Supersymmetric Born Reciprocity

by

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Declaration

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Abstract

In 1949 Max Born introduced the idea of reciprocity, that the laws of physics should be invariant under the transformation,

$$X^\mu \to P^\mu,\ P^\mu \to -X^\mu \tag{1}$$

Then a *reciprocally invariant* function is one which obeys $S(X,P) = S(P,-X)$. A *self-reciprocal* function $\mathcal{F}$ is one which obeys the eigenvalue equation,

$$S(X,P)\mathcal{F}(x) = s\mathcal{F}(x) \tag{2}$$

Born then claimed that these eigenfunctions $\mathcal{F}$ are the field operators, a contention that must be justified by comparing the solutions of $\mathcal{F}$ with the observable facts. He studied the case of the *metric operator*, $S = X^\mu X_\mu + P^\mu P_\mu$; finding eigenfunctions that involved Laguerre polynomials. The roots of these polynomials correspond with the masses of what Born thought were an infinite number of mesons.

Supersymmetry is a principle that seeks to transform bosons into fermions and vice-versa. This is done by use of a $\mathbb{Z}_2$ grading on the Poincaré algebra.

Born's theory did not have much success when related to the observed mass spectrum and so the motivation for this project is to apply supersymmetry principles to the theory to see if this alleviates these problems. This is done by modifying the metric operator to include four antisymmetric Grassmann variables and Grassmann momentum operators and then solving for the new eigenfunctions $\mathcal{F}$. Thus families of bosons and fermions have been uncovered. Also, a preliminary calculation has been made of Low's fundamental constant pertaining to an upper bound on rate of change of momentum, b, finding: $b \approx 2.61 \times 10^9$N.

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- Natalie Raymond - my beautiful girlfriend, for being there for me and providing inspiration.

Finally I would like to dedicate this thesis to my parents Brian and Sue, for doing so much for me in the past 23 years. Thank you!

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Chapter 1

Introduction

There are two major topics discussed in this thesis. The first is Max Born's principle of reciprocity. The second is the relatively new symmetry principle in particle physics that allows transformations of fermions into bosons and vice-versa. This is called supersymmetry.

In 1949 Max Born introduced his theory of reciprocity in his paper "Reciprocity Theory of Elementary Particles" [3]. Born proposed that the laws of physics should be invariant under the following transformation

$$x_\mu \to p_\mu, \; p_\mu \to -x_\mu \tag{1.1}$$

There are many simple examples in quantum and classical mechanics alike where the reciprocity transformation leaves quantities invariant. Two examples that Born [3] use are:

1. The (quantum) orbital angular momentum operators, which in three dimensions are given by
$$L_i = \epsilon_{ijk}(X_j P_k - X_k P_j). \tag{1.2}$$

2. Hamilton's canonical equations of motion in classical mechanics, given by
$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p_x} \\ \frac{dp_x}{dt} &= -\frac{\partial H}{\partial x} \end{aligned}$$

One of Born's initial applications of reciprocity was to set

$$\hbar = ag \tag{1.3}$$

where a has units of position and g has units of momentum. Therefore it is possible to measure position in terms of a and momentum in terms of g. This is the convention I have used in this thesis. The theory of reciprocity as developed by Born [3] and then much later by Low [19, 18] is the topic considered in chapter 2. The group aspect of Low's theory is also studied further in chapter 3 after some general group theory is developed in the same chapter.

Supersymmetry is a very interesting yet so far unconfirmed theory involving the transformations of bosons into fermions and vice-versa. It involves anticommuting coordinates known as Grassmann variables. Some basic theory behind Grassmann variables is developed in section 4.1. Supersymmetry involves a $\mathbb{Z}_2$ graded entity known as a Lie superalgebra. The even subalgebra of this entity is the corresponding Lie algebra. The theory of Lie algebras is developed in chapter 3 and then applied to superalgebras in section 4.2.

The original work of this thesis is the focus of chapter 5. It is the supersymmetric extension of the theory of reciprocity. A discussion of the results of Born's theory and the supersymmetric extension is given in chapter 6.

1.1 Notation

Most of the analysis in this thesis is done in four-dimensions. Quantities in four-dimensions are indicated by possessing indices from the middle of the Greek alphabet, for example μ and ν. These indices run from 0 to 3. Indices from the beginning of the Greek alphabet (α, β and γ) will run through $1, 2, ..., 2m$ with the value of m depending on the given situation. We have the covariant four-vectors

x_μ of position and p_μ of momentum, such that

$$x_0 = ct, \ p_0 = E/c$$
$$x_1 = x, \ p_1 = p_x$$
$$x_2 = y, \ p_2 = p_y$$
$$x_3 = z, \ p_3 = p_z.$$

The corresponding contravariant vectors are obtained by use of the raising metric. Throughout this thesis, I have used the following metric in Minkowski space, unless otherwise specified:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.4}$$

And so we see that

$$x^\mu = \eta^{\mu\nu} x_\nu, \ x_\mu = \eta_{\mu\nu} x^\nu$$
$$p^\mu = \eta^{\mu\nu} p_\nu, \ p_\mu = \eta_{\mu\nu} p^\nu,$$

then for each component:

$$\begin{aligned} x^0 &= -x_0, & p^0 &= -p_0 \\ x^1 &= x_1, & p^1 &= p_1 \\ x^2 &= x_2, & p^2 &= p_2 \\ x^3 &= x_3, & p^3 &= p_3. \end{aligned}$$

The *proper time* is defined to be equal to the square of the position four-vector:

$$(c\tau)^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{1.5}$$

and is an invariant quantity. Using the convention of working in a system of natural units so that the speed of light, c and $\hbar$ both have a value of unity, then the interval between two events is defined to be:

$$(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2,$$

allowing three cases to characterise the proper time:

$$
\begin{aligned}
(d\tau)^2 &> 0 \quad && \text{timelike} \\
(d\tau)^2 &= 0 \quad && \text{lightlike} \\
(d\tau)^2 &< 0 \quad && \text{spacelike.}
\end{aligned}
$$

In this thesis, only the timelike and lightlike cases are considered.

Quantities in three-dimensions are indicated by having indices from the middle of the Roman alphabet, for example i, j and k. These indices all run from 1 to 3. The metric for three-dimensional quantities is simply the identity matrix in three-dimensions. Einstein summation is implied, so that repeated indices are summed over.

The commutator of two quantities is denoted by square brackets and defined by:

$$[A, B] = AB - BA. \tag{1.6}$$

In a similar manner we define the anticommutator, denoted by curly brackets:

$$\{A, B\} = AB + BA. \tag{1.7}$$

The Levi-Civita symbol, ϵ_{ijk}, seen in equation 1.2 is defined as:

$$
\epsilon_{ijk} = \begin{cases}
+1 \text{ for } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\
-1 \text{ for } (i,j,k) \in \{(1,3,2),(2,1,3),(3,2,1)\} \\
0 \text{ otherwise.}
\end{cases}
$$

1.2 Position and Momentum Operators

We take position and momentum operators X^μ, P_ν in position space, so that

$$X^\mu \left|\psi\right\rangle = x^\mu \left|\psi\right\rangle, \;\; P_\nu \left|\psi\right\rangle = p_\nu \left|\psi\right\rangle, \;\; P_\nu \to -i\frac{\partial}{\partial x^\nu} \tag{1.8}$$

(where x^μ, p_ν are the eigenvalues of the respective operators acting on a wave-function ψ) then these operators represent the *canonically conjugate* position and momentum variables of a particle [4] and they obey the well known Heisenberg commutation relations

$$[X^\mu, X^\nu] = 0 \tag{1.9}$$

$$[P_\mu, P_\nu] = 0 \tag{1.10}$$

$$\left[X^i, P_j\right] = iI\delta^i_j \tag{1.11}$$

$$\left[X^0, P_0\right] = -iI. \tag{1.12}$$

In these equations, I is the identity matrix.

Chapter 2

Born's Theory of Reciprocity

In this chapter the scene of Born's reciprocity theory is set. Most of the work done on reciprocity has been done by Born himself [3], Landé[15, 16] and more recently, Caianello [5] and Low [19]. In section 2.1, I illustrate Born's principle of Reciprocity, in particular explaining the terms *reciprocal invariant* and *self-reciprocal*. In sections 2.2 and 2.3, I use Born's theory to formulate an equation to determine particle rest masses. In section 2.4, I develop the solution of this equation, including much work not included in Born's paper [3]. Section 2.5 gives the results of this theory as applied to the particle mass spectrum.

2.1 Definition

The basic idea of reciprocity theory of elementary particles put forward by Born [3] is that the laws of physics should be invariant under the transformation

$$X_\mu \to P_\mu,\ P_\mu \to -X_\mu. \tag{2.1}$$

This is only one example of the possible transformations we could use. It represents a rotation in the phase plane of the quantity $X+iP$ by 90 degrees (multiply $X+iP$ by i), but we could rotate any number of ways. This method is convenient however, so for the remainder of the thesis, this will be the definition used for reciprocity.

Any quantity that is of the form

$$S(X,P) = S(P,-X) \tag{2.2}$$

is said to be *reciprocally invariant.* Also, any functional F that satisfies

$$\langle x|\, S(X,P)\,|F\rangle = s\,\langle x|F\rangle \tag{2.3}$$

is said to be *self-reciprocal.* We want to show that $F(x)$ is its own Fourier transform,

$$F(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\; e^{-ixp}dx \tag{2.4}$$

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(p)\; e^{ixp}dp. \tag{2.5}$$

Indeed, this is an alternate definition of reciprocity [3, 15]. We have

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}(p)e^{ipx}dp \\ S(x,-i\frac{\partial}{\partial x})F(x) &= sF(x). \end{aligned} \tag{2.6}$$

$S(X,P)$ acting on the eigenfunction $F(x)$ yields

$$\begin{aligned} S(X,P)F(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}(p)S(x,-i\frac{\partial}{\partial x})e^{ipx}dp \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S(i\frac{\partial}{\partial p},p)\mathcal{F}(p)e^{ipx}dp \end{aligned}$$

also,

$$S(X,P)F(x) = s\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}(p)e^{ipx}dp$$

from the eigenvalue equation. Equating these gives us

$$\int_{-\infty}^{\infty} S(i\frac{\partial}{\partial p},p)\mathcal{F}(p)e^{ipx}dp = \int_{-\infty}^{\infty} \mathcal{F}(p)e^{ipx}dp.$$

But $S(X,P)$ is reciprocally invariant, and both sides of these equations involve the same integral. So we can apply the reciprocity transformation and write

$$S(p,-i\frac{\partial}{\partial p})\mathcal{F}(p) = s\mathcal{F}(p). \tag{2.7}$$

We can see that this equation is of exactly the same form as equation 2.6, so that we can say that $F(x)$ is indeed its own Fourier transform. From now on I will not make the distinction between $F(p)$ and $\mathcal{F}(p)$, I will refer to $\mathcal{F}(p)$ as $F(p)$.

Low [19] explains Born's reciprocity principle as the extension of the usual four degrees of freedom of space-time to eight degrees of freedom of space-time and momentum-energy. This is as opposed to other contemporary theories that tack on non-observable dimensions to space-time to try and explain experimental results. These additional dimensions then need to somehow be explained, especially with respect to their non-observability. Low predicts the existence of a universal upper bound on the rate of change of momentum, the momentum analogue to the universal upper bound on rate of change of position: the speed of light. Some of Low's theory is discussed in sections 2.2 and 3.9.

2.2 Reciprocity Applied to Particle Rest Masses

Bosons are generally supposed to have wave functions that obey equations of the form

$$P^{\mu}P_{\mu}\left|\psi\right\rangle = \kappa^{2}\left|\psi\right\rangle \tag{2.8}$$

where the constant κ is proportional to the rest mass of a particle. Born used as the constant of proportionality, $g/c = \hbar/ac$, so that

$$\mu = (\hbar/ac)\kappa \tag{2.9}$$

is the actual rest mass of observed particles. We can write equation 2.8 in the form

$$\langle x|F(P^{\mu})|\psi\rangle = 0 \tag{2.10}$$

where $F(P^{\mu}) = P^{\mu}P_{\mu} - \kappa^{2}$. But we could just as easily write

$$F(P^{\mu}) = F_1(P^{\mu})(P^{\mu}P_{\mu} - \kappa^{2}) \tag{2.11}$$

where $F_1(P^{\mu})$ has no roots. But if $F_1(P^{\mu})$ is itself of the form

$$F_1(P^{\mu}) = F_2(P^{\mu})(P_{\mu} - \kappa_1^{2}) \tag{2.12}$$

where $F_2(P^{\mu})$ has no roots then there will be another set of solutions to equation 2.8 corresponding to theoretical rest masses of a different set of particles. We can continue this process indefinitely, so that choosing the function F to have roots $\kappa_1, \kappa_2, \kappa_3, ...$ will produce a wave equation (equation 2.8) representing simultaneously particles with different rest masses.

Finding this function F (which has so far been arbitrary) through use of reciprocity was exactly the purpose of Born's reciprocity paper [3]. We can do this by solving equation 2.3 using an arbitrary reciprocal invariant. Then, when we have roots $\kappa_1, \kappa_2, \kappa_3, ...$ we need to determine the constant of proportionality (in other words, we need to evaluate a, because the values of c and $\hbar$ are of course, well known). To do that we need to identify one of our theoretical particles, (for instance, κ_1) with a particle of known mass, μ_1. Once we know the value of a then it is a simple matter to evaluate the rest masses of the particles corresponding to all of our theoretical particles κ_i.

An alternative method of evaluating the constant a was put forth by Low [19]. He proposed the existence of a new kind of fundamental constant; a universal upper bound on rate of change of momentum. This constant was denoted in Low's literature by b. Born's minimum length constant is then defined in terms of Low's maximum rate of change of momentum by

$$a = \sqrt{\frac{\hbar c}{b}}. \tag{2.13}$$

Low also wrote measures for time, momentum and energy in terms of the three constants $\hbar$, c and b in a similar manner [19, 18]:

$$\begin{aligned}
\lambda_t &= \sqrt{\frac{\hbar}{bc}} \\
\lambda_p &= \sqrt{\frac{\hbar b}{c}} \\
\lambda_E &= \sqrt{\hbar bc}.
\end{aligned}$$

Defining these quantities allows us to make all measurements of time, position, momentum and energy in units of λ_t, a, λ_p and λ_E respectively and so the measurements are dimensionless.

2.3 Formulating the Field Equation

According to Born's paper, the solutions of equation 2.3 represent all possible self-reciprocal scalars and tensors. Born initially said that the function S is arbitrary, but then he chose to use the simplest relativistically and reciprocally invariant function, what he called the *metric operator*:

$$S(X,P) = X^\mu X_\mu + P^\mu P_\mu. \tag{2.14}$$

In accordance with equation 2.3, we apply this operator in position space (we could equally well do this in momentum space, indeed Born himself used momentum space), to the self-reciprocal wave function. This gives us the Field equation:

$$\begin{aligned}(-\frac{\partial^2}{\partial x^\mu \partial x_\mu} + x^\mu x_\mu)F &= sF \quad (2.15)\\ (-\Box + x^2 - s)F &= 0. \quad (2.16)\end{aligned}$$

2.4 Solving the Field Equation

The best way to solve this equation is to use 4-dimensional polar coordinates r, θ, ϕ, ω, with $x'^0 = r$, $x'^1 = \omega$, $x'^2 = \theta$, $x'^3 = \phi$. So our first task is to find the d'Alembertian using these coordinates. To do this, we need to know the four-dimensional polar coordinate metric for Minkowski space. Born's reciprocity paper skips straight from the formulation of the field equation (equation 2.15) to the radial equation (equation 2.31) to the solutions of the radial equation. However to make this paper more self-contained I have included the derivation of the radial equation, as I do not believe all of the steps involved are entirely obvious.

2.4.1 The Four-Dimensional Polar Coordinate Metric

For simplicity, let $\sigma = ict$, so that for Minkowski space we have a Euclidean metric, $\eta_{\mu\nu} = \eta^{\mu\nu} = \delta^\mu_\nu$. We want to make the following change of variables:

$$\begin{aligned}x^0 = \sigma &\rightarrow r\cos\omega\\ x^1 = x &\rightarrow r\sin\omega\sin\theta\cos\phi\\ x^2 = y &\rightarrow r\sin\omega\sin\theta\sin\phi\\ x^3 = z &\rightarrow r\sin\omega\cos\theta\end{aligned}$$

where $0 \leq r \leq \infty$, $0 \leq \omega \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. Now we need to find the metric $\eta'_{\mu\nu}$ in polar coordinates. The components of this new metric are given by:

$$\eta'_{\gamma\epsilon} = \eta_{\mu\nu}\frac{\partial x^{\mu}}{\partial x'^{\gamma}}\frac{\partial x^{\nu}}{\partial x'^{\epsilon}}. \tag{2.17}$$

Clearly, this metric has the form of a diagonal matrix, with $\eta'_{\mu\nu} = \eta'^{\mu\nu}$. Therefore we can work out the diagonal components:

$$\begin{aligned}
\eta'_{00} &= \eta_{\mu\nu}\frac{\partial x^{\mu}}{\partial x'^{0}}\frac{\partial x^{\nu}}{\partial x'^{0}} \\
&= \eta_{00}(\frac{\partial \sigma}{\partial r})^2 + \eta_{11}(\frac{\partial x}{\partial r})^2 + \eta_{22}(\frac{\partial y}{\partial r})^2 + \eta_{33}(\frac{\partial z}{\partial r})^2 \\
&= \cos^2\omega + \sin^2\omega(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta) \\
&= 1 \\
\eta'_{11} &= \eta_{00}(\frac{\partial \sigma}{\partial \omega})^2 + \eta_{11}(\frac{\partial x}{\partial \omega})^2 + \eta_{22}(\frac{\partial y}{\partial \omega})^2 + \eta_{33}(\frac{\partial z}{\partial \omega})^2 \\
&= r^2\sin^2\omega + r^2\cos^2\omega(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta) \\
&= r^2 \\
\eta'_{22} &= \eta_{00}(\frac{\partial \sigma}{\partial \theta})^2 + \eta_{11}(\frac{\partial x}{\partial \theta})^2 + \eta_{22}(\frac{\partial y}{\partial \theta})^2 + \eta_{33}(\frac{\partial z}{\partial \theta})^2 \\
&= 0 + r^2\sin^2\omega(\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta) \\
&= r^2\sin^2\omega \\
\eta'_{33} &= \eta_{00}(\frac{\partial \sigma}{\partial \phi})^2 + \eta_{11}(\frac{\partial x}{\partial \phi})^2 + \eta_{22}(\frac{\partial y}{\partial \phi})^2 + \eta_{33}(\frac{\partial z}{\partial \phi})^2 \\
&= 0 + r^2\sin^2\omega\sin^2\theta(\sin^2\phi + \cos^2\phi) \\
&= r^2\sin^2\omega\sin^2\theta.
\end{aligned}$$

From this we can write the Minkowski metric in four-dimensional polar coordinates, which will now be denoted as $\eta_{\mu\nu}$:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\omega & 0 \\ 0 & 0 & 0 & r^2\sin^2\omega\sin^2\theta \end{pmatrix}. \tag{2.18}$$

We can see from this that the separation between two events in four-dimensional polar coordinates is

$$d\tau^2 = dr^2 + r^2 d\omega^2 + r^2\sin^2\omega d\theta^2 + r^2\sin^2\omega\sin\theta d\phi^2 \tag{2.19}$$

and from [2] we have the four-dimensional volume element

$$dV = r^3 \sin^2 \omega \sin \theta dr d\omega d\theta d\phi. \tag{2.20}$$

2.4.2 Connection Coefficients for the d'Alembertian

Using notation common to tensor mechanics, the d'Alembertian is written as:

$$\Box\Phi = \Phi^{;\mu}{}_{\mu} = \frac{\partial \Phi^{,\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\nu}\Phi^{,\nu}. \tag{2.21}$$

The next step is to derive the relevant connection coefficients. In general,

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2\eta_{\sigma\lambda}}(\eta_{\lambda\mu,\nu} + \eta_{\lambda\nu,\mu} - \eta_{\mu\nu,\lambda}). \tag{2.22}$$

In our case, there will only be a few non-zero derivatives of $\eta_{\mu\nu}$. To be exact, there are six non-zero derivatives. These are:

$$\begin{aligned}
\eta_{11,0} &= 2r \\
\eta_{22,0} &= 2r \sin^2 \omega \\
\eta_{22,1} &= 2r^2 \sin \omega \cos \omega \\
\eta_{33,0} &= 2r \sin^2 \omega \sin^2 \theta \\
\eta_{33,1} &= 2r^2 \sin \omega \cos \omega \sin^2 \theta \\
\eta_{33,2} &= 2r^2 \sin^2 \omega \sin \theta \cos \theta.
\end{aligned}$$

There are only six connection coefficients that we are interested in. These are:

$$\begin{aligned}
\Gamma^1_{10} &= \frac{1}{2\eta_{11}}(\eta_{11,0}) \\
&= \frac{1}{r} \\
\Gamma^2_{20} &= \frac{1}{2\eta_{22}}(\eta_{22,0}) \\
&= \frac{1}{r} \\
\Gamma^2_{21} &= \frac{1}{2\eta_{22}}(\eta_{22,1}) \\
&= \cot\omega \\
\Gamma^3_{30} &= \frac{1}{2\eta_{33}}(\eta_{33,0}) \\
&= \frac{1}{r} \\
\Gamma^3_{31} &= \frac{1}{2\eta_{33}}(\eta_{33,1}) \\
&= \cot\omega \\
\Gamma^3_{32} &= \frac{1}{2\eta_{33}}(\eta_{33,2}) \\
&= \cot\theta.
\end{aligned}$$

2.4.3 The d'Alembertian in Polar Coordinates

Now that we have the connection coefficients, we can expand equation 2.21:

$$\begin{aligned}
\Phi^{;\mu}_{\ \mu} &= \frac{\partial}{\partial r}(\frac{\partial\Phi}{\partial r}) + \frac{\partial}{\partial\omega}(\frac{1}{r^2}\frac{\partial\Phi}{\partial\omega}) + \frac{\partial}{\partial\theta}(\frac{1}{r^2\sin^2\omega}\frac{\partial\Phi}{\partial\theta}) \\
&\quad + \frac{\partial}{\partial\phi}(\frac{1}{r^2\sin^2\omega\sin^2\theta}\frac{\partial\Phi}{\partial\phi}) + \Gamma^1_{10}\Phi^{,0} + \Gamma^2_{20}\Phi^{,0} + \Gamma^2_{21}\Phi^{,1} \\
&\quad + \Gamma^3_{30}\Phi^{,0} + \Gamma^3_{31}\Phi^{,1} + \Gamma^3_{32}\Phi^{,2} \\
&= \frac{\partial^2\Phi}{\partial r^2} + \frac{3}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\omega^2} + \frac{2\cot\omega}{r^2}\frac{\partial\Phi}{\partial\omega} + \frac{1}{r^2\sin^2\omega}\frac{\partial^2\Phi}{\partial\theta^2} \\
&\quad + \frac{\cot\theta}{r^2\sin^2\omega}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\omega\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \\
&= \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\frac{\partial\Phi}{\partial r}) + \frac{1}{r^2\sin^2\omega}\frac{\partial}{\partial\omega}(\sin^2\omega\frac{\partial\Phi}{\partial\omega}) \\
&\quad + \frac{1}{r^2\sin^2\omega\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\omega\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \\
&\equiv \Box_r - \frac{1}{2}\frac{L^{\mu\nu}L_{\nu\mu}}{r^2}
\end{aligned}$$

where $\Box_r$ is the radial term of the d'Alembertian and $\frac{1}{2}L^{\mu\nu}L_{\nu\mu}$ is the angular component of the d'Alembertian; the four-dimensional analogue to the three-dimensional operator representing the square of the magnitude of the orbital angular momentum [4]; and

$$L^{\mu\nu} \equiv (X^{\mu}P^{\nu} - X^{\nu}P^{\mu}). \tag{2.23}$$

2.4.4 The Solution of the Field Equation

We want to look for a separable solution of equation 2.15 in the form

$$F(x) = F_k(R)Y_k(\theta, \phi, \omega) \tag{2.24}$$

where $R \equiv r^2$. The angular component of the solution, $Y_k(\theta, \phi, \omega)$ is a four-dimensional harmonic. Taking the d'Alembertian and breaking it up into its radial and angular components produces

$$\Box = \Box_r - \frac{\mathbf{L}^2}{r^2}. \tag{2.25}$$

When operating on a harmonic, this operator has the well known property that (see for example, [2]):

$$\mathbf{L}^2 Y_k(\theta, \phi, \omega) = k(k+2)Y_k(\theta, \phi, \omega) \tag{2.26}$$

in four-dimensions. Therefore equation 2.15 becomes:

$$\begin{aligned} (-\Box + r^2)F_k(R)Y_k(\theta, \phi, \omega) &= sF_k(R)Y_k(\theta, \phi, \omega) && (2.27) \\ (-\Box_r + \frac{\mathbf{L}^2}{r^2} + r^2)F_k(R)Y_k(\theta, \phi, \omega) &= sF_k(R)Y_k(\theta, \phi, \omega). && (2.28) \end{aligned}$$

The radial terms commute with the angular term and vice versa, so after dividing by $Y_k(\theta, \phi, \omega)$ we can write

$$(-\Box_r + \frac{k(k+2)}{r^2} + r^2)F_k(R) = sF_k(R). \tag{2.29}$$

Rearranging this gives:

$$\frac{d^2F_k(R)}{dr^2} + \frac{3}{r}\frac{dF_k(R)}{dr} + (s - r^2 - \frac{k(k+2)}{r^2})F_k(R) = 0. \tag{2.30}$$

Recalling that $F(x) = F(p)$ we want to change to momentum space and then apply the change of variables: $M \equiv p^{\mu}p_{\mu} = p^2$.

$$\begin{aligned}
\frac{dF}{dp} &= \frac{dF}{dM}\frac{dM}{dp} \\
&= 2p\frac{dF}{dM} \\
\frac{d^2F}{dp^2} &= 2\frac{dF}{dM} + 2p\frac{d^2F}{dM^2}\frac{dM}{dp} \\
&= 2\frac{dF}{dM} + 4p^2\frac{d^2F}{dM^2}.
\end{aligned}$$

Now we can substitute this into equation 2.30:

$$\begin{aligned}
4M\frac{d^2F}{dM^2} + 2\frac{dF}{dM} + \frac{3}{p}(2p\frac{dF}{dM}) + (s - M - \frac{k(k+2)}{M})F &= 0 \\
\frac{d^2F_k}{dM^2} + \frac{2}{M}\frac{dF_k}{dM} - \frac{1}{4}(1 - \frac{s}{M} + \frac{k(k+2)}{M^2})F_k &= 0. \qquad (2.31)
\end{aligned}$$

Born [3] gives the solution of this eigenvalue problem. With boundary conditions $F_k \to 0$ as $M \to \infty$, the solution is

$$s = 2(2n-k),\ F_k = M^{\frac{k}{2}}e^{-\frac{M}{2}}L_n^{(k+1)}(M), \qquad (2.32)$$

where $L_n^{(k)}(M)$ is the kth derivative of the Laguerre polynomial $L_n(M)$ of order $n \geq k+1$.

2.5 Boson Rest Masses Derived from the Field Equation

According to equation 2.10 if we can solve for $F(M) = 0$ then we can find the values of κ_i representing the theoretical rest masses of our particles. Therefore we need to find the roots of the equation

$$F(\kappa^2) = 0 \qquad (2.33)$$

and solve for the values of κ at each root. But we have the solution in equation 2.32:

$$F_k(\kappa^2) = \kappa^k e^{-\frac{\kappa^2}{2}}L_n^{(k+1)}(\kappa^2). \qquad (2.34)$$

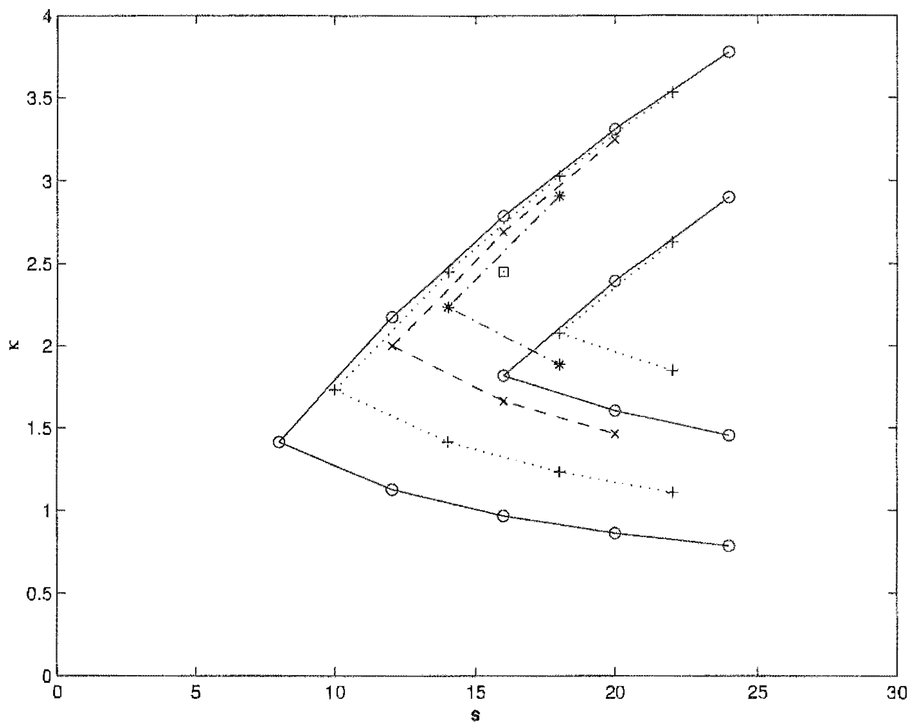

Figure 2.1: Born's theoretical mass spectrum

This will be zero when either $\kappa^k = 0 \;\Rightarrow \kappa = 0$ or when $L_n^{(k+1)}(\kappa^2) = 0$. The root $\kappa = 0$ is trivial and applies to photons of vanishing mass. The other values we are interested in are those of

$$L_n^{(k+1)}(\kappa^2) = 0. \tag{2.35}$$

This leads us to Born's theoretical mass spectrum for bosons with an infinite number of values for κ_i. I have reproduced Born's spectrum in figure 2.1 [3], for $n = 1, 2, ..., 6$. This graph plots κ_i against s. The unbroken lines all correspond to values of $k = 0$, the dotted lines correspond to $k = 1$, the dashed lines correspond to $k = 2$ and the square is one value of κ with $k = 3$.

2.6 Self-Reciprocal Functions for Half-Integral Spin

Born also tried to determine self-reciprocal eigenfunctions for elementary particles with half-integral spin. He did this by factorizing F again into two factors:

$$F = A_k Z_k$$

where A_k is a factor depending on $\eta = \alpha_k p^k$; α_k are the Dirac matrices. These are briefly explained in section 4.2.2[1]. Z_k is a generalized spherical harmonic, depending on θ, ϕ and ω:

$$Z_k = (L + 4 + 2k)Y_k$$

Y_k being an ordinary harmonic. Then we have three cases for the form of the solution. The first case is the trivial

$$\eta^k = 0, \;\; k = 1, 2, 3, ...$$

which Born took as the vanishing rest masses of neutrinos. The other two cases depend on the eigenvalues λ of the Dirac matrices. They lead to two separate equations, the roots of which correspond to an infinite number of theoretical masses κ_i of fermions:

$$\begin{aligned} \lambda = +1 : \quad & \kappa L_n^{(k+2)}(\kappa^2) \pm (n-k-1)^{\frac{1}{2}} L_n^{(k+1)}(\kappa^2) = 0 \\ \lambda = -1 : \quad & \kappa\{L_n^{(k+1)}(\kappa^2) - L_n^{(k+2)}(\kappa^2)\} \pm (n+1)^{\frac{1}{2}} L_n^{(k+1)}(\kappa^2) = 0. \end{aligned}$$

This development was a necessary part of Born's theory [3]. However, it is the author's contention that the application of supersymmetry to reciprocity will negate the need for a separate development of fermions and bosons in the theory.

[1] For a better explanation see for example http://mathworld.wolfram.com/DiracMatrices.html

Chapter 3

Groups

This chapter starts with a brief introduction to groups, including several definitions used throughout this thesis. This is largely based on [12, 11]. It then moves on to discuss a particular class of groups, called *Lie groups.* These are discussed in section 3.4. The algebra of Lie groups is discussed in section 3.5. Following this are some important examples of Lie groups, in particular the Lorentz and Poincaré groups discussed in section 3.8. For more reference into these topics see for example, [13]. Finally, another example of a Lie group is discussed in section 3.9. Most of the work that has been done on the canonical group has been done by Low [19, 18].

3.1 Definition

A set S of elements forms a *semigroup* if to any two elements x and y of S taken in a particular order there is associated a unique 3rd element of S, called their product and denoted by xy. A semigroup is called a *monoid* if there exists a neutral element, denoted by e. A monoid is called a *group*, G if for every element x of G, there exists an inverse element. The product of the group does not have to be multiplication, it is simply an operation associated with the group structure. Therefore, given a set and a product there are four necessary properties that will constitute a group:

- Two elements x and y of G must have a unique product, xy, that is contained within the group.

- Associative law must hold: $(xy)z = x(yz)$ (M2).
- There must be an identity or neutral element, denoted by e (M3).
- Every element x must have an inverse, denoted by x^{-1} (M4).

Note that M1, the commutative product law does not have to hold for groups. It may hold in a specific example, but by no means is it necessary to form a group. The *order* of a finite group G is simply the number of elements in that group, denoted by $|G|$.

3.2 Examples

Here are a few simple examples of groups:

- The set of all real numbers except for 0 under multiplication. This is only a group if 0 is not included in the group as 0 has no inverse. The set of all real numbers under addition (this time including 0) will also form a group.
- Similarly the set of all complex numbers forms a group under addition, and will form a group under multiplication if $0 + 0i$ is excluded.
- The set of all two-dimensional vectors, under vector addition. The neutral element is $(0, 0)$, the inverse of (x, y) is simply $(-x, -y)$, the product is uniquely defined and contained within the group and is clearly associative. This is an example of an *abelian group*, see section 3.3.1.
- The permutation groups: an example of a permutation group is as follows: Take three elements, 1, 2 and 3 which constitute all the elements of the group (so the group is of order 3). Now, the action is to map one element into another. One such mapping could be

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 3 \\ 3 &\rightarrow 2 \end{aligned}$$

call this x. Define y as the mapping

$$\begin{aligned} 1 &\rightarrow 3 \\ 2 &\rightarrow 1 \\ 3 &\rightarrow 2 \end{aligned}$$

Now, the product xy is simply defined as making the first map and then the second map. So xy would be the mapping

$$\begin{aligned} 1 &\rightarrow 3 \\ 2 &\rightarrow 2 \\ 3 &\rightarrow 1 \end{aligned}$$

Note that this product is associative (if a third mapping z was defined, then $(xy)z = x(yz)$), but not commutative. For two general permutations, the product may be commutative but does not have to be, just as for any other group. The inverse of a mapping is simply the backward mapping, and the identity (or neutral) permutation is the mapping of a number into itself:

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 2 \\ 3 &\rightarrow 3 \end{aligned}$$

3.3 Other Group Definitions

3.3.1 Abelian Groups

As stated above, the group product is not necessarily commutative, but may be in some instances. If it is the case that the group product is commutative (so that, for elements x and y, $xy = yx$), the group is said to be an *abelian group*. It is usual to use addition notation for abelian groups rather than multiplication notation. Therefore an abelian group has the following properties:

- Two elements x and y must have a unique sum (or product) $x + y$ that is contained within the group.

- The sum is commutative: $x + y = y + x$ (A1).
- The sum is associative, so for three elements, x, y and z, $(x + y) + z = x + (y + z)$ (A2).
- There is a neutral element, called the "zero", such that $x + 0 = 0 + x = x$ (A3).
- Each element x has an inverse element, $-x$, called the "negative" (A4).

3.3.2 Generators

Any group of finite elements has the property that all of its elements can be expressed in terms of a set of *generators* of that group. An infinite group may or may not have a set of generators. For finite groups the set of all group elements is itself a set of generators. However if it is the case that it is possible to express some generators in terms of other generators or their inverses, then we can omit the former generators and still have a set of generators. If no elements of the set of generators may be expressed in terms of the other elements in the set then we have an *independent* set of generators (some literature calls this the *minimal set* of generators). In some cases we may have more than one independent set of generators.

Once we have a set of generators we need to express other group elements in terms of this set. These relations are called the *defining relations* of the set of generators. It is often useful to describe a group simply by giving a set of generators together with the defining relations of these generators.

3.3.3 Direct Products and Semidirect Products

Take two groups, A and B with respective elements a_1, a_2 and b_1, b_2. Consider the set of ordered pairs (a_i, b_i) to be elements of a third group C. The *direct product* of this group is defined as

$$(c_1 c_2) = (g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2). \tag{3.1}$$

This product is unique, it is an element of the group and it is clearly associative, as the products a_1a_2 and b_1b_2 are themselves associative. The neutral element is simply (e_a, e_b) where e_a and e_b are the respective neutral elements of the two groups. The inverse of (a, b) is of course, (a^{-1}, b^{-1}) and so the direct product of two groups does indeed form a group itself. The direct product of groups A and B is usually denoted by

$$C = A \times B. \tag{3.2}$$

The direct product of two groups can be abelian but only if both the groups are themselves abelian. The direct product of abelian groups A and B is usually called the *direct sum* and is denoted by $A \oplus B$. We could denote the direct product of abelian groups either way but it is more convenient to denote it by the direct sum as addition connotates commutativity.

The order of the direct product is simply the (usual) product of the orders of each of the two groups, $|C| = |A||B|$. If one or both of the groups has infinite order, then the direct product also has infinite order.

The *semidirect product* differs from the direct product when the groups A and B do not commute with each other. These groups must commute to form a direct product $A \otimes B$. If they do not, then we can form the semidirect product, $A \otimes_s B$ such that

$$C = A \otimes_s B \tag{3.3}$$

then this implies that

$$\begin{aligned} (c_1c_2) &= (a_1b_1)(a_2b_2) \\ &= a_1(b_1a_2b_1^{-1})b_1b_2 \\ &= (a_1b_1a_2b_1^{-1}, b_1b_2). \end{aligned}$$

3.3.4 Homomorphisms

A *mapping,* θ takes an object a from a set, called the *object space,* A and associates it with an *image,* b in another set, called the *image space,* B. This has the following notation:

$$\theta : A \to B, \ a\theta = b. \tag{3.4}$$

A mapping must always have a unique image for each and every object. However, the reverse is not true: an image may have two objects associated with it. For example, for objects x and images y, $y = x^2$ is a proper mapping as every object has a unique image associated with it. $y^2 = x$ is not a proper mapping as there are two images associated with each object.

A mapping where every image in B has at least one object associated with it is called *onto*. If every image has only one object associated with it, then the mapping is said to be 1-1. If this is the case, then $a_1\theta = a_2\theta \Rightarrow a_1 = a_2$, and if the mapping is onto as well, then an inverse mapping exists, denoted by θ^{-1}.

A group *homomorphism* is a mapping that preserves the group structure. The image of the product of two group elements must be equal to the product of the images of the two elements. So, for elements $g_1, g_2 \in G$, then $\theta : G \to H$ is a homomorphism if

$$(g_1 g_2)\theta = (g_1\theta)(g_2\theta) \tag{3.5}$$

where both sides of this equation are elements in H. Clearly, a homomorphism must map the neutral element of G into the neutral element of H. If other elements of G map into the neutral element of H then the set of all of these objects is called the *kernel* of the homomorphism, and this kernel is a subgroup of G.

Homomorphisms that are onto and 1-1 have their own special names. A homomorphism that is onto is called an *epimorphism*, a homomorphism that is 1-1 is called a *monomorphism*, and a homomorphism that is both 1-1 and onto is called an *isomorphism.* Therefore for a homomorphism to have an inverse homomorphism it must in fact be an isomorphism. Indeed every isomorphism has an inverse homomorphism, which is itself an isomorphism.

Isomorphisms preserve the inverses of elements and the identity of a group. Isomorphisms have their own notation: $G \cong H$ indicates that an isomorphism exists pairing off the elements of G and H. Groups that are isomorphic to each other have the same group structure. This means that for theoretical purposes, they are the same abstract group, and so we can discuss the group structure in terms of either one of the isomorphic groups.

A homomorphism where the image space is identical to the object space (θ :

$A \to A$), so that θ maps A into itself is called an *endomorphism*. An endomorphism that is also an isomorphism is called an *automorphism*, so that an automorphism θ is an isomorphism that maps A into itself.

An example of an isomorphism is the mapping of real integers j into the real even integers, $2j$. The group operation could for instance be addition. This is an isomorphism as every even number has one and only one number associated with it, with respect to the mapping. An example of an automorphism is the mapping of all real integers into their negatives. Again we will take the group operation as addition. This mapping is clearly an isomorphism, and as both the image space and the object space are the integers, it is in fact, an automorphism, albeit a fairly trivial one.

3.3.5 Cosets, Invariant Subgroups and Quotient Groups

If we take H as a subgroup of G with $g \in G$ then the subgroup of G defined by gh for some $h \in H$ is said to be a *left coset* of H. Similarly the subgroup defined by hg is a *right coset* of H. If the order of G is a finite number, then $|G| = a|H|$ where a is the number of cosets in H, called the *index* of H in G. If the following is true

$$gHg^{-1} = H, \ \forall\ g \in G;\ g \notin H \tag{3.6}$$

then H is said to be an *invariant subgroup* of G, otherwise known as a *normal subgroup* of G. In other words, an invariant subgroup of G is a subgroup in which all of the left cosets are right cosets. In particular, all subgroups of an abelian group are invariant.

The set of cosets of the invariant subgroup H of G is called the *Quotient group* of H in G, denoted by G/H. The elements of G/H form a group with H as the identity element, and we have

$$(Ha)(Hb) = H(ab) \tag{3.7}$$

for elements $a, b \in G$. A *simple group* is a group that has no invariant subgroups. A *semi-simple group* is a group that has invariant subgroups that are not abelian.

3.4 Lie Groups

Lie groups were invented by Sophus Lie in the late 19th century, mainly for the purpose of solving differential equations. Since then they have been used in many areas of mathematics and physics, and are of specific importance in particle physics. Lie groups possess the same properties as ordinary groups except they additionally have the property that the group operations are continuous. Alternatively, we could say that they depend on a set of continuous parameters; the number of which is not necessarily related to the dimension of the vector space on which the Lie group operates.

Lie groups have associated with them a Lie Algebra discussed in section 3.5. As Lie groups' operations can be thought of as linear transformations acting on a vector space, we can represent the group by a set of linear matrices homomorphic to the group. The group operation for these sets of matrices is then simply matrix multiplication. There may be more than one appropriate representation for each Lie group. In section 3.6 I discuss some examples of Lie groups, with the groups' corresponding algebra discussed in section 3.7.

3.4.1 Lie Group Generators

The generators of a Lie group can be defined in several different ways. For instance, [9] uses a clever representation of groups and [29] uses one parameter subgroups. These are both standard theory and in this section I will follow the more general convention used in [13].

Suppose we have a set of linear transformations f^i on a space (for instance a vector space) of n variables x^i, $i = 1, 2, ..., n$, such that

$$x'^i = f^i(x^1, x^2, ..., x^n; \alpha^1, \alpha^2, ..., \alpha^r) \tag{3.8}$$

where α^ρ is a set of r independent parameters that is the smallest set needed to completely define the properties of the transformations. We assume the transformations f^i satisfy the necessary conditions to form a group. Writing this more compactly we have

$$x' = f(x;\alpha). \tag{3.9}$$

Corresponding to each transformation we have the inverse transformation defined by

$$x = f(x';\alpha^{-1}) \tag{3.10}$$

and we have the identity transformation

$$x = f^e(x;0) \tag{3.11}$$

where we can arbitrarily choose $\alpha^\rho = 0,\ \forall\rho = 1, 2, ..., r$ in the identity transformation. We can thus define the infinitesimal change in x as

$$dx = \left.\frac{\partial f(x;\alpha)}{\partial\alpha^\sigma}\right|_{\alpha=0} \delta\alpha^\sigma \tag{3.12}$$

and we define $u^i_\sigma(x)$ such that

$$u^i_\sigma(x) \equiv \left.\frac{\partial f^i(x;\alpha)}{\partial\alpha^\sigma}\right|_{\alpha=0} \tag{3.13}$$

then we use this to express the *generators* X_σ of the group as

$$X_\sigma = -iu^i_\sigma(x)\frac{\partial}{\partial x^i}. \tag{3.14}$$

3.5 Lie Algebras

The generators of a Lie group form a closed algebra, called a *Lie algebra* [13]. If we have generators X_a, X_b, and X_c belonging to the Lie algebra of a specific group, then these generators obey the following commutation relations:

$$\begin{aligned} [X_a, X_a] &= 0 \\ [X_a + X_b, X_c] &= [X_a, X_c] + [X_b, Z] \\ [X_a,[X_b,X_c]] + [X_b,[X_c,X_a]] + [X_c,[X_a,X_b]] &= 0 \\ [X_a, X_b] &= -[X_b, X_a] \end{aligned} \tag{3.15}$$

where equation 3.15 is known as the *Jacobi* identity of the Lie group. In fact we can say that any vector space that satisfies the Jacobi identity forms a Lie algebra.

Lie algebras are often more convenient to discuss than their corresponding groups because they form a vector space. Therefore the basis elements of the Lie algebra determine the properties of the algebra and hence the group.

3.5.1 Structure Constants

From Lie's second theorem (see, for example, [13]), we have the following result for the generators of the Lie group (given by equation 3.13)

$$[X_a, X_b] = f^c_{ab} X_c \tag{3.16}$$

where f^c_{ab} are constants known as the *structure constants* of the Lie group. If the structure constants of an algebra vanish then the elements of the algebra commute and hence we have an abelian Lie algebra. Structure constants themselves provide an alternate representation of a Lie group, called the *adjoint* representation. From the Jacobi identity (equation 3.15) we can see that the structure constants obey the following relations: Lie's third theorem,

$$f^e_{ad} f^d_{bc} + f^e_{bd} f^d_{ca} + f^e_{cd} f^d_{ab} = 0 \tag{3.17}$$

3.6 Examples of Lie Groups

- $\mathcal{GL}(n, \mathbb{C})$: The general linear group: the group of all complex non-singular $n \times n$ matrices (the group of all invertible matrices). These matrices perform linear transformations on a vector space X with elements $(x_1, x_2, x_3, ...x_n)$. An important subgroup is the subgroup with unit determinant - $\mathcal{SL}(n, \mathbb{C})$. All of the other groups studied below are subgroups of the general linear group.

- $\mathcal{U}(n)$: The group of all unitary $n \times n$ matrices: the group of all matrices that satisfy $U^\dagger U = I$, where I is the identity matrix. This group leaves the norm $(|x_1|^2 + |x_2|^2 + |x_3|^2 + ... + |x_n|^2)$ invariant. And similarly to $\mathcal{SL}(n, \mathbb{C})$ in $\mathcal{GL}(n, \mathbb{C})$, so is $\mathcal{SU}(n)$ - the group of unitary matrices with unit determinant - an important subgroup of $\mathcal{U}(n)$. We can also have the group $\mathcal{U}(n, m)$: the group of pseudo-unitary matrices which acts on the $(n+m)$ dimensional vector space and leaves $(|x_1|^2+|x_2|^2+|x_3|^2+...+|x_n|^2)-(|y_1|^2+|y_2|^2+|y_3|^2+...+|y_n|^2)$ invariant.

- $\mathcal{O}(n)$: The group of all orthogonal matrices: the group of all matrices A that satisfy $AA^T=I$. This group is the real subgroup of $\mathcal{U}(n)$. We can also have $\mathcal{O}(n,m)$, the real subgroup of $\mathcal{U}(n,m)$. And again we have the group $\mathcal{SO}(n)$ as the subgroup of $\mathcal{O}(n)$ with unit determinant.

- $\mathcal{S}p(2n)$: The group of all symplectic $n\times n$ matrices. This group, introduced by Weyl in 1938 [13] preserves anti-symmetric products. For example, the group $\mathcal{S}p(2,\mathbb{R})$ preserves the quantity $|u|^2-|v|^2$ for spinors $\left(\begin{smallmatrix}u\\v\end{smallmatrix}\right)$. This group is sometimes called $\mathcal{SL}(2,\mathbb{R})$ or $\mathcal{SU}(1,1)$ [13], and it is a simple group.

3.7 Examples of Lie Algebras

The following are the Lie algebras that generate the Lie groups discussed in section 3.6 [9].

- $gl(n)$: All $n\times n$ matrices.

- $u(n)$: All anti-Hermitian $n\times n$ matrices. That is, matrices M that satisfy $M=-M^\dagger$.

- $o(n)$: All anti-symmetric $n\times n$ matrices, sometimes called *skew-symmetric.* These are matrices A that satisfy $A=-A^T$. Note that this requires all diagonal elements to be zero. In terms of the individual elements of matrix A, we have $a_{ij}=-a_{ji}$.

- $sp(2n)$: All matrices A which satisfy $A^TJ+JA=0$ where

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$

with I as the $n\times n$ identity matrix.

3.8 The Lorentz and Poincaré Groups

The Lorentz group is the group of transformations that leaves the line element $\tau^2 = t^2 - \mathbf{x}^2$ invariant. We have for *Lorentz transformations* on (flat) Minkowski space x^μ

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{3.18}$$

where $\Lambda^\mu{}_\nu$ is called a *Lorentz tensor.* We can define this transformation as

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{3.19}$$

and the inverse transformation

$$(\Lambda^\mu{}_\nu)^{-1} \equiv \Lambda_\nu{}^\mu = \frac{\partial x^\mu}{\partial x'^\nu} \tag{3.20}$$

and from this we can see that $\Lambda^\mu{}_\nu$ satisfies

$$\begin{aligned}\Lambda^\mu{}_\sigma \Lambda_\nu{}^\sigma &= \delta^\mu_\nu \\ \delta^\mu_\nu &= \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}\end{aligned}$$

A *Poincaré transformation* is an inhomogeneous Lorentz transformation. This means that it includes translations of a four-vector x^μ:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{3.21}$$

where a^μ is a constant tensor. The set of all Poincaré transformations (inhomogeneous transformations) is known as the *Poincaré group*, while the set of all Lorentz transformations (homogeneous transformations) is known as the *Lorentz group*, and this is clearly a subgroup of the Poincaré group. The theory behind both of these groups is extensive and well covered in the literature (see for example, [13]). Accordingly I will provide only a brief summary of this theory.

As elements of the Lorentz group leave invariant the quantity $\tau^2 = t^2 - x^2 - y^2 - z^2$ the group is often represented by $\mathcal{O}(1,3)$. The group consists of all matrices Λ that satisfy equation 3.18. These matrices have the property that

$$det(\Lambda) = \pm 1$$

and only six independent parameters are needed to define the group properties. Taking $det(\Lambda) = +1$ gives us the group $\mathcal{SO}(1,3)$. An invariant subgroup of the Lorentz group is this group $\mathcal{SO}(1,3)$ with the additional condition that the transformations are *orthochronous transformations*, defined by

$$\Lambda^0{}_0 > 1.$$

This is called the *proper Lorentz group*, and is generated by the Lie algebra of pseudo-anti-symmetric matrices $so(1,3)$.

A subgroup of the proper Lorentz group is $\Lambda_r \cong \mathcal{SO}(3)$. This describes the set of rotations in three-dimensional space. This group is generated by the Lie algebra $so(3)$ which is the set of 3×3 anti-symmetric matrices with unit determinant.

We also want to look at the set of proper Lorentz boosts Λ_b. These matrices are all symmetric, and they also have three independent parameters. However, the set of all matrices Λ_b does not form a subgroup because the product of two boosts does not result in another boost. In other words, if we were to make one rotation followed by a second rotation, the result is another rotation and hence the set of rotations does form a (sub-) group. But if we were to make one boost followed by a second boost then the result is not necessarily a third boost, and so the set of all boosts does not form a subgroup. However, we can write a general Lorentz transformation in terms of a rotation and a boost:

$$\Lambda = \Lambda_r \Lambda_b.$$

We now want to find the group properties in terms of its algebra, by looking at its generators.

3.8.1 Generators of the Proper Lorentz Group

According to [13] we can write Λ_r and Λ_b in terms of the generators L_i and K_i that are the generators for pure rotations and pure boosts respectively. Then, a general Lorentz transformation will be of the form:

$$\Lambda = exp\left[-i\sum_{i=1}^{3}(\theta_i L_i + \eta_i K_i)\right].$$

When thought of as operators acting on Minkowski space, these generators take the infinitesimal form [13]

$$\begin{aligned} L_i &= -i\epsilon_{ijk}x_j\frac{\partial}{\partial x_k} \\ K_i &= -i\left[t\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial t}\right]. \end{aligned}$$

The Lorentz generators also satisfy the following commutation relations [13]

$$\begin{aligned} [L_i, L_j] &= i\epsilon_{ijk}L_k \\ [L_i, K_j] &= i\epsilon_{ijk}K_k \\ [K_i, K_j] &= -i\epsilon_{ijk}L_k. \end{aligned}$$

We can combine these six generators succinctly by using the four-dimensional analogue to the standard three-dimensional angular momentum operators, defined in equation 2.23. Then in position space we have:

$$L_{\mu\nu} = -i\left[x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right]. \tag{3.22}$$

We can now express our previous generators in terms of the angular momentum operators by

$$L_i = \epsilon_{ijk}L_{kj} \tag{3.23}$$

$$K_i = L_{0i}. \tag{3.24}$$

3.8.2 The Group of Translations in Four-Dimensions

According to [18], the Poincaré group can be represented by the semidirect product

$$\mathcal{P} = \mathcal{SO}(1,3) \otimes_s \mathcal{T}(4). \tag{3.25}$$

We have already discussed some properties of the group $\mathcal{SO}(1,3)$, but now we are adding the group of translations in four-dimensions, $\mathcal{T}(4) \cong \mathbb{R}^4$ under addition.

This group is abelian, and its elements T can be written in the matrix form

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.26}$$

The generators of translations are simply the momentum operators; which we recall are given in position space through the correspondence principle as

$$P_\nu = -i\frac{\partial}{\partial x^\nu}. \tag{3.27}$$

3.8.3 Generators of the Poincaré Group

The Lorentz generators are given in equation 3.22. Following the convention used in [13] to write the most general generators of the Lorentz group in the form,

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \tag{3.28}$$

where $L_{\mu\nu}$ operates on coordinate variables and the operators $S_{\mu\nu}$ operate on all other variables not applicable to $L_{\mu\nu}$. The set of generators $\{J_{\mu\nu}, P_\nu\}$ have the following well known defining relations

$$[P_\mu, P_\nu] = 0 \tag{3.29}$$

$$[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \tag{3.30}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\rho\nu} - \eta_{\nu\sigma}J_{\rho\mu}). \tag{3.31}$$

3.9 The Canonical Group

Low [19, 18] defines the *Canonical group* or *Canonical Relativistic group* as the semidirect product

$$\mathcal{C}(1,3) = \mathcal{U}(1,3) \otimes_s \mathcal{H}(1,3). \tag{3.32}$$

We have already seen that $\mathcal{U}(1,3)$ is the pseudo-unitary group of space-time. This is the homogeneous analogue in Reciprocity theory of the homogeneous group

$\mathcal{SO}(1,3)$ of the Poincaré group of special relativity [19]. The group $\mathcal{H}(1,3)$ is the *Weyl-Heisenberg group*, and is the inhomogeneous part of the canonical group, in analogue to the group of translations in the Poincaré group.

The Weyl-Heisenberg group's nine generators $\{X^\mu, P^\mu, I\}$ satisfy the Lie algebra given by the Heisenberg commutation relations of equations 1.9 to 1.12. Low uses the principle of Reciprocity to combine the line elements corresponding to position and momentum space. Recall that the line element of position space is given by equation 1.5:

$$\tau^2 = c^2T^2 - (X_1)^2 - (X_2)^2 - (X_3)^2 \tag{3.33}$$

and the line element of momentum space is

$$m^2 = \frac{E^2}{c^2} - (P_1)^2 - (P_2)^2 - (P_3)^2 \tag{3.34}$$

so then we have Low's reciprocally invariant line element

$$s^2 = \frac{\tau^2}{c^2} + \frac{m^2}{b^2} = T^2 - \frac{\mathbf{X}\cdot\mathbf{X}}{c^2} + \frac{E^2}{(bc)^2} - \frac{\mathbf{P}\cdot\mathbf{P}}{b^2}. \tag{3.35}$$

The Canonical Relativistic group keeps this line element invariant as well as the Heisenberg commutation relations. Additionally it has 25 generators as opposed to the Poincaré algebra that has only 10 generators.

An alternate definition of the Canonical group according to Low [18] is the semidirect product

$$\mathcal{C}(1,3) = \mathcal{SU}(1,3) \otimes_s \mathcal{O}s(1,3). \tag{3.36}$$

In this case, the $\mathcal{O}s(1,3)$ group is the *Oscillator group*. For a detailed discussion of the group theory of the Canonical group, refer to Low's articles [19, 18].

The most interesting aspects of this group are that it fixes several problems of the Poincaré group with respect to experimental evidence. For instance, the Heisenberg commutation relations are not included in the Poincaré algebra and therefore they must be added to the theory. Also, the Canonical group suggests that the individual line elements τ^2 and m^2 need not themselves be invariant; rather the line element suggested by the reciprocity principle: $s^2 = \tau^2 + m^2$ is the element that is kept invariant. This will only be seen in the case where the rates of change of position and momentum approach the limits c and b respectively. When this is *not* the case, the physical space involving eight degrees of freedom actually decomposes into the usual space-time and momentum-energy. This is a similar argument used to explain the space-time of special relativity decomposing into the Newtonian, space degrees of freedom and time degree of freedom, in the non-relativistic case.

3.9.1 Casimir Operators of the Canonical Group

Casimir operators are the maximal set of all operators that commute with the generators of a Lie group. In other words, the Casimir operators are invariant under group transformations [13]. For the Canonical group, Low [18] found two Casimir operators, one of which is the line element that we have said is invariant under reciprocity. However, there was an extra term in this Casimir operator not included in equation 3.35. This term is only important in quantum systems when we take $\hbar \neq 0$. In this case the Hermitian metric is not invariant as the extra term in the line element allows time-like states and null states to transform into each other. This extra term involves the generator of the $\mathcal{U}(1)$ algebra that appears in $\mathcal{O}s(1,3)$; for a discussion of this refer to Low's papers [19, 18].

Chapter 4

Supersymmetry

Section 4.1 extends the position and momentum theory of section 1.2 to anticommuting variables, called *Grassmann variables.* In section 4.2 I introduce the idea of supersymmetry; first by looking at a *Lie superalgebra* and then by giving an example of a Lie superalgebra in section 4.2.2. In section 4.2.3 the idea of using Grassmann variables as coordinates is developed, in an eight-dimensional manifold called *superspace.*

4.1 Grassmann Variables

The arguments in this section follow that of [14], but summarized to stay within the scope of this thesis.

Grassmann coordinates satisfy anticommutation relations that are the analogue to the commutation relations of equations 1.9, 1.10 and 1.11. The general property of Grassmann variables θ^α is

$$\left\{\theta^\alpha, \theta^\beta\right\} = 0 \tag{4.1}$$

and in particular, for $\alpha, \beta = 1, 2$ for example

$$(\theta^1)^2 = (\theta^2)^2 = \left\{\theta^1, \theta^2\right\} = 0. \tag{4.2}$$

Equation 4.1 is true for any even number of Grassmann coordinates, by defining $\alpha, \beta = 1, 2, 3, ..., 2m$.

The metric for these Grassmannian variables is $\epsilon_{\alpha\beta}$, the totally anticommutating tensor, with

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.3)$$

Therefore

$$\theta_1 = \theta^2,\ \theta_2 = -\theta^1 \quad (4.4)$$

$$(\theta)^2 = \theta^\alpha\theta_\alpha = \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta = 2\theta^1\theta^2. \quad (4.5)$$

For four Grassmannian variables, the metric is

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (4.6)$$

and so on for more Grassmannian variables.

4.1.1 Grassmannian Derivatives and Taylor Series'

The derivative for Grassmann variables is simply given by

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta^\beta_\alpha. \quad (4.7)$$

The product rule for differentiation of functions of Grassmann variables differs from the usual product rule in that there is a minus sign in between terms:

$$\frac{d}{d\theta}(AB) = B\frac{dA}{d\theta} - A\frac{dB}{d\theta}. \quad (4.8)$$

Expanding a wavefunction $\psi(\theta)$ in a Grassmannian Taylor series will yield an exact, terminating sequence, as all higher order terms vanish. For two Grassmannian variables this is given by:

$$\psi(\theta) = A + B_1\theta^1 + B_2\theta^2 + C\theta^1\theta^2. \quad (4.9)$$

For $2m$ Grassmannian variables, the Taylor series is given by

$$\psi(\theta) = c_0 + \sum_{p=1}^{2m} c_{i_1...i_p}\theta^{i_1}...\theta^{i_p}. \quad (4.10)$$

4.1.2 Supernumbers

The Grassmann algebra is defined by equation 4.1, with generators θ^α. The elements of this algebra are called *supernumbers*, ζ. We can write the basis elements of the Grassmann algebra as all the possible non-zero combinations (products) of the generators. If we have $\alpha = 1, 2, ..., 2m$ then there are $2m$ generators and 2^{2m} basis elements. These elements are [14]

$$1, \theta^1, ..., \theta^{2m}, \theta^1\theta^2, ..., \theta^{2m-1}\theta^{2m}, ..., \theta^\alpha\theta_\alpha. \tag{4.11}$$

We can see that these basis elements themselves are not necessarily anticommuting quantities. Looking at the commutator of $\theta^1\theta^2$ with θ^3 produces:

$$\begin{aligned}
[\theta^1\theta^2, \theta^3] &= \theta^1\theta^2\theta^3 - \theta^3\theta^1\theta^2 \\
&= \theta^1\theta^2\theta^3 + \theta^1\theta^3\theta^2 \\
&= \theta^1\theta^2\theta^3 - \theta^1\theta^2\theta^3 \\
&= 0
\end{aligned}$$

and so $\theta^1\theta^2$ commutes with θ^3. In fact, $\theta^1\theta^2$ will commute with any basis element of the Grassmann algebra, and hence we label it even. There are other even basis elements as well, and these elements commute with all other elements. The remaining basis elements can be labelled odd, and these elements will anticommute among other odd elements but commute with even elements, as required in a Lie superalgebra; see section 4.2.1.

Any supernumber can be written in terms of an expansion of the basis of the Grassmann algebra [14]:

$$\zeta = c_0 + \sum_{p=1}^{2m} c_{i_1...i_p}\theta^{i_1}...\theta^{i_p}. \tag{4.12}$$

We can then write the elements of the Grassmann algebra as a sum of its odd and even components:

$$\zeta = \zeta_e + \zeta_o \tag{4.13}$$

with

$$\begin{aligned}\zeta_e &= c_o + \sum_{p=1}^{m} c_{i_1 \ldots i_{2p}} \theta^{i_1} \ldots \theta^{i_{2p}} &(4.14)\\ \zeta_o &= \sum_{p=0}^{m-1} c_{i_1 \ldots i_{2p+1}} \theta^{i_i} \ldots \theta^{i_{2p+1}}. &(4.15)\end{aligned}$$

We can see that in the same manner as the basis elements of the Grassmann algebra, the even supernumbers commute with all other supernumbers, and the odd supernumbers anticommute among themselves. This is required for a Lie superalgebra (section 4.2.1).

4.1.3 Grassmann Operators

Express Grassmann coordinate operators as Θ^α, with eigenvalues θ^α when operating on a wavefunction $\psi(\theta)$, ie

$$\Theta^\alpha |\psi\rangle = \theta^\alpha |\psi\rangle \tag{4.16}$$

and define Π_α as the Grassmannian analogue of momentum operators with eigenvalues π_α, ie

$$\Pi_\alpha |\psi\rangle = \pi_\alpha |\psi\rangle, \ \Pi_\alpha \to -i\frac{\partial}{\partial \theta^\alpha}. \tag{4.17}$$

From equation 4.1, these Grassmann coordinate and momentum operators satisfy the following anticommutation relations

$$\begin{aligned}\{\Theta^\alpha, \Theta^\beta\} &= 0 &(4.18)\\ \{\Pi_\alpha, \Pi_\beta\} &= 0 &(4.19)\\ \{\Theta^\alpha, \Pi_\beta\} &= i\delta^\alpha_\beta &(4.20)\end{aligned}$$

where equation 4.8 has been used in equation 4.20. These are the Grassmannian equivalent of the Heisenberg commutation relations.

4.2 Supersymmetry

Supersymmetry is an extension of the Standard Model of particle physics that includes transformations of fermions into bosons and vice-versa. So far there is no experimental evidence that supersymmetry is present in nature; however the maths of the theory is very elegant and many physicists believe it is only a matter of time before experimentalists confirm its existence.

The Standard Model of particle physics involves unitary irreducible representations of the Poincaré group [18]. The supersymmetry algebra is a $\mathbb{Z}_2$ *graded* extension of the Poincaré algebra. The Poincaré group is discussed in section 3.8 and the meaning of the word graded is explained in section 4.2.1. We label the extra generators of graded Lie algebras by Q_α^n where $n = 1, 2, ..., N$. The simplest version of supersymmetry has $N = 1$, and from now on we will take $N = 1$. Supersymmetry also seeks to include gravity in a local version of supersymmetry called *supergravity*.

Particles predicted by the Standard Model and observed experimentally will all have a *superpartner* under the supersymmetry theory. These pairs of particles will have the same quantum numbers as each other but they will differ in spin by $\frac{1}{2}$. For example, the photon with spin 1 would have a supersymmetric partner called the *photino* with spin $\frac{1}{2}$. None of these superpartners have been found experimentally and it is hypothesized that the reason for this is because the masses of the superpartners are too large for modern particle accelerators to detect. Indeed, there exist theories that predict superparticles contribute to the dark matter of the galaxy.

4.2.1 Lie Superalgebras

A *Lie superalgebra* L, which is a type of *graded Lie algebra*, is an algebra that includes the idea of odd and even elements. This graded lie algebra can be written as the sum of its even and odd elements:

$$L = L_e + L_o \tag{4.21}$$

where L_e is the set of even elements in L and L_o is the set of odd elements in L. Odd elements anticommute among themselves, but commute with even elements. Even elements commute with all other elements. So we have the defining relations of the superalgebra:

$$\begin{aligned} [A_i, A_j] &= A_i A_j - A_j A_i \\ [A_i, B_j] &= A_i B_j - B_j A_i \\ [B_i, B_j] &= B_i B_j + B_j B_i, \end{aligned}$$

where the elements $A_i \in L_e$ and $B_i \in L_o$. Introducing the function [9]

$$\sigma(X) = \begin{cases} 0 & X \in L_e \\ 1 & X \in L_o \end{cases} \tag{4.22}$$

for any $X \in L$ allows us to combine the defining relations into the single expression

$$[X, Y] = XY - (-1)^{\sigma(X)\sigma(Y)} YX. \tag{4.23}$$

Note that this will be a commutator unless both X and Y are odd, in which case it will be an anti-commutator, as desired. We also have the following properties:

$$\sigma([X, Y]) = |\sigma(X) - \sigma(Y)| \tag{4.24}$$

so that a Lie bracket is odd unless both of the elements are even or both of the elements are odd, and

$$[X, [Y, Z]] + [Z, [X, Y]] + (-1)^{\sigma(X)\sigma(Y)} [Y, [Z, X]] = 0 \tag{4.25}$$

which is the Jacobi identity for the superalgebra (see equation 3.15). This grading structure is called a $\mathbb{Z}_2$ grading structure [31].

4.2.2 The Super-Poincaré Algebra

The literature varies with regard to how to write the generators and relations of the super-Poincaré group. I will follow the method of [31]. In addition to the usual generators of the Poincaré group, we add the new generators Q_α that satisfy anticommutation relations, we add a generator for internal symmetry, S, and we also take Q_α to be Majorana spinors ($Q_\alpha = C_{\alpha\beta}\overline{Q}^\beta$, where C is the *charge conjugation matrix* defined in equation 4.30). Thus we have the set of generators for the super-Poincaré group as $\{P_\nu, J_{\mu\nu}, S, Q_\alpha\}$. Now, these generators obey the following defining relations in addition to the defining relations of the Poincaré group, listed in section 3.8.3, for the $N = 1$ supersymmetry:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma_\mu C)_{\alpha\beta} P^\mu \tag{4.26}$$

$$[Q_\alpha, P_\mu] = 0 \tag{4.27}$$

$$[Q_\alpha, J_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha^{\ \beta} Q_\beta \tag{4.28}$$

$$[Q_\alpha, S] = i(\gamma_5)_\alpha^{\ \beta} Q_\beta. \tag{4.29}$$

These relations involve the Dirac matrices, the Pauli spin matrices and the charge conjugation matrix. The Pauli spin matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can define the Dirac matrices in terms of the Pauli spin matrices. Note that the method I use here (from [31]) is only one representation of the relevant algebra. There are many ways of representing this algebra. The method used here is but one. And so, we have:

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where all of the elements in the above matrix are 2×2 matrices. We can now write the charge conjugation matrix as,

$$C = i\gamma^0\gamma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.30}$$

4.2.3 Superspace and Superfields

Superspace is obtained by adding four fermionic (Grassmann) coordinates to the usual coordinates of space-time. Hence we have eight dimensions, and we take our coordinates to be

$$z^M = (x^\mu, \theta^\alpha), \tag{4.31}$$

with the index M running over the bosonic indices μ and the fermionic indices α [28]. Srivastava [28] (and many others) uses for his Grassmann coordinates, *dotted and undotted spinors* [25]: Weyl spinors $\theta^\alpha = \binom{\theta^1}{\theta^2}$ and $\overline{\theta}_{\dot\alpha} = \binom{\overline{\theta}_{\dot 1}}{\overline{\theta}_{\dot 2}}$. In my notation I have used $\alpha = 1, 2, 3, 4$ to produce the four Grassmann variables. These superspace coordinates obey relations of the form in equation 4.23:

$$z^M z^N = (-1)^{\sigma(z^M)\sigma(z^N)} z^N z^M \tag{4.32}$$

and we can then define transformations in superspace to be *supertranslations* (see [28, 25].

We obtain a *superfield* by making a Taylor expansion of $F(z) \equiv F(x^\mu, \theta^\alpha)$ [28] and then requiring the coefficients of the Taylor expansion to be fields rather than numbers. We then have a superfield as a combination or *multiplet* of fields [14]. Ryder [25] classifies each field as a scalar or a spinor (bosonic or fermionic). If we have the superfield [14]

$$F(x,\theta) = F_0(x) + \sum_{p=1}^{n} F_{i_1 \ldots i_p} \theta^{i_1} \ldots \theta^{i_p} \tag{4.33}$$

then we have the fields

$$
\begin{aligned}
F_0(x) &\rightarrow 1 \text{ scalar} \\
F_\alpha(x) &\rightarrow 4 \text{ spinors} \\
F_{\alpha\beta}(x)\ (\alpha \neq \beta) &\rightarrow 6 \text{ scalars} \\
F_{\alpha\beta\gamma}(x)\ (\alpha \neq \beta \neq \gamma \neq \alpha) &\rightarrow 4 \text{ spinors} \\
F_{1234} &\rightarrow 1 \text{ scalar.}
\end{aligned}
$$

This means there are eight bosonic and eight fermionic fields that make up the multiplet. In other words, there are eight bosonic and eight fermionic degrees of freedom [25].

Chapter 5

Supersymmetric Reciprocity

This chapter continues on from the work done by Born, outlined in chapter 2. First we modify the metric operator (equation 2.14), by adding Grassmann variables

$$S(X, P, \Theta, \Pi) = X^{\mu}X_{\mu} + P^{\mu}P_{\mu} + \Theta^{\alpha}\Theta_{\alpha} + \Pi^{\alpha}\Pi_{\alpha}. \tag{5.1}$$

Then proceed in much the same way as in Born's theory. In equation 5.1 I have purposely not defined the values for α. This is because α depends on how many Grassmann variables we want to include in the operator. As a mathematical exercise, we can use any even number of Grassmann variables. I used two Grassmann variables in section 5.1 and four Grassmann variables in section 5.2.

5.1 Reciprocity in two Grassmannian Variables

In this section, letters from the beginning of the Greek alphabet (namely α, β) will run through the numbers 1,2; and as always, letters from the middle of the Greek alphabet (μ, ν) will run through the numbers 0,1,2,3. The metric operator becomes

$$\begin{aligned} S(X, P, \Theta, \Pi) &= X^{\mu}X_{\mu} + P^{\mu}P_{\mu} + \Theta^{\alpha}\Theta_{\alpha} + \Pi^{\alpha}\Pi_{\alpha} && (5.2) \\ &= X^{\mu}X_{\mu} + P^{\mu}P_{\mu} + 2\Theta^{1}\Theta^{2} + 2\Pi_{1}\Pi_{2} && (5.3) \\ S(x, \theta) &= x^{\mu}x_{\mu} - \frac{\partial^2}{\partial x_{\mu}\partial x^{\mu}} + 2\theta^{1}\theta^{2} - 2\frac{\partial^2}{\partial\theta^{1}\partial\theta^{2}} && (5.4) \end{aligned}$$

after the substitutions $p_{\mu} \to -i\frac{\partial}{\partial x^{\mu}}$, $\pi_{\alpha} \to -i\frac{\partial}{\partial\theta^{\alpha}}$ have been made.

We want to find the family of self-reciprocal functions that arises from this reciprocal invariant. Equation 2.3 becomes

$$S(x,\theta)F(x,\theta) = sF(x,\theta) \tag{5.5}$$

We can expand the self-reciprocal function $F(x,\theta)$ in a Grassmannian Taylor series, of two Grassmann variables (see equation 4.9) recalling that this is in fact an exact, terminating series.

$$F(x,\theta) = A + B_1\theta^1 + B_2\theta^2 + C\theta^1\theta^2 \tag{5.6}$$

where A, B_α and C are all functions of x. Notice that when the reciprocal invariant defined by equation 5.4 acts on the function given in equation 5.6, some interesting things happen. It leads to the following coupled equations:

$$(-\Box + x^\mu x_\mu)A + 2C = sA \tag{5.7}$$

$$(-\Box + x^\mu x_\mu)B = sB \tag{5.8}$$

$$(-\Box + x^\mu x_\mu)C + 2A = sC. \tag{5.9}$$

We can see that equation 5.8 is exactly the same as Born's differential equation (equation 2.15) and hence it can be solved in exactly the same manner. This is not true for equations 5.7 and 5.9. Adding these two equations yields

$$(-\Box + x^\mu x_\mu)(A+C) = (s-2)(A+C) \tag{5.10}$$

and then subtracting equation 5.9 from equation 5.7 yields

$$(-\Box + x^\mu x_\mu)(A-C) = (s+2)(A-C). \tag{5.11}$$

Note that equations 5.10 and 5.11 are similar to equation 2.15; they differ only in the eigenvalue. So we have the solution:

$$F_k = M^{\frac{k}{2}} e^{-\frac{M}{2}} L_n^{(k+1)}(M) \tag{5.12}$$

and three sets of values for the quantum number s, given by

$$A + C: \quad s = 2(2n-k) + 2 \tag{5.13}$$

$$B: \quad s = 2(2n-k) \tag{5.14}$$

$$A - C: \quad s = 2(2n-k) - 2. \tag{5.15}$$

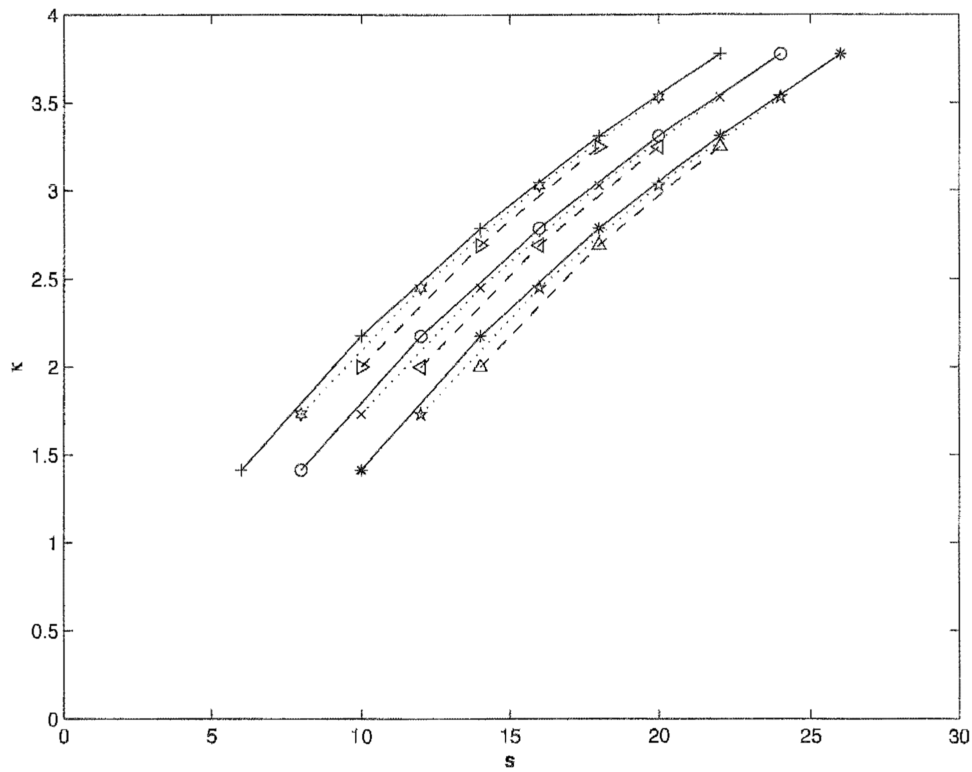

Figure 5.1: Theoretical mass spectrum with two Grassmann coordinates

As the solution corresponding to equation 5.13 comes from odd elements of the Grassmann Taylor series given in equation 5.6 we identify this set of solutions as belonging to a fermionic family. Similarly, both equation 5.13 and equation 5.15 involve even elements of the Grassmann Taylor series, so we identify this set of solutions as belonging to two bosonic families.

Figure 5.1 is a plot of the theoretical mass spectrum corresponding to two Grassmann coordinates. This is a supersymmetric version of Born's spectrum, given in figure 2.1. However, where multiple solutions of the Laguerre polynomial exist I have only taken the greatest solution. Then we have the unbroken lines corresponding to $k = 0$, the dotted lines correspond to $k = 1$ and the dashed lines correspond to $k = 2$. We can see that there are indeed three families of the same spectrum; Born's original spectrum has been shifted two units of s to the left and to the right of the original spectrum.

5.2 Reciprocity in four Grassmannian Variables

Now we take $\alpha, \beta = 1, 2, 3, 4$, so that there are four Grassmann position and four Grassmann momentum operators. Then the metric operator becomes

$$\begin{aligned} S(X, P, \Theta, \Pi) &= X^2 + P^2 + 2\Theta^1\Theta^2 + 2\Theta^3\Theta^4 + 2\Pi^1\Pi^2 + 2\Pi^3\Pi^4 && (5.16) \\ S(x, \theta) &= x^2 - \Box + 2\theta^1\theta^2 + 2\theta^3\theta^4 - 2\frac{\partial^2}{\partial\theta^1\partial\theta^2} - 2\frac{\partial^2}{\partial\theta^3\partial\theta^4}. && (5.17) \end{aligned}$$

Again we want to apply this to equation 5.5, so again, expand the self-reciprocal function in a Grassmannian Taylor series, this time of four variables:

$$\begin{aligned} F(x, \theta) \;=\; & c_0 + c_1\theta^1 + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + c_5\theta^1\theta^2 + c_6\theta^3\theta^4 + c_7\theta^1\theta^3 \\ & + c_8\theta^1\theta^4 + c_9\theta^2\theta^3 + c_{10}\theta^2\theta^4 + c_{11}\theta^2\theta^3\theta^4 + c_{12}\theta^1\theta^3\theta^4 \\ & + c_{13}\theta^1\theta^2\theta^4 + c_{14}\theta^1\theta^2\theta^3 + c_{15}\theta^1\theta^2\theta^3\theta^4. \end{aligned}$$

This is equivalent to the expansion of equation 4.33 but written out explicitly in terms of the coefficients (fields) $c_n(x)$, to clarify the following argument.

From equation 5.5 this leads to a set of twelve coupled equations in coefficients of powers of θ:

$$\begin{aligned} (-\Box + x^2 - s)c_0 + 2c_5 + 2c_6 &= 0 \\ (-\Box + x^2 - s)c_1 + c_{12} &= 0 \\ (-\Box + x^2 - s)c_2 + c_{11} &= 0 \\ (-\Box + x^2 - s)c_3 + c_{14} &= 0 \\ (-\Box + x^2 - s)c_4 + c_{13} &= 0 \\ (-\Box + x^2 - s)c_5 + 2c_{15} + 2c_0 &= 0 \\ (-\Box + x^2 - s)c_6 + 2c_{15} + 2c_0 &= 0 \\ (-\Box + x^2 - s)c_{11} + 2c_2 &= 0 \\ (-\Box + x^2 - s)c_{12} + 2c_1 &= 0 \\ (-\Box + x^2 - s)c_{13} + 2c_4 &= 0 \\ (-\Box + x^2 - s)c_{14} + 2c_3 &= 0 \\ (-\Box + x^2 - s)c_{15} + 2c_5 + 2c_6 &= 0. \end{aligned}$$

We want to add and subtract these equations as in section 5.1 to find equations similar to those of equations 5.10 and 5.11. The result is

$$\begin{aligned}
(-\Box + x^2)(c_0 + c_{15} + c_5 + c_6) &= (s-4)(c_0 + c_{15} + c_5 + c_6) \\
(-\Box + x^2)(c_1 + c_{12}) &= (s-2)(c_1 + c_{12}) \\
(-\Box + x^2)(c_2 + c_{11}) &= (s-2)(c_2 + c_{11}) \\
(-\Box + x^2)(c_3 + c_{14}) &= (s-2)(c_3 + c_{14}) \\
(-\Box + x^2)(c_4 + c_{13}) &= (s-2)(c_4 + c_{13}) \\
(-\Box + x^2)(c_0 - c_{15}) &= s(c_0 - c_{15}) \\
(-\Box + x^2)(c_1 - c_{12}) &= (s+2)(c_1 - c_{12}) \\
(-\Box + x^2)(c_2 - c_{11}) &= (s+2)(c_1 - c_{11}) \\
(-\Box + x^2)(c_3 - c_{14}) &= (s+2)(c_1 - c_{14}) \\
(-\Box + x^2)(c_4 - c_{13}) &= (s+2)(c_1 - c_{13}) \\
(-\Box + x^2)(c_0 + c_{15} - c_5 - c_6) &= (s+4)(c_0 + c_{15} - c_5 - c_6).
\end{aligned}$$

The operations which were used on equations 5.18 to 5.18 to produce these new eigenvalue equations should be apparent. Notice that the only way in which the operators of these equations differ from Born's field equation (if they differ at all) is in the presence of a number ± 2 or ± 4. As these are just a number they commute with the eigenvector and hence they change the value of the eigenvalue. Instead of Born's value for s, we now have five families: three bosonic type families and two fermionic type families. These families differ from one another in their value of s:

$$\begin{aligned}
\text{Bosonic:} \quad & s = 2(2n-k) + 4 \\
\text{Fermionic:} \quad & s = 2(2n-k) + 2 \\
\text{Bosonic:} \quad & s = 2(2n-k) \\
\text{Fermionic:} \quad & s = 2(2n-k) - 2 \\
\text{Bosonic:} \quad & s = 2(2n-k) - 4,
\end{aligned}$$

where I have labelled families according to which types of field they involve.

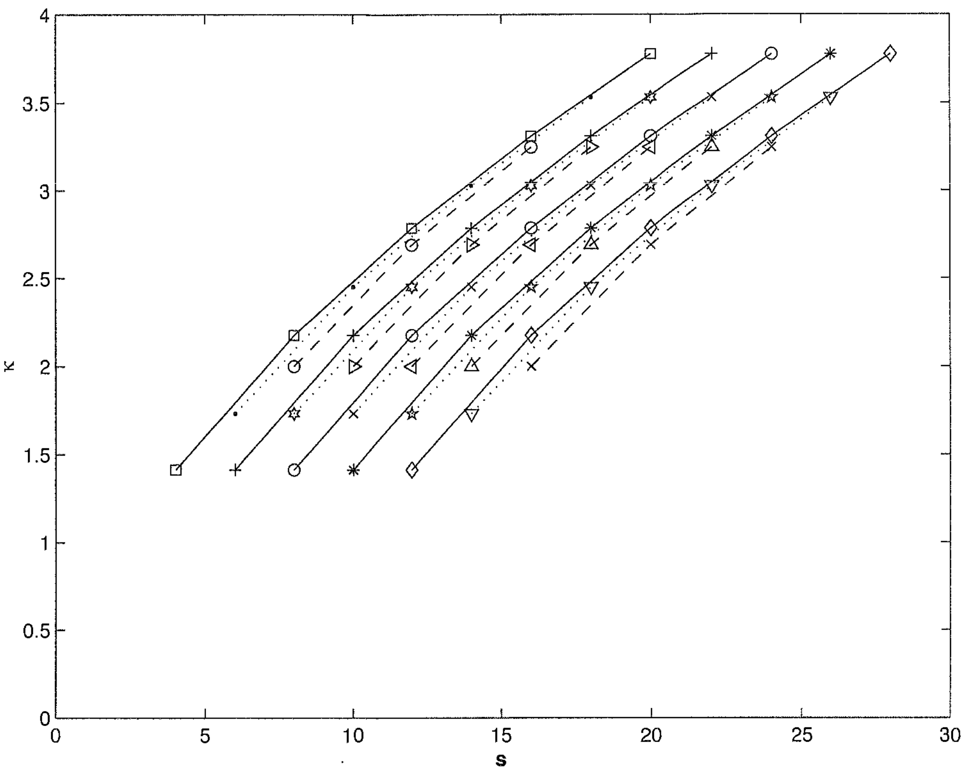

Figure 5.2: Theoretical mass spectrum with four Grassmann coordinates

Bosonic families only involve scalar fields, while fermionic families only involve spinor fields, as per section 4.2.3. It is interesting to note that the case $s-2(2n-k)$ now corresponds to a bosonic type family, whereas in the previous case with two Grassmann coordinates, the $s = 2(2n - k)$ case corresponded to a fermionic type family. We can see these five families in figure 5.2. Again the unbroken lines correspond to $k = 0$, the dotted lines correspond to $k = 1$ and the dashed lines correspond to $k = 2$.

Although I have not proved this to be true, we can now generalize from our position to say that if we add $2m$ Grassmann coordinates to the usual space-time metric operator then the result is the production of an extra $2m$ families of fermions and bosons. This will result in us having $(m + 1)$ families of bosons and m families of fermions. Each family has the same values of κ_i; the theoretical masses of an infinite number of elementary particles. However, each family has a different value of s corresponding to each theoretical mass. The significance of this is yet to be determined.

5.3 Normalizing the Wave Function

We want to find $\psi_k(M)$ such that

$$\langle \psi_k | \psi_{k'} \rangle = \delta^k_{k'} \tag{5.18}$$

where $\psi_k(M)$ is the normalized solution of the field equation, equation 2.3. So we need to find the value of $\langle F_k | F_k \rangle$, where $F_k(M)$ is given in equation 2.32. Then

$$\psi_k(M) = \sqrt{\frac{F_k(M)}{\langle F_{k'} | F_{k'} \rangle}}. \tag{5.19}$$

Expanding this expression yields

$$\langle F_k | F_k \rangle = \langle M^{k/2} e^{-M/2} L_n^{(k+1)}(M) | M^{k/2} e^{-M/2} L_n^{(k+1)}(M) \rangle \,. \tag{5.20}$$

We are only interested in the case $M \equiv P^\mu P_\mu \geq 0$ so that this expression is purely real, with commuting terms. So switching back to position space, we can drop the Dirac notation and write

$$\begin{aligned} \langle F_k | F_k \rangle &\equiv I & (5.21)\\ I &= \iiiint_{-\infty}^{\infty} R^k e^{-R} (L_n^{(k+1)}(R))^2 d^4x & (5.22)\\ &= \iiiint_{-\infty}^{\infty} R^k e^{-R} (L_{n-(k+1)}^{k+1}(R))^2 d^4x. & (5.23) \end{aligned}$$

The notation used in equation 5.23, $(L_{n-(k+1)}^{k+1}(R))$ is actually that of an *associated* Laguerre polynomial, as opposed to the derivative of the ordinary Laguerre polynomials; $L_n^{(k+1)}(R)$. We have the four-dimensional volume element defined in equation 2.20, so we can define $\beta \equiv n - (k+1)$ and write

$$\begin{aligned} I &= \int_0^\pi \sin^2\omega d\omega \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty r^3 R^k e^{-R} (L_\beta^{k+1}(R))^2 dr & (5.24)\\ &= \left[\frac{1}{2}\omega - \frac{1}{4}\sin 2\omega\right]_{\omega=0}^{\pi} (2)\,(2\pi) \int_0^\infty \frac{1}{2} R^{k+1} e^{-R} (L_\beta^{k+1}(R))^2 dR & (5.25)\\ &= \left(\frac{\pi}{2}\right)(4\pi)\left(\frac{1}{2}\frac{\Gamma(n+1)}{\beta!}\right) & (5.26)\\ &= \pi^2 \frac{\Gamma(n+1)}{\Gamma(n-k)} & (5.27) \end{aligned}$$

and so we have the normalized wave function (in momentum space)

$$\psi_k(M) = \frac{M^{k/2}e^{-M/2}L_n^{(k+1)}(M)\sqrt{\Gamma(n-k)}}{\sqrt{\pi^2\Gamma(n+1)}} \tag{5.28}$$

$$= F_k(M)\sqrt{\frac{\Gamma(n-k)}{\pi^2\Gamma(n+1)}}. \tag{5.29}$$

Chapter 6

Discussion

Now that we have the mass spectra given by figures 2.1, 5.1 and 5.2 there are several questions we need answered. First of all, what is the mass scale pertinent to these spectra? Or to put this more quantitatively, what is the value of the constant a defined by Born in [3]? Also, as in figures 5.1 and 5.2 each theoretical mass corresponds to three or five values respectively of the eigenvalue s of equation 2.3, what is the significance of s? And what is the significance of having each theoretical mass corresponding to more than one value of s? Also, why does there appear to be an infinite number of particles? In this chapter I will attempt to answer these questions.

To find the value of a we only need to identify one value of κ with the actual known mass of a particle. Once we have found the value of a we can use this to match up every other theoretical mass with the actual mass of a known particle.

The analysis Born and I have used to obtain the mass spectra studied in this thesis has been restricted to integral spin particles (bosons). Therefore we could look at the spectra as if it belonged to gauge bosons, or to mesons. First we need to define a scale by matching up a theoretical mass with the actual mass of a corresponding particle, then match up other particles of similar mass and look for patterns.

Matching the lightest meson, the pion π^0 to the point $(n, k) = (2, 0)$, $\kappa = 1.414214$ provides us with a scale we can use to match up the masses of other mesons with our theoretical masses. In this case, $\frac{\mu}{\kappa} \approx 0.010476$. We can then determine the theoretical masses of other mesons, and try to match them up with the spectra, obtaining

$$
\begin{aligned}
\pi^0 &\rightarrow (2,0), & \kappa &= 1.414214 \\
\mathrm{K}^+ &\rightarrow (10,0), & \kappa &= 5.30267 \\
\eta^0 &\rightarrow (11,0), & \kappa &= 5.62875 \\
\rho^0 &\rightarrow (20,0), & \kappa &= 8.0405 \\
\omega^0 &\rightarrow (21,0), & \kappa &= 8.26904 \\
\eta^{0'} &\rightarrow (30,0), & \kappa &= 10.1121 \\
\phi &\rightarrow (33,0), & \kappa &= 10.6604
\end{aligned}
$$

where the parentheses indicate points belonging to (n, k) on the spectra corresponding to the largest value of κ pertaining to those values of n and k. There is no distinct pattern in the values of κ that seem to correspond to actual particles and those that do not. It should be noted that the matchings are all very approximate, and I have by no means proved this to be an accurate fit of the spectra. Indeed, there seems to be no reason why so many of the theoretical values of κ should not have an actual mass partner. Therefore for the theory to fit the experimental evidence, there must be some way of building into the theory a reason why only some of the data points correspond to a physical particle. This would also have to explain why there are an infinite number of theoretical masses; while as far as we can determine there are only a finite number of elementary particles.

In Born's original theory, we had $s = 2(2n - k)$, $n \geq k + 1$. Therefore s is clearly always positive. Also, for fixed values of k, s is proportional to n. In other words, s and n have the same significance for fixed values of k. After adding in the supersymmetric arguments we now have (for four Grassmann variables), the same theoretical mass arising for five different values of s. So if this theory is an accurate portrayal of nature, s would need to have some physical interpretation. For example, it might play the role of a quantum number, perhaps the colour quantum number. As yet, there is not enough information to determine the significance of s.

Assuming the model detailed above, with the point $(n,k)=(2,0)$ belonging to the pion, π^0, we can evaluate Born's minimum length constant, a. Taking the mass of the pion to be 135 MeV [1],

$$a=\frac{\kappa\hbar}{\mu c}.$$

From here it is a simple task to calculate the value of a as:

$$a\approx 2.07\times 10^{-15}\text{m}. \tag{6.1}$$

This would put a as the same order as the classical electron radius: the *Compton radius* $r_o = 2.82\times 10^{-15}\text{m}$[1]. This is very interesting, and so far the only real evidence in this thesis that might support Born's theory.

We can now evaluate Low's maximum rate of change of momentum constant, b:

$$\begin{aligned} b &= \frac{\hbar c}{a^2} \\ &\approx 7.37\times 10^{3}\text{N}. \end{aligned}$$

If we were to take a to be exactly the Compton radius, then we would evaluate b as

$$b=3.98\times 10^{3}\text{N}. \tag{6.2}$$

This intuitively seems to be several orders of magnitude too small. However, it does give us a rough guide as to the scale we should be using. Therefore, instead of matching the pion to the point $(n,k)=(2,0)$, matching the much heavier, W boson to this point produces a smaller value for a, and a larger value for b:

$$\begin{aligned} a &\approx 3.48\times 10^{-18}\text{m} \\ b &\approx 2.61\times 10^{9}\text{N}. \end{aligned}$$

This is a much more sensible value for a maximum rate of change of momentum. Also, we could naively say that gauge bosons and photons are in the same class of particle (force carriers), therefore it makes sense that the mass spectrum would belong to this single class of particles.

[1]from http://scienceworld.wolfram.com/physics/PhysicalConstants.html

We can now try to match up the Z boson to one of the theoretical masses, obtaining $\kappa_Z \approx 1.604$. Even though this corresponds quite well with the second smallest root of $(n, k) = (5, 0)$, it seems as if we could almost take the rest mass of any particle and try and argue why it should be matched to a particular theoretical mass. If it is the case that the W boson corresponds to the point (2,0), and hence the Z boson corresponds to the second smallest root of (5,0), then what is so significant about these points? Why are so many theoretical masses unmatched? This theory is unable to answer these questions. Therefore, while it may be possible that the previous arguments are correct (or at least have some merit), more research needs to be done before any definite conclusions can be drawn.

Chapter 7

Conclusion

Born's theory of reciprocity is a very interesting idea, but as yet has not much experimental evidence to support its validity. This is also true of supersymmetry. The aim of this thesis was to apply supersymmetric principles to Born's theory to try to correct the inconsistencies arising from the theory. Born's metric operator was first used to reproduce Born's theoretical mass scale for bosons, but then the metric operator was extended to become the supermetric operator by way of introducing Grassmann position and momentum variables into the operator. This new supermetric operator was then used to determine a supersymmetric version of Born's theoretical mass scale; first for two Grassmann variables and then for four. Families of bosonic and fermionic theoretical masses were produced; where in general, for an addition of $2m$ Grassmann variables, we obtained $m+1$ bosonic type families and m fermionic type families to make a total of $2m+1$ families. Each family had the same theoretical mass scale, but the eigenvalues of the self-reciprocal equation (equation 2.3) were modified so that each family was translated by an even integral number of units of s, with the last family producing a spectrum with the same eigenvalue to Born.

This result was no help in trying to fit Born's theory to the experimental data. However, a very qualitative discussion of Born's spectrum was undertaken, involving the lightest meson - the pion - as the basic guide to determine the scale factor between the theoretical mass spectrum and the actual mass spectrum. According to this scale, Born's minimum length constant, a was thus calculated to be approximately:

$$a \approx 2.07 \times 10^{-15}\text{m},$$

leading to the approximate evaluation of Low's maximum rate of change constant, b:

$$b \approx 7.37 \times 10^{3}\text{N}.$$

While the value of a seems at first glance to have physical meaning as it is of the same order as the classical electron radius, the value of b is several orders of magnitude too small. This in turn means a is too large to be taken very seriously. Also, only a finite value of theoretical masses were able to be matched up with physical particles. The theory was unable to account for the unmatched theoretical particles, nor was it able to explain the existence of an infinite number of theoretical particles.

An alternative scale was considered, matching up the theoretical mass at the point $(n, k) = (2, 0)$ to the W boson. This yielded much more realistic values for a and b:

$$\begin{aligned} a &\approx 3.48 \times 10^{-18}\text{m} \\ b &\approx 2.61 \times 10^{9}\text{N}. \end{aligned}$$

Using this scale, the Z boson was then matched to the second smallest root of (5,0). Again, this scale skipped several theoretical mass values, and we still cannot explain the supposed existence of an infinite number of theoretical particles.

The most interesting aspect of this thesis is not its conclusions but the questions that arise from it. There are several avenues of research able to be undertaken to extend this theory. First of all, a supersymmetric generalization of Born's theoretical mass scale pertaining to fermions could be performed, but it is the author's suggestion that this would run into similar problems to that of this thesis.

A more interesting and possibly more relevant direction for further research would be to study the self-reciprocal functions belonging to other reciprocal invariants. Born and myself have used only the simplest non-trivial reciprocal invariant (being the metric operator), but there is no reason why we cannot use any quantity that is reciprocally invariant. One such example is the hyper-Coulomb potential:

$$S(X, P) = \frac{1}{|X|} + \frac{1}{|P|}.$$

There is also the possibility of extending Low's theory. For example, a canonical superalgebra could be developed. Also, it would be worthwhile examining the value of b obtained in this thesis, and ascertaining its validity. If it was valid then there exists even more possibilities for research in this area.

References

[1] J. Bartels, D. Haidt, and A. Zichichi, editors. *The European Physical Journal C*. Springer Verlag, 2000.

[2] H Bateman. *Higher Transcendental Functions*, volume II. McGraw-Hill Book Company, 1953. Compiled by the Staff of the Bateman Manuscript Project.

[3] M. Born. Reciprocity theory of elementray particles. *Reviews of Modern Physics*, 21(3):463–473, July 1949.

[4] B.H. Bransden and C.J. Joachain. *Quantum Mechanics*. Pearson Education, 2000.

[5] E. R. Caianello. Is there a maximal acceleration? *Lettere al Nuovo Cimento*, 32:65–70, 1981.

[6] M. Carmeli. *Group Theory and General Relativity*. McGraw-Hill, 1977.

[7] M. Carmeli, E. Leibowitz, and N. Nissani. *Gravitation: SL(2,C) Gauge Theory and Conservation Laws*. World Scientific Publishing, 1990.

[8] S. Ferrara. *Supersymmetry*, volume 1. World Scientific Publishing and Elsevier Science Publishers, 1987.

[9] K.S. Fienberg. Schizosymmetry: An alternate approach to superfield expansions. Master's thesis, University of Tasmania, 1998.

[10] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals Series and Products*. Academic Press, 1965.

[11] F.M. Hall. *An Introduction to Abstract Algebra*, volume 2. Cambridge University Press, 1969.

[12] F.M. Hall. *An Introduction to Abstract Algebra*, volume 1. Cambridge University Press, 1972.

[13] Y. S. Kim and M. E. Noz. *Theory and Applications of the Poincaré Group.* D. Reidel Publishing Company, 1986.

[14] J.S. Kosasih. Grassmann variables in particle physics. Master's thesis, University of Tasmania, 1994.

[15] A. Lande. Sommerfeld's fine structure constant and born's reciprocity. *Physical Review*, 56:482–483, September 1939.

[16] A. Lande. The structure of electric particles and the number 137. *Physical Review*, 56:486, September 1939.

[17] R. Lidl and G. Pilz. *Applied Abstract Algebra.* Springer-Verlag, 1984.

[18] S. G. Low. Representations of the canonical group, (the semidirect product of the unitary and weyl-heisneberg groups), acting as a dynamical group on noncommutative extended phase space. *J. Phys. A*, 35:5711–5729, 2002.

[19] S.G. Low. Canonically relativistic quantum mechanics: Representations of the unitarty semidirect heisenberg group, $U(1,3) \otimes_s H(1,3)$. *J. Math. Phys.*, 38:2197–2209, May 1997.

[20] A. V. Mikhalev and G. F. Pilz. *The Concise Handbook of Algebra.* Kluhwer Academic Publishers, 2002.

[21] W. Miller. *Symmetry Groups and Their Applications.* Academic Press, 1972.

[22] W. Penrose, R. Rindler. *Spinors and Space-Time.* Cambridge University Press, 1984.

[23] P. Ramond. *Field Theory A Modern Primer.* The Benjamin/Cummings Publishing Company, Inc., 1943.

[24] L.S. Ryder. *Elementray Particles and Symmetries.* Gordon and Breach, Science Publishers, 1986.

[25] L.S. Ryder. *Quantum Field Theory.* Cambridge University Press, second edition, 1996.

[26] A. Salam and J. Strathdee. Super-gauge transformations. *Nuclear Physics B*, 76:477–482, 1974.

[27] D.H. Sattinger and O.L. Weaver. *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics.* Springer-Verlag, 1986.

[28] P. P. Srivastava. *Supersymmetry, Superfields and Supergravity: an Introduction.* IOP Publishing, 1986.

[29] J. Sumner. Lie symmetries and applications.

[30] J. Wess and B. Zumino. Supergauge transformations in four dimensions. *Nuclear Physics B*, 70:39–50, 1974.

[31] P. West. *Introduction to Supersymmetry and Supergravity.* World Scientific Publishing, 1986.