dot\alpha}{}^{\dot\beta}\,\delta_{a}{}^{b}$$

$$\partial_{\dot\alpha a}\,(\bar\theta\bar\theta)^{bc} = \delta_{a}{}^{b}\,\bar\theta_{\dot\alpha}{}^{c} + \delta_{a}{}^{c}\,\bar\theta_{\dot\alpha}{}^{b}$$

$$\partial_{\dot\alpha a}\,(\bar\theta\bar\theta)^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}{}^{\dot\beta}\,\bar\theta^{\dot\gamma}{}_{a} + \delta_{\dot\alpha}{}^{\dot\gamma}\,\bar\theta^{\dot\beta}{}_{a}$$

$$\partial_{\dot\alpha a}\,(\bar\theta^{3})^{\dot\beta b} = \frac{3}{2}\,(\bar\theta\bar\theta)^{b}{}_{a}\,\delta_{\dot\alpha}{}^{\dot\beta} + \frac{3}{2}\,(\bar\theta\bar\theta)^{\dot\beta}{}_{\dot\alpha}\,\delta_{a}{}^{b}$$

$$\partial_{\dot\alpha a}\,(\bar\theta^{4}) = -4\,(\bar\theta^{3})_{\dot\alpha a}$$

Identities:

$$(\bar\theta\bar\theta)^{ab} = \bar\theta^{\dot\alpha a}\ \bar\theta_{\dot\alpha}{}^{b}$$

$$(\bar\theta\bar\theta)^{\dot\alpha\dot\beta} = \bar\theta^{\dot\alpha a}\ \bar\theta^{\dot\beta}{}_{a}$$

$$(\bar\theta^3)^{\dot\alpha a} = (\bar\theta\bar\theta)^{\dot\alpha\dot\beta}\ \bar\theta_{\dot\beta}{}^{a} = -(\bar\theta\bar\theta)^{ab}\ \bar\theta^{\dot\alpha}{}_{b}$$

$$(\bar\theta^4) = (\bar\theta^3)^{\dot\alpha a}\ \bar\theta_{\dot\alpha a} = (\bar\theta\bar\theta)^{\dot\alpha\dot\beta}\ (\bar\theta\bar\theta)_{\dot\alpha\dot\beta}$$

$$= -(\bar\theta\bar\theta)^{ab}\ (\bar\theta\bar\theta)_{ab}$$

$$\bar\theta^{\dot\alpha a}\ \bar\theta^{\dot\beta}{}_{b} = \tfrac{1}{2}(\bar\theta\bar\theta)^{\dot\alpha\dot\beta}\ \delta^{a}_{b}\ -\tfrac{1}{2}(\bar\theta\bar\theta)^{a}{}_{b}\ \varepsilon^{\dot\alpha\dot\beta}$$

$$\bar\theta^{\dot\alpha}{}_{a}\ (\bar\theta\bar\theta)^{bc} = -\frac{1}{3}\ (\bar\theta^3)^{\dot\alpha b}\delta^{c}_{a} - \frac{1}{3}\ (\bar\theta^3)^{\dot\alpha c}\ \delta^{b}_{a}$$

$$\bar\theta^{\dot\alpha}{}_{a}\ (\bar\theta\bar\theta)^{\dot\beta\dot\gamma} = \frac{1}{3}\ (\bar\theta^3)^{\dot\beta}{}_{a}\ \varepsilon^{\dot\alpha\dot\gamma} + \frac{1}{3}\ (\bar\theta^3)^{\dot\gamma}{}_{a}\ \varepsilon^{\dot\alpha\dot\beta}$$

$$\bar\theta^{\dot\alpha}{}_{a}(\bar\theta^3)^{\dot\beta b} = -\tfrac{1}{4}\ (\bar\theta^4)\ \ \varepsilon^{\dot\alpha\dot\beta}\ \delta^{b}_{a}$$

$$(\bar\theta\bar\theta)^{\dot\alpha\dot\beta}\ (\bar\theta\bar\theta)^{ab} = 0$$

$$(\bar\theta\bar\theta)^{\dot\alpha\dot\beta}\ (\bar\theta\bar\theta)^{\dot\gamma\dot\delta} = \frac{1}{6}\ (\varepsilon^{\dot\beta\dot\gamma}\ \varepsilon^{\dot\alpha\dot\delta} + \varepsilon^{\dot\alpha\dot\gamma}\ \varepsilon^{\dot\beta\dot\delta})\ (\bar\theta^4)$$

$$(\bar\theta\bar\theta)^{ab}\ (\bar\theta\bar\theta)^{cd} = \frac{1}{6}\ (\varepsilon^{bc}\ \varepsilon^{ad} + \varepsilon^{ac}\ \varepsilon^{bd})\ (\bar\theta^4)$$

APPENDICES - REFERENCES

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