RIGID AND GENERIC STRUCTURES

by

Garn Cooper B.Sc.(Hons)

Submitted in fulfilment of the requirements
for the degree of Master of Science

University of Tasmania

February, 1993

This thesis contains no material which has been accepted for the award of any other degree or diploma in any tertiary institution, and to the best of my knowledge and belief this thesis contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

Gavin Cooper

Abstract

This thesis is an examination of infinitesimal rigidity in generic structures using linear algebra and matroid theory techniques. The structures examined are bar and joint structures in 1, 2 and 3 dimensions, and hinged panel structures. The focus of this work is a conjectured environment for higher dimensional analogues of Laman's theorem, and some light is consequently shed on the quest for a combinatorial characterisation of (generic) rigid bar and joint structures in three dimensions.

Towards completing this thesis

Don helped as supervisor,

Betty helped type it,

Kym helped computer it, and

Robyn helped illustrate it.

Contents

Introduction.

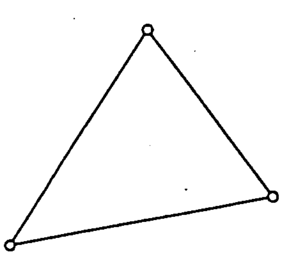
Consider a triangular framework and a square framework in a plane, for which the edges are inflexible rods which are joined at the vertices by universal joints. The latter is flexible in the plane since it can deform into the shape of a rhombus. The triangular framework is rigid in the plane since the three rods determine the relative positions of the three vertices. Similarly a tetrahedral framework in space, consisting of six rods connected at the vertices by universal joints is rigid, whereas a cube constructed the same way is flexible. A figure consisting of two triangles with a common edge is rigid in the plane but flexible in space, since one triangle can then rotate relative to the other along the common edge. Given a framework how can we tell if it is rigid or flexible in a given context?

The rigid frameworks above are structures, while the flexible ones are mechanisms, so the above is a first step in modelling both structures and mechanisms. The mathematical theory of rigidity, as it stands today, is so closely related to the disciplines of structural engineering and mechanical engineering that from a mathematician's viewpoint there are no distinct margins between the three. For example, the papers of Calladine [C1], Pellegrino [P1] [P2], and Kaveh [K7] [K8] [K9] [K10] [K11] on the mathematics behind theory of structures are by structural engineers and those of Baker [B2] [B3] [B4] [B5] [B6] and Hunt [H1] are some of their counterparts in mechanical engineering. Also Crapo [C14] and Baracs [B7] with a very mathematical approach, address unusual engineering problems and Laman [L1], an engineer, has produced a quintessential mathematical result.

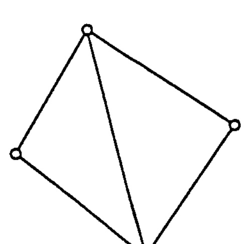
Although in the nineteenth century the distinctions between structural and mechanical engineering were not as strong as they are today, and indeed one was a branch of the other to some extent [R3], for the sake of this discussion we regard the work done then according to contemporary perceptions. From the theory of structures in the nineteenth century [C3] the greatest contributions are from the practical discipline of graphical statics [C17]. Cremona [C17] acknowledges Carl Culmann, who was appointed professor of engineering sciences at Zürich Polytechnikum in 1855, as "the ingenious and esteemed creator of graphical statics". However from graphical statics it is Maxwell's geometrical theory of reciprocal diagrams [M3] [M4], subscribed to by Rankine [R1] [R2] which has had the greatest impact on modern theory so far, leading to the work of Crapo [C13] and Whiteley [C16] [W8]. Apart from this, a method of L.Henneberg [T8] [C3] has been generalised by Tay and Whiteley [T6] [T7]. The enduring early contributions from mechanical engineering have been more in the nature of analyses of particular mechanisms [B8] [B9] [D5].

During the above investigations it became clear that it was necessary to take care with what was meant by *rigid*. A framework may fall into any one of the following classes:

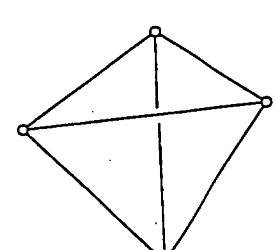
It is rigid, a proper structure. For example



in plane

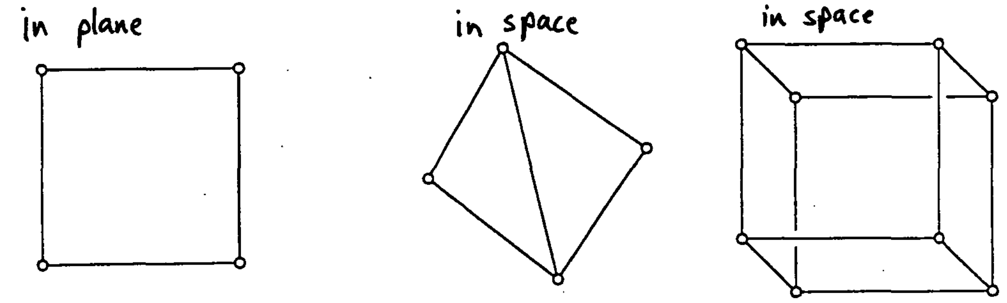


in plane

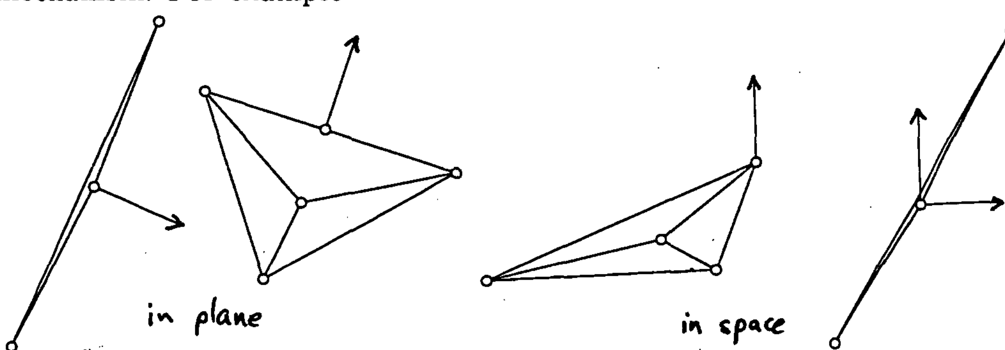


in space

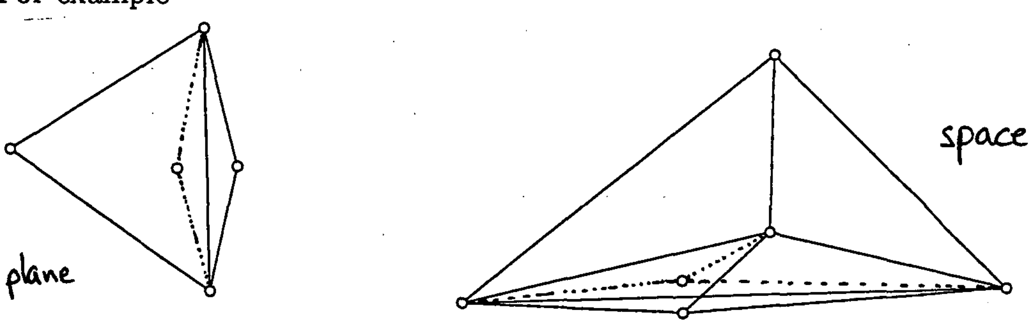
It is continuously movable, a proper mechanism. For example



It is infinitesimally moveable, a shaky structure or an immobile mechanism. For example



Or it is multistable, where the framework has two or more stable forms. For example



The concepts of rigid and continuously movable frameworks are perfectly clear, but what is this third group? It consists of things which liberal definitions of structures, such as a mechanical engineer might use, would represent as structures, but liberal definitions of mechanisms, such as a structural engineer might use, would represent as mechanisms. Such things would not be acceptable as structures to a structural engineer because they are inherently unstable, and would not be acceptable to a mechanical engineer as mechanisms, because they do not actually move. Multistable structures are actually rigid mathematically hence this class is really a subclass of rigid frameworks, but we mention them because due to deformability of all materials they are encountered in practice.

The powerful methods of graphical statics were born of an understanding of geometry, the two areas of which most influenced this practical discipline were projective geometry and the study of polyhedra. We introduce now the work by geometers in developing knowledge of abstract rigidity of polyhedra and shall return later to projective geometry. This work was not practically oriented and was not the work which influenced the workers in graphical statics. In 1766 Euler conjectured, "A closed spacial figure allows no changes, as long as it is not ripped apart" [G1] [C6] [C7], and expanded on this in a letter to Lagrange in 1770. Despite the simplicity of this conjecture, it stood for 200 years.

The first advance was made in 1813 by Cauchy [C2] who essentially proved that a convex closed polyhedral surface is rigid if its flat polygonal faces are held rigid. In 1896 Bricard [B17] showed that the only flexible octahedra had bad self intersections, so all embedded octahedra were rigid. Similar results to Cauchy's were obtained by Liebmann [L2] for analytic surfaces, and by Cohn–Vossen [C4] for the smooth case. In the 1940s Alexandrov showed that all triangulated convex polyhedral surfaces were rigid if there were no vertices in the interior of the flat natural faces. Then in 1974 Gluck [G1] using an idea of Dehn [D2] and Weyl [W4] that Cauchy's theorem was also true for the stronger infinitesimal rigidity, showed that almost all triangulated spherical surfaces were rigid. In 1976 Connelly began to have an inkling of how to settle the conjecture [C5], and he produced a counterexample [C7] [C8], which was soon refined [C6] [C8], based on a Bricard flexible octahedron.

Other researchers with a strong interest in polyhedra have been Bennett [B8], Blaschke [B10], Goldberg [G3] [G4] and Wunderlich [W21] [W22] with analyses of specific polyhedra, and more recently Sugihara [S3] and Whiteley [W8] [W12] [W13] [W16] and Kann [K3] [K4] [K5] [K6] in apparent isolation from his contemporaries. In Connelly's later work [C9] [C10] he acknowledges the advances of A.D.Alexandrov toward his result that arbitrarily triangulated convex surfaces are rigid. The work of Roth [R11] and Asimow [A1] [A2] gives some results of Alexandrov.

In these later works of Connelly [C9] [C10] he obtains results about tensegrity frameworks which have been investigated to some degree by Calladine [C1], Roth and Whiteley [R12], to a lesser extent by Baglivo and Graver [B1], and Whiteley [W10] [W15], and are mentioned in the survey articles by Crapo [C13] and Goldberg [G3].

In addition to the work which led up to Connelly's results, rigidity theory as a mathematical pastime really burgeoned in the mid to late 1970's and until about 1970 most work on bar and joint frameworks was manifestly of an engineering nature and the new mathematical results focussed on peculiar structures or mechanisms. Examples can be found in the work of Bennett [B8] [B9], Blaschke [B10], Goldberg [G2] [G3] [G4], Crapo [C14], and Wunderlich [W21] [W22]. Since Laman's 1970 result [L1] characterising a class of planar isostatic structures there has been an increase in activity which is largely attributable to two factors – namely the survey work by Grünbaum which raised many interesting questions and evidently reached a wide and receptive audience, and the formation at the University of Montreal of the structural topology research group. It seems that Janos Baracs had been collecting interesting and intractable idiosyncratic or nice-looking problems from architecture and structural engineering which were to provide food for this group. He initiated the elegant and thorough analysis of the rigidity of regular rectangular grids undertaken by Crapo and Bolker [B11] [B12] [B13] [B14] [C12] using combinatorial techniques, and later extended by Baglivo and Graver [B1], Whiteley [W10] and Recski [R6] [R7]. As well as providing problems such as the tetrahedral–octahedral truss [C14], he introduced in 1975 [B7] a new type of structure whose analysis has become one of the chief activities of researchers in this area and is one of the main topics of this thesis. Although introduced in their best known contemporary form by Baracs [B7] and developed nicely by Crapo and Whiteley [C16] [W8], hinged panel structures have actually been around a long time and particular examples have been presented and analysed by Bennett [B9], Wunderlich [W21] and Goldberg [G2] [G3] [G4]. It is apparent from the paper of Baracs [B7] and comments of Bennett [B9] that Crapo and Whiteley's result [C16] giving explicitly the conditions for the rigidity of a cycle of k panels was suspected long before it was proved. More recently Tay [T3] [T5] [T6] and White and Whiteley [W6] have generalised these structures to higher dimensions.

Due to the concurrent development of statics and projective geometry, it was realised, in 1863 by Rankine [R1] [R2] [R3], that static equilibria were projectively invariant. This is the fundamental theorem of rigidity, and proofs are given by Wunderlich [W23], Crapo and Whiteley [C16], and Wegner [W1]. The natural extension of this idea to an investigation of the effect of a polarity on a structure has been carried out by Tarnai [T1] and Whiteley [W15]. A belief in the power of projective methods was promulgated recently by Baracs [B7] and Crapo [C12], and their influence in the structural topology research group sparked a proliferation of papers with this bias, ranging from the introduction of Crapo's methods [C15] and their development [C16] [W8], applications and consequences [T3] [T5] [T6] [W5] [W6] [W10] [W13] [W14] to the more geometrical, less algebraic, work of Dandurand [D1]. Independent researchers also realised the importance of a projective approach and the most salient of these is Wegner [W1] [W2].

An obvious natural description of bar and joint structures is in terms of graphs, and consequently the tendency of the most recent work is to consider rigidity as a graph property which may carry across to structures related to that graph. This inclination has been associated with the explicit distinction of generic structures and the papers [A1] [A2] [B15] [C13] [G5] [K7] [K8] [K9] [K11] [L1] [L5] [M1] [R4] [R5] [R8] [T3] [T4] [T7] [W7] [W9] [W11] [W17] [W18] all display a consciousness of this. Bolker and Roth [B15] and Whiteley [W7] [W9] [W11] have investigated the rigidity of bipartite graphs.

Because of its close connection with graph theory we cannot be surprised at the application of matroid theory in the modelling of various types of structure, however we find the variety of these applications noteworthy. Bolker and Crapo [B11] [B12] [B13] [B14] and their successors [B1] [R6] [R7] have used a matroid defined on the diagonal braces of their gridworks. Baracs [B7], Dandurand [D1], Crapo and Whiteley [C16] [W8] and Tay [T3] [T4] have used a matroid defined on the lines (and screws [V1] [K12]) in space, based on a projective coordinatisation of these [V1] [K12]. Recski [R4] [R5] [R7] has used a matroid on the coordinates of the velocities of the joints of his bar and joint structures, which is the dependence matroid on the columns of the coordinatising matrix of his structure. By far the most common matroid associated with a bar and joint structure is called the structure geometry of the structure, and is determined by the bars of the structure. It is the dependence matroid on the rows of the coordinatising matrix of the structure and is presented in this thesis and is used in Asimow and Roth [A2], Crapo [C15], Graver [G5], Sugihara [S2] [S3], Servatius [S1], Tay [T3] [T5], and Whiteley [T7], and Lovász and Yemini [L5]. In fact Lovász and Yemini have used the theory of polymatroids, and the usual cycle matroid on the edges of a graph to prove results concerning this structure geometry.

As with any graph theoretic and combinatorial work, there is an attraction in the above for those interested in an algorithmic approach which computers can handle. Papers written from this perspective range from the early inefficient work of Kahn [K2] to those of Kaveh [K7] [K8] [K10], Recski [R7], Sugihara [S2] [S3], Lovász [L4] and Mansfield [M2] which give explicit algorithms, and to the work of Lovász and Yemini [L5], Rosenberg [R9] [R10], White and Whiteley [W5] [W6], which are simply written with such an approach in mind.

In generalising to higher dimensional structures many people have moved away from modelling real problems, and in fact the idea of structural rigidity has also been modified until it bears little resemblance to its forerunner [D3] [D4]. Gromov [G6] and Kalai [K1] have also applied rigidity theory in pure mathematics, and it has also come to be employed in scene analysis by Crapo and Whiteley [C16] [W8], Sugihara [S3] and Whiteley [W13], and in geodesy, see Whiteley [W11] [W17] and Wunderlich [W19] [W20].

Recent general work on structural rigidity has been by Asimov and Roth [A1] [A2], Crapo [C13], Recski [R7] and Graver [G5], and these would provide a basis for an understanding of the area, especially if augmented by Goldberg [G3], Connelly [C8] and Rooney and Wilson [R8].

In summary, the several outstanding general ideas in the area have been:

i) The work in graphical statics [C17] [M3] [M4] [C3] and in particular the realisation that infinitesimal rigidity is projectively invariant [R1] [R2] [C16] [W1] [W23].

ii) Cauchy's theorem [C2] and Gluck's theorem [G1] on polyhedra and Connelly's counterexample [C6].

iii) Laman's characterisation of a class of isostatic planar bar and joint structures [L1].

iv) Consequent upon ii) and iii) (and numerous other results) the explicit recognition of generic structures [C13] [L5] [G5].

In addition to ideas iii) and iv) above, and the subsequent quest for satisfactory characterisations of (generic) spatial isostatic structures, a strong influence on this thesis has been the introduction by Baracs in 1975 [B7] of what he calls articulated spatial panel structures, accompanied by an indication of the reliance of the analysis of these structures on projective geometry. A slight generalisation of these structures of Baracs, alluded to by Crapo and Whiteley [C16], is a type of structure which can't be modelled in terms of the simple bar and joint structures, and therefore from a mathematical rigidity theorist's viewpoint entails something new and fundamental. A screw hinge is a basic joint between two bodies in space, just as a joint of a planar bar and joint structure is a basic joint between two bodies in the plane [H1].

The main tools employed in this thesis come from linear algebra, matroid theory and graph theory, with a strong influence on our ideas coming from projective geometry.

The first chapter was motivated by our slight discontent with every explanation we had seen, of the fact that a rigid body in the plane has three independent rigid motions, and a rigid body in space has six independent rigid motions. We know that good explanations exist in simple dynamics, however we've not seen any among works in this area. Also, even in papers as recent as 1991 there are inaccuracies. For example the definition of a rigid structure given by Graver [G5] on page 356 is inadequate for a collinear structure in 3–space. This chapter presents a clear exposition of these basic facts.

Chapters two and four deal with well known theory about planar structures, including Lovasz and Yemini's proof of Laman's theorem and a suggestion, inspired by their paper, of a possible new avenue of proof for this result, and several results of our own. The results 2.6, 2.7, 2.13, and 4.7 are the original results suggested by Lovasz and Yemini's paper. The results 4.1 – 4.5 are from Lovasz and Yemini [L5], and the results 4.6, 4.8, 4.9 are widely known. Definitions and results 4.11 - 4.15 can be motivated by ideas from mechanical engineering [B6], and are found in Tay [T3]. Results 4.16 - 4.20 are original results of ours with 4.19 appearing in [R8].

Chapter three is an original discussion of the development of the idea of genericity, including our definition and a result relating this to its predecessors.

Dealing with spatial structures, chapter five explains the difference between the planar case and the spatial case for bar and joint structures, and chapter six introduces certain hinged panel structures. Chapter six also explains the similarities between planar bar and joint structures and these hinged panel structures. These ideas originate with Laman [L1] and Baracs [B7], although our work is based rather more on work of their successors, Lovasz and Yemini [L5] and Crapo and Whiteley [C16]. Since Laman's 1970 characterisation of generic isostatic structures in the plane, and the subsequent appearance of counterexamples to the generalisation of his theorem to spatial bar and joint frameworks, some people have assumed that generalisations don't exist, whilst others have wondered how such things might manifest themselves. In 1981 Tay [T2] [T3] gave a generalisation which has nowhere been explicitly acknowledged as such. We conjecture that a version of this theorem holds for these hinged panel structures, presaged by those studied by Baracs [B7] and Crapo and Whiteley [C16]. This conjecture is similar to a particular case of this general theorem of Tay's and is amenable to being proven in the same manner. The original work is 5.6, 5.7 and chapter six.

Chapter seven consists of speculations, including a dicussion of the projective viewpoint and a list of obvious and routine natural extensions of work presented earlier and less obvious interesting things. The reason some of the ideas in this chapter have not been developed more fully is that they are consequences of a *conjecture* and not an established fact, and most of the work represented by this thesis consisted in attempts to prove this, rather than develop speculations. We hope the effect of the thoughts in chapter seven is to convince one of the worth of the ideas in chapter six.

Graph Theory and Matroid Theory Preamble.

This thesis assumes some knowledge of graph theory and matroid theory. Here is a list of items from graph theory and matroid theory which are used in the thesis. For more background than is mentioned here refer to Bondy and Murty [B16] for graph theory, and Oxley [O1] for matroid theory.

Definition 0.1: A *graph* G(V,E) is a finite non–empty set V(G) whose elements are called *vertices*, and a list E(G) of unordered pairs of elements of V(G) called *edges*. An edge e and a vertex v are *incident* iff $v \in e$. A graph is *simple* iff every edge is a pair of distinct vertices and no two edges are identical. A graph G is *complete* iff every pair of vertices is an edge. We denote the complete graph with n vertices by K_n.

Definition 0.2: A *walk* of a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. It is a *closed path* if $v_0 = v_n$. A minimal closed path is called a *cycle*. A graph is *connected* iff every two vertices are joined by a walk.

Definition 0.3: A *directed* graph is a finite non–empty set V(G) of *vertices*, and a list E(G) of ordered pairs of elements of V(G). The elements of E(G) are called *directed edges.* The first element of a directed edge is called its *head,* and the second its *tail.* Clearly any graph may be made into a directed graph by ordering each of its edges.

Definition 0.4: Let G be a directed graph where $V(G)=\{v_1,\dots,v_n\}$ and $E(G)=\{e_1,\dots e_m\}$. Set $a_{ij}=\begin{cases}1 & \text{if } v_i \text{ is the head of } e_j,\\ -1 & \text{if } v_i \text{ is the tail of } e_j,\\ 0 & \text{otherwise.}\end{cases}$

The matrix (a_{ij}) is called the *incidence matrix* of G.

Definition 0.5: A *matroid* M(E) on a set E is an integer valued function, *rank,* on 2^E which obeys;

i) $0\le rk(A)\le |A| \quad \forall\, A\subset E$,

ii) $A\subset B\subset E \Rightarrow rk(A)\le rk(B)$,

iii) $rk(A)+rk(B)\le rk(A\cup B)+rk(A\cap B)$

Definition 0.6: A *polymatroid* on a set E is an integer valued function, *rank,* on 2^E which obeys;

i) $rk(\emptyset)=0$,

ii) $A\subset B\subset E \Rightarrow rk(A)\le rk(B)$,

iii) $rk(A)+rk(B)\le rk(A\cup B)+rk(A\cap B)$.

Definition 0.7: The *independent sets* of a matroid are the elements of the set $\{A: rk(A)=|A|\}=\mathcal{I}$

Theorem 0.8: The independent sets satisfy;

i) $\emptyset \in \mathcal{I}$

ii) $I \in \mathcal{I}$ and $J \subset I \Rightarrow J \in \mathcal{I}$

iii) if $I, J \in \mathcal{I}$ and $|I| = |J| + 1$ then $\exists\ i \in I$ s.t. $J \dot{\cup} i \in \mathcal{I}$ □

Definition 0.9: The *circuits* of a matroid are the elements of the set $\{C : rk(C) = rk(C \backslash e) = |C| - 1\ \forall\ e \in C\} = \mathscr{C}$.

Theorem 0.10: The circuits satisfy;

i) If $X, Y \in \mathscr{C}$ and $X \neq Y$ then X isn't a subset of Y,

ii) If X and Y are distinct members of $\mathscr{C}$, $a \in X \cap Y$, and $b \in X \backslash Y$, then there exists $C \in \mathscr{C}$ such that $b \in C \subset (X \cup Y) \backslash a$. □

Definition 0.11: The *closure operator* of a matroid on E is the function $cl : 2^E \to 2^E$ defined by $cl(A) = \{a : rk(a \cup A) = rk(A)\}$.

Theorem 0.12: The closure operator satisfies, for $X, Y \subset E$ and $x, y \in E$;

i) $X \subset cl(X)$,

ii) $Y \subset X \Rightarrow cl(Y) \subset cl(X)$,

iii) $cl(X) = cl(cl(X))$,

iv) If $y \notin cl(X)$ and $y \in cl(X \cup x)$ then $x \in cl(X \cup y)$. □

Examples 0.13: i) *Free matroids* where $rk(A) = |A|$ if $A \subset E$.

ii) $M_{E(G)}$, a matroid on the edges of a graph G, where $\mathscr{C} = \{A$: the edges of A form a cycle of the graph$\}$.

iii) The *dependence matroid* on the columns (or on the rows) of a matrix has for E the collection of columns, and $rk(A) = \dim\langle A \rangle$. In this case E is a subset of a vector space, and $cl(A) = \langle A \rangle \cap E$.

Definition 0.14: Two matroids on E_1 and E_2 respectively are *isomorphic* iff $\exists$ a bijection between E_1 and E_2 which preserves the rank function.

Theorem 0.15: For a graph G, $M_{E(G)}$ is isomorphic to the dependence matroid on the incidence matrix of G (however it is directed). $\square$

Definition 0.16: We say M_1 is a *weak map image* of M_2 iff $rk_1(A) \leq rk_2(A)$ $\forall$ $A \subset E$, and write $M_1(E) \leq M_2(E)$.

Definition 0.17: Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2. The *direct sum* of M_1 and M_2, written $M_1 \oplus M_2$, is the matroid on $E_1 \cup E_2$ with $rk_{12}(A) = rk_1(A \cap E_1) + rk_2(A \cap E_2)$.

Defining Rigidity.

In this chapter we arrive at a definition (1.18) for rigid structures by considering physical requirements from the real world and how these relate to the model we first establish. Because we can only build structures in 1, 2 or 3 dimensions, we are really only interested in the cases of this definition where the dimension is 1, 2 or 3, although the abstractions of this section extend readily to higher dimensions. Having thus completed the aim of this chapter we include several general concepts for use in subsequent chapters.

In the following, $\mathbb{R}^\ell$ denotes the ℓ dimensional vector space over the field of real numbers, generated by the basis

$$\{(1,0,\dots,0),\dots,(0,\dots,1,\dots,0),\dots,(0,\dots,0,1)\} \ ;$$

with inner product $$(a_1,\dots,a_\ell)\cdot(b_1,\dots,b_\ell) = \sum_{i=1}^{\ell} a_i b_i \ .$$

Definition 1.1: The *affine span* of n points $p_1,\dots,p_n$ of $\mathbb{R}^\ell$ is $\{ \Sigma\alpha_i p_i \mid \Sigma\alpha_i = 1\}$. Despite the apparent significance of the order in which these points are arranged, it is easy to show that under any rearrangement of the order of these points the same affine span results.

Definition 1.2: *A bar and joint structure in* $\mathbb{R}^\ell$, S, consists of a simple connected graph G(V,E) where $E(G)\neq\emptyset$, and an injective mapping χ: $V(G) \to \mathbb{R}^\ell$. We will identify V(G) with the set of the first n positive integers, and write χ_i instead of $\chi(i)$, and χ_i is called a *joint* of S. If $(i,j) \in E(G)$ then the unordered pair $\{\chi_i,\chi_j\}$ is called a *bar* of S. G is called the *graph underlying* S. The set of all joints of S is denoted J(S) and the set of bars of S is denoted B(S), and when necessary S(G) denotes a structure which graph G underlies.

The reason we study bar and joint structures is that they are a mathematical abstraction of physical structures designed and built by engineers, so we model the behaviour of real structures with our mathematical bar and joint structures.

The following definition is imposed by the fact that the length of every bar of a physical structure scarcely varies, and in our simple model we can demand that every bar of a bar and joint structure be rigid.

Definition 1.3: An *admissible motion* of a bar and joint structure in $\mathbb{R}^\ell$ is the image of a mapping $\mu : V(G) \to \mathbb{R}^\ell$ which satisfies $(\mu_i-\mu_j)\cdot(\chi_i-\chi_j)=0$, $\forall\ (i,j) \in E(G)$, where $\cdot$ denotes inner product and $\mu_i = \mu(i)$.

If μ_i is viewed as an instantaneous velocity assignment, then this condition requires that the difference in the velocities of the joints of any one bar at any instant, must be a vector perpendicular to that bar. That is, each bar is rigid.

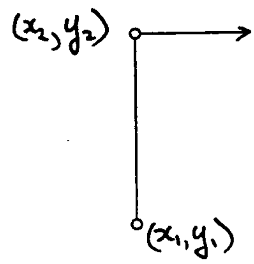
In accordance with the interpretation of μ_i as a velocity we will denote

$$\chi_i \text{ by } \begin{cases} (x_i,y_i,z_i) & \text{if } \ell = 3 \\ (x_i,y_i) & \text{if } \ell = 2 \\ (x_i) & \text{if } \ell = 1 \end{cases}, \quad \text{and } \mu_i \text{ by } \begin{cases} (\dot{x}_i,\dot{y}_i,\dot{z}_i) & \text{if } \ell = 3 \\ (\dot{x}_i,\dot{y}_i) & \text{if } \ell = 2 \\ (\dot{x}_i) & \text{if } \ell = 1 \end{cases}.$$

Although we are using notation here which in other places denotes a derivative w.r.t. time, in this case $\dot{x}_i$ cannot always be regarded as a derivative of x_i.

Example 1.4:

when $t=0$:



$$x_1(t)=0=y_1(t)=\dot{y}_1(t)=\dot{x}_1(t)=x_2(t)$$
$$y_2(t)=1,\ \dot{y}_2(t)=0,\ \dot{x}_2(t)=\begin{cases}0 \text{ if } t\neq 1\\ 1 \text{ if } t=1\end{cases}$$

Evidently $\dot{x}_2(t)$ isn't the derivative of $x_2(t)$, yet $\forall\ t\in(-\infty,\infty)$, we have $(x_1(t)-x_2(t))(\dot{x}_1(t)-\dot{x}_2(t))+(y_1(t)-y_2(t))(\dot{y}_1(t)-\dot{y}_2(t))=0$.

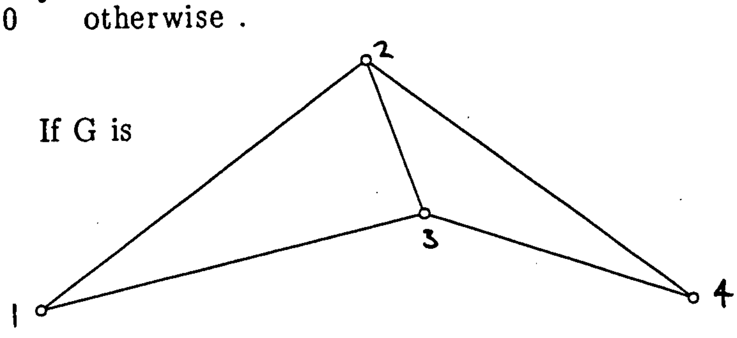
Since for every bar of a structure, S, we have by the last definition an orthogonality condition, overall we have a system of $m = |B(S)|$ equations which we can write as $$A_S\underline{u} = 0,$$

$$\text{where } \underline{u} = \begin{cases}(\dot{x}_1,\ldots,\dot{x}_n,\dot{y}_1,\ldots,\dot{y}_n,\dot{z}_1,\ldots,\dot{z}_n) & \text{if } \ell = 3\\ (\dot{x}_i,\ldots,\dot{x}_n,\dot{y}_1,\ldots,\dot{y}_n) & \text{if } \ell = 2\ ,\\ (\dot{x}_i,\ldots,\dot{x}_n) & \text{if } \ell = 1\end{cases}$$

and A_S is an $m\times\ell n$ matrix with

$$a_{\mu\nu}=\begin{cases}x_i-x_j & \text{if the } \mu^{th} \text{ bar is (i,j) and } \nu = i\\ y_i-y_j & \text{if the } \mu^{th} \text{ bar is (i,j) and } \nu = i+n \text{ and } \ell\geq 2\\ z_i-z_j & \text{if the } \mu^{th} \text{ bar is (i,j) and } \nu = i+2n \text{ and } \ell \geq 3\\ 0 & \text{otherwise .}\end{cases}$$

Example 1.5: If G is



and underlies both S_1, a bar and joint structure in $\mathbb{R}^2$, and S_2, a bar and joint structure in $\mathbb{R}^3$,

$$\text{then } A_{S_1} = \begin{bmatrix} x_1-x_2 & x_2-x_1 & 0 & 0 & y_1-y_2 & y_2-y_1 & 0 & 0 \\ x_1-x_3 & 0 & x_3-x_1 & 0 & y_1-y_3 & 0 & y_3-y_1 & 0 \\ 0 & x_2-x_3 & x_3-x_2 & 0 & 0 & y_2-y_3 & y_3-y_2 & 0 \\ 0 & x_2-x_4 & 0 & x_4-x_2 & 0 & y_2-y_4 & 0 & y_4-y_2 \\ 0 & 0 & x_3-x_4 & x_4-x_3 & 0 & 0 & y_3-y_4 & y_4-y_3 \end{bmatrix},$$

and $A_{S_2} =$

$$\begin{bmatrix} x_1-x_2 & x_2-x_1 & 0 & 0 & y_1-y_2 & y_2-y_1 & 0 & 0 & z_1-z_2 & z_2-z_1 & 0 & 0 \\ x_1-x_3 & 0 & x_3-x_1 & 0 & y_1-y_3 & 0 & y_3-y_1 & 0 & z_1-z_3 & 0 & z_3-z_1 & 0 \\ 0 & x_2-x_3 & x_3-x_2 & 0 & 0 & y_2-y_3 & y_3-y_2 & 0 & 0 & z_2-z_3 & z_3-z_2 & 0 \\ 0 & x_2-x_4 & 0 & x_4-x_2 & 0 & y_2-y_4 & 0 & y_4-y_2 & 0 & z_2-z_4 & 0 & z_4-z_2 \\ 0 & 0 & x_3-x_4 & x_4-x_3 & 0 & 0 & y_3-y_4 & y_4-y_3 & 0 & 0 & z_3-z_4 & z_4-z_3 \end{bmatrix}.$$

Definition 1.6: The matrix A_S is the *coordinatising matrix* of S. Every column of A_S corresponds to one coordinate of a joint of S, and every row corresponds to a bar. The null space of A_S is denoted by N_S.

Lemma 1.7: N_S is a vector space consisting of all admissible motions of the structure S.

Proof: Elementary linear algebra. □

Lemma 1.8: If J(S) affinely spans $\mathbb{R}^\ell$, then N_S has a subspace of dimension $\frac{\ell(\ell+1)}{2}$.

Proof: $\ell = 1$: N_S contains (1,.. ..,1) which is a translation of S.

$\ell = 2$: N_S contains (1,.. ..,1,0,.. ..,0) and (0,.. ..,0,1,.. ..,1) which are translations of the structure in directions along the coordinate axes, and $(y_1,\ldots,y_n,-x_1,\ldots,-x_n)$ which is a rotation of the structure about the origin. These three vectors are independent.

$\ell = 3$: N_S contains (1,.. ..,1,0,.. ..,0,0,.. ..,0), (0,.. ..,0,1,.. ..,1,0,.. ..,0) and (0,.. ..,0,0,.. ..,0,1,.. ..,1) (or $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$ for convenience) which are translations of the structure in the directions along the coordinate axes, and $(0,\ldots,0,z_1,\ldots,z_n,-y_1,\ldots,-y_n) = \underline{u}_4$, $(z_1,\ldots,z_n,0,\ldots,0,-x_1,\ldots,-x_n) = \underline{u}_5$ and

$(y_1,\ldots\ldots,y_n,-x_1,\ldots\ldots,-x_n,0,\ldots\ldots,0) = \underline{u}_6$, which are rotations of the structure about the coordinate axes.

Let $\underline{u}_7 = (0,\ldots\ldots,0,0,z_2-z_1,\ldots\ldots,z_n-z_1,0,y_1-y_2,\ldots\ldots,y_1-y_n) = \underline{u}_4-\underline{u}_2z_1+\underline{u}_3y_1$, $\underline{u}_8 = (0,z_2-z_1,\ldots\ldots z_n-z_1,0,\ldots\ldots,0,0,x_1-x_2,\ldots\ldots,x_1-x_n) = \underline{u}_5-\underline{u}_1z_1+\underline{u}_3x_3$, $\underline{u}_9 = (0,y_2-y_1,\ldots\ldots,y_n-y_1,0,x_1-x_2,\ldots\ldots,x_1-x_n,0,\ldots\ldots,0) = \underline{u}_6-\underline{u}_1y_1+\underline{u}_2x_1$. By considering the first coordinates of vectors in $\{\underline{u}_1,\underline{u}_2,\underline{u}_3,\underline{u}_7,\underline{u}_8,\underline{u}_9\}$, respectively 1,0,0,0,0, and 0, and using symmetry arguments we deduce that this set of vectors is linearly independent if and only if $\{\underline{u}_7,\underline{u}_8,\underline{u}_9\}$ is linearly independent.

Suppose J(S) is a collinear set. Since χ is injective, no two joints have the same coordinates and w.l.o.g. $z_1 \neq z_2$. Since J(S) is collinear $z_1 \neq z_i \ \forall\, i = 2\ldots\ldots n$. For the same reason all the directions of the lines determined by pairs of joints will be the same, so

$$\alpha_1 = \frac{y_2-y_1}{z_2-z_1} = \frac{y_i-y_1}{z_i-z_1} \quad \text{and} \quad \alpha_2 = \frac{x_1-x_2}{z_2-z_1} = \frac{x_1-x_i}{z_i-z_1} \quad \forall\, i = 2\ldots\ldots n.$$

So $\underline{u}_9 = \alpha_1\underline{u}_8 + \alpha_2\underline{u}_7$ and $\{\underline{u}_9,\underline{u}_8,\underline{u}_7\}$ is dependent. Conversely, suppose $\{\underline{u}_9,\underline{u}_8,\underline{u}_7\}$ is dependent. Then without loss of generality $\underline{u}_9 = \alpha_1\underline{u}_8 + \alpha_2\underline{u}_7$. Comparing coordinates, we must have

$$\alpha_1 = \frac{y_2-y_1}{z_2-z_1} = \frac{y_i-y_1}{z_i-z_1} \quad \text{and} \quad \alpha_2 = \frac{x_1-x_2}{z_2-z_1} = \frac{x_1-x_i}{z_i-z_1} \quad \forall\, i = 2\ldots\ldots n.$$

Since the directions of the lines containing joint 1 and each of the other joints are the same, J(S) is collinear. That is $\{\underline{u}_1,\underline{u}_2,\underline{u}_3,\underline{u}_7,\underline{u}_8,\underline{u}_9\}$ are independent if and only if J(S) isn't collinear. □

Definition 1.9: Regardless of its dimension, the space generated by $\{\underline{u}_1,\underline{u}_2,\underline{u}_3,\underline{u}_4,\underline{u}_5,\underline{u}_6\}$ is called the *space of rigid motions* of S (in $\mathbb{R}^3$). Similarly the space generated by $\{(1,\ldots\ldots,1,0,\ldots\ldots,0),(0,\ldots\ldots,0,1,\ldots\ldots,1),(y_1,\ldots\ldots,y_n,-x_1,\ldots\ldots-x_n)\}$ is called the *space of rigid motions* of S (in $\mathbb{R}^2$), and the space generated by$\{(1,\ldots\ldots,1)\}$ is called the *space of rigid motions* of S (in $\mathbb{R}^1$). We denote this space by R_S.

Definition 1.10: If S is a structure in $\mathbb{R}^\ell$, then the codimension of R_S in N_S is called the *degree of freedom* of S, and is denoted by f(S).

If, to a bar and joint structure in $\mathbb{R}^\ell$, we add bars, we can only increase the rank of the system and only diminish the dimension of the space of admissible motions. When we have added all possible bars to a bar and joint structure in $\mathbb{R}^\ell$ (so that the underlying graph of our structure is now a complete graph) we have a structure which has the least amount of admissible motions of all structures with these joints. We know from lemma 1.8 that we cannot keep adding bars until there are no admissible motions, so how can we define rigidity? At least we know that $\dim(N_S)$ for a structure which isn't "rigid" is necessarily greater than $\dim(N_{S'})$ for a "rigid" structure with the same set of joints.

In summary we have

Comment 1.11: If a structure with underlying graph complete is "rigid", there may be other structures having the same set of joints which are "rigid". However if a structure with a complete graph underlying it is not "rigid", then there is no structure with the same set of joints which is "rigid".

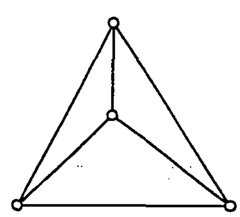
An examination of structures on complete graphs will lead us to a definition for rigidity.

A question which arises is this: "Why do we consider rigidity in spaces of different dimension, when perhaps it might suffice to embed all structures in a space of sufficiently high dimension, thereby avoiding the complication of considering structures in spaces of different dimensions?", and although a complete answer to this question requires more information, we can at this stage demonstrate simply the necessity of our approach with these examples.

Example 1.12: A little consideration will lead to a belief that the structure in example 1.5 built in the plane and with joints constrained to move in the plane will be "rigid", whereas if the joints were allowed to move freely in space, the structure would flex. In the former case the only admissible motions are rigid motions, whereas in the latter case in addition to the six dimensional space of rigid motions, the structure also has the admissible motion $(0,0,0,(y_1-y_2)(z_1-z_3)-(y_1-y_3)(z_1-z_2),0,0,0,(z_1-z_2)(x_1-x_3)-(z_1-z_3)(x_1-x_2),$ $0,0,0,(x_1-x_2)(y_1-y_3)-(x_1-x_3)(y_1-y_2)) = \underset{\sim}{f}$. Thus in the former case $\dim(N_S) = 3$ but in the latter case $\dim(N_S) = 7$.

Example 1.13: Consider the structures S_1' obtained from S_1 in example 1.5, and S_2' and S_1'' obtained from S_2 in example 1.5 by adding a bar (so that now we are dealing with structures with K_4 as the underlying graph).

If $J(S_1')=\{(0,0),(1,1),(2,0),(1,2)\}$, $J(S_1'')=\{(0,0,0),(1,1,0),(2,0,0),(1,2,0)\}$, $J(S_2)=\{(0,0,0),(1,1,0),(1,-1,0),(0,1,1)\}$, and G is

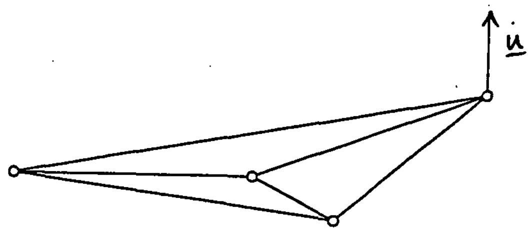


$$\text{then } A_{S_1'} = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 2 & -2 \\ -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dim(N_{S_1'}) = 3,$$

$$A_{S_1''} = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dim(N_{S_1''}) = 7, \text{ and}$$

$$A_{S_2'} = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -2 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \dim(N_{S_2'}) = 6.$$

In some sense S_1' and S_1'' appear identical but although S_1' is "rigid", S_1'' has an admissible motion:



$\underline{u} = (0,0,0,0,0,0,0,0,0,0,0,1)$;

which S_1' cannot have, and is therefore not "rigid".

These examples suggest that the affine span of the joints of a structure may be a suitable environment in which to consider the "rigidity" of that structure. We return to our examination of complete graphs in order to consolidate this suggestion.

Lemma 1.14: In $\mathbb{R}^2$, if K_n underlies S and the joints of S affinely span $\mathbb{R}^2$, then $\dim N_S = 3$.

Proof: We use induction on the number of joints, n. If $n = 3$, J(S) is a non-collinear triangle and we have three independent equations in six unknowns, so that $\dim N_S = 3$. Now suppose that $\dim N_{S(K_n)} = 3$ whenever $J(S(K_n))$ span $\mathbb{R}^2$. Any structure $S(K_{n+1})$ where $n>3$ and the joints affinely span $\mathbb{R}^2$ can be obtained by adding a joint, (x_{n+1},y_{n+1}), and n bars to a structure $S(K_n)$ whose joints affinely span $\mathbb{R}^2$. Clearly $\dim N_{S(K_{n+1})} = \dim N_{S(K_n)} + \begin{cases}2\\1\\0\end{cases}$, since $A_{S(K_{n+1})}$ has two more columns than $A_{S(K_n)}$, and for every matrix A we know that the number of columns of A is equal to the sum of the rank of A and the dimension of the null space of A. Now since $J(S(K_{n+1}))$ span $\mathbb{R}^2$, $\exists\ (x_1,y_1), (x_2,y_2) \in J(S(K_n))$ (and $J(S(K_{n+1}))$) s.t. (x_1,y_1), (x_2,y_2) and (x_{n+1},y_{n+1}) are not collinear.

So the two equations:

$$(x_1-x_{n+1})(\dot{x}_1-\dot{x}_{n+1}) + (y_1-y_{n+1})(\dot{y}_1-\dot{y}_{n+1}) = 0$$

and

$$(x_2-x_{n+1})(\dot{x}_2-\dot{x}_{n+1}) + (y_2-y_{n+1})(\dot{y}_2-\dot{y}_{n+1}) = 0$$

which are amongst the n added to the system for $S(K_n)$ to make it the system for $S(K_{n+1})$, are independent. Clearly these equations are also independent of the equations for $S(K_n)$ since they involve the extra variables $\dot{y}_{n+1}$ and $\dot{x}_{n+1}$, so the rank of the system increases by exactly 2 when the extra bars are added, and $\dim N_{S(K_{n+1})} = \dim N_{S(K_n)} = 3$. So by induction $\dim N_{S(K_n)} = 3$ if $J(S(K_n))$ span $\mathbb{R}^2$. □

Also we have:

Lemma 1.15: In $\mathbb{R}^3$, if $J(S(K_n))$ affinely span $\mathbb{R}^3$, then $\dim N_{S(K_n)} = 6$.

Proof: Exactly analogous to the previous proof. □

It is a well known and simple result that in $\mathbb{R}^1$ a structure S is "rigid" if and only if its underlying graph is connected. In fact the analysis of structures in $\mathbb{R}^1$ beyond this result is generally assumed to be too simple to hold any intrinsic interest, or to give any useful insight into higher dimensional analysis. This is certainly true for the type of results which remain in this chapter, and for this reason we make no more observations about $\mathbb{R}^1$, except to say that all our general results expressed in terms of $\mathbb{R}^\ell$ are true in $\mathbb{R}^1$.

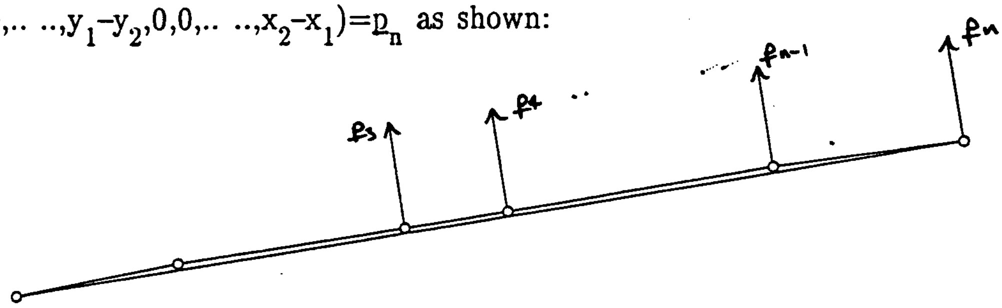
In the physical situations where our theory is used we know that structures which are "rigid" in the sense we would like to define, are exactly those structures where $f(S)=0$, that is where the space of admissible motions is identical with the space of rigid motions. We can use the last three lemmas to express this mathematically for structures whose joints affinely span the space $\mathbb{R}^{\ell}$ in which they are being considered. For complete structures the space of rigid motions and the space of admissible motions are identical iff (by lemmas 1.14 and 1.15) $N_S = \frac{\ell(\ell+1)}{2}$, and by lemma 1.11 we can extend this idea to all structures whose joints affinely span the space $\mathbb{R}^{\ell}$ in which they are being considered.

Definition 1.16: A structure S in $\mathbb{R}^{\ell}$ whose joints affinely span $\mathbb{R}^{\ell}$, is *rigid* if and only if $\dim N_S = \frac{\ell(\ell+1)}{2}$.

What shall we do in the case of a structure whose joints don't span $\mathbb{R}^{\ell}$? Such structures are divided into two classes; those which have more joints than ℓ ($|J(S)| > \ell$), and the others ($|J(S)| \le \ell$). We deal first with the former class:

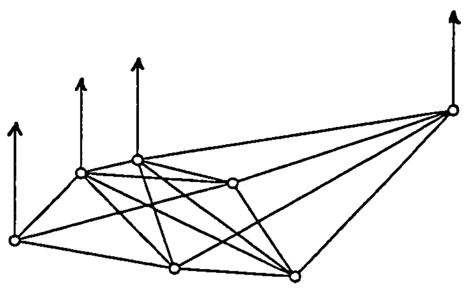
Lemma 1.17: If for a structure S whose joints don't affinely span $\mathbb{R}^{\ell}$, $|J(S)| > \ell$, then $\dim N_S \neq \frac{\ell(\ell+1)}{2}$.

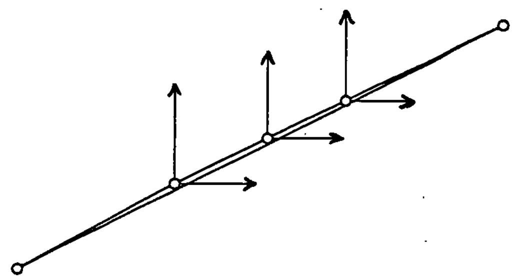
Proof: The only structures whose joints don't span $\mathbb{R}^2$ are collinear. In $\mathbb{R}^2$ if J(S) are collinear, then N_S has a subspace of dimension $|J(S)| + 1$, since N_S contains $(1,\dots,1,0,\dots,0)$, $(0,\dots,0,1,\dots,1)$ and $(y_1,\dots,y_n,-x_1,\dots,-x_n)$, the three rigid motions of lemma 1.8, and $(0,0,y_1-y_2,\dots,0,0,0,x_2-x_1,\dots,0)=\underline{p}_3,\dots,$ $(0,0,\dots,y_1-y_2,0,0,\dots,x_2-x_1)=\underline{p}_n$ as shown:



which are seen to be independent.

Similarly in $\mathbb{R}^3$ if the joints of S are coplanar, then N_S has a subspace of dimension $|J(S)| + 3$, and if the joints of S are collinear then N_S has a subspace of dimension $2|J(S)| + 1$.



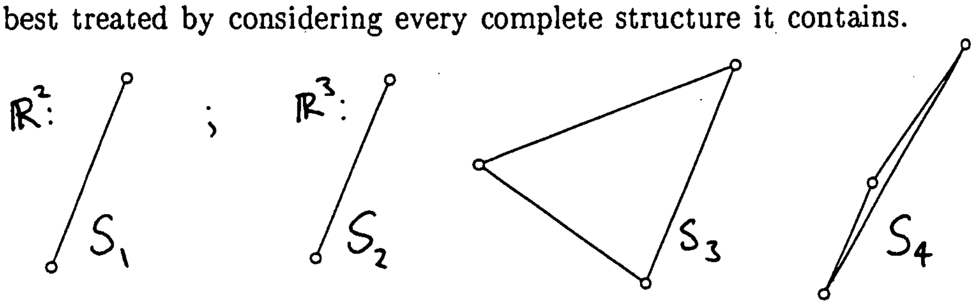


□

From observing the behaviour of structures built by engineers, we know that collinear structures in $\mathbb{R}^2$ and $\mathbb{R}^3$, with K_n as their underlying graph, where $n \geq 3$, are not "rigid" in the strictest sense, as they admit "infinitesimal motions" (in the literature such structures are described as rigid but not infinitesimally rigid [A2]). Similarly coplanar structures in $\mathbb{R}^3$ with K_n as underlying graph with $n \geq 4$, are manifestly not "rigid" in the sense we want to define.

Hence we can extend our definition to include this class of structures also.

We finally consider structures where $\ell \geq |J(S)|$ and J(S) doesn't span $\mathbb{R}^\ell$. Due to the low number of such structures for $\ell = 2$ and 3, this class is perhaps best treated by considering every complete structure it contains.



We show the relevant information about these four structures in a table:

	$\dim(N_S)$	equal $\frac{\ell(\ell+1)}{2}$?	engineers information about rigidity. RIGID?	is definition 1.16 adequate?
S_1	3	Yes	Yes	Yes
S_2	5	No	Yes	No
S_3	6	Yes	Yes	Yes
S_4	6	Yes	No	No.

So in combining this information with our previous comments we realise that for $\ell \leq 3$ we want a bar and joint structure S in $\mathbb{R}^\ell$ to be rigid if and only if $\dim(N_S) = \frac{\ell(\ell+1)}{2}$, unless in $\mathbb{R}^3$ S is S_4 or S_2, where the latter is rigid, but the former is not. Summing up we say

Definition 1.18: A bar and joint structure S in $\mathbb{R}^\ell$ is *rigid* if and only if either

i) J(S) spans $\mathbb{R}^\ell$ and $\dim N_S = \frac{\ell(\ell+1)}{2}$,

or ii) S is S_1 in $\mathbb{R}^2$,

or iii) S is S_2 or S_3 in $\mathbb{R}^3$.

Having arrived at the desired object of this section we now include a comment concerning higher dimensions. Most of the generalisation of this section to $\ell \geq 4$ is immediately clear, however the section between lemma 1.17

and definition 1.18, which deals with structures S in $\mathbb{R}^{\ell}$ with $\ell \geq |J(S)|$ and J(S) not affinely spanning $\mathbb{R}^{\ell}$, will necessarily require some extra work because the explicit treatment used here is not generally possible since for a given ℓ the number of such complete structures is $\frac{\ell(\ell-1)}{2}$. There is also the added consideration that engineers do not give us information about real physical structures for $\ell \geq 4$, and the lack of this intuitive aid requires that the mathematical essence of these be inferred from the lower dimensional cases. For example the question of whether K_5 built in $\mathbb{R}^5$ with joints spanning $\mathbb{R}^4$ is rigid according to the abstraction of what engineers tell us about structures in $\mathbb{R}^2$ and $\mathbb{R}^3$, can be answered in the affirmative by analogy with K_4 built in $\mathbb{R}^4$ with joints spanning $\mathbb{R}^3$ and K_3 built in $\mathbb{R}^3$ with joints spanning $\mathbb{R}^2$. Despite the possible benefits such a general approach may confer in terms of giving greater insight into this class of exceptions, we don't pursue this here because it is only of peripheral interest relative to the motivation of this work, namely the behaviour of physical structures.

An appropriate extension of our current meaning of rigidity into higher dimensions leads naturally to conjectures such as:

(i) If $J(S(K_n))$ span $\mathbb{R}^{\ell}$, then $S(K_n)$ is rigid in any space of dimension $k \geq \ell$ if and only if $n-1 = \ell$; and

(ii) If $S(K_n)$ is rigid in $\mathbb{R}^{\ell}$ $\forall$ $\ell \geq n-1$, then the joints of $S(K_n)$ span $\mathbb{R}^{n-1}$; however in the remainder of this thesis we shall restrict ourselves to working within spaces of dimension less than four.

We finish this chapter with several concepts which hold generally in $\mathbb{R}^{\ell}$, specific cases of which will be used in later chapters.

Referring back to definition 1.6 we have:

Definition 1.19: A set of bars of S is *independent* if the corresponding rows of A_S are linearly independent. Since there is a matroid on the rows of A_S, this definition induces a matroid on the bars of S. The matroid (see 0.7 0.8 0.13 iii) thus induced on the bars of S, or equivalently on the rows of A_S, is called the *structure geometry* of S and is denoted D_S.

Definition 1.20: A rigid bar and joint structure in $\mathbb{R}^\ell$, whose structure geometry is a free matroid, is called an *isostatic* structure. A rigid structure which is not isostatic is *hyperstatic.*

Definition 1.21: If S is a bar and joint structure in $\mathbb{R}^\ell$ and $\chi_i,\chi_j \in J(S)$ then we say $\{\chi_i,\chi_j\}$ is an *implicit bar* of S iff $\{\chi_i,\chi_j\}$ is not a bar of S but $(\mu_i-\mu_j)\cdot(\chi_i-\chi_j)=0$ for every admissible motion μ.

We use the simple notation $S\cup\{\chi_i,\chi_j\}$ for the structure obtained from S by adding an extra edge (i,j) to its underlying graph and retaining the same injective mapping χ into $\mathbb{R}^\ell$.

Lemma 1.22: If S is a bar and joint structure in $\mathbb{R}^\ell$ and $\{\chi_i,\chi_j\}$ is an implicit bar of S then $f(S)=f(S\cup\{\chi_i,\chi_j\})$.

Proof: Immediate consequence of the definition. □

Lemma 1.23: S is rigid iff $\{\chi_i,\chi_j\}$ is either a bar of S or an implicit bar of S, $\forall\ \chi_i\neq\chi_j\in J(S)$ provided J(S) affinely spans $\mathbb{R}^\ell$ or the joints are affinely independent.

Proof: Although this lemma holds more generally than in $\mathbb{R}^2$, we only prove it in the planar case. The reason for this is that the proof for $\mathbb{R}^3$ is exactly analogous, but more cumbersome.

Suppose S is rigid and $\{\chi_i, \chi_j\}$ is not a bar of S. Every admissible motion $\underline{v}$ is expressible as a linear combination of a basis of N_S;

$\underline{v}=\alpha_1(1,\ldots\ldots,1,0,\ldots\ldots,0)+\alpha_2(0,\ldots\ldots,0,1,\ldots\ldots,1)+\alpha_3(-y_1,\ldots\ldots,-y_n,x_1,\ldots\ldots,x_n)$, so if $(\mu_i-\mu_j)\cdot(\chi_i-\chi_j)=0$ for each of these basis vectors then $\{\chi_i,\chi_j\}$ is an implicit bar.

$(1,\ldots\ldots,1,0,\ldots\ldots,0)$: $(x_i-x_j)(1-1)+(y_i-y_j)(0-0)=0$ O.k.

$(0,\ldots\ldots,0,1,\ldots\ldots,1)$: $(x_i-x_j)(0-0)+(y_i-y_j)(1-1)=0$ O.k.

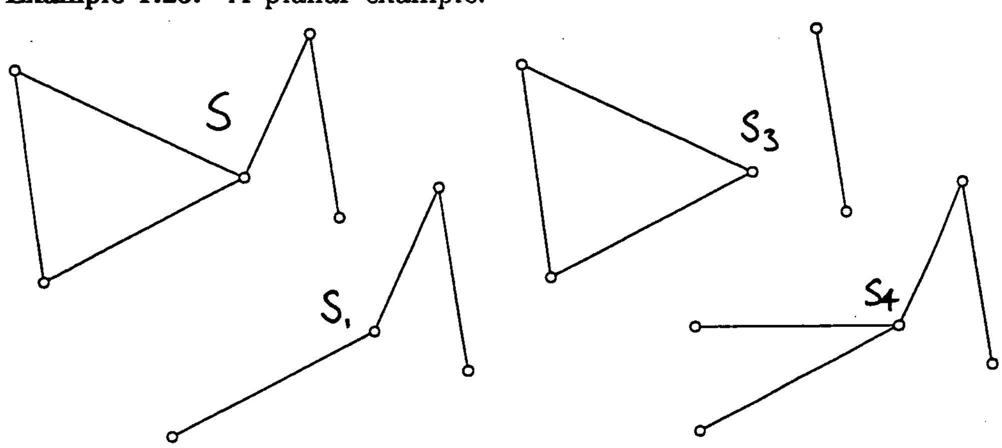
$(-y_1,\ldots\ldots,-y_n,x_1,\ldots\ldots,x_n)$: $(x_i-x_j)(y_j-y_i)+(y_i-y_j)(x_i-x_j)=0$ O.k.

Conversely if T is the complete structure on J(S), then f(S)=f(T) by the preceding lemma, and f(T)=0 by lemma 1.14, so f(S)=0 and S is rigid. □

Still thinking more generally than $\mathbb{R}^2$ we begin to think of structures as consisting of "simpler" structures joined to form a larger structure.

Definition 1.24: Suppose S is a bar and joint structure with graph G(V,E), and injection $\chi:V(G)\to\mathbb{R}^\ell$. A *substructure* of S is a structure S′, with graph G′(V′,E′) and injection $\chi':V'(G')\to\mathbb{R}^\ell$, s.t. G′ is a subgraph of G, and $\chi'=\chi|_{V'}$.

Example 1.25: A planar example.



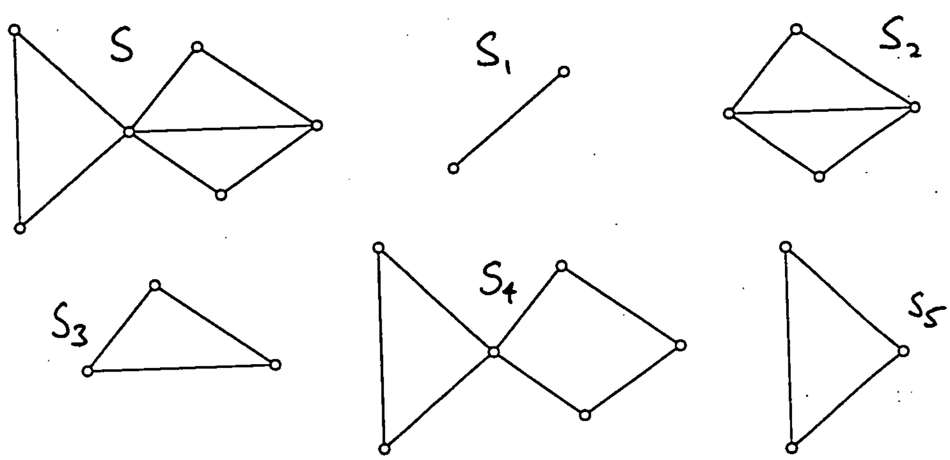
S_1 is a substructure of S. S_3 is not a structure and thus not a substructure of S. S_4 is not a substructure of S as $\chi_1\neq\chi_1'$.

Lemma 1.26: If S' is a substructure of S, then $D_{S'}$ is the restriction of D_S to the bars of S'. Conversely every restriction of D_S to a connected subset $B(S')$ of $B(S)$, is the structure geometry of the substructure S' of S.

Proof: We see this by looking at the rows of A_S and $A_{S'}$. □

Definition 1.27: A *maximal rigid substructure* of S is a rigid substructure of S which is a substructure of no rigid substructure of S other than itself.

Example 1.28: Another planar example.



S_1, S_2, S_3, S_4, S_5 are substructures of S. S_4 is not a rigid substructure of S, but S_1, S_2, S_3, and S_5 all are. S_1 and S_3 are not maximal rigid substructures of S, but S_2 and S_5 are. In fact S_2 and S_5 are the only maximal rigid substructures of S.

Rigidity in $\mathbb{R}^2$.

This technical chapter is part of an examination of bar and joint structures in $\mathbb{R}^2$. We introduce some properties of the structure geometry in $\mathbb{R}^2$, and also some lemmas designed for use in chapter 4 where our presentation of structures in $\mathbb{R}^2$ is continued in a restricted environment. Also theorem 2.13 is important to our investigation in chapters 5 and 6 of structures in $\mathbb{R}^3$.

Definition 2.1: A *planar bar and joint structure* is a bar and joint structure in $\mathbb{R}^2$. Except when we explicitly state otherwise, throughout chapter two we shall always mean "bar and joint structure in $\mathbb{R}^2$" when we say "bar and joint structure", or "structure".

We bring to the reader's attention non-ambiguous notation abuses at theorem 2.10, lemma 2.12, theorem 2.13, and lemma 4.15, which consist of using graph terminology for structures, and of a looseness in the use of the $\cup$ symbol. In addition we sometimes talk of a complete structure on a certain set of joints, by which we mean the structure with those joints which has a bar between every pair of joints. Also on page 35 the vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{v}_S$ & $\underline{m}_S$ have had an unexplained rearrangement of co-ordinates.

Lemma 2.2: i) $f(S)+rk(A_S)=2n-3$.

ii) $0 \leq f(S) \leq 2n-3$.

iii) $0 \leq rk(A_S) \leq 2n-3$.

iv) S is rigid iff $f(S)=0$.

Proof: Refer to definitions 1.18 and 1.10.

i) From linear algebra we know that, for a matrix A_S, we have

$$rk(A_S) + dim(N_S) = \text{the number of columns of } A_S.$$

Since $dim(N_S)=dim(R_S)+[\text{codimension of } R_S \text{ in } N_S]$, we have

$$[\text{codimension of } R_S \text{ in } N_S]+dim(R_S)+rk(A_S)=2n$$

$$[\text{codimension of } R_S \text{ in } N_S]+3+rk(A_S)=2n$$

$$f(S)+rk(A_S)=2n-3.$$

ii) & iii) Since rank is never negative and codimension is never negative, these follow directly from i).

iv) From definition 1.16 or 1.18 we know S is rigid iff $dim(N_S)=3$. That is, iff $3=[\text{the codimension of } R_S \text{ in } N_S]+dim(R_S)=f(S)+3$. That is, iff $f(S)=0$. □

Theorem 2.3: $rk(A_S) \leq \min\{|B(S)|, 2|J(S)|-3\}$

Proof: A_S has $|B(S)|$ rows and $2|J(S)|$ columns, and $dim(N_S) \geq 3$. □

Lemma 2.4: If S is rigid then $|B(S)| \geq 2|J(S)|-3$.

Proof: S rigid $\Rightarrow rk(A_S)=2|J(S)|-3 \Rightarrow |B(S)| \geq 2|J(S)|-3$, since $|B(S)|$ is the number of rows of A_S. □

Theorem 2.5: Any two of the following conditions together imply the third:

i) D_S is free.

ii) S is rigid.

iii) $|B(S)|=2|J(S)|-3$.

Proof: i & ii $\Rightarrow$ iii: S is rigid so $rk(A_S)=2|J(S)|-3$, and B(S) is independent so $rk(A_S)=|B(S)|$.

i & iii $\Rightarrow$ ii: B(S) is independent, so $rk(A_S)=|B(S)|=2|J(S)|-3$. So S is rigid.

ii & iii $\Rightarrow$ i: S is rigid, so $rk(A_S)=2|J(S)|-3=|B(S)|$. So $B(S)$ is independent. □

The next two simple results were suggested by the work in [L5] and are related to results 4.3 and 4.7.

Theorem 2.6: Let $B(S)=B_1\dot{\cup}\ldots\dot{\cup}B_k$ be a partition of the bars of S so that $B_i=B(S_i)$ is the set of bars of a substructure S_i of S. Then

$$f(S)=2|J(S)|-3-\sum_{i=1}^{k}(2|J(S_i)|-3-f(S_i)) \quad \text{iff} \quad D_S=D_{S_1}\oplus\ldots\oplus D_{S_k}.$$

Proof: $\sum_{i=1}^{k}(2|J(S_i)|-3-f(S_i))=2|J(S)|-3-f(S)$ iff $\sum_{i=1}^{k}rk(A_{S_i})=rk(A_S)$ iff

$$D_S=D_{S_1}\oplus\ldots\oplus D_{S_k}.$$ □

Corollary 2.7: Let $B(S)=B_1\dot{\cup}\ldots\dot{\cup}B_k$ be a partition of the bars of S so that $B_i=B(S_i)$ is the set of bars of a substructure S_i of S. Then

$$f(S)=2|J(S)|-3-\sum_{i=1}^{k}(2|J(S_i)|-3) \quad \text{iff} \quad D_S=D_{S_1}\oplus\ldots\oplus D_{S_k} \text{ and } S_i \text{ is rigid } \forall\, i=1\ldots k.$$

Proof: S_i is rigid iff $f(S_i)=0$. □

Of course these two results are only useful for those structures for which such partitions exist. Now we present some technical results for use in chapters 4, 5 and 6.

Now suppose we have a rigid substructure S, of some structure, with $J(S)=\{(x_1,y_1),\ldots,(x_n,y_n)\}$. We define a three dimensional vector space $M_S^{ij}=\{(\alpha_1+y_i\alpha_3,\alpha_1+y_j\alpha_3,\alpha_2-x_i\alpha_3,\alpha_2-x_j\alpha_3):\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}\}$ for fixed $1\leq i\neq j\leq n$, with addition $(a_1,a_2,a_3,a_4)+(b_1,b_2,b_3,b_4)=(a_1+b_1,a_2+b_2,a_3+b_3,a_4+b_4)$ and scalar multiplication $\gamma(a_1,a_2,a_3,a_4)=(\gamma a_1,\gamma a_2,\gamma a_3,\gamma a_4)$. Consider the bijective linear transformation to M_S^{ij} from N_S defined by

$$B_S^{ij}:\underline{m}=\alpha_1\underline{u}_1+\alpha_2\underline{u}_2+\alpha_3\underline{u}_3 \mapsto (\alpha_1+y_i\alpha_3,\alpha_1+y_j\alpha_3,\alpha_2-x_i\alpha_3,\alpha_2-x_j\alpha_3),$$

where $\underline{u}_1=(1,\dots,1,0,\dots,0,0,\dots,0)$, $\underline{u}_2=(0,\dots,0,1,\dots,1,0,\dots,0)$, $\underline{u}_3=(y_1,\dots,y_n,-x_1,\dots,-x_n,0,\dots,0)$, with the extra zeroes corresponding to vertices of the structure of which S is a substructure. Summing up,

Lemma 2.8: The bijective linear transformation $B_S^{ij}:N_S\to M_S^{ij}$ exists, for any two joints i and j of a rigid substructure S of some larger structure.

Proof: Discussion above. □

Lemma 2.9: If $B(S)=B(S_1)\cup B(S_2)$ and $B(T)=B(T_1)\cup B(S_2)$, S_1 and T_1 are rigid substructures of S and T respectively, and $J(S_1)\cap J(S_2)=J(T_1)\cap J(S_2)\neq\emptyset$, then $f(S)=f(T)$.

Proof: We demonstrate the existence of a bijective linear transformation between N_S and N_T, using the fact that if the velocities of two joints of a rigid body are known, then the velocities of all the joints of the rigid body are consequently known, as shown in the previous lemma.

For every $\underline{m}_S\in N_S$ we define a unique $\underline{v}_S=(0,\dots,0,a_{r+1},\dots,a_{2n})$ where $\underline{m}_S=(a_1,\dots,a_{2n})$ where the last $|J(S_1)|=2n-r$ entries correspond to joints of S_1. Similarly for each $\underline{m}_T\in N_T$ define a $\underline{v}_T$.

Define a bijective linear transformation $L'':N_{S_1}\to M_{S_1}^{ij}\to M_{T_1}^{ij}\to N_{T_1}$ by

$$L''=\begin{cases}(B_{T_1}^{ik})^{-1}\circ I^{kj}\circ B_{S_1}^{ij} & \text{if } J(S_1)\cap J(S_2)=\{i\} \text{ and } j\in J(S_1) \text{ and } k\in J(T_1).\\ (B_{T_1}^{ij})^{-1}\circ I^{jj}\circ B_{S_1}^{ij} & \text{if } \{i,j\}\subset J(S_1)\cap J(S_2).\end{cases}$$

where $I^{kj}:M_{S_1}^{ij}\to M_{T_1}^{ik}$ is defined by $(\alpha_1+y_i\alpha_3,\alpha_1+y_j\alpha_3,\alpha_2-x_i\alpha_3,\alpha_2-x_j\alpha_3) \mapsto (\alpha_1+y_i\alpha_3,\alpha_1+y_k\alpha_3,\alpha_2-x_i\alpha_3,\alpha_2-x_k\alpha_3)$.

Now define $L':\{\underline{u}_S=\underline{m}_S-\underline{v}_S:\underline{m}_S\in N_S\}\to\{\underline{u}_T=\underline{m}_T-\underline{v}_T:\underline{m}_T\in N_T\}$ by $L'(a_1,\ldots,a_r,0,\ldots,0)=(a_1,\ldots,a_r,0,\ldots,0)$.

It isn't clear that L' is well defined. It may be that there is a $\underline{u}_S$ s.t. $L'(\underline{u}_S)$ doesn't exist. If this is not a problem then L' is a bijective linear transformation. We now show that L' is well defined.

Consider the orthogonality conditions which originally gave rise to each of the matrices A_S and A_T. First we have, common to S and T, all the equations derived from the bars of S_2. Call this system 1. Call the equations derived from the bars of T_1 system 2, and the equations derived from the bars of S_1 system 3.

We consider two cases: i) where $|J(S_1)\cap J(S_2)|\geq 2$, and ii) where $|J(S_1)\cap J(S_2)|=1$. In case i) we know from the preceding lemma, that systems 2 and 3 have three dimensional solution spaces, and the solutions can be expressed in terms of x_i,x_j,y_i,y_j, where(x_i,y_i) and $(x_j,y_j)\in J(S_1)\cap J(S_2)$ and $J(T_1)\cap J(S_2)$. Since for both structures (S and T) the remaining equations are identical (system 1) and contain x_i,x_j,y_i,y_j, we see that for every $\underline{u}_S$ there does exist a corresponding $\underline{u}_T$, and we are therefore assured that L' is well defined. Case ii) is similar: since every structure has at least two vertices, S_1 and T_1 contain (x_j,y_j) and (x_k,y_k) respectively, and using this information we apply the same argument as in case i) to assure ourselves that L' is well defined in this case also.

In either case there exists a bijective linear transformation $L:N_S\to N_T$ defined by $L(\underline{m}_S)=L(\underline{u}_S+\underline{v}_S)=L'(\underline{u}_S)+L''(\underline{v}_S)=\underline{u}_T+\underline{v}_T=\underline{m}_T$. So $\dim(N_S)=\dim(N_T)$ and $f(S)=f(T)$. □

Theorem 2.10: Let $S_1, \ldots \ldots, S_k$ be rigid substructures of S s.t. $B(S)=B(S_1)\cup \ldots \ldots \cup B(S_k)$, and let $T_1, \ldots \ldots, T_k$ be rigid substructures of T s.t. $B(T)=B(T_1)\cup \ldots \ldots \cup B(T_k)$. Further let S′ be the set of joints of S which are in more than one S_i, and T′ be the set of joints of T which are in more than one T_i, and $J(T_i)\cap T'=J(S_i)\cap S'$ $\forall$ i=1.. ..k. Then f(S)=f(T).

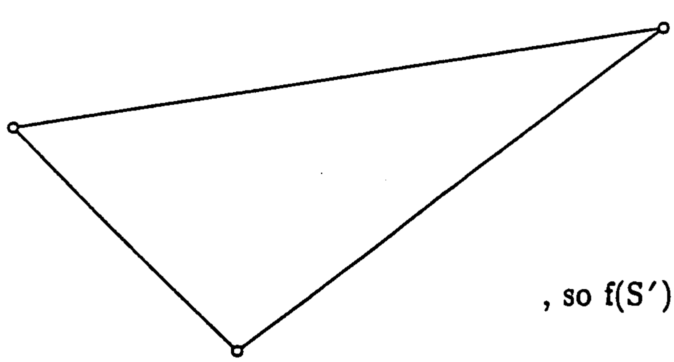
Proof: Starting with S, we invoke lemma 2.9 k times, on each occasion replacing one S_i by the corresponding T_i without altering the degree of freedom, until we are left with T, giving $f(S) = f(S_1\cup S_2\cup \ldots \ldots \cup S_{k-1}\cup S_k) = f(T_1\cup S_2\cup \ldots \ldots \cup S_{k-1}\cup S_k) = \ldots \ldots = f(T_1\cup T_2\cup \ldots \ldots \cup T_{k-1}\cup S_k) = f(T_1\cup T_2\cup \ldots \ldots \cup T_{k-1}\cup T_k) = f(T)$. □

We construct a structure S′ from S by choosing two joints, (x_1,y_1) and (x_2,y_2), of S and adding to S an extra joint (x,y) not collinear with (x_1,y_1) and (x_2,y_2), and the two bars $\{(x_1,y_1),(x,y)\}$ and $\{(x_2,y_2),(x,y)\}$.

Lemma 2.11: Then S is rigid iff S′ is rigid.

Proof: Suppose S′ is rigid. Then $rk(A_{S'})=2(|J(S)|+1)-3$. Since A_S is just $A_{S'}$ with two rows and two columns removed, $rk(A_S)\leq 2|J(S)|-3$, so S is rigid.

Conversely suppose S is rigid. Then by theorem 2.10 S′ has the same degree of freedom as



, so f(S′)=0. □

Lemma 2.12: If structures S_1 and S_2 are rigid, and $|J(S_1)\cap J(S_2)|>1$, then the union of S_1 and S_2, denoted by T, is rigid.

Proof: $f(T)=f(S_1\cup S(K_{J(S_1)\cap J(S_2)}))$ by lemma 2.9, and $f(S_1\cup S(K_{J(S_1)\cap J(S_2)}))=0$ by lemmas 1.22 and 1.23, since S_1 is rigid and all bars of $S(K_{J(S_1)\cap J(S_2)})$ are implicit bars of, or bars of S_1. So $f(T)=0$.

Theorem 2.13: For any planar bar and joint structure, the maximal rigid substructures are bar disjoint, and there is a unique partition of B(S) into $\dot{\bigcup}_i B(S_i)$ where the S_is are exactly the maximal rigid substructures.

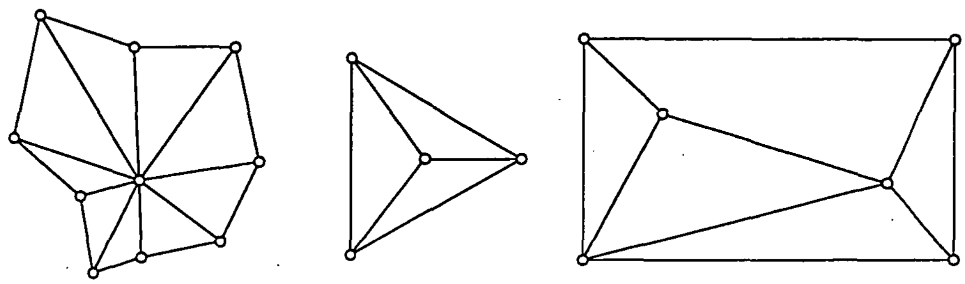
Proof: Clearly every bar is in a maximal rigid substructure.

Suppose a bar b is in two maximal rigid substructures, S and S′. Then $|J(S)\cap J(S')|>1$ and by lemma 2.12 $S\cup S'=T$ is rigid, contradicting the maximality of the rigidity of S and S′. Therefore every bar is in exactly one maximal rigid substructure. □

The properties of the structure geometry of a structure, S, are related to its rigidity characteristics. Since the structure geometry of an isostatic structure is free, its only basis consists of the set of all its bars. If S is rigid then the bases of D_S are the structure geometries of the isostatic substructures S_i of S for which $J(S)=J(S_i)$. As an isostatic structure S is rigid with D_S free, we see that the removal of any bar decreases the rank of D_S by one, and therefore the resulting structure cannot be rigid since its rank, being even, cannot be $2n-3$ for any n. A hyperstatic structure contains redundant bars. To understand flats of the structure geometry, consider a structure S′ as a substructure of the "complete structure $S(K_n)$" on those joints. If S′ is rigid then $cl(D_{S'})=D_{S(K_n)}$ where $J(S(K_n))=J(S')$ since if we add to S′ all the bars of $S(K_n)\backslash S'$ the rank cannot increase. If $S_1, \ldots, S_k$ are the maximal rigid substructures of S′, then $cl(D_{S'})\supset D_{S(K_{n_1})}\oplus \ldots \oplus D_{S(K_{n_k})}$, where $J(S(K_{n_i}))=J(S_i)$ $\forall\ i=1 \ldots k$.

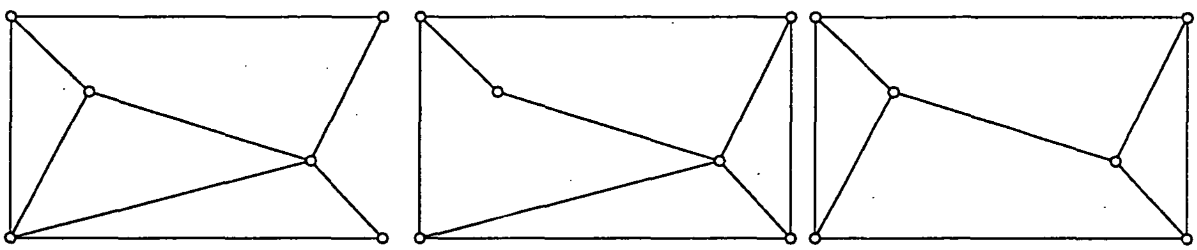
By looking at examples of circuits of D_S we see that some are rigid and some are not.

Example 2.14: Rigid circuits



S

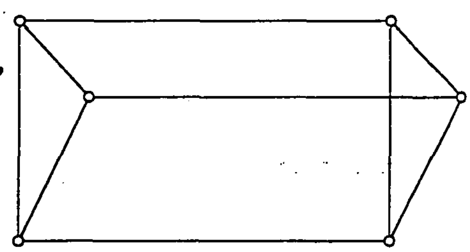
Bases of S:



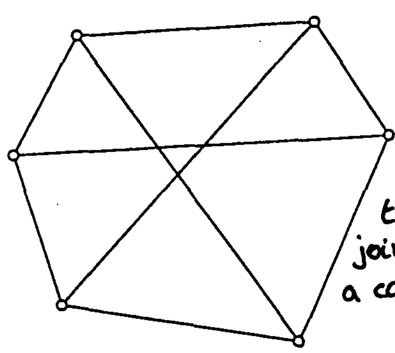
S'

Non–rigid circuits

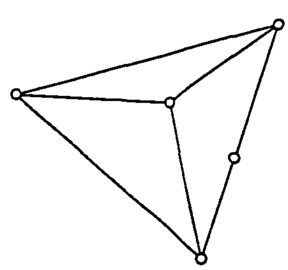
all apparently parallel lines here → are parallel.



S''



these six joints lie on a conic.

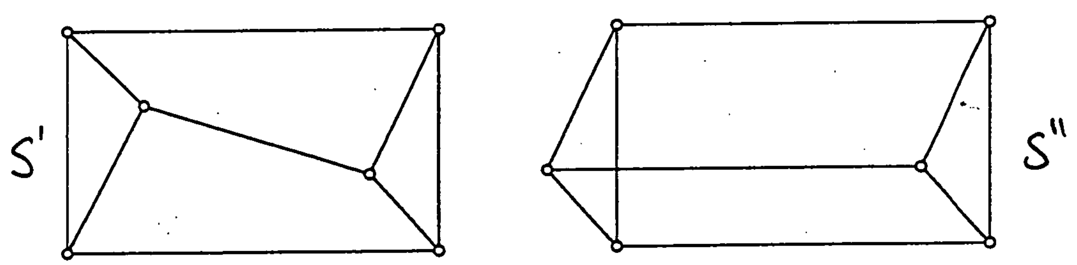


Those circuits which are rigid are necessarily minimally hyperstatic (i.e. every structure obtained from such a circuit by removing one bar, is isostatic) since every one bar deletion has the same rank, but any two bar deletion doesn't, and so every one bar deletion is a basis.

Genericity.

In this chapter we review the concept of generic structures by introducing our own definition and comparing to it the definitions offered by other people. Although this concept is independent of dimension and we accordingly give all definitions and results in this chapter a general setting, for simplicity our examples will be two dimensional.

Example 3.1: Consider S' and S'' from example 2.14. They have the same underlying graph but one is rigid and the other is not.



This example prompts the following definition:

Definition 3.2: Suppose we have a bar and joint structure S, in $\mathbb{R}^{\ell}$, with underlying graph G. If G is a single edge then S is *generic*, otherwise S is *generic* iff

i) $rk(A_S) \geq rk(A_T)$ $\forall$ structures T in $\mathbb{R}^{\ell}$ s.t. G underlies T, and

ii) every substructure S' of S is *generic.*

Lemma 3.3: For two generic structures S_1 and S_2 in $\mathbb{R}^\ell$ with the same underlying graph, the obvious bijection, I, between the bars is a matroid isomorphism.

Proof: Suppose I is not a matroid isomorphism. Then without loss of generality there exists a substructure S_1' of S_1 where the bars of the corresponding substructure, $I(S_1')=S_2'$ of S_2, are dependent, but $B(S_1')$ is independent. Therefore $rk(A_{S_1'})=|B(S_1')|=|B(S_2')|>rk(A_{S_2'})$, and so S_2 cannot be generic. □

Lemma 3.4: For two structures, S_1 and S_2 in $\mathbb{R}^\ell$, if S_1 and S_2 have the same underlying graph and S_1 is generic, then $D_{S_1} \geq D_{S_2}$.

Proof: Let S_2' be a substructure of S_2 s.t. $B(S_2')$ is independent, and let $S_1'=I(S_2')$ where I is the obvious bijection between S_1 and S_2 as in the previous lemma (it is not necessarily a matroid isomorphism in this case). Then $rk(A_{S_1'})\geq rk(A_{S_2'})=|B(S_2')|=|B(S_1')|$ and so $B(S_1')$ is independent. So D_{S_2} is a weak map image of D_{S_1}. □

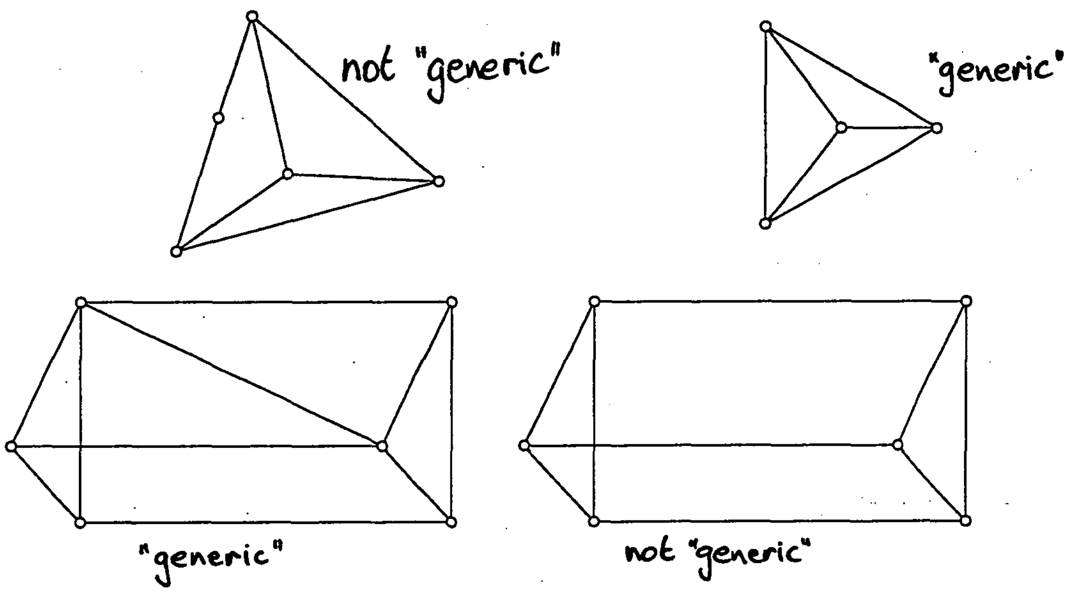
This concept is introduced to eliminate the type of degeneracy in the positions of the joints which we encountered in the last example, where S' is generic and S'' is not.

In 1979 Crapo [C13] defined *generic structure*; "A structure in such geometric position that it has maximum possible rank, given its topological makeup.".

We interpret this as meaning, for a structure S in $\mathbb{R}^\ell$: "If S has underlying graph G and $rk(A_S)\geq rk(A_T)$ $\forall$ structures T in $\mathbb{R}^\ell$ s.t. G underlies T, then S is *generic*."

This definition neatly divides structures into two classes; one whose members have more motions (i.e. lower rank) than other structures with the same graph, owing to a special arrangement of their joints, and one consisting of all other structures. This does seem a sensible distinction to make, but there is a slight drawback in that this definition allows a generic structure to have non–generic substructures, and therefore precludes the existence of a matroid isomorphism between generic structures with the same underlying graph.

Example 3.5: A planar example.

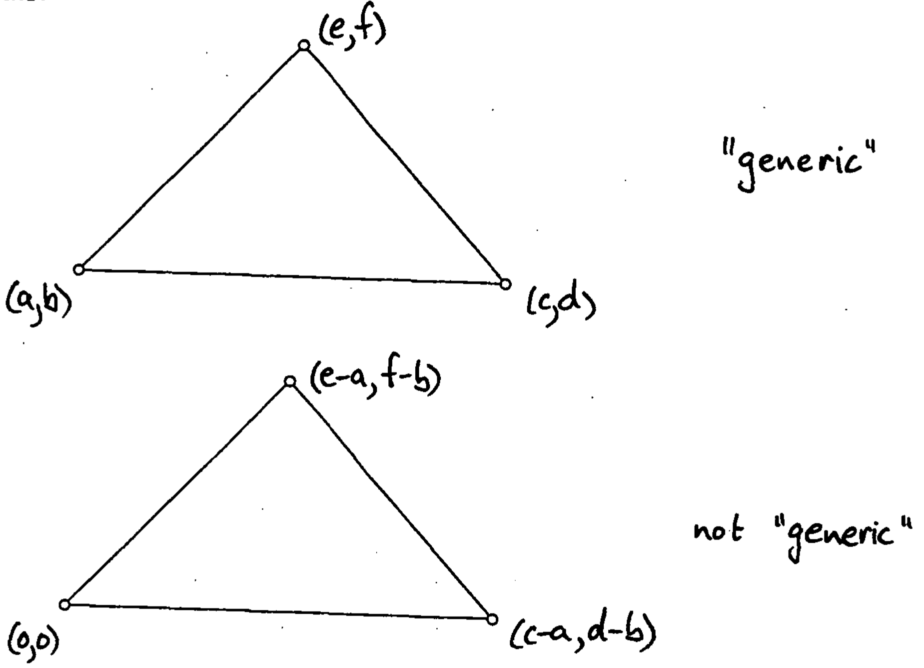


Thus in a sense this definition allows too many generic structures.

A more useful definition is used by Lovász and Yemini [L5] in 1982: "A structure S is *generic* if the coordinates of its joints are algebraically independent over the rational field."

This has the useful hereditary property of our definition, however it allows a generic structure and a non–generic structure with identical physical interpretations as real frameworks:

Example 3.6: a, b, c, d, e, and f are algebraically independent over the rationals.



Hence this definition defines too small a class of structures, by eliminating many structures which we think of as generic when realised physically.

Graver [G5] chooses the middle ground between these last two definitions. We paraphrase his definition: "Consider the determinant of each minor of A_S as the coordinates of the joints vary over all of $\mathbb{R}$. A structure is *generic* if all the nontrivial minors of A_S, i.e. the minors with determinants that are not identically zero, have nonzero determinants."

Theorem 3.7: Graver's definition for generic structures is equivalent to ours, and every structure which is generic by Lovász and Yemini's definition is generic by our definition.

Proof: A structure S in $\mathbb{R}^{\ell}$ is not generic in our sense

$\Leftrightarrow$ There exists a substructure S' of S s.t. $rk(A_{S'})<rk(A_T)$ for some structure T in $\mathbb{R}^{\ell}$ with the same underlying graph as S'.

$\Leftrightarrow$ $A_{S'}$ has a non–trivial minor with a zero determinant.

($\Leftrightarrow$ S is not generic in Graver's sense.)

$\Rightarrow$ There exists a polynomial (namely that determinant) which has the coordinates of the joints as roots.

$\Leftrightarrow$ S is not generic in the sense of Lovász and Yemini. □

Any results about the rigidity of structures which are generic in the sense of Lovász and Yemini, will also hold for our generic structures, because rigidity is defined in terms of rank, which is left invariant by matroid isomorphisms. Therefore as convenience dictates, we can use our definition, or Graver's definition, or even Lovász and Yemini's, in work concerning generic structures, and when we make general statements about generic structures we can safely include structures which are generic by our definition but not by Lovász and Yemini's.

If a structure is not generic, then by the definition of generic it has a dependent substructure, and so is itself dependent. Thus every non–generic structure contains a circuit. If we look back at our examples 2.14 we find that the circuits which were not rigid were not generic either. We have:

Theorem 3.8: Every rigid circuit in D_S is generic.

Proof: Suppose we have a rigid non–generic circuit B(C). Then $rk(A_C)<rk(A_T)$ for some T with the same underlying graph as C.

But $rk(A_C)=2|J(C)|-3<rk(A_T)$ is impossible. □

Theorem 3.9: Every minimal non–generic structure S in $\mathbb{R}^\ell$ is a circuit in D_S and every non–generic circuit in $\mathbb{R}^\ell$ is minimal non–generic.

Proof: Suppose we have a minimal non–generic structure C. Then since C is minimal non–generic, $rk(A_C)<rk(A_T)$ for some T with the same underlying graph as C. Also since C is minimal non–generic $rk(A_{C\backslash b})=rk(A_{T\backslash b'})$ since $rk(A_{T\backslash b'})\geq rk(A_T)-1$. Since b was chosen arbitrarily, C is a circuit.

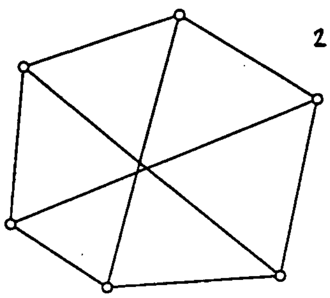
Conversely, every non–generic circuit is minimal non–generic since every substructure of a circuit is independent and therefore generic. □

The existence of the matroid isomorphism between generic structures in $\mathbb{R}^\ell$ with the same underlying graph shows us that the rigidity characteristics of a generic structure in $\mathbb{R}^\ell$ depend only on its underlying graph. Because of this we can regard the generic properties of structures in $\mathbb{R}^\ell$ as properties of graphs, and once we know G and ℓ we should be able to determine the rigidity properties of all generic structures in $\mathbb{R}^\ell$ with G as underlying graph. Therefore we speak of $A_{\ell G}$, $D_{\ell G}$, $N_{\ell G}$, $R_{\ell G}$, and $f_\ell(G)$ (or ℓ–degree of freedom) meaning respectively the coordinatising matrix, structure geometry, space of admissible motions, space of rigid motions, and degree of freedom of an arbitrary generic realisation in $\mathbb{R}^\ell$ of G.

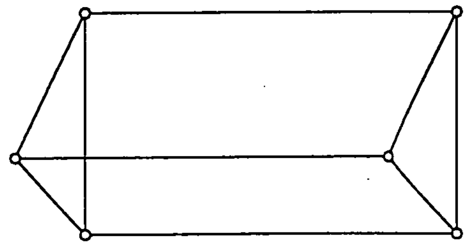
Definition 3.10: We say that a graph G is *ℓ–stiff* iff all the generic structures in $\mathbb{R}^\ell$ with underlying graph G, are rigid. Also if S(G) is an isostatic structure in $\mathbb{R}^\ell$ we call G an *ℓ–isostatic* graph.

Referring back to definition 1.18 this means: G is 2–stiff iff $\dim(N_{2G})=3$ and; G is 3–stiff iff $\dim(N_{3G})=6$ or G is a single edge (which incidentally is ℓ–stiff $\forall$ ℓ).

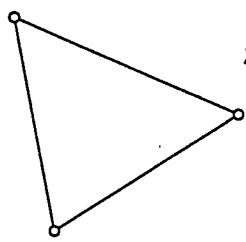
Examples:



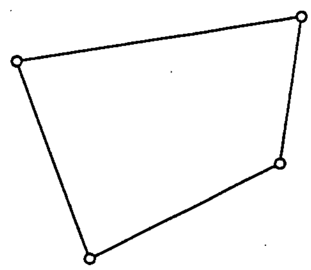
2-stiff
not 3-stiff



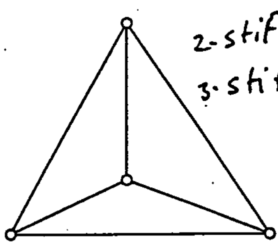
2-stiff
not 3-stiff



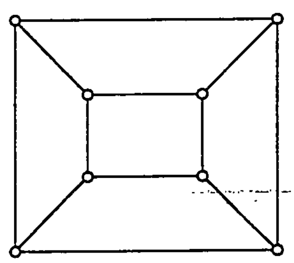
2-stiff
3-stiff



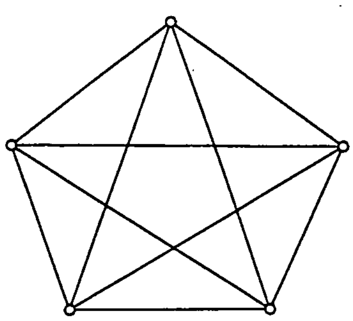
not 2-stiff
not 3-stiff



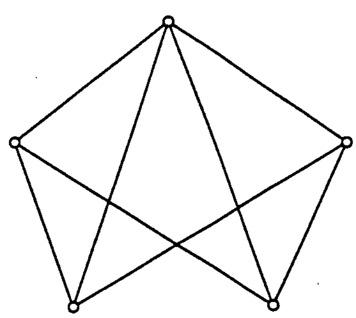
2-stiff
3-stiff



not 2-stiff
not 3-stiff



2-stiff
3-stiff



2-stiff
not 3-stiff

Generic Rigidity in $\mathbb{R}^2$.

The focus of this chapter is Laman's theorem. We give Lovász and Yemini's proof of this result and consequently some properties of the generic structure geometry in $\mathbb{R}^2$. We discuss some other proofs of Laman's theorem including a perspective which may give rise to another, more easily generalisable, proof.

To begin we raise the question of circuits in D_{2G}. Looking back once more to example 2.14 we see that all our examples of such circuits are rigid. Is this generally true?

To answer this question is not simple, but consequent upon the answer we find that structure geometries of generic structures in $\mathbb{R}^2$ have some nice properties which structure geometries of non–generic structures lack. For the rest of this chapter we shall say nothing more about non–generic structures, and shall devote ourselves entirely to generic structures. For this reason we will abandon reference to structures, bars and joints, and talk only of graphs, edges and vertices, where it must be understood that we mean generic structures with such graphs underlying them. We shall use the symbols G, A_{2G}, $f_2(G)$, N_{2G}, D_{2G}, 2–*stiff* and maximal 2–stiff subgraph, where before we used S, A_S, f(S), N_S, D_S, *rigid* and maximal rigid substructure. Invoking earlier results about a more general class of structures will not create problems since anything which is true for structures in general is certainly true for generic structures.

Now we look at some ideas and results by Lovász and Yemini, with their origins in polymatroid theory. For a less detailed, and thus in some ways clearer exposition, see [L3] [L5]. Also, we shall revert to their definition of generic structure:

Let G be a graph of a structure, with $E(G)=\{e_1,\ldots,e_m\}$. If we arbitrarily direct G, then the incidence matrix of G [0.4] is given by

$$a_{ij}=\begin{cases}1 & \text{if } i \text{ is the head of } e_j.\\ -1 & \text{if } i \text{ is the tail of } e_j.\\ 0 & \text{otherwise.}\end{cases}$$

(We shall be interested in the independence of the columns of (a_{ij}), and the way we direct the graph doesn't affect this independence since an opposite direction of an edge results in a column of (a_{ij}) which is different only by a factor of -1).

If $i \mapsto (x_i,y_i)$ gives a realisation where all the x_is and y_is are algebraically independent (i.e. the structure is generic according to Lovasz and Yemini), let $\underline{x}=(x_1,\ldots,x_n)$, $\underline{y}=(y_1,\ldots,y_n)$, and $\underline{a}_j=(a_{1j},\ldots,a_{nj})$. Here $\underline{a}_j$ is the j^{th} column of (a_{ij}), and therefore corresponds to the j^{th} edge of G.

Also let $A_i=\{\ (\lambda\underline{a}_i,\mu\underline{a}_i)\ :\ \lambda,\mu\ \in\ \mathbb{R}\ \}$ for $i=1..m$, and let $H=\{(\underline{z},\underline{z}'):(\underline{z}\cdot\underline{y})-(\underline{z}'\cdot\underline{x})=0\}$

Lemma 4.1: If X is the union of some A_is, then

$\langle(X\cap H)\cup A_i\rangle\cap\langle(X\cap H)\cup A_j\rangle\subseteq H \Rightarrow \langle(X\cap H)\cup A_i\rangle\cap\langle(X\cap H)\cup A_j\rangle\subseteq\langle X\cap H\rangle\ \forall\ 1\leq i,j\leq m.$

Proof: Consider $\underline{a}$, an arbitrary element of $\langle(X\cap H)\cup A_i\rangle$. Then, since $\{(\underline{a}_i,\underline{0}),(\lambda_i\underline{a}_i,\mu_i\underline{a}_i)\}$ is a basis of A_i, we know that $\underline{a}=\alpha(\underline{a}_i,\underline{0})+\sum_j\beta_j(\lambda_j\underline{a}_j,\mu_j\underline{a}_j)$, where $(\lambda_j\underline{a}_j,\mu_j\underline{a}_j)\in\langle X\cap H\rangle\ \forall\ j$. If also $\underline{a}\in H$, then from the definition of H, we have $((\alpha\underline{a}_i+\sum_j\beta_j\lambda_j\underline{a}_j)\cdot\underline{y})-((\sum_j\beta_j\mu_j\underline{a}_j)\cdot\underline{x})=0$, or $\alpha(\underline{a}_i\cdot\underline{y})+\sum_j\beta_j[\lambda_j(\underline{a}_j\cdot\underline{y})-\mu_j(\underline{a}_j\cdot\underline{x})]=0.$ However since $(\lambda_j\underline{a}_j,\mu_j\underline{a}_j)\in H$ we know that $\lambda_j(\underline{a}_j\cdot\underline{y})-\mu_j(\underline{a}_j\cdot\underline{x})=0$, and thus we have $\alpha(\underline{a}_i\cdot\underline{y})=0$. Since by genericity, no two entries of $\underline{y}$ are the same, we know that $\underline{a}_i \cdot \underline{y} \neq 0$, so we must have $\alpha=0$, and so $\underline{a}(=\sum_j \beta_j(\lambda_j \underline{a}_j, \mu_j \underline{a}_j))$ is in $\langle X \cap H \rangle$.

Thus we have seen that if $\underline{a} \in \langle (X \cap H) \cup A_i \rangle$ and $\underline{a} \in H$, then $\underline{a} \in \langle X \cap H \rangle$, from which the result follows. □

Lemma 4.2: $\dim\langle H \cap A_i : i=1..m \rangle = \min \sum_{i=1}^{k} (2\dim\langle \underline{a}_r : r \in N_i \rangle - 1)$, where the minimum is taken over all partitions $N_1, \ldots, N_k$ of $\{1, \ldots, m\}$ into non empty subsets.

Proof: Consider any partition $N_1, \ldots, N_k$ of $\{1, \ldots, m\}$. Let $Q = \bigcup_{i=1}^{m} (A_i \cap H)$ and $\bigcup_{r \in N_i} (A_r \cap H) = Q_i \ \forall\, i=1..k$. Now $\forall\, i=1..k$ we have:

Consider any $r \in N_i$. Since by genericity no two entries of $\underline{y}$ are the same, we know that $\underline{a}_r \cdot \underline{y} \neq 0$, so $(\underline{a}_r, \underline{0}) \notin H$. Since $\langle Q \rangle \subseteq H$ we see that $(\underline{a}_r, \underline{0}) \notin \langle Q \rangle$, but $(\underline{a}_r, \underline{0}) \in \langle \bigcup_{r \in N_i} A_r \cup Q \rangle$, so

$$1 + \dim\langle Q \rangle \leq \langle \bigcup_{r \in N_i} A_r \cup Q \rangle \qquad (*)$$

But by modularity of rank

$$\dim\langle A_r : r \in N_i \rangle + \dim\langle Q \rangle = \dim\langle \bigcup_{r \in N_i} A_r \cup Q \rangle + \dim\langle \langle A_r : r \in N_i \rangle \cap \langle Q \rangle \rangle, \text{ so}$$

$\dim\langle \langle A_r : r \in N_i \rangle \cap \langle Q \rangle \rangle \leq \dim\langle A_r : r \in N_i \rangle - 1$. From the definition of Q_i we know that $\dim\langle Q_i \rangle = \dim\langle \langle A_r : r \in N_i \rangle \cap \langle Q \rangle \rangle$ so we have $\dim\langle Q_i \rangle \leq \dim\langle A_r : r \in N_i \rangle - 1$.

Also by submodularity we have:

$$\dim\langle H \cap A_i : i=1..m \rangle = \dim\langle Q \rangle \leq \sum_{i=1}^{k} \dim\langle Q_i \rangle \qquad (**)$$

Together these last two inequalities imply:

$$\dim\langle H \cap A_i : i=1..m \rangle \leq \sum_{i=1}^{k} (2\dim\langle \underline{a}_r : r \in N_i \rangle - 1) \text{ for every partition.}$$

Thus to complete the proof it suffices to show the existence of a partition for which (*) and (**) are equalities.

Consider the relation: $i \equiv j \Leftrightarrow A_i \subseteq \langle A_j \cup Q \rangle$

i) $\equiv$ is symmetric.

ii) $i \equiv j \Leftrightarrow (\underline{a}_i,\underline{0}) \in \langle(\underline{a}_j,\underline{0}) \cup Q\rangle$ (since $A_j = \langle(\underline{a}_j,\underline{0}), A_j \cup H\rangle$)

$\Leftrightarrow$ there exists a circuit C_{ij} s.t. $\{(\underline{a}_i,\underline{0}),(\underline{a}_j,\underline{0})\} \subseteq C_{ij} \subseteq Q \cup \{(\underline{a}_i,\underline{0}),(\underline{a}_j,\underline{0})\}$

$\Leftrightarrow (\underline{a}_j,\underline{0}) \in \langle(\underline{a}_i,\underline{0}) \cup Q\rangle \Leftrightarrow j \equiv i$.

So $\equiv$ is reflexive.

iii) $i \equiv j$ & $j \equiv k \Rightarrow$ There exist circuits C_{ij} and C_{jk} $(=C_{ij})$ s.t.

$\{(\underline{a}_i,\underline{0}),(\underline{a}_j,\underline{0})\} \subseteq C_{ij} \subseteq Q \cup \{(\underline{a}_i,\underline{0}),(\underline{a}_j,\underline{0})\}$ and

$\{(\underline{a}_k,\underline{0}),(\underline{a}_j,\underline{0})\} \subseteq C_{kj} \subseteq Q \cup \{(\underline{a}_k,\underline{0}),(\underline{a}_j,\underline{0})\}$.

Since $(\underline{a}_j,\underline{0}) \in C_{ij} \cap C_{jk}$ and $(\underline{a}_i,\underline{0}) \in C_{ij} \backslash C_{jk}$, then by strong circuit exchange there exists a circuit C s.t. $(\underline{a}_i,\underline{0}) \in C \subseteq C_{ij} \cup C_{jk} \backslash (\underline{a}_j,\underline{0})$. This means there exists a circuit C s.t. $(\underline{a}_i,\underline{0}) \in C \subseteq Q \cup \{(\underline{a}_k,\underline{0}),(\underline{a}_i,\underline{0})\}$, so $i \equiv k$. So $\equiv$ is transitive.

Thus $\equiv$ is an equivalence relation and defines a partition.

For this partition $\dim\langle\langle A_r : r \in N_i\rangle \cup Q\rangle = \dim\langle A_r \cup Q\rangle$ for any $r \in N_i$. Also since $\dim A_r = 2$, but A_r is not in Q and $A_r \cap Q \neq \emptyset$, we have $\dim\langle Q\rangle + 1 = \dim\langle A_r \cup Q\rangle$. Together these imply $\dim\langle Q\rangle + 1 = \dim\langle\langle A_r : r \in N_i\rangle \cup Q\rangle\ \forall\ i = 1..k$. That is we have equality in equation (*).

Now consider $i \not\equiv j$ and suppose there exists a circuit $C \subseteq Q$ s.t. $A_i \cap H \in C$ and $A_j \cap H \in C$. Choose any $w \in C$. Then $w = A_x \cap H$ for some $x \in \{1,\ldots,k\}$. Let $X = (\bigcup_{r=1}^{m} A_r) \backslash A_x$. Then $\langle Q \cup A_i\rangle = \langle(Q\backslash w) \cup A_i\rangle$ since $w \in C \subseteq Q$

$= \langle(X \cap H) \cup A_i\rangle$, and similarly $\langle Q \cup A_j\rangle = \langle(X \cap H) \cup A_j\rangle$. Now $\langle Q \cup A_j\rangle \cap \langle Q \cup A_i\rangle \subseteq H$ (since $i \not\equiv j$), so invoking lemma 4.1 on X, A_i and A_j, we have $\langle Q \cup A_j\rangle \cap \langle Q \cup A_i\rangle \subseteq \langle X \cap H\rangle = \langle((\bigcup_{r=1}^{m} A_r) \backslash A_x) \cap H\rangle$. Since this is true for all $w \in C$, this shows that $\langle Q \cup A_j\rangle \cap \langle Q \cup A_i\rangle = \emptyset$, but $A_i \cap H \in C$ and we have a contradiction. Therefore the initial supposition about the nature of C was wrong, and for all $C \subseteq Q$ we have, $A_i \cap H \in C \Rightarrow A_j \cap H \notin C$ if $i \not\equiv j$. Therefore Q is a direct sum of $Q_1,\ldots,Q_k$ and so we have, for this partition, equality in equation (**) also, and the assertion is true. □

Theorem 4.3: The 2–degree of freedom of a graph G with n vertices is

$$2n-3-\min\sum_{i=1}^{k}(2|V(G_i)|-3)$$ where the minimum extends over all partitions of the edges of G: $E(G)=E(G_1)\dot{\cup}\ldots\dot{\cup}E(G_k)$.

Proof: In general $f_2(G) = 2n-3-\dim(\text{rowspace of } A_{2G}) = 2n-3-\dim\langle((\underline{a}_i\cdot\underline{x})\underline{a}_i,(\underline{a}_i\cdot\underline{y})\underline{a}_i): i=1..m\rangle$. Since the entries of $\underline{x}$ and $\underline{y}$ are algebraically independent over the rationals, we can apply the previous lemma to get $f_2(G)=2n-3-\min\sum_{i=1}^{k}(2\dim\langle\underline{a}_r:r\in N_i\rangle-1)$, where the minimum is taken over all partitions $N_1,\ldots,N_k$ of $\{1,\ldots,m\}$ into non–empty subsets.

Now by considering the isomorphism (see 0.14 and 0.15) between the usual cycle matroid (see 0.13 ii) on the edges of the graph G, and the dependence matroid on the incidence matrix of G, it is clear that $rkE(G_i)=\dim\langle\underline{a}_r:r\in N_i\rangle$, and so

$$f_2(G)=2n-3-\min\sum_{i=1}^{k}(2rkE(G_i)-1) \qquad (\dagger)$$

where rk is the rank function of the matroid on the edges G, and the minimum is taken over all partitions of the edges of G.

Now the partition minimising the right hand side of equation (†) is automatically such that each G_i is a connected subgraph. This follows from the fact that if G_e is not connected, then the partition with each component of G_e regarded as a separate subgraph of the partition will yield a smaller sum than the partition where just G_e as a whole is considered as a subgraph.

$rkE(G_i)=|V(G_i)|-1$ since G_i is connected.

Thus equation (†) becomes $2n-3-\min\sum_{i=1}^{k}(2|V(G_i)|-3)$. □

A corollary gives us a characterisation of 2–stiff graphs:

Corollary 4.4: A graph G is 2–stiff iff $\sum_{i=1}^{k}(2|V(G_i)|-3)\geq 2|V(G)|-3$ holds for every system of subgraphs G_i s.t. $E(G)=E(G_1)\dot{\cup}\dots\dot{\cup}E(G_k)$.

Proof: G is 2–stiff iff $f_2(G)=0$. □

We now return to our look at some generic properties of the structure geometry.

Theorem 4.5: D_{2G} is free iff $2|V(Y)|-3\geq|Y|$ holds for every $Y\subseteq E(G)$.

Proof: Now D_{2G} is free iff $f_2(G)=2n-3-m$, where $m=|E(G)|$. By corollary 4.4, this is equivalent to the condition that for every system of subgraphs $G_1,\dots,G_k$ s.t. $E(G)=E(G_1)\cup\dots\cup E(G_k)$, we have $\sum_{i=1}^{k}(2|V(G_i)|-3)\geq m$ (1)

We show that this is equivalent to the condition in the theorem:

Suppose equation (1) holds. Let H be an arbitrary subgraph of G. Choose $G_1=H$ and $G_2,\dots,G_k$ to be the subgraphs consisting of one edge of $E(G)\setminus E(H)$ each. Then equation (1) implies that $2|V(G_1)|-2+(k+1)\geq m$. But since $k=m-|E(H)|+1$, we obtain $2|V(H)|-3\geq|E(H)|$.

Conversely, suppose $2|V(Y)|-3\geq|Y|$ is true $\forall$ $Y\subseteq E(G)$. Then $\sum_{i=1}^{k}(2|V(G_i)|-3)\geq\Sigma|E(G_i)|\geq|E(G)|$ follows for arbitrary subgraphs $G_1,\dots,G_k$ s.t. $E(G)=E(G_1)\cup\dots\cup E(G_k)$, and equation (1) is true. □

Now we can answer our question regarding generic circuits.

Corollary 4.6: Every circuit in D_{2G} is 2–stiff.

Proof: Suppose C is a circuit which is not 2–stiff. Since C is minimally dependent the previous theorem implies that $2|V(C)|-3<|C|$. Also since C is not 2–stiff $rk(C)<2|V(C)|-3$.

Combining these we have $|C|-1=\mathrm{rk}(C)<2|V(C)|-3<|C|$ which is not possible. So every circuit is 2–stiff. □

A consequence of this corollary is that D_{2G} is a direct sum of the D_{2G_i}s, where the G_is are the maximal 2–stiff subgraphs of G. This immediately tells us that

Theorem 4.7: $f_2(G)=2n-3-\sum_{i=1}^{k}(2|V(G_i)|-3)$, where $E(G)=E(G_1)\dot{\cup}\ldots\dot{\cup}E(G_k)$ is the partition of G into maximal 2–stiff subgraphs.

Proof: Corollary 2.7 and the discussion above. □

Theorem 4.8: If G is a subgraph of K_n with the same vertices, then G is 2–stiff iff $\mathrm{cl}_2(G)=K_n$.

Proof: If G is 2–stiff then $\mathrm{rk}(A_{2G})=2|V(G)|-3=\mathrm{rk}(D_{2K_n})$, and so $\mathrm{cl}_2(G)=K_n$. If G is not 2–stiff then $\mathrm{rk}(D_{2G})\neq 2|V(G)|-3=\mathrm{rk}(D_{2K_n})$, and so $\mathrm{cl}_2(G)\neq K_n$. □

Furthermore, since rigidity is defined in terms of rank, $\mathrm{cl}_2(G)$ exhibits the same rigidity characteristics and mechanical behaviour as any basis of D_{2G}.

Now we present the result of Laman [L1] mentioned in the introduction, a characterisation of 2–isostatic graphs. This is essentially equivalent to theorems 4.4 and 4.5.

Laman's theorem 4.9: A graph G is 2–isostatic iff

i) $|E(G)|=2|V(G)|-3$ and

ii) $|E(H)|\leq 2|V(H)|-3$ for every subgraph H of G.

Proof: G is 2–isostatic $\Leftrightarrow$ G is 2–stiff and independent. Then ii) follows by theorem 4.5, and then i) follows by ii) and lemma 2.4.

Conversely, if i) and ii) hold, then E(G) is independent by ii) and theorem 4.5, and G is 2–stiff by i) and lemma 2.5. □

There exist two other proofs of this theorem. The first, by Laman [L1] (1970), is an induction proof on the number of vertices of the graph. The second, by Asimow and Roth [A2] (1979), is an induction proof on the number of edges of the graph. Both these proofs entail the construction, from a structure with known properties, of another larger structure whose properties depend on the method of construction and the properties of the original structure. Lovâsz and Yemini [L5] (1982), whose proof is the one presented here, have avoided this approach, and have revealed a complicated geometrical structure underlying the problem, which they have used in its solution, thereby revealing new results (2.6 2.7 4.3 4.4 4.7), and suggesting a possible fourth method of proof. Noting that Laman's theorem follows directly from theorem 4.7 we have:

Conjecture 4.10: Theorem 4.7 may be provable using an induction proof on the number of maximal 2–stiff subgraphs of G.

For reasons which will become evident in the next section, we tried without success produce such a proof. Here are some consequences of this attempt.

We introduce a new concept, the introduction of which, along with other subsequent ideas, entails adding edges to a graph G to obtain another graph G′. We encounter difficulties with this because if we try to add an edge between two vertices of G when there is already an edge between these two vertices, then the resulting graph G' is not simple, and therefore is not a graph underlying any structure. We remedy this by only adding an edge between two vertices if there isn't one there already. Since we may not know in advance whether or not an edge exists where we try to add one, when we say "add an edge" we mean "add an edge if one doesn't already exist".

Definition 4.11: The *relative 2–degree of freedom between two vertices*, 1 and 2 of a graph G is denoted and defined by $f_2(1,2)=rk(A_{G'})-rk(A_G)$, where G' is the graph obtained from G by adding an extra edge (1,2).

Also the *relative 2–degree of freedom between a vertex* 1 and *an edge* $e_2=(2,3)$, of a graph G is denoted and defined by $f_2(1,e_2)=f_2(e_2,1)=rk(A_{G'})-rk(A_G)$, where G' is the graph obtained from G by adding two extra edges, (1,2) and (1,3).

Also the *relative 2–degree of freedom between two edges*, $e_0=(1,2)$ and $e_1=(3,4)$ of a graph G is denoted and defined by $f_2(e_0,e_1)=rk(A_{G'})-rk(A_G)$, where G' is the graph obtained from G by adding:

i) three edges (1,3),(2,4),(2,3) if $|\{1,2,3,4\}|=4$, or

ii) one edge $(\{0,1\}\backslash i,\{2,3\}\backslash j)$ if i=j for some $i\in\{0,1\}$ and $j\in\{2,3\}$.

Finally the *2–relative degree of freedom between two 2–stiff subgraphs*, G_0 and G_1 where $e_0=(1,2)\in E(G_0)$ and $e_1=(3,4)\in E(G_1)$ of a graph G is denoted and defined by $f_2(G_0,G_1)=rk(A_{G'})-rk(A_G)$, where G' is the graph obtained from G by adding:

i) three edges (1,3),(2,4),(2,3) if $|\{1,2,3,4\}|=4$, or

ii) one edge $(\{0,1\}\backslash i,\{2,3\}\backslash j)$ if i=j for some $i\in\{0,1\}$ and $j\in\{2,3\}$.

Clearly the third of these definitions is a special case of the fourth definition.

It is a straightforward consequence of the unit increasing property of the rank function that:

$$0 \le f_2(1,2) \le 1$$
$$0 \le f_2(e_1,2) \le 2$$
$$0 \le f_2(e_1,e_2) \le 3 \ \forall \ e_1,e_2,1,2 \text{ in the graph}$$

s.t. these relative 2–degrees of freedom are defined.

Lemma 4.12:

If $e_0=(1,2)$ and $e_1=(3,4)$, then $f_2(1,3)=k \Rightarrow k \le f_2(e_0,e_1) \le k+2$ and $k \le f(e_0,3) \le k+1$. Also

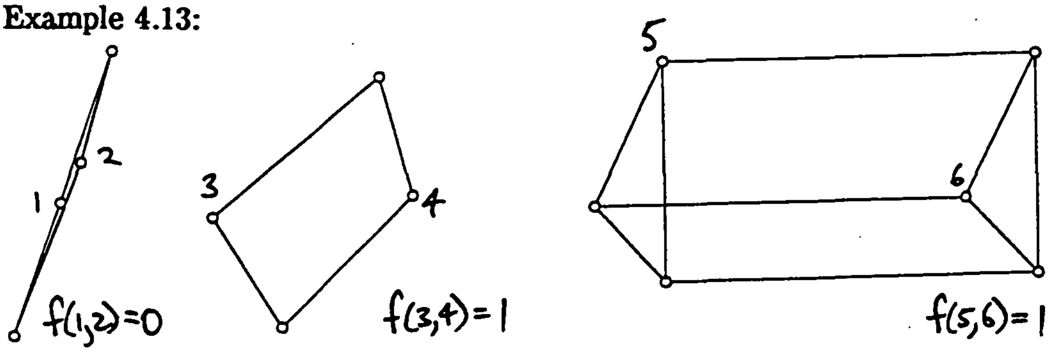
$$f_2(e_0,e_1)=k \Rightarrow \left.\begin{matrix} 0 \text{ if } k \ne 3 \\ 1 \text{ if } k=3 \end{matrix}\right\} \le f_2(1,3) \le \begin{cases} 1 \text{ if } k \ne 0 \\ 0 \text{ if } k=0 \end{cases} \text{ and } \left.\begin{matrix} k-1 & \text{if } k \ne 0 \\ 0 & \text{if } k=0 \end{matrix}\right\} \le f_2(e_0,3) \le \begin{cases} 2 \text{ if } k=3 \\ k \text{ if } k \ne 3. \end{cases}$$

Finally $f_2(e_0,3)=k \Rightarrow \left.\begin{matrix} 1 \text{ if } k=2 \\ 0 \text{ if } k \ne 2 \end{matrix}\right\} \le f_2(1,3) \le \begin{cases} 1 \text{ if } k=2 \\ k \text{ if } k \ne 2 \end{cases}$ and $k \le f\ (e\ ,e\ ) \le k+1$.

Proof: Straightforward consequence of the unit increasing property of the rank function. □

The reason this concept of relative 2–degree of freedom won't work nicely for non–generic structures is that it is an inherently mechanical concept which lacks relevance in some non–generic structures involving collinearity.

Example 4.13:



Intuitively the relative 2–degree of freedom between 1 and 2 would be expected to be greater than zero.

Also we can extend our original definition of 2–degree of freedom for disconnected graphs.

Definition 4.14: If a graph G has k disconnected components $G_1, \ldots, G_k$, then it has *2-degree of freedom* given by $f_2(G)=3(k-1)+\sum_{i=1}^{k} f_2(G_i)$. For connected graphs this is equivalent to our original definition.

Lemma 4.15: Suppose e_0 and e_1 are edges of G, and G_0 is the maximal 2–stiff subgraph of G which contains e_0. Then $f_2(e_0,e_1)=0$ iff G_0 contains e_1.

Proof: If e_1 is in G_0, then any edges added between the vertices of e_0 and e_1 will be in the rowspace of A_{2G}. So $f_2(e_1,e_0)=0$.

Conversely if $e_1 \notin G_0$ then we have two cases to consider;

either i) e_0 and e_1 share a common vertex $(e_0=(1,2), e_1=(1,3))$

or ii) not.

Case i) Suppose the graph G′ obtained from G by adding the edge a=(2,3) has the same 2–degree of freedom as G (i.e. $rk(A_{G'})=rk(A_G)$). this means that a is contained in a circuit, C, of G′. C is 2–stiff and C\a is 2–stiff. If G_1 is the maximal 2–stiff subgraph containing e_1, then by theorem 2.10, $f_2(E(G_0) \cup E(G_1) \cup E(C\backslash a)) = f_2(e_0 \cup e_1 \cup a) = 0$. Therefore G_0 and G_1 are not maximal 2–stiff subgraphs, contradicting our supposition that $rk(A_G)=rk(A_{G'})$, so $f_2(e_0,e_1) \neq 0$.

Case ii) This is essentially identical with case i) but with three edges playing the role of a in case i). Because of this, and because we later need to refer only to case i), we omit the details. □

A proof of this lemma which didn't assume the 2–stiffness of circuits, would furnish us with the mentioned fourth proof of Laman's theorem.

Given a graph G of a structure, we are going to derive a new graph G'' having one vertex fewer, the same number of edges, and possibly the same number of maximal 2–stiff subgraphs. To motivate this derivation we suggest, in general non–technical terms, thinking of a graph in terms its maximal 2–stiff subgraphs rather than its edges, as the fundamental components. If we choose a vertex from each of two maximal 2–stiff components, add an edge between these two vertices and then contract it, it seems likely that the 2–degree of freedom of the resulting graph will differ from that of the initial graph. We will want to know the exact extent of this change.

Construction 4.16: Start with a graph G of a structure.

i) Choose two vertices 1 and 2 $\epsilon V(G)$ s.t. 1 is contained in only one maximal 2–stiff subgraph, and $f_2(1,2)=1$. This last condition ensures that 1 and 2 are not in the same maximal 2–stiff subgraph.

ii) Choose an edge $e_1=(1,3)$ of the maximal 2–stiff subgraph G_1 containing the vertex 1.

iii) Derive G' from G by adding two edges a=(1,2) and b=(2,3). If $f_2(e_1,2)=1$, we see that $rk(A_G)+1=rk(A_{G'})$ and if $f_2(e_1,2)=2$, we see that $rk(A_G)+2=rk(A_{G'})$. i.e.

$$f_2(G)\text{-}f_2(G')=\begin{cases}1 \text{ if } f_2(e_1,2)=1\\ 2 \text{ if } f_2(e_1,2)=2.\end{cases} \qquad (1)$$

The 2–stiff subgraph of G' with edges $E(G_1)\cup\{a,b\}$ we call G_1'.

iv) Consider the graph G'' obtained from G by adding just edge (1,2) and contracting it. Using lemma 2.9 to compare G'' with G we find that

$$f_2(G'')=f_2(G') \qquad (2)$$

(Since if $E(G'')=E(G_1)\cup E(G_r)$ then $E(G')=E(G_1')\cup E(G_r)$, G_1 and G_1' are 2–stiff, and $V(G_1)\cap V(G_r)=V(G_1')\cap V(G_r)$)

v) From equations (1) and (2) we have $f_2(G)-f_2(G'')=\begin{cases}1 \text{ if } f_2(e_1,2)=1\\ 2 \text{ if } f_2(e_1,2)=2,\end{cases}$ and we know that G has one more vertex than G'' but has the same number of edges. Furthermore, if we call all the edges by the same names in G and G'', we see that every subset of edges of G which forms a maximal 2–stiff subgraph of G, must form a 2–stiff subgraph of G'', since we have done nothing to affect the internal structure of each 2–stiff subgraph. □

Lemma 4.17: For the number of maximal 2–stiff subgraphs in G'' to equal the number of maximal 2–stiff subgraphs in G we require at least that $f_2(e_1,e_j)=3\ \forall$ e_j s.t. $e_j=(2,4)$ for some $4\in V(G)$.

Proof: Suppose $f_2(e_1,e_2)<3$ for some $e_2=(2,6)$ (and $f_2(e_1,e_j)\geq 1\ \forall\ e_j$ s.t. $e_j=(2,4)$ for some $4\in V(G)$). Then vertex 3 is not vertex 6 since otherwise our resulting graph G'' would have a doubled edge and hence not be the graph of any structure. Now we perform our construction and obtain

$$f_2(G)-f_2(G'')=\begin{cases}1 \text{ if } f_2(e_1,e_2)=1\\ 2 \text{ if } f_2(e_1,e_2)=2\end{cases} \tag{3}$$

Consider G^* obtained by adding to G' the edge c=(3,6), and G^{**} obtained by adding to G'' the edge c=(3,6). By theorem 2.10 $f_2(G^{**})=f_2(G^*)$. Also by the definition of $f_2(e_1,e_2)$ we know $f_2(G)-f_2(G^{**})=\begin{cases}1 \text{ if } f_2(e_1,e_2)=1\\ 2 \text{ if } f_2(e_1,e_2)=2.\end{cases}$ Combining these two with equation three we have $f_2(G'')-f_2(G^{**})=0$ if $1\leq f_2(e_1,e_2)\leq 2$. This implies that edge c is in the rowspace of $A_{G''}$, so that $f_2(e_1,e_2)=0$ in G''. Hence by the preceding lemma, e_0 and e_1 are in the same maximal 2–stiff subgraph of G''. This violates the desired property of conservation of the number of maximal 2–stiff subgraphs, and our lemma is established. □

Now we know that if we start our construction with e_0 and e_1 which satisfy $f_2(e_0,e_1)=3$, then $f_2(G)=f_2(G'')+2$, and the number of maximal 2–stiff subgraphs of G may or may not equal the number of maximal 2–stiff subgraphs of G'', but if we start our construction with e_0 and e_1 s.t. $f_2(e_0,e_1)<3$ then the number of maximal 2–stiff subgraphs of G will not equal the number of maximal 2–stiff subgraphs of G''.

Theorem 4.18 (4.7): The 2–degree of freedom of a graph G with n vertices is $2n-3-\sum_{i=1}^{k}(2|V(G_i)|-3)$ where $E(G)=E(G_1)\dot{\cup}\ldots\dot{\cup}E(G_k)$ is the partition of the edges of G into maximal 2–stiff subgraphs.

Proof: We use induction on the number of maximal 2–stiff subgraphs, using construction 4.16.

k=1: $f_2(G)=0=2n-3-(2|V(G)|-3)$ o.k.

Suppose the result is true for every graph with k maximal 2–stiff subgraphs. Consider a graph G_T with k=1 maximal 2–stiff subgraphs. The graph $G_T\backslash G_{k+1}$ obtained from G_T by deleting every edge of the maximal 2–stiff subgraph G_{k+1}, along with all the requisite vertices, has k maximal 2–stiff subgraphs, so $f_2(G_T\backslash G_{k+1})=2|V(G_T\backslash G_{k+1})|-3-\sum_{i=1}^{k}(2|V(G_i)|-3)$. Now consider the graph F consisting of the two disconnected components $G_T\backslash G_{k+1}$ and a copy of G_{k+1} called G_{k+1}^{copy}.

$f_2(F)=f_2(G_T\backslash G_{k+1})+3$, Since G_{k+1}^{copy} is 2–stiff.

$=2|V(G_T\backslash G_{k+1})|-\sum_{i=1}^{k}(2|V(G_i)|-3)$

$=2(|V(G_T\backslash G_{k+1})|+|V(G_{k+1}^{copy})|)-3-\Sigma(2|V(G_i)|)$

Now commencing with our graph F, we use our construction process $|V(G_T\backslash G_{k+1})\cap V(G_{k+1}^{copy})|$ times until F becomes G_T. Every time we join two vertices of F we lose two 2–degrees of freedom, since every time we execute our construction, the number of maximal 2–stiff subgraphs doesn't change. Thus $f_2(G_T)=2|V(G_T)|-3-\sum_{i=1}^{k}(2|V(G_i)|-3)$ and by induction the result is again established. □

From this theorem it is straightforward to establish corollary 4.6, theorem 4.5, and Laman's theorem 4.9, thus if it could be independently proven that lemma 4.15 is true we would have an independent proof of these results.

We finish this chapter on planar rigidity by giving two results about the 2–degree of freedom of a graph in terms of its maximal 2–stiff subgraphs. Non–generic examples of this type of structure were looked at by Grünbaum and Shephard [G7], and generic examples were looked at by Rooney and Wilson [R8], who stated the following theorem.

Theorem 4.19: If J is the number of vertices of G which are in more than one maximal 2–stiff subgraph, and ℓ_j is the number of maximal 2–stiff subgraphs with j vertices in more than one maximal 2–stiff subgraph, then $f_2(G)=2J-3-\sum_{i=1}(2i-3)\ell_i$.

Proof: $f_2(G)=2n-3-\sum_{i=1}^{k}(|V(G_i)|-3)$. Now relabel this partition into G_is so that all the G_is with j vertices in more than one maximal 2–stiff subgraph are $G_{j1},\ldots$ $\ldots,G_{j\ell_j}$. Then $f_2(G)=2n-3-\sum_{j=1}\sum_{i=1}^{\ell_j}(2|V(G_{ji})|-3)$.

But $\sum_{i=1}^{\ell_j}(2|V(G_{ji})|-3)=2j\ell_j-3\ell_j+2\sum_{i=1}^{\ell_j}(|V(G_{ji})|-3)=(2j-3)\ell_j+2\sum_{i=1}^{\ell_j}(|V(G_{ji})|-j)$, so

$f_2(G)=2n-3-\sum_{j=1}(2j-3)\ell_j-\sum_{j=1}\sum_{i=1}^{\ell_j}(2|V(G_{ji})|-2j)=2[n-\sum_{j=1}\sum_{i=1}^{\ell_j}(|V(G_{ji})|-j)]-3-\sum_{j=1}(2j-3)\ell_j$.

But $n-\sum_{j=1}\sum_{i=1}^{\ell_j}(|V(G_{ji})|-j)=n-\sum_{j=1}((\sum_{i=1}^{\ell_j}|V(G_{ji})|)-j\ell_j)=n-\sum_{i=1}^{k}|V(G_i)|+\sum_{j=1}j\ell_j=J$ so

$f_2(G)=2J-3-\sum_{j=1}(2j-3)\ell_j$ as required. □

There is a similar formula giving the 2–degree of freedom of a graph in terms of the vertices of G which are in j maximal 2–stiff subgraphs.

Theorem 4.20: If G has k maximal 2–stiff subgraphs, and m_j is the number of vertices of G in j maximal 2–stiff subgraphs, then $f_2(G)=3(k-1)-\sum_{i=1}(2i-2)m_i$.

Proof: The number of 2–degrees of freedom of k disconnected bodies is 3(k–1). Now if we use our construction process to derive G from joining these k disconnected subgraphs then each join must reduce the 2–degree of freedom by two, and since the number of such joins is $\sum_{i=1}(i-1)m_i$, the result follows. □

Rigidity in $\mathbb{R}^3$.

This chapter is an examination of bar and joint structures in $\mathbb{R}^3$, and the results presented mirror those of chapter 2 as far as possible. We highlight the point at which the dissimilarity between bar and joint structures in $\mathbb{R}^2$ and bar and joint structures in $\mathbb{R}^3$ occurs and give the usual counterexample to the obvious analogue of Laman's theorem for bar and joint structures in $\mathbb{R}^3$.

Definition 5.1: A *spatial bar and joint structure* is a bar and joint structure in $\mathbb{R}^3$. Throughout this chapter we shall mean "spatial bar and joint structure" when we say "bar and joint structure" or "structure".

Again we note non–ambiguous notation abuses at 5.10 6.15, and 6.16. These consist of using graph terminology for structures and vice versa, and shouldn't lead to confusion. Also on page 66 vectors $\underline{u}_1,\underline{u}_2,\underline{u}_3,\underline{u}_4,\underline{u}_5,\underline{u}_6,\underline{v}_S$ & $\underline{m}_S$ have had an unexplained rearrangement of co–ordinates.

Lemma 5.2: If S is a non–collinear structure, and T is a collinear structure, then

i) $f(S)+rk(A_S)=3n-6$	$f(T)+rk(A_T)=3n-5$.
ii) $0\leq f(S)\leq 3n-6$	$0\leq f(T)\leq 3n-5$.
iii) $0\leq rk(A_S)\leq 3n-6$	$0\leq rk(A_T)\leq 3n-5$.
iv) S is rigid iff $f(S)=0$	T is rigid iff $f(T)=0$.

Proof: If T is collinear then we assert without proof that $\dim(R_T)=5$, not 6, since a collinear structure in space has one rotational degree of freedom less than a non-collinear structure in space. The remainder of this proof is exactly analogous to the proof of lemma 2.2. □

Theorem 5.3: Unless S is a single bar, $rk(A_S) \leq \min\{|B(S)|, 3|J(S)|-6\}$.

Proof: A_S has $|B(S)|$ rows and $3|J(S)|$ columns, and unless S is a collinear triangle $\dim(N_S) \geq 6$. If S is a collinear triangle, then $|J(S)|=|B(S)|=3$, and $rk(A_S)=2$ and the result is clear. □

Lemma 5.4: If a structure S is rigid then $|B(S)| \geq 3|J(S)|-6$

Proof: If a structure S, which is not a single bar, is rigid then $rk(A_S)=3|J(S)|-6$ and consequently $|B(S)| \geq 3|J(S)|-6$ since B(S) is the number of rows of A_S. A single bar has $|B(S)|=1 \geq 0=3|J(S)|-6$. □

Theorem 5.5: Unless S is a single bar, any two of the following conditions imply the third:

i) B(S) is independent in D_S.

ii) S is rigid.

iii) $3|J(S)|-6=|B(S)|$.

Proof: i & ii ⇒ iii: S is rigid so $rk(A_S)=3|J(S)|-6$, and B(S) is independent, so $rk(A_S)=|B(S)|$.

i & iii ⇒ ii: B(S) is independent, so $rk(A_S)=|B(S)|=3|J(S)|-6$. So S is rigid.

iii & ii ⇒ i: S is rigid, so $rk(A_S)=3|J(S)|-6=|B(S)|$. So B(S) is independent. □

Theorem 5.6: Let $B(S)=B_1\dot{\cup}\ldots\dot{\cup}B_k$ be a partition of the bars of S so that $B_i=B(S_i)$ is the set of bars of a substructure S_i of S. Then

$$f(S)=3|J(S)|-6-\sum_{i=1}^{k}(3|J(S_i)|-6-f(S_i)) \text{ iff } D_S=D_{S_1}\oplus\ldots\oplus D_{S_k}.$$

Proof: $\sum_{i=1}^{k}(3|J(S_i)|-6-f(S_i))=3|J(S)|-6-f(S)$ iff $\sum_{i=1}^{k}rk(A_{S_i})=rk(A_S)$ iff

$$\bigoplus_{i=1}^{k} D_{S_i}=D_S.$$ □

Corollary 5.7: Let $B(S)=B_1\dot{\cup}\ldots\dot{\cup}B_k$ be a partition of the bars of S so that $B_i=B(S_i)$ is the set of bars of a substructure S_i of S. Then

$$f(S)=3|J(S)|-6-\sum_{i=1}^{k}(3|J(S_i)-6) \text{ iff}$$

both $D_S=D_{S_1}\oplus\ldots\oplus D_{S_k}$ and S_i is rigid $\forall$ i=1.. ..k.

Proof: Each S_i is rigid iff $f(S_i)=0$. □

Now suppose we have a rigid substructure S which is not a single bar, of some other structure, with $J(S)=\{(x_1,y_1,z_1),\ldots,(x_n,y_n,z_n)\}$. We define a six dimensional vector space $M_S^{ijk}=\{(\alpha_1+y_i\alpha_5+z_i\alpha_6,\ \alpha_1+y_j\alpha_5+z_j\alpha_6,\ \alpha_1+y_k\alpha_5+z_k\alpha_6,\ \alpha_2-z_i\alpha_4-x_i\alpha_5,\ \alpha_2-z_j\alpha_4-x_j\alpha_5,\ \alpha_2-z_k\alpha_4-x_k\alpha_5,\ \alpha_3+y_i\alpha_4-x_i\alpha_6,\ \alpha_3+y_j\alpha_4-x_j\alpha_6,\ \alpha_3+y_k\alpha_4-x_k\alpha_6) : \alpha_1,\ldots,\alpha_6 \in \mathbb{R}\}$, with addition $(a_1,\ldots,a_9)+(b_1,\ldots,b_9) = (a_1+b_1,\ldots,a_9+b_9)$, and scalar multiplication $\gamma(a_1,\ldots,a_9) = (\gamma a_1,\ldots,\gamma a_9)$, where i, j & k are between 1 and n.

We consider the transformation B_S^{ijk} to M_S^{ijk} from N_S defined by $\alpha_1\underline{u}_1+\ldots+\alpha_6\underline{u}_6 \mapsto (\alpha_1+y_i\alpha_5+z_i\alpha_6,\ \alpha_1+y_j\alpha_5+z_j\alpha_6,\ \alpha_1+y_k\alpha_5+z_k\alpha_6,\ \alpha_2-z_i\alpha_4-x_i\alpha_5,\ \alpha_2-z_j\alpha_4-x_j\alpha_5,\ \alpha_2-z_k\alpha_4-x_k\alpha_5,\ \alpha_3+y_i\alpha_4-x_i\alpha_6,\ \alpha_3+y_j\alpha_4-x_j\alpha_6,\ \alpha_3+y_k\alpha_4-x_k\alpha_6)$, is called B_S^{ijk}, where $\underline{u}_1=(1,\ldots,1,0,\ldots,0,0,\ldots,0,0,\ldots,0)$,

$\underline{u}_2=(0,\ldots,0,1,\ldots,1,0,\ldots,0,0,\ldots,0)$,

$\underline{u}_3=(0,\ldots,0,0,\ldots,0,1,\ldots,1,0,\ldots,0)$,

$\underline{u}_4=(0,\ldots\ldots,0,-z_1,\ldots\ldots,-z_n,y_1,\ldots\ldots,y_n,0,\ldots\ldots,0)$,

$\underline{u}_5=(y_1,\ldots\ldots,y_n,-x_1,\ldots\ldots,-x_n,0,\ldots\ldots,0,0,\ldots\ldots,0)$,

$\underline{u}_6=(z_1,\ldots\ldots,z_n,0,\ldots\ldots,0,-x_1,\ldots\ldots,-x_n,0,\ldots\ldots,0)$ with the extra zeroes corresponding to vertices of the structure of which S is a substructure.

Lemma 5.8: For any three non-collinear joints i, j and k of a rigid substructure S, which is not a single bar, of some larger structure, B_S^{ijk} is a bijective linear transformation.

Proof: Discussion above. □

Lemma 5.9: If S and T are two structures where $B(S)=B(S_1)\cup B(S_2)$, $B(T)=B(T_1)\cup B(S_2)$, S_1 and T_1 are rigid substructures of S and T respectively, but are not single bars, and $J(S_1)\cap J(S_2)=J(T_1)\cap J(S_2)\neq\emptyset$, then $f(S)=f(T)$

Proof: We demonstrate the existence of a bijective linear transformation between N_S and N_T, using the fact that if the velocities of three non-collinear joints of a rigid body are known, then the velocities of all the joints of the rigid body are consequently known, as shown in the previous lemma.

For every $\underline{m}_S\in N_S$ we define a unique $\underline{v}_S=(0,\ldots\ldots,0,a_{r+1},\ldots\ldots,a_{2n})$ where $\underline{m}_S=(a_1,\ldots\ldots,a_r,a_{r+1},\ldots\ldots,a_{2n})$ where the last $|J(S_1)|=2n-r$ entries correspond to joints of S_1. Similarly for each $\underline{m}_T\in N_T$ we define a $\underline{v}_T$.

Define a bijective linear transformation $L'':N_{S_1}\rightarrow M_{S_1}\rightarrow M_{T_1}\rightarrow N_{T_1}$ by

$$L''=\begin{cases} (B_{T_1}^{ih\ell})^{-1}\circ I^{jkh\ell}\circ B_{S_1}^{ijk} \text{ if } J(S_1)\cap J(S_2)=\{i\},\ j,k\in J(S_1),\ h,\ell\in J(T_1). \\ (B_{T_1}^{ij\ell})^{-1}\circ I^{jkj\ell}\circ B_{S_1}^{ijk} \text{ if } J(S_1)\cap J(S_2)=\{i,j\},\ k\in J(S_1),\ \ell\in J(T_1). \\ (B_{T_1}^{ijk})^{-1}\circ I^{jkjk}\circ B_{S_1}^{ijk} \text{ if } \{i,j,k\}\subset J(S_1)\cap J(S_2). \end{cases}$$

where $I^{jkh\ell}:M_{S_1}^{ijk}\rightarrow M_{T_1}^{ih\ell}$ is defined by; $(\alpha_1+y_i\alpha_5+z_i\alpha_6,\ \alpha_1+y_j\alpha_5+z_j\alpha_6,$ $\alpha_1+y_k\alpha_5+z_k\alpha_6,\ \alpha_2-z_i\alpha_4-x_i\alpha_5,\ \alpha_2-z_j\alpha_4-x_j\alpha_5,\ \alpha_2-z_k\alpha_4-x_k\alpha_5,\ \alpha_3+y_i\alpha_4-x_i\alpha_6,$

$\alpha_3+y_j\alpha_4-x_j\alpha_6$, $\alpha_3+y_k\alpha_4-x_k\alpha_6) \mapsto (\alpha_1+y_i\alpha_5+z_i\alpha_6$, $\alpha_1+y_h\alpha_5+z_h\alpha_6$, $\alpha_1+y_\ell\alpha_5+z_\ell\alpha_6$, $\alpha_2-z_i\alpha_4-x_i\alpha_5$, $\alpha_2-z_h\alpha_4-x_h\alpha_5$, $\alpha_2-z_\ell\alpha_4-x_\ell\alpha_5$, $\alpha_3+y_i\alpha_4-x_i\alpha_6$, $\alpha_3+y_h\alpha_4-x_h\alpha_6$, $\alpha_3+y_\ell\alpha_4-x_\ell\alpha_6)$.

Now define $L':\{\underline{u}_S=\underline{m}_S-\underline{v}_S:\underline{m}_S\in N_S\}\to\{\underline{u}_T=\underline{m}_T-\underline{v}_T:\underline{m}_T\in N_T\}$ by $L'(a_1,\ldots,a_r,0,\ldots,0)=(a_1,\ldots,a_r,0,\ldots,0)$.

It isn't clear that L' is well defined. It may be that there is a $\underline{u}_S$ s.t. $L'(\underline{u}_S)$ doesn't exist. If this is not a problem, then L' is a bijective linear transformation. We show now that L' is well defined.

Consider the systems of orthogonality conditions which originally gave rise to the matrices A_S and A_T. First we have, common to S and T, all the equations derived from the bars of S_2. Call this system 1. Let system 2 be the system of equations derived from the bars of T_1, and system 3 be the system of equations derived from the bars of S_1.

We consider three cases; i) where $|J(S_1)\cap J(S_2)|\geq 3$, ii) where $|J(S_1)\cap J(S_2)|=2$, and iii) where $|J(S_1)\cap J(S_2)|=1$. In case i) we know systems 2 and 3 have 6–dimensional solution spaces, and the solutions can be expressed in terms of $x_i,x_j,x_k,y_i,y_j,y_k,z_i,z_j,z_k$, where $(x_i,y_i,z_i),(x_j,y_j,z_j)\&(x_k,y_k,z_k)\in J(S_1)\cap J(S_2)$ and $J(S_1)\cap J(S_2)$, from the lemma 5.8. Since the remainder of each system is identical (system 1) and contains $x_i,x_j,x_k,y_i,y_j,y_k,z_i,z_j,z_k$, we see that for every $\underline{u}_S$ there does exist a corresponding $\underline{u}_T$, and we are therefore assured that L' is well defined.

Case ii) is similar: since S_1 and T_1 are not collinear they contain, say, $(x_{k_1},y_{k_1},z_{k_1})$ and $(x_{k_2},y_{k_2},z_{k_2})$ respectively, and we can apply the argument from case i). Case iii) is similar: since S_1 and T_1 are not collinear, S_1 contains $(x_{k_1},y_{k_1},z_{k_1})$ and $(x_{j_1},y_{j_1},z_{j_1})$, and T_1 contains $(x_{k_2},y_{k_2},z_{k_2})$ and $(x_{j_2},y_{j_2},z_{j_2})$, and we can apply the argument from case i).

In each case there exists a bijective linear transformation $L:N_S \to N_T$ defined by $L(\underline{m}_S)=L(\underline{u}_S+\underline{v}_S)=L'(\underline{u}_T)+L''(\underline{v}_T)=\underline{u}_T+\underline{v}_T=\underline{m}_T$. So $\dim(N_S)=\dim(N_T)$ and $f(T)=f(S)$. □

Theorem 5.10: Let $S_1,\ldots,S_k$ be rigid substructures of S, each of at least three vertices, s.t. $B(S)=B(S_1)\cup\ldots\cup B(S_k)$, and let $T_1,\ldots,T_k$ be rigid substructures of T, each of at least three vertices, s.t. $B(T)=B(T_1)\cup\ldots\cup B(T_k)$. Further let S' be the set of joints of S which are in more than one S_i, and let T' be the set of joints of T which are in more than one T_i, and $J(T_i)\cap T'=J(S_i)\cap S'$ $\forall$ i=1..k. Then $f(S)=f(T)$.

Proof: Starting with S, we invoke lemma 5.9 k times, on each occasion replacing one S_i by the corresponding T_i without altering the degree of freedom, until we are left with T: $f(S) = f(S_1\cup S_2\cup\ldots\cup S_{k-1}\cup S_k) = f(T_1\cup S_2\cup\ldots\cup S_{k-1}\cup S_k)$ $=\ldots= f(T_1\cup T_2\cup\ldots\cup T_{k-1}\cup S_k) = f(T_1\cup T_2\cup\ldots\cup T_{k-1}\cup T_k) = f(T)$. □

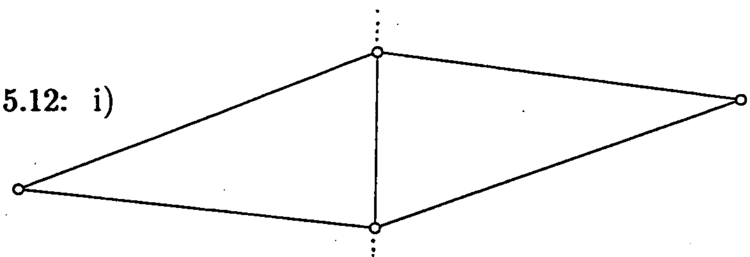
Lemma 5.11: If a structure S' is obtained from S by choosing three joints of S, $\chi_i=(x_i,y_i,z_i)$, $\chi_j=(x_j,y_j,z_j)$ and $\chi_k=(x_k,y_k,z_k)$ which are not collinear, and adding to S an extra joint $(x_{n+1},y_{n+1},z_{n+1})=\chi_{n+1}$ not coplanar with the first three joints, and the three bars (χ_i,χ_{n+1}), (χ_j,χ_{n+1}) and (χ_k,χ_{n+1}), then S is rigid iff S' is rigid.

Proof: Suppose S' is rigid. Then $rk(A_{S'})=3(|J(S)|+1)-6$. Since A_S is simply $A_{S'}$ with three rows and three columns removed, we have $rk(A_S)\geq 3|J(S)|-6$, so S is rigid.

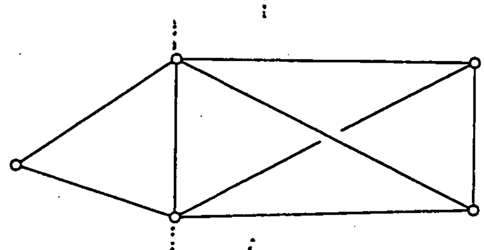
Conversely suppose S is rigid. Then by the theorem 5.10, S' has the same degree of freedom as the (obviously rigid) structure consisting of the four joints $\chi_i,\chi_j,\chi_k,\chi_{n+1}$ and the six possible bars between them, so $f(S')=0$. □

Until this point there are few significant differences between the planar and the spatial cases. The main differences have been with the small, low dimensional, degenerate exceptions. To be exact, all results and definitions from 2.1 to 2.11 have direct analogues for spatial structures, as we have demonstrated. However we note at this stage that lemma 2.12 and theorem 2.13 have no direct analogues for spatial bar and joint structures.

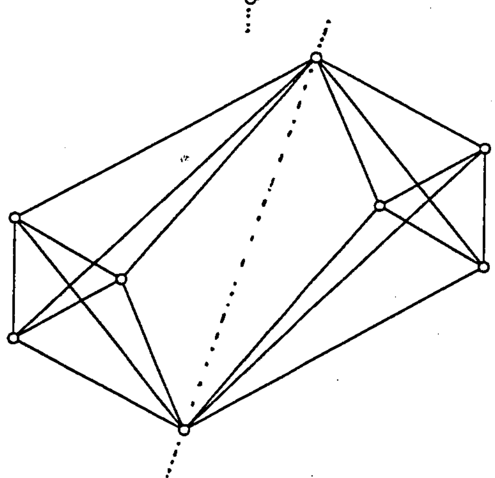
Example 5.12: i)



ii)



iii)



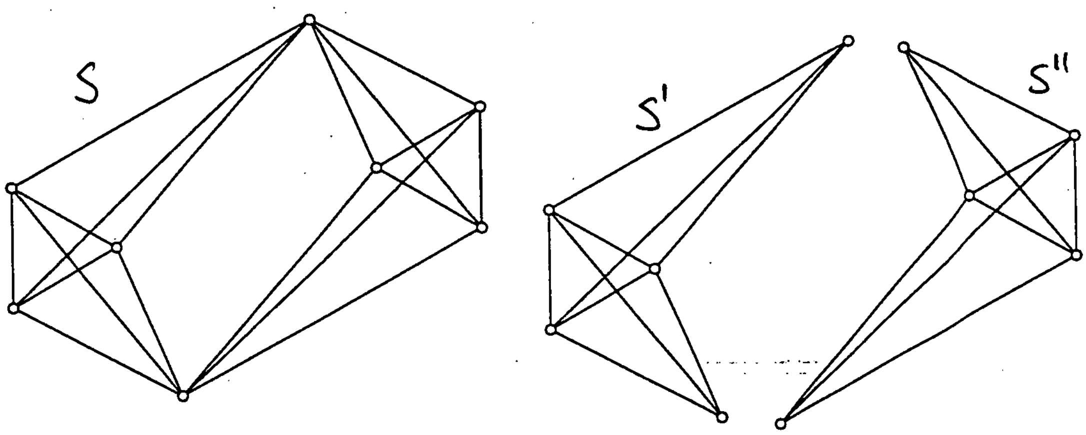
□

Since the concept of genericity, like the concept of implicit bar, is independent of the dimension of the space in which our structures lie we simply reiterate that references to A_{3G}, $f_3(G)$, N_{3G}, D_{3G} and *3-stiff* are references to graphs, or generic structures only, while references to A_S, $f(S)$, N_S, D_S and *rigid* apply more generally to any structures.

While results so far in $\mathbb{R}^3$ have been very similar to those in $\mathbb{R}^2$, when we raise the question of generic properties of spatial bar and joint structures we have reached a point where significant differences between the planar and spatial cases become evident. We know from the last example, that for some structures there is no partition of their bars into maximal rigid substructures because maximal rigid substructures are not bar disjoint. For this reason when we look at theorem 4.6 we immediately realise that it can have no analogue in $\mathbb{R}^3$. The structure of example 5.12 also provides a counterexample to analogues of other results in the plane, including theorems 4.4, 4.5, and 4.7:

Theorem 5.13: Not every circuit in D_{3G} is 3–stiff.

Proof: Consider once more the structure S from the previous example. It can flex about the dotted hinge line, so it is not rigid.



Therefore $rk(A_S)<3\times8-6=18=|B(S)|$, and so B(S) is dependent and contains a circuit C. Now each substructure, S' and S'', is independent so doesn't contain C. Therefore C must be partly in each substructure of S, and cannot be rigid. □

Comment 5.14: In general it is false that: D_{3G} is free iff $3|V(Y)|-6\geq|Y|$ holds for every $Y\subset E(G)$.

Proof: The structure S from the previous proof satisfies the condition in this proposition, but contains a circuit. □

Comment 5.15: In general it is false that: G is 3–isostatic iff

i) $|E(G)|=3|V(G)|-6$ and

ii) $|E(H)|\leq 3|V(H)|-6$ for every subgraph $H\subset G$.

Proof: Same as previous refutation. □

This has led to comments of the following nature:

"However, the 3–dimensional analogues of [two theorems including Laman's] are simply not true." Recski [R7] p244;

"...Laman's result does not extend to dimension three or higher." Graver [G5] p362;

which can be construed as misleading.

More cautious people have simply stated facts:

"In spite of considerable effort on the part of several people, the problem of extending Laman's theorem to higher dimensions is still open." Lovász & Yemini [L5] p98; at about the same time that Tay [T2] extended Laman's theorem to higher dimensions in his PhD thesis.

We believe that we have discovered a natural environment for other possible higher dimensional analogues of Laman's theorem (our ideas are very similar to and complementary to Tay's), along with a class of spatial bar and joint structures for which the underlying graph G is 3–isostatic iff

i) $|E(G)|=3|V(G)|-6$ and

ii) $|E(H)|\leq 3|V(H)|-6$ for every subgraph $H\subset G$, both hold.

Although, due to our inability to prove our conjecture, this belief is based mainly on mechanical impressions, an unreliable foundation for belief, we present our idea in the next chapter.

Hinged Panel Structures.

We introduce a new type of structure in space, called a hinged panel structure, and show its similarity to planar bar and joint structures and its dissimilarity to bar and joint structures in $\mathbb{R}^3$. This type of structure is slightly different from the hinged panel structure introduced by Baracs and developed a little further by Crapo & Whiteley, and while a familiarity with their structures might be useful in understanding these, it should be remembered that they are not the same. Our hinged panel structures are made from rigid panels joined by hinges. Each rigid panel has exactly two hinges on it. Each hinge may have any number of panels attached to it. A hinge is best thought of as a hinge. A panel should be considered as the simplest rigid body which keeps the relative positions of its two hinges fixed. Hence the only property of a panel, apart from incidence with its two hinges, is that the two hinges on it cannot move relative to each other. In accordance with convention we will ignore the unpleasant physical impossibility of hinges and panels passing through each other.

Definition 6.1: A *hinged panel structure*, R, consists of a simple graph H(V,E) and an injection χ:V(H)$\to\{\{\gamma(a,b,c)+(d,e,f):\gamma\in\mathbb{R}:\}a,b,c,d,e,f\in\mathbb{R}\}$. We will identify V(H) with the set of the first n positive integers and write χ_i instead of $\chi(i)$. If i$\in$V(H) we call χ_i a *hinge* of R, and if (i,j)$\in$E(H) we call $\{\chi_i,\chi_j\}$ a *panel* of R. We call H the *graph underlying* R.

Now how can we model the fundamental property of a panel; namely that its two hinges cannot move relative to each other? Of the many choices at this point we make a simplification. As a prelude to this we detail the following construction of a graph G, from a graph H:

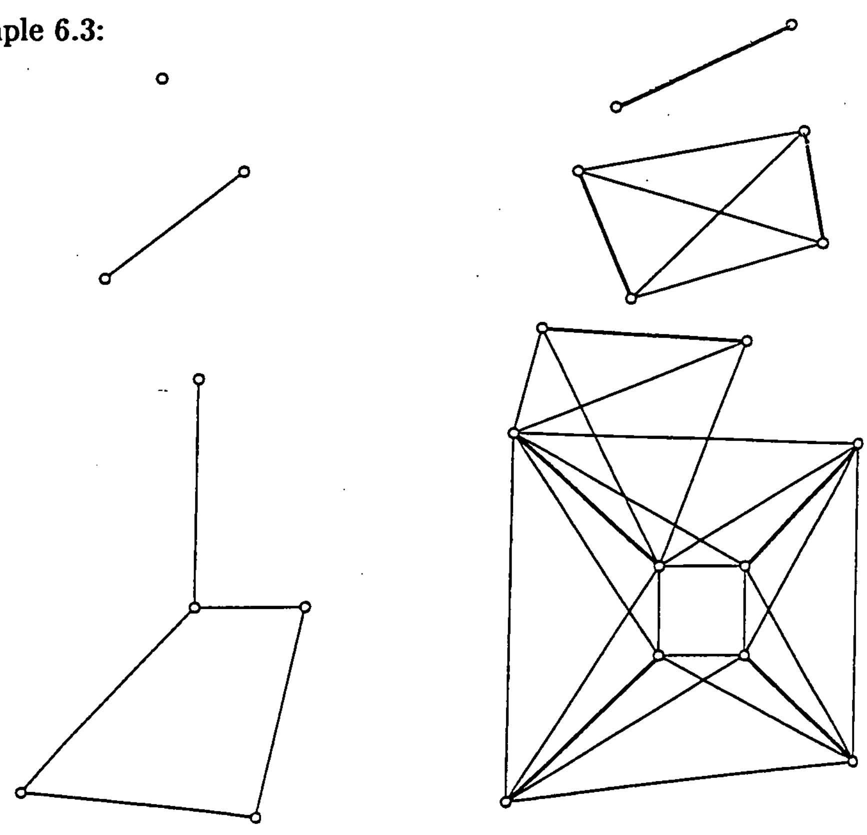
Construction 6.2: Start with a graph H.

i) For each vertex i of H create two vertices i_1, i_2 of G and an edge (i_1,i_2) of G between them s.t. $|V(G)|=2|V(H)|$. Now all the vertices of G are defined.

ii) For each edge (i,j) of H create four edges of G as follows: If $i \mapsto (i_1,i_2)$ and $j \mapsto (j_1,j_2)$ during stage i) of our construction, then (i_1,j_2), (i_1,j_1), (i_2,j_1) and (i_2,j_2) are edges of G and we have finished our construction of G. The resulting graph we call also $\Gamma(H)$.

iii) Clearly $|V(G)|=2|V(H)|$ and $|E(G)|=4|E(H)|+|V(H)|$.

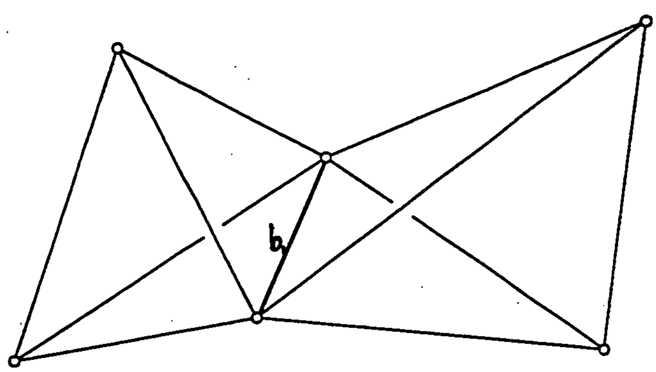
Example 6.3:



Now, in line with Tay [T5], we introduce a class of spatial bar and joint structures called simple hinged panel structures:

Definition 6.4: A *simple hinged panel structure* R with *underlying graph* H, is a spatial bar and joint structure with underlying graph G=Γ(H). If (i,j) is an edge of G corresponding in the construction to a vertex of H, then the bar of R which it underlies is called a *hinge*. If p is a set of six edges of G corresponding in the construction to an edge H, then the six bars of R which they underly is called a *panel*.

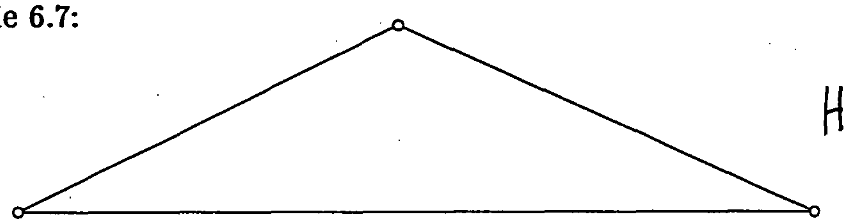
Example 6.5:

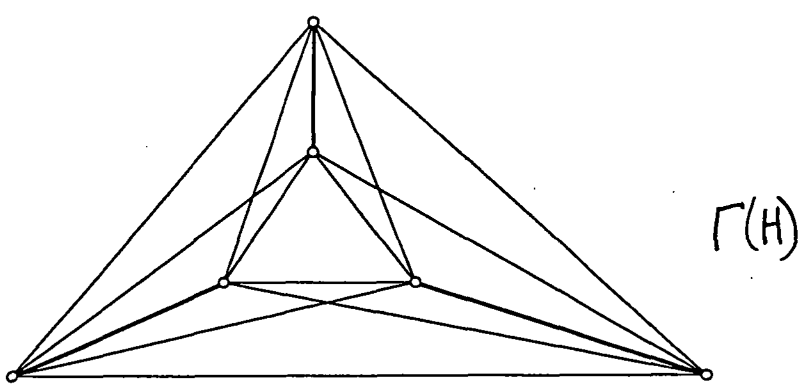


The bar b_1 is clearly functioning as a hinge, and we use bars in this way to model hinges. This has allowed us a very simple way to model the property of a panel that its two hinges cannot move relative to each other. By making a panel consist of, in addition to its two hinge bars, simply the other four bars between the vertices of these two. However, if in a hinged panel structure, we have a panel with coplanar hinges, then such a panel is not rigid (definition 1.18), but we require all panels to be rigid. Although simple hinged panel structures can be easily modified to account for this inadequacy, we will more easily understand these structures if at first we avoid this complication. Since simple hinged panel structures are bar and joint structures, we shall overcome this problem by dealing with generic simple hinged panel structures since by definition 3.26 none of these has a panel with coplanar hinges. To simplify things in a sensible fashion commensurate with the first chapters, we shall deal henceforth with generic simple hinged panel structures and use the graph notation.

Definition 6.6: A graph H is *HP–stiff* iff $G=\Gamma(H)$ is stiff and *HP–isostatic* iff G is 3–isostatic. A *maximal HP–stiff subgraph* of H is a HP–stiff subgraph of H which is a subgraph of no HP–stiff subgraph of H other than itself.

Example 6.7:



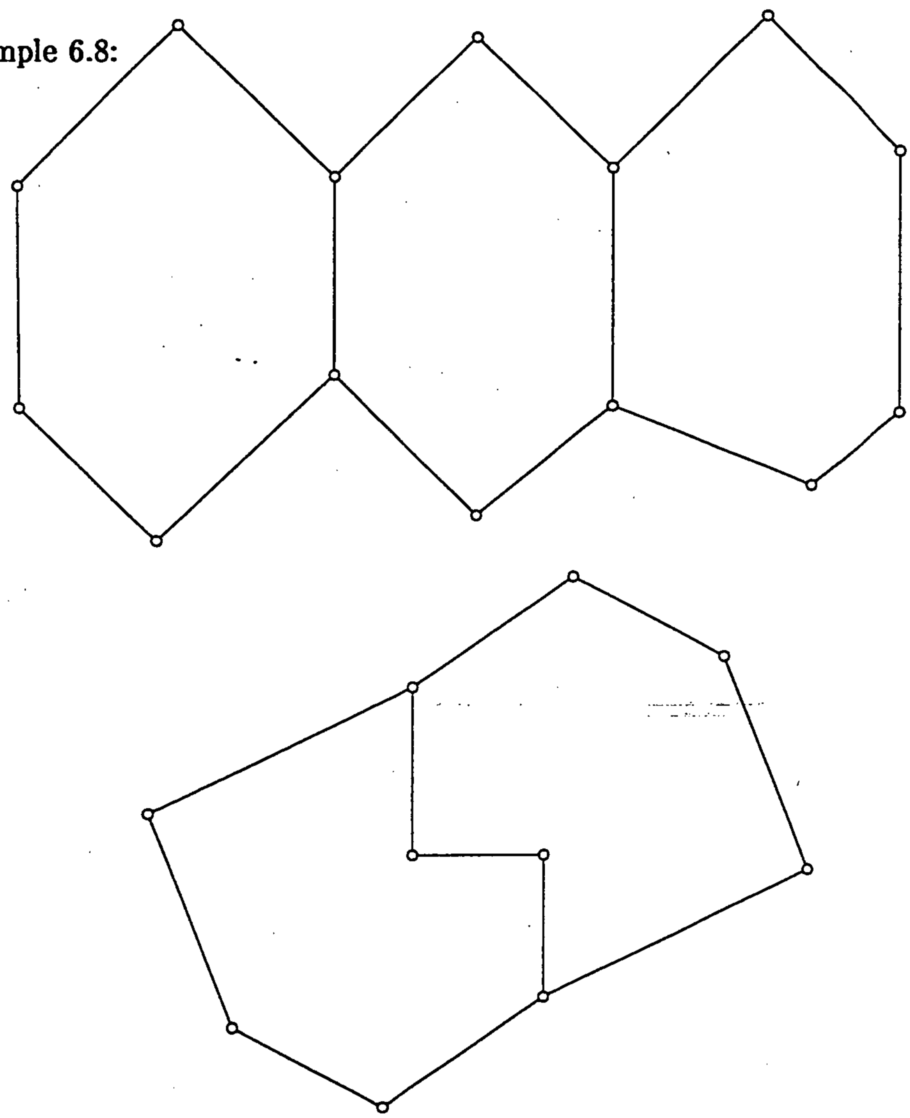


H is HP–stiff since $G=\Gamma(H)$ is K_6. Also since $rk(A_S)=12<15=|E(G)|$ we find that the edges of G are dependent.

In fact when a simple hinged panel structure is just a cycle of panels; i.e. every hinge has exactly two panels attached; then it is a structure about which a great deal is understood. Crapo and Whiteley [C16] showed that for generic structures of this type, cycles of size three, four, and five are HP–stiff and dependent, a cycle of size six is HP–stiff and independent, and cycles of size seven or greater are independent but not HP–stiff. They gave explicitly the conditions for genericity.

While some structures can be understood by viewing them as many cycles joined in various ways, many cannot be treated this way:

Example 6.8:



H_1

H_2

Since H_1 consists of three "independent" cycles of six we can easily see that it is HP–stiff. But what of H_2? Is it HP–stiff? Is it independent?

Now we compare the generic properties of planar bar and joint structures with the generic properties of simple hinged panel structures.

Planar bar and joint	**Simple hinged panel**
[n points moving freely in the plane have 2n degrees of freedom.]	[n bars moving freely in space have 5n degrees of freedom.]
A bar in the plane joins at most two joints, and in doing so removes at most 1 degree of freedom.	A panel in space joins at most two hinges, and in doing so removes at most 4 degrees of freedom.
In the underlying graph a joint is represented by a vertex and a bar is represented by an edge.	In the underlying graph a hinge is represented by a vertex and a panel is represented by an edge.
A triangle is the largest cycle which is 2–isostatic.	A hexagon is the largest cycle which is HP–isostatic.
The dimension of the space of rigid motions is three.	The dimension of the space of rigid motions is six.
If a graph G, is 2–stiff then $\lvert E(G)\rvert \geq 2\lvert V(G)\rvert - 3$.	If a graph H, is HP–stiff then $4\lvert E(H)\rvert \geq 5\lvert V(H)\rvert - 6$.

We justify this last claim.

Lemma 6.9: If H is HP–stiff then $4\lvert E(H)\rvert \geq 5\lvert V(H)\rvert - 6$.

Proof: If $G=\Gamma(H)$ is 3–stiff then by lemma 5.4, $\lvert E(G)\rvert \geq 3\lvert V(G)\rvert - 6$. But by 6.2 iii) we know $\lvert V(G)\rvert = 2\lvert V(H)\rvert$ and $\lvert E(G)\rvert = 4\lvert E(H)\rvert + \lvert V(H)\rvert$ and the result follows upon substitution. □

In the same way theorem 2.5 has an analogue.

Theorem 6.10: Any two of the following conditions imply the third:

i) H is HP–stiff.

ii) $D_{\Gamma(H)}$ is free.

iii) $4|E(H)|=5|V(H)|-6$.

Proof: As in the last proof, we simply substitute the equations $|V(G)|=2|V(H)|$ and $|E(G)|=4|E(H)|+|V(H)|$ from construction 6.2 iii) into theorem 5.5 and this result follows. □

So far so good, but what about an analogue for theorems 2.3 and 5.3?

$\mathrm{rk}(A_{\Gamma(H)}) \leq \min\{|E(G)|,3|V(G)|-6\}$, by theorem 5.3

$=\min\{4|E(H)|+|V(H)|,6|V(H)|-6\}$, by construction 6.2

$\neq\min\{4|E(H)|,5|V(H)|-6\}$, in general.

Although this looks sad, we should not expect these complicated structures to be too simple. In fact this difference is very easily resolved.

Theorem 6.11: $\mathrm{rk}(A_{\Gamma(H)})-|V(H)|\leq\min\{4|E(H)|,5|V(H)|-6\}$.

Proof: Discussion above. □

Suppose we have a generic simple hinged panel structure R with underlying graph H. It is old hat that associated with the graph $G=\Gamma(H)$ we have a matroid on its edges (the structure geometry), however using the rows of the coordinatising matrix A_R, we observe a polymatroid on the ground set $V(H)\cup E(H)$, induced as follows. If $A\subset V(H)\cup E(H)$, then $\mathrm{rk}_p(A)$ is the rank of the rows of A_R corresponding to the hinges and panels of R (see definition 6.4) which vertices and edges in A underly. In this polymatroid edges of H have rank six and vertices of H have rank one.

We know that the matroid called the structure geometry in space, differs from the matroid called the structure geometry in the plane, but is this polymatroid sufficiently similar to the planar structure geometry to allow the possibility of analogous results to theorems 4.4, 4.5, and 4.7 for simple hinged panel structures? Possibly, however because we do not wish to become embroiled in polymatroid theory, we do not present analogues of 4.7, and we only introduce enough polymatroid concepts to indicate that feasible parallels may exist between matroid properties of planar bar and joint structures and polymatroid properties of simple hinged panel structures.

Definition 6.12: A set A of edges of a graph H is *polyindependent* iff $\Gamma(A)$ is an independent set of edges of the structure geometry $D_{3\Gamma(H)}$. A set A of edges of a graph H is *polydependent* iff $\Gamma(A)$ is a dependent set of edges of $D_{3\Gamma(H)}$. A *polycircuit* C of a graph H is a polydependent subset of E(H), all of whose proper subsets are polyindependent.

Conjecture 6.13: All polycircuits are HP–stiff.

Conjecture 6.14: H is polyindependent iff $5|V(Y)|-6\geq 4|E(Y)|\ \forall\ E(Y)\subset E(H)$.

It was essentially at this point in the development of the theory of spatial bar and joint structures that we realised their dissimilarity to planar bar and joint structures. Up until this point the theory for hinged panel structures has been similar to that for both planar and spatial bar and joint structures. Why should we expect the behaviour of hinged panel structures, from this point on, to be more like that of planar bar and joint than spatial bar and joint structures?

The answer comes in two parts. First a mathematical indication, and then a physical one. It is generally impossible to partition the edges of a graph into maximal 3–stiff subgraphs because an edge can be in two maximal 3–stiff subgraphs (example 5.12), however for simple hinged panel structures there is an analogue for 2.12 and 2.13:

Theorem 6.15: If G_1 and G_2 are 3–stiff subgraphs of G, and $|V(G_1) \cap V(G_2)| > 2$, then $G = G_1 \cup G_2$ is 3–stiff.

Proof: $f_3(G) = f_3(G_1 \cup K_{V(G_1) \cap V(G_2)})$ by lemma 5.9, and $f_3(G_1 \cup K_{V(G_1) \cap V(G_2)}) = 0$ by lemmas 1.22 and 1.23. □

Corollary 6.16: For any graph H, the maximal HP–stiff subgraphs are bar disjoint, and the partition of the edges into its maximal HP–stiff subgraphs is unique.

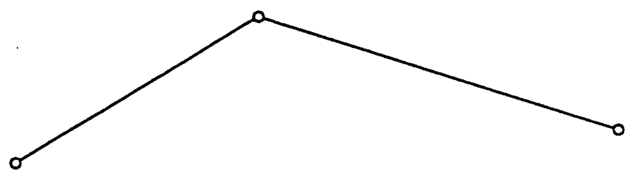
Proof: Clearly every edge is in a maximal HP–stiff subgraph.

Suppose an edge e is in two maximal HP–stiff subgraphs, H_1 and H_2. Then $|V(\Gamma(H_1)) \cap V(\Gamma(H_2))| = 4 > 2$ so by lemma 6.15 $H_1 \cup H_2$ is HP–stiff, contradicting the maximality of the HP–stiffness of H_1 and H_2. Therefore every edge is in exactly one maximal HP–stiff subgraph. □

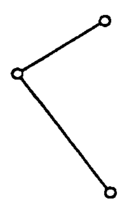
Thus it is always possible to partition the edges of a graph into maximal HP–stiff subgraphs, and, as in the planar bar and joint case the partition into maximal 2–stiff subgraphs of a graph G yielded the $\min \Sigma 2|V(G_i)| - 3$ where the minimum extended over all partitions of the edges of G, so we conjecture that the partition into maximal HP–stiff subgraph of a graph H will yield $\min \Sigma 5|V(H_i)| - 6$ where the minimum is taken over all partitions of the edges of H.

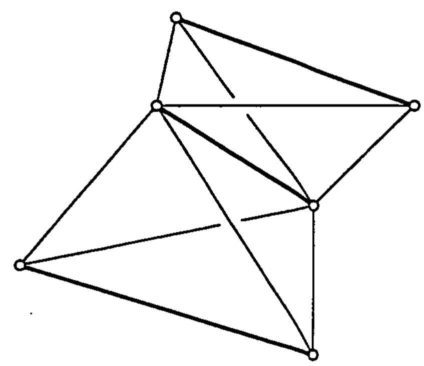
Secondly we examine the nature of the joints of each of the three types of structure by considering the following example.

Example 6.17: Let G be



and let S_1 be a planar bar and joint structure, S_2 be a spatial bar and joint structure, and S_3 be a simple hinged panel structure, each with G as its underlying graph.





a) $f(S_1)=1=f_2(G)$ b) $f(S_2)=3=f_3(G)$ c) $f(S_3)=1=f_3(\Gamma(G))$

In mechanical engineering parlance ([H1] p5–13) the joints exhibited here are known as lower pairs with a) a turning or revolute pair; b) a spherical or globe pair; and c) a turning pair. Also engineers say the "number of freedoms" of a turning pair is 1 and the "number of freedoms" of a spherical pair is 3. Note that in structures of all three types we are not always dealing with simple pairs, but sometimes with many members meeting at a joint. Nevertheless from a mechanical viewpoint it may be more natural to expect similarities between planar bar and joint structures and hinged panel structures than between the former and spatial bar and joint structures.

Since 4.2 – 4.8 are equivalent in the sense that each is a simple corollary of each of the others, we just state here the analogue of 4.8 in the expectation that if one can prove it then one can prove the analogues of 4.2 – 4.7. Because of the simplicity of the language of graphs, we present this conjecture in the same form as theorem 4.8 and comment 5.15, with our usual understanding that in addition to being a statement about graphs it is a statement about (generic) simple hinged panel structures. We also believe that a corresponding result may be true for certain more general hinged panel structures.

Conjecture 6.18: A graph H is HP–isostatic iff

i) $4|E(H)|=5|V(H)|-6$ and

ii) $4|E(F)|=5|V(F)|-6$ for every subgraph F of H.

Because of the resemblance this conjecture bears to Laman's theorem, we look to the proofs of Laman's theorem for approaches to a proof of this.

Laman's original induction proof relies on building every 2–isostatic structure with k vertices from 2–isostatic structures with k–1 vertices, in a simple way. It seems unnatural to try his technique here, and for simple hinged panel structures it would require a vastly more complicated method to build every 3–isostatic structure with k vertices from 3–isostatic structures with k–4 vertices.

Asimow and Roth's proof relies on simple accessible properties of planar structures which don't generalise in a straightforward way to simple hinged panel structures. It seems unnatural to try their technique here as it is tailor–made for the planar case.

Lovász and Yemini's proof seems at first a little more promising, as it involves polymatroid theory and here we are dealing with a polymatroid. After attempting without complete success to use their ideas, it seems likely that they will only be effective for simple hinged panel structures, and won't be easily adaptable to more general hinged panel structures.

In the last half of the previous chapter we proposed a possible fourth method of proof which generalises more simply to higher dimensions than these three.. We didn't make it work for planar bar and joint structures, but if true, it may be possible to apply the idea to hinged panel structures and their higher dimensional analogues. We give an indication of how this idea might develop.

Definition 6.19: The *relative 3–degree of freedom between two vertices* 1 and 2 of a graph H is denoted and defined by $f_3(1,2)=rk(A_{3G'})-rk(A_{3G})$, where H′ is a graph obtained from H by adding an edge (1,2), and $G=\Gamma(H)$ and$G'=\Gamma(H')$.

Also the *relative 3–degree of freedom between a vertex* 1, *and an edge* $p_2=(2,3)$ of a graph H, is denoted and defined by $f_3(1,p_2)=rk(A_{3G'})-rk(A_{3G})$, where $G'=\Gamma(H')$, $G=\Gamma(H)$, and H′ is obtained from H by adding two edges (1,2) and (1,3).

Also the *relative 3–degree of freedom between two edges* $p_0=(1,2)$ and $p_1=(3,4)$ of a graph H is denoted and defined by $f_3(p_1,p_0)=rk(A_{3G'})-rk(A_{3G})$ where $G'=\Gamma(H')$, $G=\Gamma(H)$ and H′ is a graph obtained from H by adding:

i) edges (1,3) and (2,4) if $1\neq3\neq2$ and $1\neq4\neq2$.

ii) one edge $(\{1,2\}\backslash i,\{3,4\}\backslash j)$ if i=j for some $i\in\{1,2\}$, $j\in\{3,4\}$.

Finally the *relative 3–degree of freedom between two HP–stiff subgraphs* H_0 and H_1 where $(1,2)\in E(H_0)$ and $(3,4)\in E(H_1)$ of a graph H is denoted and defined by $f_3(H_1,H_0)=rk(A_{3G'})-rk(A_{3G})$ where $G'=\Gamma(H')$, $G=\Gamma(H)$ and H′ is a graph obtained from H by adding:

i) edges (1,3) and (2,4) if $1\neq3\neq2$ and $1\neq4\neq2$.

ii) one edge $(\{1,2\}\backslash i,\{3,4\}\backslash j)$ if i=j for some $i\in\{1,2\}$, $j\in\{3,4\}$.

The third definition of these is a special case of the fourth.

It can be shown, using the unit-increasing property of the rank function and some basic knowledge of the dependence of some simple small spatial bar and joint structures, that

$$0 \leq f_3(1,2) \leq 4$$
$$0 \leq f_3(e_1,2) \leq 5$$
$$0 \leq f_3(e_1,e_2) \leq 6 \quad \forall\ e_1,e_2,1,2 \text{ in a graph } H$$

s.t. all these relative 3-degrees of freedom are defined.

Lemma 6.20: If $e_0=(1,2)$ and $e_1=(3,4)$, then

$f_3(1,3)=k \Rightarrow k \leq f_3(e_0,3) \leq k+1$ and $k \leq f_3(e_0,e_1) \leq k+2$.

Also $f_3(e_0,e_1)=k \Rightarrow \left.\begin{matrix} 0 \text{ if } k<2 \\ k-2 \text{ if } k \geq 2 \end{matrix}\right\} \leq f_3(1,3) \leq \begin{cases} k \text{ if } k<4 \\ 4 \text{ if } k \geq 4, \end{cases}$ and

$\left.\begin{matrix} 0 \text{ if } k=0 \\ k-1 \text{ if } k \neq 0 \end{matrix}\right\} \leq f_3(e_0,3) \leq \begin{cases} k \text{ if } k \neq 6 \\ 5 \text{ if } k=6. \end{cases}$

Finally $f_3(e_0,3)=k \Rightarrow \left.\begin{matrix} k-1 \text{ if } k \neq 0 \\ 0 \text{ if } k=0 \end{matrix}\right\} \leq f_3(1,3) \leq \begin{cases} 4 \text{ if } k=5 \\ k \text{ if } k \neq 5, \end{cases}$ and

$k \leq f_3(e_0,e_1) \leq k+1$.

Proof: Straightforward consequence of the unit-increasing property of the rank function. □

Conjecture 6.21: Suppose e_0 and e_1 are edges of H and H_0 is the maximal HP-stiff subgraph of H which contains e_0. Then $f_3(e_0,e_1)=0$ iff H_0 contains e_1.

Tay [T3] has proved, using a projective approach, a similar lemma (theorem 4.3 in his paper) which may imply this. If this conjecture is true then we may proceed in an analogous fashion to the way we did in the second part of chapter four, from lemma 4.15 onwards, and to realise analogous results including conjecture 6.18. In particular our construction process and consequent proof by induction on the number of maximal 2-stiff subgraphs, generalise in this way.

An analogous proposition for spatial bar and joint structures would be more complicated since some bars are in more than one maximal rigid substructure.

If we compare conjecture 6.18 with comment 5.15, for simple hinged panel structures, using the equations in construction 6.2 iii) and definition 6.6, we easily realise that these two are identical for simple hinged panel structures, and we have a class of spatial bar and joint structures within which the conditions in comment 5.15 may be a valid characterisation of 3–isostatic graphs. Perhaps we can extend our class of bar and joint structures which obey this proposition by replacing any panel of a simple hinged panel structure with any 3–isostatic structure which contains those four vertices, as in lemma 5.8, ending up with a 3–isostatic structure which obeys the conditions in the proposition.

Conclusions.

This chapter consists of speculations, including a discussion of the projective viewpoint and a list of possible extensions of work presented earlier.

In chapter 6 we defined hinged panel structures, and in order to get a handle on these we also introduced simple hinged panel structures. There were many possible simplifications available at that point, and many ways to model hinged panel structures, both general and simple. The choice of model must depend mostly on what you want to do with it, and ours was chosen because it was familiar and we were trying for a simple introduction. Given that one is normally trying to produce new ideas, there are good arguments for choosing a projective model, like for instance that of Crapo and Whiteley [C16].

First, the result telling us that rigid structures remain rigid under projective transformations [R1] [R2] [C16] [W1] [W23], allowing an appropriate mechanical interpretation of points at infinity in extended euclidean space, which is fundamental to this subject, is naturally best tackled using a projective approach (compare the proof of this by Wunderlich [W23] with that by Crapo and Whiteley [C16]). In other words structures generally have in some sense an inherently projective nature.

Secondly, as is evident by our definition, we believe that it is the line taken by the hinge, that is crucial. That is, no matter where the hinges of a simple hinged panel structure were placed on the hinge lines of the associated hinged panel structure, the behaviour would be the same. Although this may simply be a consequence of the projective invariance of statics, the projective description of lines in space is appealing and tractable.

Thirdly, associated unavoidably with the projective description for the lines in space is a description for screws [K12] [V1] in space, and there is no reason not to include screw hinges [H1] in our definition of hinged panel structures, although simple hinged panel structures cannot be modelled this way. It is possible to build such things, and we conjecture that the same results would hold for them, since it seems that an ordinary hinge joint is a screw joint with zero pitch.

Finally, and incidentally, for anyone interested in attempting to examine our conjecture, or something similar, for simple hinged panel structures using Lovász and Yemini's techniques, there is a reason to consider projective models. The proof by Lovász and Yemini of Laman's theorem relies on several fortuitous coincidences, one of which is the fact that the *incidence matrix* (0.4) of the graph underlying the structure (and so also the cycle matroid of the graph, (0.13 ii)) can be neatly used to describe the *coordinatising matrix* (1.6) of the structure. The arrangement of this coincidence for simple hinged panel structures was ultimately effected using a development of Crapo and Whiteley's [C16] projective model.

We believe that these hinged panel structures are not contrived, but useful and natural objects to study, and we have presented as a major part of this thesis, detailed explanations of only one reason for this belief. Another reason is that they are similar to the *articulated panel structures* introduced by Baracs [B7], an engineer who in his paper suggests construction techniques for buildings designed using these ideas, so that since a practical man invented them they must be useful. Although our structures are not the same as these, ours are at least as general in that any of his structures can be expressed in terms of ours:

In Rooney and Wilson [R8], two possible representations (for kinematic systems) are given, called the "direct graph representation" and the "interchange graph representation". In the former the vertices of the graph correspond to the joints of the system and the edges of the graph correspond to the links of the system. This is appropriate only for systems with binary links. In the interchange graph representation the vertices of the graph correspond to the links of the system and the edges of the graph correspond to the joints of the system. This is appropriate only for systems with binary joints. In this work we have used the direct graph representation and dealt solely with those structures (including our hinged panel structures) for which this is appropriate. Crapo and Whiteley [C16] have used the interchange graph representation and dealt with a type of hinged panel structure for which this is appropriate. We allow two hinges per panel and any number of panels per hinge, whereas Crapo and Whiteley allow only two panels per hinge and any number of hinges per panel. Rooney and Wilson [R8] give an example of how a joint like one of ours with three panels, may be "expanded" into a succession of joints with only two panels if we allow panels with more than two hinges. In a similar fashion a more complicated link (e.g. a panel from a Crapo and Whiteley hinged panel structure) may be expressed in terms of binary links (e.g. panels from our hinged panel structures) if joints are allowed with more than two links. This can theoretically be done by replacing each m–ary panel by a rigid (or isostatic if independence is important) structure which has m hinges. In practice this is a contrived procedure, however no less contrived than the converse one shown by Rooney and Wilson. Neither of these two representations comfortably encompasses the most general type of structure, which would allow both links with more than two joints and joints with more than two links. Rooney and Wilson suggest a hypergraph representation which overcomes this.

A third reason is that these ideas constitute more information with which to tackle the search for a combinatorial characterisation of (generic) rigid spatial bar and joint structures. It seems likely that there are no isostatic spatial bar and joint structures which disobey the conditions in comment 5.15, and that the only structures which obey these conditions but are not isostatic (they clearly can't have hyperstatic subgraphs), are not rigid due to one or more hinge type arrangements like those in example 5.12. This seems reasonable because if we have a graph G of a spatial bar and joint structure, with two maximal 3–stiff subgraphs G_1 and G_2, we know that these two maximal 3–stiff subgraphs can be joined in one of three ways:

i) $E(G_1) \cap E(G_2)=\{\}$ and $|V(G_1) \cap V(G_2)|=1$,

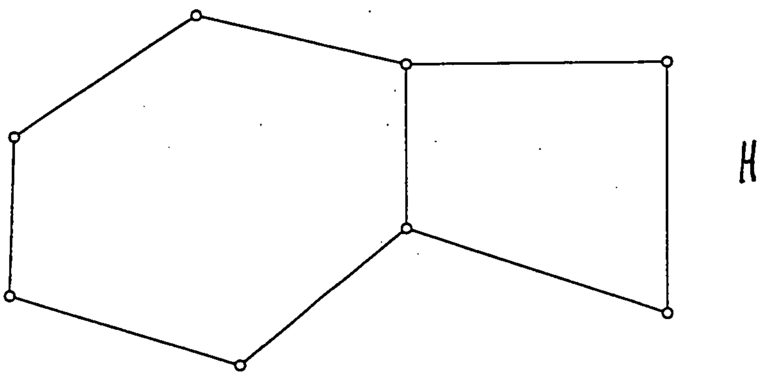
ii) $E(G_1) \cap E(G_2)=\{\}$ and $|V(G_1) \cap V(G_2)|=2$,

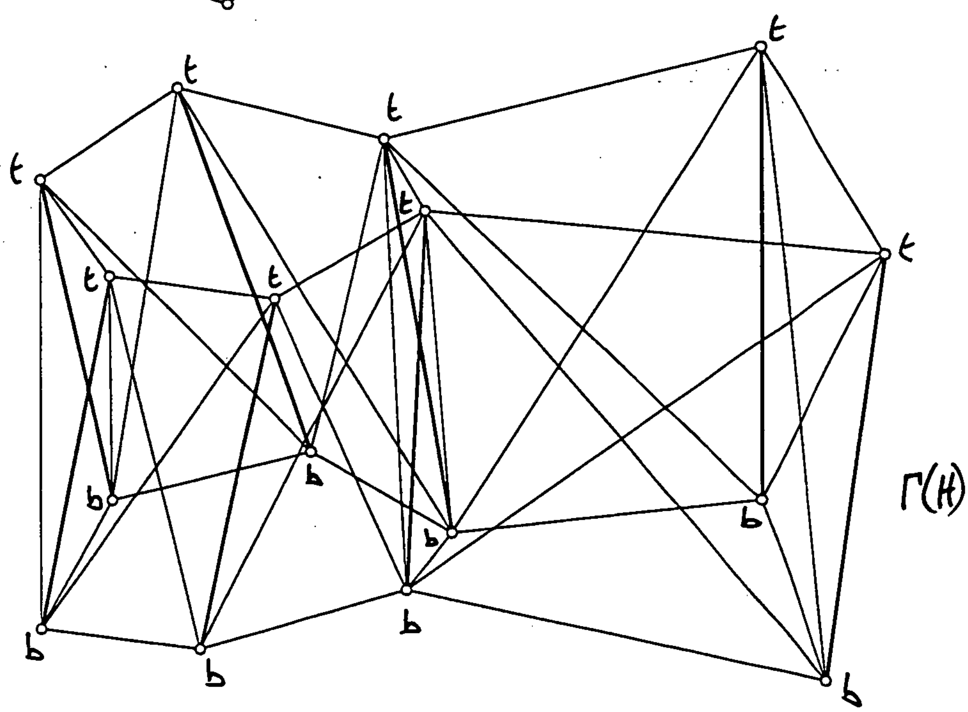
iii) $|E(G_1) \cap E(G_2)|=1$ and $|V(G_1) \cap V(G_2)|=2$,. It is the second case here which to some extent prohibits a general version of Laman's theorem for spatial bar and joint structures.

Thus any non–rigid structures which obey the conditions in comment 5.15 can be viewed as hinged panel structures which aren't rigid, and so we should have an exact condition on them from conjecture 6.18. If we can refine this condition and translate it into spatial bar and joint terms, then we have characterised a set of exceptions to comment 5.15, and if these are the only exceptions, as we suspect, then we have characterised (generic) rigid spatial bar and joint structures. Perhaps something along these lines will be fruitful.

Finally, once more on the practical side, it would be very easy to design complex spatial bar and joint trusses for which the critical forms [T8] are easily predicted and avoided. Such a truss would consist of two equicardinal sets of joints, one set in each of two parallel planes, with enough various bars joining them to ensure the rigidity of the structure, see Crapo [C14]. When designing our truss based on a simple hinged panel structure we commence with a graph H which we know to be HP–stiff. Then we construct a graph G=Γ(H) using construction 6.2. At step i) in the construction of G when we create two vertices for every one of H, we label one of these vertices with a t (for top) and the other with a b (for bottom), so that when G is constructed, half the vertices are labelled t and the other half are labelled b. If we realise G as a spatial bar and joint structure where the vertices labelled t go to a coplanar set of joints, and the vertices labelled b go to another coplanar set of joints, with these two planes parallel and distinct.

Example:





We conjecture that it is only necessary to keep track of the lines taken by each of the edges bt to then decide on the structure's rigidity. Using this technique it should be easy to design isostatic and hyperstatic trusses both generic and non–generic.

The remainder of this thesis is devoted to indicating where one might proceed from here. Some extensions of work in this thesis are immediately evident.

The work of chapter one is extendable to higher dimensions. If this is done patterns emerge in the inconvenient little exceptions, and these may consequently be better understood. The results 2.9, 2.10, 2.11, 5.9, 5.10, and 5.11 have many kindred results which can presumably be treated the same way as these. Similarly relative degree of freedom (4.11, 6.19) can be more generally defined so that we can find the relative degree of freedom of any set of components of a given graph, of a planar bar and joint structure, or a hinged panel structure. Also there exist other ways of expressing the construction 4.16 and these may be more helpful than the one presented.

The form of some higher dimensional analogues for Laman's theorem is also now clear. They require initially an understanding of the appropriate type of structure for which they might hold. Namely structures where the relative degree of freedom between any two edges/panels/bars with a common vertex/hinge/joint, is at most one.

All these directions should be routine once the hinged panel structures are thoroughly understood.

Finally, in addition to all the abovementioned thoughts, there is an appealing idea which is potentially fruitful. Suggested by the natural polymatriod description of hinged panel structures, is the possibility that a polymatroid description for bar and joint structures in the plane might be useful as well as natural. For example any bar and joint structure in the plane has many hypergraph/polymatroid representations, where an edge of a hypergraph which represents a structure must be a 2–stiff subgraph. If this is done then every hypergraph/polymatroid representation lies between the two extremes of the original graph/matroid description, and the hypergraph with every edge a maximal 2–stiff subgraph.

Reference List.

[A1] L.Asimow and B.Roth, The Rigidity of Graphs, *Trans. Amer. Math. Soc.* 245 (1978) 279–289.

[A2] L.Asimow and B.Roth, The Rigidity of Graphs II, *J. Math. Anal. Appl.* 68 (1979) 171–190.

[B1] J.A.Baglivo and J.E.Graver, "Incidence and Symmetry in Design and Architecture", Cambridge University Press, Cambridge 1983.

[B2] J.E.Baker, Limit Positions of Spatial Linkages via Connectivity Sum Reduction, *J. Mech. Design* 101 (1979) 504–508.

[B3] J.E.Baker, A Compendium of Line–Symmetric Four–Bars, *J. Mech. Design* 101 (1979) 509–514.

[B4] J.E.Baker, An Analysis of the Bricard Linkages, *Mechanism and Machine Theory* 15 (1980) 267–286.

[B5] J.E.Baker, On Structures and Linkages, *Structural Topology* 5 (1980) 39–44.

[B6] J.E.Baker, On Mobility and Relative Freedoms in Multiloop Linkages and Structures, *Mechanism and Machine Theory* 16 (1981) 583–596.

[B7] J.J.Baracs, Rigidity of Articulated Spatial Panel Structures, *Bulletin International Assoc. for Shell and Spatial Structures* 59, vol XVI–3, *Madrid* (1975), 37–52.

[B8] G.T.Bennett, Deformable Octahedra, *Proc. London Math. Soc. Series 2* 10 (1912) 309–343.

[B9] G.T.Bennett, The Skew Isogram Mechanism, *Proc. London Math. Soc.* 13 (1913) 151–176.

[B10] W.Blaschke, Wackelige Achtflache, *Math. Z.* 6 (1920) 85–93.

[B11] E.D.Bolker, Bracing Grids of Cubes, *Environment and Planning B*, 4 (1977) 157–172.

[B12] E.D.Bolker, Bracing Rectangular Frameworks II, *SIAM J. Appl. Math.* 36 (1979) no. 3, 491–508.

[B13] E.D.Bolker and H.Crapo, How to Brace a One–Story Building?, *Environment and Planning B*, 4 (1977) 125–152.

[B14] E.D.Bolker and H.Crapo, Bracing Rectangular Frameworks I, *SIAM J. Appl. Math.* 36 (1979) no. 3, 473–490.

[B15] E.D.Bolker and B.Roth, When is a Bipartite Graph a Rigid Framework?, *Pacific Journal of Mathematics* 90 (1980) no. 1, 27–43.

[B16] J.A.Bondy and U.S.R.Murty, "Graph Theory With Applications", Macmillan, London; North–Holland, New York 1976.

[B17] R.Bricard, Mémoire sur la théorie de l'octaèdre articulé, *J. de Math. pur. app.* 3 (1897) 113–148.

[C1] C.R.Calladine, Buckminster Fuller's "Tensegrity" Structures and Clerk Maxwell's Rules for the Construction of Stiff Frames, *Inter. J. Solids Structures*, 14 (1978) 161–172.

[C2] A.L.Cauchy, Deuxième mémoire sur les polygones et les polyèdres, *J. Ecole Polytechnique Ser. 9* 19 (1813) 87–89.

[C3] T.M.Charlton, "A History of Theory of Structures in the Nineteenth Century", University Press, Cambridge 1982.

[C4] S.E.Cohn–Vossen, Die Verbiegung von Flächen im Grossen, *Fortschr. Math. Wiss.* 1 (1936) 33–76.

[C5] R.Connelly, An Immersed Polyhedral Surface which Flexes, *Indiana Univ. Math. J.* 25 (1976) 965–972.

[C6] R.Connelly, A Counterexample to the Rigidity Conjecture for Polyhedra, *Inst. Hautes Etudes Sci. Publ. Math.* 47 (1978) 333–338.

[C7] R.Connelly, A Flexible Sphere, vol.1 no.3 *Math. Intelligencer*, August 1978 130–131.

[C8] R.Connelly, The Rigidity of Polyhedral Surfaces, *Mathematics Magazine* 52 (1979) no. 5, 275–283.

[C9] R.Connelly, The Rigidity of Certain Cabled Frameworks and the Second–Order Rigidity of Arbitrarily Triangulated Convex Surfaces, *Adv. in Math.* 37 (1980) no. 3, 272–299.

[C10] R.Connelly, Rigidity and Energy, *Invent. Math.* 66 (1982) no. 1, 11–33.

[C11] H.R.Cox, On the Synthesis and Analysis of Simply Stiff Frameworks, *Proc. London Math. Soc.* 40 (1936) 203–216.

[C12] H.Crapo, More on the Bracing of One–Story Buildings, *Environment and planning B*, 4 (1977) 153–156.

[C13] H.Crapo, Structural Rigidity, *Structural Topology* 1 (1979) 26–45.

[C14] H.Crapo, The Tetrahedral–Octahedral Truss, *Structural Topology* 7 (1982) 51–60.

[C15] H.Crapo, The Combinatorial Theory of Structures. in "Matroid Theory (Szeged 1982)" *Colloq. Math. Soc. Janos Bolyai, 40* (1985) 107–213.

[C16] H.Crapo and W.Whiteley, Statics of Frameworks and Motions of Panel Structures: a Projective Geometric Introduction, *Structural Topology* 6 (1982) 43–82.

[C17] L.Cremona, "Graphical Statics", English Translation, Oxford U. Press, London 1890.

[D1] A.Dandurand, The Rigidity of Compound Spatial Grids, *Structural Topology* 10 (1984) 41–56.

[D2] M.Dehn, Über die Starrheit Konvexer Polyeder, *Math. Ann.* 77 (1916) 466–473.

[D3] M.Deza and I.G.Rosenberg, Intersection and Distance Patterns, *Util. Math.* 25 (1984) 191–214.

[D4] M.Deza and N.M.Singhi, Rigid Pentagons in Hypercubes, *Graphs Combin.* 4 (1988) no. 1, 31–42.

[D5] A.C.Dixon, On Certain Deformable Frameworks, *Messenger Math.* 29 (1899) 1–21.

[G1] H.Gluck, Almost All Simply Connected Closed Surfaces Are Rigid, in "Geometric Topology", *Lecture Notes in Math., no. 438*, Springer–Verlag, Berlin (1975) 225–239.

[G2] M.Goldberg, A six-plate linkage in three dimensions, *Mathematical Gazette* 58 (1974) 287–289.

[G3] M.Goldberg, Unstable Polyhedral Structures, *Mathematics Magazine* 51 (1978) no. 3, 165–170.

[G4] M.Goldberg, A Mathematical Autobiography, *Structural Topology* 4 (1980) 57–60.

[G5] J.E.Graver, Rigidity Matroids, *SIAM J. Disc. Math.* 4 (1991) no. 3, 355–368.

[G6] M.Gromow, "Partial Differential Relations", Springer–Verlag, Berlin New York, 1986.

[G7] B.Grünbaum and G.C.Shephard, Rigid Plate Frameworks, *Structural Topology* 14 (1988) 1–8.

[H1] K.H.Hunt, "Kinematic Geometry of Mechanisms", Clarendon Press, Oxford 1978.

[K1] G.Kalai, Rigidity and the Lower Bound Theorem.I, *Invent. Math.* 88 (1987) no. 1, 125–151.

[K2] P.J.Kahn, Counting Types of Rigid Frameworks, *Invent. Math.* 55 (1979) 299–308.

[K3] E.Kann, A New Method For Infinitesimal Rigidity of Surfaces with K>0, *J. Differential Geometry*, 4 (1970) 5–12.

[K4] E.Kann, An Elementary Proof of a Finite Rigidity Problem by Infinitesimal Rigidity Methods, *Proc. Amer. Math. Soc.* 60 (1976) 252–258.

[K5] E.Kann, Glidebending of General Caps: An Infinitesimal Treatment, *Proc. Amer. Math. Soc.* 84 (1982) 247–255.

[K6] E.Kann, Infinitesimal Rigidity of Almost–Convex Oriented Polyhedra of Arbitrary Euler Characteristic, *Pacific J. Math.* 144 (1990) no. 1, 71–103.

[K7] A.Kaveh, Graph Theoretical Methods for Efficient Flexibility Analysis of Planar Trusses, *Comput. and Structures* 23 (1986) no. 4, 305–319.

[K8] A.Kaveh, A Combinatorial Study of the Rigidity of Planar Structures, *Acta Mech.* 62 (1986) no. 1–4, 189–196.

[K9] A.Kaveh, Topology and Skeletal Structures. *Z. Angew. Math. Mech.* 68 (1988) no. 8, 347–353.

[K10] A.Kaveh, Suboptimal Cycle Bases of Graphs for the Flexibility Analysis of Skeletal Structures, *Comput. Methods Appl. Mech. Engrg* 71 (1988) no. 3, 259–271.

[K11] A.Kaveh, Topological Properties of Skeletal Structures, *Comput. and Structures* 29 (1988) no. 3, 403–411.

[K12] F.Klein, "Elementary Mathematics from an Advanced Standpoint. Geometry", Macmillan, London 1939.

[L1] G.Laman, On Graphs and Rigidity of Plane Skeletal Structures, *J. Engineering Math.* 4 (1970) no. 4, 331–340.

[L2] H.Liebmann, Über die Verbiegung der geschlossen Flächen positiver Krümmung, *Math. Ann.* 53 (1900) 81–112.

[L3] L.Lovász, Flats in Matroids and Geometric Graphs, in "Combinatorial Surveys", *Proc. 6th British Combinatorial Conf., P. Cameron, ed.*, Academic Press, New York (1977) 45–89.

[L4] L.Lovász, Matroid Matching and some Applications, *J. Combin. Th. Ser. B*, 28 (1980) 208–236.

[L5] L.Lovász and Y.Yemini, On Generic Rigidity in the Plane, *SIAM J. Alg. Disc. Meth.* 3 (1982) no. 1, 91–98.

[M1] H.Maehara, On Graver's Conjecture Concerning the Rigidity Problem of Graphs, *Discrete Comput. Geom.* 6 (1991) no. 4, 339–342.

[M2] A.Mansfield, On the Computational Complexity of a Rigidity Problem, *IMA J. Appl. Math.* 27 (1981) no. 4, 423–429.

[M3] J.C.Maxwell, On Reciprocal Figures & Diagrams of Forces, *Phil. Mag. Series 4*, 27 (1864) 250–261.

[M4] J.C.Maxwell, On Reciprocal Figures, Frames & Diagrams of Forces, *Trans. Royal Soc.* Edinburgh 26 (1869–1872) 1–40.

[O1] J.G.Oxley, "Matroid Theory", Oxford University Press, Oxford 1992.

[P1] S.Pellegrino, On the Rigidity of Triangulated Hyperbolic Paraboloids, *Proc. Roy. Soc. London Ser. A* 418 (1988) no. 1855, 425–452.

[P2] S.Pellegrino and C.R.Calladine, Matrix Analysis of Statically and Kinematically Indeterminate Frameworks, *Int. J. Solids Struct.* 22 (1986) 409–428.

[R1] W.J.M.Rankine, On the Application of Barycentric Perspective to the Transformation of Structures, *Phil. Mag. Series 4*, 26 (1863) 387–388.

[R2] W.J.M.Rankine, Principle of the Equilibrium of Polyhedral Frames, *Phil. Mag. Series 4*, 27 (1864) 92.

[R3] W.J.M.Rankine, "A Manual of Applied Mechanics.", C.Griffin, London 1876.

[R4] A.Recski, A Network Theory Approach to the Rigidity of Skeletal Structures. Part 1. Modelling and Interconnection, *Discrete Applied Mathematics* 7 (1984) 313–324.

[R5] A.Recski, A Network Theory Approach to the Rigidity of Skeletal Structures. Part 2. Laman's Theorem and Topological Formulae, *Discrete Applied Mathematics* 8 (1984) 63–68.

[R6] A.Recski, Bracing Cubic Grids – A Necessary Condition, *Discrete Math.* 73 (1989) no. 1–2, 199–206.

[R7] A.Recski, "Matroid Theory with Applications in Electric Network Theory and in Statics", Springer–Verlag, Berlin New York, 1989.

[R8] J.Rooney and R.J.Wilson, The Mobility of a Graph, in "Graph Theory, Singapore 1983", Lecture Notes in Math. 1073, Springer, Berlin–New York, (1984) 135–149.

[R9] I.Rosenberg, Structural Rigidity I: Foundations and Rigidity Criteria, *Annals Discrete Math.* 8 (1980) 143–161.

[R10] I.Rosenberg, Structural Rigidity II: Almost Infinitesimally Rigid Bar Frameworks, *Discrete Appl. Math.* 13 (1986) no. 1, 41–59.

[R11] B.Roth, Rigid and Flexible Frameworks, *Amer. Math. Monthly* 88 (1981) 6–21.

[R12] B.Roth and W.Whiteley, Tensegrity Frameworks, *Trans. Amer. Math. Soc.* 265 (1981) no. 2, 419–446.

[S1] B.Servatius, On the Two–Dimensional Generic Rigidity Matroid and Its Dual, *Journal of Combinatorial Theory, Series B* 53 (1991) 106–113.

[S2] K.Sugihara, On Some Problems in the Design of Plane Skeletal Structures, *SIAM J. Alg. Disc. Meth.* 4 (1983) 355–362.

[S3] K.Sugihara, A Unifying Approach to Descriptive Geometry and Mehanisms, *Disc. Appl. Math.* 5 (1983) no. 3, 313–328.

[T1] T.Tarnai, Duality Between Plane Trusses and Grillages, *Internat. J. Solids & Structures* 25 (1989) no. 12, 1395–1409.

[T2] T.–S.Tay, Review: Rigidity Problems in Bar and Joint Frameworks & Linkages of Rigid Bodies, *Structural Topology* 8 (1983) 33–36.

[T3] T.–S.Tay, Rigidity of Multigraphs I, Linking Rigid Bodies in n–Space, *J. Combin. Theory Ser. B* 36 (1984) no. 1, 95–112.

[T4] T.–S.Tay, Rigidity of Multigraphs II, in "Graph Theory, Singapore 1983", Lecture Notes in Math. 1073, Springer, Berlin–New York (1984) 129–134.

[T5] T.–S.Tay, Linking (n–2)–Dimensional Panels in n–Space. II. (n–2,2)–Frameworks and Body & Hinge Strutures, *Graphs Combin.* 5 (1989) no. 3, 245–273.

[T6] T.–S.Tay and W.Whiteley, Recent Advances in the Generic Rigidity of Structures, *Structural Topology* 9 (1984) 31–38.

[T7] T.–S.Tay and W.Whiteley, Generating Isostatic Frameworks, *Structural Topology* 11 (1985) 21–68.

[T8] S.Timoshenko and S.Young, "Theory of Structures", Second Edition, McGraw–Hill, New York 1965.

[V1] O.Veblen and J.W.Young, "Projective Geometry, vol. 1", Blaisdell Publishing Company, 1938.

[W1] B.Wegner, On the Projective Invariance of Shaky Structures in Euclidean Space, *Acta Mech.* 53 (1984) no. 3–4, 163–171.

[W2] B.Wegner, Infinitesimal Rigidity of Cone–like and Cylinder–like Frameworks, *Acta Mech.* 57 (1985) no. 3–4, 253–259.

[W3] D.J.A.Welsh, "Matroid Theory", Academic Press, London 1976.

[W4] H.Weyl, Über die Starrheit der Eiflächen und Konvexen Polyeder, *Sitzungs–Berichte, Preuss. Akad. Wiss.* (1917) 250–266.

[W5] N.White and W.Whiteley, The Algebraic Geometry of Stresses in Frameworks, *SIAM J. Algebraic discrete Methods* 4 (1983) no. 4, 481–511.

[W6] N.White and W.Whiteley, The Algebraic Geometry of Motions of Bar–and–Body Frameworks, *SIAM J. Alg. Disc. Methods* 8 (1987) no. 1, 1–32.

[W7] W.Whiteley, Motions of Bipartite Frameworks, *Structural Topology* 3 (1979) 62–63.

[W8] W.Whiteley, Motions and Stresses of Projected Polyhedra, *Structural Topology* 7 (1982) 13–38.

[W9] W.Whiteley, Motions of Trusses and Bipartite Frameworks, *Structural Topology* 7 (1982) 61–68.

[W10] W.Whiteley, Cones, Infinity and 1–story Buildings, *Structural Topology* 8 (1983) 53–70.

[W11] W.Whiteley, Infinitesimal Motions of a Bipartite Framework, *Pacific J. Math.* 110 (1984) no. 1, 233–255.

[W12] W.Whiteley, Infinitesimally Rigid Polyhedra I. Statics of Frameworks, *Trans. Amer. Math. Soc.* 285 (1984) no. 2, 431–465.

[W13] W.Whiteley, A Correspondence Between Scene Analysis and Motions of Frameworks, *Discrete Appl. Math.* 9 (1984) no. 3, 269–295.

[W14] W.Whiteley, The Projective Geometry of Rigid Frameworks, in "Finite Geometries" (Winnipeg, Man., 1984) Lecture Notes in Pure and Applied Math. 103 (1984) 353–370.

[W15] W.Whiteley, Rigidity and Polarity I: Statics of Sheet Structures, *Geom. Dedicata* 22 (1987) no. 3, 329–362.

[W16] W.Whiteley, Infinitesimally Rigid Polyhedra II. Modified Spherical Frameworks, *Trans. Amer. Math. Soc.* 306 (1988) no. 1, 115–139.

[W17] W.Whiteley, The Union of Matroids and the Rigidity of Frameworks, *SIAM J. Disc. Math.* 1 (1988) no. 2, 237–255.

[W18] W.Whiteley, Vertex Splitting in Isostatic Frameworks, *Structural Topology* 16 (1990) 23–30.

[W19] W.Wunderlich, Gefährliche Annahmen der Trilateration und bewegliche Fachwerke I, *Z. Angew. Math. Mech.* 57 (1977) 297–304.

[W20] W.Wunderlich, Gefährliche Annahmen der Trilateration und bewegliche Fachwerke II, *Z. Angew. Math. Mech.* 57 (1977) 363–368.

[W21] W.Wunderlich, Snapping and Shaky Antiprisms, *Math. Mag.* 52 (1979) no. 4, 235–236.

[W22] W.Wunderlich, Eine Familie von geschlossenen gleichflächigen Polyedern, die fast beweglich sind, *Elem. Math.* 41 (1986) no. 4, 88–98.

[W23] W.Wunderlich, Projective Invariance of Shaky Structures, *Acta Mech.* 42 (1982) 171–181.