Theorems of Birkhoff Type in Pseudovarieties and E-varieties of Regular Semigroups

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To Leigh, Agesilaus and Eleanor

I declare that this thesis contains no material which has been accepted for the award of a degree or diploma by the University or any other institution, and that, to the best of my knowledge and belief, it contains no material previously published or written by another person except when due reference is made in the text of the thesis.

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Abstract

This thesis is concerned with the problem of being able to use, or generalize, Birkhoff's fundamental theorems for classes of algebras which do not form varieties - particularly in pseudovarieties and e-varieties. After giving an introduction to these areas in Chapter 1, we first look at pseudovarieties, focusing on certain generalized varieties.

Let $\mathcal{Com}$, $\mathcal{Nil}$, and $\mathcal{N}$ denote the generalized varieties of all commutative, nil, and nilpotent semigroups respectively. For a class $\mathcal{W}$ of semigroups let $\mathcal{L}(\mathcal{W})$ and $\mathcal{G}(\mathcal{W})$ denote respectively the lattices of all varieties and generalized varieties of semigroups contained in $\mathcal{W}$. Almeida has shown that the mapping $\mathcal{L}(\mathcal{Nil} \cap \mathcal{Com}) \cup \{\mathcal{Nil} \cap \mathcal{Com}\} \to \mathcal{G}(\mathcal{N} \cap \mathcal{Com})$ given by $\mathcal{W} \mapsto \mathcal{W} \cap \mathcal{N}$ is an isomorphism, and asked whether the extension of this mapping to $\mathcal{L}(\mathcal{Nil}) \cup \{\mathcal{Nil}\}$ is also an isomorphism.

In Chapter 2 we consider this question. In Section 2.2 we show that the extension is not surjective. Non-injectivity is then established in Sections 2.4 - 2.6; this involves analysing sequences of words of unbounded lengths derived from the defining identities of certain nil varieties. Results of a more general nature are also given, in Section 2.3, involving the question of when two arbitrary semigroup varieties possess the same set of nilpotent semigroups.

In Chapter 3 we turn to the problem of establishing analogues of Birkhoff's theorems for e-varieties. In Section 3.1 Auinger's Birkhoff-style theory for locally inverse e-varieties is expanded, to obtain a unified theory for e-varieties of locally inverse or of E-solid semigroups - that is, for the entire lattice of e-varieties in which nonmonogenic bifree objects exist. In addition an alternative unification, based on the techniques used by Kaďourek and Szendrei to describe a Birkhoff-style theory for E-solid e-varieties, is given in Section 3.2.

In Section 3.3 we show that trifree objects on at least three generators exist

in an e-variety **V** of regular semigroups if and only if **V** is locally E-solid; this extends Kaďourek's work on the existence of trifree objects in locally orthodox e-varieties and generalizes Yeh's result on the existence of bifree objects.

In conclusion, a theory of "n-free" objects is outlined in Section 3.4, indicating how analogues of the concept of a free object can be defined for any e-variety.

The results presented in Sections 2.4 - 2.6 appear in [12]. The results of Chapter 3 will appear in [13].

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Contents

Chapter 1

Introduction

The classes of finite semigroups and regular semigroups are important examples of classes of semigroups that do not form varieties or admit free objects, and are therefore without the direct advantages of Birkhoff's fundamental theorems of universal algebra relating varieties, identities, and free objects. For both cases the definition of a variety of algebras, as a class of algebras of a given type closed under taking homomorphic images, subalgebras and direct products, is weakened: a *pseudovariety* of semigroups is a class of finite semigroups closed under taking homomorphic images, subsemigroups and finite direct products; and an *e-variety* of regular semigroups is a class of regular semigroups closed under taking homomorphic images, regular subsemigroups and direct products. Various theories have been developed in both cases, by considering links with varieties, or by devising analogues of the notions of identity or free object that allow for "Birkhoff-style" theorems.

This thesis is concerned with the problem of being able to use, or generalize, Birkhoff's fundamental theorems for classes of algebras which do not form varieties - particularly in pseudovarieties and e-varieties. We first look at pseudovarieties, focusing on a link with varieties by which Birkhoff-style properties can be used for pseudovarieties. We then turn to the problem of developing Birkhoff-style theories for e-varieties of regular semigroups. This chapter gives an introduction to these areas, with preliminary results.

In Chapter 2, we investigate certain generalized varieties, which are used in the study of pseudovarieties. In [6], Ash defined a generalized variety as a directed union of varieties, and proved that a class of algebras is a generalized variety if and only if it is closed under the formation of homomorphic images, subalgebras, arbitrary powers and finite direct products. Ash showed that generalized varieties provide a link between varieties and pseudovarieties: a class of algebras is a pseudovariety if and only if it consists of the finite members of some generalized variety. Several authors including Almeida and Reilly [4], Almeida [1], Pastijn [31], and Pastijn and Trotter [32] have investigated pseudovarieties and therefore generalized varieties from this point of view.

In Chapter 2 we look at this connection between pseudovarieties and varieties in a special case. Let $\mathcal{C}om$ denote the variety of all commutative semigroups, let $\mathcal{N}il$ denote the generalized variety of all nil semigroups, and let $\mathcal{N}$ denote the generalized variety of all nilpotent semigroups. For any class $\mathcal{W}$ of semigroups let $\mathcal{L}(\mathcal{W})$ denote the lattice of all varieties of semigroups contained in $\mathcal{W}$, and let $\mathcal{G}(\mathcal{W})$ denote the lattice of all generalized varieties of semigroups contained in $\mathcal{W}$. Almeida [1] has shown that the mapping

$$\mathcal{L}(\mathcal{N}il \cap \mathcal{C}om) \cup \{\mathcal{N}il \cap \mathcal{C}om\} \to \mathcal{G}(\mathcal{N} \cap \mathcal{C}om)$$

given by $\mathcal{W} \mapsto \mathcal{W} \cap \mathcal{N}$ is an isomorphism, and asked ([3, Problem 10]) whether the extension of this mapping to $\mathcal{L}(\mathcal{N}il) \cup \{\mathcal{N}il\}$ is also an isomorphism.

In Section 2.2 we show that this extension is not surjective, and proceed to consider the question of injectivity. We first give some more general results in Section 2.3, involving the question of when two arbitrary semigroup varieties possess the same set of nilpotent semigroups. The question of when two nil semigroup varieties have the same set of nilpotent semigroups is more complex. In Sections 2.4 - 2.6 two varieties $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{N}il)$ are defined and used to show that the extension described above is not injective. These varieties have a remarkable property: there is an infinite sequence of words with unbounded lengths, derived from the defining identities of $\mathcal{U}$ and $\mathcal{V}$, such that for any set X, with $|X| \geq 3$, the terms of the sequence constitute a congruence class of both the fully invariant congruence on the free semigroup X^+ which corresponds to $\mathcal{U}$ and the fully invariant congruence on X^+ which corresponds to $\mathcal{V}$.

Chapter 3 deals with analogues of free objects for e-varieties of regular semigroups. The concept of an e-variety of regular semigroups was introduced independently by Hall [20], and Kaďourek and Szendrei [26]. Kaďourek and Szendrei used the term *bivariety*, and considered only classes of orthodox semigroups. With this restriction they gave definitions of bifree objects, biidentities and bi-invariant congruences, and were able to generalize Birkhoff's theorems. Yeh [43] investigated the existence of bifree objects in arbitrary classes of regular semigroups, and proved a necessary and sufficient result; namely that non-monogenic bifree objects exist in an e-variety $\mathbf{V}$ if and only if $\mathbf{V}$ is contained in either the e-variety of all E-solid semigroups or the e-variety of all locally inverse semigroups.

In [7] and [8], Auinger considered classes of locally inverse semigroups, and was able to extend the results of Kaďourek and Szendrei from [26]. In a paper [27] yet to appear, Kaďourek and Szendrei also extended the results of [26], to classes of E-solid semigroups. So analogues of Birkhoff's theorems hold for the entire lattice of e-varieties in which non-monogenic bifree objects exist. But the two approaches are quite different. In Section 3.1 we expand Auinger's approach, to obtain a unified theory for e-varieties of locally inverse or of E-solid semigroups. In Section 3.2 we give an alternative unification, based on the techniques of Kaďourek and Szendrei in [27].

In [25], Kaďourek considered classes of locally orthodox semigroups. By Yeh's results, bifree objects do not always exist in these classes. Generalizing the ideas used to develop the theory of bifree objects, Kaďourek defined *trifree objects*. He showed that trifree objects exist in every e-variety of locally orthodox semigroups. However, he also showed that other Birkhoff-type theorems do not hold in this context. In Section 3.3 we extend Kaďourek's work to classes of locally E-solid semigroups. In fact we show that trifree objects exist in an e-variety $\mathbf{V}$ of regular semigroups if and only if $\mathbf{V}$ consists of locally E-solid semigroups.

In conclusion, a theory of "n-free" objects is outlined in Section 3.4, indicating how analogues of the concept of a free object can be defined for any e-variety.

This chapter is meant as an introduction to our subject - that is, pseudovarieties of finite semigroups and e-varieties of regular semigroups, with the underlying common theme of using or generalizing Birkhoff's theory of varieties, free objects and identities. We therefore start with a section on universal algebra, proceeding to discuss pseudovarieties in Section 1.2, and e-varieties in Section 1.3.

1.1 Universal algebra.

Although we will be mainly using semigroups in this thesis, sometimes we consider extra operations on these semigroups, and so we begin by briefly reviewing the definitions of subalgebra, direct product, homomorphism, and congruence for algebras in general. This leads to the theory of varieties, identities, and free objects, and to Birkhoff's theorems relating these concepts. For undefined notation and terminology see the book by Burris and Sankappanavar [11].

Throughout this section let τ be a fixed type of algebras, with set O of operation symbols. All algebras will be considered to be of type τ unless otherwise stated. We fix $\mathcal{C}$ as a class of algebras of type τ.

For every $n \geq 0$ let $O_n = \{f \in O : f \text{ has arity } n\}$. For every $f \in O$ we write f^A for the corresponding operation on an algebra A.

For this section let X be a nonempty set of distinct objects called *variables* or *letters* such that $X \cap O_0 = \emptyset$. The set X is sometimes called an alphabet.

1.1.1 Subalgebras, direct products, homomorphisms, and congruences.

Let A and B be two algebras. The algebra B is a *subalgebra* of A, written $B \leq A$, if $B \subseteq A$ (as sets) and for every $f \in O$ the operation f^B is the restriction of f^A to B.

The *direct product* $A = \prod_{i \in I} A_i$ of a family $\{A_i : i \in I\}$ of algebras is such that

$$f^A(a_1, \ldots, a_n)(i) = f^{A_i}(a_1(i), \ldots, a_n(i))$$

for all $i \in I$ whenever $f \in O_n$ and $a_1, \ldots, a_n \in \prod_{i \in I} A_i$. The product A is said to be a *direct power* when all the A_i coincide.

An equivalence relation ρ on an algebra A is a *congruence* if whenever $f \in O_n$ and $a_i, b_i \in A$ satisfy $(a_i, b_i) \in \rho$ for $1 \leq i \leq n$ we have

$$(f^A(a_1, \ldots, a_n), f^A(b_1, \ldots, b_n)) \in \rho.$$

The *quotient algebra* A/ρ is such that $f^{A/\rho}(a_1\rho, \ldots, a_n\rho) = f^A(a_1, \ldots, a_n)\rho$ for every $a_1, \ldots, a_n \in A$ and $f \in O_n$.

A *homomorphism* from A to B is a mapping $\alpha : A \to B$ which satisfies $f^A(a_1, \ldots, a_n)\alpha = f^B(a_1\alpha, \ldots, a_n\alpha)$ for each $f \in O_n$ and $a_1, \ldots, a_n \in A$.

Notation 1.1.1 Let A and B be two algebras. The *kernel* of a homomorphism $\alpha : A \to B$ is the congruence $\alpha \circ \alpha^{-1} = \{(a,b) \in A \times A : a\alpha = b\alpha\}$. Given a congruence ρ on A, the homomorphism $A \to A/\rho$ given by $a \mapsto a\rho$, $a \in A$, is denoted by $\rho^\sharp$. For a relation ρ on A let $\rho^{-1} = \{(a,b) : (b,a) \in \rho\}$. Let $\langle\rho\rangle$ denote the congruence on A generated by ρ; that is, the least congruence on A that contains ρ. □

Result 1.1.2 ([11]) *Let $\alpha : A \to B$ and $\beta : A \to C$ be two homomorphisms such that β is surjective and $\beta \circ \beta^{-1} \subseteq \alpha \circ \alpha^{-1}$. Then there is a homomorphism $C \to B$ given by $a\beta \mapsto a\alpha$, $a \in A$.* □

1.1.2 Varieties and identities.

Recall that $\mathcal{C}$ is defined as a fixed class of algebras of type τ.

Definition 1.1.3 We write:

$I(\mathcal{C})$ for the class of all isomorphic images of members of $\mathcal{C}$;

$H(\mathcal{C})$ for the class of all homomorphic images of members of $\mathcal{C}$;

$S(\mathcal{C})$ for the class of all subalgebras of members of $\mathcal{C}$;

$P(\mathcal{C})$ for the class of all direct products of members of $\mathcal{C}$;

$P_f(\mathcal{C})$ for the class of all finite direct products of members of $\mathcal{C}$;

$Pow(\mathcal{C})$ for the class of all direct powers of members of $\mathcal{C}$. □

A *variety* is a class of algebras (of type τ) closed under H, S and P. Let $V(\mathcal{C})$ denote the variety generated by $\mathcal{C}$, which is the intersection of all varieties containing $\mathcal{C}$. We write $V(A)$ when $\mathcal{C}$ has only one member A.

Theorem 1.1.4 (Tarski) $V(\mathcal{C}) = HSP(\mathcal{C})$. □

The set $T(X)$ of *terms* (of type τ) over X is the smallest set such that

- $X \cup O_0 \subseteq T(X)$, and
- if $p_1, \ldots, p_n \in T(X)$ and $f \in O_n$ then $f(p_1, \ldots, p_n) \in T(X)$.

The *term algebra* (of type τ) over X is the algebra $T(X)$, for which $f^{T(X)}(p_1, \ldots, p_n)$ is the term $f(p_1, \ldots, p_n)$ whenever $f \in O_n$ and $p_1, \ldots, p_n \in T(X)$. So $T(X)$ is generated by the set X.

An *identity* (of type τ) over X is a pair (p, q), also written $p = q$, where $p, q \in T(X)$. We write Id_X for the set of all identities over X. We may regard Id_X as being a binary relation over $T(X)$ or a set of equations over X.

An identity (u, v) is said to be *trivial* if u and v are the same element of $T(X)$, and *nontrivial* otherwise.

An algebra A *satisfies* an identity $p = q$ over X, written $A \models p = q$, if $p\alpha = q\alpha$ for every homomorphism $\alpha : T(X) \to A$. The class $\mathcal{C}$ satisfies an identity $p = q$, and we write $\mathcal{C} \models p = q$, if $A \models p = q$ for every $A \in \mathcal{C}$.

We write $\mathrm{Id}_X(\mathcal{C})$ for the set of all identities satisfied by $\mathcal{C}$ (or $\mathrm{Id}_X(A)$ if $\mathcal{C} = \{A\}$). We sometimes write simply $\mathrm{Id}(\mathcal{C})$ or $\mathrm{Id}(A)$. If every member of a set Σ of identities is satisfied by a class $\mathcal{C}$ (or algebra A) we say that $\mathcal{C}$ (or A) satisfies Σ, and write $\mathcal{C} \models \Sigma$ (or $A \models \Sigma$).

A congruence ρ on an algebra A is *fully invariant* if $(a\alpha, b\alpha) \in \rho$ whenever α is an endomorphism of A and $(a, b) \in \rho$. For a set Σ of identities let $\Theta(\Sigma)$ denote the fully invariant congruence on $T(X)$ generated by Σ. We write $\Sigma \models p = q$ if $(p, q) \in \Theta(\Sigma)$.

1.1.3 Free objects and Birkhoff's theorems.

In this section we define free objects and state the fundamental theorems of Birkhoff's 1935 paper [9] connecting varieties, identities and free objects. The first of these establishes that a class of algebras is a variety if and only if it is equationally defined. Given a set Σ of identities, let $[\Sigma]$ be the class of all algebras that satisfy Σ.

Theorem 1.1.5 ([9]) *If $\mathcal{V}$ is a variety and the set X is denumerable then $\mathcal{V} = [\mathrm{Id}_X(\mathcal{V})]$; and conversely, for a set Σ of identities the class $[\Sigma]$ is a variety.* □

Notice that for a class $\mathcal{C}$ of algebras (of type τ) we have $\mathrm{Id}_X(\mathcal{C}) = \mathrm{Id}_X(V(\mathcal{C}))$, which is a fully invariant congruence on $T(X)$ by the next theorem of Birkhoff. We often write $\rho(\mathcal{C})$ for $\mathrm{Id}_X(\mathcal{C})$.

Theorem 1.1.6 ([9]) *Suppose X is denumerable. The lattice of varieties of algebras of type τ is antiisomorphic to the lattice of fully invariant congruences on $T(X)$ (with respect to $\subseteq$) via the mutually inverse mappings*

$$\mathcal{V} \mapsto \mathrm{Id}_X(\mathcal{V}) \text{ and } \rho \mapsto [\rho]. \qquad \square$$

An algebra F, together with a mapping $\iota : X \to F$, is said to have the *universal mapping property for $\mathcal{C}$ over X* if for every $S \in \mathcal{C}$ and mapping $\phi : X \to S$ there is a unique homomorphism $\varphi : F \to S$ such that $\iota\varphi = \phi$.

Theorem 1.1.7 ([9]) *The algebra $T(X)/\mathrm{Id}_X(\mathcal{C})$, together with the natural injection $\iota : X \to T(X)/\mathrm{Id}_X(\mathcal{C})$, has the universal mapping property for $\mathcal{C}$ over X.* □

We write $F_{\mathcal{C}}(X) = T(X)/\mathrm{Id}_X(\mathcal{C})$, and call this algebra the *$\mathcal{C}$-free algebra on X*. So $F_{\mathcal{C}}(X) = F_{V(\mathcal{C})}(X)$. We usually assume that $X \subseteq F_{\mathcal{C}}(X)$.

Theorem 1.1.8 ([9]) *We have $F_{\mathcal{C}}(X) \in ISP(\mathcal{C})$. In particular, if $\mathcal{C}$ is a variety then $F_{\mathcal{C}}(X) \in \mathcal{C}$. If $\mathcal{C}$ is a variety and X is denumerable then $\mathcal{C} = V(F_{\mathcal{C}}(X))$.*
□

Remark 1.1.9 By Theorems 1.1.7 and 1.1.8, if $\mathcal{C}$ is a variety then the members of $\mathcal{C}$ are precisely the homomorphic images of the $\mathcal{C}$-free algebras. □

A semigroup is an algebra $(S, \cdot)$ of type (2) where $\cdot$ is associative. We write $a \cdot b$, or ab, so that a semigroup satisfies the identity $x(yz) = (xy)z$. An algebra $(S, \cdot, ')$ of type (2,1), where $\cdot$ is associative, is called a *unary semigroup*. An algebra $(S, \cdot, s)$ of type (2,2), where $\cdot$ is associative, is called a *binary semigroup*.

We will use the terms "semigroup homomorphism", "unary semigroup homomorphism", and "binary semigroup homomorphism" to distinguish homomorphisms between algebras of these types. We often simply use the term "homomorphism" when the meaning is clear. The same applies to the terms "congruence", "variety", "identity", etc.

Notation 1.1.10 For any nonempty set X, let X^+ denote the free semigroup on X. We will call the members of X *letters* or *variables*, and the members of X^+ *words*. The free monoid X^* on X is obtained by adjoining the empty word 1 to X^+ as an identity. □

1.2 Pseudovarieties and generalized varieties.

Classes of finite algebras do not usually form varieties, and so a *pseudovariety* is defined to be a class of algebras (all of the same type) closed under taking homomorphic images, subalgebras, and finite direct products. The term "pseudovariety" was introduced, for semigroups and monoids, by Eilenberg and Schützenberger in [16]. Pseudovarieties of semigroups and monoids are fundamental to automata and formal language theory, and most of the early results concerning pseudovarieties were motivated by the applications in these areas (see Eilenberg's book [15]).

In the 1980's Reiterman [36] and Ash [6] began investigating pseudovarieties from a universal algebraic perspective. Reiterman proved an analogue for pseudovarieties of Birkhoff's Theorem 1.1.5, establishing that a class of finite algebras is a pseudovariety if and only if it is "pseudoequationally" defined. This result has led to a Birkhoff-style theory for pseudovarieties, which we briefly review in Section 1.2.1.

In Section 1.2.2 we discuss Ash's work from [6]. In this paper Ash considered the concept of pseudovariety for algebras of arbitrary type, and described a certain kind of connection with varieties, namely *generalized varieties*, which involves the characterization of pseudovarieties in terms of nets of identities.

There are several other characterizations of pseudovarieties apart from those already mentioned. For instance, they may be described in terms of filters of congruences (see [37]) and varieties of languages (see [15]). See Almeida's book [3] as a reference for the contents of this section.

1.2.1 Pseudoidentities and free profinite algebras.

Of course pseudovarieties do not usually admit free objects, as these algebras are usually infinite. For example, the smallest semigroup variety that contains the pseudovariety $\mathbf{G}$ of all finite groups is the variety of all semigroups; and so $F_{\mathbf{G}}(A) = A^+$, the free semigroup on A, for every alphabet A. So the free object $F_{\mathbf{V}}(A)$ is often too general to be useful for the study of a pseudovariety $\mathbf{V}$ - for example, a finite A-generated homomorphic image of $F_{\mathbf{G}}(A)$ need not be in $\mathbf{G}$.

An alternative candidate for a concept of "free" object for pseudovarieties is the relatively free profinite algebra. These algebras can be large and somewhat unwieldy, but behave quite like the usual free objects, as will be seen below. For more details concerning relatively free profinite semigroups and monoids see the survey [5] by Almeida and Weil.

Throughout the remainder of this section let $\mathbf{V}$ be a fixed pseudovariety.

A partially ordered set I is *directed* if each pair of elements has a common upper bound. A *directed system of algebras* $(S_i)_{i\in I}$ is a family of algebras indexed by a directed poset I such that:

- whenever $i \geq j$ in I there exists a homomorphism $\varphi_{ij} : S_i \to S_j$,
- φ_{ii} is the identity map on S_i,
- if $i \geq j \geq k$ in I then $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$.

The *projective limit* of a directed system of algebras $(S_i)_{i\in I}$ is the following subalgebra of the direct product $\prod_{i\in I} S_i$:

$$\varprojlim(S_i)_{i\in I} = \{(x_i)_{i\in I} \in \prod_{i\in I} S_i : x_i\varphi_{i,j} = x_j \text{ whenever } i \geq j\}.$$

We consider all finite algebras to be endowed with the discrete topology. Then all homomorphisms of finite algebras are continuous, and all finite algebras are compact. An algebra is said to be *profinite* if it is a projective limit of finite algebras. An algebra is said to be *pro*-$\mathbf{V}$ if it is a projective limit of elements of $\mathbf{V}$. Notice that every member of $\mathbf{V}$ is pro-$\mathbf{V}$.

Let A be a finite set. A profinite algebra S is said to be *A-generated* if there exists a mapping $\mu : A \to S$ such that $\langle A\mu\rangle$ is a dense subalgebra of S. Let $\mathbf{V}_A$ be the class of all A-generated members of $\mathbf{V}$, with isomorphic algebras identified; thus $\mathbf{V}_A = \{A^+/\theta : \theta \in \Theta_A\mathbf{V}\}$, where A^+ denotes the free semigroup on A and $\Theta_A\mathbf{V}$ is the set of all congruences θ on A^+ for which $A^+/\theta \in \mathbf{V}$.

The set $\Theta_A\mathbf{V}$ is a directed poset with respect to $\supseteq$ (the common upper bound for $\theta_1, \theta_2 \in \Theta_A\mathbf{V}$ is $\theta_1 \cap \theta_2$), and it follows that the class $\mathbf{V}_A$ is a directed system of algebras. We write $\hat{F}_A(\mathbf{V})$ for the projective limit of $\mathbf{V}_A$.

Of course $\hat{F}_A(\mathbf{V})$ is usually infinite. In fact $\hat{F}_A(\mathbf{V}) \in \mathbf{V}$ if and only if $\mathbf{V}_A$ (or equivalently $F_A(\mathbf{V})$) is finite, and in this case $\hat{F}_A(\mathbf{V}) = F_A(\mathbf{V})$. However, as the following details show, the free pro-$\mathbf{V}$ algebra $\hat{F}_A(\mathbf{V})$ behaves quite like free objects in some respects, and Birkhoff-style theorems have been proved in this setting.

Theorem 1.2.1

(i) There exists a mapping $\iota_{\mathbf{V}} : A \to \hat{F}_A(\mathbf{V})$ such that the subsemigroup of $\hat{F}_A(\mathbf{V})$ generated by $A\iota_{\mathbf{V}}$ is dense in $\hat{F}_A(\mathbf{V})$ and is isomorphic to $F_A(\mathbf{V})$.

(ii) The algebra $\hat{F}_A(\mathbf{V})$ is the free pro-$\mathbf{V}$ algebra over A*: if S is a pro-$\mathbf{V}$ algebra and $\varphi : A \to S$ is a mapping then there exists a unique continuous homomorphism $\overline{\varphi} : \hat{F}_A(\mathbf{V}) \to S$ such that $\iota_{\mathbf{V}}\overline{\varphi} = \varphi$.*

(iii) A finite algebra S is in $\mathbf{V}$ if and only if S is a continuous homomorphic image of $\hat{F}_A(\mathbf{V})$ for some A. □

A *pro-$\mathbf{V}$-identity* is a pair (x, y), also written $x = y$, of members of $\hat{F}_A(\mathbf{V})$ for some A. When $|A| = n$ we say that $x = y$ is an n-variable pro-$\mathbf{V}$-identity. If $\mathbf{V}$ is the pseudovariety of all finite semigroups then pro-$\mathbf{V}$-identities are called *proidentities* or *pseudoidentities.* A pro-$\mathbf{V}$ algebra S is said to *satisfy* the pro-$\mathbf{V}$-identity $x = y$ if $x\alpha = y\alpha$ for each continuous homomorphism $\alpha : \hat{F}_A(\mathbf{V}) \to S$.

Let Σ be a set of pro-$\mathbf{V}$-identities (not necessarily involving a bounded number of variables). A class $\mathbf{W}$ of pro-$\mathbf{V}$ algebras is said to satisfy Σ if each element of $\mathbf{W}$ satisfies each pro-$\mathbf{V}$-identity in Σ; we write $\mathbf{W} \models \Sigma$. The class of all finite algebras which satisfy Σ is denoted by $[\Sigma]_{\mathbf{V}}$.

The first part of the next result is a consequence of Theorem 1.2.1. The second is from [33].

Theorem 1.2.2

(i) If $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$ then the mapping $\iota_{\mathbf{W}}$ induces a continuous surjective homomorphism $\pi : \hat{F}_A(\mathbf{V}) \to \hat{F}_A\mathbf{W}$ such that $\iota_{\mathbf{W}} = \iota_{\mathbf{V}}\pi$.

(ii) For all $x, y \in \hat{F}_A(\mathbf{V})$ we have $\mathbf{W} \models x = y$ if and only if $x\pi = y\pi$. $\square$

The following result is Reiterman's analogue of Birkhoff's Theorem 1.1.5, and states that pseudovarieties are exactly the "pseudoequationally" defined classes.

Theorem 1.2.3 ([36]) *Let $\mathbf{V}$ be a pseudovariety, and let $\mathbf{W} \subseteq \mathbf{V}$. Then $\mathbf{W}$ is a pseudovariety if and only if $\mathbf{W} = [\Sigma]_{\mathbf{V}}$ for some set Σ of pro-$\mathbf{V}$-identities.*
$\square$

Remark 1.2.4 If $\mathcal{V}$ is a variety we call any pair (p, q) of members of $F_{\mathcal{V}}(X)$ a $\mathcal{V}$-identity. Then Theorem 1.1.5 can be stated in the form of Theorem 1.2.3: given a variety $\mathcal{V}$, a class $\mathcal{W} \subseteq \mathcal{V}$ is a variety if and only if $\mathcal{W}$ is defined by a set of $\mathcal{V}$-identities. $\square$

Almeida gave an analogue of Birkhoff's Theorem 1.1.6, for which we need to consider an infinite number of variables.

If A and B are two finite sets with $|A| = |B|$ then $\hat{F}_A(\mathbf{V})$ and $\hat{F}_B(\mathbf{V})$ are isomorphic. If the cardinality of the set A is of interest, we write $\hat{F}_n(\mathbf{V})$ instead of $\hat{F}_A(\mathbf{V})$ if $|A| = n$. We may assume $\hat{F}_n(\mathbf{V}) \subseteq \hat{F}_{n+1}(\mathbf{V})$ for every $n \geq 1$. Let $\hat{F}_w(\mathbf{V})$ be the union (that is, inductive limit) of the topological algebras $\hat{F}_n\mathbf{V}$, $n \geq 1$. Thus we may assume that any set Σ of pro-$\mathbf{V}$-identities is contained in $\hat{F}_w(\mathbf{V}) \times \hat{F}_w(\mathbf{V})$.

A set Σ of pro-$\mathbf{V}$-identities is said to be *strongly closed* if

- Σ is a fully invariant congruence on $\hat{F}_w(\mathbf{V})$;
- for every $u, v \in \hat{F}_w(\mathbf{V})$ such that $u\Sigma \neq v\Sigma$ there exists a clopen union F of classes of Σ for which $u\Sigma \subseteq F$ and $v\Sigma \subseteq \hat{F}_w(\mathbf{V}) \setminus F$.

The set of all strongly closed sets of pro-$\mathbf{V}$-identities, denoted by $PI\mathbf{V}$, is a complete lattice (under inclusion). We write $\mathcal{P}s\mathbf{V}$ for the lattice of all subpseudovarieties of $\mathbf{V}$. For a subpseudovariety $\mathbf{W}$ of $\mathbf{V}$, let $\mathrm{Pid}_{\mathbf{V}}\mathbf{W}$ be the set of all pro-$\mathbf{V}$-identities that are satisfied by $\mathbf{W}$.

Theorem 1.2.5 ([2]) *The lattice $\mathcal{P}s\mathbf{V}$ is antiisomorphic to $PI\mathbf{V}$ via the mutually inverse correspondences $\mathbf{W} \mapsto \mathrm{Pid}_{\mathbf{V}}\mathbf{W}$ and $\Sigma \mapsto [\Sigma]_{\mathbf{V}}$.* □

Remark 1.2.6 Theorem 1.2.3 was originally given in terms of implicit operations, which are defined as follows.

For a finite set A an *A-ary implicit operation* on a pseudovariety $\mathbf{V}$ is a family $\pi = (\pi_S)_{S\in\mathbf{V}}$ such that

- π_S is a function $S^A \to S$ for each $S \in \mathbf{V}$;
- given any homomorphism $\varphi : S \to T$ for $S, T \in \mathbf{V}$, and $(s_a)_{a\in A} \in S^A$ we have $(s_a)_{a\in A}\pi_S\varphi = (s_a\varphi)_{a\in A}\pi_T$.

The set of all A-ary implicit operations on $\mathbf{V}$ forms a topological algebra that is isomorphic to the free pro-$\mathbf{V}$ algebra $\hat{F}_A(\mathbf{V})$. □

1.2.2 Generalized varieties.

In [6] Ash gave the name *generalized varieties* to classes satisfying the conditions of the next result. A *filter* over a set I is a family of subsets of I closed under taking supersets and finite intersections.

Theorem 1.2.7 ([6]) *The following are equivalent for a class $\mathcal{C}$ of algebras (of type τ).*

(i) $\mathcal{C}$ is closed under H, S, P_f and Pow.

(ii) $\mathcal{C} = HSP_fPow(\mathcal{C})$.

(iii) $\mathcal{C}$ *is the union of some directed family of varieties.*

(iv) There exists a filter F *over* Id_X *such that*

$$A \in \mathcal{C} \Leftrightarrow \mathrm{Id}_X(A) \in F$$

for all algebras A *(of type* τ*).* □

Remark 1.2.8

- Note that condition (ii) of Theorem 1.2.7 implies that for every class $\mathcal{C}$ of algebras the class $Gen(\mathcal{C}) = HSP_f Pow(\mathcal{C})$ is the least generalized variety containing $\mathcal{C}$.
- For a finite class $\mathcal{C}$ of algebras the class $HSP_f(\mathcal{C})$ is the least pseudovariety containing $\mathcal{C}$. (See [22].) □

For a class $\mathcal{C}$ of algebras, let $\mathcal{C}^F$ denote the class of all finite members of $\mathcal{C}$. A pseudovariety $\mathbf{V}$ is said to be *equational* if $\mathbf{V} = \mathcal{V}^F$ for some variety $\mathcal{V}$. The pseudovariety $\mathbf{B} = [x^2 = x]^F$ of all bands (that is, semigroups of idempotents) is an example of a equational pseudovariety.

An example of a pseudovariety that is not equational is the class $\mathbf{N}$ of all finite nilpotent semigroups. For every $m \geq 1$ the class $\mathcal{N}_m$ of all m-nilpotent semigroups is the variety of semigroups with zero for which each product of m elements is zero. So

$$\mathcal{N}_m = [x_1 \ldots x_m y = y x_1 \ldots x_m = x_1 \ldots x_m],$$

and

$$\mathbf{N} = \Big(\bigcup_{m \geq 1} \mathcal{N}_m\Big)^F = \bigcup_{m \geq 1} \mathcal{N}_m^F.$$

The smallest equational pseudovariety that contains $\mathbf{N}$ is the pseudovariety of all finite semigroups. The same is true for the pseudovariety $\mathbf{G}$ of all finite groups, which can also be represented as a union of equational pseudovarieties: we have $\mathbf{G} = \bigcup_{m\geq 1}[x^m y = yx^m = y]^F$.

In general, unions of equational pseudovarieties do not form pseudovarieties. However, as will be stated in Result 1.2.9, every pseudovariety is the union of a directed family of equational pseudovarieties. As shown by Eilenberg and Schützenberger in [16] for pseudovarieties of semigroups and monoids, and Ash in [6] for pseudovarieties of algebras of arbitrary type, if the algebraic type of a pseudovariety $\mathbf{V}$ is finite then the family may be chosen to be a chain, and there exists a sequence $(\varepsilon_n)_{n\geq 1}$ of identities such that $\mathbf{V} = \bigcup_{k\geq 1}[\varepsilon_n : n \geq k]^F$.

The next result is Ash's characterization of pseudovarieties as the finite parts of generalized varieties, which thus connects varieties and pseudovarieties via generalized varieties: a generalized variety is a directed union of varieties, and the finite members of a generalized variety form a pseudovariety. This construction has proved to be very useful in transferring information about varieties to pseudovarieties; see for example Almeida and Reilly [4], Almeida [1], Pastijn [31], Pastijn and Trotter [32].

Notice that the operators $\bigcup$ and ^{F} commute; that is, $\bigcup_{\lambda\in\Lambda}\mathcal{V}_\lambda^F = (\bigcup_{\lambda\in\Lambda}\mathcal{V}_\lambda)^F$ for a family $\{\mathcal{V}_\lambda : \lambda \in \Lambda\}$ of classes of algebras.

Theorem 1.2.9 ([6]) *A class $\mathbf{V}$ of algebras is a pseudovariety if and only if $\mathbf{V} = \mathcal{V}^F$ for some generalized variety $\mathcal{V}$. In particular, if $\mathbf{V}$ is a pseudovariety then $\mathbf{V} = (Gen(\mathbf{V}))^F$.* □

1.3 E-varieties of regular semigroups.

We begin this section on e-varieties with some preliminary notation and results from semigroup theory. We then introduce e-varieties in Section 1.3.2. The concept of an e-variety was introduced by both Hall [20] and Kaďourek and Szendrei

[26], independently. In Section 1.3.2 we discuss some of the material contained in [20], which concerns the whole lattice of e-varieties and describes connections with varieties of unary semigroups. In Section 1.3.3 some background information about relevant e-varieties is given. We then review the Birkhoff-style results obtained by Kadourek and Szendrei [26], for orthodox e-varieties alone, in Section 1.3.4. This work involves an alternative concept of free object that can be used for e-varieties, namely "bifree objects". In Section 1.3.5 we discuss Yeh's work in [43], where he established that nonmonogenic bifree objects exist precisely in E-solid or locally inverse e-varieties; and in Section 1.3.6 we give an account of the extensions of the results of [26] for orthodox e-varieties to the lattices of all E-solid e-varieties and all locally inverse e-varieties, by Kadourek and Szendrei [27] and Auinger [7, 8] respectively. Finally, in Section 1.3.7 we discuss Kadourek's work in [25], in which he broadened the concept of a bifree object, and thus gave partial Birkhoff-style results for locally orthodox e-varieties.

1.3.1 Preliminaries.

As a general reference for semigroup theory, the reader is referred to the book [23] by Howie.

For a semigroup S without an identity, let S^1 denote S with an identity adjoined. If S has an identity let $S^1 = S$.

Green's relations on a semigroup S are given by

$$\begin{aligned}
\mathcal{L} &= \{(a,b) : S^1a = S^1b\},\\
\mathcal{R} &= \{(a,b) : aS^1 = bS^1\},\\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R},\\
\mathcal{D} &= \mathcal{L} \vee \mathcal{R},
\end{aligned}$$

and they are equivalence relations. For $\mathcal{A} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}\}$ and $x \in S$ we write $A_x = \{y \in S : x\ \mathcal{A}\ y\}$.

Let S be a semigroup. An element $y \in S$ is said to be an *inverse* of an element $x \in S$ if $xyx = x$ and $yxy = y$. We denote the set of all inverses of an element $x \in S$ by $V(x)$. A semigroup S is said to be *regular* if $V(x) \neq \emptyset$ for every $x \in S$.

For a subset A of S let $\langle A \rangle$ be the subsemigroup of S generated by A.

For a subsemigroup R of a regular semigroup S, let R_c denote the subsemigroup generated by the conjugates of R in S^1;

$$R_c = \langle xrx' : r \in R, x \in S^1, x' \in V(x) \rangle.$$

For $n \geq 2$ define $R_{c^n} = (R_{c^{n-1}})_c$. We say that R is *self-conjugate* if $R_c \subseteq R$.

Clearly $R \subseteq R_c$, and hence $R_{c^{n-1}} \subseteq R_{c^n}$ for all $n \geq 2$. So $\bigcup_{n=1}^{\infty} R_{c^n}$ is a self-conjugate subsemigroup of S. Moreover, it is the least self-conjugate subsemigroup of S that contains R.

The set of idempotents of S is denoted by $E(S)$, or simply by E. The *core* of S is $\langle E \rangle$ and is denoted by $C(S)$, or C if the context is clear. The semigroup $\bigcup_{n=1}^{\infty} C_{c^n}$ is called the *self-conjugate core* of S, and is denoted by $C_\infty(S)$, or simply C_∞.

The next results are well known.

Result 1.3.1 ([23]) *Let $\varphi : S \to T$ be a surjective homomorphism of regular semigroups. For every $e \in E(T)$ there exists $f \in E(S)$ such that $f\varphi = e$.* □

Result 1.3.2 ([23]) *Let $\varphi : S \to T$ be a surjective homomorphism of regular semigroups. If $a \in T$ and $b \in V(a)$ then there exist $c \in S$ and $d \in V(c)$ such that $a = c\varphi$ and $b = d\varphi$.* □

Result 1.3.3 ([23]) *Let ρ be a binary relation on a semigroup S. If $a, b \in S$ then $(a, b) \in \langle \rho \rangle$ if and only if either $a = b$ or for some $m \geq 0$ there exist*

$r_i, s_i \in S^1$ and $(d_i, e_i) \in \rho \cup \rho^{-1}$ for $0 \leq i \leq m$ such that $a = r_0 d_0 s_0$, $b = r_m e_m s_m$, and $r_{i-1} e_{i-1} s_{i-1} = r_i d_i s_i$ for $1 \leq i \leq m$. $\square$

Result 1.3.4 ([14]) *Suppose that a, b are elements of a semigroup S. Then $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a \cap E(S) \neq \emptyset$. If this is so then $H_{ab} = R_a \cap L_b$.* $\square$

Result 1.3.5 ([23]) *If H is an $\mathcal{H}$-class in a semigroup S then either $H^2 \cap H = \emptyset$ (so H contains no idempotents) or $H^2 = H$ and H is a subgroup of S. Therefore an element a of S is in a subgroup of S if and only if $a \ \mathcal{H} \ a^2$.* $\square$

Notation 1.3.6 If k lies in a subgroup H of a semigroup S then the inverse of k in H will always be denoted by k^{-1}, and the identity of H by k°. $\square$

We note the following well known result.

Result 1.3.7 *Let $\varphi : S \to T$ be a homomorphism. If a is in a subgroup of S then $a\varphi$ is in a subgroup of T, and $a^{-1}\varphi = (a\varphi)^{-1}$.* $\square$

We will conclude this section with some notes on the absolutely free unary semigroup and the absolutely free group. Let X be a nonempty set.

Notation 1.3.8 Let Y denote the set $X \cup \{(,)'\}$. By [18], the free unary semigroup $F_{\mathcal{U}}(X)$ on X, with unary operation $'$, can be seen as the least subsemigroup F of the free semigroup Y^+ such that $X \subseteq F$ and $(w)' \in F$ whenever $w \in F$. We write $w' = (w)'$ and denote the set $\{x' : x \in X\} \subseteq F_{\mathcal{U}}(X)$ by X'. Thus the set $\overline{X} = X \cup X'$ is a subset of $F_{\mathcal{U}}(X)$, and X' is a disjoint bijective copy of X. $\square$

The class of all groups, considered as unary semigroups $(G, \cdot, ')$ where $V(u) = \{u'\}$ for all $u \in G$, forms a variety $\mathcal{G}$ of (regular) unary semigroups. We denote the free object on X in this variety by $(F_{\mathcal{G}}(X), \cdot, ')$.

The fully invariant unary semigroup congruence $\rho(\mathcal{G})$ on $F_{\mathcal{U}}(X)$ corresponding to the variety $\mathcal{G}$ is the least unary semigroup congruence on $F_{\mathcal{U}}(X)$ containing the relation

$$\{(u, uu'u), (u', u'uu'), ((u')', u), (uu', vv') : u, v \in F_{\mathcal{U}}(X)\}.$$

For a word $u \in F_{\mathcal{U}}(X)$, the *group-reduced form* $\overline{u}$ of u is the word in the free monoid $\overline{X}^*$ obtained from u by applying the rules $(vw)' \to w'v'$, $(v')' \to v$, and $xx', x'x \to 1$ for $u, v \in F_{\mathcal{U}}(X)$ and $x \in X$. The set $\{\overline{u} : u \in F_{\mathcal{U}}(X)\}$ of all *group-reduced words on* X, together with the multiplication given by $\overline{u} \cdot \overline{v} = \overline{uv}$, is well-known as a model for the free group $F_{\mathcal{G}}(X)$ on X, and

$$\rho(\mathcal{G}) = \{(u, v) \in F_{\mathcal{U}}(X) \times F_{\mathcal{U}}(X) : \overline{u} = \overline{v}\}.$$

1.3.2 E-varieties and regular unary semigroup varieties.

A subsemigroup of a regular semigroup need not be regular, and thus classes of regular semigroups do not necessarily form varieties of semigroups (that is, of type (2) algebras). In particular the class of all regular semigroups, considered as type (2) algebras, is not a variety.

Some classes of regular semigroups have been studied as varieties; not as varieties of semigroups as such, but of unary semigroups (that is, as varieties of type (2,1) instead of type (2)). Examples are the varieties of groups, inverse semigroups and completely regular semigroups. An *inverse semigroup* is a semigroup in which every element has a unique inverse. A *completely regular semigroup* is a semigroup which is a union of groups. For inverse semigroups and groups the unary operation $x \mapsto x'$ takes an element x to its unique inverse. For completely regular semigroups x' is the group inverse of x.

In 1989-90 Hall [20], and independently Kaďourek and Szendrei [26], introduced the concept of an *existence variety* or *e-variety* of regular semigroups as a class of regular semigroups closed under taking homomorphic images, regular subsemigroups and direct products. Examples of e-varieties include the class of all groups, and the classes of all regular, inverse, and completely regular semigroups. In particular, the varieties mentioned above of groups, inverse semigroups, and completely regular semigroups coincide with the corresponding e-varieties, since in these cases the unary operations are uniquely specified.

In [20] Hall developed links between e-varieties and varieties of regular unary semigroups, which we will briefly outline. We write $(S,')$ to denote a semigroup S with a unary operation $'$. A *regular unary semigroup* is a unary semigroup $(S,')$ that satisfies the identities $xx'x = x$ and $x'xx' = x'$. A unary operation of this kind is called an *inverse unary operation.* The class of all regular unary semigroups thus forms a variety, which is denoted by $\mathcal{RUS}$. For any regular semigroup S, a unary operation $'$ can be selected (by the axiom of choice) such that s' is an inverse of s for each $s \in S$. Thus $'$ is an inverse unary operation on S, and $(S,') \in \mathcal{RUS}$.

Result 1.3.9 ([20]) *For a given e-variety* $\mathbf{V}$ *the class*

$$\mathbf{V}' = \{(S,') \in \mathcal{RUS} : S \in \mathbf{V}\}$$

is a variety of regular unary semigroups. □

Hall showed that not every subvariety of $\mathcal{RUS}$ is of the form $\mathbf{V}'$ for an e-variety $\mathbf{V}$.

Notice that for an e-variety $\mathbf{V}$ and $S \in \mathbf{V}$ there may be many copies of S in $\mathbf{V}'$, each with a different inverse unary operation. In fact, in the free regular unary semigroup $F_{\mathcal{RUS}}(X)$ the elements $x', (x')', \ldots$ are distinct for every $x \in F_{\mathcal{RUS}}(X)$ (see [34]). For every e-variety $\mathbf{V}$ of inverse semigroups we have $\mathbf{V}' = \mathbf{V}$, as inverses are unique. There are completely regular semigroups however in which inverses are not unique, and so for example the e-variety $\mathbf{CR}$ of all completely regular semigroups is not equal to $\mathbf{CR}'$.

Let $\mathbf{RS}$ denote the e-variety of all regular semigroups. Hall obtained the following analogue for e-varieties of Birkhoff's Theorem 1.1.5.

Theorem 1.3.10 **([20])** *For an e-variety* $\mathbf{V}$*, a set* Σ *of* $\mathcal{RUS}$*-identities is a basis for the identities of* $\mathbf{V}'$ *if and only if*

$$\begin{aligned}\mathbf{V} \;=\; \{S \in \mathbf{RS} :\ & \textit{there exists an inverse unary operation}\ ' \textit{ on } S \\ & \textit{such that } (S,') \models \Sigma\}\end{aligned}$$

and

$$\mathbf{V} = \{S \in \mathbf{RS} : (S,') \models \Sigma \textit{ for every inverse unary operation}\ ' \textit{ on } S\}.$$

□

Under the conditions of Theorem 1.3.10 the e-variety $\mathbf{V}$ is said to be *strongly determined* by Σ. Hall considered a regular semigroup S to satisfy a set Σ of $\mathcal{RUS}$-identities if for every inverse unary operation $'$ on S the regular unary semigroup $(S,')$ satisfies Σ. Hall called the class $[\Sigma]_e = \{S \in \mathbf{RS} : S \models \Sigma\}$ an *equational class.* When $[\Sigma]_e$ is an e-variety Σ is said to *weakly determine* the class $[\Sigma]_e$.

By Theorem 1.3.10 every e-variety $\mathbf{V}$ is an equational class, and is both weakly and strongly determined by each basis of the identities of $\mathbf{V}'$. However, not every equational class is an e-variety, as is demonstrated in [20].

1.3.3 Some relevant e-varieties.

In this section we introduce some important e-varieties and provide background and results.

An *E-solid* (regular) semigroup is a regular semigroup such that whenever $e, f, g \in E(S)$ satisfy $e \mathcal{L} f \mathcal{R} g$ there exists $h \in E(S)$ for which $e \mathcal{R} h \mathcal{L} g$.

By Hall (see the supplement to [42]) and Trotter [39] we have:

Result 1.3.11 *The following conditions are equivalent for a regular semigroup S:*

(i) S is E-solid,

(ii) ef lies in a subgroup of S for every $e, f \in E(S)$,

(iii) $C(S)$ is completely regular,

(iv) $C_\infty(S)$ is completely regular. □

Remark 1.3.12 By Result 1.3.11(ii), every E-solid semigroup S admits a partial binary operation $^{-1}$ given by $(e, f) \mapsto (ef)^{-1}$, the $\mathcal{H}$-related inverse of ef, for all $e, f \in E(S)$. □

If S is a regular semigroup then $C(S)$ is a regular subsemigroup of S by [17], and for every $e \in E(S)$ the *local submonoid* eSe is regular. For a class $\mathbf{V}$ of regular semigroups, Hall [20] defined the classes

$$L\mathbf{V} = \{S \in \mathbf{RS} : eSe \in \mathbf{V} \text{ for every } e \in E(S)\}$$

and

$$C\mathbf{V} = \{S \in \mathbf{RS} : C(S) \in \mathbf{V}\}.$$

Hall proved that if $\mathbf{V}$ is an e-variety then $L\mathbf{V}$ and $C\mathbf{V}$ are also e-varieties. The class $\mathbf{B} = [x = x^2]$ of all bands is a semigroup variety, and hence an e-variety. Therefore the class $\mathbf{O} = C\mathbf{B}$ of all *orthodox* semigroups is an e-variety. By Result 1.3.11 the class $\mathbf{ES}$ of all E-solid regular semigroups coincides with $C\mathbf{CR}$, and thus $\mathbf{ES}$ is also an e-variety. The classes $L\mathbf{I}$, $L\mathbf{O}$ and $L\mathbf{ES}$ of all *locally inverse, locally orthodox* and *locally E-solid* semigroups respectively are also e-varieties.

Notation 1.3.13 We mostly use bold capitals to denote e-varieties, and script capitals to denote the corresponding varieties of regular unary semigroups described in Result 1.3.9. So for example we write $L\mathcal{I} = (L\mathbf{I})'$ and $\mathcal{ES} = (\mathbf{ES})'$. $\square$

Definition 1.3.14 The *sandwich set* $S(a,b)$ of elements a, b of a regular semigroup S is the set $S(a,b) = bV(ab)a$. $\square$

The following are due to Nambooripad [30] or are easily deduced (see [39]).

Result 1.3.15 *Let S be a regular semigroup. Suppose $a, b \in S$, $a' \in V(a)$, $b' \in V(b)$ and $e, f \in E(S)$. Then*

(i) $S(e,f) \subseteq V(ef) \cap E(S)$,

(ii) $S(a,b) = S(a'a, bb')$,

(iii) $b'S(a,b)a' \subseteq V(ab)$,

(iv) $aS(a,b)b = \{ab\}$,

(v) $S(e,e) = \{e\}$,

(vi) if $\varphi : S \to T$ is a homomorphism of regular semigroups then $S(a,b)\varphi \subseteq S(a\varphi, b\varphi)$. $\square$

Nambooripad also proved the following characterization of locally inverse semigroups in terms of their sandwich sets.

Result 1.3.16 ([30], Theorem 7.6) *A regular semigroup S is locally inverse if and only if $S(a,b)$ is a singleton, denoted $s(a,b)$, for every $a, b \in S$.* $\square$

The following list gives some classes of regular semigroups which form e-varieties. Figure 1.1 arranges these as they appear in the lattice of all e-varieties.

T	trivial semigroups	**CR**	completely regular semigroups
G	groups	L**CR**	locally completely regular semigroups
I	inverse semigroups	L**I**	locally inverse semigroups
ES	E-solid semigroups	L**ES**	locally E-solid semigroups
O	orthodox semigroups	L**O**	locally orthodox semigroups
RS	regular semigroups		

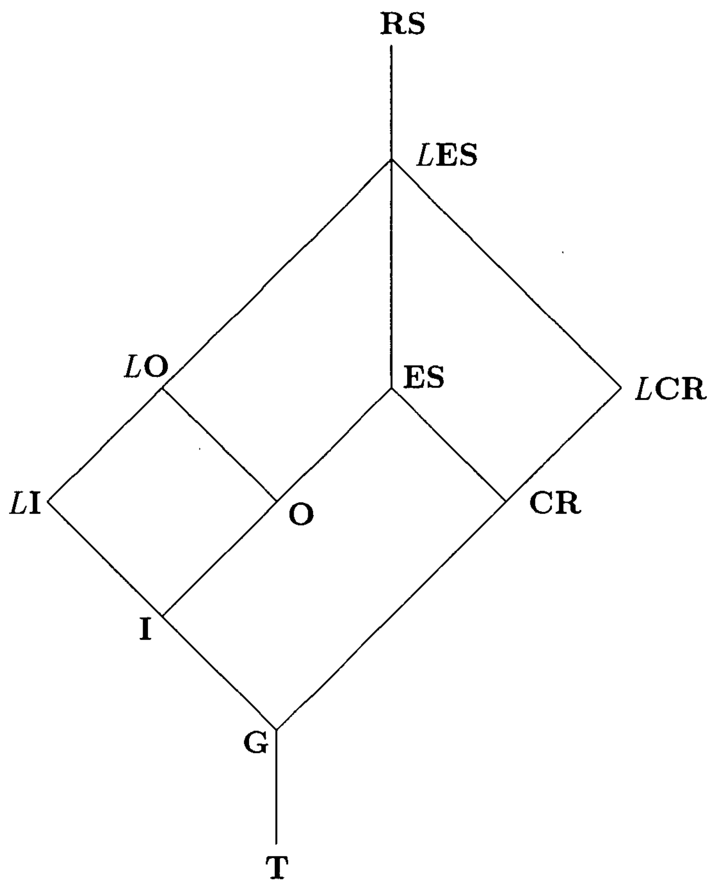

Figure 1.1: Some classes of regular semigroups which form e-varieties.

Remark 1.3.17 By Result 1.3.16 a second binary operation s, given by $(a, b) \mapsto s(a, b)$, can be defined on a locally inverse semigroup S, and S becomes a type (2,2) algebra.

By the axiom of choice, for any regular semigroup S a binary operation s may be defined such that $s(a, b) \in S(a, b)$ for all $a, b \in S$. Such an operation will be called a *sandwich operation.* $\square$

Result 1.3.18 ([43]) *Let $\varphi : S \to T$ be a homomorphism of locally inverse semigroups. Then $s(a, b)\varphi = s(a\varphi, b\varphi)$.* $\square$

We will need the next well-known result.

Result 1.3.19 ([23]) *A regular semigroup S is inverse if and only if its idempotents commute.* $\square$

1.3.4 Orthodox e-varieties.

Whereas in [20] Hall considered arbitrary e-varieties, independently Kaďourek and Szendrei considered the same concept just for classes of orthodox semigroups, and were able to prove analogues of Birkhoff's fundamental theorems for varieties. They used the term *bivariety* for an e-variety of orthodox semigroups, and developed notions of biidentities and bifree objects, as follows.

Notation 1.3.20 Let X be a nonempty set, with a disjoint bijective copy $X' = \{x' : x \in X\}$. We write $\overline{X} = X \cup X'$. $\square$

A *biidentity* over X is a pair (u, v), also written $u = v$, where u, v are members of the free semigroup $(\overline{X})^+$. The following definition was given in [26] for orthodox semigroups.

Definition 1.3.21 ([26]) For a regular semigroup S, a mapping $\phi : \overline{X} \to S$ is *matched* whenever $x'\phi$ is an inverse of $x\phi$ for each $x \in X$. $\square$

An orthodox semigroup S is said to *satisfy* the biidentity $u = v$ for $u, v \in (\overline{X})^+$ if whenever $\phi : \overline{X} \to S$ is a matched mapping, and $\theta : (\overline{X})^+ \to S$ is the unique homomorphism extending ϕ, we have $u\theta = v\theta$.

For a nonempty set X and class $\mathbf{V}$ of orthodox semigroups, let $\mathrm{bId}_X(\mathbf{V})$ (also written $\rho(\mathbf{V}, X)$, or simply $\rho(\mathbf{V})$) be the set of all biidentities over X that are satisfied by $\mathbf{V}$. Kačourek and Szendrei gave a generating relation for $\mathrm{bId}_X(\mathbf{O})$.

The original definition for orthodox semigroups of a biinvariant congruence found in [26] is equivalent to the one given below, which is from [27].

Definition 1.3.22 ([26]) For $u, p, q \in (\overline{X})^+$ and $x \in X$ let $u(x \to p, x' \to q)$ denote the word in $(\overline{X})^+$ obtained from u by substituting p for all occurrences of x, and q for all occurrences of x'. A congruence ρ on $(\overline{X})^+$ is said to be closed under *regular substitution* if

$$u \;\rho\; v,\;\; p \;\rho\; pqp,\;\; q \;\rho\; qpq \Rightarrow u(x \to p, x' \to q) \;\rho\; v(x \to p, x' \to q)$$

whenever $u, v, p, q \in (\overline{X})^+$ and $x \in X$.

A congruence ρ on $(\overline{X})^+$ is said to be *biinvariant* if $\rho(\mathbf{O}, X) \subseteq \rho$ and ρ is closed under regular substitution. □

The following definition was originally given in [26] for orthodox semigroups.

Definition 1.3.23 ([26]) For a class $\mathbf{V}$ of regular semigroups, a semigroup $F \in \mathbf{V}$ together with a matched mapping $\iota : \overline{X} \to F$ is a *bifree object* on X in $\mathbf{V}$ if, for any $S \in \mathbf{V}$ and matched mapping $\phi : \overline{X} \to S$, there is a unique homomorphism $\theta : F \to S$ satisfying $\iota\theta = \phi$. □

Remark 1.3.24 ([43]) If S is a member of an e-variety $\mathbf{V}$ and $A \subseteq S$ is such that $A \cap V(a) \neq \emptyset$ for every $a \in A$, then there is a matched map $\phi : \overline{X} \to A$ for some set X. Suppose that a bifree object (F, ι) on X exists for $\mathbf{V}$, and let

$\theta : F \to S$ be such that $\iota\theta = \phi$. Then $F\theta$ is a regular subsemigroup of S that contains A and is contained in every regular subsemigroup of S that contains A; that is, $F\theta$ is the least regular subsemigroup of S that contains A.

Thus if an e-variety $\mathbf{V}$ is to admit a bifree object, then for every $S \in \mathbf{V}$ and $A \subseteq S$ such that $A \cap V(a) \neq \emptyset$ for all $a \in A$ there must exist a least regular subsemigroup of S that contains A. □

Using the next result, Kad̆ourek and Szendrei showed that the conclusion of Remark 1.3.24 is true for orthodox semigroups.

Result 1.3.25 ([35]) *If S is a regular semigroup then S is orthodox if and only if $V(a)V(b) \subseteq V(ba)$ for all $a, b \in S$.* □

Corollary 1.3.26 ([26]) *If S is an orthodox semigroup and $A \subseteq S$ is such that $A \cap V(a) \neq \emptyset$ for every $a \in A$ then $\langle A \rangle$ is a regular subsemigroup of S.* □

From Corollary 1.3.26, Kad̆ourek and Szendrei proved the following analogue of Birkhoff's Theorem 1.1.7.

Theorem 1.3.27 ([26]) *Let $\mathbf{V}$ be a class of orthodox semigroups closed under taking regular subsemigroups and direct products. For every nonempty set X there exists a bifree object F on X in $\mathbf{V}$, and F is isomorphic to $(\overline{X})^+/\rho(\mathbf{V}, X)$.*
□

Notation 1.3.28 For a class $\mathcal{C}$ of semigroups, let $S_r(C)$ be the class of all regular subsemigroups of members of $\mathcal{C}$. □

Kad̆ourek and Szendrei proved the following generalization of Theorem 1.1.4.

Theorem 1.3.29 ([26]) *For a class $\mathbf{V}$ of orthodox semigroups the e-variety generated by $\mathbf{V}$ is the class $HS_rP(\mathbf{V})$.* □

The next result contains analogues of Birkhoff's Theorems 1.1.5 and 1.1.6. Given a set Σ of biidentities, we write $[\Sigma]_b$ for the class of all orthodox semigroups that satisfy Σ.

Theorem 1.3.30 ([26]) *A class* $\mathbf{V}$ *of orthodox semigroups is an e-variety if and only if there exists a set* Σ *of biidentities such that* $\mathbf{V} = [\Sigma]_b$. *If* X *is denumerable then the mappings*

$$\mathbf{V} \mapsto \rho(\mathbf{V}, X) \text{ and } \rho \mapsto [\rho]_b$$

define an antiisomorphism between the lattice of all e-varieties of orthodox semigroups and the lattice of all biinvariant congruences on $(\overline{X})^+$. □

Remark 1.3.31 ([29]) We may also speak of a regular semigroup (not necessarily orthodox) satisfying a set of biidentities. The proof of Theorem 1.3.30 also shows that the class of all regular semigroups which satisfy a given set of biidentities is an e-variety. However, the converse does not hold. For example, in [8] Auinger showed that every biidentity satisfied by the e-variety of all completely simple semigroups (that is, completely regular semigroups without zero and with only one $\mathcal{D}$-class) is also satisfied by every regular semigroup. □

Kadourek and Szendrei noted the following result.

Result 1.3.32 ([26]) *A congruence* ρ *on* $(\overline{X})^+$ *is biinvariant if and only if* $\rho(\mathbf{O}, X) \subseteq \rho$ *and the congruence* $\rho/\rho(\mathbf{O}, X)$ *is a fully invariant semigroup congruence on* $(\overline{X})^+/\rho(\mathbf{O}, X)$. □

1.3.5 Bifree objects in e-varieties.

Yeh, using the term *e-free*, considered the concept of a bifree object in an arbitrary e-variety in [43]. Recall from Remark 1.3.24 that if an e-variety $\mathbf{V}$ is to admit a bifree object, then for every $S \in \mathbf{V}$ and $A \subseteq S$ such that $A \cap V(a) \neq \emptyset$ for all $a \in A$ there must exist a least regular subsemigroup of S that contains A. With the following result, Yeh extended Corollary 1.3.26.

Result 1.3.33 ([43]) *Suppose that $S \in \mathbf{ES} \cup \mathcal{LI}$. Let $A \subseteq S$ be such that $A \cap V(a) \neq \emptyset$ for every $a \in A$. Then there is a least regular subsemigroup of S that contains A.* □

The next result gives details of the constructions of these least regular subsemigroups.

Result 1.3.34 ([43]) *Under the conditions of Result 1.3.33, let T be the least regular subsemigroup of S containing A. Then $T = \bigcup_{i \geq 0} T_{2i+1}$, where*

$$\begin{aligned} T_0 &= A \\ T_1 &= \langle T_0 \rangle \\ &\vdots \\ T_{2i} &= T_{2i-1} \cup T'_{2i-1} \\ T_{2i+1} &= \langle T_{2i} \rangle \\ &\vdots \end{aligned}$$

and T'_{2i-1} is defined for every $i \geq 1$ as follows.

If S is E-solid then $T'_{2i-1} = \{(ef)^{-1} : e, f \in E(T_{2i-1})\}$ for every $i \geq 1$; the subsemigroup T is the closure of A under the operations $\cdot$ and $^{-1}$, where $^{-1}$ is the partial binary operation on S described in Remark 1.3.12.

If S is locally inverse then $T'_{2i-1} = \{s(a, b) : a, b \in T_{2i-1}\}$ for every $i \geq 1$; here T is the closure of A under the operations $\cdot$ and s, where s is the binary operation on S described in Remark 1.3.17. □

The bifree object on X in $\mathbf{V}$ is unique up to isomorphism if it exists; we denote it by $bF_{\mathbf{V}}(X)$. In [43], Yeh gave an example of a semigroup that is both locally completely regular and locally orthodox, for which Result 1.3.33 fails. He also showed that if S is a regular semigroup with a subset A, where $A \cap V(a) \neq \emptyset$ for all $a \in A$, such that there is no least regular subsemigroup of S containing A, then no e-variety that contains S admits a bifree object on any set $|X|$ with $X \geq |A|$. In Yeh's example the set A has two members, and as a consequence he proved the following remarkable result.

Theorem 1.3.35 (**[43]**) *For an e-variety* $\mathbf{V}$ *and set* X *with* $|X| \geq 2$ *the bifree object* $bF_{\mathbf{V}}(X)$ *exists if and only if* $\mathbf{V} \subseteq \mathbf{ES}$ *or* $\mathbf{V} \subseteq L\mathbf{I}$. □

Monogenic bifree objects do exist in e-varieties not contained in $\mathbf{ES}$ or $L\mathbf{I}$. In [10] it is proved that monogenic bifree objects exist in every sub e-variety of $L\mathbf{CR}$.

Remark 1.3.36 If a class $\mathbf{V}$ of regular semigroups is closed under S_r and P then, by the proof of Result 1.3.9, the corresponding class $\mathcal{V} = \mathbf{V}'$ of regular unary semigroups is closed under S and P. Therefore by Theorem 1.1.8 the free object $(F_{\mathcal{V}}(X), ')$ exists for every nonempty set X and, considered as a type (2) algebra, is a member of $\mathbf{V}$.

Recall the assumption $X \subseteq F_{\mathcal{V}}(X)$. We write $X' = \{x' : x \in X\} \subseteq F_{\mathcal{V}}(X)$ and $\overline{X} = X \cup X'$. □

Result 1.3.37 (**[43]**) *Suppose that* $\mathbf{V}$ *is a class of regular semigroups closed under* P *and* S_r*, and such that* $\mathbf{V} \subseteq \mathbf{ES}$ *or* $\mathbf{V} \subseteq L\mathbf{I}$*. Then the least regular subsemigroup of* $F_{\mathbf{V}'}(X)$ *containing the set* $\overline{X}$ *is bifree on* X *in* $\mathbf{V}$. □

Yeh proved the following result, which extends Theorem 1.3.29.

Theorem 1.3.38 (**[43]**) *Let* $\mathcal{C}$ *be a class of* E*-solid or of locally inverse semigroups. Then* $HS_rP(\mathcal{C})$ *is the smallest e-variety containing* $\mathcal{C}$. □

1.3.6 E-solid and locally inverse e-varieties.

In this section we outline the work of Auinger in [7] and [8] on locally inverse semigroups and Kadourek and Szendrei in [27] on E-solid semigroups. In both cases concepts of biidentity and biinvariant congruence were defined, and the Birkhoff-style results previously obtained by Kadourek and Szendrei in [26] for orthodox semigroups were generalized. So a Birkhoff-style theory holds for the whole lattice of e-varieties that admit nonmonogenic bifree objects, but via two very different approaches.

By Results 1.3.34 and 1.3.37, the bifree locally inverse semigroup on X is the closure in the free regular unary locally inverse semigroup $F_{\mathcal{LI}}(X)$ of the set $\overline{X}$ under the operations $\cdot$ and s; and Auinger defined biidentities for locally inverse semigroups to be formal equalities between members of $F_{(2,2)}(\overline{X})$, the free type (2,2) algebra on the set $\overline{X}$ that is a semigroup with respect to one of its operations. A locally inverse semigroup S satisfies the biidentity $u = v$ if $u\theta = v\theta$ whenever $\phi : \overline{X} \to S$ is a matched mapping and $\theta : F_{(2,2)}(\overline{X}) \to S$ is the unique extension of ϕ.

Auinger generalized all the Birkhoff-style results previously obtained for orthodox semigroups. In particular, he described a generating relation for the congruence $\rho(L\mathbf{I}, X)$ on $F_{(2,2)}(\overline{X})$ which is equal to the set of all biidentities satisfied by $L\mathbf{I}$, and is such that $F_{(2,2)}(\overline{X})/\rho(L\mathbf{I}, X)$ is the bifree locally inverse semigroup. His definition of a biinvariant congruence is a direct generalization of Definition 1.3.22, with $(\overline{X})^+$ and $\rho(\mathbf{O}, X)$ replaced with $F_{(2,2)}(\overline{X})$ and $\rho(L\mathbf{I}, X)$ respectively.

We now turn to the theory for E-solid semigroups developed by Kadourek and Szendrei in [27]. By Results 1.3.34 and 1.3.37 the bifree E-solid semigroup is the closure in $F_{\mathcal{ES}}(X)$ of $\overline{X}$ under the operations $\cdot$ and ${}^{-1}$. Let $F_{\mathcal{U}}(X)$ be the free unary semigroup on X, with unary operation $'$. Let $F'^{\infty}(X)$ be the least subsemigroup U of $F_{\mathcal{U}}(X)$ such that $\overline{X} \subseteq U$ and the set $\{u \in U : \overline{u} = 1\}$ is a unary subsemigroup of U, where $\overline{u}$ is the group-reduced form of u. By Result 1.3.11, a regular semigroup S is E-solid if and only if the self-conjugate core of S is completely regular, and in this setting Kaďourek and Szendrei considered a biidentity to be a pair $u = v$, where $u, v \in F'^{\infty}(X)$. They showed that any matched mapping $\phi : \overline{X} \to S$, where S is an E-solid semigroup, has a (unique) extension $\theta : F'^{\infty}(X) \to S$, such that whenever $\overline{u} = 1$ the element $u\theta$ lies in the self-conjugate core of S and $u'\theta$ is the $\mathcal{H}$-related inverse of $u\theta$. An E-solid semigroup S *satisfies* a biidentity $u = v$ if whenever $\phi : \overline{X} \to S$ is matched, and θ is the extension described above of ϕ to $F'^{\infty}(X)$, we have $u\theta = v\theta$.

Kaďourek and Szendrei also generalized all the Birkhoff-style results previously obtained for orthodox semigroups. In particular, they gave a generating relation for the set $\rho(\mathbf{ES}, X)$ of all biidentities satisfied by $\mathbf{ES}$, which is such that $F'^{\infty}(X)/\rho(\mathbf{ES}, X)$ is the bifree E-solid semigroup. Their definition of a biinvariant congruence is also a direct generalization of Definition 1.3.22, with $(\overline{X})^{+}$ and $\rho(\mathbf{O}, X)$ replaced with $F'^{\infty}(X)$ and $\rho(\mathbf{ES}, X)$ respectively.

1.3.7 Locally orthodox e-varieties.

In [25], Kaďourek investigated e-varieties of locally orthodox semigroups. By Result 1.3.35, bifree objects do not generally exist for these e-varieties, so Kaďourek developed a theory of trifree objects and triidentities. With bifree objects the nonexistence of the usual free object on X is compensated for by extending the alphabet to $\overline{X}$. In the case of trifree objects, the alphabet is further extended by the addition of certain sandwich elements.

Firstly, Kaďourek gave an extension of Result 1.3.25, which was vital to Kaďourek and Szendrei's original work on classes of orthodox semigroups.

Result 1.3.39 **([25])** *A regular semigroup S is locally orthodox if and only if*

$$V(b).S(p,q).V(a) \subseteq V(ab)$$

whenever $p, q \in S$ and $a \in Sp$, $b \in qS$. $\square$

Corollary 1.3.40 **([25])** *If A, C are subsets of a locally orthodox semigroup S such that*

(i) $A \cap V(a) \neq \emptyset$ for every $a \in A$,

(ii) $C \subseteq \bigcup\{S(a,b) : a, b \in A\}$, and

(iii) $C \cap S(a,b) \neq \emptyset$ for every $a, b \in A$,

then the subsemigroup of S generated by $A \cup C$ is regular. $\square$

Notation 1.3.41 Recall the definition of the set $\overline{X}$ from Notation 1.3.20. Let $X_1 = \{s(x,y) : x, y \in \overline{X}\}$ be a set of (distinct) labels disjoint from $\overline{X}$, and let $\overline{X}_1 = \overline{X} \cup X_1$. $\square$

Kaďourek defined a *triidentity* to be a pair (u, v), also written $u = v$, where u, v are members of the free semigroup $(\overline{X}_1)^+$.

Definition 1.3.42 **([25])**

- A *tied* mapping is a mapping $\phi : \overline{X}_1 \to S$ where S is a regular semigroup, $x'\phi \in V(x\phi)$ for every $x \in X$, and $s(x,y)\phi \in S(x\phi, y\phi)$ for every $x, y \in \overline{X}$.

- A *trifree object* for a class $\mathbf{V}$ of regular semigroups is a pair (S, ι), where $\iota : \overline{X}_1 \to S$ is a tied mapping, such that for any $T \in \mathbf{V}$ and tied mapping $\phi : \overline{X}_1 \to T$ there is a unique homomorphism $\theta : S \to T$ such that $\iota\theta = \phi$. $\square$

Kadourek proved an analogue of Theorem 1.3.27: the trifree object on X exists in any e-variety $\mathbf{V}$ of locally orthodox semigroups, and is isomorphic to $(\overline{X}_1)^+/\rho(\mathbf{V},X)$, where $\rho(\mathbf{V},X)$ is the congruence on $(\overline{X}_1)^+$ consisting of all triidentities satisfied in $\mathbf{V}$.

However, other Birkhoff-style theorems do not hold for locally orthodox semigroups. For example, Kadourek proved that if $\mathbf{V}$ is an e-variety of locally orthodox semigroups then $\mathbf{V}$ is exactly the class of locally orthodox semigroups which satisfy all triidentities satisfied by $\mathbf{V}$; but the converse is not true: for a set Σ of triidentities, the class $[\Sigma]_t$ of all locally orthodox semigroups which satisfy each member of Σ need not be an e-variety. Moreover, Theorem 1.3.38 fails in this setting: for a class $\mathbf{V}$ of locally orthodox semigroups the class $HS_rP(\mathbf{V})$ need not be an e-variety. Kadourek did prove, however, that if X is an infinite set and $\mathbf{V}$ is a class of locally orthodox semigroups such that $[\rho(\mathbf{V},X)]_t$ is an e-variety then $[\rho(\mathbf{V},X)]_t = HS_rP(\mathbf{V})$ is the smallest e-variety containing $\mathbf{V}$.

Chapter 2

Almeida's Generalized Variety Problem

2.1 Introduction.

This chapter consists of an answer to a problem posed in Almeida's book [3], which concerns generalized varieties of nil and nilpotent semigroups.

All varieties mentioned in this chapter will be assumed to be semigroup varieties. Consider the following varieties:

- $\mathcal{S}$, the variety of all semigroups;
- $\mathcal{C}om = [xy = yx]$, the variety of all commutative semigroups;
- $\mathcal{N}il_n = [x^n y = yx^n = x^n]$, the variety of all nil semigroups of index $n \geq 1$; and
- $\mathcal{N}_n = [x_1 \ldots x_n y = yx_1 \ldots x_n = x_1 \ldots x_n]$, the variety of all nilpotent semigroups of index $n \geq 1$.

Notice that $\mathcal{N}il_n$ and $\mathcal{N}_n$ consist of semigroups with zero. For $u \in X^+$ we write $u = 0$ as an abbreviation of $ux = xu = u$ where x is a variable not occurring in u, so that for example $\mathcal{N}il_n = [x^n = 0]$.

Let $\mathcal{N}il$ denote the class of all nil semigroups, and $\mathcal{N}$ denote the class of all nilpotent semigroups. Then

$$\mathcal{N}il = \bigcup_{n \geq 1} \mathcal{N}il_n \text{ and } \mathcal{N} = \bigcup_{n \geq 1} \mathcal{N}_n.$$

Observe that $\mathcal{N}il$ and $\mathcal{N}$ are generalized varieties, although they are not varieties. For any class $\mathcal{W}$ of semigroups let $\mathcal{L}(\mathcal{W})$ denote the lattice of all varieties contained in $\mathcal{W}$, and $\mathcal{G}(\mathcal{W})$ denote the lattice of all generalized varieties contained in $\mathcal{W}$.

Almeida proved the following result.

Theorem 2.1.1 ([1]) *$\mathcal{L}(\mathcal{N}il \cap \mathcal{C}om) \cup \{\mathcal{N}il \cap \mathcal{C}om\}$ is isomorphic to $\mathcal{G}(\mathcal{N} \cap \mathcal{C}om)$ via the correspondence*

$$\mathcal{W} \mapsto \mathcal{W} \cap \mathcal{N}, \quad \mathcal{W} \in \mathcal{L}(\mathcal{N}il \cap \mathcal{C}om) \cup \{\mathcal{N}il \cap \mathcal{C}om\}. \qquad \square$$

The motivation for this result was the question of the structure of the lattice $\mathcal{L}(\mathcal{C}om)$. In [1] Almeida established an embedding

$$\mathcal{L}(\mathcal{C}om) \setminus \{\mathcal{C}om\} \to \mathcal{L}(\mathcal{G} \cap \mathcal{C}om) \times N \times \mathcal{L}(\mathcal{N}il \cap \mathcal{C}om)$$

of semilattices for the meet operation, where N is the semilattice of natural numbers under $\leq$ and $\mathcal{G} = \bigcup_{n \geq 1}[x^n y = y = yx^n]$ is the generalized variety of all groups. The lattice $\mathcal{L}(\mathcal{G} \cap \mathcal{C}om)$ is isomorphic to the lattice of natural numbers ordered by division, and so the structure of $\mathcal{L}(\mathcal{N}il \cap \mathcal{C}om)$ became of interest. Information about $\mathcal{G}(\mathcal{N} \cap \mathcal{C}om)$ is known from $\mathcal{L}(\mathcal{N} \cap \mathcal{C}om)$, and can be transferred to $\mathcal{L}(\mathcal{N}il \cap \mathcal{C}om)$ by Theorem 2.1.1.

As a consequence of Result 1.2.9, Almeida (also in [1]) obtained the following information about the lattice of pseudovarieties of commutative semigroups from these results. Let $\mathbf{G} = \mathcal{G}^F$, $\mathbf{N} = \mathcal{N}^F$, and $\mathbf{Com} = \mathcal{C}om^F$ denote respectively the pseudovarieties of all finite groups, finite nilpotent semigroups and finite commutative semigroups. There is an embedding

$$\mathcal{P}s\mathbf{Com} \to \mathcal{P}s(\mathbf{G} \cap \mathbf{Com}) \times (N \cup \{\infty\}) \times \mathcal{P}s(\mathbf{N} \cap \mathbf{Com})$$

of meet semilattices (note that $\mathcal{N}il^F = \mathbf{N}$, by [15, Proposition 9.2]). The lattices $\mathcal{G}(\mathcal{G} \cap \mathcal{C}om)$ and $\mathcal{P}s(\mathbf{G} \cap \mathbf{Com})$ are isomorphic, as are the lattices $\mathcal{G}(\mathbf{N} \cap \mathbf{Com})$ and $\mathcal{P}s(\mathbf{N} \cap \mathbf{Com})$ (both via the mapping $\mathcal{V} \mapsto \mathcal{V}^F$), and information about $\mathcal{G}(\mathcal{G} \cap \mathcal{C}om)$ is known from $\mathcal{L}(\mathcal{G} \cap \mathcal{C}om)$.

In [3, Problem 10], Almeida asked whether the mapping

$$\alpha : \mathcal{W} \mapsto \mathcal{W} \cap \mathcal{N}, \quad \mathcal{W} \in \mathcal{L}(\mathcal{N}il) \cup \{\mathcal{N}il\},$$

defines an isomorphism between $\mathcal{L}(\mathcal{N}il) \cup \{\mathcal{N}il\}$ and $\mathcal{G}(\mathcal{N})$ - that is, whether Theorem 2.1.1 can be extended to non-commutative semigroups.

Notice that $\mathcal{N}il \notin \mathcal{L}(\mathcal{N}il)$ as $\mathcal{N}il$ is not a variety; but $\mathcal{N} \in \mathcal{G}(\mathcal{N})$. Since $\mathcal{N} \subseteq \mathcal{N}il$ then $\mathcal{N}il\alpha = \mathcal{N}$.

We begin by showing in Section 2.2 that α does not map onto $\mathcal{G}(\mathcal{N})$. We then consider the question of whether α is injective. First in Section 2.3 we give some more general results, involving the question of when two arbitrary semigroup varieties (not necessarily nil) have the same set of nilpotent semigroups. The remainder of the chapter is devoted to the proof that α is not injective.

2.2 The mapping α is not surjective.

From this point onwards X will denote a fixed denumerable set.

Lemma 2.2.1 *Let $\mathcal{W} = \bigcup_{\gamma\in\Gamma} \mathcal{W}_\gamma$, where $\{\mathcal{W}_\gamma : \gamma \in \Gamma\}$ is a directed family of varieties. Let $\mathcal{U} \in \mathcal{L}(\mathcal{W})$. Then $\mathcal{U} \subseteq \mathcal{W}_\gamma$ for some $\gamma \in \Gamma$.*

Proof: Let $\mathcal{U} \in \mathcal{L}(\mathcal{W})$. Then $F_{\mathcal{U}}(X)$, the free object on X in the variety $\mathcal{U}$, is a member of $\mathcal{W}_\gamma$ for some $\gamma \in \Gamma$, and this implies that $\mathcal{U} \subseteq \mathcal{W}_\gamma$, by Theorem 1.1.8. □

Corollary 2.2.2 *Every variety $\mathcal{U} \in \mathcal{L}(\mathcal{N}il)$ is a member of $\mathcal{L}(\mathcal{N}il_m)$ for some m.* □

We will need the next two results. Recall that for a class $\mathcal{W}$ of semigroups the variety generated by $\mathcal{W}$ is denoted by $V(\mathcal{W})$.

Result 2.2.3 ([4]) *If $\mathcal{W}$ is a generalized variety of nilpotent semigroups then $\mathcal{W} = V(\mathcal{W}) \cap \mathcal{N}$.* □

Result 2.2.4 ([4]) *If $\mathcal{W}$ is a generalized variety of nilpotent semigroups such that $\mathcal{N} \cap \mathcal{C}om \not\subseteq \mathcal{W}$ then $V(\mathcal{W}) \subseteq \mathcal{N}il$.* □

We can now show that α does not map onto $\mathcal{G}(\mathcal{N})$. Recall that for a variety $\mathcal{V}$ we often denote the fully invariant congruence on X^+ corresponding to $\mathcal{V}$ by $\rho(\mathcal{V})$.

Theorem 2.2.5 *The image of the mapping α is the set*

$$\{\mathcal{W} \in \mathcal{G}(\mathcal{N}) : \mathcal{N} \cap \mathcal{C}om \not\subseteq \mathcal{W}\}.$$

Proof: Let $\mathcal{U} \in \mathcal{L}(\mathcal{N}il)$. Then $\mathcal{U} \in \mathcal{L}(\mathcal{N}il_m)$ for some m by Corollary 2.2.2. If $\mathcal{N} \cap \mathcal{C}om \subseteq \mathcal{U} \cap \mathcal{N}$ then $\mathcal{N} \cap \mathcal{C}om \subseteq \mathcal{N}il_m$. But $\mathcal{N} \cap \mathcal{C}om = \bigcup_{n\geq 1}(\mathcal{N}_n \cap \mathcal{C}om)$, so that we now have $\mathcal{N}_{m+1} \cap \mathcal{C}om \subseteq \mathcal{N}il_m$. This means that for every $x \in X^+$

$$\begin{aligned}
\{ry^m s : r, s \in X^*, y \in X^+\} &= x^m \rho(\mathcal{N}il_m) \\
&\subseteq x^m \rho(\mathcal{N}_{m+1} \cap \mathcal{C}om) \\
&= \{x^m\},
\end{aligned}$$

which is false; and so $\mathcal{N} \cap \mathcal{C}om \not\subseteq \mathcal{U} \cap \mathcal{N}$.

Now suppose that $\mathcal{W} \in \mathcal{G}(\mathcal{N})$ is such that $\mathcal{N} \cap \mathcal{C}om \not\subseteq \mathcal{W}$. By Results 2.2.4 and 2.2.3, we have $V(\mathcal{W}) \in \mathcal{L}(\mathcal{N}il)$ and $\mathcal{W} = V(\mathcal{W}) \cap \mathcal{N} = V(\mathcal{W})\alpha$. □

2.3 Semigroup varieties whose nilpotent parts coincide.

The question of the injectivity of α remains of interest however. First we take a look at the more general question of when two semigroup varieties have the same set of nilpotent semigroups.

For $u = x_1 \dots x_n$ where $x_i \in X$, the *length* $|u|$ of u is defined to be n, and the *content* $c(u)$ of u is the set $\{x_1, \dots, x_n\}$.

Suppose that $\mathcal{U}$ is a variety of semigroups. Let $B(\mathcal{U})$ denote the set of all words $u \in X^+$ for which there exists r such that $|y| < r$ for all $y \in u\rho(\mathcal{U})$. We say that $B(\mathcal{U})$ is the set of all *bounded words for* $\mathcal{U}$, and that the set $U(\mathcal{U}) = X^+ \setminus B(\mathcal{U})$ is the set of all *unbounded words for* $\mathcal{U}$. Let

$$\mathrm{BId}(\mathcal{U}) = \{u = v \in \mathrm{Id}(\mathcal{U}) : u \in B(\mathcal{U})\},$$

the set of all *bounded identities for* $\mathcal{U}$. For $u \in B(\mathcal{U})$ let

$$B_{\mathcal{U}}(u) = \max\{|v| : v \in u\rho(\mathcal{U})\}.$$

Remark 2.3.1

- $\mathrm{BId}(\mathcal{U}) = \{u = v \in \mathrm{Id}(\mathcal{U}) : u, v \in B(\mathcal{U})\}$.
- If $\mathcal{U}, \mathcal{V}$ are two varieties for which $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V})$ then $B(\mathcal{U}) = B(\mathcal{V})$, and for every $u \in B(\mathcal{U})$ we have $u\rho(\mathcal{U}) = u\rho(\mathcal{V})$ and $B_{\mathcal{U}}(u) = B_{\mathcal{V}}(u)$.
- The set $U(\mathcal{U})$ is a subsemigroup of X^+ that is closed under endomorphisms of X^+.
- If $u \in B(\mathcal{U})$ then $c(v) = c(u)$ for all $v \in u\rho(\mathcal{U})$.
- If $u \in B(\mathcal{U})$ then $B_{\mathcal{U}}(u)$ exists. □

As the following theorem shows, the nilpotent part of a variety of semigroups is completely determined by its bounded identities.

Theorem 2.3.2 *Let $\mathcal{U}, \mathcal{V}$ be varieties. Then*

$$\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N} \Leftrightarrow \mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V}).$$

Proof: Suppose that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$. Then $\rho(\mathcal{U} \cap \mathcal{N}_m) = \rho(\mathcal{V} \cap \mathcal{N}_m)$ for every m by Theorem 1.1.6. Let $u = v \in \mathrm{BId}(\mathcal{U})$. Let $r = B_{\mathcal{U}}(u) + 1$. Then $u\rho(\mathcal{U}) = u\rho(\mathcal{U} \cap \mathcal{N}_r) = u\rho(\mathcal{V} \cap \mathcal{N}_r)$. Therefore $|y| < r$ for all $y \in u\rho(\mathcal{V})$, and $u\rho(\mathcal{U}) = u\rho(\mathcal{V} \cap \mathcal{N}_r) = u\rho(\mathcal{V})$. Hence $u = v \in \mathrm{BId}(\mathcal{V})$. So $\mathrm{BId}(\mathcal{U}) \subseteq \mathrm{BId}(\mathcal{V})$, and a dual argument shows that $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V})$.

Now suppose that $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V})$. Then $U(\mathcal{U}) = U(\mathcal{V})$. Let $m \geq 1$ and $u \in X^+$. If $u \in U(\mathcal{U})$, or if $u \in B(\mathcal{U})$ and $B_{\mathcal{U}}(u) \geq m$, then

$$(u, u^m) \in \rho(\mathcal{U} \cap \mathcal{N}_m) \cap \rho(\mathcal{V} \cap \mathcal{N}_m);$$

so $u\rho(\mathcal{U} \cap \mathcal{N}_m)$ and $u\rho(\mathcal{V} \cap \mathcal{N}_m)$ are the zeros of $X^+/\rho(\mathcal{U} \cap \mathcal{N}_m)$ and $X^+/\rho(\mathcal{V} \cap \mathcal{N}_m)$ respectively. Otherwise, we have $u \in B(\mathcal{U})$ and $B_{\mathcal{U}}(u) < m$, and in this case $u\rho(\mathcal{U} \cap \mathcal{N}_m) = u\rho(\mathcal{U}) = u\rho(\mathcal{V}) = u\rho(\mathcal{V} \cap \mathcal{N}_m)$ (by Remark 2.3.1).

It now follows that for all $m \geq 1$ the semigroups $X^+/\rho(\mathcal{U} \cap \mathcal{N}_m)$ and $X^+/\rho(\mathcal{V} \cap \mathcal{N}_m)$ are isomorphic, and $\rho(\mathcal{U} \cap \mathcal{N}_m) = \rho(\mathcal{V} \cap \mathcal{N}_m)$. Therefore $\mathcal{U} \cap \mathcal{N}_m = \mathcal{V} \cap \mathcal{N}_m$ for all m by Theorem 1.1.6, and hence $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$. □

By Remark 2.3.1 we have the following corollary.

Corollary 2.3.3 *If $\mathcal{U}, \mathcal{V}$ are varieties such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ then $B(\mathcal{U}) = B(\mathcal{V})$, and $u\rho(\mathcal{U}) = u\rho(\mathcal{V})$ and $B_{\mathcal{U}}(u) = B_{\mathcal{V}}(u)$ for all $u \in B(\mathcal{U})$.* □

The next result gives a useful characterization of the varieties which have the same nilpotent semigroups as a given variety $\mathcal{U}$.

Lemma 2.3.4 *Let $\mathcal{U}$ be a variety. A variety $\mathcal{V}$ satisfies $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ if and only if for every $u \in U(\mathcal{U})$ there exists $\Omega(u) \subseteq U(\mathcal{U})$ such that $\{|v| : v \in \Omega(u)\}$ is unbounded, and*

$$\mathcal{V} = [\mathrm{BId}(\mathcal{U}) \cup \{u = v : u \in U(\mathcal{U}), v \in \Omega(u)\}].$$

Proof: Suppose that $\mathcal{V}$ is a variety such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$. Then $U(\mathcal{U}) = U(\mathcal{V})$ by Corollary 2.3.3. For every $u \in U(\mathcal{U})$ let $\Omega(u) = u\rho(\mathcal{V})$. Then

$$\{u = v : u \in U(\mathcal{U}), v \in \Omega(u)\} \subseteq \mathrm{Id}(\mathcal{V}).$$

Also, $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V}) \subseteq \mathrm{Id}(\mathcal{V})$ by Theorem 2.3.2, so that

$$\mathcal{V} \subseteq [\mathrm{BId}(\mathcal{U}) \cup \{u = v : u \in U(\mathcal{U}), v \in \Omega(u)\}].$$

Now suppose that $u = v \in \mathrm{Id}(\mathcal{V})$. If $u \in B(\mathcal{V})$ then $u = v \in \mathrm{BId}(\mathcal{V}) = \mathrm{BId}(\mathcal{U})$. If $u \in U(\mathcal{V}) = U(\mathcal{U})$ then $v \in \Omega(u)$, and therefore

$$[\mathrm{BId}(\mathcal{U}) \cup \{u = v : u \in U(\mathcal{U}), v \in \Omega(u)\}] \subseteq \mathcal{V}.$$

For the converse, suppose that for every $u \in U(\mathcal{U})$ we have a set $\Omega(u) \subseteq U(\mathcal{U})$ such that $\{|v| : v \in \Omega(u)\}$ is unbounded. Let

$$\Sigma = \mathrm{BId}(\mathcal{U}) \cup \{u = v : u \in U(\mathcal{U}), v \in \Omega(u)\},$$

and let $\mathcal{V} = [\Sigma]$.

Suppose that $u \in X^+$, and let $\mathcal{V} \models u = v$. By Result 1.3.3, for some $m \geq 0$ there exist $r_i, s_i \in X^*$, an endomorphism φ_i of X^+, and $(d_i, e_i) \in \Sigma \cup \Sigma^{-1}$ for every i, $0 \leq i \leq m$, such that

$$\begin{aligned} u &= r_0(d_0\varphi_0)s_0, \\ v &= r_m(e_m\varphi_m)s_m, \end{aligned}$$

and $r_{i-1}(e_{i-1}\varphi_{i-1})s_{i-1} = r_i(d_i\varphi_i)s_i$ for every i, $1 \leq i \leq m$.

If $(d_i, e_i) \in \mathrm{BId}(\mathcal{U})$ for every i then $\mathrm{BId}(\mathcal{U}) \models u = v$, and so $\mathcal{U} \models u = v$. Otherwise, let j be the least i such that $(d_i, e_i) \notin \mathrm{BId}(\mathcal{U})$. So $d_j \in U(\mathcal{U})$.

If $j = 0$ then $u = r_0(d_0\varphi_0)s_0 \in U(\mathcal{U})$ by Remark 2.3.1. If $j \geq 1$ then $\mathrm{BId}(\mathcal{U}) \models u = r_{j-1}(e_{j-1}\varphi_{j-1})s_{j-1} = r_j(d_j\varphi_j)s_j$. Since $d_j \in U(\mathcal{U})$ then again $u \in U(\mathcal{U})$.

Therefore if $u \in X^+$ and $\mathcal{V} \models u = v$ then either $u \in U(\mathcal{U})$ or $\mathcal{U} \models u = v$. So for $u \in X^+$ either $u \in U(\mathcal{U})$ or $u\rho(\mathcal{V}) \subseteq u\rho(\mathcal{U})$.

Clearly $U(\mathcal{U}) \subseteq U(\mathcal{V})$. Suppose that $u \in U(\mathcal{V})$. From the above observation, either $u \in U(\mathcal{U})$ or $u\rho(\mathcal{V}) \subseteq u\rho(\mathcal{U})$. But the conditions $u \in U(\mathcal{V})$ and $u\rho(\mathcal{V}) \subseteq u\rho(\mathcal{U})$ imply that $u \in U(\mathcal{U})$; so that $u \in U(\mathcal{U})$ in both cases. Therefore $U(\mathcal{U}) = U(\mathcal{V})$, and hence $B(\mathcal{U}) = B(\mathcal{V})$.

If $u = v \in \mathrm{BId}(\mathcal{U})$ then $u \in B(\mathcal{U}) = B(\mathcal{V})$ and $u = v \in \mathrm{Id}(\mathcal{V})$, so that $u = v \in \mathrm{BId}(\mathcal{V})$. Conversely, let $u = v \in \mathrm{BId}(\mathcal{V})$. Since $u \in B(\mathcal{V}) = B(\mathcal{U})$ then from the observation above we conclude that $u\rho(\mathcal{V}) \subseteq u\rho(\mathcal{U})$ so that $u = v \in \mathrm{BId}(\mathcal{U})$. Hence $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V})$, and so $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ by Theorem 2.3.2. □

For a variety $\mathcal{U}$ let

$$\Sigma(\mathcal{U}) = \mathrm{BId}(\mathcal{U}) \cup \{u = v : u, v \in U(\mathcal{U})\}.$$

Let $\overline{\mathcal{U}} = [\Sigma(\mathcal{U})]$.

Theorem 2.3.5 *Let $\mathcal{U}$ be a variety. Then $\overline{\mathcal{U}} \cap \mathcal{N} = \mathcal{U} \cap \mathcal{N}$, and*

$$\overline{\mathcal{U}} = \bigcap\{\mathcal{V} \in \mathcal{L}(\mathcal{S}) : \mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}\}.$$

Proof: For every $u \in U(\mathcal{U})$ define $\Omega(u) = U(\mathcal{U})$. Then

$$\Sigma(\mathcal{U}) = \mathrm{BId}(\mathcal{U}) \cup \{u = v : v \in \Omega(u), u \in U(\mathcal{U})\}$$

so by Lemma 2.3.4 we have $\overline{\mathcal{U}} \cap \mathcal{N} = \mathcal{U} \cap \mathcal{N}$.

Suppose that $\mathcal{V}$ is a variety such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$. Let $u = v \in \mathrm{Id}(\mathcal{V})$. If $u = v \in \mathrm{BId}(\mathcal{V})$ then $u = v \in \mathrm{BId}(\mathcal{U}) \subseteq \Sigma(\mathcal{U})$ by Theorem 2.3.2. Otherwise $u, v \in U(\mathcal{V}) = U(\mathcal{U})$, so that $u = v \in \Sigma(\mathcal{U})$. Therefore $\mathrm{Id}(\mathcal{V}) \subseteq \Sigma(\mathcal{U})$, and hence $\overline{\mathcal{U}} \subseteq \mathcal{V}$. □

We can now characterize those varieties which are such that no other variety has the same set of nilpotent semigroups.

Theorem 2.3.6 *For a variety $\mathcal{U}$ the following are equivalent.*

- $\mathcal{U} = \mathcal{V}$ *whenever* $\mathcal{V}$ *is a variety such that* $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$.
- $U(\mathcal{U}) = \emptyset$.

Proof: Suppose that $\mathcal{U}$ is a variety which is such that if $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ for some variety $\mathcal{V}$ then $\mathcal{U} = \mathcal{V}$. Suppose that $u \in U(\mathcal{U})$. Let $c(u) = \{x_1, \ldots, x_n\}$, and let $Y = \{y_1, \ldots, y_n\} \subseteq X \setminus c(u)$ be such that $|Y| = n$. There exists an endomorphism φ of X^+ such that $x_j\varphi = y_j$ for all j, $1 \leq j \leq n$. Write $v = u\varphi$, so that $v \in U(\mathcal{U})$ by Remark 2.3.1. For every $w \in U(\mathcal{U})$ let

$$\Omega(w) = \{w, w^2, w^3, \ldots\} \subseteq U(\mathcal{U}),$$

again by Remark 2.3.1. Let $\Sigma = \mathrm{BId}(\mathcal{U}) \cup \{w = z : w \in U(\mathcal{U}), z \in \Omega(w)\}$. By Lemma 2.3.4 we have $\mathcal{U} \cap \mathcal{N} = [\Sigma] \cap \mathcal{N}$. Since $u, v \in U(\mathcal{U})$ then $(u, v) \in \Sigma(\mathcal{U}) \subseteq \rho(\overline{\mathcal{U}})$. Since $c(w) = c(z)$ whenever $w = z \in \mathrm{BId}(\mathcal{U})$, then $c(w) = c(z)$ whenever $(w, z) \in \rho([\Sigma])$, and so $(u, v) \notin \rho([\Sigma])$. Now $\overline{\mathcal{U}}$ and $[\Sigma]$ are two distinct varieties (by Theorem 1.1.6) such that $\overline{\mathcal{U}} \cap \mathcal{N} = \mathcal{U} \cap \mathcal{N} = [\Sigma] \cap \mathcal{N}$, contradicting our assumption. Therefore $U(\mathcal{U}) = \emptyset$.

Conversely, suppose $\mathcal{U}$ is a variety for which $U(\mathcal{U}) = \emptyset$. Then $B(\mathcal{U}) = X^+$ and $\mathrm{BId}(\mathcal{U}) = \mathrm{Id}(\mathcal{U})$. If $\mathcal{V}$ is a variety such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ then $\mathrm{BId}(\mathcal{V}) = \mathrm{BId}(\mathcal{U})$ by Theorem 2.3.2. Hence $B(\mathcal{V}) = B(\mathcal{U}) = X^+$, so that $\mathrm{Id}(\mathcal{V}) = \mathrm{BId}(\mathcal{V}) = \mathrm{Id}(\mathcal{U})$. Thus $\mathcal{U} = \mathcal{V}$. □

Remark 2.3.7 We have shown that the mapping $\beta : \mathcal{L}(\mathcal{S}) \to \mathcal{G}(\mathcal{N})$ given by $\mathcal{V}\beta = \mathcal{V} \cap \mathcal{N}$ is not injective; although it is surjective by Result 2.2.3. Clearly β respects $\cap$.

By Corollary 2.3.3 we have $B(\mathcal{U}) = B(\mathcal{V}) \subseteq B(\mathcal{U} \vee \mathcal{V})$ and $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V}) \subseteq \mathrm{Bid}(\mathcal{U} \vee \mathcal{V})$. However, $\mathrm{BId}(\mathcal{U})$ need not coincide with $\mathrm{Bid}(\mathcal{U} \vee \mathcal{V})$, as the following example shows. Let $\mathcal{U} = [xyx = xy^2x]$ and $\mathcal{V} = [xyx = (xyx)^2]$. Then

$$\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V}) = \{(w, w) \in X^+ \times X^+ : \text{ if } |w| \geq 3 \text{ then } |c(w)| = |w|\},$$

and so $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ by Theorem 2.3.2. Let $x, y \in X$, where $x \neq y$. Then $xyx \in U(\mathcal{U}) = U(\mathcal{V})$. It is easy to see using Result 1.3.3 that for $v \in X^+$ and $x, y \in X$, with $v \neq xyx$, we have $\mathcal{V} \models xyx = v$ if and only if $v = xwxzx$ for some $w, z \in X^+$, and $\mathcal{U} \models xyx = v$ if and only if $v = xy^kx$ for some $k > 1$. Therefore $(xyx)\rho(\mathcal{U} \vee \mathcal{V}) = \{xyx\}$. So $xyx \in B(\mathcal{U} \vee \mathcal{V}) \setminus B(\mathcal{U})$. Therefore $\mathrm{Bid}(\mathcal{U} \vee \mathcal{V}) \neq \mathrm{BId}(\mathcal{U})$; and hence $(\mathcal{U} \vee \mathcal{V}) \cap \mathcal{N} \neq \mathcal{U} \cap \mathcal{N} = (\mathcal{U} \cap \mathcal{N}) \vee (\mathcal{V} \cap \mathcal{N})$ by Theorem 2.3.2. That is, the mapping β does not respect $\vee$. □

2.4 Nil varieties whose nilpotent parts coincide.

If $\mathcal{U} \in \mathcal{L}(\mathcal{N}il)$ then $\mathcal{U} \in \mathcal{L}(\mathcal{N}il_m)$ for some m by Corollary 2.2.2, and so

$$U(\mathcal{N}il_m) = \{rx^m s : x \in X^+, r, s \in X^*\} \subseteq U(\mathcal{U}).$$

Now, by Theorem 2.3.6 there exists a variety $\mathcal{V} \neq \mathcal{U}$ for which $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$; but although $U(\mathcal{N}il_m) \subseteq U(\mathcal{U}) = U(\mathcal{V})$ by Theorem 2.3.2, this variety $\mathcal{V}$ need not be nil.

We begin with the following lemma, which shows that if a pair $\mathcal{U}, \mathcal{V}$ of nil varieties are to satisfy $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ they must have

$$\{m : \mathcal{U} \in \mathcal{L}(\mathcal{N}il_m)\} = \{m : \mathcal{V} \in \mathcal{L}(\mathcal{N}il_m)\}.$$

For a variety $\mathcal{U} \in \mathcal{L}(\mathcal{N}il)$, let $\mathrm{nil}(\mathcal{U}) = \min\{m : \mathcal{U} \in \mathcal{L}(\mathcal{N}il_m)\}$.

Lemma 2.4.1 *Let $\mathcal{U} \in \mathcal{L}(\mathcal{N}il)$. Then* $\min\{|w| : w \in U(\mathcal{U})\} = \mathrm{nil}(\mathcal{U})$. *If $\mathcal{V} \in \mathcal{L}(\mathcal{N}il)$ is such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ then* $\mathrm{nil}(\mathcal{U}) = \mathrm{nil}(\mathcal{V})$.

Proof: Write $r = \mathrm{nil}(\mathcal{U})$. Let $u \in U(\mathcal{U})$ have $|u| = \min\{|w| : w \in U(\mathcal{U})\}$. There exists $v \in u\rho(\mathcal{U})$ such that $|v| \geq r$.

Choose $x \in X$. There exists an endomorphism φ of X^+ such that $y\varphi = x$ for all $y \in X$. Then $u\varphi = x^{|u|}$ and $v\varphi = x^{|v|}$, and consequently

$$\mathcal{U} \models x^{|u|} = x^{|v|} = x^r = 0.$$

That is, $\mathcal{U} \in \mathcal{L}(\mathcal{N}il_{|u|})$. So $\min\{|w| : w \in U(\mathcal{U})\} \geq r = \mathrm{nil}(\mathcal{U})$. Conversely, $\mathcal{U} \in \mathcal{L}(\mathcal{N}il_r)$ implies that $x^r \in U(\mathcal{U})$ for every $x \in X$, and so

$$\min\{|w| : w \in U(\mathcal{U})\} \leq r = \mathrm{nil}(\mathcal{U}).$$

Thus $\min\{|w| : w \in U(\mathcal{U})\} = \mathrm{nil}(\mathcal{U})$.

If $\mathcal{V} \in \mathcal{L}(\mathcal{N}il)$ is such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ then we have shown that $\min\{|w| : w \in U(\mathcal{V})\} = \mathrm{nil}(\mathcal{V})$. But $U(\mathcal{U}) = U(\mathcal{V})$ by Corollary 2.3.3, and so $\mathrm{nil}(\mathcal{U}) = \min\{|w| : w \in U(\mathcal{U})\} = \min\{|w| : w \in U(\mathcal{V})\} = \mathrm{nil}(\mathcal{V})$. □

Notice that if $\mathcal{U} \in \mathcal{L}(\mathcal{N}il_m \cap \mathcal{C}om)$ then

$$u \in U(\mathcal{U}) \Leftrightarrow \mathcal{U} \models u = 0$$

(if $u \in U(\mathcal{U})$ and $c(u) = c(v)$ for all $v \in u\rho(\mathcal{U})$ then there exists a word $w \in u\rho(\mathcal{U})$ and a variable $x \in X$ that occurs at least m times in w, so that $x^m \in u\rho(\mathcal{U})$ since $\mathcal{U}$ is commutative; and on the other hand if $u \in U(\mathcal{U})$ and there exists $v \in u\rho(\mathcal{U})$ and $x \in c(v) \setminus c(u)$ then we may substitute x^m for x in v, and again obtain $x^m \in u\rho(\mathcal{U})$).

Therefore for $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{N}il_m \cap \mathcal{C}om)$ such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ we have $B(\mathcal{U}) = B(\mathcal{V})$, with

$$u \in B(\mathcal{U}) \Rightarrow u\rho(\mathcal{U}) = u\rho(\mathcal{V})$$

by Corollary 2.3.3 and

$$u \in U(\mathcal{U}) \Rightarrow \mathcal{U}, \mathcal{V} \models u = 0.$$

As in the proof of Theorem 2.3.2, this means that $\mathcal{U} = \mathcal{V}$. In view of Lemma 2.4.1, we have just shown that the mapping described in Theorem 2.1.1 is injective.

On the other hand, if $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{N}il_m)$ are such that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ then again $B(\mathcal{U}) = B(\mathcal{V})$ and

$$u \in B(\mathcal{U}) \Rightarrow u\rho(\mathcal{U}) = u\rho(\mathcal{V});$$

also if $u \in U(\mathcal{U})$ is such that there exists $v \in u\rho(\mathcal{U})$ with $c(u) \neq c(v)$ then $\mathcal{U} \models u = 0$. However, if $c(u) = c(v)$ for all $v \in u\rho(\mathcal{U})$ then perhaps $\mathcal{U} \not\models u = 0$. A relevant question here then, in the noncommutative case, is this: when does there exist a sequence of words, all with the same content, and with unbounded lengths, that contain no subwords of the form x^m, and hence are not equal to 0 under the laws of $\mathcal{N}il_m$?

We now introduce two varieties $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{N}il_5)$ in order to address this question; and we show that $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ but $\mathcal{U} \neq \mathcal{V}$, so the mapping α is not injective.

Let

$$\mathcal{U} = [x^5 = 0, x^3yz^3x = (yx)^3zx^3yx] \tag{2.1}$$

and

$$\mathcal{V} = [x^5 = 0, x^3yz^3x = (yx)^3zx^3yx, x^3yz^3x^2 = (yx)^3zx^3(yx)^2]. \tag{2.2}$$

Then $\mathcal{U}$ and $\mathcal{V}$ are members of $\mathcal{L}(\mathcal{N}il)$. Notice that for $\mathcal{W} \in \{\mathcal{U}, \mathcal{V}\}$ we have

$$\begin{aligned} U(\mathcal{W}) &= \{r(x^3yz^3x)s : r, s \in X^* \text{ and } x, y, z \in X^+\} \cup U(\mathcal{N}il_5) \\ &= \{u \in X^+ : \text{ there exists a nontrivial identity } v = w \in \mathrm{Id}(\mathcal{W}) \\ &\qquad \text{such that } u = rvs \text{ for } r, s \in X^*\}; \end{aligned}$$

and hence $B(\mathcal{U}) = B(\mathcal{V})$. Moreover, if $u \in B(\mathcal{U})$ then $u\rho(\mathcal{U}) = \{u\} = u\rho(\mathcal{V})$, so that $\mathrm{BId}(\mathcal{U}) = \mathrm{BId}(\mathcal{V})$. Hence by Theorem 2.3.2 we have the following result.

Theorem 2.4.2 *The varieties $\mathcal{U}$ and $\mathcal{V}$ as defined above satisfy*

$$\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}. \qquad \square$$

Let $A = \{a, b, c\} \subset X$ where a, b and c are distinct. Note that whenever the identity $x^3yz^3x = (yx)^3zx^3yx$ (see (2.1) and (2.2)) is applied, it can be reapplied nontrivially. Thus from the word a^3bc^3a we obtain first $(ba)^3ca^3ba$, then $(cba)^3a(ba)^3cba$, $(acba)^3ba(cba)^3acba$ and so on. With this in mind, we inductively define sequences $(a_n)_{n\geq 0}$ and $(A_n)_{n\geq 0}$ in A^+ as follows:

$$\begin{aligned} a_0 &= a, \\ a_1 &= ba, \\ a_2 &= cba, \text{ and} \\ a_n &= a_{n-3}a_{n-1} \text{ for } n \geq 3; \end{aligned}$$

$$\begin{aligned} A_0 &= a^3bc^3a, \\ A_1 &= (ba)^3ca^3ba, \text{ and} \\ A_n &= (a_n)^3a_{n-2}(a_{n-1})^3a_n \text{ for } n \geq 2. \end{aligned}$$

The identities which define the varieties $\mathcal{U}$ and $\mathcal{V}$ can now be rewritten:

$$\mathcal{U} = [x^5 = 0, A_0 = A_1] \text{ and } \mathcal{V} = [x^5 = 0, A_0 = A_1, A_0a = A_1ba].$$

It follows that $A_n \in (A_0)\rho(\mathcal{U})$ for every $n \geq 0$. The bulk of the proof that $\mathcal{U} \neq \mathcal{V}$ is concerned with showing that in fact

$$(A_0)\rho(\mathcal{U}) = (A_0)\rho(\mathcal{V}) = \{A_n : n \geq 0\}.$$

This equality is proved from the following two results. Firstly, for $n \geq 0$ the word A_n has no factor of the form x^5, $x \in A^+$. Some bound on the size of powers appearing in A_n is obviously needed since otherwise, regardless of the choice of the number m such that $\mathcal{U}, \mathcal{V} \subseteq \mathcal{N}il_m$, a simple modification of the proof of Theorem 2.4.2 would lead to the conclusion $\mathcal{U} = \mathcal{V}$. Secondly, it is shown that if $A_n = r(x^3yz^3x)s$ for $n \geq 2$, $r, s \in A^*$ and $x, y, z \in A^+$ then $r = s = 1$, $x = a_n$, $y = a_{n-2}$ and $z = a_{n-1}$; that is, there is only one way to reapply the identity. (These properties of A_n are actually proved for the word A_na; however, they are obvious consequences of the corresponding results: $(A_0a)\rho(\mathcal{U}) = \{A_na : n \geq 0\}$, A_na has no factor of the form x^5, $x \in A^+$, and if $A_na = r(x^3yz^3x)s$ for $r, s \in A^*$ and $x, y, z \in A^+$ then $r = 1$, $s = a$, $x = a_n$, $y = a_{n-2}$ and $z = a_{n-1}$.) The combination of the terms a_n, a_{n-1} and a_{n-2} which forms A_n was chosen with this property in mind. Although for $n \geq 0$ the word a_n does contain squares, it has no factor of the form x^3, $x \in A^+$. This explains the choice of the terms $(a_n)^3$ and $(a_{n-1})^3$ in A_n. There are many other cubes in A_n, but the fact that a_n also appears at the end of A_n ensures that the correct cubes are selected, and leaves no choice for the other terms.

2.5 The sequences $(a_n)_{n\geq 0}$ and $(A_n)_{n\geq 0}$.

We begin by giving a number of lemmas establishing properties of the sequences $(a_n)_{n\geq 0}$ and $(A_n)_{n\geq 0}$. Most proofs require direct verification for the first few terms of the sequence $(a_n)_{n\geq 0}$, so for convenience the first ten terms are listed below.

$$
\begin{aligned}
a_0 &= a \\
a_1 &= ba \\
a_2 &= cba \\
a_3 &= acba \\
a_4 &= baacba \\
a_5 &= cbabaacba \\
a_6 &= acbacbabaacba \\
a_7 &= baacbaacbacbabaacba \\
a_8 &= cbabaacbabaacbaacbacbabaacba \\
a_9 &= acbacbabaacbacbabaacbabaacbaacbacbabaacba
\end{aligned}
$$

For $j, k \geq 0$ and $x_i \in X^+$ we write $\prod_{i=j}^{k} x_i$ for the (noncommutative) product $x_j \dots x_k$. The relative ordering of j and k will always be clear, and for notational convenience the following rule is set: whenever such a product $\prod_{i=j}^{k} x_i$ is written in the form $x\left(\prod_{i=j'}^{k'} x_i\right) y$ for $x, y \in X^*$,

- if $j \leq k$ then $j' > k' \Leftrightarrow \prod_{i=j'}^{k'} x_i = 1$, and
- if $k \leq j$ then $k' > j' \Leftrightarrow \prod_{i=j'}^{k'} x_i = 1$.

The first result details some basic facts. For a real number r, the symbol $[r]$ is used to denote the integer part of r.

Lemma 2.5.1

(i) Let $n \geq 3$. If $2 \leq k \leq n-1$ then $a_n = \left(\prod_{i=n-3}^{k-2} a_i\right) a_k$.

(ii) Let $n \geq 3$. If $1 \leq k \leq [n/3]$ then $a_n = a_{n-3k}\left(\prod_{i=k}^{1} a_{n+2-3i}\right)$.

(iii) Let $m, n \geq 0$. Then $m \leq n \Leftrightarrow |a_m| \leq |a_n|$.

(iv) *Let $m, n \geq 0$. Then $m \equiv n \pmod 3 \Leftrightarrow a_m$ and a_n have the same initial letter.*

Proof: (i) and (ii) follow easily by induction on n. Part (iii) is clear from part (i). Part (iv) is clear from part (ii) since $a_0 = a$, $a_1 = ba$ and $a_2 = cba$. □

A word $u \in X^+$ is a *left* (respectively *right*) *factor* of $v \in X^+$ if $v = uw$ (respectively $v = wu$) for some $w \in X^*$.

The next lemma gives information about words which are both left factors of a_n and right factors of a_m for some n, m.

Lemma 2.5.2 *Let $n \geq 3$ and $m \geq 0$. If $a_n = uv$ and $a_m = wu$ for $u, v \in A^+$ and $w \in A^*$ then $u = a_{n-3k}$ for some k, $1 \leq k \leq [n/3]$, and hence $v = \prod_{i=k}^{1} a_{n+2-3i}$.*

Proof: We will prove by induction on m that $u = a_{n-3k}$ for $1 \leq k \leq [n/3]$, and then $v = \prod_{i=k}^{1} a_{n+2-3i}$ by Lemma 2.5.1(ii).

The cases $m = 0, 1, 2$ are easily checked. Let $m \geq 3$, and suppose that $a_n = uv$ and $a_m = wu$ where $n \geq 3$, $u, v \in A^+$ and $w \in A^*$. If $w = 1$ then $a_n = a_m v$. Then $m = n - 3k$ for some k, $1 \leq k \leq [n/3]$, by Lemma 2.5.1(iii) and (iv), so that $u = a_{n-3k}$ as desired. Suppose therefore that $w \in A^+$. From $wu = a_{m-3}a_{m-1}$ we take the following cases: (a) $w = a_{m-3}x$, $x \in A^*$, and (b) $a_{m-3} = wx$, $x \in A^+$.

Case (a): If $w = a_{m-3}x$, $x \in A^*$, then $a_{m-1} = xu$. By the induction hypothesis applied to a_n and a_{m-1} there exists k, $1 \leq k \leq [n/3]$, such that $u = a_{n-3k}$.

Case (b): If $a_{m-3} = wx$, $x \in A^+$, then $u = xa_{m-1}$. We will show that this case cannot occur. By the induction hypothesis applied to $a_n = x(a_{m-1}v)$ and a_{m-3} there exists k, $1 \leq k \leq [n/3]$, such that $x = a_{n-3k}$. Then $a_{m-3} = wa_{n-3k}$, and hence $n - 3k < m - 3$ by Lemma 2.5.1(iii). We consider two cases.

(b)(1): Suppose that $k = 1$. Then the inequality $n - 3k < m - 3$ becomes $n < m$. Since $a_n = xa_{m-1}v$, from Lemma 2.5.1(iii) we obtain the contradiction $m - 1 < n < m$.

(b)(2): Suppose that $k \geq 2$. We have $a_n = a_{n-3k} \left(\prod_{i=k}^{1} a_{n+2-3i}\right)$ by Lemma 2.5.1(ii). Since also $a_n = a_{n-3k}a_{m-1}v$, then

$$a_{m-1}v = \prod_{i=k}^{1} a_{n+2-3i} = a_{n+2-3k} \prod_{i=k-1}^{1} a_{n+2-3i}. \tag{2.3}$$

Let $\ell = \frac{m-n}{3} + k - 1$. Since $m \equiv n \ (mod\ 3)$ by (2.3), then ℓ is an integer. Since $k \leq [n/3]$ then $\ell \leq [\frac{m-1}{3}]$, and we obtain $\ell \geq 1$ from $n - 3k < m - 3$. Therefore $a_{m-1} = a_{n+2-3k} \prod_{i=\ell}^{1} a_{m+1-3i}$ by Lemma 2.5.1(ii). Eq.(2.3) then yields $\left(\prod_{i=\ell}^{1} a_{m+1-3i}\right) v = \prod_{i=k-1}^{1} a_{n+2-3i}$. However, this implies that $m \equiv n + 1 \ (mod\ 3)$, contrary to the former conclusion $m \equiv n \ (mod\ 3)$. □

The following lemma shows that a_n has no right factor of the form x^2, $x \in A^+$.

Lemma 2.5.3 *Let $n \geq 0$. If $a_n = uv^2$ for $u, v \in A^*$ then $v = 1$.*

Proof: If $0 \leq n \leq 8$ the result is easily verified. The proof for $n \geq 9$ is by induction on n. Let $n \geq 9$ and suppose that $a_n = uv^2$ where $u, v \in A^*$. Then $uv^2 = a_{n-3}a_{n-1}$. We consider the following cases:

(a) $v = wa_{n-1}$, $w \in A^*$;

(b) $a_{n-1} = wv$, $v = xw$, $w \in A^+$, $x \in A^+$; and

(c) $a_{n-1} = wv^2$, $w \in A^*$.

Case (a): If $v = wa_{n-1}$, $w \in A^*$, then $a_{n-3} = uvw = uwa_{n-1}w$, which contradicts Lemma 2.5.1(iii).

Case (b): If $a_{n-1} = wv$ and $v = xw$ for $w, x \in A^+$ then $a_{n-3} = ux$ and $a_{n-1} = wxw$. By Lemma 2.5.2 applied to $a_{n-1} = w(xw) = (wx)w$ we have $w = a_{n-1-3k}$ for some $k \geq 1$.

(b)(1): If $k = 1$ then $w = a_{n-4}$, and hence $a_{n-1} = a_{n-4}xa_{n-4}$. Since $a_{n-1} = a_{n-4}a_{n-5}a_{n-6}a_{n-4}$, then $x = a_{n-5}a_{n-6} = a_{n-8}(a_{n-6})^2$. The induction hypothesis applied to $a_{n-3} = ux$ now gives the contradiction $a_{n-6} = 1$.

(b)(2): If $k \geq 2$ then

$$\begin{aligned} |x| &= |a_{n-1}| - 2|a_{n-1-3k}| \\ &\geq |a_{n-1}| - 2|a_{n-7}| \\ &= |a_{n-7}a_{n-8}a_{n-9}a_{n-7}a_{n-2}| - 2|a_{n-7}| \\ &> |a_{n-2}|, \end{aligned}$$

and this provides the contradiction $|a_{n-2}| < |a_{n-3}|$ since $a_{n-3} = ux$.

Case (c): By the induction hypothesis applied to a_{n-1} we have $v = 1$. $\square$

The next lemma plays a significant part in our investigation of the word $A_n = (a_n)^3 a_{n-2}(a_{n-1})^3 a_n$.

Lemma 2.5.4 *Let $n \geq 5$ and $m \geq 1$, and let $n-2 \leq p_i, q_i \leq n$ for $1 \leq i \leq m$. If $\prod_{i=1}^m a_{p_i} = (\prod_{i=1}^m x_i a_{q_i})x$ for $x \in A^*$ and $x_i \in A^*$, $1 \leq i \leq m$, then $x = 1$.*

Proof: The proof is by induction on m. For the first step, suppose that $a_p = ua_qx$ where $u, x \in A^*$ and $n-2 \leq p, q \leq n$. If $u = 1$ then $p = q$ by Lemma 2.5.1(iv), so that $x = 1$. We may therefore assume that $u \in X^+$. Then $q \in \{p-2, p-1\}$. We have $a_{p-3}a_{p-1} = ua_qx$, so that either (a) $x = wa_{p-1}$, $w \in A^*$, or (b) $a_{p-1} = wx$, $w \in A^+$.

Case (a): If $x = wa_{p-1}$, $w \in A^*$, then $a_{p-3} = ua_qw$, which contradicts Lemma 2.5.1(iii).

Case (b): If $a_{p-1} = wx$, $w \in A^+$, then $a_{p-3}w = ua_q$.

(b)(1): If $q = p - 1$ then $a_q = wx$. Hence $a_p = uwx^2$, and thus $x = 1$ by Lemma 2.5.3.

(b)(2): If $q = p - 2$ then $a_{p-3}w = ua_{p-2}$. Either $w = va_{p-2}$, $v \in A^*$, or $a_{p-2} = vw$, $v \in A^+$.

If $w = va_{p-2}$, $v \in A^*$, then $a_{p-1} = va_{p-2}x$, and from the case $q = p - 1$ we obtain $x = 1$.

If $a_{p-2} = vw$, $v \in A^+$, then $a_{p-3} = uv$. Therefore, since a_{p-1} and w have the same initial letter, Lemma 2.5.2 applied to a_{p-2} and a_{p-3} gives the contradiction $p - 1 \equiv p \ (mod\ 3)$. This completes the case $m = 1$.

Suppose now that $m \geq 2$, and that $\prod_{i=1}^{m} a_{p_i} = (\prod_{i=1}^{m} x_i a_{q_i})\, x$ where $n - 2 \leq p_i, q_i \leq n$ and $x_i \in A^*$ for $1 \leq i \leq m$, and $x \in A^*$. Either (a′) $a_{p_1} = x_1 a_{q_1} y$, $y \in A^+$, or (b′) $x_1 a_{q_1} = a_{p_1} y$, $y \in A^*$.

Case (a′): If $a_{p_1} = x_1 a_{q_1} y$, $y \in A^+$, the induction hypothesis applied to a_{p_1} gives the contradiction $y = 1$.

Case (b′): If $x_1 a_{q_1} = a_{p_1} y$, $y \in A^*$, then $\prod_{i=2}^{m} a_{p_i} = y\,(\prod_{i=2}^{m} x_i a_{q_i})\, x$, and $x = 1$ by induction. □

Although the word a_n does not have right factors of the form x^2, $x \in A^+$, it does have left factors of this form. The following result concerns a particular case.

Lemma 2.5.5 *Let $n, m \geq 0$. If $a_n = (uv)^2 w$ and $a_m = xv$ for $u, v, w \in A^+$ and $x \in A^*$ then $uy \neq wz$ for all $y, z \in A^*$.*

Proof: The proof is by induction on n. It is easily checked that the result holds when $0 \leq n \leq 9$. Let $n \geq 10$ and suppose that $a_n = (uv)^2 w$, $a_m = xv$ and $uy = wz$ where $m \geq 0$, $u, v, w \in A^+$ and $x, y, z \in A^*$. Then $a_{n-3}a_{n-1} = (uv)^2 w$, and we have either:

(a) $w = ta_{n-1}$, $t \in A^*$;

(b) $a_{n-1} = tw$, $uv = rt$, $t \in A^+$, $r \in A^+$; or

(c) $a_{n-1} = tuvw$, $t \in A^*$.

Case (a): If $w = ta_{n-1}$, $t \in A^*$, then $a_{n-3} = (uv)^2 t$, $a_m = xv$ and $uy = t(a_{n-1}z)$. If $t = 1$ then Lemma 2.5.3 gives the contradiction $uv = 1$. If $t \in A^+$ then the induction hypothesis is contradicted.

Case (b): If $a_{n-1} = tw$ and $uv = rt$ for $t, r \in A^+$ then $a_{n-3} = rtr$. By Lemma 2.5.2 applied to $a_{n-3} = r(tr) = (rt)r$ there exists $k \geq 1$ such that $r = a_{n-3-3k}$ and $tr = \prod_{i=k}^{1} a_{n-1-3i}$.

(b)(1): Suppose first that $k = 1$. Then $r = a_{n-6}$ and $tr = a_{n-4}$. Since $a_{n-4} = a_{n-7}a_{n-8}a_{n-6}$, then $t = a_{n-7}a_{n-8}$. Hence $uv = rt = a_{n-6}a_{n-7}a_{n-8}$, so that $a_n = (uv)^2 w = (a_{n-6}a_{n-7}a_{n-8})^2 w$. Also, $a_n = a_{n-3}a_{n-1} = a_{n-6}(a_{n-4})^2 a_{n-2}$ $= a_{n-6}(a_{n-7}a_{n-8}a_{n-6})^2 a_{n-2}$ and thus $w = a_{n-6}a_{n-2}$.

Since $uv = a_{n-6}a_{n-7}a_{n-8}$, then either $a_{n-6} = us$, $s \in A^*$, or $u = a_{n-6}s$, $s \in A^+$. If $a_{n-6} = us$, $s \in A^*$, then $v = sa_{n-7}a_{n-8} = sa_{n-10}(a_{n-8})^2$, and Lemma 2.5.3 applied to $a_m = xv$ provides the contradiction $a_{n-8} = 1$. If $u = a_{n-6}s$, $s \in A^+$, then $sv = a_{n-7}a_{n-8}$. Since $w = a_{n-6}a_{n-2}$, the equation $uy = wz$ gives $sy = a_{n-2}z$. Thus a_{n-2} and a_{n-7} have the same initial letter, which contradicts Lemma 2.5.1(iv).

(b)(2): If $k \geq 2$ then $\left|\prod_{i=k-1}^{1} a_{n-1-3i}\right| \geq |a_{n-4}| > |a_{n-3-3k}| = |r|$. Therefore, since

$$tr = \prod_{i=k}^{1} a_{n-1-3i} = a_{n-1-3k} \prod_{i=k-1}^{1} a_{n-1-3i},$$

there exists $s \in A^+$ such that $t = a_{n-1-3k}s$ and

$$sr = \prod_{i=k-1}^{1} a_{n-1-3i}. \tag{2.4}$$

Then $a_{n-1} = a_{n-1-3k}sw$. Since $a_{n-1} = a_{n-1-3k}\left(\prod_{i=k}^{1} a_{n+1-3i}\right)$, then

$$\prod_{i=k}^{1} a_{n+1-3i} = sw. \tag{2.5}$$

By (2.4) and (2.5) the initial letters of a_{n-1} and a_{n+1} are the same, which contradicts Lemma 2.5.1(iv).

Case (c): If $a_{n-1} = tuvw$, $t \in A^*$, then $a_{n-3}t = uv$. Consequently $a_{n-1} = ta_{n-3}tw$, and Lemma 2.5.4 gives the contradiction $tw = 1$. This completes the proof. $\square$

We now turn our attention to the sequence $(A_n)_{n\geq 0}$. The next lemma identifies the factors of the word $A_n = (a_n)^3 a_{n-2}(a_{n-1})^3 a_n$ of the form x^3, $x \in A^+$.

Lemma 2.5.6 *Let $1 \leq \ell \leq 8$ and $n \geq 5$. Let $(p_1, \ldots, p_\ell)$ be a subsequence of*

$$(n_1, \ldots, n_8) = (n, n, n, n-2, n-1, n-1, n-1, n)$$

such that if $p_1 = n_i$ then $p_{j+1} = n_{j+i}$ for $1 \leq j < \ell$. Let

$$\prod_{i=1}^{\ell} a_{p_i} = wx^3v \tag{2.6}$$

and $a_{p_1} = wu$ where $w, v \in A^*$ and $u, x \in A^+$. Then either:

(i) $\ell \geq 3$, $(p_1, p_2, p_3) = (n, n, n)$, $w = 1$, $x = a_n$ and $v = \prod_{i=4}^{\ell} a_{p_i}$;

(ii) $\ell \geq 3$, $(p_1, p_2, p_3) = (n-1, n-1, n-1)$, $w = 1$, $x = a_{n-1}$ and $v = \prod_{i=4}^{\ell} a_{p_i}$;

(iii) $\ell \geq 5$, $p_2 = n-2$, $x = ua_{n-2}t$, $a_{n-4} = tu$ and $v = ua_{n-2}\prod_{i=6}^{\ell} a_{p_i}$ for some $t \in A^+$;

(iv) $\ell \geq 4$, $p_1 = n-2$, $x = ut$, $a_{n-1} = tu$ and $v = u\prod_{i=5}^{\ell} a_{p_i}$ for some $t \in A^+$; or

(v) $\ell \geq 4$, $p_2 = n-2$, $x = ra_{n-4}t$, $a_{n-2} = tr$ and $v = r\prod_{i=5}^{\ell} a_{p_i}$ for some $r \in A^*$ and $t \in A^+$.

Proof: The first step is to show that the cases $\ell = 1, 2$ contradict the assumption $x \in A^+$. The case $\ell = 1$ is proved first. The case $\ell = 2$ is then clear, since for every choice of (p_1, p_2) there exists $m \geq 0$ and $y, z \in A^*$ such that $a_m = y a_{p_1} a_{p_2} z$: indeed for any $m \geq 7$ we have

$$\begin{aligned} a_m &= a_{m-3} a_{m-1} \\ &= a_{m-6} (a_{m-4})^2 a_{m-2} \\ &= a_{m-6} a_{m-7} (a_{m-5} a_{m-4}) a_{m-2} \\ &= a_{m-6} a_{m-7} (a_{m-5} a_{m-7}) a_{m-5} a_{m-2}. \end{aligned}$$

Suppose then that $\ell = 1$. We prove by induction on $n \geq 3$ that if $a_n = wx^3v$ for $w, x, v \in A^*$ then $x = 1$. The initial cases $n = 3, 4, 5$ are easily verified.

Let $a_n = wx^3v$ where $w, x, v \in A^*$ and $n \geq 6$. Then $a_{n-3}a_{n-1} = wx^3v$, and we consider the following cases:

(a) $v = ya_{n-1}$, $y \in A^*$;

(b) $a_{n-1} = yv$, $x = zy$, $y \in A^+$, $z \in A^*$;

(c) $a_{n-1} = yxv$, $x = zy$, $y \in A^+$, $z \in A^*$;

(d) $a_{n-1} = yx^2v$, $x = zy$, $y \in A^+$, $z \in A^+$; and

(e) $a_{n-1} = yx^3v$, $y \in A^*$.

Case (a): If $v = ya_{n-1}$, $y \in A^*$, then $a_{n-3} = wx^3y$, and hence $x = 1$ by induction.

Case (b): In this case $a_{n-3} = wz(yz)^2$, and Lemma 2.5.3 gives the contradiction $yz = 1$.

Case (c): In this case $a_{n-3} = wz(yz)$ and $a_{n-1} = (yz)yv$. Therefore yv and a_{n-1} have the same initial letter. Hence Lemma 2.5.2 applied to a_{n-1} and a_{n-3} gives the contradiction $n - 1 \equiv n + 1$ $(mod\ 3)$.

Case (d): We have $a_{n-3} = wz$ and $a_{n-1} = (yz)^2yv$, so this case is contradicted by Lemma 2.5.5.

Case (e): The induction hypothesis applied to a_{n-1} shows that $x = 1$. This completes the case $\ell = 1$.

Now suppose that $3 \leq \ell \leq 8$. Then

$$u \prod_{i=2}^{\ell} a_{p_i} = x^3 v \tag{2.7}$$

by (2.6) since $a_{p_1} = wu$.

We proceed by considering cases based on the relative lengths of the left factors of (2.7), obtaining in each case a reduced form of the equation, for which we then repeat the procedure if necessary. Most cases are shown to be impossible; those that remain are identified as the cases (i)-(v) given in the statement of the lemma.

We first distinguish the following cases, taken from (2.7):

(a) $u = x^3 d$, $d \in A^*$;

(b) $u = x^2 d$, $x = de$, $d \in A^*$, $e \in A^+$;

(c) $u = xd$, $x = de$, $d \in A^+$, $e \in A^+$;

(d) $u = x$; and

(e) $x = ud$, $d \in A^+$.

Case (a): We have $a_{p_1} = wu = wx^3 d$, which contradicts the case $\ell = 1$.

Case (b): We have $a_{p_1} = wx^2 d = wd(ed)^2$, and Lemma 2.5.3 provides the contradiction $ed = 1$.

Case (c): In this case (2.7) gives

$$\prod_{i=2}^{\ell} a_{p_i} = edev, \tag{2.8}$$

and we have $a_{p_1} = wded$. From (2.8) we take three cases:

(1) $e = a_{p_2} f$, $f \in A^*$;

(2) $a_{p_2} = ef$, $d = fg$, $f \in A^+$, $g \in A^*$; or

(3) $a_{p_2} = edf$, $f \in A^+$.

(c)(1): In this case $a_{p_1} = wda_{p_2}fd$, and Lemma 2.5.4 gives the contradiction $fd = 1$.

(c)(2): We have $a_{p_1} = wfgefg = wfga_{p_2}g$. Thus $p_2 < p_1$, and Lemma 2.5.4 shows that $g = 1$. Then $a_{p_2} = ed$, and hence $\prod_{i=3}^{\ell} a_{p_i} = ev$ by (2.8). Therefore $p_2 = p_3$ by Lemma 2.5.1(iv). This means that $p_1 > p_2 = p_3$, which contradicts any choice of the sequence $(p_1, \ldots, p_\ell)$.

(c)(3): In this case we obtain $f\prod_{i=3}^{\ell} a_{p_i} = ev$ from (2.8), and we have $a_{p_2} = (ed)f$ and $a_{p_1} = wd(ed)$. Therefore a_{p_2} and f have the same initial letter, and Lemma 2.5.2 applied to a_{p_2} and a_{p_1} gives the contradiction $p_2 \equiv p_2 + 2$ $(mod\ 3)$. This concludes case (c).

Case (d): In this case $a_{p_1} = wx$, and (2.7) gives $\prod_{i=2}^{\ell} a_{p_i} = x^2v$. Either (1) $a_{p_2} = xd$, $d \in A^+$, or (2) $x = a_{p_2}d$, $d \in A^*$.

(d)(1): If $a_{p_2} = xd$, $d \in A^+$, then $d\prod_{i=3}^{\ell} a_{p_i} = xv$. Therefore a_{p_2} and d have the same initial letter, and Lemma 2.5.2 applied to a_{p_2} and a_{p_1} yields the contradiction $p_2 \equiv p_2 + 2$ $(mod\ 3)$.

(d)(2): If $x = a_{p_2}d$, $d \in A^*$, then $a_{p_1} = wa_{p_2}d$. Thus $p_2 \leq p_1$, and from Lemma 2.5.4 we find that $d = 1$. Therefore $x = a_{p_2}$, so that $\prod_{i=3}^{\ell} a_{p_i} = a_{p_2}v$, and hence $p_2 = p_3$. We now have $p_1 \geq p_2 = p_3$, and this implies that $p_1 = p_2 = p_3$. Hence $(p_1, p_2, p_3) \in \{(n, n, n), (n-1, n-1, n-1)\}$, and we have $\prod_{i=4}^{\ell} a_{p_i} = v$. The definition of case (d) gives $u = x$, and therefore this describes cases (i) and (ii).

Case (e): We have $a_{p_1} = wu$, and (2.7) gives

$$\prod_{i=2}^{\ell} a_{p_i} = d(ud)^2 v. \tag{2.9}$$

Either:

(1) $d = a_{p_2}a_{p_3}e$, $e \in A^*$;

(2) $d = a_{p_2}e$, $a_{p_3} = ef$, $e \in A^+$, $f \in A^+$; or

(3) $a_{p_2} = de$, $e \in A^*$.

(e)(1): We obtain $\prod_{i=4}^{\ell} a_{p_i} = e(ua_{p_2}a_{p_3}e)^2 v$ from (2.9), and we have $a_{p_1} = wu$. If $\ell \leq 6$ then Lemma 2.5.4 shows that $a_{p_3} = 1$, a contradiction. Therefore we may assume that $\ell \geq 7$. Either $eu = a_{p_4}f$, $f \in A^*$, or $a_{p_4} = euf$, $f \in A^+$. If $eu = a_{p_4}f$, $f \in A^*$, then $\prod_{i=5}^{\ell} a_{p_i} = fa_{p_2}a_{p_3}a_{p_4}fa_{p_2}a_{p_3}ev$, and again Lemma 2.5.4 shows that $a_{p_3} = 1$, a contradiction. If $a_{p_4} = euf$, $f \in A^+$, then

$$f\prod_{i=5}^{\ell} a_{p_i} = a_{p_2}a_{p_3}eua_{p_2}a_{p_3}ev. \tag{2.10}$$

Observe that $p_4 < p_2 = p_5 + 1$ since $\ell \geq 7$. Thus $|f| < |a_{p_2}|$ since $a_{p_4} = euf$, and hence by (2.10) there exists $g \in A^+$ such that $a_{p_2} = fg$. Therefore

$$\prod_{i=5}^{\ell} a_{p_i} = ga_{p_3}eua_{p_2}a_{p_3}ev. \tag{2.11}$$

Lemma 2.5.2 applied to a_{p_2} and a_{p_4} shows that $g = ha_{p_2-1} = ha_{p_5}$ for some $h \in A^*$, so that $|a_{p_5}| \leq |g|$. By (2.11), there exists k, with $|k| = |h|$, such that $g = a_{p_5}k$. Then $a_{p_2} = fa_{p_5}k$, and Lemma 2.5.4 shows that $k = h = 1$. Thus $g = a_{p_5}$, so that $\prod_{i=6}^{\ell} a_{p_i} = a_{p_3}eua_{p_2}a_{p_3}ev$. Then $p_3 = p_6$, which is impossible, and case (e)(1) is completed.

Cases (e)(2) and (e)(3) are clearly similar:

- in case (e)(2) we have $f\prod_{i=4}^{\ell} a_{p_i} = (ua_{p_2}e)^2 v$, $a_{p_1} = wu$, $a_{p_3} = ef$ and $x = ua_{p_2}e$ where $e, f \in A^+$; and

- in case (e)(3) we have $e\prod_{i=3}^{\ell} a_{p_i} = (ud)^2 v$, $a_{p_1} = wu$, $a_{p_2} = de$ and $x = ud$ where $d \in A^+$ and $e \in A^*$.

We combine these two cases to form the following case:

$$r\prod_{i=j+1}^{\ell} a_{p_i} = (ust)^2 v, \tag{2.12}$$

$a_{p_1} = wu$, $a_{p_j} = tr$ and $x = ust$ where $t \in A^+$, $j \in \{2,3\}$ and

- $j = 2 \Rightarrow s = 1$ and $r \in A^*$,
- $j = 3 \Rightarrow s = a_{p_2}$ and $r \in A^+$.

Note that the case $j = 2$ corresponds to the case (e)(3), with $r = e$ and $t = d$, and the case $j = 3$ corresponds to the case (e)(2), with $r = f$ and $t = e$. This case is labelled (e)(4), and the following subcases are considered:

(I) $r = ustuy$, $y \in A^*$;

(II) $r = uy$, $stu = yz$, $y \in A^+$, $z \in A^+$;

(III) $r = u$;

(IV) $u = ry$, $a_{p_{j+1}} = yz$, $y \in A^+$, $z \in A^+$; or

(V) $u = ra_{p_{j+1}}y$, $y \in A^*$.

(e)(4)(I): We have $a_{p_1} = wu$ and $a_{p_j} = tustuy$, and (2.12) shows that $y\prod_{i=j+1}^{\ell} a_{p_i} = stv$. If $j = 2$ then $s = 1$, so that $a_{p_2} = (tu)^2 y$ and

$$y \prod_{i=3}^{\ell} a_{p_i} = tv. \tag{2.13}$$

If $y = 1$ then Lemma 2.5.3 applied to a_{p_2} gives the contradiction $tu = 1$. Therefore $y \in A^+$. In view of (2.13), Lemma 2.5.5 is now contradicted by a_{p_2} and a_{p_1}.

If $j = 3$ then $s = a_{p_2}$. Thus $a_{p_3} = tua_{p_2}tuy$, and Lemma 2.5.4 gives the contradiction $tuy = 1$.

(e)(4)(II): In this case (2.12) gives

$$\prod_{i=j+1}^{\ell} a_{p_i} = zstv, \tag{2.14}$$

and we have $a_{p_1} = wu$, $a_{p_j} = tuy$ and $stu = yz$.

If $j = 2$ then $s = 1$, so that $tu = yz$. Then $a_{p_2} = tuy = yzy$. By (2.14), a_{p_3} and z have the same initial letter. Therefore Lemma 2.5.2 applied to $a_{p_2} = y(zy) = (yz)y$ yields $p_3 \equiv p_2 + 2$ $(mod\ 3)$, which contradicts any choice of the sequence $(p_1, \ldots, p_\ell)$.

If $j = 3$ then $a_{p_3} = tuy$ and $s = a_{p_2}$. Eq.(2.14) becomes

$$\prod_{i=4}^{\ell} a_{p_i} = za_{p_2}tv, \tag{2.15}$$

and we have $a_{p_2}tu = yz$. From the last equation we take the cases (A) $a_{p_2} = yh$, $h \in A^+$, and (B) $y = a_{p_2}h$, $h \in A^*$.

(e)(4)(II)(A) If $a_{p_2} = yh$, $h \in A^+$, then $z = htu$ and (2.15) becomes

$$\prod_{i=4}^{\ell} a_{p_i} = htua_{p_2}tv. \tag{2.16}$$

Thus h and a_{p_4} have the same initial letter. From Lemma 2.5.2, with $a_{p_2} = yh$ and $a_{p_3} = (tu)y$, we see that $p_4 \equiv p_2 + 2$ $(mod\ 3)$ and $h = ma_{p_2-1}$ for some $m \in A^*$. Since $t \in A^+$, Lemma 2.5.4 applied to (2.16) shows that $\ell \geq 5$. Observe that $p_4 \equiv p_2 + 2$ $(mod\ 3)$ implies that $p_2 = n$, $p_3 = n-2$ and $p_4 = p_5 = n-1$. Thus $h = ma_{n-1}$, $a_n = yh$ and $a_{n-2} = tuy$, and (2.16) becomes

$$(a_{n-1})^2 \prod_{i=6}^{\ell} a_{p_i} = htua_ntv.$$

Since $|a_{n-1}| \leq |h|$, there exists $o \in A^*$ with $|o| = |m|$ such that $h = a_{n-1}o$. Then $a_n = ya_{n-1}o$, and from Lemma 2.5.4 we find $o = m = 1$. Thus $h = a_{n-1}$, and hence $a_{n-1}\prod_{i=6}^{\ell} a_{p_i} = tua_ntv$. However, $a_{n-2} = tuy$, and Lemma 2.5.1(iv) gives the contradiction $n-1 \equiv n-2$ $(mod\ 3)$.

(e)(4)(II)(B) If $y = a_{p_2}h$, $h \in A^*$, then $tu = hz$ and $a_{p_3} = tua_{p_2}h$. Thus $p_2 < p_3$, and $h = 1$ by Lemma 2.5.4. Consequently $y = a_{p_2}$ and $z = tu$, so that (2.15) becomes $\prod_{i=4}^{\ell} a_{p_i} = tuytv = a_{p_3}tv$. Then $p_3 = p_4$, so that $p_2 < p_3 = p_4$. This means that $p_2 = n-2$ and $p_3 = p_4 = p_5 = n-1$. Then $y = a_{n-2}$, $a_{n-1} = tua_{n-2}$ and $a_{n-1}\prod_{i=6}^{\ell} a_{p_i} = tv$. Thus $tu = a_{n-4}$, and hence $v = ua_{n-2}\prod_{i=6}^{\ell} a_{p_i}$.

Since $j = 3$, we have $x = ua_{n-2}t$. We have now described case (iii), and this completes case (e)(4)(II).

(e)(4)(III): We have $a_{p_1} = wu$ and $a_{p_j} = tu$, and (2.12) gives

$$\prod_{i=j+1}^{\ell} a_{p_i} = stustv. \tag{2.17}$$

If $j = 2$ then $s = 1$ and $a_{p_2} = tu$, and (2.17) becomes $\prod_{i=3}^{\ell} a_{p_i} = a_{p_2}tv$. This means that $p_2 = p_3 = p_4$, and consequently $p_1 = n-2$ and $p_2 = p_3 = p_4 = n-1$. Hence $a_{n-1} = tu$, and thus $u\prod_{i=5}^{\ell} a_{p_i} = v$. Since in this case $x = ut$, we have described case (iv).

If $j = 3$ then $s = a_{p_2}$. We have $a_{p_3} = tu$, and (2.17) becomes

$$\prod_{i=4}^{\ell} a_{p_i} = a_{p_2}a_{p_3}a_{p_2}tv.$$

Thus $p_2 = p_4$ and $p_3 = p_5$, which is impossible. This completes case (e)(4)(III).

(e)(4)(IV): In this case $a_{p_1} = wry$, $a_{p_j} = tr$ and $a_{p_{j+1}} = yz$, and we obtain

$$z\prod_{i=j+2}^{\ell} a_{p_i} = strystv = sa_{p_j}ystv = \begin{cases} a_{p_2}ytv & \text{if } j = 2, \\ a_{p_2}a_{p_3}ya_{p_2}tv & \text{if } j = 3 \end{cases}$$

from (2.12). Then z and a_{p_2} have the same initial letter, and hence Lemma 2.5.2, with $a_{p_{j+1}} = yz$ and $a_{p_1} = (wr)y$, shows that $p_2 \equiv p_{j+1} + 2 \ (mod\ 3)$ and that $z = ma_{p_{j+1}-1}$ for some $m \in A^*$. It follows from $p_2 \equiv p_{j+1} + 2 \ (mod\ 3)$ that either $p_2 = n-2$ or $p_{j+1} = n-2$.

(e)(4)(IV)(A) Let $p_2 = n-2$. Then $p_3 = p_{j+1} = p_{j+2} = n-1$, so that $z = ma_{n-2}$, $a_{n-1} = yz$ and

$$z\prod_{i=j+2}^{\ell} a_{p_i} = \begin{cases} a_{n-2}ytv & \text{if } j = 2, \\ a_{n-2}a_{n-1}ya_{n-2}tv & \text{if } j = 3. \end{cases}$$

Therefore, since $|a_{n-2}| \leq |z|$, there exists $o \in A^*$ with $|o| = |m|$ such that $z = a_{n-2}o$. Then $a_{n-1} = ya_{n-2}o$, and Lemma 2.5.4 shows that $o = m = 1$.

Hence $z = a_{n-2}$ and $y = a_{n-4}$, and thus

$$\prod_{i=j+2}^{\ell} a_{p_i} = \begin{cases} a_{n-4}tv & \text{if } j = 2, \\ a_{n-1}a_{n-4}a_{n-2}tv = (a_{n-1})^2 tv & \text{if } j = 3. \end{cases}$$

Therefore, since $p_{j+2} = n - 1$,

$$a_{n-2} \prod_{i=j+3}^{\ell} a_{p_i} = \begin{cases} tv & \text{if } j = 2, \\ a_{n-2}a_{n-1}tv & \text{if } j = 3. \end{cases}$$

If $j = 2$ then $a_{n-2} = a_{p_2} = tr$, and we have $r \prod_{i=5}^{\ell} a_{p_i} = v$. Since $u = ry = ra_{n-4}$, then $x = ra_{n-4}t$, and we have described case (v). If $j = 3$ then $\prod_{i=6}^{\ell} a_{p_i} = a_{n-1}tv$. However, $p_6 = n$ and so we obtain the contradiction $n \equiv n - 1$ $(mod\ 3)$.

(e)(4)(IV)(B) If $p_{j+1} = n - 2$ then $p_2 = p_j = n$ and $p_{j+2} = p_{j+3} = n - 1$. In this case $z = ma_{n-3}$, $a_{n-2} = yz$, $a_n = tr$ and

$$z \prod_{i=j+2}^{\ell} a_{p_i} = \begin{cases} a_n ytv & = & a_{n-3}a_{n-1}ytv & \text{if } j = 2, \\ (a_n)^2 ya_n tv & = & a_{n-3}a_{n-1}a_n ya_n tv & \text{if } j = 3. \end{cases}$$

Therefore, since $|a_{n-3}| \leq |z|$, there exists $o \in X^*$ with $|o| = |m|$ such that $z = a_{n-3}o$. Then $a_{n-2} = ya_{n-3}o$, and Lemma 2.5.4 shows that $o = m = 1$. Therefore $z = a_{n-3}$ and $y = a_{n-5}$, and we obtain

$$a_{n-1} \prod_{i=j+4}^{\ell} a_{p_i} = \begin{cases} a_{n-5}tv & \text{if } j = 2, \\ a_n a_{n-5} a_n tv & \text{if } j = 3. \end{cases}$$

This gives the required contradiction: if $j = 2$ then $n - 1 \equiv n - 5$ $(mod\ 3)$, and if $j = 3$ then $n - 1 \equiv n$ $(mod\ 3)$.

(e)(4)(V): We have $a_{p_1} = wra_{p_{j+1}}y$ and $a_{p_j} = tr$, and (2.12) shows that

$$\prod_{i=j+2}^{\ell} a_{p_i} = ystra_{p_{j+1}}ystv = ysa_{p_j}a_{p_{j+1}}ystv.$$

Lemma 2.5.4 applied to a_{p_1} shows that $y = 1$, and hence

$$\prod_{i=j+2}^{\ell} a_{p_i} = sa_{p_j}a_{p_{j+1}}stv.$$

If $j = 2$ then $s = 1$, and $\prod_{i=4}^{\ell} a_{p_i} = a_{p_2}a_{p_3}tv$. Thus $p_2 = p_4$ and $p_3 = p_5$, which is impossible.

If $j = 3$ then $s = a_{p_2}$, and $\prod_{i=5}^{\ell} a_{p_i} = a_{p_2}a_{p_3}a_{p_4}a_{p_2}tv$. As above we find $p_2 = p_5$, which is impossible. This concludes case (e)(4), and hence case (e), which ends the proof. □

2.6 The varieties $\mathcal{U}$ and $\mathcal{V}$ are distinct.

First recall the definition of the varieties $\mathcal{U}$ and $\mathcal{V}$:

$$\mathcal{U} = [x^5 = 0, x^3yz^3x = (yx)^3zx^3yx]$$

and

$$\mathcal{V} = [x^5 = 0, x^3yz^3x = (yx)^3zx^3yx, x^3yz^3x^2 = (yx)^3zx^3(yx)^2].$$

Recall that $A = \{a, b, c\} \subset X$ where X is denumerable. Since $a_n = a_{n-3}a_{n-1}$, for $n \geq 1$ the identity $x^3yz^3x = (yx)^3zx^3yx$ can be applied to the word $A_na = (a_n)^3a_{n-2}(a_{n-1})^3a_na$ in two obvious ways, giving either $A_{n-1}a$ or $A_{n+1}a$. The first part of the following lemma shows that there is no other way. The second part gives the critical result that A_na contains no factor of the form x^5, $x \in A^+$.

The analogues of these two results for the word A_n follow as immediate consequences:

- if $A_n = wx^5y$ for $n \geq 0$ and $w, x, y \in A^*$ then $x = 1$, and
- if $A_n = w(x^3yz^3x)u$ for $n \geq 2$, $x, y, z \in A^+$ and $w, u \in A^*$ then $w = u = 1$, $x = a_n$, $y = a_{n-2}$, and $z = a_{n-1}$.

Lemma 2.6.1

(i) Suppose that $n \geq 0$. If $A_na = w(x^3yz^3x)u$ for $x, y, z \in A^+$ and $w, u \in A^*$ then $w = 1$, $u = a$ and:

$n = 0 \Rightarrow x = a,\ y = b,\ and\ z = c;$

$n = 1 \Rightarrow x = ba,\ y = c,\ and\ z = a;\ and$

$n \geq 2 \Rightarrow x = a_n,\ y = a_{n-2},\ and\ z = a_{n-1}.$

(ii) For $n \geq 0$, if $A_n a = wx^5y$ for $w, x, y \in A^*$ then $x = 1$.

Proof: (i): The cases $0 \leq n \leq 4$ are easily checked. Let $A_n a = w(x^3yz^3x)u$ where $n \geq 5$, $x, y, z \in A^+$ and $w, u \in A^*$. Then $xu = va$ for some $v \in A^*$, and hence $A_n = wx^3(yz^3v)$. It follows from Lemma 2.5.6 that either:

(a) $w = 1$, $x = a_n$ and $yz^3v = a_{n-2}(a_{n-1})^3 a_n$;

(b) $x = a_{n-1}$ and $yz^3v = a_n$;

(c) $x = da_{n-2}e$, $a_{n-4} = ed$ and $yz^3v = da_{n-2}a_n$ for some $d, e \in A^+$;

(d) $x = de$, $a_{n-1} = ed$ and $yz^3v = da_n$ for some $d, e \in A^+$; or

(e) $x = da_{n-4}e$, $a_{n-2} = ed$ and $yz^3v = da_{n-1}a_n$ for some $d \in A^*$ and $e \in A^+$.

Cases (b), (c) and (d) can be combined to form case (f): $x = de$, $a_{n-1} = ed$ and $yz^3v = da_n$ for some $d \in A^*$ and $e \in A^+$; so only cases (a), (e) and (f) need to be considered.

Case (a): Lemma 2.5.6 can be applied to $a_{n-2}(a_{n-1})^3 a_n = yz^3v$, and in this case the only possibilities are the cases (ii) and (iv) given in the statement of the lemma. We thus obtain the following cases:

(1) $y = a_{n-2}$, $z = a_{n-1}$ and $v = a_n$; and

(2) $z = fg$, $a_{n-1} = gf$ and $v = fa_n$ for $f, g \in A^+$.

(a)(1): In this case $a_n u = xu = va = a_n a$, and hence $u = a$. This gives the required result.

(a)(2): We have $a_n u = xu = va = f a_n a$. Since $a_{n-1} = gf$ then $|f| < |a_n|$. Therefore there exists $h \in A^+$ such that $a_n = fh$, and hence $a_n a = hu$. Thus a_n and h have the same initial letter, and from Lemma 2.5.2, with $a_n = fh$ and $a_{n-1} = gf$, we obtain the contradiction $n \equiv n + 2 \ (mod\ 3)$.

Case (e): In this case $a_{n-4}e(yz^3v) = a_{n-4}e(da_{n-1}a_n) = (a_{n-1})^2 a_n$, which contradicts Lemma 2.5.6 applied to $(a_{n-1})^2 a_n$.

Case (f): In this case $e(yz^3v) = e(da_n) = a_{n-1}a_n$, which contradicts Lemma 2.5.6 applied to $a_{n-1}a_n$. This concludes the proof.

(ii): Again the cases $0 \leq n \leq 4$ are easy to check, and we assume that $n \geq 5$. Let $A_n a = wx^5y$ where $w, x, y \in A^*$, and suppose that $x \neq 1$. Then $xy = ua$ for some $u \in A^*$. Thus $A_n = wx^4u = wx^3(xu)$, and as in part (i) we obtain the following possibilities from Lemma 2.5.6:

(a′) $x = a_n$ and $xu = a_{n-2}(a_{n-1})^3 a_n$;

(e′) $x = da_{n-4}e$, $a_{n-2} = ed$ and $xu = da_{n-1}a_n$ for some $d \in A^*$ and $e \in A^+$; and

(f′) $x = de$, $a_{n-1} = ed$ and $xu = da_n$ for some $d \in A^*$ and $e \in A^+$.

Case (a′): We have $a_{n-2}(a_{n-1})^3 a_n = xu = a_n u$, which gives the contradiction $n - 2 \equiv n \ (mod\ 3)$.

Case (e′): We have $d(a_{n-4}ed)a_n = da_{n-1}a_n = xu = (da_{n-4}e)u$, so that $da_n = u$. Consequently $(da_{n-4}e)y = xy = ua = da_n a$, which gives the contradiction $n - 4 \equiv n \ (mod\ 3)$.

Case (f′): We have $da_n = xu = (de)u$, so that $a_n = eu$. Since $a_{n-1} = ed$, we thus obtain the contradiction $n \equiv n - 1 \ (mod\ 3)$. □

The next result shows that the members of X^+ that are equivalent to the word $A_0 a = a^3bc^3a^2$ under the laws of $\mathcal{U}$ are precisely the words $A_n a$, $n \geq 0$.

Theorem 2.6.2 $(A_0a)\rho(\mathcal{U}) = \{A_na : n \geq 0\}$.

Proof: Let $\rho(\mathcal{U})' = \{(x^5, x^5y), (x^5, yx^5), (x^3yz^3x, (yx)^3zx^3yx) : x, y, z \in X^+\}$, so that $\rho(\mathcal{U})$ is the congruence on X^+ generated by $\rho(\mathcal{U})'$. It is clear that $A_na \in (A_0a)\rho(\mathcal{U})$ for all $n \geq 0$.

Let $t \in (A_0a)\rho(\mathcal{U})$. By Result 1.3.3, for some $m \geq 0$ there exist $r_i, s_i \in X^*$ and $(d_i, e_i) \in \rho(\mathcal{U})' \cup (\rho(\mathcal{U})')^{-1}$ for $0 \leq i \leq m$ such that $A_0a = r_0d_0s_0$, $t = r_me_ms_m$ and $r_{i-1}e_{i-1}s_{i-1} = r_id_is_i$ for $1 \leq i \leq m$. We will show by induction that $r_i = 1$, $s_i = a$ and $(d_i, e_i) \in \{(A_{n+1}, A_n), (A_n, A_{n+1}) : n \geq 0\}$ for all i, $1 \leq i \leq m$. This will imply that $t = A_na$ for some $n \geq 0$, and conclude the proof.

Let $i = 0$. If $d_0 = rx^5s$ where $x \in X^+$ and $r, s \in X^*$ then $A_0a = r_0(rx^5s)s_0$, and Lemma 2.6.1(ii) gives the contradiction $x = 1$. Therefore $d_0 = x^3yz^3x$ for $x, y, z \in X^+$. Then $A_0a = r_0(x^3yz^3x)s_0$, and from Lemma 2.6.1(i) we obtain $r_0 = 1$, $s_0 = a$ and $d_0 = A_0$, which implies that $e_0 = A_1$.

Suppose now that $1 \leq i \leq m$. By induction we have $r_{i-1} = 1$, $s_{i-1} = a$ and $e_{i-1} = A_n$ for some $n \geq 0$. Then $r_id_is_i = A_na$. As above, Lemma 2.6.1(ii) shows that $d_i = x^3yz^3x$ for some $x, y, z \in X^+$, and Lemma 2.6.1(i) then shows that $r_i = 1$, $s_i = a$, and:

$n = 0 \Rightarrow x = a$, $y = b$, and $z = c$;

$n = 1 \Rightarrow x = ba$, $y = c$, and $z = a$; and

$n \geq 2 \Rightarrow x = a_n$, $y = a_{n-2}$, and $z = a_{n-1}$.

Thus $d_i = A_n$, so that either (a) $e_i = A_{n+1}$ or (b) $n \geq 1$ and $e_i = u^3vw^3u$ where $u, v, w \in X^+$ and

$n = 1 \Rightarrow ba = vu$, $c = w$, and $a = u$, and

$n \geq 2 \Rightarrow a_n = vu$, $a_{n-2} = w$, and $a_{n-1} = u$.

Case (a): If $e_i = A_{n+1}$ then $(d_i, e_i) = (A_n, A_{n+1})$, as desired.

Case (b): If $n = 1$ then $e_i = u^3vw^3u = a^3bc^3a = A_0$. If $n \geq 2$ then $a_n = va_{n-1}$, so that $v = a_{n-3}$. Then $e_i = u^3vw^3u = (a_{n-1})^3 a_{n-3}(a_{n-2})^3 a_{n-1} = A_{n-1}$, and in both cases $(d_i, e_i) = (A_n, A_{n-1})$, as desired. □

Corollary 2.6.3 $(A_0)\rho(\mathcal{U}) = (A_0)\rho(\mathcal{V}) = \{A_n : n \geq 0\}$.

Proof: The result $(A_0)\rho(\mathcal{U}) = \{A_n : n \geq 0\}$ follows directly from Theorem 2.6.2. A straightforward modification of the proof of Theorem 2.6.2 establishes $(A_0)\rho(\mathcal{V}) = \{A_n : n \geq 0\}$. □

The proof that $\mathcal{U} \neq \mathcal{V}$ can now be given.

Theorem 2.6.4 *The varieties $\mathcal{U}$ and $\mathcal{V}$ are distinct.*

Proof: Clearly $A_1ba \in (A_0a)\rho(\mathcal{V})$, but if $A_1ba \in (A_0a)\rho(\mathcal{U})$ then Theorem 2.6.2 shows that $A_1ba = A_na$ for some $n \geq 0$. Then $A_na = (ba)^3ca^3(ba)^2$, and Lemma 2.6.1(i) shows that $ba = a$, which is impossible. Therefore $(A_0a)\rho(\mathcal{U}) \neq (A_0a)\rho(\mathcal{V})$, and the result follows from Theorem 1.1.6. □

Remark 2.6.5

- It is also of interest to reformulate Almeida's question in terms of semigroups of some fixed nil index: that is, for a given $n \geq 1$ is the mapping $\mathcal{W} \mapsto \mathcal{W} \cap \mathcal{N}$, $\mathcal{W} \in \mathcal{L}(\mathcal{N}il_n)$, an isomorphism onto $\mathcal{G}(\mathcal{N} \cap \mathcal{N}il_n)$? In this case it follows as in Section 2.2 that the mapping is surjective for all $n \geq 1$, but not injective for $n \geq 5$, as shown by the pair of varieties described above. This question remains open concerning smaller values of n.

- A semigroup S is said to be *locally finite* if every finitely generated subsemigroup of S is finite. A class $\mathcal{C}$ of semigroups is said to be locally finite if every member of $\mathcal{C}$ is locally finite. It is easy to see that $\mathcal{N}$ is locally finite.

The following was brought to the attention of the author by M. Volkov. An alternative method of establishing the non-injectivity of the mapping would be to produce a variety $\mathcal{V} \in \mathcal{L}(\mathcal{N}il)$ which, although not itself locally finite, were such that its locally finite members form a variety $\mathcal{U}$. Then $\mathcal{U} \cap \mathcal{N} = \mathcal{V} \cap \mathcal{N}$ since $\mathcal{N}$ is locally finite, although $\mathcal{U} \neq \mathcal{V}$. However, by a result of Sapir [28, Theorem 3.13], the locally finite members of a finitely based variety $\mathcal{V} \in \mathcal{L}(\mathcal{N}il)$ form a subvariety of $\mathcal{V}$ only if $\mathcal{V}$ is locally finite, and thus this method cannot work for a finitely based variety. □

Chapter 3

Biidentities

In this chapter we look at e-varieties. We begin in Section 3.1 by expanding the techniques of Auinger [7, 8] for locally inverse e-varieties to provide a unified Birkhoff-style theory for the whole lattice of e-varieties for which nonmonogenic bifree objects exist; that is, for e-varieties contained in **ES** or $L\mathbf{I}$.

We then give an alternative approach to this material in Section 3.2, based on the techniques of Kaďourek and Szendrei in [27] for E-solid e-varieties. When the paper [13] was originally submitted for publication, it included the material presented in Section 3.2. The referee made suggestions that led to this material being replaced with the contents of Section 3.1. Both approaches have been incorporated into this thesis, because of their substantial difference.

In Section 3.3 we consider locally E-solid e-varieties. In Section 3.3.1 we show that the results of Kaďourek [25] concerning the existence of trifree objects in locally orthodox e-varieties can be extended: trifree objects exist in every e-variety of locally E-solid semigroups. In Section 3.3.2 we construct an example, modelled on the example with which Yeh [43] proved that nonmonogenic bifree objects exist precisely in E-solid and locally inverse e-varieties, that enables us to prove that in fact trifree objects exist in an e-variety $\mathbf{V}$ of regular semigroups if and only if $\mathbf{V}$ consists of locally E-solid semigroups.

Finally, in Section 3.4 we outline a theory of "n-free" objects, indicating how analogues of the concept of a free object can be defined for any e-variety.

3.1 E-varieties of E-solid or locally inverse semigroups.

First, recall from Notation 1.3.6 that if an element k lies in a subgroup of a semigroup S then the unique $\mathcal{H}$-related inverse of k is denoted by k^{-1} and the unique idempotent $\mathcal{H}$-related to k is denoted by k°. Recall also that the sandwich set $S(a,b)$ of elements a,b of a regular semigroup S is the set $bV(ab)a$.

Lemma 3.1.1 *Let S be a regular semigroup and suppose that k lies in a subgroup of S. Then $S(k,k) = \{k^\circ\}$. Furthermore, if T is a regular subsemigroup of S and $k \in T$ then $k^{-1}, k^\circ \in T$ and $k^{-1}\ \mathcal{H}\ k^\circ\ \mathcal{H}\ k$ in T.*

Proof: We have $S(k,k) = S(k^{-1}k, kk^{-1}) = S(k^\circ, k^\circ) = \{k^\circ\}$ by Result 1.3.15(ii),(v).

Now consider a regular subsemigroup T of S with $k \in T$. Let $(k^2)' \in V(k^2) \cap T$. Then $k(k^2)'k \in S(k,k) \cap T$, so that $k^\circ = k(k^2)'k \in T$. Also,

$$k(k^2)'k^2 = k^{-1}k^2(k^2)'k^2 = k^{-1}k^2 = k = k^2(k^2)'k^2k^{-1} = k^2(k^2)'k,$$

and hence $k^\circ = k(k^2)'k\ \mathcal{H}\ k$ in T.

Let $k' \in V(k) \cap T$. Then

$$k^\circ k' k^\circ = k^{-1}kk'kk^{-1} = k^{-1}kk^{-1} = k^{-1},$$

so that $k^{-1} \in T$. Moreover, $k^{-1}\ \mathcal{H}\ k^\circ$ in T since we also have $k^\circ = kk^{-1} = k^{-1}k$. □

We start by fixing a particular construction of the least regular subsemigroups in E-solid and locally inverse semigroups, taken from Result 1.3.34.

Construction 3.1.2 Suppose that A is a subset of an E-solid semigroup U. Let $a' \in V(a)$ for each $a \in A$, and write $A' = \{a' : a \in A\}$. If $e, f \in E(U)$ then ef is in a subgroup of U by Result 1.3.11; and by Result 1.3.34, the least regular subsemigroup of U containing $A \cup A'$ is $P = \bigcup_{i \geq 0} P_{2i+1}$, where

$$
\begin{aligned}
P_0 &= A \cup A' \\
P_1 &= \langle P_0 \rangle \\
&\vdots \\
P_{2i} &= \{(ef)^{-1} : e, f \in E(P_{2i-1})\} \cup P_{2i-1} \\
P_{2i+1} &= \langle P_{2i} \rangle \\
&\vdots
\end{aligned}
$$

Let s be a sandwich operation on U (see Remark 1.3.17) *such that $s(u,v) = v(uv)^{-1}u$ whenever uv is a group element of U, and $s(u,v) \in P$ whenever $u, v \in P$.* □

Construction 3.1.3 For any locally inverse semigroup V with subset A, let $a' \in V(a)$ for all $a \in A$ and write $A' = \{a' : a \in A\}$. Again by Result 1.3.34, the least regular subsemigroup of V containing $A \cup A'$ is $Q = \bigcup_{i \geq 0} Q_{2i+1}$, where

$$
\begin{aligned}
Q_0 &= A \cup A' \\
Q_1 &= \langle Q_0 \rangle \\
&\vdots \\
Q_{2i} &= \{s(a,b) : a, b \in Q_{2i-1}\} \cup Q_{2i-1} \\
Q_{2i+1} &= \langle Q_{2i} \rangle \\
&\vdots
\end{aligned}
$$

Remember that $s(u,v)$ is the unique member of $S(u,v)$ for each $u, v \in V$; *so the sandwich operation s is also an operation on Q.* □

Recall that by a *binary semigroup* is meant a type $(2,2)$ algebra where one of the binary operations is associative. We will write simply (S,s) to denote a binary semigroup $(S,\cdot,s)$.

As in Notation 1.3.20, let X be a nonempty set, with a disjoint bijective copy $X' = \{x' : x \in X\}$. We write $\overline{X} = X \cup X'$. Let $(F_{(2,2)}(\overline{X}), s)$ denote the free binary semigroup on $\overline{X}$. Recall that for a set Y the free semigroup and free monoid on Y are denoted by Y^+ and Y^* respectively.

Every word $u \in F_{(2,2)}(\overline{X})$ can be factorized uniquely as

$$u = u_0 w_1 u_1 \ldots w_n u_n \tag{3.1}$$

where $u_i \in (\overline{X})^*$ for $0 \le i \le n$, and $w_i = s(a_i, b_i)$ for some $a_i, b_i \in F_{(2,2)}(\overline{X})$, $1 \le i \le n$.

The semigroup $F_{(2,2)}(\overline{X})$ is embedded in the free semigroup F on the alphabet consisting of the set $\overline{X}$ together with the symbols "$s($", "," and ")"; and so, given a word $u \in F_{(2,2)}(\overline{X})$, we may define the length $|u|$ of u to be the usual length of u considered as a member of F.

We denote the inverse unary operation on the free group $F_{\mathcal{G}}(X)$ by $^{-1}$. We may assume $\overline{X} \subseteq F_{\mathcal{G}}(X)$.

Define a sandwich operation s on $F_{\mathcal{G}}(X)$ by $s(a,b) = 1$ for all $a, b \in F_{\mathcal{G}}(X)$. Then there is a binary semigroup homomorphism $F_{(2,2)}(\overline{X}) \to F_{\mathcal{G}}(X)$ extending the natural injection $\overline{X} \to F_{\mathcal{G}}(X)$. We denote the image under this homomorphism of a word $u \in F_{(2,2)}(\overline{X})$ by $\overline{u}$, so that if $u = u_0 w_1 u_1 \ldots w_n u_n$ is the factorization (3.1) of u as described above then $\overline{u}$ is the usual group-reduced form of $u_0 \ldots u_n \in (\overline{X})^+$. In particular, we have $\overline{s(u,v)} = 1$ for every $u, v \in F_{(2,2)}(\overline{X})$. The congruence on $F_{(2,2)}(\overline{X})$ associated with this homomorphism is the least group congruence on $F_{(2,2)}(\overline{X})$, and is denoted by σ. Let $R(X) = \{u \in F_{(2,2)}(\overline{X}) : \overline{u} = 1\}$.

We now construct a subsemigroup of $F_{(2,2)}(\overline{X})$ in the same manner as the semigroups P and Q of constructions 3.1.2 and 3.1.3 respectively.

Construction 3.1.4 Let $W(X) = \bigcup_{i \geq 0} W_{2i+1}$, where

$$\begin{aligned} W_0 &= \overline{X} \\ W_1 &= \langle W_0 \rangle \\ &\vdots \\ W_{2i} &= \{s(a,b) : a, b \in W_{2i-1} \cap R(X)\} \cup W_{2i-1} \\ W_{2i+1} &= \langle W_{2i} \rangle \\ &\vdots \end{aligned}$$

□

So $W(X)$ is the least subsemigroup W of $F_{(2,2)}(\overline{X})$ such that $\overline{X} \subseteq W$ and $s(u,v) \in W$ whenever $u, v \in W \cap R(X)$. Let $K(X) = W(X) \cap R(X)$.

Let us now define a unary operation $'$ on $W(X)$, to be such that $\overline{w'} = (\overline{w})^{-1}$ for every $w \in W(X)$. We use induction on $|w|$. For the initial case $|w| = 1$, we already have w' for each $w \in X$, and we define $(w')' = w$. Suppose that $|w| \geq 2$. If $w = s(u,v)$ for $u, v \in K(X)$, let $w' = w$. Then $\overline{w'} = 1 = (\overline{w})^{-1}$. Otherwise, let $w = u_0 w_1 u_1 \ldots w_n u_n$ be the factorization of w as in (3.1). Define $u \in W(X)$ to be $u = x$ if $u_0 = xy$ for some $x \in \overline{X}$ and $y \in (\overline{X})^*$, and $u = w_1$ if $u_0 = 1$. Let $v \in F_{(2,2)}(\overline{X})$ be such that $w = uv$. Then $u, v \in W(X)$ and $|u|, |v| < |w|$ so we may assume that $u', v' \in W(X)$ are defined, with $\overline{u'u} = \overline{vv'} = 1$. Now $s(u'u, vv') \in W(X)$, and we define $w' = v's(u'u, vv')u'$. Clearly $\overline{w'} = (\overline{w})^{-1}$.

Consider the partial binary semigroup congruence η on $W(X)$ generated by

$$\{(xx'x, x), (x'xx', x') : x \in X\} \quad \cup$$

$$\{(s(u,v), s(u,v)s(u,v)), (uv, us(u,v)v), (s(u,v), vv's(u,v)u'u) : u, v \in K(X)\}.$$

(That is; η respects the partial binary operation s on $W(X)$.)

Remark 3.1.5 As a binary relation on $F_{(2,2)}(\overline{X})$, the congruence η is contained in the least group congruence σ on $F_{(2,2)}(\overline{X})$. Therefore if ρ is a congruence on $W(X)$ such that $\eta \subseteq \rho \subseteq \sigma$ then $\overline{w} = 1$ whenever $w\rho \in E(W(X)/\rho)$. □

The following results detail properties of the congruence η.

Lemma 3.1.6 *Let $u, v \in K(X)$. If $u^*\eta \in V(u\eta)$ and $v^*\eta \in V(v\eta)$ for some $u^*, v^* \in W(X)$ then $s(u,v)\eta \in S(u^*u\eta, vv^*\eta)$.*

Proof: We have

$$\begin{aligned} s(u,v)u^*uvv^*s(u,v)\eta &= vv's(u,v)u'u(u^*uvv^*)vv's(u,v)u'u\eta \\ &= vv's(u,v)u'uvv's(u,v)u'u\eta \\ &= s(u,v)\eta. \end{aligned}$$

Also,

$$\begin{aligned} vv^*s(u,v)u^*u\eta &= vv^*vv's(u,v)u'uu^*u\eta \\ &= vv's(u,v)u'u\eta \\ &= s(u,v)\eta, \end{aligned}$$

so that

$$\begin{aligned} u^*uvv^*s(u,v)u^*uvv^*\eta &= u^*us(u,v)vv^*\eta \\ &= u^*uvv^*\eta. \end{aligned}$$

Therefore $s(u,v)\eta \in V(u^*uvv^*\eta)$, and hence

$$\begin{aligned} s(u,v)\eta &= vv^*s(u,v)u^*u\eta \\ &\in vv^*\eta V(u^*uvv^*\eta)u^*u\eta \\ &= S(u^*u\eta, vv^*\eta). \end{aligned}$$

□

Lemma 3.1.6 gives enough information to show that $W(X)/\eta$ is regular.

Lemma 3.1.7

(i) If $w \in W(X)$ then $w'\eta \in V(w\eta)$; that is, the semigroup $W(X)/\eta$ is regular.

(ii) Suppose that $\theta : W(X) \to S$ is a semigroup homomorphism such that $x'\theta \in V(x\theta)$ for all $x \in X$, and $s(u,v)\theta \in S(u\theta, v\theta)$ whenever $u, v \in K(X)$. Then $w'\theta \in V(w\theta)$ for all $w \in W(X)$, and $\eta \subseteq \theta \circ \theta^{-1}$.

Proof: (i) We use induction on $|w|$, for $w \in W(X)$. The initial case $|w| = 1$ is clear. Let $|w| \geq 2$. If $w = s(u,v)$ then $w' = w$ by definition, and $w\eta \in E(W(X)/\eta)$. Therefore $w'\eta = w\eta \in V(w\eta)$. Otherwise by definition $w' = v's(u'u, vv')u'$ where $w = uv$, $u \in \overline{X} \cup \{s(a,b) : a, b \in K(X)\}$, and $u, v \in W(X)$. Since $|u|, |v| < |w|$, we have $u'\eta \in V(u\eta)$ and $v'\eta \in V(v\eta)$. So by Lemma 3.1.6

$$\begin{aligned} s(u'u, vv')\eta &\in S(u'u\eta, vv'\eta) \\ &= vv'\eta V(u'uvv'\eta)u'u\eta. \end{aligned}$$

Therefore

$$\begin{aligned} ww'w\eta &\in uvv'vv'\eta V(u'uvv'\eta)u'uu'uv\eta \\ &= uu'uvv'\eta V(u'uvv'\eta)u'uvv'v\eta \\ &= \{uu'uvv'v\eta\} \\ &= \{w\eta\}. \end{aligned}$$

Also, writing $s(u'u, vv')\eta = vv'yu'u\eta$ for $y\eta \in V(u'uvv'\eta)$,

$$\begin{aligned} w'ww'\eta &= v'vv'yu'uu'uvv'vv'yu'uu'\eta \\ &= v'vv'yu'uu'\eta \\ &= w'\eta. \end{aligned}$$

Thus $w'\eta \in V(w\eta)$.

(ii) We again use induction on $|w|$ for $w \in W(X)$. The initial case $|w| = 1$ is clear. Suppose that $|w| \geq 2$. If $w = s(u, v)$ then $w' = w$, and $w\theta \in E(S)$. Therefore $w'\theta = w\theta \in V(w\theta)$. Otherwise $w' = v's(u'u, vv')u'$ where $w = uv$, $u \in \overline{X} \cup \{s(a, b) : a, b \in K(X)\}$, and $u, v \in W(X)$. Since $|u|, |v| < |w|$, we have $u'\theta \in V(u\theta)$ and $v'\theta \in V(v\theta)$. Now

$$\begin{aligned} s(u'u, vv')\theta &\in S(u'u\theta, vv'\theta) \\ &= S(u\theta, v\theta), \end{aligned}$$

so that $w'\theta \in V(w\theta)$ by Result 1.3.15(iii). This implies that $\eta \subseteq \theta \circ \theta^{-1}$. □

The following corollary is evident from Lemma 1.3.15(ii), Lemma 3.1.7(i), and Lemma 3.1.6.

Corollary 3.1.8 *If $u, v \in K(X)$ then $s(u, v)\eta \in S(u\eta, v\eta)$.* □

Notice that by Lemma 3.1.7 and Corollary 3.1.8 the congruence η is the least binary semigroup congruence ρ on $W(X)$ such that: (i) $x'\rho \in V(x\rho)$ for all $x \in X$, (ii) $s(u, v)\rho \in S(u\rho, v\rho)$ for all $u, v \in K(X)$, and (iii) $w'\rho \in V(w\rho)$ for all $w \in W(X)$.

The next lemma is now a natural consequence of the similarity of constructions 3.1.2, 3.1.3 and 3.1.4.

Lemma 3.1.9 *Let Y denote the E-solid semigroup U or the locally inverse semigroup V of constructions 3.1.2 and 3.1.3, with A, A' and s as given in the constructions. Let T denote the regular subsemigroup P or Q respectively. Let $\alpha : F_{(2,2)}(A \cup A') \to Y$ be the binary semigroup homomorphism with respect to the operations s such that $a\alpha = a$ for all $a \in A \cup A'$. Then $T = \{w\alpha : w \in W(A)\}$.*

Proof: Define α_1 to be the restriction of α to $W(A)$ and consider constructions 3.1.2, 3.1.3 and 3.1.4. We have $(A \cup A')\alpha_1 = A \cup A' \subseteq T$ and $W_1\alpha_1 \subseteq T$.

Assuming that $W_{2i-1}\alpha_1 \subseteq T$ for $i \geq 1$, it follows, since s is a binary operation on T, that $W_{2i}\alpha_1 \subseteq T$. Hence $W(A)\alpha_1 \subseteq T$.

Since $a'\alpha_1 \in V(a\alpha_1)$ for all $a \in A$, and $s(u,v)\alpha_1 \in S(u\alpha_1, v\alpha_1)$ for all $u, v \in K(A)$, then by Lemma 3.1.7(ii) the subsemigroup $W(A)\alpha_1$ of Y is regular. But T is the least regular subsemigroup of Y that contains $A \cup A'$, so $T = W(A)\alpha_1$.□

Recall that the free regular unary E-solid semigroup is denoted by $F_{\mathcal{ES}}(X)$, and the free regular unary locally inverse semigroup is denoted by $F_{\mathcal{LI}}(X)$. (See Notation 1.3.13 and the notation given in Remark 1.3.36.) Recall that the bifree object on a set X in an e-variety $\mathbf{V}$ is denoted by $bF_{\mathbf{V}}(X)$. By Result 1.3.37 the least regular subsemigroups of $F_{\mathcal{ES}}(X)$ and $F_{\mathcal{LI}}(X)$ containing the set $\overline{X}$ are isomorphic to $bF_{\mathbf{ES}}(X)$ and $bF_{\mathbf{LI}}(X)$ respectively, and we therefore have the following corollary to Lemma 3.1.9.

Corollary 3.1.10 *In the statement of Lemma 3.1.9, if $A = X$, $A' = X'$, and $Y = F_{\mathcal{ES}}(X)$ or $Y = F_{\mathcal{LI}}(X)$ then $\{w\alpha : w \in W(X)\}$ is isomorphic to $bF_{\mathbf{ES}}(X)$ or $bF_{\mathbf{LI}}(X)$ respectively.* □

Therefore there exist (binary semigroup) congruences $\alpha_1 \circ {\alpha_1}^{-1}$ on $W(X)$ such that $W(X)/(\alpha_1 \circ {\alpha_1}^{-1})$ is bifree on X in **ES** or *LI*. We will introduce two such congruences after a preliminary lemma.

For $u, v \in K(X)$, let $\overline{s}(u,v) = s(x,x)$ where $x = s(u,u)vs(u,u)$. So $\overline{s}(u,v) \in W(X)$. Recall that σ is the least group congruence on $F_{(2,2)}(\overline{X})$.

Lemma 3.1.11 *Let ρ be a congruence on $W(X)$ such that $\eta \subseteq \rho \subseteq \sigma$.*

(i) $E(W(X)/\rho) = \{s(w,w)\rho : w \in K(X)\}$.

(ii) Each idempotent in each local submonoid of the semigroup $W(X)/\rho$ is of the form $\overline{s}(u,v)\rho$ for some $u, v \in K(X)$.

Proof: (i) If $w \in K(X)$ then $s(w,w)\eta \in S(w\eta, w\eta)$ by Corollary 3.1.8. Therefore $s(w,w)\rho \in S(w\rho, w\rho) \subseteq E(W(X)/\rho)$ by Result 1.3.15(i),(vi) since $\eta \subseteq \rho$. Conversely if $w\rho \in E(W(X)/\rho)$ then $w \in K(X)$ by Remark 3.1.5, and as above $s(w,w)\rho \in S(w\rho, w\rho)$. But $S(w\rho, w\rho) = \{w\rho\}$ by Result 1.3.15(v).

(ii) Let $e \in E(W(X)/\rho)$, and let $f \in E(e \cdot W(X)/\rho \cdot e)$. By part (i) there exist $u \in K(X)$ and $v \in W(X)$ such that $e = s(u,u)\rho$ and $f = s(u,u)vs(u,u)\rho$. Since $f \in E(W(X)/\rho)$ then $s(u,u)vs(u,u) \in K(X)$ by Remark 3.1.5. This means that $v \in K(X)$. Thus $f = s(u,u)vs(u,u)\rho = \overline{s}(u,v)\rho$ by part (i). □

Let $\rho_{\mathbf{ES}}$ denote the partial binary semigroup congruence on $W(X)$ generated by

$$\begin{aligned} \eta \ &\cup \ \{(w, ws(w,w)), (w, s(w,w)w) : w \in K(X)\} \\ &\cup \ \{(s(u,v), vs(uv,uv)(uv)'s(uv,uv)u) : u, v \in K(X)\}, \end{aligned}$$

and let $\rho_{\mathbf{LI}}$ denote the partial binary semigroup congruence on $W(X)$ generated by

$$\eta \cup \{(\overline{s}(u,v)\overline{s}(u,w), \overline{s}(u,w)\overline{s}(u,v)) : u, v, w \in K(X)\}.$$

Our aim is to show that the factor semigroups $W(X)/\rho_{\mathbf{ES}}$ and $W(X)/\rho_{\mathbf{LI}}$ are isomorphic to $bF_{\mathbf{ES}}(X)$ and $bF_{\mathbf{LI}}(X)$ respectively. We begin with some preliminary lemmas. Recall that the self-conjugate core of a regular semigroup S is denoted $C_\infty(S)$.

Lemma 3.1.12

(i) $C_\infty(W(X)/\rho_{\mathbf{ES}}) \subseteq \{w\rho_{\mathbf{ES}} : w \in K(X)\}$.

(ii) Let $\theta : W(X) \to S$ *be a semigroup homomorphism, where* S *is regular and* $\eta \subseteq \theta \circ \theta^{-1}$. *Then* $w\theta \in C_\infty(S)$ *for every* $w \in K(X)$.

Proof: (i) Observe that $\rho_{\mathbf{ES}}$, considered as a binary relation on $F_{(2,2)}(\overline{X})$, is contained in the group congruence σ. Therefore

$$E(W(X)/\rho_{\mathbf{ES}}) = \{s(w,w)\rho_{\mathbf{ES}} : w \in K(X)\}$$

by Lemma 3.1.11(i), and hence $C(W(X)/\rho_{\mathbf{ES}}) \subseteq \{w\rho_{\mathbf{ES}} : w \in K(X)\}$. In addition, $\overline{uv} = 1$ whenever $u\rho_{\mathbf{ES}} \in V(v\rho_{\mathbf{ES}})$, and consequently

$$C_\infty(W(X)/\rho_{\mathbf{ES}}) \subseteq \{w\rho_{\mathbf{ES}} : w \in K(X)\}.$$

(ii) The proof is by induction on $|w|$, for $w \in K(X)$. The result is trivial for the initial case $|w| = 1$, so assume that $|w| \geq 2$, and that $u\theta \in C_\infty(S)$ whenever $u \in K(X)$ satisfies $|u| < |w|$. Let $w = u_0w_1u_1 \dots w_nu_n$ be the factorization of w according to (3.1). Now $\overline{u_0 \dots u_n} = \overline{w} = 1$, so if $|u_0 \dots u_n| < |w|$ then $(u_0 \dots u_n)\theta \in C_\infty(S)$ by the induction hypothesis, and hence $w\theta \in C_\infty(S)$. Otherwise $w = u_0 \dots u_n \in \langle\overline{X}\rangle$. In this case let $w = y_1 \dots y_m$, where $y_1, \dots, y_m \in \overline{X}$. Then there exists i for which $y_i = y'_m$ if $y_m \in X$, and $y_i = x$ if $y_m = x'$ for some $x \in X$; and $\overline{y_1 \dots y_{i-1}} = \overline{y_{i+1} \dots y_{m-1}} = 1$. Then $(y_1 \dots y_{i-1})\theta$, $(y_{i+1} \dots y_{m-1})\theta \in C_\infty(S)$ by the induction hypothesis; and hence $w\theta \in C_\infty(S)$. □

Remark 3.1.13

(i) Lemma 3.1.12 parts (i) and (ii) together show that

$$C_\infty(W(X)/\rho_{\mathbf{ES}}) = \{w\rho_{\mathbf{ES}} : w \in K(X)\}.$$

(ii) Suppose that S is a regular semigroup and $\theta : W(X) \to S$ is a semigroup homomorphism such that $\eta \subseteq \theta \circ \theta^{-1}$ and $u\theta$ lies in a subgroup of S whenever $u \in K(X)$. Let $u \in K(X)$. Now Corollary 3.1.8 shows that $s(u,u)\theta \in S(u\theta, u\theta)$, and therefore, by Lemma 3.1.1, we have $s(u,u)\theta = (u\theta)^\circ$. Thus if also $v \in K(X)$ then $uv\theta$ lies in a subgroup of S, and

$s(uv,uv)\theta = (uv\theta)^\circ$. Then $s(uv,uv)(uv)'s(uv,uv)\theta = (uv\theta)^{-1}$ by the proof of Lemma 3.1.1, and we have proved that $\rho_{\mathbf{ES}} \subseteq \theta \circ \theta^{-1}$ if and only if $s(u,v)\theta = v\theta(uv\theta)^{-1}u\theta$ for every $u, v \in K(X)$. □

Lemma 3.1.14

(i) The semigroup $W(X)/\rho_{\mathbf{ES}}$ is a member of **ES**.

*(ii) The semigroup $W(X)/\rho_{L\mathbf{I}}$ is a member of L***I**.

Proof: (i) If $w\rho_{\mathbf{ES}} \in C_\infty(W(X)/\rho_{\mathbf{ES}})$ then $w \in K(X)$ by Lemma 3.1.12. Then $w\rho_{\mathbf{ES}}$ is $\mathcal{H}$-related to the idempotent $s(w,w)\rho_{\mathbf{ES}}$, so $w\rho_{\mathbf{ES}}$ is in a subgroup of $W(X)/\rho_{\mathbf{ES}}$ (by Result 1.3.5). The result now follows from Result 1.3.11(iv).

(ii) By Lemma 3.1.11 a typical idempotent in $W(X)/\rho_{L\mathbf{I}}$ is of the form $s(u,u)\rho_{L\mathbf{I}}$ for $u \in K(X)$, and a typical idempotent in the local submonoid

$$s(u,u)\rho_{L\mathbf{I}}.W(X)/\rho_{L\mathbf{I}}.s(u,u)\rho_{L\mathbf{I}}$$

is $\bar{s}(u,v)\rho_{L\mathbf{I}}$ for $v \in K(X)$. Therefore, by Result 1.3.19, the definition of $\rho_{L\mathbf{I}}$ shows that $W(X)/\rho_{L\mathbf{I}} \in L\mathbf{I}$. □

We should note the following.

Remark 3.1.15 If $u, v \in K(X)$ then, as in the proof of Lemma 3.1.14, $uv\rho_{\mathbf{ES}}$ lies in a subgroup of $W(X)/\rho_{\mathbf{ES}}$. Then $s(u,v)\rho_{\mathbf{ES}} = v\rho_{\mathbf{ES}}(uv\rho_{\mathbf{ES}})^{-1}u\rho_{\mathbf{ES}}$ by Remark 3.1.13. By Corollary 3.1.8, Lemma 3.1.14, and Result 1.3.18 we have $s(u,v)\rho_{L\mathbf{I}} = s(u\rho_{L\mathbf{I}}, v\rho_{L\mathbf{I}})$. □

The main result of this section can now be proved. Recall Definition 1.3.21, which defines a matched mapping.

Theorem 3.1.16 *Let $\mathbf{V} = \mathbf{ES}$ or $\mathbf{V} = L\mathbf{I}$. Let $\iota_{\mathbf{V}} : \overline{X} \to W(X)/\rho_{\mathbf{V}}$ be given by $x\iota_{\mathbf{V}} = x\rho_{\mathbf{V}}$, $x \in \overline{X}$. Then $(W(X)/\rho_{\mathbf{V}}, \iota_{\mathbf{V}})$ is the bifree object in* **V** *on X.*

Proof: Certainly $\iota_{\mathbf{V}}$ is matched, and $W(X)/\rho_{\mathbf{V}} \in \mathbf{V}$ by Lemma 3.1.14. Suppose that $S \in \mathbf{V}$, and let $\phi : \overline{X} \to S$ be a matched mapping. If $\mathbf{V} = \mathbf{ES}$ let s be a sandwich operation on S satisfying $s(a,b) = b(ab)^{-1}a$ whenever $a, b \in S$ are such that ab is a group element of S; if $\mathbf{V} = \mathbf{LI}$ let s be the sandwich operation on S. Let $\overline{\varphi} : F_{(2,2)}(\overline{X}) \to S$ be the binary semigroup homomorphism with respect to s that extends ϕ. Let φ_1 be the restriction of $\overline{\varphi}$ to $W(X)$. By Lemma 3.1.7(ii), we have $\eta \subseteq \varphi_1 \circ (\varphi_1)^{-1}$.

Suppose $\mathbf{V} = \mathbf{LI}$. If $u, v, w \in K(X)$ then $s(u,u)\varphi_1 \in E(S)$ and

$$s(u,u)vs(u,u)\varphi_1, s(u,u)ws(u,u)\varphi_1 \in s(u,u)\varphi_1.S.s(u,u)\varphi_1.$$

Then $\overline{s}(u,v)\varphi_1, \overline{s}(u,w)\varphi_1 \in E(s(u,u)\varphi_1.S.s(u,u)\varphi_1)$. Since $S \in \mathbf{LI}$ then Result 1.3.19 shows that $\overline{s}(u,v)\overline{s}(u,w)\varphi_1 = \overline{s}(u,w)\overline{s}(u,v)\varphi_1$; that is, $\rho_{\mathbf{LI}} \subseteq \varphi_1 \circ (\varphi_1)^{-1}$.

Suppose $\mathbf{V} = \mathbf{ES}$. If $w \in K(X)$ then $w\varphi_1 \in C_\infty(S)$ by Lemma 3.1.12(ii), and then $w\varphi_1$ lies in a subgroup of S by Lemma 1.3.11 since $S \in \mathbf{ES}$. Therefore for $u, v \in K(X)$ the element $uv\varphi_1$ is in a subgroup of S and so

$$s(u,v)\varphi_1 = s(u\varphi_1, v\varphi_1) = v\varphi_1(uv\varphi_1)^{-1}u\varphi_1$$

by the definition of the operation s on S. Hence $\rho_{\mathbf{ES}} \subseteq \varphi_1 \circ (\varphi_1)^{-1}$ by Remark 3.1.13(ii).

So $\rho_{\mathbf{V}} \subseteq \varphi_1 \circ (\varphi_1)^{-1}$, and hence (by Result 1.1.2) there is a semigroup homomorphism $\varphi : W(X)/\rho_{\mathbf{V}} \to S$ given by $w\rho_{\mathbf{V}}\varphi = w\varphi_1$, $w \in W(X)$. If $x \in \overline{X}$ then $x\iota_{\mathbf{V}}\varphi = x\rho_{\mathbf{V}}\varphi = x\varphi_1 = x\overline{\varphi} = x\phi$, and so $\iota_{\mathbf{V}}\varphi = \phi$.

Say $\theta : W(X)/\rho_{\mathbf{V}} \to S$ is a semigroup homomorphism satisfying $\iota_{\mathbf{V}}\theta = \phi$. We show by induction on $|w|$ that $w\rho_{\mathbf{V}}\theta = w\rho_{\mathbf{V}}\varphi$ for all $w \in W(X)$. If $w \in \overline{X}$ then certainly $w\rho_{\mathbf{V}}\theta = w\rho_{\mathbf{V}}\varphi$. Suppose that $|w| \geq 2$. If $w \in \langle \overline{X} \rangle$ then again $w\rho_{\mathbf{V}}\theta = w\rho_{\mathbf{V}}\varphi$. If $w = s(u,v)$ for $u, v \in K(X)$ then, using the induction hypothesis,

$$s(u,v)\rho_{\mathbf{LI}}\theta \quad = \quad s(u\rho_{\mathbf{LI}}\theta, v\rho_{\mathbf{LI}}\theta)$$

$$
\begin{aligned}
&= s(u\rho_{LI}\varphi, v\rho_{LI}\varphi) \\
&= s(u,v)\rho_{LI}\varphi
\end{aligned}
$$

by Remark 3.1.15 and Result 1.3.18; and

$$
\begin{aligned}
s(u,v)\rho_{\mathbf{ES}}\theta &= (v\rho_{\mathbf{ES}}(uv\rho_{\mathbf{ES}})^{-1}u\rho_{\mathbf{ES}})\theta \\
&= v\rho_{\mathbf{ES}}\theta(uv\rho_{\mathbf{ES}}\theta)^{-1}u\rho_{\mathbf{ES}}\theta \\
&= v\rho_{\mathbf{ES}}\varphi(uv\rho_{\mathbf{ES}}\varphi)^{-1}u\rho_{\mathbf{ES}}\varphi \\
&= s(u,v)\rho_{\mathbf{ES}}\varphi
\end{aligned}
$$

by Remark 3.1.15 and Result 1.3.7. Therefore, in view of the factorization of w according to (3.1), we have $w\rho_{\mathbf{V}}\theta = w\rho_{\mathbf{V}}\varphi$ for all $w \in W(X)$. So $\theta = \varphi$, and the result is proved. □

Let $\mathbf{V} = \mathbf{ES}$ or $\mathbf{V} = L\mathbf{I}$. By Theorem 3.1.16, for any semigroup $S \in \mathbf{V}$ each matched mapping $\phi : \overline{X} \to S$ extends uniquely (by the universal property of bifree objects, and via the mapping $\iota_{\mathbf{V}}$) to a semigroup homomorphism $\theta : W(X) \to S$ such that $\theta \circ \theta^{-1} \supseteq \rho_{\mathbf{V}}$. We call θ the $\mathbf{V}$-*extension* of ϕ. In the E-solid case, if $u, v \in K(X)$ then $uv\theta$ lies in a subgroup of S, and $s(u,v)\theta = v\theta(uv\theta)^{-1}u\theta$. In the locally inverse case, if $u, v \in K(X)$ then $s(u,v)\theta = s(u\theta, v\theta)$. (See Remark 3.1.15.)

We define a *biidentity* to be a pair (u, v), also written $u = v$, of words $u, v \in W(X)$. Then $S \in \mathbf{V}$ is said to $\mathbf{V}$-*satisfy* a biidentity $u = v$ if for any matched mapping $\phi : \overline{X} \to S$ and its $\mathbf{V}$-extension $\theta : W(X) \to S$ we have $u\theta = v\theta$. A biidentity $u = v$ is said to be $\mathbf{V}$-satisfied by a class $\mathbf{C} \subseteq \mathbf{V}$ if $u = v$ is $\mathbf{V}$-satisfied by each member of $\mathbf{C}$. For any set Σ of biidentities, let $[\Sigma]_{\mathbf{V}}$ denote the class of all members of $\mathbf{V}$ that $\mathbf{V}$-satisfy all biidentities in Σ. Given a class $\mathbf{C}$ of semigroups such that $\mathbf{C} \subseteq \mathbf{V}$, let

$$\rho_{\mathbf{V}}(\mathbf{C}, X) = \{(u,v) \in W(X) \times W(X) : u = v \text{ is } \mathbf{V}\text{-satisfied by } \mathbf{C}\}.$$

Then $\rho_{\mathbf{V}}(\mathbf{C}, X)$ is a congruence on $W(X)$ for every $\mathbf{C} \subseteq \mathbf{V}$; and in particular $\rho_{\mathbf{V}}(\mathbf{V}, X) = \rho_{\mathbf{V}}$.

We can give a version of the definition of biinvariant congruence in this setting. As in Definition 1.3.22, for $u, p, q \in W(X)$ and $pq \in R(X)$, let

$$u(x \to p, x' \to q)$$

denote the word in $F_{(2,2)}(\overline{X})$ obtained from u by substituting p for all occurrences of x, and q for all occurrences of x'. A simple induction on $i \geq 1$ such that $u \in W_{2i+1}$ (recall the construction 3.1.4 of $W(X) \subseteq F_{(2,2)}(\overline{X})$) shows that the condition $pq \in R(X)$ implies that $u(x \to p, x' \to q) \in W(X)$.

A congruence ρ on $W(X)$ is said to be closed under *regular substitution* if $u \ \rho \ v$, $p \ \rho \ pqp$, $q \ \rho \ qpq$, and $pq \in R(X)$ imply $u(x \to p, x' \to q) \ \rho \ v(x \to p, x' \to q)$. A congruence ρ on $W(X)$ is said to be $\mathbf{V}$-*biinvariant* whenever $\rho_{\mathbf{V}} \subseteq \rho$ and ρ is closed under regular substitution.

As explained in Section 1.3.6, analogues of Birkhoff's fundamental theorems have been provided for e-varieties of E-solid semigroups (by Kaďourek and Szendrei [27]) and for e-varieties of locally inverse semigroups (by Auinger [7]). These analogues respectively rely on the very different semigroups $F'^{\infty}(X)$ and $F_{(2,2)}(\overline{X})$ in their characterizations of bifree objects, biinvariant congruences and biidentities.

In Theorem 3.1.16 we have characterized the bifree E-solid and bifree locally inverse semigroups on X as images of the one semigroup $W(X)$. In the next theorem we present further unified analogues of Birkhoff-style results for e-varieties of E-solid or locally inverse semigroups.

Theorem 3.1.17 *Let* $\mathbf{V} = \mathbf{ES}$ *or* $\mathbf{V} = \mathbf{LI}$.

(i) In any class $\mathbf{C} \subseteq \mathbf{V}$ *closed under taking regular subsemigroups and direct products there exists a bifree object on any nonempty set* X, *and it is isomorphic to* $W(X)/\rho_{\mathbf{V}}(\mathbf{C}, X)$.

(ii) A class $\mathbf{C} \subseteq \mathbf{V}$ is an e-variety if and only if there exists a set Σ of biidentities such that $\mathbf{C} = [\Sigma]_{\mathbf{V}}$. In particular, if $\mathbf{C}$ is an e-variety then $\mathbf{C} = [\rho_{\mathbf{V}}(\mathbf{C}, X)]_{\mathbf{V}}$.

(iii) The mappings between the lattice of all e-varieties contained in $\mathbf{V}$ and the lattice of all $\mathbf{V}$-biinvariant congruences on $W(X)$ that are defined by

$$\mathbf{C} \mapsto \rho_{\mathbf{V}}(\mathbf{C}, X) \text{ and } \rho \mapsto [\rho]_{\mathbf{V}}$$

are mutually inverse order-reversing bijections. A congruence ρ on $W(X)$ is $\mathbf{V}$-biinvariant if and only if $\rho_{\mathbf{V}} \subseteq \rho$ and $\rho/\rho_{\mathbf{V}}$ is a fully invariant congruence on $W(X)/\rho_{\mathbf{V}}$, and so there are also mutually inverse order-reversing bijections between the lattice of all e-varieties contained in $\mathbf{V}$ and the lattice of all fully invariant congruences on $W(X)/\rho_{\mathbf{V}}$.

Proof: The E-solid cases of parts (ii) and (iii) appear in [27] with $F'^{\infty}(X)$ used instead of $W(X)$; but the analogous proofs may be directly applied here, for both the E-solid and locally inverse cases. For part (i), write $\rho_{\mathbf{V}}(\mathbf{C}) = \rho_{\mathbf{V}}(\mathbf{C}, X)$. A proof analogous to the first part of the proof of Theorem 2.5 of [27] shows that $W(X)/\rho_{\mathbf{V}}(\mathbf{C}) \in \mathbf{C}$, and we need only prove that $W(X)/\rho_{\mathbf{V}}(\mathbf{C})$ is bifree on X for $\mathbf{C}$.

Suppose that $S \in \mathbf{C}$, and let $\phi : \overline{X} \to S$ be a matched mapping. Let $\theta : W(X) \to S$ be the $\mathbf{V}$-extension of ϕ. Then $u\theta = v\theta$ whenever $(u, v) \in \rho_{\mathbf{V}}(\mathbf{C})$, and hence (by Result 1.1.2) there is a semigroup homomorphism $\varphi : W(X)/\rho_{\mathbf{V}}(\mathbf{C}) \to S$ defined by $w\rho_{\mathbf{V}}(\mathbf{C})\varphi = w\theta$ for all $w \in W(X)$. Moreover, we have $x\phi = x\rho_{\mathbf{V}}(\mathbf{C})\varphi$ for all $x \in \overline{X}$.

Now suppose that $\psi : W(X)/\rho_{\mathbf{V}}(\mathbf{C}) \to S$ is a semigroup homomorphism satisfying $x\phi = x\rho_{\mathbf{V}}(\mathbf{C})\psi$ for all $x \in \overline{X}$. Recall the construction 3.1.4 of $W(X)$. We have $x\rho_{\mathbf{V}}(\mathbf{C})\varphi = x\phi = x\rho_{\mathbf{V}}(\mathbf{C})\psi$ for all $x \in \overline{X} = W_0$, and hence $w\rho_{\mathbf{V}}(\mathbf{C})\varphi =$

$w\rho_{\mathbf{V}}(\mathbf{C})\psi$ for all $w \in W_1$. Suppose inductively that $w\rho_{\mathbf{V}}(\mathbf{C})\varphi = w\rho_{\mathbf{V}}(\mathbf{C})\psi$ for all $w \in W_{2i-1}$, where $i \geq 1$, and consider $u, v \in W_{2i-1} \cap R(X)$.

Since $\rho_{\mathbf{V}} \subseteq \rho_{\mathbf{V}}(\mathbf{C})$, there is a semigroup homomorphism $\alpha_{\mathbf{V}} : W(X)/\rho_{\mathbf{V}} \to W(X)/\rho_{\mathbf{V}}(\mathbf{C})$ given by $w\rho_{\mathbf{V}}\alpha_{\mathbf{V}} = w\rho_{\mathbf{V}}(\mathbf{C})$, $w \in W(X)$.

Now, if $\mathbf{V} = L\mathbf{I}$ then $s(u,v)\rho_{\mathbf{V}}(\mathbf{C}) = s(u\rho_{\mathbf{V}}(\mathbf{C}), v\rho_{\mathbf{V}}(\mathbf{C}))$ by Remark 3.1.15. Therefore

$$\begin{aligned} s(u,v)\rho_{\mathbf{V}}(\mathbf{C})\varphi &= s(u\rho_{\mathbf{V}}(\mathbf{C})\varphi, v\rho_{\mathbf{V}}(\mathbf{C})\varphi) \\ &= s(u\rho_{\mathbf{V}}(\mathbf{C})\psi, v\rho_{\mathbf{V}}(\mathbf{C})\psi) \\ &= s(u,v)\rho_{\mathbf{V}}(\mathbf{C})\psi \end{aligned}$$

by the induction hypothesis.

Suppose that $\mathbf{V} = \mathbf{ES}$, and write $\alpha = \alpha_{\mathbf{ES}}$. Then

$$\begin{aligned} s(u,v)\rho_{\mathbf{ES}}(\mathbf{C})\varphi &= s(u,v)\rho_{\mathbf{ES}}\alpha\varphi \\ &= v\rho_{\mathbf{ES}}\alpha\varphi(uv\rho_{\mathbf{ES}}\alpha\varphi)^{-1}u\rho_{\mathbf{ES}}\alpha\varphi \\ &= v\rho_{\mathbf{ES}}(\mathbf{C})\varphi(uv\rho_{\mathbf{ES}}(\mathbf{C})\varphi)^{-1}u\rho_{\mathbf{ES}}(\mathbf{C})\varphi \\ &= v\rho_{\mathbf{ES}}(\mathbf{C})\psi(uv\rho_{\mathbf{ES}}(\mathbf{C})\psi)^{-1}u\rho_{\mathbf{ES}}(\mathbf{C})\psi \\ &= s(u,v)\rho_{\mathbf{ES}}(\mathbf{C})\psi \end{aligned}$$

by the induction hypothesis.

Thus $w\rho_{\mathbf{V}}(\mathbf{C})\varphi = w\rho_{\mathbf{V}}(\mathbf{C})\psi$ for all $w \in W_{2i}$, and hence for all $w \in W_{2i+1}$; so that $\varphi = \psi$. □

3.2 An alternative approach to e-varieties of E-solid or locally inverse semigroups.

In this section we give an alternative unification of the methods of [7, 8] and [27]. Instead of using $F_{(2,2)}(\overline{X})$, as in [7], which is better suited to the locally inverse case, we use the semigroup $F'^{\infty}(X)$ of [27], which is better suited to the E-solid case.

Notation 3.2.1 Any inverse unary operation denoted by $^{-1}$ on a regular semigroup S will always be assumed to be such that if k lies in a subgroup of S then $k \, \mathcal{H} \, k^{-1}$. □

In this section we will be using the free unary semigroup $F_{\mathcal{U}}(X)$ (as described in Notation 1.3.8) instead of the free binary semigroup $F_{(2,2)}(\overline{X})$, and so we will use some notation defined for Section 3.1 to label slightly different objects.

We need to reconsider constructions 3.1.2 and 3.1.3. Here we replace the sandwich operations with inverse unary operations.

Construction 3.2.2 Suppose that U is an E-solid semigroup, with an inverse unary operation $^{-1}$ (so u^{-1} is the $\mathcal{H}$-related inverse of u whenever u lies in a subgroup of U) and subset A. Let $A^{-1} = \{a^{-1} : a \in A\}$. By Result 1.3.34, the least regular subsemigroup of U containing $A \cup A^{-1}$ is $P = \bigcup_{i \geq 0} P_{2i+1}$, where

$$\begin{aligned}
P_0 &= A \cup A^{-1} \\
P_1 &= \langle P_0 \rangle \\
&\vdots \\
P_{2i} &= \{(ef)^{-1} : e, f \in E(P_{2i-1})\} \cup P_{2i-1} \\
P_{2i+1} &= \langle P_{2i} \rangle \\
&\vdots
\end{aligned}$$

□

Observe that, apart from $P_0 = A \cup A^{-1}$, the construction of P is independent of the choice of the operation $^{-1}$ since $(ef)^{-1}$ will always be the group inverse of ef. The same is true for the locally inverse semigroup Q constructed below.

Construction 3.2.3 Let V be a locally inverse semigroup with inverse unary operation $^{-1}$ and subset A, and write $A^{-1} = \{a^{-1} : a \in A\}$. Again by Result 1.3.34, the least regular subsemigroup of V containing $A \cup A^{-1}$ is $Q = \bigcup_{i \geq 0} Q_{2i+1}$, where

$$\begin{aligned}
Q_0 &= A \cup A^{-1} \\
Q_1 &= \langle Q_0 \rangle \\
&\vdots \\
Q_{2i} &= \{s(a,b) : a, b \in Q_{2i-1}\} \cup Q_{2i-1} \\
Q_{2i+1} &= \langle Q_{2i} \rangle \\
&\vdots
\end{aligned}$$

$\square$

For $u \in F_{\mathcal{U}}(X)$ let $\overline{u}$ be the usual group-reduced form of u, as described at the end of Section 1.3.1. Let $\hat{R}(X) = \{u \in F_{\mathcal{U}}(X) : \overline{u} = 1\}$.

Define $bF(X)$ to be the least subsemigroup W of $F_{\mathcal{U}}(X)$ such that $\overline{X} \subseteq W$ and $W \cap \hat{R}(X)$ is a unary subsemigroup of W. Then $bF(X)$ is the semigroup $F'^{\infty}(X)$ of [27], and can be constructed in the same manner as the semigroups P and Q of constructions 3.2.2 and 3.2.3 respectively.

Construction 3.2.4 We have $bF(X) = \bigcup_{i \geq 0} F_{2i+1}$, where

$$\begin{aligned}
F_0 &= \overline{X} \\
F_1 &= \langle F_0 \rangle \\
&\vdots \\
F_{2i} &= \{u' : u \in F_{2i-1} \cap \hat{R}(X)\} \cup F_{2i-1} \\
F_{2i+1} &= \langle F_{2i} \rangle \\
&\vdots
\end{aligned}$$

$\square$

Let $\hat{K}(X) = \{u \in bF(X) : \overline{u} = 1\}$. Consider the partial unary semigroup congruence $\hat{\eta}$ on $bF(X)$ generated by

$$\{(uu'u, u), (u'uu', u') : u \in X \cup \hat{K}(X)\}.$$

(That is; $\hat{\eta}$ is the least semigroup congruence on $bF(X)$ that contains $\{(uu'u, u), (u'uu', u') : u \in X \cup \hat{K}(X)\}$ and respects the partial unary operation $'$ on $bF(X)$.)

Remark 3.2.5 As a binary relation on $F_{\mathcal{U}}(X)$, $\hat{\eta}$ is contained in the least group congruence $\hat{\sigma} = \{(u, v) : \overline{u} = \overline{v}\}$ on $F_{\mathcal{U}}(X)$ (recall the discussion in Section 1.3.1). Hence if ρ is a congruence on $bF(X)$ such that $\hat{\eta} \subseteq \rho \subseteq \hat{\sigma}$ then $u \in \hat{K}(X)$ whenever $u\rho \in E(bF(X)/\rho)$. □

The next few results are exact analogues of results from Section 3.1.

Lemma 3.2.6 *The semigroup $bF(X)/\hat{\eta}$ is regular.*

Proof: If $w \in F_0$ then $w\hat{\eta}$ has an inverse in $bF(X)/\hat{\eta}$. Let $i \geq 1$ and assume that there is an inverse in $bF(X)/\hat{\eta}$ of $w\hat{\eta}$ for every $w \in F_{2i-2}$. Consider $u \in F_{2i}$. If $u = v'$ for some $v \in F_{2i-1}$ with $\overline{v} = 1$ then $v\hat{\eta} \in V(u\hat{\eta})$, so let $u = u_1 \ldots u_n$ where $u_1, \ldots, u_n \in F_{2i-2}$. By the induction assumption we may suppose that $n > 1$, and that $u_1\hat{\eta}$ and $u_2\hat{\eta}$ have inverses, say $u_1'\hat{\eta}$ and $u_2'\hat{\eta}$, in $bF(X)/\hat{\eta}$. Also, since $(u_1'u_1u_2u_2')\hat{\eta}$ is a product of idempotents then $\overline{u_1'u_1u_2u_2'} = 1$ by Remark 3.2.5 and so $(u_1'u_1u_2u_2')\hat{\eta}$ has an inverse, say $w\hat{\eta}$, in $bF(X)/\hat{\eta}$. But now $(u_2'wu_1')\hat{\eta}$ is an inverse of $(u_1u_2)\hat{\eta}$ in $bF(X)/\hat{\eta}$. We may repeat the argument to show that the product $u_1u_2u_3\hat{\eta}$ has an inverse in $bF(X)/\hat{\eta}$, and eventually this process leads to an inverse of $u\hat{\eta}$ in $bF(X)/\hat{\eta}$. □

Lemma 3.2.7 *Let W denote the E-solid semigroup U or the locally inverse semigroup V of constructions 3.2.2 and 3.2.3, with A and $^{-1}$ as given in the* *constructions. Let T denote the regular subsemigroup P or Q respectively. Suppose ${}^{-1}$ is also an inverse unary operation on T, and let $\alpha : F_{\mathcal{U}}(A) \to W$ be the unary semigroup homomorphism with respect to ${}^{-1}$ such that $a\alpha = a$ for all $a \in A$. Then $T = \{u\alpha : u \in bF(A)\}$.*

Proof: Define α_1 to be the restriction of α to $bF(A)$ and consider constructions 3.2.2, 3.2.3 and 3.2.4. We have $(A \cup A')\alpha_1 = A \cup A^{-1} \subseteq T$ and $F_1\alpha_1 \subseteq T$. Assuming that $F_{2i-1}\alpha_1 \subseteq T$ for $i \geq 1$, it follows, since ${}^{-1}$ is an inverse unary operation on T, that $F_{2i}\alpha_1 \subseteq T$. Hence $bF(A)\alpha_1 \subseteq T$. Again since ${}^{-1}$ is an inverse unary operation on T, for each $v \in A \cup \hat{R}(A)$ we have $u'\alpha_1 \in V(u\alpha_1)$. Therefore, by Lemma 3.2.6, $bF(A)\alpha_1$ is a regular subsemigroup of T. But T is the least regular subsemigroup of W that contains $(A \cup A')\alpha_1$, so $T = bF(A)\alpha_1$. □

By Result 1.3.37 we have the following corollary.

Corollary 3.2.8 *In the statement of Lemma 3.2.7, if $A = X$ and $W = F_{\mathcal{ES}}(X)$ or $W = F_{\mathcal{LI}}(X)$ then $\{u\alpha : u \in bF(X)\}$ is isomorphic to $bF_{\mathbf{ES}}(X)$ or $bF_{\mathbf{LI}}(X)$ respectively.* □

Let $\hat{\rho}_{\mathbf{ES}}$ denote the partial unary semigroup congruence on $bF(X)$ generated by

$$\hat{\eta} \cup \{((u')', u), (uu', u'u) : u \in \hat{K}(X)\}.$$

Note that if $u \in \hat{K}(X)$ then $u\hat{\rho}_{\mathbf{ES}}$ lies in a subgroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$ (by Result 1.3.5), and $(u\hat{\rho}_{\mathbf{ES}})^{-1} = u'\hat{\rho}_{\mathbf{ES}}$.

The next result is a consequence of Remark 2.5(a) and Lemma 1.2(ii) of [40]. For a congruence ρ on a regular semigroup S we write

$$\ker\rho = \bigcup\{e\rho : e \in E(S)\}.$$

Result 3.2.9 ([40]) *Let ξ denote the least group congruence on $F_{\mathcal{ES}}(X)$. Then*

$$\ker\xi = C_\infty(F_{\mathcal{ES}}(X)). \qquad \square$$

The following theorem is due to Kadourek and Szendrei [27, Theorem 2.5]. Since our proof is straightforward, is quite different to that of [27], and is considered later in the text, we present it here.

Theorem 3.2.10 *The semigroups $bF_{\mathbf{ES}}(X)$ and $bF(X)/\hat{\rho}_{\mathbf{ES}}$ are isomorphic.*

Proof: Let $^{-1}$ be an inverse unary operation on $F_{\mathcal{ES}}(X)$ as used in construction 3.2.2 for Corollary 3.2.8. Let $\alpha : F_{\mathcal{U}}(X) \to F_{\mathcal{ES}}(X)$ be the unary semigroup homomorphism with respect to $^{-1}$ such that $x\alpha = x$ for each $x \in X$. By Corollary 3.2.8, Lemma 3.2.7 gives $bF_{\mathbf{ES}}(X) \cong \{u\alpha : u \in bF(X)\} = P$, where P is the regular subsemigroup of $F_{\mathcal{ES}}(X)$ as constructed in 3.2.2, with $A = X$.

Recall that the natural inverse unary operation on $F_{\mathcal{ES}}(X)$ is denoted by $'$, and the fully invariant congruence on $F_{\mathcal{U}}(X)$ corresponding to the variety $\mathcal{ES}$ is denoted by $\rho_{\mathcal{ES}}$. Let $\theta : (F_{\mathcal{ES}}(X),') \to (F_{\mathcal{G}}(X),')$ be the unary semigroup homomorphism such that $x\theta = x$ for all $x \in X$. So $\theta \circ \theta^{-1}$ is the least group congruence on $F_{\mathcal{ES}}(X)$, and $u\rho_{\mathcal{ES}}\theta = \overline{u}$ for all $u \in F_{\mathcal{U}}(X)$. By Result 3.2.9 we have $\ker\theta = C_\infty(F_{\mathcal{ES}}(X))$.

Now let $u \in \hat{R}(X)$. Then $u\rho_{\mathcal{ES}}\theta = 1$, and hence $u\rho_{\mathcal{ES}} \in \ker\theta = C_\infty(F_{\mathcal{ES}}(X))$. Therefore by Result 1.3.11 the element $u\rho_{\mathcal{ES}}$ is in a subgroup of $F_{\mathcal{ES}}(X)$. Since $(F_{\mathcal{ES}}(X),')$ is the free regular unary E-solid semigroup and $F_{\mathcal{U}}(X)\alpha \in \mathcal{ES}$, we have $\rho_{\mathcal{ES}} \subseteq \alpha\circ\alpha^{-1}$. Result 1.3.5 now shows that $u\alpha$ is in a subgroup of $F_{\mathcal{U}}(X)\alpha$. By the definition of the operation $^{-1}$ on $F_{\mathcal{ES}}(X)$, this means that $u'\alpha = (u\alpha)^{-1}$ is the $\mathcal{H}$-related inverse of $u\alpha$ in $F_{\mathcal{U}}(X)\alpha$. This implies that $\hat{\rho}_{\mathbf{ES}} \subseteq \alpha \circ \alpha^{-1}$, and consequently (by Result 1.1.2) there is a semigroup homomorphism φ : $bF(X)/\hat{\rho}_{\mathbf{ES}} \to P$ given by $(u\hat{\rho}_{\mathbf{ES}})\varphi = u\alpha$ for all $u \in bF(X)$.

By Remark 3.2.5, whenever $u\hat{\rho}_{\mathbf{ES}}, v\hat{\rho}_{\mathbf{ES}} \in E(bF(X)/\hat{\rho}_{\mathbf{ES}})$ we have $uv \in \hat{K}(X)$. The definition of $\hat{\rho}_{\mathbf{ES}}$ then shows that $uv\hat{\rho}_{\mathbf{ES}}$ lies in a subgroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$, and thus $bF(X)/\hat{\rho}_{\mathbf{ES}} \in \mathbf{ES}$ by Lemma 1.3.11. Therefore there is a semigroup homomorphism $\psi : P \to bF(X)/\hat{\rho}_{\mathbf{ES}}$ such that $x\psi = x\hat{\rho}_{\mathbf{ES}}$ for each

$x \in \overline{X}$.

Recall the construction 3.2.4 of $bF(X)$. Let T be the least regular subsemigroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$ containing $\{x\hat{\rho}_{\mathbf{ES}} : x \in \overline{X}\}$, constructed as in 3.2.2. Then $w\hat{\rho}_{\mathbf{ES}} \in T$ for every $w \in F_1 = \langle \overline{X} \rangle$. Let $i \geq 1$ and suppose inductively that $w\hat{\rho}_{\mathbf{ES}} \in T$ for every $w \in F_{2i-1}$. Consider $u \in F_{2i-1} \cap \hat{K}(X)$. Then $u\hat{\rho}_{\mathbf{ES}}$ is in a subgroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$, and $u'\hat{\rho}_{\mathbf{ES}} = (u\hat{\rho}_{\mathbf{ES}})^{-1}$. By Lemma 3.1.1 the element $u\hat{\rho}_{\mathbf{ES}}$ is in a subgroup of T, and $u'\hat{\rho}_{\mathbf{ES}} \in T$. So $w\hat{\rho}_{\mathbf{ES}} \in T$ for every $w \in F_{2i+1}$. Thus $bF(X)/\hat{\rho}_{\mathbf{ES}} = T$. Now both $bF(X)/\hat{\rho}_{\mathbf{ES}}$ and P have the form of the semigroup of construction 3.2.2, and consequently, in view of Result 1.3.5, the homomorphisms φ and ψ are mutually inverse. □

Notation 3.2.11 For $u, v \in \hat{K}(X)$ write $s(u,v) = v(uv)'u$. Let $\overline{s}(u,v) = s(w,w)$ where $w = s(u,u)vs(u,u)$. □

Note that $s(u,v)\hat{\eta} \in S(u\hat{\eta}, v\hat{\eta})$ for every $u, v \in \hat{K}(X)$. We have the following analogue of Lemma 3.1.11.

Lemma 3.2.12 *Let ρ be a congruence on $bF(X)$ such that $\hat{\eta} \subseteq \rho \subseteq \hat{\sigma}$.*

(i) $E(bF(X)/\rho) = \{s(u,u)\rho : u \in \hat{K}(X)\}$.

(ii) Each idempotent in each local submonoid of the semigroup $bF(X)/\rho$ is of the form $\overline{s}(u,v)\rho$ for some $u, v \in \hat{K}(X)$.

Proof: : (i) If $u\rho \in E(bF(X)/\rho)$ then $u \in \hat{K}(X)$ by Remark 3.2.5. Since $s(u,u)\hat{\eta} \in S(u\hat{\eta}, u\hat{\eta})$ then $s(u,u)\rho \in S(u\rho, u\rho) = \{u\rho\}$ by Result 1.3.15(v),(vi). The result easily follows.

(ii) See the proof of Lemma 3.1.11(ii). □

Corollary 3.2.8 indicates the existence of congruences ρ on $bF(X)$ such that $bF(X)/\rho$ is isomorphic to $bF_{\mathbf{LI}}(X)$; but these congruences rely on the choice of inverse unary operations on $F_{\mathcal{LI}}(X)$, and are not easily described.

Consider the partial unary semigroup congruence $\hat{\rho}_{LI}$ on $bF(X)$ generated by

$$\hat{\eta} \cup \{(\overline{s}(a,p)\overline{s}(a,q), \overline{s}(a,q)\overline{s}(a,p)) : a,p,q \in \hat{K}(X)\}.$$

Lemma 3.2.13 *We have $bF(X)/\hat{\rho}_{LI} \in$ **LI**. Hence $s(u,v)\hat{\rho}_{LI} = s(u\hat{\rho}_{LI}, v\hat{\rho}_{LI})$ for every $u,v \in \hat{K}(X)$.*

Proof: Since by Result 1.3.19 a semigroup S is inverse if and only if the idempotents of S commute, Lemma 3.2.12 shows that $bF(X)/\hat{\rho}_{LI} \in$ **LI**. □

It appears that $bF(X)/\hat{\rho}_{LI}$ might be isomorphic to $bF_{LI}(X)$, but in fact it can be shown that for every $u \in \hat{K}(X)$ the class $u'\hat{\rho}_{LI}$ is not a member of the least regular subsemigroup of $bF(X)/\hat{\rho}_{LI}$ that contains the set $\{x\hat{\rho}_{LI} : x \in \overline{X}\}$, constructed as in 3.2.3. However, we construct below a subsemigroup $\hat{W}(X)$ of $bF(X)$ which is isomorphic to the semigroup $W(X)$ of the previous section and, when used instead of $bF(X)$, provides results analogous to those obtained above. In particular, $\hat{W}(X)/\rho^W_{\mathbf{ES}}$ is isomorphic to $bF_{\mathbf{ES}}(X)$, where $\rho^W_{\mathbf{ES}}$ is the restriction of $\hat{\rho}_{\mathbf{ES}}$ to $\hat{W}(X)$. Moreover, $\hat{W}(X)/\rho^W_{LI}$ is isomorphic to $bF_{LI}(X)$, where ρ^W_{LI} is the restriction to $\hat{W}(X)$ of the congruence $\hat{\rho}_{LI}$ defined above.

Construction 3.2.14 Let $\hat{W}(X) = \bigcup_{i \geq 0} \hat{W}_{2i+1}$, where

$$\begin{aligned}
\hat{W}_0 &= \overline{X} \\
\hat{W}_1 &= \langle \hat{W}_0 \rangle \\
&\vdots \\
\hat{W}_{2i} &= \{s(a,b) : a,b \in \hat{W}_{2i-1} \cap \hat{K}(X)\} \cup \hat{W}_{2i-1} \\
\hat{W}_{2i+1} &= \langle \hat{W}_{2i} \rangle \\
&\vdots
\end{aligned}$$

□

Note that $\hat{W}(X) \subseteq bF(X)$, and that in light of the next Lemma the statements of Lemma 3.2.7 and Corollary 3.2.8 remain true when $bF(X)$ is replaced with $\hat{W}(X)$.

If ρ is a congruence on $bF(X)$ then ρ^W will denote $\rho \cap (\hat{W}(X) \times \hat{W}(X))$. Of course $\hat{W}(X)/\rho^W$ is embedded in $bF(X)/\rho$. As previously mentioned, we will write $\rho^W_{\mathbf{ES}}$ and $\rho^W_{\mathbf{\mathit{LI}}}$ for the restrictions to $\hat{W}(X)$ of $\hat{\rho}_{\mathbf{ES}}$ and $\hat{\rho}_{LI}$ respectively.

Lemma 3.2.15 *The semigroup $\hat{W}(X)/\hat{\eta}^W$ is regular.*

Proof: If $w \in \hat{W}_0$ then $w\hat{\eta}^W$ has an inverse in $\hat{W}(X)/\hat{\eta}^W$. Let $i \geq 1$ and assume that there is an inverse in $\hat{W}(X)/\hat{\eta}^W$ of $w\hat{\eta}^W$ for every $w \in \hat{W}_{2i-2}$. Consider $u \in \hat{W}_{2i}$. If $u = s(a,b)$ for $a, b \in \hat{W}_{2i-1}$ then $u\hat{\eta}^W \in V(u\hat{\eta}^W)$, so let $u = u_1 \dots u_n$ where $u_1, \dots, u_n \in \hat{W}_{2i-2}$. By the induction assumption we may suppose that $n > 1$, and that $u_1\hat{\eta}^W$ and $u_2\hat{\eta}^W$ have inverses, say $u_1{}'\hat{\eta}^W$ and $u_2{}'\hat{\eta}^W$, in $\hat{W}(X)/\hat{\eta}^W$. Then

$$(u_1{}'u_1)\hat{\eta}^W, (u_2u_2{}')\hat{\eta}^W \in E(\hat{W}(X)/\hat{\eta}^W)$$

so $\overline{u_1{}'u_1} = \overline{u_2u_2{}'} = 1$ by Remark 3.2.5. Therefore $s(u_1{}'u_1, u_2u_2{}') \in \hat{W}(X)$. But

$$s(u_1{}'u_1, u_2u_2{}')\hat{\eta} \in S(u_1{}'u_1\hat{\eta}, u_2u_2{}'\hat{\eta}) = S(u_1\hat{\eta}, u_2\hat{\eta})$$

by Result 1.3.15(ii), so that

$$u_2's(u_1{}'u_1, u_2u_2{}')u_1'\hat{\eta} \in V(u_1u_2\hat{\eta})$$

by Result 1.3.15(iii). Thus $u_2's(u_1{}'u_1, u_2u_2{}')u_1'\hat{\eta}^W$ is an inverse of $(u_1u_2)\hat{\eta}^W$ in $\hat{W}(X)/\hat{\eta}^W$. This argument may be repeated to show that $u_1u_2u_3\hat{\eta}^W$ has an inverse in $\hat{W}(X)/\hat{\eta}^W$, and eventually this process leads to an inverse of $u\hat{\eta}^W$ in $\hat{W}(X)/\hat{\eta}^W$. □

Since $(uv)' \notin W(X)$ for $u, v \in W(X)$, the next statement is not immediately obvious.

Lemma 3.2.16 *If $u, v \in \hat{K}(X) \cap \hat{W}(X)$ then $s(u,v)\hat{\eta}^W \in S(u\hat{\eta}^W, v\hat{\eta}^W)$.*

Proof: Let $u, v \in \hat{K}(X) \cap \hat{W}(X)$. By Lemma 3.2.15 there exist $u^*, v^* \in \hat{W}(X)$ such that $u^*\hat{\eta}^W \in V(u\hat{\eta}^W)$ and $v^*\hat{\eta}^W \in V(v\hat{\eta}^W)$. Then

$$\begin{aligned} s(u,v)\hat{\eta} &= v(uv)'u\hat{\eta} \\ &= vv^*v(uv)'uu^*u\hat{\eta} \\ &= vv^*s(u,v)u^*u\hat{\eta}. \end{aligned}$$

Since $s(u,v)\hat{\eta} \in S(u\hat{\eta}, v\hat{\eta})$ then $v^*s(u,v)u^*\hat{\eta} \in V(uv\hat{\eta})$ by Result 1.3.15(iii); and hence $v^*s(u,v)u^*\hat{\eta}^W \in V(uv\hat{\eta}^W)$. Therefore

$$\begin{aligned} s(u,v)\hat{\eta}^W &\in v\hat{\eta}^W V(uv\hat{\eta}^W)u\hat{\eta}^W \\ &= S(u\hat{\eta}^W, v\hat{\eta}^W). \end{aligned}$$

□

We now have the following analogue of Lemma 3.2.12.

Lemma 3.2.17 *Let ρ be a congruence on $\hat{W}(X)$ such that $\hat{\eta}^W \subseteq \rho \subseteq \hat{\sigma}^W$.*

(i) $E(\hat{W}(X)/\rho) = \{s(w,w)\rho : w \in \hat{K}(X) \cap \hat{W}(X)\}$.

(ii) Each idempotent in each local submonoid of the semigroup $\hat{W}(X)/\rho$ is of the form $\bar{s}(u,v)\rho$ for some $u, v \in \hat{K}(X) \cap \hat{W}(X)$. □

The next observation will be needed presently.

Remark 3.2.18 Let $u \in \hat{K}(X)$. Then $u\hat{\rho}_{\mathbf{ES}}$ is in a subgroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$ and $(u\hat{\rho}_{\mathbf{ES}})^{-1} = u'\hat{\rho}_{\mathbf{ES}}$.

Suppose that $u\hat{\rho}_{\mathbf{ES}} = w\hat{\rho}_{\mathbf{ES}}$ for some $w \in \hat{W}(X)$. Since $\{y\hat{\rho}_{\mathbf{ES}} : y \in \hat{W}(X)\}$ is a regular subsemigroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$ then by Lemma 3.1.1 the element $w\rho^W_{\mathbf{ES}}$ is in a subgroup of $\hat{W}(X)/\rho^W_{\mathbf{ES}}$ and there exists $w^* \in \hat{W}(X)$ such that $w^*\rho^W_{\mathbf{ES}} = (w\rho^W_{\mathbf{ES}})^{-1}$ and $u'\hat{\rho}_{\mathbf{ES}} = (u\hat{\rho}_{\mathbf{ES}})^{-1} = w^*\hat{\rho}_{\mathbf{ES}}$. □

The next result states, as was previously indicated, that both $\hat{W}(X)/\rho^W_{\mathbf{ES}}$ and $bF(X)/\hat{\rho}_{\mathbf{ES}}$ are bifree E-solid semigroups on X.

Theorem 3.2.19 *The semigroups $\hat{W}(X)/\rho^W_{\mathbf{ES}}$, $bF(X)/\hat{\rho}_{\mathbf{ES}}$ and $bF_{\mathbf{ES}}(X)$ are isomorphic.*

Proof: Recall the constructions 3.2.4 and 3.2.14, of $bF(X)$ and $\hat{W}(X)$ respectively. We show by induction on i that for every $u \in F_{2i+1}$ there exists $w \in \hat{W}(X)$ such that $u\hat{\rho}_{\mathbf{ES}} = w\hat{\rho}_{\mathbf{ES}}$. For the first step, we have $\hat{W}_1 = F_1$. Suppose that $i \geq 1$, and let $u \in F_{2i-1} \cap \hat{K}(X)$. Then $u\hat{\rho}_{\mathbf{ES}} = w\hat{\rho}_{\mathbf{ES}}$ for some $w \in \hat{W}(X)$ by the induction hypothesis. Therefore Remark 3.2.18 shows that there exists $w^* \in \hat{W}(X)$ such that $u'\hat{\rho}_{\mathbf{ES}} = w^*\hat{\rho}_{\mathbf{ES}}$. It now follows that for every $u \in F_{2i+1}$ there exists $w \in \hat{W}(X)$ such that $u\hat{\rho}_{\mathbf{ES}} = w\hat{\rho}_{\mathbf{ES}}$. Thus $\hat{W}(X)/\rho^W_{\mathbf{ES}} \cong bF(X)/\hat{\rho}_{\mathbf{ES}}$, and the result follows from Theorem 3.2.10. □

Before moving on, we note the following information.

Lemma 3.2.20 If $u, v \in \hat{K}(X) \cap \hat{W}(X)$ then $uv\rho^W_{\mathbf{ES}}$ lies in a subgroup of $\hat{W}(X)/\rho^W_{\mathbf{ES}}$, and $s(u,v)\rho^W_{\mathbf{ES}} = v\rho^W_{\mathbf{ES}}(uv\rho^W_{\mathbf{ES}})^{-1}u\rho^W_{\mathbf{ES}}$.

Proof: If $u, v \in \hat{K}(X) \cap \hat{W}(X)$ then $uv \in \hat{K}(X)$. As in Remark 3.2.18 the element $uv\hat{\rho}_{\mathbf{ES}}$ lies in a subgroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$, and $uv\rho^W_{\mathbf{ES}}$ lies in a subgroup of $\hat{W}(X)/\rho^W_{\mathbf{ES}}$. Also, there exists $w^* \in \hat{W}(X)$ such that $(uv\hat{\rho}_{\mathbf{ES}})^{-1} = w^*\hat{\rho}_{\mathbf{ES}}$ and $(uv\rho^W_{\mathbf{ES}})^{-1} = w^*\rho^W_{\mathbf{ES}}$. Then

$$\begin{aligned} s(u,v)\hat{\rho}_{\mathbf{ES}} &= v(uv)'u\hat{\rho}_{\mathbf{ES}} \\ &= v\hat{\rho}_{\mathbf{ES}}(uv\hat{\rho}_{\mathbf{ES}})^{-1}u\hat{\rho}_{\mathbf{ES}} \\ &= vw^*u\hat{\rho}_{\mathbf{ES}}, \end{aligned}$$

and hence $s(u,v)\rho^W_{\mathbf{ES}} = vw^*u\rho^W_{\mathbf{ES}} = v\rho^W_{\mathbf{ES}}(uv\rho^W_{\mathbf{ES}})^{-1}u\rho^W_{\mathbf{ES}}$. □

Let us now consider the bifree locally inverse semigroup. Recall the congruence $\hat{\rho}_{LI}$.

Theorem 3.2.21 *The semigroups $bF_{\mathbf{LI}}(X)$ and $\hat{W}(X)/\rho^W_{\mathbf{LI}}$ are isomorphic.*

Proof: Let $^{-1}$ denote the inverse unary operation on $F_{L\mathcal{I}}(X)$ as used in construction 3.2.3 for Corollary 3.2.8, and let $\alpha : F_{\mathcal{U}}(X) \to F_{L\mathcal{I}}(X)$ be the unary semigroup homomorphism with respect to this operation such that $x\alpha = x$ for each $x \in X$. The proofs of Corollary 3.2.8 and Lemma 3.2.7 hold when $\hat{W}(X)$ is substituted for $bF(X)$, so we have $bF_{\mathbf{LI}}(X) \cong \{w\alpha : w \in \hat{W}(X)\} = Q$, where Q is the regular subsemigroup of $F_{L\mathcal{I}}(X)$ constructed as in 3.2.3, with $A = X$.

Notice that for $u, v \in \hat{R}(X)$ we have

$$\begin{aligned} s(u,v)\alpha &= v(uv)'u\alpha \\ &= v\alpha(uv\alpha)^{-1}u\alpha \\ &\in S(u\alpha, v\alpha). \end{aligned}$$

So if $a, p, q \in \hat{K}(X)$ then $s(a,a)\alpha \in E(F_{L\mathcal{I}}(X))$ and

$$\overline{s}(a,p)\alpha, \overline{s}(a,q)\alpha \in E(s(a,a)\alpha \cdot F_{L\mathcal{I}}(X) \cdot s(a,a)\alpha).$$

Now Result 1.3.19 shows that $\rho_{\mathbf{LI}} \subseteq \alpha \circ \alpha^{-1}$ since $F_{\mathcal{U}}(X)\alpha \in L\mathcal{I}$, and hence (by Result 1.1.2) there is a semigroup homomorphism $\hat{W}(X)/\rho^W_{\mathbf{LI}} \to Q$ given by $w\rho^W_{\mathbf{LI}} \mapsto w\alpha$, $w \in \hat{W}(X)$.

Since $\hat{\eta}^W \subseteq \rho^W_{\mathbf{LI}}$, then $\hat{W}(X)/\rho^W_{\mathbf{LI}}$ is isomorphic to a regular subsemigroup of $bF(X)/\hat{\rho}_{\mathbf{LI}}$, which is a member of $L\mathbf{I}$ by Lemma 3.2.13, and hence $\hat{W}(X)/\rho^W_{\mathbf{LI}} \in L\mathbf{I}$. Therefore there is a semigroup homomorphism $Q \to \hat{W}(X)/\rho^W_{\mathbf{LI}}$ such that $x \mapsto x\rho^W_{\mathbf{LI}}$, $x \in \overline{X}$.

Recall the construction 3.2.14 of $\hat{W}(X)$. The inductive method used to show that the least regular subsemigroup of $bF(X)/\hat{\rho}_{\mathbf{ES}}$ that contains $\{x\hat{\rho}_{\mathbf{ES}} : x \in \overline{X}\}$ is $bF(X)/\hat{\rho}_{\mathbf{ES}}$ itself can be used to show the analogous result for $\hat{W}(X)/\rho^W_{\mathbf{LI}}$, as follows.

Let T be the least regular subsemigroup of $\hat{W}(X)/\rho^W_{\mathbf{LI}}$ containing the set $\{x\rho^W_{\mathbf{LI}} : x \in \overline{X}\}$, constructed as in 3.2.3. Then $w\rho^W_{\mathbf{LI}} \in T$ whenever $w \in \hat{W}_1 =$

$\langle \overline{X} \rangle$. Suppose that $w\rho_{LI}^W \in T$ for every $w \in \hat{W}_{2i-1}$, where $i \geq 1$, and let $a, b \in \hat{W}_{2i-1} \cap \hat{K}(X)$. Let $v \in \hat{W}(X)$ be such that $v\rho_{LI}^W \in V(ab\rho_{LI}^W)$. Then

$$bva\rho_{LI}^W \in S(a\rho_{LI}^W, b\rho_{LI}^W) = \{s(a\rho_{LI}^W, b\rho_{LI}^W)\},$$

so that $bva\rho_{LI}^W \in T$ by the definition of T since $a\rho_{LI}^W, b\rho_{LI}^W \in T$ by the induction hypothesis. Further, we have $bva\hat{\rho}_{LI} = s(a\hat{\rho}_{LI}, b\hat{\rho}_{LI}) = s(a,b)\hat{\rho}_{LI}$, and hence $s(a,b)\rho_{LI}^W = bva\rho_{LI}^W \in T$. Thus $\hat{W}(X)/\rho_{LI}^W = T$, and by an argument analogous to that used in the proof of Theorem 3.2.10 it now follows that $bF_{LI}(X) \cong \hat{W}(X)/\rho_{LI}^W$. □

We can proceed exactly as in Section 3.1 to describe a Birkhoff-style theory in this setting, as follows. Let $\mathbf{V} = \mathbf{ES}$ or $\mathbf{V} = L\mathbf{I}$. The (partial) unary semigroup homomorphism $(\rho_{\mathbf{V}}^W)^\sharp$ maps $\hat{W}(X)$ onto $bF_{\mathbf{V}}(X)$. So for any semigroup $S \in \mathbf{V}$, each matched mapping $\phi : \overline{X} \to S$ extends uniquely (by the universal property of bifree objects) to a semigroup homomorphism $\theta : \hat{W}(X) \to S$ such that $\theta \circ \theta^{-1} \supseteq \rho_{\mathbf{V}}^W$. In the E-solid case, if $u, v \in \hat{K}(X) \cap \hat{W}(X)$ then $uv\theta$ lies in a subgroup of S, and $s(u,v)\theta = v(uv)'u\theta = v\theta(uv\theta)^{-1}u\theta$ by Lemma 3.2.20. In the locally inverse case, if $u, v \in \hat{K}(X) \cap \hat{W}(X)$ then $s(u,v)\theta = s(u\theta, v\theta)$ (since $s(u,v)\rho_{LI}^W = s(u\rho_{LI}^W, v\rho_{LI}^W)$ by the last paragraph in the proof of Theorem 3.2.21).

With the obvious changes, all the theory set out in Section 3.1 can be duplicated, and an analogue of Theorem 3.1.17 also holds in this context. The advantage of the approach of Section 3.1 over that given in this section is clear: here we cannot explicitly give generators of the congruences relating to the bifree objects, but must rely on the restrictions of congruences on $bF(X)$.

3.3 E-varieties of locally E-solid semigroups.

Recall that $L\mathbf{ES}$ denotes the e-variety of all locally E-solid semigroups. In [25] Kaďourek established that trifree objects exist in all e-varieties of locally orthodox semigroups. In Section 3.3.1 we establish the existence of trifree objects in e-varieties of locally E-solid semigroups, and show in Section 3.3.2 that trifree objects on at least three generators exist in an e-variety $\mathbf{V}$ if and only if $\mathbf{V} \subseteq L\mathbf{ES}$. This also generalizes Yeh's result [43] which states that bifree objects on at least two generators exist in an e-variety $\mathbf{V}$ if and only if $\mathbf{V} \subseteq \mathbf{ES}$ or $\mathbf{V} \subseteq L\mathbf{I}$.

3.3.1 The existence of trifree objects in locally E-solid e-varieties.

First we generalize Kaďourek's results, which we have stated in Section 1.3.7, to classes of locally E-solid semigroups.

A semigroup S without zero is said to be *completely simple* if S is completely regular and has only one $\mathcal{D}$-class.

Result 3.3.1 ([19],Theorem 7) *For every locally E-solid semigroup S there exists a least congruence ρ on S such that S/ρ is locally inverse. For every $e \in E(S)$ the class $e\rho$ is a completely simple subsemigroup of S.* □

Recall from Notation 1.3.6 that if k is a member of a subgroup of a semigroup S then k° denotes the unique idempotent $\mathcal{H}$-related to k.

We have the following characterizations of locally E-solid semigroups.

Theorem 3.3.2 *For a regular semigroup S the following are equivalent:*

(i) S *is locally E-solid,*

(ii) *if* $e \in E(S)$ *and* $f, g \in E(eSe)$ *then* $S(fg, fg) \cap V(gf) \neq \emptyset$,

(iii) *if* $e \in E(S)$ *and* $f, g \in E(eSe)$ *then* $S(fg, fg) \subseteq V(gf)$, *and*

(iv) *if* $a \in Sx$ *and* $b \in yS$ *for* $x, y \in S$ *then* $V(b)S(k, k)V(a) \subseteq V(ab)$ *for every* $k \in bV(b)S(x, y)V(a)a$.

Proof: (i) $\Rightarrow$ (iv): Let $S \in$ **LES**. Suppose that $a \in Sx$ and $b \in yS$ for $x, y \in S$, and let $k = bb'ua'a$ where $u \in S(x,y)$, $a' \in V(a)$ and $b' \in V(b)$. Consider $v \in V(ab)$ and $h = bva \in S(a,b)$, and let ρ be the least locally inverse congruence on S. Then $h\rho \in S(a\rho, b\rho) = \{s(a\rho, b\rho)\}$. But $(b'ua')\rho \in V(ab\rho)$ by Result 1.3.39, so

$$\begin{aligned} k\rho &= (bb'ua'a)\rho \\ &\in b\rho V(ab\rho)a\rho \\ &= S(a\rho, b\rho) \\ &= \{h\rho\}. \end{aligned}$$

Hence h and k lie in the same completely simple subsemigroup D of S by Result 3.3.1. This means that every $\mathcal{H}$-class of D is a group and that the elements $h, k, hk^\circ, hk^\circ h \in D$ are all $\mathcal{D}$-related in S. Now by Result 1.3.4 we have $h \;\mathcal{H}\; hk^\circ h$. Since $khk = bb'ua'a(bva)bb'ua'a = bb'ua'abb'ua'a = k^2$, then

$$(hk^\circ h)^2 = hk^\circ hk^\circ h = hk^{-1}khkk^{-1}h = hk^{-1}k^2k^{-1}h = hk^\circ h.$$

Therefore $h = hk^\circ h$ since h and $hk^\circ h$ are $\mathcal{H}$-related idempotents. Note that $ahb = ab$ by Result 1.3.15(iv). Also, if $a^* \in V(a)$ and $b^* \in V(b)$ then $bb^*k = k = ka^*a$, and hence $bb^*k^\circ = bb^*kk^{-1} = k^\circ = k^\circ a^*a$, so that

$$ab(b^*k^\circ a^*)ab = ahb(b^*k^\circ a^*)ahb = ahk^\circ hb = ahb = ab$$

and $b^*k^\circ a^*(ab)b^*k^\circ a^* = b^*k^\circ a^*$. Therefore $b^*k^\circ a^* \in V(ab)$; so that we have $V(b)S(k,k)V(a) \subseteq V(ab)$, as required, since $S(k,k) = \{k^\circ\}$ by Lemma 3.1.1.

(iv) $\Rightarrow$ (iii): Let $e \in E(S)$ and $f, g \in E(eSe)$. Note that

$$S(fg, fg) = fS(fg, fg)g$$

and

$$fg = feg = ffegg \in fV(f)S(e,e)V(g)g,$$

so that, since $g \in Se$ and $f \in eS$, we have

$$S(fg, fg) = fS(fg, fg)g \subseteq V(f)S(fg, fg)V(g) \subseteq V(gf).$$

(iii) $\Rightarrow$ (ii) trivially.

(ii) $\Rightarrow$ (i): Let $e \in E(S)$ and $f, g \in E(eSe)$. Then $S(fg, fg) \cap V(gf) \neq \emptyset$ and $S(gf, gf) \cap V(fg) \neq \emptyset$, so we may choose $w \in V((fg)^2)$ and $z \in V((gf)^2)$ such that $u = fgwfg \in V(gf)$ and $v = gfzgf \in V(fg)$. Then

$$\begin{aligned} gf &= gfugf \\ &= gf(fgwfg)gf \\ &= g(fgvfg)w(fgvfg)f \\ &= gfgvuvfgf \\ &= gfg(gfzgf)u(gfzgf)fgf \\ &\in (gf)^2S(gf)^2, \end{aligned}$$

so that $gf\ \mathcal{H}\ (gf)^2$. Therefore $eSe \in \mathbf{ES}$ by Result 1.3.11(i). □

Corollary 3.3.3 *Let $S \in$ **LES**. If $a \in Sx$ and $b \in yS$ for $x, y \in S$, and $k \in bV(b)S(x, y)V(a)a$, then k lies in a subgroup of S and $S(k, k) = \{k^\circ\}$.*

Proof: The proof that (i) implies (iv) of Lemma 3.3.2 shows that k lies in a subgroup of S, and so $S(k, k) = \{k^\circ\}$ by Lemma 3.1.1. □

Recall that by Result 1.3.5 an element k of a regular semigroup S is in a subgroup of S if and only if $k\ \mathcal{H}\ k^2$, and that in this case the unique $\mathcal{H}$-related inverse of k is denoted k^{-1}.

The next result gives the existence of least regular subsemigroups in locally E-solid semigroups. Note that the subsemigroup R is the closure of $A \cup C$ in S under the operations of multiplication and taking group inverses.

Lemma 3.3.4 *Let S be a locally E-solid semigroup. Suppose that $A, C \subseteq S$ are such that*

(i) $A \cap V(a) \neq \emptyset$ for every $a \in A$,

(ii) $C \subseteq \bigcup\{S(a,b) : a, b \in A\}$, and

(iii) $C \cap S(a,b) \neq \emptyset$ for every $a, b \in A$.

Then the semigroup $R = \bigcup_{i \geq 0} R_{2i+1}$ is the least regular subsemigroup of S containing $A \cup C$, where

$$\begin{aligned}
R_0 &= A \cup C \\
R_1 &= \langle R_0 \rangle \\
&\vdots \\
R_{2i} &= \{r^{-1} : r \in R_{2i-1}, r \,\mathcal{H}\, r^2\} \cup R_{2i-1} \\
R_{2i+1} &= \langle R_{2i} \rangle \\
&\vdots
\end{aligned}$$

Proof: We first prove by induction that for every $r \in R$ we can find $a, b \in A$ such that $r \in aR \cap Rb$. If $r \in C$ then $r = aub$ for some $a, b \in A$ and $u \in V(ba)$, so that the result holds for all $r \in A \cup C$ and hence for all $r \in R_1 = \langle A \cup C \rangle$. Now assume that $i \geq 1$ and that if $r \in R_{2i-1}$ then $r \in aR \cap Rb$ for some $a, b \in A$. If $r \in R_{2i}$ then either $r \in R_{2i-1}$, or $r = s^{-1} = ss^{-3}s$ for some $s \in R_{2i-1}$ with $s \,\mathcal{H}\, s^2$. In both cases $r \in aR \cap Rb$ for some $a, b \in A$ by the induction hypothesis, and hence the same holds for all $r \in R_{2i+1} = \langle R_{2i} \rangle$.

The members of $A \cup C$ have inverses in $A \cup C$. Assume inductively that the members of R_{2i-2} have inverses in R, where $i \geq 1$, and consider $r \in R_{2i}$. If

$r = s^{-1}$ for $s \in R_{2i-1}$ with $s \mathcal{H} s^2$ then $s \in V(r) \cap R$. So let $r = r_1 \dots r_n$ where $r_1, \dots, r_n \in R_{2i-2}$; if $n = 1$ then r has an inverse in R. Suppose that $n \geq 2$ and that if $t_1, \dots, t_{n-1} \in R_{2i-2}$ then $V(t_1 \dots t_{n-1}) \cap R \neq \emptyset$. Let $t = r_1 \dots r_{n-1}$ and let $t' \in V(t) \cap R$ and $r_n' \in V(r_n) \cap R$. We may assume $t \in Ra$ and $r_n \in bR$ for some $a, b \in A$. Let $k = r_n r_n' c t' t$, where $c \in S(a,b) \cap C$. Then $k \in R$, and Corollary 3.3.3 shows that k lies in a subgroup of S and $S(k,k) = \{k^\circ\}$. Consequently $k^{-1} \in R$, so that $k^\circ \in R$, and hence $r_n' k^\circ t' \in V(r) \cap R$ by Lemma 3.3.2. Thus R is regular, and is clearly the least regular subsemigroup of S containing $A \cup C$.
□

For the next part we recall X and $\overline{X}$ as given in Notation 1.3.20, and $\overline{X}_1$ as given in Notation 1.3.41. The following definition is Definition 1.3.42, given again for convenience.

Definition 3.3.5 ([25])

- A *tied* mapping is a mapping $\phi : \overline{X}_1 \to S$ where S is a regular semigroup, $x'\phi \in V(x\phi)$ for every $x \in X$, and $s(x,y)\phi \in S(x\phi, y\phi)$ for every $x, y \in \overline{X}$.
- A *trifree object* for a class $\mathbf{V}$ of regular semigroups is a pair (S, ι), where $\iota : \overline{X}_1 \to S$ is a tied mapping, such that for any $T \in \mathbf{V}$ and tied mapping $\phi : \overline{X}_1 \to T$ there is a unique homomorphism $\theta : S \to T$ such that $\iota\theta = \phi$. □

As we have seen, the construction of the least regular subsemigroup of a locally E-solid semigroup containing sets of the form $A \cup C$, as described in Lemma 3.3.4, is not as simple as that for locally orthodox semigroups, where the subsemigroup $\langle A \cup C \rangle$ is itself regular. This gives rise to complications; for example, in the locally orthodox case tied mappings can be naturally extended to the free semigroup $\overline{X}_1{}^+$, but for locally E-solid semigroups we require a more artificial setting. There is a semigroup $T(X)$ for which some of the results of Sections 3.1 and 3.2 can be reproduced, with triidentities being equations between members of $T(X)$, but we do not discuss triidentities here. The description is complicated, and Kadourek's proof that there is no Birkhoff-type theorem connecting e-varieties of locally orthodox semigroups with sets of triidentities also shows that there is no such theorem for e-varieties of locally E-solid semigroups. Instead, we construct trifree objects in e-varieties of locally E-solid semigroups as subsemigroups of the free objects in certain varieties.

When Yeh constructed the bifree locally inverse and E-solid semigroups as subsemigroups of the corresponding free regular unary semigroups, he used *paired* mappings instead of matched mappings. As stated in Definition 1.3.21, a mapping $\phi : \overline{X} \to S$, where S is a regular semigroup, is said to be matched if $x'\phi \in V(x\phi)$ for every $x \in X$. A mapping $\phi : \overline{X} \to S$, where S is a regular semigroup, is said to be paired if it is matched and $x'\phi = y'\phi$ whenever $x\phi = y\phi$ for $x, y \in X$. The extra condition was required in order to obtain unary semigroup homomorphisms into S extending ϕ, via an inverse unary operation on S. But Hall (private communication, see [24]) has shown that the two definitions are equivalent for classes of regular semigroups closed under taking finite direct products. This follows from the fact that if $\phi : \overline{X} \to S$ is matched then the mapping $\overline{\phi} : \overline{X} \to S \times S$ given by $x\overline{\phi} = (x\phi, x'\phi)$, $x'\overline{\phi} = (x'\phi, x\phi)$, $x \in X$, is paired.

The material presented in this section appears in [13]. In the version of this paper that was first submitted for publication, a generalization of the concept of a paired mapping was used to define tied mappings for locally E-solid semigroups, for the purpose of locating trifree objects as subsemigroups of the corresponding free regular unary semigroups. The referee offered the idea of using semigroups with many unary operations, which allows the use of Kadourek's definition (3.3.5) of a tied mapping.

Definition 3.3.6 Let Λ be a set and $\mathbf{V}$ an e-variety. By a *regular Λ-unary semigroup* is meant a semigroup S together with a set $\{i_\lambda : \lambda \in \Lambda\}$ of inverse unary operations on S. $\square$

The variety of all regular Λ-unary semigroups is denoted $\mathbf{RUS}^\Lambda$. A straightforward generalization of the proof of Result 1.3.9 shows that for an e-variety $\mathbf{V}$ the class

$$\{(S, (i_\lambda)_{\lambda\in\Lambda}) \in \mathbf{RUS}^\Lambda : S \in \mathbf{V}\}$$

is a variety of regular Λ-unary semigroups; we denote it by $\mathbf{V}^\Lambda$.

Consider a fixed $\Lambda = X \cup \overline{X}^2$, and write $\mathcal{LES} = L\mathbf{ES}^\Lambda$. Let

$$(F_{\mathcal{LES}}(X), ({}^x)_{x\in X}, ({}^{yz})_{y,z\in\overline{X}})$$

be the free object on X in $\mathcal{LES}$; here ^{p} denotes the unary operation $u \mapsto u^p$, $u \in F_{\mathcal{LES}}(X)$, for all $p \in X \cup \overline{X}^2$.

Here we redefine $\overline{X}_1$. Let $I(X) = \{x^x : x \in X\} \subseteq F_{\mathcal{LES}}(X)$. Then X' can be identified with $I(X)$ by writing $x' = x^x$ for each $x \in X$, and we also write $\overline{X} = X \cup I(X)$. Let $s(y,z) = z(yz)^{yz}y$ for all $y, z \in \overline{X}$, and write $\overline{X}_1 = \overline{X} \cup \{s(y,z) : y, z \in \overline{X}\}$.

Suppose that $\mathbf{V}$ is a sub e-variety of $L\mathbf{ES}$, and write $\mathcal{V} = \mathbf{V}^\Lambda$. Let

$$(F_{\mathcal{V}}(X), ({}^x)_{x\in X}, ({}^{yz})_{y,z\in\overline{X}})$$

be the free object on X in $\mathcal{V}$. Then $F_{\mathcal{V}}(X) \in L\mathbf{ES}$ so by Lemma 3.3.4 there is a least regular subsemigroup R of $F_{\mathcal{V}}(X)$ such that $\overline{X}_1 \subseteq R$, constructed as in (6). Let $\iota : \overline{X}_1 \to R$ be the natural injection. Then ι is a tied mapping.

Theorem 3.3.7 *The semigroup R, together with the mapping ι, is the trifree object on X in $\mathbf{V}$.*

Proof: Let $S \in \mathbf{V}$, and let $\phi : \overline{X}_1 \to S$ be a tied mapping. Let ϕ' be the restriction of ϕ to X. For every $x \in X$ choose an inverse unary operation ^{x} on

S satisfying $(x\phi)^x = x^x\phi$. Now for every $y, z \in \overline{X}$ we have $s(y,z)\phi = z\phi w_{yz} y\phi$ for some $w_{yz} \in V(y\phi z\phi)$. So for every $y, z \in \overline{X}$ we can choose an inverse unary operation ^{yz} on S such that $(y\phi z\phi)^{yz} = w_{yz}$. Then

$$\overline{S} = (S, ({}^x)_{x\in X}, ({}^{yz})_{y,z\in\overline{X}}) \in \mathcal{V},$$

so there is a Λ-unary semigroup homomorphism $\varphi' : F_{\mathcal{V}}(X) \to \overline{S}$ extending ϕ'. Let φ be the restriction of φ' to R. Then φ extends ϕ: if $x \in X$ then $x\varphi = x\varphi' = x\phi' = x\phi$ and $x^x\varphi = x^x\varphi' = (x\varphi')^x = (x\phi)^x = x^x\phi$; and if $y, z \in \overline{X}$ then

$$\begin{aligned} s(y,z)\varphi &= s(y,z)\varphi' \\ &= (z(yz)^{yz}y)\varphi' \\ &= z\varphi'(yz\varphi')^{yz}y\varphi' \\ &= z\phi(y\phi z\phi)^{yz}y\phi \\ &= z\phi w_{yz} y\phi \\ &= s(y,z)\phi. \end{aligned}$$

It remains to show the uniqueness of φ. Suppose that $\theta : R \to S$ is a semigroup homomorphism extending ϕ. Then $u\varphi = u\theta$ for every $u \in \overline{X}_1\iota = R_0$, and hence for every $u \in R_1 = \langle R_0 \rangle$. Let $i \geq 1$, and assume inductively that $u\varphi = u\theta$ for every $u \in R_{2i-1}$. Let $u \in R_{2i}$. If $u \in R_{2i-1}$ then $u\varphi = u\theta$ by the induction assumption. Otherwise, $u = r^{-1}$ for some $r \in R_{2i-1}$ with $r \;\mathcal{H}\; r^2$, and in this case $r\varphi = r\theta$ so that $u\varphi = u\theta$ (by Result 1.3.7). Thus $u\varphi = u\theta$ for every $u \in R_{2i+1} = \langle R_{2i} \rangle$, and the proof is completed. □

Remark 3.3.8 For an e-variety $\mathbf{V} \subseteq \mathbf{ES}$ or $\mathbf{V} \subseteq \mathcal{L}\mathbf{I}$, it is easy to see that the least regular subsemigroup T of $F_{\mathbf{V}^\Lambda}(X)$ containing $\overline{X}$ is isomorphic to the least regular subsemigroup of $F_{\mathbf{V}'}(X)$ containing $\overline{X}$ (recall Result 1.3.9 for the notation $\mathbf{V}'$), and is therefore bifree in $\mathbf{V}$ by Result 1.3.37. If $\mathbf{V} \subseteq \mathbf{ES}$ then for $x, y \in \overline{X}$ the element $s(x,y) \in F_{\mathbf{V}^\wedge}(X)$ is not necessarily a member of T, and in this case the bifree and trifree objects in $\mathbf{V}$ on X are not isomorphic. However if $\mathbf{V}$ is contained in $L\mathbf{I}$ then the bifree and trifree objects in $\mathbf{V}$ on X are isomorphic. $\square$

3.3.2 Only locally E-solid e-varieties admit trifree objects.

The aim in this section is to show that trifree objects on three or more generators exist in an e-variety $\mathbf{V}$ only if $\mathbf{V} \subseteq L\mathbf{ES}$.

Let C_2 denote the Rees matrix semigroup

$$\mathcal{M}^\circ(\langle 1 \rangle, 2, 2, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}).$$

Result 3.3.9 ([21]) *For an e-variety* $\mathbf{V}$ *we have* $\mathbf{V} \subseteq \mathbf{ES}$ *if and only if* $C_2 \notin \mathbf{V}$. $\square$

The next result is well-known, and the proof is easy.

Result 3.3.10 *For an e-variety* $\mathbf{V}$ *we have* $\mathbf{V} \subseteq L\mathbf{I}$ *if and only if* $\mathbf{V}$ *does not contain* R^1*, the two element right (equivalently, left) zero semigroup with adjoined identity.* $\square$

For a class $\mathcal{C}$ of regular semigroups let $\mathbf{V}(\mathcal{C})$ be the e-variety generated by $\mathcal{C}$.

In [43], Yeh constructed a regular semigroup in $\mathbf{V}(C_2, R^1)$ with a subset $A = \{e, f\}$, where e and f are idempotents, which has no least regular subsemigroup containing A. Yeh used this together with Results 3.3.9 and 3.3.10 to prove that non-monogenic bifree objects exist in an e-variety $\mathbf{V}$ if and only if $\mathbf{V} \subseteq \mathbf{ES}$ or $\mathbf{V} \subseteq L\mathbf{I}$.

Following Yeh's method, we now proceed to establish an analogous result for $L\mathbf{ES}$. First we extend Result 3.3.9 by showing that for an e-variety $\mathbf{V}$ we have $\mathbf{V} \subseteq L\mathbf{ES}$ if and only if ${C_2}^1 \notin \mathbf{V}$, where ${C_2}^1$ is C_2 with an identity adjoined. We then construct a regular semigroup in $\mathbf{V}({C_2}^1, R^1) = \mathbf{V}({C_2}^1)$ with subsets A, C that satisfy the conditions of Lemma 3.3.4, but which has no least regular subsemigroup containing $A \cup C$. As a consequence of these results we prove that trifree objects on three or more generators exist in an e-variety $\mathbf{V}$ if and only if $\mathbf{V} \subseteq L\mathbf{ES}$.

For an e-variety $\mathbf{V}$, let $\mathbf{V}_M$ be the e-variety generated by the monoids of $\mathbf{V}$.

Lemma 3.3.11 *For an e-variety* $\mathbf{V}$ *we have*

$$\begin{aligned} \mathbf{V}_M &= \{S : S \textit{ is a regular subsemigroup of a monoid of } \mathbf{V}\} \\ &= \{S \in \mathbf{RS} : S^1 \in \mathbf{V}\}. \end{aligned}$$

Proof: Let $\mathbf{W} = \{S : S$ is a regular subsemigroup of a monoid of $\mathbf{V}\}$. Then $\{S \in \mathbf{RS} : S^1 \in \mathbf{V}\} \subseteq \mathbf{W}$. Conversely, if $S \in \mathbf{W}$ then there is a monoid $T \in \mathbf{V}$, with identity $e \in T$, such that S is a regular subsemigroup of T. But then $S \cup \{e\}$ is a monoid in $\mathbf{V}$, so that $\mathbf{W} \subseteq \{S \in \mathbf{RS} : S^1 \in \mathbf{V}\}$; and thus $\mathbf{W} = \{S \in \mathbf{RS} : S^1 \in \mathbf{V}\}$.

Clearly $\mathbf{W}$ is closed under taking regular subsemigroups, and also $\mathbf{W} \subseteq \mathbf{V}$. Suppose that $f : S \to U$ is a surjective homomorphism, for $S \in \mathbf{W}$. If $S = S^1$ then U is a monoid in $\mathbf{V}$, and hence $U \in \mathbf{W}$. If S is not a monoid but is a subsemigroup of a monoid $T \in \mathbf{V}$, with identity e, then f may be extended to $f^1 : S^1 \to U^1$ where $S^1 = S \cup \{e\}$ with e as identity and $U^1 = U \cup \{f^1(e)\}$ with $f^1(e)$ as identity. Since $S^1 \in \mathbf{V}$ then $U^1 \in \mathbf{V}$, and therefore $\mathbf{W}$ is closed under homomorphisms.

Now let $\{S_i : i \in I\}$ be a family from $\mathbf{W}$. For each $i \in I$ let e_i be the identity for S_i^1. Then $\prod_{i \in I} S_i^1$ is a monoid in $\mathbf{V}$, with identity $(e_i)_{i \in I}$. But $\prod_{i \in I} S_i$ is a regular subsemigroup of $\prod_{i \in I} S_i^1$, and is therefore a member of $\mathbf{W}$. Hence $\mathbf{W}$ is an e-variety, and it follows that $\mathbf{W} = \mathbf{V}_M$. $\square$

Corollary 3.3.12 *An e-variety* $\mathbf{V} \not\subseteq L\mathbf{ES}$ *if and only if* $C_2{}^1 \in \mathbf{V}$.

Proof: Recall from Section 1.3.3 that the class

$$L\mathbf{V} = \{S \in \mathbf{RS} : eSe \in \mathbf{V} \text{ for all } e \in E(S)\}$$

is an e-variety. Note that $LL\mathbf{V} = L\mathbf{V}$. Clearly $\mathbf{V} \not\subseteq L\mathbf{ES}$ if and only if $L\mathbf{V} \not\subseteq L\mathbf{ES}$.

If $S \in L\mathbf{V} \setminus L\mathbf{ES}$ then $eSe \in \mathbf{V} \setminus \mathbf{ES}$ for some $e \in E(S)$, and so $eSe \in \mathbf{V}_M \setminus \mathbf{ES}$. Thus $L\mathbf{V} \not\subseteq L\mathbf{ES}$ implies $\mathbf{V}_M \not\subseteq \mathbf{ES}$; conversely $\mathbf{V}_M \not\subseteq \mathbf{ES}$ implies $\mathbf{V} \not\subseteq \mathbf{ES}$, and hence $L\mathbf{V} \not\subseteq L\mathbf{ES}$. Therefore $\mathbf{V} \not\subseteq L\mathbf{ES}$ if and only if $\mathbf{V}_M \not\subseteq \mathbf{ES}$. Finally, $\mathbf{V}_M \not\subseteq \mathbf{ES}$ if and only if $C_2 \in \mathbf{V}_M$, and $C_2 \in \mathbf{V}_M$ if and only if $C_2{}^1 \in \mathbf{V}$ by Lemma 3.3.11. $\square$

Recall from Notation 1.1.10 that for a nonempty set X the free semigroup and free monoid on X are denoted X^+ and X^* respectively.

We now introduce the example. Let $A = \{a, b, c, d, e, f\}$ be a set of distinct variables. Let $A^+ \cup \{0\}$ be the free semigroup A^+ with an adjoined zero. So $x \cdot 0 = 0 \cdot x = 0$ for all $x \in A^+$.

Define

$$\begin{aligned} T &= \{0, a, b, c, d, db, ad, ab, e, ec, ae, ac, f, fb, fc, fe, af, \\ &\qquad afb, afc, afe, df, dfb, dc, de, adf, adfb, adc, ade\} \qquad (3.2) \\ &\subseteq \{a, b, c, d, e, f\}^+ \cup \{0\}. \end{aligned}$$

It can be routinely checked that T is the subsemigroup of $A^+ \cup \{0\}$ generated by $\{a, b, c, d, e, f, 0\}$ subject to the relations

$$\begin{array}{lllll}
aa = a & bf = 0 & cf = f & ee = e & adb = ab \\
ba = 0 & ca = 0 & da = d & ef = f & aec = ac \\
bb = b & cb = 0 & dd = d & fa = f & dec = dc \\
bc = 0 & cc = c & ea = e & fd = f & dfc = dc \\
bd = d & cd = 0 & eb = 0 & ff = f & dfe = de \\
be = 0 & ce = e & ed = 0 & & fec = fc
\end{array}$$

This semigroup has been constructed with significant help from J. Almeida and his computer program for constructing semigroups from their presentations. Given that T is a semigroup we easily see that T is a regular combinatorial semigroup of order 28, with 7 $\mathcal{D}$-classes as pictured below (the $*$ denotes that an element is an idempotent).

*b	*a	*c

*d	*db
*ad	ab

*e	*ec
*ae	ac

*f	fb	*fc	*fe
*af	afb	afc	*afe
*df	*dfb	dc	de
*adf	adfb	adc	ade

*0

Our first aim is to show that $T \in \mathbf{V}({C_2}^1)$. To this end we need some preliminary results.

A *semilattice* is a commutative semigroup of idempotents. A semigroup S is a *local semilattice* if eSe is a semilattice for every $e \in E(S)$.

Result 3.3.13 ([21]) *The e-variety generated by C_2 is precisely the class of all regular local semilattices.* $\square$

See Trakhtman in [38, Theorem 20.4] or Trotter [41] for a proof of the following well known result.

Result 3.3.14 *A basis of identities for the semigroup variety generated by C_2 is given by*

$$x^3 = x^2, x(yx)^2 = xyx, xyxzx = xzxyx. \qquad \square$$

Let X be a nonempty set, with $Y \subseteq X$. For a word $u \in X^+$ define $u_Y \in Y^*$ to be the word obtained from u by deleting all occurrences of variables not in Y.

For an e-variety $\mathbf{V}$ let $\mathbf{V}^M = \mathbf{V}(\{S^1 : S \in \mathbf{V}\})$. The proof of [3, Lemma 7.2.1] is easily generalized to give the next result.

Lemma 3.3.15 *Suppose that $\mathbf{V}$ is an e-variety and X is a nonempty set. Let $u, v \in X^+$. Then the class $\mathbf{V}^M$ satisfies the (semigroup) identity $u = v$ if and only if $\mathbf{V}$ satisfies the (semigroup) identity $u_Y = v_Y$ for every subset Y of X.* $\square$

We now give the main lemma of this section.

Lemma 3.3.16 *The semigroup T is a member of $\mathbf{V}({C_2}^1)$.*

Proof: It follows from Lemma 3.3.11 that $\mathbf{V}(C_2) \subseteq \mathbf{V}({C_2}^1)_M$, and that therefore any $W \in \mathbf{V}(C_2)$ is such that $W^1 \in \mathbf{V}({C_2}^1)$. Hence

$$\mathbf{V}({C_2}^1) = \mathbf{V}(\{W^1 : W \in \mathbf{V}(C_2)\}) = \mathbf{V}(C_2)^M.$$

Let X be a denumerable set of variables. By Lemma 3.3.15, an identity $u = v$ on X is now valid in the e-variety $\mathbf{V}({C_2}^1)$ if and only if the identity $u_Y = v_Y$ holds in the e-variety $\mathbf{V}(C_2)$ for any set $Y \subseteq X$.

By Result 3.3.13, the e-variety $\mathbf{V}(C_2)$ is precisely the class of all regular local semilattices. By Result 3.3.14 a basis of identities for the semigroup variety generated by C_2 is given by

$$x^3 = x^2, x(yx)^2 = xyx, xyxzx = xzxyx.$$

Conversely, it is easy to see that any regular semigroup that satisfies these identities is locally a semilattice. Hence these identities form a basis of biidentities for $\mathbf{V}(C_2)$. In particular, if $u = v$ is a semigroup identity for $\mathbf{V}(C_2)$, where $u = x_1 \dots x_n$ and $v = y_1 \dots y_m$ for $x_1, \dots, x_n, y_1, \dots, y_m \in X$, then the following are true:

- $x_1 = y_1$,
- $x_n = y_m$,
- $\{x_1, \dots, x_n\} = \{y_1, \dots, y_m\}$, and
- $\{x_i x_{i+1} : 1 \leq i \leq n-1\} = \{y_i y_{i+1} : 1 \leq i \leq m-1\}$.

Suppose that $T \notin \mathbf{V}({C_2}^1)$. Then there is a semigroup identity $u = v$, where $u = x_1 \dots x_n$ and $v = y_1 \dots y_m$ for $x_1, \dots, x_n, y_1, \dots, y_m \in X$, that is satisfied by ${C_2}^1$ but not by T. Let $\phi : X^+ \to T$ be a semigroup homomorphism such that $u\phi \neq v\phi$.

Observe that $D_f \cup \{0\}$ is an ideal of T, where D_f is the $\mathcal{D}$-class of f, and that $T/(D_f \cup \{0\})$ and $T \setminus \{a,b,c\}$ are regular local semilattices. Therefore the semigroups $T/(D_f \cup \{0\})$ and $T \setminus \{a,b,c\}$ satisfy the identity $u = v$, and consequently $\{a,b,c\} \cap \{x_1\phi, \ldots, x_n\phi, y_1\phi, \ldots, y_m\phi\} \neq \emptyset$ and $u\phi, v\phi \in D_f \cup \{0\}$. We may assume $u\phi \neq 0$.

Notice that if the variable a occurs in $t \in T$, as displayed in (3.2), then $t = as$ for some $s \in T^1$ such that a does not occur in s. Let $\psi : X^+ \to T^1$ be the semigroup homomorphism given by

$$x\psi = \begin{cases} x\phi & \text{if} \quad a \text{ does not occur in } x\phi \\ w & \text{if} \quad x\phi = aw \end{cases}$$

for all $x \in X$.

Notice that

$$xa = \begin{cases} x & \text{if} \quad x \in \{a,d,e,f\} \\ 0 & \text{if} \quad x \in \{b,c\}, \end{cases}$$

and so there is no pair x_i, x_{i+1} for which $x_{i+1}\phi$ begins with a and $x_i\phi$ ends with b or c. Hence

$$u\phi = \begin{cases} a.u\psi & \text{if} \quad a \text{ appears in } x_1\phi \\ u\psi & \text{if} \quad a \text{ does not appear in } x_1\phi. \end{cases}$$

Therefore $u\psi \neq 0$, and hence if $v\psi = 0$ then $u\psi \neq v\psi$. If $v\phi \neq 0$ then we may repeat the above for v; and then, since $x_1 = y_1$, we have $u\phi = u\psi$ if and only if $v\phi = v\psi$. Consequently $u\psi \neq v\psi$. Otherwise, $v\phi = 0$ and $v\psi \neq 0$. Since $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_m\}$, then $y_i\phi \neq 0$ for all i. It follows that $y_i\phi$ ends with b or c, and $y_{i+1}\phi$ begins with a for some i. But $y_i y_{i+1} = x_j x_{j+1}$ for some j, so this is a contradiction. Thus in all cases we have $u\psi \neq v\psi$.

Therefore, if $Y = \{x \in X : x\phi \neq a\}$, then $u_Y = v_Y$ is an identity satisfied by $\mathbf{V}({C_2}^1)$ and $\psi : X^+ \to T$ is a homomorphism such that $u_Y\psi = u\psi \neq v\psi = v_Y\psi$; and so we may assume that the range of ϕ excludes a.

Observe also that

$$bx = \begin{cases} x & \text{if} \quad x \in \{b, d\} \\ 0 & \text{if} \quad x \in \{a, c, e, f\} \end{cases}$$

and

$$cx = \begin{cases} x & \text{if} \quad x \in \{c, e, f\} \\ 0 & \text{if} \quad x \in \{a, b, d\}. \end{cases}$$

By a dual of the above argument we may assume the range of ϕ also excludes b and c. But then ϕ maps into $T \setminus \{a, b, c\} \in \mathbf{V}(C_2)$, and this provides the contradiction since $u = v$ is satisfied by $\mathbf{V}(C_2)$. □

Consider the following subsets of T:

$$\begin{aligned} O &= \{a, b, d, db, ad, ab\}, \\ U &= \{0, c, e, ec, ae, ac, fc, afc, dc, adc, fe, afe, de, ade\}, \\ V &= \{0, f, af, df, adf, fb, afb, dfb, adfb\}. \end{aligned}$$

Let $R = \{u, v\}$ be a right zero semigroup (so R satisfies the identity $xy = y$) and consider the regular subsemigroup

$$P = \{(x, 1), (y, u), (z, v) : x \in O, y \in U \cup V, z \in V\}$$

of $T \times R^1$. Let $Q = P/I$, where I is the ideal $\{(0, u), (0, v)\}$ of P. The $\mathcal{D}$-classes of Q are given below in Figure 3.1.

It follows from Lemma 3.3.11 that $R^1 \in \mathbf{V}({C_2}^1)$, and therefore $Q \in \mathbf{V}({C_2}^1)$ by Lemma 3.3.16. Consider the subsets

$$A = \{(a, 1), (b, 1), (c, u)\} \text{ and } C = \{(d, 1), (e, u), 0\}$$

of Q. Observe that $(d, 1) \in S((a, 1), (b, 1))$ and $(e, u) \in S((a, 1), (c, u))$; and that $S(p, q) = \{0\}$ for all $(p, q) \in (A \times A) \setminus \{((a, 1), (b, 1)), ((a, 1), (c, u))\}$ with $p \neq q$. Then A and C satisfy the conditions of Lemma 3.3.4 (with Q in place of S).

*(b,1)	*(a,1)	*(c,u)

*(d,1)	*(db,1)
*(ad,1)	(ab,1)

*(e,u)	*(ec,u)
*(ae,u)	(ac,u)

*(f,v)	(fb,v)	*(f,u)	(fb,u)	*(fc,u)	*(fe,u)
*(af,v)	(afb,v)	*(af,u)	(afb,u)	(afc,u)	*(afe,u)
*(df,v)	*(dfb,v)	*(df,u)	*(dfb,u)	(dc,u)	(de,u)
*(adf,v)	(adfb,v)	*(adf,u)	(adfb,u)	(adc,u)	(ade,u)

*0

Figure 3.1: The $\mathcal{D}$-classes of the semigroup Q.

Lemma 3.3.17 *There is no least regular subsemigroup of Q that contains $A \cup C$.*

Proof: Let

$$T_1 = \{(x,1),(y,u) : x \in O, y \in U \cup V\}$$

and

$$T_2 = \{(x,1),(y,u),(z,v) : x \in O, y \in U, z \in V\}.$$

Write $Q_1 = T_1/I$ and $Q_2 = T_2/I$. Then Q_1 and Q_2 are regular subsemigroups of Q containing $A \cup C$. However,

$$T_1 \cap T_2 = \{(x,1),(y,u) : x \in O, y \in U\}$$

so that $(de,u) \in Q_1 \cap Q_2$; but the inverses of de in T are f, fb, af, afb, and none of these is a member of U. Therefore $Q_1 \cap Q_2$ is not regular. $\square$

We conclude this section with the main result.

Theorem 3.3.18 *Suppose that* $\mathbf{V}$ *is an e-variety and* X *is a set with* $|X| \geq 3$. *There is a trifree object in* $\mathbf{V}$ *on* X *if and only if* $\mathbf{V} \subseteq \mathbf{\mathit{LES}}$.

Proof: Suppose $\mathbf{V} \not\subseteq \mathbf{\mathit{LES}}$. Then ${C_2}^1 \in \mathbf{V}$ by Corollary 3.3.12, and hence $Q \in \mathbf{V}$. Suppose there is a trifree object (F, ι) on X in $\mathbf{V}$. Let x, y, z be distinct members of X. We may consider a mapping $\varphi : \overline{X}_1 \to Q$ which is tied and satisfies

- $w\varphi = w'\varphi = s(w, w)\varphi = s(w', w')\varphi = \begin{cases} (a, 1) & \text{if} \quad w = x \\ (b, 1) & \text{if} \quad w = y \\ (c, u) & \text{if} \quad w = z \end{cases}$
- $s(x, y)\varphi = s(x', y')\varphi = (d, 1)$; $s(x, z)\varphi = s(x', z')\varphi = (e, u)$
- $s(w, t)\varphi = 0$ for all other $w, t \in \{x, y, z, x', y', z'\}$.

Let $\overline{\varphi} : F \to Q$ be the unique homomorphism satisfying $\iota\overline{\varphi} = \varphi$. Then $F\overline{\varphi}$ is a regular subsemigroup of Q containing $A \cup C$; moreover, the above argument shows that any regular subsemigroup of Q that contains $A \cup C$ also contains $F\overline{\varphi}$. This contradicts Lemma 3.3.17. $\square$

Remark 3.3.19 In [25] Kadourek claimed, without proof, the statement of Theorem 3.3.18. $\square$

3.4 A theory of n-varieties.

The theory for locally E-solid semigroups that was developed in the previous section can be extended in such a way as to obtain a suitable notion of a "free object" for every e-variety. Given a regular semigroup S and $3 \leq n < \infty$, we say that subsets $A_2, A_3, \ldots, A_n$ of S satisfy property $(*_n)$ if

(i) $A_2 \cap V(a) \neq \emptyset$ for all $a \in A_2$, and

(ii) $A_2 \subseteq \ldots \subseteq A_n$;

and if for $3 \leq j \leq n$

(iii) every $a \in A_j$ is a member of $S(u_1 \ldots u_k, v_1 \ldots v_\ell)$ for some $k, \ell < j$ and $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in A_{j-1}$, and

(iv) $S(u_1 \ldots u_k, v_1 \ldots v_\ell) \cap A_j \neq \emptyset$ for every $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in A_{j-1}$, where $k, \ell < j$.

If subsets $A_2, A_3, \ldots$ of a regular semigroup S satisfy property $(*_n)$ for every n, $3 \leq n < \infty$, then define $A_\infty = \bigcup_{n \geq 3} A_n$, and say that $A_2, A_3, \ldots$ satisfy property $(*_\infty)$.

For every n, $3 \leq n \leq \infty$, let $\mathbf{V}_n$ be the class of all regular semigroups S for which the following is true: whenever $A_2, A_3, \ldots, A_n$ are subsets of S that satisfy property $(*_n)$, the closure of A_n under the operations of multiplication and taking group inverses is a regular subsemigroup of S.

Lemma 3.4.1 *Suppose that S is a regular semigroup, with subsets $A_2, A_3, \ldots$ which satisfy property $(*_\infty)$. Then $\langle A_\infty \rangle$ is regular.*

Proof: Let $a = a_1 \ldots a_m$ where $a_1, \ldots, a_m \in A_\infty$. Notice that $V(a_i) \cap A_\infty \neq \emptyset$ for each i, $1 \leq i \leq m$. Suppose that $a_1' \in V(a_1) \cap A_\infty$. If $m = 1$ then $a_1' \in V(a) \cap A_\infty$, so assume that $m \geq 2$ and proceed by induction. There exists $b' \in V(a_2 \ldots a_m) \cap \langle A_\infty \rangle$. Let $n \geq m - 1$ be such that $a_1, \ldots, a_m \in A_n$. Then there exists $c \in S(a_1, a_2 \ldots a_m) \cap A_{n+1}$, and $b'ca_1' \in V(a) \cap \langle A_\infty \rangle$ by Lemma 1.3.15(iii). □

By Lemma 3.4.1, the class $\mathbf{V}_\infty$ contains all regular semigroups, and hence $\mathbf{V}_\infty = \mathbf{RS}$. So for each e-variety we can obtain an analogue of a free object, as follows.

Recall the Definition 3.3.6 of a regular Λ-unary semigroup, and the discussion which follows. Let $\Gamma_2 = \overline{X} \times \overline{X}$, and for all $n \geq 3$ let

$$\Gamma_n = \{(u_1 \dots u_k, v_1 \dots v_\ell) : u_1, \dots, u_k, v_1, \dots, v_\ell \in \Gamma_{n-1} \text{ and } k, \ell < n\} \cup \Gamma_{n-1}$$

and $\Lambda_n = X \cup \Gamma_n$. Let $(F_{\mathcal{RUS}^{\Lambda_n}}(X), ({}^x)_{x \in X}, ({}^\gamma)_{\gamma \in \Gamma_n})$ be the free object on X in the variety $\mathcal{RUS}^{\Lambda_n}$. As in Section 3.3.1, we identify the set $I(X) = \{x^x : x \in X\}$ with X'. Let $X_2 = X \cup I(X)$. For $n \geq 3$ let

$$X_n = X_{n-1} \cup \{s(u_1 \dots u_k, v_1 \dots v_\ell) : u_1, \dots, u_k, v_1, \dots, v_\ell \in X_{n-1} \text{ and } k, \ell < n\},$$

where $s(u_1 \dots u_k, v_1 \dots v_\ell) = v_1 \dots v_\ell (u_1 \dots u_k v_1 \dots v_\ell)^{u_1 \dots u_k, v_1 \dots v_\ell} u_1 \dots u_k$.

Definition 3.4.2 Let $n \geq 3$. A mapping $\phi : X_n \to S$, where S is a regular semigroup, will be called *n-tied* if

(i) $x'\phi \in V(x\phi)$ for every $x \in X$, and

(ii) $s(u_1 \dots u_k, v_1 \dots v_\ell)\phi \in S(u_1\phi \dots u_k\phi, v_1\phi \dots v_\ell\phi)$ for every $k, \ell < n$ and $u_1 \dots u_k, v_1 \dots v_\ell \in X_{n-1}$. □

Let $3 \leq n \leq \infty$. By an *n-free object* for a class $\mathbf{V}$ of regular semigroups is meant a pair (S, ι), where $\iota : X_n \to S$ is an n-tied mapping, such that for any $T \in \mathbf{V}$ and n-tied mapping $\phi : X_n \to T$ there is a unique homomorphism $\theta : S \to T$ such that $\iota\theta = \phi$.

Let $\mathbf{V}$ be a class of regular semigroups closed under taking regular subsemigroups and direct products, and contained in $\mathbf{V}_n$ for some $3 \leq n \leq \infty$. Then F, the free object on X in the variety $\mathbf{V}^{\Lambda_n}$, exists. As usual, we assume that $X_n \subseteq F$. Let $\iota : X_n \to F$ be the natural injection. Then R, the closure of X_n in F under multiplication and taking group inverses, is the least regular subsemigroup of F containing X_n; and a straightforward modification of the proof of Lemma 3.3.7 shows that R, together with the mapping ι, is the n-free object in $\mathbf{V}$ on X.

Remark 3.4.3 When $n = 3$ the notions of n-tied mappings and n-free objects coincide with the notions of tied mappings and trifree objects as described in Section 3.3.1. $\square$

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