UNIVERSITY OF TASMANIA
DEPARTMENT OF MATHEMATICS

APPROXIMATION SOLUTIONS OF WAVE PROBLEMS

by

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Submitted in partial fulfilment of the requirement for the

MASTER OF SCIENCE STUDIES

1991

SUPERVISOR: DR.J.D.DONALDSON

ACKNOWLEDGMENT

First and foremost I wish to acknowledge Dr. J.D. Donaldson for his encouragement, supervision and guidance which has made this thesis possible. I would like to thank Dr. D.F. Paget, Mr. D. Fearnly- Sander, Mr. S. Wotherspoon, Dr. T. Stokes and Mr. M. Bulmer for many valuable illiminating discussions. I would also like to thank Dr. K. Hill, Math. comp. lab. manager, for his contribution in computing. Finally, I wish to thank Prof. Alex Lazenby, the former vice conselor of University of Tasmania, Prof. D. Elliot, Prof R. Lidl and all Mathematics department staffs, Dr. Bruce Scott and Mrs. Robin Bowden from AIDAB, whose led me to undertake the degree.

ABSTRAC

In the time dependent situations, the partial differential equations the most clossely associated with the wave propagation are of hyperbolic type. Their role in the study of non-linear wave propagation is becoming increasingly important; and the knowledge of the properties of their solution is of considerable value when applications to the physical situations are to be made. Non linearity in wave occur in the evolution of discontinuous solutions from initial data propagates along their characteristics. To obtain accuracy in numerical integrations, small intervals and difference formulas are convinient immediatly after crossing the characteristic curve. This work is intended to discussed several numerical solutions, for the two dimensional non-linear wave equations. The methods will be used involve successive approximation, characteristics, and finite difference methods.

Contents

PREFACE

Partial differential equations as mathematical models of physical and engineering problems involve *continuous functions* defined on a *continuous domain.* Analytic methods for solving *hyperbolic equations* are limited to well behaved problems, in which the partial differential equations involve constant coefficients and regular boundaries. Often the solution leads to indefinite integrals and infinite series in terms of special functions. For more complex situations *numerical methods* which use discrete values of both the function and the domain have been found to be extremely successful.

Discretizations involve two steps. First, the continuous domain is replaced by a mesh of discrete points within the domain. Second, the governing equations which are continuous formulations are replaced by a finite system of equations which is related to the original infinite system.

Basically, discretizations of partial differential equations may be accomplished by using one of three methods. The first involves expressing the partial differential equation in terms of its characteristic coordinates and integrating along the characteristic directions; this is called *the method of characteristic.* The second method approximates the original partial differential equation by a finite difference equation, and is known as *the finite difference method.* The third, *the finite element method*, uses direct approximation to the solution by a function from a finite dimensional space of functions. This thesis will deal with these approximation methods to solve *second-order hyperbolic equations* involving *initial value problems* and *boundary value problems.* .

The characteristics of a hyperbolic equation will be discussed in Chapter1. Based on the number of characteristics, of which there may be two, one or no *real characteristic*, we may classify the types of a partial differential equation to *parabolic*, *elliptic* and *hyperbolic.* The classification is a useful concept because the general theory and methods of solution usually apply only to a given class of equations.

In physical applications the initial value problem is well known as *the Cauchy problem.* If the Cauchy problem involves boundary conditions it will become *the Cauchy Gaursat problem.* These problems which describe propagation problems in mathematical models will be formulated in section 1.3. We will discuss the difficulties that arise when the Cauchy problem is formulated for a non-hyperbolic equation. However the Cauchy problem is well posed for hyperbolic equation.

The coordinates of a second order hyperbolic equation, when written in canonical form, are the *characteristic directions.* Characteristic coordinates are the *natural* coordinates of the system in the sense that, in terms of these coordinates, the equation is much more simplified, and can often be integrated directly. *The methods of successive approximation* will be found useful to solve a Cauchy problem for hyperbolic equations in the canonical form. Further if the canonical form of a hyperbolic equation is linear, *the Riemann's method* will give the solution of the Cauchy problem in terms of prescribed *Cauchy data.*

In physical applications, the wave equation is the prototype and most important example of hyperbolic equation. Mathematical models for some two-dimensional wave problems will be given in Chapter 2. We will discuss *Cauchy problems* involving *steady state boundaries* and *moving boundaries.* The analytical solutions, if any, will also be discussed.

Chapter 3 will deal with *quasi-linear* second order partial differential equations. Since the equation is non-linear in the dependent variable and its first derivatives, the type of equation is dependent on the solution. Generally it is impossible to transform the equation into a canonical form. Therefore methods discussed in Chapter 1 are not applicable to this type of equation. Hence we seek other methods to solve it, such as *the method of characteristics.* Basically, the method is similar to those methods discussed in Chapter 1 in the sense that we deal with integration problems along the characteristic curves. However in the method of characteristics we keep the original coordinates while former methods use the characteristic curves as coordinates. The characteristic curves are represented by non-linear ordinary equations involving first derivatives of the dependent variables.

Using *the method of characteristics*, the Cauchy problem for the quasi-linear second order partial differential equation will reduce to a *characteristic system*, involving a system of non-linear ordinary differential equations which have to be solved simultaneously, using iteration calculation processes. We will explore some numerical procedures to approximate the grid points of the characteristic curves, and then find numerical solutions of the partial differential equations at these grid points.

The finite difference equation as the approximation equation is not unique; it is dependent on the configuration of the discretization of the continuous domain. Several *finite difference methods* will be described in Chapter 4. The boundary conditions usually determine the methods suitable for solving a particular problem.

Chapter 1

CAUCHY'S PROBLEMS

1.1 Introduction

Most problems in physics and engineering may be classified into three physical categories: *equilibrium*, *diffusion* and *propagation problems.* The governing equations of each type are partial differential equations which differ in character. In comparison with geometrical terminology they may be classified as *parabolic*, *elliptic* and *hyperbolic.* Such classifications are a useful because the general theory and methods of solutions usually apply only to a given class of equations.

In section 1.2 , the *characteristic curves* of *a linear second order equation* will be introduced to classify the type of the equation. Knowledge of the characteristics is useful in the development and understanding of numerical solutions. Of particular significance is that in the case of *hyperbolic* equations there are two *real characteristics directions* at each point. By an appropriate choice of coordinates the original hyperbolic equation may be transformed into one in which the independent coordinates are the characteristic directions. Characteristic coordinates are the *natural* coordinates of the system in the sense that, in terms of these coordinates, the equation is much simplified. This often results in forms which may be integrated directly.

Section 1.3 will discuss the *Cauchy problem*, the problem of determining the solution of a second order partial differential equation with data prescribed at initial points or along an initial curve. We shall discuss the difficulties that arise when the Cauchy problem is formulated for non-hyperbolic equations. Section 1.4 will show that Cauchy problem for the hyperbolic equation in canonical form, reduces to solving an *integral equation* over a region bounded by the initial curve and characteristic lines.

The length of the initial curve is of importance. In the Cauchy problems, the initial curve is assumed to be infinite. If it is not then we may impose conditions at the end points. Disturbance moving along the characteristics will be reflected or transmitted at the boundaries. In the case of a semi-infinite initial curves, if beside the initial curve a set of data are given along a characteristic curve we will have *Goursat problems,* the problems will be discussed in sections 1.5.

Furthermore by assuming that the initial curve is monotonic decreasing and the coefficient functions involving the first derivatives is continuous and satisfies the *Lipschitz condition*, the Cauchy problem for hyperbolic equation in canonical form is well posed. *Picard's procedure* provides the *method of successive approximations* to solve the Cauchy and the Cauchy Goursat problems. The methods will be given in section 1.6.

Finally in section 1.7, *Riemann's method* will be used to deal with Cauchy problems for linear hyperbolic equations. Based on *Green's theorem* the surface integral problems are reduced to line integrals along the directions parallel to the boundary.

1.2 Classification of Equations

In discussing second order partial different equations, the most important and frequently occurring in physical situations, are *the wave equation*, *the heat equation* and *the Laplace equation.* These types of equations are different in their characteristic, hence we may classify them into three different type of equations. The classification is reflected by the analytic character of their solutions which is dependent on the type of boundary conditions necessary to determine their solutions.

The three different types of second order partial different equation on which the wave equation, the heat equation and the Laplace equation classified into, are ;

$$u_{xx} - u_{yy} = \psi_1(x, y, u, u_x, u_y), \tag{1.1}$$

$$u_{yy} = \psi_3(x, y, u, u_x, u_y), \tag{1.2}$$

and

$$u_{xx} + u_{yy} = \psi_2(x, y, u, u_x, u_y). \tag{1.3}$$

In geometrical terminology, these three *canonical forms* are known as the *hyperbolic*, *parabolic* and the *elliptic equations*, respectively.

In this chapter we will deal with a general form of a second order linear differential equation in two independent variables, say x and y,

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = g(x, y, u, u_x, u_y). \tag{1.4}$$

It is assumed that the coefficients a, b, c, g are *real-valued* and *continuously differentiable* on a region D of the xy-plane.

Difficulties may occur when we use the equation (1.4) as a model of a physical problem, since we may have a variety of initial and boundary value problems. However we shall establish that using an appropriate transformation, the equation (1.4) may be reduced to one of the three canonical forms above. Hence we can classify the second order linear differential equation (1.4) into hyperbolic equation, parabolic equation and elliptic equation. The classification is essential, since often the general theory and methods of solution is applicable only in a particular type of equation.

It is useful to write

$$L[u] = a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} \tag{1.5}$$

where L is the operator consisting of the second order differential operators in the left hand side of the equation (1.4). This is called *the principal part* of the differential equation (1.4). Further, we introduce the real function Δ defined on D by

$$\Delta(x,y) = (b(x,y))^2 - a(x,y)c(x,y). \tag{1.6}$$

This function is called *the discriminant* of the equation (1.4).

The classification of the equation (1.4) can be introduced simply by considering the effect of performing a change of independent variables. We intend to prove that the sign of the discriminant (1.6) is unchanged under continuous second order differentiable one to one real transformations of variables x and y. Let

$$\xi = \xi(x,y) \qquad\qquad \eta = \eta(x,y) \tag{1.7}$$

be real-valued functions and continuous second order differentiable on D such that *the Jacobian*

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0. \tag{1.8}$$

The functions map the region D of the xy-plane onto a region D^* of the $\xi\eta$-plane.

Using the chain rule, calculating the first and second derivatives of u results in

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}[\xi_x\eta_y + \xi_y\eta_x] + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned}$$

Substituting these into the equation (1.4) we have

$$A(\xi,\eta)u_{\xi\xi} + 2B(\xi,\eta)u_{\xi\eta} + C(\xi,\eta)u_{\eta\eta} = G(\xi,\eta,u,u_\xi,u_\eta) \tag{1.9}$$

as the representation of the equation (1.4) in $\xi\eta$ variables, where

$$\begin{aligned}
A(\xi,\eta) &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\
B(\xi,\eta) &= a\xi_x\eta_x + b[\xi_x\eta_y + \xi_y\eta_x] + c\xi_y\eta_y
\end{aligned}$$

$$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

$$G(\xi,\eta) = g - [a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}]u_\xi - [a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy}]u_\eta$$

Referring (1.1), (1.2)and (1.3) we are now able to classify the second order linear differential equation (1.4) as follows.

The partial differential equation (1.4) is *hyperbolic, parabolic* or *elliptic* if there exists transformations (1.7) such that the equation (1.4) becomes

$$u_{\xi\xi} - u_{\eta\eta} = G_1(\xi,\eta,u,u_\xi,u_\eta), \tag{1.10}$$

$$u_{\eta\eta} = G_2(\xi,\eta,u,u_\xi,u_\eta), \tag{1.11}$$

or

$$u_{\xi\xi} + u_{\eta\eta} = G_3(\xi,\eta,u,u_\xi,u_\eta), \tag{1.12}$$

respectively.

The discriminant of the principal part in the representation (1.9) is

$$\Delta^* = [B(\xi,\eta)]^2 - A(\xi,\eta)C(\xi,\eta). \tag{1.13}$$

Then using (1.6), the equation (1.13) yields

$$\Delta^* = \left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right]^2 \Delta. \tag{1.14}$$

Since the Jacobian (1.8) does not vanish, the equation (1.14) shows that the discriminants Δ and Δ^* are both positive (or zero or negative) at corresponding points. Therefore, the equation (1.4) can be classified according to the sign of its discriminant. The classification will not change under such change of variables. From this result we may conclude that we can achieve the canonical form (1.10), that (1.4) is a hyperbolic equation, if and only if

$$b^2 - ac > 0. \tag{1.15}$$

We can achieve the canonical form (1.11), that (1.4) is a parabolic equation, if and only if

$$b^2 - ac = 0. \tag{1.16}$$

We can achieve the canonical form 1.12, that (1.4) is an elliptic equation, if and only if

$$b^2 - ac < 0. \tag{1.17}$$

The problem now is how to determine ξ and η in (1.7) such that the equation (1.4) is reduced to one of the three canonical forms above. The problem is straight forward if the discriminant Δ has the same sign *everywhere* on the domain D, either positive, negative or zero. To do that consider an equation

$$a{\Phi_x}^2 + 2b\Phi_x\Phi_y + c{\Phi_y}^2 = 0. \tag{1.18}$$

Dividing by Φ_y^2 leads to a quadratic equation

$$a(\frac{\Phi_x}{\Phi_y})^2 + 2b\frac{\Phi_x}{\Phi_y} + c = 0. \tag{1.19}$$

It is a quadratic equation in $\frac{\Phi_x}{\Phi_y}$; hence we have

$$\frac{\Phi_{1x}}{\Phi_{1y}} = \frac{-b + \sqrt{b^2 - ac}}{a} \tag{1.20}$$

and

$$\frac{\Phi_{2x}}{\Phi_{2y}} = \frac{-b - \sqrt{b^2 - ac}}{a}. \tag{1.21}$$

Φ_1 and Φ_2 are independent solutions of (1.18) if (1.20) and (1.21) are distinct.

We only deal with the hyperbolic type so that we wish to reduce (1.4) into the canonical form (1.10). Taking the transformation (1.7) to be

$$\xi = \Phi_1(x, y), \qquad \eta = \Phi_2(x, y). \tag{1.22}$$

we have in $A(\xi, \eta) = C(\xi, \eta) = 0$. Furthermore since the Jacobian (1.8) does not vanish, (1.14) gives

$$\begin{aligned} \Delta = b^2 - ac &= \frac{1}{[\frac{\partial(\xi,\eta)}{\partial(x,y)}]^2}\Delta^* \\ &= \frac{1}{[\frac{\partial(\xi,\eta)}{\partial(x,y)}]^2}(B^2 - AC). \end{aligned}$$

Since $\Delta > 0$ and $A = C = 0$ then the last equation gives $B \neq 0$. Substituting these into (1.9) and neglecting the factor $2B$, the equation (1.9) becomes

$$u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta). \tag{1.23}$$

To show that this equation also represents a hyperbolic equation such as (1.10), take

$$\tilde{\xi} = \xi + \eta, \qquad \tilde{\eta} = \xi - \eta$$

then the equation (1.23) becomes

$$u_{\tilde{\xi}\tilde{\xi}} - u_{\tilde{\eta}\tilde{\eta}} = \tilde{\psi}(\tilde{\xi}, \tilde{\eta}, u, u_{\tilde{\xi}}, u_{\tilde{\eta}}),$$

Hence the canonical forms of the hyperbolic equation can be represented by (1.10) or (1.23).

We shall analyse the solution of equation (1.18) by investigating the level curves

$$\Phi_1(x, y) = k_1 \tag{1.24}$$

and

$$\Phi_2(x, y) = k_2 \tag{1.25}$$

where k_1 and k_2 are arbitrary constants. Differentiating these we have

$$\Phi_{1x}dx + \Phi_{1y}dy = 0,$$

and

$$\Phi_{2x}dx + \Phi_{2y}dy = 0.$$

Hence along each level curves we have

$$\frac{dy}{dx} = \frac{-\Phi_{1x}}{\Phi_{1y}}$$

and

$$\frac{dy}{dx} = \frac{-\Phi_{2x}}{\Phi_{2y}}.$$

Substituting these into the equations (1.20) and (1.21), respectively, we obtain

$$\frac{dy}{dx} = \frac{b(x,y) + \sqrt{b^2(x,y) - a(x,y)c(x,y)}}{a(x,y)} \tag{1.26}$$

$$\frac{dy}{dx} = \frac{b(x,y) - \sqrt{b^2(x,y) - a(x,y)c(x.y)}}{a(x,y)} \tag{1.27}$$

These two distinct ordinary differential equations are known as the *characteristic equations*. The solution of the characteristic equations define *two* families of *characteristic curves* in the domain D. Using the notation in the equation (1.22) , we have at each point of D exactly one curve $\xi = k_1$ and exactly one curve $\eta = k_2$. Since the Jacobian does not vanish on D, then the level curves have distinct slope at each point. The characteristic curves of the same family do not intersect, so they can be used as *a basis for the coordinate grid.* See figure1.1. Along these curves the simplified form of the original equation may be used to obtain a solution.

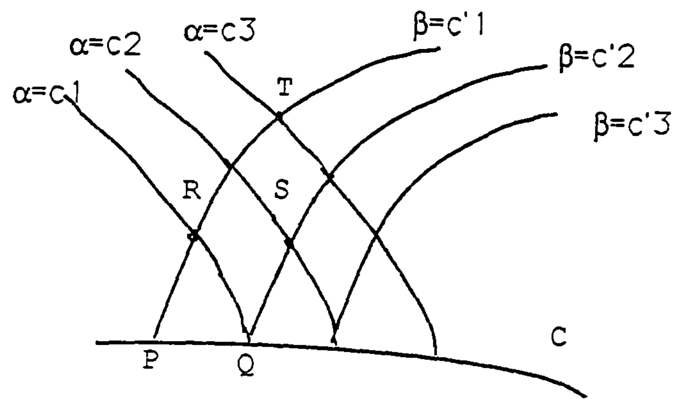


Fig.1.1: The Characteristic curves

1.3 The Cauchy's Data

Mathematical formulations of physical problems often leads to the problems of determining the solution of partial differential equations satisfying some conditions along a given curve. The problem are called *initial value problem*, while the conditions are called the *initial conditions,* and the given curve is called the *initial curve*. Time is often considered as one of the independent variables, and the term *initial values* refers to the fact that the data are assigned at the initial time. In this case the problem is sometimes called the *Cauchy problem*, while the values at the initial conditions are also known as the *Cauchy data.*

We will utilize the Cauchy problem widely to describe the *wave propagation problem.* By knowing the initial state of the system, we wish to predict the subsequent behaviour of the system. In this section we will discuss the Cauchy problem for the general linear second order equation (1.4). Suppose the partial differential equation (1.4) is defined over a continuous domain D. We will show that the type of the equation has a significant influence when we deal with a Cauchy problem.

To formulate a Cauchy problem for the equation (1.4) we need initial conditions. Initial conditions involve the value of u and u_n which are given along an initial curve, say C, where u_n denotes *the normal derivative*, the derivative in the direction normal to the curve C. See figure 1.2.

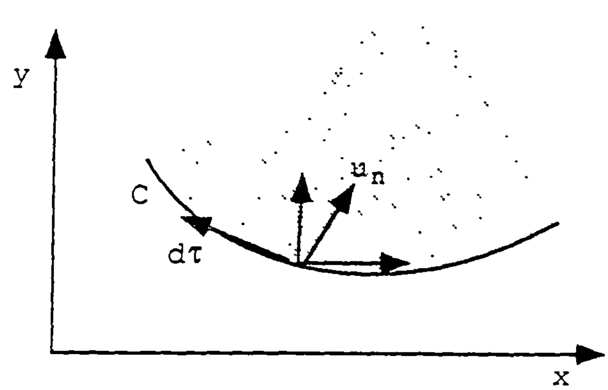


Figure 1.2:The Cauchy problem

Suppose the initial curve C is defined parametrically by

$$x = x(\tau), \qquad\qquad y = y(\tau) \tag{1.28}$$

for a real variable τ, and satisfies

$$((x'(\tau))^2 + (y'(\tau))^2) \neq 0 \tag{1.29}$$

with $x' = \frac{dx}{d\tau}$ and $y' = \frac{dy}{d\tau}$.
Then the Cauchy problem can be defined as follows : given continuously differentiable functions ϖ and ω, determine a solution of the equation (1.4) such that the initial conditions

$$u(x(\tau), y(\tau)) = \varpi(\tau), \qquad u_n(x(\tau), y(\tau)) = \omega(\tau) \tag{1.30}$$

are satisfied along the initial curve (1.28). The functions x, y, ϖ and ω constitute the Cauchy data of the problem.

It is possible to represent the Cauchy problem where the initial conditions involve derivatives with respect to an axis. Differentiating u with respect to τ leads to

$$\frac{du}{d\tau} = u_x\frac{dx}{d\tau} + u_y\frac{dy}{d\tau} \tag{1.31}$$

where $\frac{dx}{d\tau}$ and $\frac{dy}{d\tau}$ respectively are the direction cosines of the tangent to the curve C with slope $\frac{dy}{dx}$. Furthermore by using the first initial condition in (1.30) we have

$$u_x x' + u_y y' = \varpi' \tag{1.32}$$

along the curve C. The direction cosines of the normal $\mathbf{n}$ to curve C are $-y'$ and x', hence we have

$$-u_y x' + u_x y' = \sqrt{[(x')^2 + (y')^2]}\omega. \tag{1.33}$$

Since (1.29) holds, then these last two equations determine uniquely u_x and u_y along the given curve C. Hence instead of prescribing the values of u and u_n along the given curve C, the values of u, u_x and u_y may be prescribed along the curve C. In this case, the initial conditions (1.30) may be replaced by [8]

$$u(x(\tau), y(\tau)) = \varpi(\tau), \quad u_x(x(\tau), y(\tau)) = \omega_1(\tau), \quad u_y(x(\tau), y(\tau)) = \omega_2(\tau) \tag{1.34}$$

where ϖ, ω_1 and ω_2 are given smooth functions. Therefore the Cauchy problem can be expressed as follows: given continuously differentiable functions x, y, ϖ,

ω_1 and ω_2, determine a solution of the equation (1.4) satisfying the conditions in (1.34). Further we can calculate all first derivatives in any direction not tangential to C.

The Cauchy data now consists of the values of the functions x, y, ϖ, ω_1 and ω_2. However ϖ, ω_1 and ω_2 cannot be assigned arbitrarily. From the equations (1.32) and (1.33), it follows that the relation

$$\omega_1(\tau)x'(\tau) + \omega_2(\tau)y'(\tau) = \varpi'(\tau) \tag{1.35}$$

must hold along the curve C. The normal derivative of u along the initial curve C is then given by

$$u_n(\tau) = \frac{-\omega_1(\tau)x'(\tau) + \omega_2(\tau)y'(\tau)}{[(x(\tau))^2 + (y(\tau))^2]^{\frac{1}{2}}}. \tag{1.36}$$

1.4 Hyperbolic Equations

The Cauchy problem formulated for equation (1.4), in the previous section, is too general. The hypothesis on the coefficients are too weak, and there may be no solution of equation (1.4). Furthermore the type of equation must be taken into account, and it is important to know whether the initial curve is a characteristic curve. In the present section it will be shown that Cauchy problem is well posed for the hyperbolic equation.

Consider the hyperbolic equation in the canonical form,

$$\frac{\partial^2 u}{\partial x \partial y} = G(x, y, u, u_x, u_y). \tag{1.37}$$

Since equation (1.37) is in canonical form, then the characteristics are

$$x = k_1, \qquad y = k_2 \tag{1.38}$$

where k_1 and k_2 are arbitrary constants. As a result we can use either x or y to replace τ as the parameter in the Cauchy data (1.34). Suppose the non parametric representation for the initial curve C is given by

$$y = \Psi(x) \tag{1.39}$$

which is assumed to be *invertible and strictly monotonic.* Then the Cauchy data (1.34) can be written as

$$u(x, \Psi(x)) = \varpi(x, y), \quad u_x(x, \Psi(x)) = \omega_1(y), \quad u_y(x, \Psi(x)) = \omega_2(x). \tag{1.40}$$

The Cauchy data *can not* be assigned arbitrarily along one of the characteristics curves. Since along the characteristic line $y = k_1$, for example, the equation (1.37) becomes

$$\frac{d(u_y)}{dx} = G(x, k_1, u, \frac{du}{dx}, u_y)$$

which is an ordinary differential equation for u and u_y in their dependence on the variable x. Hence if the Cauchy data are assigned along one of the characteristics line $y = k_1$, then they should satisfy the last relation, even so we can not expect the solution to be unique. This shows that the initial curve *cannot* be one of the characteristic curves. Therefore we impose the hypothesis on the initial curve (1.39) that should nowhere be tangent to a characteristic.

Consider first the Cauchy problem for the homogeneous equation,

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \tag{1.41}$$

which satisfies the initial conditions (1.40).

Writing (1.41) as

$$\frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = 0,$$

we see that u_x is independent of y, hence we have

$$u_x = \Phi(x)$$

for an arbitrary function Φ. Integrating this over x gives

$$u(x, y) = \phi(x) + \psi(y). \tag{1.42}$$

where $\phi(x)$ is the anti derivative of $\Phi(x)$. Differentiating (1.42) with respect to x gives

$$\phi'(x) = u_x(x) = \frac{u_x(x)}{2} + \frac{u_x(x)}{2},$$

and since from (1.31) we have

$$u_x = \frac{du}{dx} - u_y \frac{dy}{dx}$$

then

$$\phi'(x) = \frac{u_x(x)}{2} + \frac{\frac{du}{dx} - u_y \frac{dy}{dx}}{2}.$$

Multiplying this by dx and integrating the result, leads to

$$\phi(x) = \frac{u(x)}{2} + \frac{1}{2}\int u_x(x)dx - \frac{1}{2}\int u_y(\Psi(x))dy. \qquad (1.43)$$

Similarly differentiating (1.42) with respect to y gives

$$\psi'(y) = u_y(y) = \frac{u_y(y)}{2} + \frac{u_y(y)}{2}$$

and from (1.31) we have

$$u_y = \frac{du}{dy} - u_x \frac{dx}{dy}$$

then

$$\psi'(y) = \frac{u_y(y)}{2} + \frac{\frac{du}{dy} - u_x \frac{dx}{dy}}{2}.$$

Multiplying this by dy and integrating the result, leads to

$$\psi(y) = \frac{u(y)}{2} + \frac{1}{2}\int u_y(y)dy - \frac{1}{2}\int u_x(\Psi^{-1}(x))dx. \qquad (1.44)$$

Substituting (1.43) and (1.44) into (1.42) results in

$$u(x,y) = \frac{1}{2}[u(x) + u(y)] + \frac{1}{2}\int_{\Psi^{-1}(x)}^{x} u_x(\xi)d\xi + \frac{1}{2}\int_{\Psi(x)}^{y} u_y(\eta)d\eta$$

and using the initial conditions (1.34), this leads to

$$u(x,y) = \frac{1}{2}[\varpi(x) + \varpi(y)] + \frac{1}{2}\int_{\Psi^{-1}(x)}^{x} \omega_1(\xi)d\xi + \frac{1}{2}\int_{\Psi(x)}^{y} \omega_2(\eta)d\eta \qquad (1.45)$$

This is the solution of the Cauchy problem for the homogeneous equation (1.41).

In the case of the non homogeneous equation, consider first when the function G is independent of u and its first derivatives,

$$\frac{\partial^2 u}{\partial x \partial y} = G(x,y). \qquad (1.46)$$

To find the solution of the Cauchy problem for the equation (1.46) we can use the linearity property of the problem. Suppose $v(x, y)$ is the solution of the non-homogeneous equation (1.46) where the Cauchy data vanish on the initial curve, i.e. they satisfy the homogeneous initial conditions, $\varpi = \omega_1 = \omega_2 = 0$. Suppose $w(x, y)$ is the solution of the homogeneous equation (1.41) with the initial conditions (1.34). Then the solution of the non-homogeneous equation (1.46) satisfying the initial conditions (1.34) is $u = v + w$. The solution w was found to be (1.45). However finding the solution v is becomes finding the solution of the Cauchy problem for the non-homogeneous equation (1.46) satisfying the initial conditions

$$u(x, \Psi(x)) = 0, \quad u_x(x, \Psi(x)) = 0, \quad u_y(x, \Psi(x)) = 0. \tag{1.47}$$

Suppose we wish to find the solution at a particular point $R = (x, y)$ not lying on the initial curve C (1.39). Say $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are the intersection points of the initial curve and the two characteristic lines through R, such that $x_1 = \Psi^{-1}(y), y_1 = y$ and $x_2 = x, y_2 = \Psi(x)$. See figure 1.3.

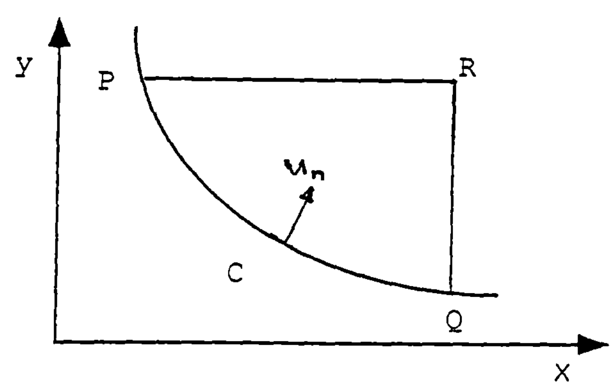


Figure 1.3:The Cauchy problem: hyperbolic equation

Consider (x^*, y^*) an arbitrary point on the segment PQ. Keeping y constant, integration (1.46) over x from x^* to x gives

$$u_y(x, y) - u_y(x^*, y^*) = \int_{x^*}^{x} G(\xi, y) d\xi.$$

The initial conditions (1.47) implies that $u_y(x^*, x^*) = 0$ then we have

$$u_y(x,y) = \int\limits_{x^*}^{x} G(\xi, y) d\xi.$$

Furthermore keeping x constant, integrating the last equation over y from y^* to y yields

$$u(x,y) - u(x,y^*) = \int\limits_{y^*}^{y} \int\limits_{x^*}^{x} G(\xi,\eta) d\xi d\eta.$$

Again the initial conditions (1.47) implies that $u(x^*, y^*) = 0$ then we have

$$u(x,y) = \int\limits_{y^*}^{y} \int\limits_{x^*}^{x} G(\xi,\eta) d\xi d\eta. \tag{1.48}$$

Since (x^*, y^*) is an arbitrary point on PQ, then the double integral in the right-hand side is an integral over a region, say Σ, bounded by the two characteristic lines through $R = (x,y)$, $\xi = x$ and $\eta = y$, and the arc PQ, the segment of the initial curve C, which is intercepted by the characteristic lines. Hence we have

$$u(x,y) = \iint\limits_{\Sigma} G(\xi,\eta) d\xi d\eta. \tag{1.49}$$

as the solution of the Cauchy problem for the non-homogeneous equation (1.46) satisfying the homogeneous initial conditions (1.47).

Adding (1.45) and (1.48) gives

$$u(x,y) = \frac{1}{2}[\varpi(x) + \varpi(y)] + \frac{1}{2}\int_{x_1}^{x} \omega_1(\xi) d\xi + \frac{1}{2}\int_{y_2}^{y} \omega_2(\eta) d\eta + \iint\limits_{\Sigma} G(\xi,\eta) d\xi d\eta. \tag{1.50}$$

This is the solution of Cauchy problem for the non homogeneous equation (1.46) satisfies the initial conditions (1.40). However, since we have $dy = 0$ on PR and $dx = 0$ on QR then the lines integrals in the last equation can be written as

$$\begin{aligned}
&\frac{1}{2}\int_{x_1}^{x} \omega_1(\xi) d\xi + \frac{1}{2}\int_{y_2}^{y} \omega_2(\eta) d\eta \\
&= \frac{1}{2}\int_{x_1}^{x} \omega_1(\xi) d\xi + \frac{1}{2}\int_{x}^{x_2} \omega_1(\xi) d\xi - \frac{1}{2}\int_{y_1}^{y} \omega_2(\eta) d\eta - \frac{1}{2}\int_{y}^{y_2} \omega_2(\eta) d\eta \\
&= -\frac{1}{2}\int_{P}^{Q} \omega_2(\eta) d\eta - \omega_1(\xi) d\xi.
\end{aligned}$$

As a result, the solution (1.50) can be written as

$$u(x,y) = \frac{1}{2}[u(P) + u(Q)] - \frac{1}{2}\int\limits_{PQ} \omega_2(\eta)d\eta - \omega_1(\xi)d\xi + \iint\limits_{\Sigma} G(\xi,\eta)d\xi d\eta. \quad (1.51)$$

This last result may be generalized to a solution of the equation (1.37). By assuming that the values of u, u_x and u_y are known, then the value of the function G in right-hand side of (1.37) can be considered as given. Hence The solution of the Cauchy problem for the equation (1.37) at the point R is [9]

$$\begin{aligned} u(x(R), y(R)) &= \frac{1}{2}[u(P) + u(Q)] - \frac{1}{2}\int\limits_{PQ} (\omega_2(\eta)d\eta - \omega_1(\xi)d\xi) \\ &\quad + \iint\limits_{\Sigma} G(\xi, \eta, u, u_\xi, u_\eta)d\xi d\eta. \end{aligned} \quad (1.52)$$

The solutions (1.51) and (1.50) show us that the Cauchy data along the *arc* PQ together with values of the given function G over the region bounded by Σ are sufficient to determine the solution of the Cauchy problem at the point R. For this reason, the region together with the arc PQ is called the *domain of dependence* of the solution u with respect to the point R, while PQ is called the *segment of determination.* However the Cauchy data along the segment PQ can influence the solution only in the region bounded by β characteristic curve through the point P, $x = x(P)$, and α characteristic curve through the point Q, $y = y(Q)$. Such region is called the *range of influence*, denoted by $\Re$, with respect to the Cauchy data along the segment PQ, see figure 1.4.

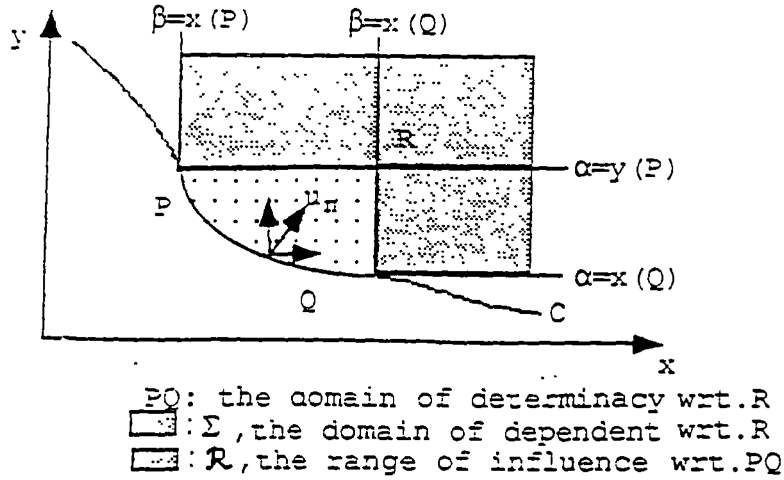


Figure 1.4:The Domain of dependence and the range of influence

1.5 Goursat Problems

We have mentioned before that the characteristic curves of the hyperbolic equation (1.37)

$$\frac{\partial^2 u}{\partial x \partial y} = G(x, y, u, u_x, u_y)$$

are straight lines (1.38)

$$x = k_1, \qquad y = k_2.$$

Assume that the inital curve C

$$y = \Psi(x) \tag{1.53}$$

is *monotonically increasing* curve and intersects the axes at the origin. Suppose that besides the Cauchy data (1.34) along the given curve (1.53), we also have the value of u along a characteristic, say the x-axis. The problem of determining the solution of the equation (1.37) subject to these mixed boundary conditions above is called the *Goursat problem* [9]. In the case when the initial curve (1.53) is reduced to y axis, hence we have data along the system co-ordinate on which are the characteristic curves of the equation (1.37). The problem of determining the solution of the equation (1.37) subject to the two characteristic curves is called *characteristic Goursat problem.*

Let the point $R = (x, y)$ lie in the region above the x-axis and below the curve (1.39). Say $P = (x_1, y_1)$ the intersection point of the characteristic line through R and the initial curve C. Hence $x_1 = \Psi^{-1}(y)$ and $y_1 = y$. Denote by Q and S respectively, the projection points of R and P onto x-axis, i.e. $Q = (x, 0)$ and $S = (x_1, 0)$. Hence the straight lines PR, RQ, QS and SP form a rectangular region , say Σ. See figure 1.5.

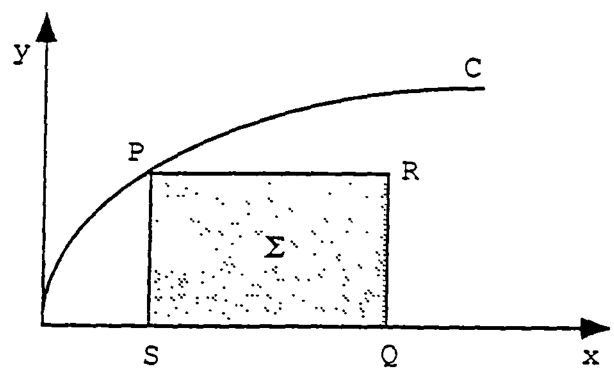


Figure 1.5:The Gaursat problem

Suppose the function $G(x, y, u, u_x, u_y)$ in the equation (1.37) is continuous at all points in the region Σ and satisfies *the Lipschitz condition* in all bounded sub-rectangles r of Σ. Integrating u_{xy} over Σ reduces to a line integration along the boundary of Σ, ie. straight lines PR, RQ, QS and SP. Hence

$$\iint_{\Sigma} u_{\xi\eta} d\xi_d\eta = \int_{x_1}^{x} u_x(\xi) d\xi + \int_{y}^{0} u_y(\eta) d\eta + \int_{x}^{x_1} u_x(\xi) d\xi + \int_{0}^{y} u_y(\eta) d\eta,$$

that is

$$\iint_{\Sigma} u_{\xi\eta} d\xi_d\eta = u(R) - u(P) + u(S) - u(Q). \tag{1.54}$$

Hence

$$u(R) = u(P) - u(S) + u(Q) + \iint_{\Sigma} G(\xi, \eta, u, u_\xi, u_\eta) d\xi_d\eta \tag{1.55}$$

The last equation shows that the solution at R depends only on Cauchy data along the *arc* OP of the initial curve and data along the segment OQ of the x-axis, where O is the origin.

In the next chapter we will deal with the simple case of the Goursat problem in which the monotonic curve (1.53) is replaced by an axis, say the x-axis, then u_n, the normal derivative, in the initial condition becomes u_y.

1.6 Methods of Successive Approximations

The previous section shows us that solving the Cauchy problem for hyperbolic equation (1.37) leads to solving the integrodifferential equation (1.51) over the domain of dependence. Now we wish to establish that the Cauchy problem is well posed for the hyperbolic equation (1.37) when the initial curve (1.39) is a monotonic curve. It is possible to assume that the Cauchy data vanish along the initial curve. Hence we only need to deal with the integrodifferential

$$u(x,y) = \iint\limits_{\Sigma} G(\xi, \eta, u, u_\xi, u_\eta) d\xi d\eta \tag{1.56}$$

Furthermore there is no lost of generality if we suppose that the initial curve (1.39) is the straight line

$$x + y = 0, \tag{1.57}$$

since a suitable change of the characteristic coordinates such as

$$x^* = y(x), \qquad y^* = -y$$

or

$$x^* = -x, \qquad y^* = x(y)$$

will transform (1.39) into (1.57).

Using Picard's procedures we may solve the equation (1.56) by constructing a sequence of successive approximations u^n by the formula

$$u^{n+1}(x,y) = \iint\limits_{\Sigma} G(\xi, \eta, u^n, u^n_\xi, u^n_\eta) d\xi d\eta, \tag{1.58}$$

where

$$u_x^{n+1}(x,y) = \int_{-x}^{y} G(x,\eta,u^n,u_x^n,u_\eta^n)d\eta, \tag{1.59}$$

and

$$u_y^{n+1}(x,y) = \int_{-y}^{x} G(\xi,y,u^n,u_\xi^n,u_y^n)d\xi. \tag{1.60}$$

These last two first derivative formulas are found by differentiating (1.58) with respect to x and y respectively. The first initial guess may be taken to be

$$u_0(x,y) = 0. \tag{1.61}$$

The iteration will converge , that is the limit

$$u = \lim_{n\to\infty} u^n = \sum_{n=0}^{\infty} [u^{n+1} - u^n] \tag{1.62}$$

exist and satisfies the integrodifferential (1.50), if the function G is continuous at all points in a region $R = \{(x,y)|x_0 < x < x_1,\ \ y_0 < y < y_1\}$ for all values of x, y, u, u_x, u_y, and to satisfies *the Lipschitz condition*

$$\begin{aligned} &\left|G(x,y,u^{n+1},u_x^{n+1},u_y^{n+1}) - G(x,y,u^n,u_x^n,u_y^n)\right| \\ &\leq M[|u^{n+1}-u^n| + |u_x^{n+1}-u_x^n| + |u_y^{n+1}-u_y^n|] \end{aligned} \tag{1.63}$$

for some positive number M.

Meanwhile, for the Goursat problem we can generate a sequence of successive approximations by

$$u_{n+1}(R) = u(P) - u(S) + u(Q) + \iint_{\Sigma} G(\xi,\eta,u_n,u_{\xi_n},u_{\eta_n})d\xi_d\eta \tag{1.64}$$

with the initial approximation

$$u_0(R) = u(P) - u(S) + u(Q). \tag{1.65}$$

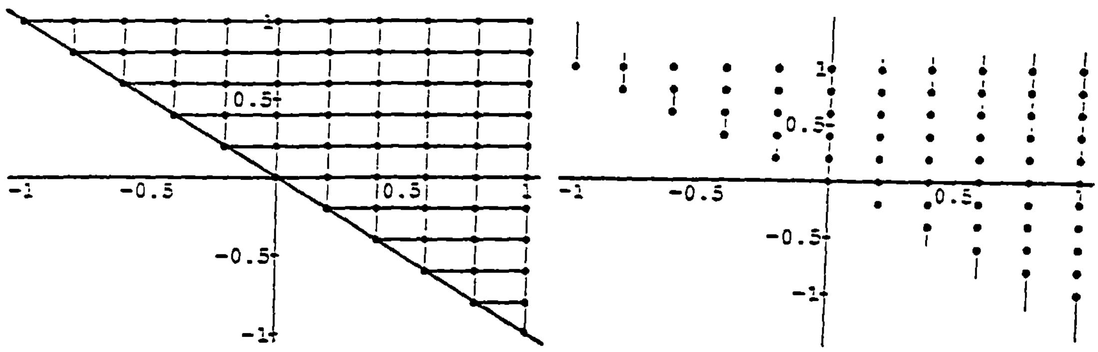
As particular example consider the Cauchy problem for

$$\frac{\partial^2 u}{\partial x \partial y} = x + y + u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \tag{1.66}$$

satisfying

$$u(x, y(x)) = -x + y, \qquad y(x) = -x, \qquad -1 \leq x \leq 1. \tag{1.67}$$

The characteristic lines and the solution at the intersection points of the characteristic lines are depicted by figures 1.6.a and 1.6.b respectively.



[a] Characteristic grids

[b] The solution
The vertical lines indicate the solution, they are $\frac{1}{20}$ from the actual size.

Figure 1.6 : Successive approximations

1.7 Riemann's Methods

In the previous sections, the characteristics has been used as the coordinate system of hyperbolic equations in the canonical form. The Cauchy problem is reduced to an integrodifferential equation over the domain of dependence. In this section we will derive the *Riemann's method* of solutions of *linear* hyperbolic equations. The method presents the solution in a manner depending explicitly on prescribed Cauchy data, and using *Green's theorem,* the surface integrals are reduced to line integrals along the directions parallel to the given boundary.

Recall the hyperbolic equation in the canonical form (1.37)

$$\frac{\partial^2 u}{\partial x \partial y} = G(x, y, u. u_x, u_y)$$

Assume the function G in the hyperbolic equation (1.37) be continuous at all points on the region$R = \{(x,y)|x_0 < x < x_1,\ \ y_0 < y < y_1\}$ for all values of x, y, u, u_x, u_y, and satisfy *the Lipschitz condition* (1.63) in all bounded sub rectangles r of R. Assuming that the equation (1.37) is linear, we may write

$$\frac{\partial^2 u}{\partial x \partial y} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y). \tag{1.68}$$

Using the linear operator notation, L, the left hand side of (1.68) can be written as

$$L[u] = \frac{\partial^2 u}{\partial x \partial y} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + fu. \tag{1.69}$$

In order to find a solution of the Cauchy problem for (1.68), we shall use *Green's Theorem* in the form

$$\iint\limits_{\Sigma} (U_x + V_y)dxdy = \int\limits_{\Gamma} \mathrm{n}.(U,V)d\sigma, \tag{1.70}$$

where the line integration is evaluated in the counter-clockwise direction over the closed contour Γ bounding the region of integration Σ. The parameter σ is the arc length of the curve Γ, while $\mathbf{n}$ is the unit *normal* to the curve. The integrand in the left-hand side of equation (1.70) is the *divergence* of the vector (U, V). Hence the aim is to set up such a divergence expression involving the linear operator L. For this purpose we introduce an operator $M[v]$, for $v = v(x, y)$, an arbitrary continuously differentiable function. The operator $M[v]$ is defined such that $vL[u] - uM[v]$ becomes a divergence of the vector (U, V) say, that is

$$vL[u] - uM[v] = U_x + V_y. \tag{1.71}$$

The operator M satisfying equations (1.71) *the adjoint operator* to the operator L. If $L = M$, then L is said to be *self-adjoint.* Considering the terms in the operator $L[u]$, we may write

$$v\frac{\partial^2 u}{\partial x \partial y} - u\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right),$$

$$vd\frac{\partial u}{\partial x} + u\frac{\partial (vd)}{\partial x} = \frac{\partial}{\partial x}(uvd),$$

and

$$ve\frac{\partial u}{\partial y} + u\frac{\partial (ve)}{\partial y} = \frac{\partial}{\partial y}(uve). \tag{1.72}$$

Hence the operator $M[v]$ must have the form

$$M[v] = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial (vd)}{\partial x} - \frac{\partial (ve)}{\partial y} + vf. \tag{1.73}$$

Furthermore from equation (1.69), (1.71) and (1.73) and using the properties in the equation (1.72) we have

$$U = uvd - u\frac{\partial v}{\partial y}, \tag{1.74}$$

and

$$V = uve + v\frac{\partial u}{\partial x}. \tag{1.75}$$

Notice that the representation of U and V is not unique, but it must satisfy the Green's Theorem.

Integrating the equation (1.71) gives

$$\iint_{\Sigma} (vL[u] - uM[v])dxdy = \iint_{\Sigma} [U_x + V_y]\, dxdy,$$

and using *Green's Theorem*[17], this equation becomes

$$\iint_{\Sigma} (vL[u] - uM[v])dxdy = \int_{\Gamma} \mathbf{n}.(U, V)d\sigma \tag{1.76}$$

where the line integration is evaluated in the counter-clockwise direction over the closed contour Γ bounding the region of integration Σ. The parameter σ is the arc length of the curve Γ, while $\mathbf{n}$ is the unit *normal* to the curve. Suppose Γ has the parametric representation

$$x = x(\sigma), \qquad y = y(\sigma) \tag{1.77}$$

then the unit normal $\mathbf{n}$ is given by

$$\mathbf{n} = \left(\frac{dy}{d\sigma}, -\frac{dx}{d\sigma}\right). \tag{1.78}$$

Substituting this into (1.76) results

$$\iint_{\Sigma} (vL[u] - uM[v])dxdy = \int_{\Gamma} (Udy - Vdx). \tag{1.79}$$

Suppose the Cauchy data are given along initial curve C with the parametric representation

$$x = x(\tau) \qquad\qquad y = y(\tau). \tag{1.80}$$

which is assumed to be *monotonically decreasing.* We wish to find the solution of the Cauchy problem at the point $R = (\xi, \eta)$ above the initial curve. Choose the region of integration Σ in (1.79) to be *the domain of dependence* of the solution of the Cauchy problem with respect to the point R. Thus the contour Γ is a closed curve, enclosing the area Σ, consisting of the characteristic segment PR , *arc* PQ and characteristic segment QR. (See figure 1.7)

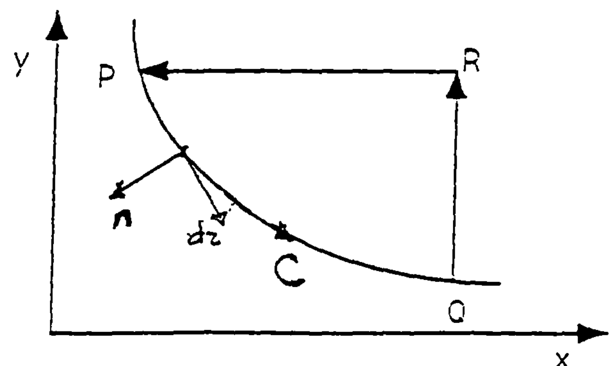


Figure 1.7: Riemann's Methods

Since we have $dx = 0$ on QR and $dy = 0$ on PR, then equation (1.79) may be written as

$$\iint_{\Sigma} vL[u] - uM[v]dxdy = \int_{PQ} (Udy - Vdx) + \int_{QR} Udy - \int_{RP} Vdx. \tag{1.81}$$

Integrating (1.75) gives

$$\begin{aligned}\int_{RP} Vdx &= \int_{RP} (uve + v\frac{\partial u}{\partial x})dx \\ &= \int_{RP} (v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x})dx + \int_{RP} (uve - u\frac{\partial v}{\partial x})dx \\ &= [uv]_R^P + \int_{RP} u\left(ve - \frac{\partial v}{\partial x}\right)dx\end{aligned}$$

$$= [uv]_P - [uv]_R + \int\limits_{RP} u\left(ve - \frac{\partial v}{\partial x}\right) dx.$$

Substituting this into equation (1.81) we get

$$\begin{aligned}[uv]_R &= [uv]_P + \int_{RP} u(ve - \frac{\partial v}{\partial x})dx - \int_{QR} u(vd - \frac{\partial v}{\partial y})dy - \int_{PQ} (Udy - Vdx) \\ &+ \iint\limits_{\Sigma} (vL[u] - uM[v])dxdy. \end{aligned} \tag{1.82}$$

Now our next aim is to select the function $v(x, y)$ such that (1.82) will represents the solution of the Cauchy problem in terms of Cauchy data along the *arc* PQ. Since we wish to eliminate u from the integral over Σ on (1.82) we have to choose $v(x, y)$ to be a solution of

$$M[v] = 0. \tag{1.83}$$

and satisfies

$$\frac{\partial v}{\partial x} = ve \tag{1.84}$$

along PR (when $y = \eta$),

$$\frac{\partial v}{\partial y} = vd \tag{1.85}$$

along QR (when $x = \xi$) and also

$$v(x, y) = 1 \tag{1.86}$$

at the point R (when $x = \xi, y = \eta$). Such a function satisfying (1.83), (1.84), (1.85) and (1.86) is called the *Riemann-Green function.*.

By (1.86), the equation (1.82), at the point R, is reduced to

$$\begin{aligned}[u]_R &= [uv]_P + \int_{RP} u(ve - \frac{\partial v}{\partial x})dx - \int_{QR} u(vd - \frac{\partial v}{\partial y})dy - \int_{PQ} (Udy - Vdx) \\ &+ \iint\limits_{\Sigma} (vL[u] - uM[v])dxdy. \end{aligned} \tag{1.87}$$

The second and the third terms on the right-hand side are vanish by (1.84) and (1.85)

$$\int_{RP} u(ve - \frac{\partial v}{\partial x})dx = \int_{RP} u(ve - ve)dx = 0, \tag{1.88}$$

$$\int_{QR} u(vd - \frac{\partial v}{\partial y})dy = \int_{QR} u(vd - vd)dy = 0. \tag{1.89}$$

Furthermore using (1.74) and (1.75) we have

$$-\int_{PQ} (Udy - Vdx) = -\int_{PQ} uv[d(x,y)dy - e(x,y)dx] + \int_{PQ} \left(u\frac{\partial v}{\partial y}dy + v\frac{\partial u}{\partial x}dx \right), \tag{1.90}$$

while (1.83) gives

$$\iint_{\Sigma} (vL[u] - uM[v])dxdy = \iint_{\Sigma} vL[u]dxdy. \tag{1.91}$$

Substituting (1.88),(1.89), (1.90) and (1.91) into (1.87) we get

$$\begin{aligned} [u]_R &= [uv]_P - \int_{PQ} uv[d(x,y)dy - e(x,y)dx] + \int_{PQ} \left(u\frac{\partial v}{\partial y}dy + v\frac{\partial u}{\partial x}dx \right) \\ &\quad + \iint_{\Sigma} (vL[u])\, dxdy. \end{aligned}$$

Since $L[u] = g(x,y)$ the last equation can be written as

$$\begin{aligned} [u]_R &= [uv]_P - \int_{PQ} uv[d(x,y)dy - e(x,y)dx] + \int_{PQ} \left(u\frac{\partial v}{\partial y}dy + v\frac{\partial u}{\partial x}dx \right) \\ &\quad + \iint_{\Sigma} (vg)\, dxdy. \end{aligned} \tag{1.92}$$

Hence if the value of $\frac{\partial u}{\partial x}$ is given along the curve C, the equation (1.92) can be used to find the value of u at the point R.

Suppose $\frac{\partial u}{\partial y}$ is given along the curve C. Since

$$\begin{aligned} [uv]_Q - [uv]_P &= \int_{PQ} \left(\frac{\partial(uv)}{\partial x}dx + \frac{\partial(uv)}{\partial y}dy \right) \\ &= \int_{PQ} \left((u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x})dx + (u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y})dy \right) \end{aligned}$$

we have

$$[uv]_P = [uv]_Q - \int_{PQ} (u\frac{\partial v}{\partial x}dx + v\frac{\partial u}{\partial y}dy) - \int_{PQ} (v\frac{\partial u}{\partial x}dx + u\frac{\partial v}{\partial y}dy).$$

Substituting this into the equation (1.92) we get

$$\begin{aligned} [u]_R &= [uv]_Q - \int_{PQ} uv(d(x,y)dy - e(x,y)dx) - \int_{PQ} \left(u\frac{\partial v}{\partial x}dx + v\frac{\partial u}{\partial y}dy \right) \\ &\quad + \iint_{\Sigma} (vg)\, dxdy \end{aligned} \tag{1.93}$$

Hence if the value of $\frac{\partial u}{\partial y}$ is given along the curve C, the equation (1.93) can be used to find the value of u at the point R. Furthermore, adding the equations (1.92) and (1.93) produces

$$\begin{aligned}[u]_R &= \frac{1}{2}([uv]_P - [uv]_Q) - \int_{PQ} uv(d(x,y)dy - e(x,y)dx) \\ &\quad -\frac{1}{2}\int_{PQ} v\left(\frac{\partial u}{\partial y}dy - \frac{\partial u}{\partial x}dx\right) - \frac{1}{2}\int_{PQ} u\left(\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial x}dy\right) + \iint_{\Sigma} (vg)\,dxdy.\end{aligned} \tag{1.94}$$

These results will be found useful in the discussion of the one dimensional wave equation in Chapter 2.

1.8 Discussions and Conclusions

The values of the solution of the Cauchy problem for the hyperbolic equation in canonical form at a particular point does not depend on all Cauchy data in the initial curve but only on the segment of dependence, the segment of the curve intercepted by the characteristic lines through the point, and the value of the given function over the domain of dependence, bounded by the characteristic line and the segment of dependence.

Further if beside the initial curve the data are given along a characteristic curve then the problem becomes the Goursat problem. The solution of the Goursat problem at a particular point lying on the region between the characteristic curve the initial curve, depend on the data along the characteristic and initial curves.

Under the assumption that the hyperbolic equation satisfies the Lipschitz condition, the Cauchy problem for the hyperbolic equation in canonical form is a stable problem. The methods of successive approximation generates a sequence of approximate solutions which is converges to the exact solution .

Finally by introducing the *Riemann-Green function*, *Riemann's method* presents the solution of the Cauchy problem for linear hyperbolic equation, in a manner depending explicitly on prescribed initial conditions.

Chapter 2

ONE DIMENSIONAL WAVE PROBLEMS

2.1 Introduction

In this chapter we will discuss problems involving one dimensional wave equations. There are several physical situations which can be described by one dimensional wave equations, such as transverse vibrations of a string, longitudinal vibration in a beam, and longitudinal sound waves.

In order to provide an intuitive feeling for expected properties of a solution and also what appropriate set of initial or boundary condition may be applied, we begin this chapter with derivation of the wave equation for a simple physical situation, the linear model of the vibrating string . Non homogeneous wave equations are found by applying driving forces. After constructing the linearized model for the motion of a vibrating string, we will give a characterization of the solutions of the model equations from which many properties of waves can be deduced.

In section 2.4 the *D'Alembert solution* for the homogeneous one dimensional wave equations will be derived. This section will illustrate the use of the D'Alembert solution to describe the motion of an infinite string. In the case of a semi-infinite string, we have to take into account a disturbance reflected at the boundary; therefore a modification of the D'Alembert solution must be considered. Space-time interpretations will be discussed using different initial conditions.

In the case of finite string problems, the wave is no longer a travelling wave, but *a standing wave*. In section 2.6, the methods of separation of variables will be used. This method leads to a solution which can be interpreted as an infinite sum of simple vibrations, which describes a standing wave. In this section we shall analyse the simple and intuitive system involving wave motion, transverse vibration of a tightly stretched , elastic string. We will examine some physical systems, which have different types of boundaries.

The Riemann method which was discussed in the previous chapter, will be extended to *the Riemann-Volterra Solution* in section 2.7.

2.2 Vibrating Strings: A Linear model

Consider an elastic string of negligible thickness tightly stretched between end points $x = 0$ and $x = L$. Suppose there are no external transverse forces such as gravity acting on the string, so that the forces acting on the string is only due to a tension force. The equilibrium position of the string is the interval $0 \leq x \leq L$. Suppose that the string is distorted and then at a certain time say $y = 0$, it is released and allowed to vibrate only in the direction perpendicular to x-axis, that is no displacement along the x-axis. Hence the motion of the string takes place in a fixed plane. The problem is determining the deflection of the string, say u, at any point and any time $y > 0$, that is finding $u(x, y)$ for $0 \leq x \leq L$ and $y > 0$.

We shall construct a model for the motion of the string under the action of the tension force, say $T(x, y)$. Since the string is elastic, then the tension force T in the string offers no resistance to bending. Hence the tension at each particle of the string is tangential to the curve of the string. Denoting θ the angle between the curve of the string and the x-axis, the components of the tension force in the direction x and u are $T_x = T cos\theta$ and $T_u = T sin\theta$.

Consider the motion of PQ, a small element of the string, of length ds, in the segment $[x, x + \Delta x]$ as shown in figure 2.1.

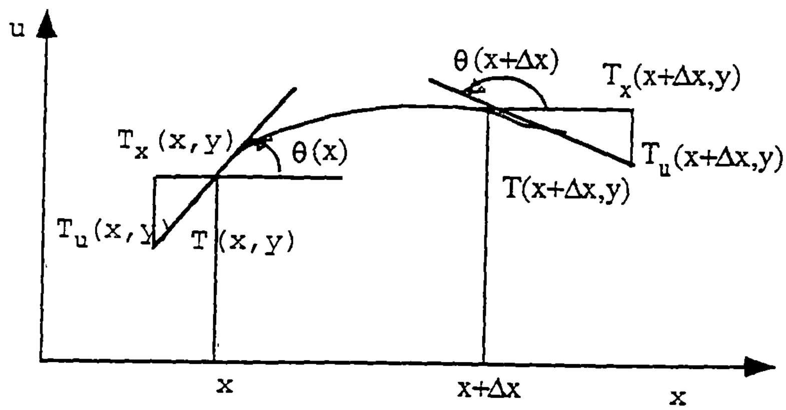


Figure 2.1:A linear model of string

Referring to the figure 2.1, the forces acting at the end point P are

$$T_u(x, y) = T(x, y) \sin\theta(x),$$

$$T_x(x, y) = T(x, y) \cos\theta(x)$$

and the forces acting at the end point Q are

$$T_u(x + \Delta x, y) = T(x + \Delta x, y) \sin\theta(x + \Delta x),$$

$$T_x(x + \Delta x, y) = T(x + \Delta x, y) \cos\theta(x + \Delta x).$$

The net force in the u direction, T_u, is given by

$$T_u = T(x + \Delta x, y) \sin\theta(x + \Delta x) - T(x, y) \sin\theta(x).$$

However since there is no motion in the x-axis direction then

$$T_u(x, y) = T_x(x + \Delta x, y) = constant$$

such that the net force in the x direction is

$$T_x(x, y) = T(x + \Delta x, y) \cos\theta(x + \Delta x) - T(x, y) \cos\theta(x) = 0$$

hence

$$\frac{\partial}{\partial x}(T cos\theta) = 0. \tag{2.1}$$

The net forces acting on PQ is

$$\begin{aligned} T(x,y) &= T_u(x,y) + T_x(x,y) \\ &= T(x+\Delta x, y)\sin\theta(x+\Delta x) - T(x,y)\sin\theta(x), \end{aligned} \tag{2.2}$$

Applying the *Newton's second law* gives

$$\frac{T(x+\Delta x, y)\sin\theta(x+\Delta x) - T(x,y)\sin\theta(x)}{\Delta x} = \rho\Delta x\frac{\partial^2 u}{\partial y^2}$$

where $\rho(x)$ is the mass per unit length of the string. Dividing by Δx and taking limits as $\Delta x \to 0$ yields

$$\frac{\partial}{\partial x}(Tsin\theta) = \rho\frac{\partial^2 u}{\partial y^2}, \tag{2.3}$$

Since $tan\theta = \frac{\partial u}{\partial x}$ then

$$sin\theta = \frac{\partial u}{\partial x}cos\theta,$$

hence

$$\begin{aligned} \frac{\partial}{\partial x}(Tsin\theta) &= \frac{\partial}{\partial x}(Tcos\theta\frac{\partial u}{\partial x}) \\ &= Tcos\theta\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) + \frac{\partial u}{\partial x}\frac{\partial}{\partial x}(Tcos\theta), \end{aligned}$$

and recalling (2.1) results

$$\frac{\partial}{\partial x}(Tsin\theta) = Tcos\theta\frac{\partial^2 u}{\partial x^2}.$$

Substituting this into (2.3) we get

$$Tcos\theta\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial y^2}. \tag{2.4}$$

Furthermore for small deflection $\theta \to 0$, $cos\theta \approx 1$ and hence

$$\frac{\partial}{\partial x}(Tsin\theta) \approx T\frac{\partial^2 u}{\partial x^2}. \tag{2.5}$$

Using this, the equation (2.4) may be replaced by

$$T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial y^2}. \tag{2.6}$$

This equation govern the vibrating of the string with small deflection.

Consider now

$$\rho = \frac{mass}{length} = \frac{m}{l}$$

and

$$T = mass \times acceleration = \frac{mass \times length}{time^2} = \frac{ml}{t^2}$$

therefore,

$$\frac{T}{\rho} = \frac{ml}{t^2}\frac{l}{m} = (\frac{l}{t})^2.$$

Hence $\frac{T}{\rho}$ has the dimensions of a velocity squared. Since T and ρ are positive, we can put

$$\frac{T}{\rho} = c^2. \tag{2.7}$$

where the real constant c represents a velocity. Using this, the equation (2.6) may also be expressed as

$$\frac{\partial^2 u}{\partial y^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2.8}$$

It is well known as the *one dimensional wave equation,* which arises in many other physical problem such as one-dimensional compressible fluid flow. Since $c \neq 0$ the wave equation (2.8) may be written in the *homogeneous* form

$$\frac{\partial^2 u}{\partial x^2} - c^{-2}\frac{\partial^2 u}{\partial y^2} = 0 \tag{2.9}$$

2.3 Driving Forces

The wave equation (2.9) may be generalize by applying a driving force G along the string at any value of x and y. If the force G is a function of x and y we have the *non-homogeneous wave equation*

$$\frac{\partial^2 u}{\partial x^2} - c^{-2}\frac{\partial^2 u}{\partial y^2} = G(x,y). \tag{2.10}$$

The force may also depend on u its first derivatives; in this case

$$\frac{\partial^2 u}{\partial x^2} - c^{-2}\frac{\partial^2 u}{\partial y^2} = G(x,y,u,u_x,u_y). \tag{2.11}$$

Notice that this equation is the hyperbolic equation in the canonical form (1.37). Using (1.22),

$$\xi = \Phi_1(x,y) \qquad\qquad \eta = \Phi_2(x,y), \tag{2.12}$$

the equation (2.11) will be transformed into the canonical form (1.37)

$$\frac{\partial^2 u}{\partial x \partial y} = G(x,y,u,u_x,u_y). \tag{2.13}$$

From section 1.2 we know that the functions Φ_1 and Φ_2 in (2.12) are chosen to be the characteristic curves of the equation.

To solve the Cauchy problem for the non-homogeneous wave equation (2.11) we transform first (2.11) into the canonical form (2.13) and then we may use the method of successive approximations (see section 1.7). If the function G is linear in the first derivatives, then we may use the Riemann's method (see section 1.8). Some examples will be given later in section 2.8.

2.4 Travelling Waves

In order to clarify the properties of transverse vibrations of the string consider first the *Cauchy problem* for the wave equation

$$u_{yy} = c^2 u_{xx} \tag{2.14}$$

for $-\infty < x < \infty$ and $0 < y < \infty$.

According to Newtonian mechanics, the natural condition to impose are the prescription of the initial displacement and the initial velocity. That is, initially at $y = 0$ we prescribe the shape and the velocity of the string to be given functions $r(x)$ and $s(x)$ respectively

$$u(x,0) = r(x), \qquad u_y(x,0) = s(x). \tag{2.15}$$

In this case, the functions $r(x)$ and $s(x)$ are defined over $-\infty < x < \infty$. Notice that the set of *initial conditions* (2.15) is a special case of Cauchy data (1.30) when the initial curve C (1.39) is the straight line $y = 0$, i.e. x-axis, such that the normal derivative u_n becomes u_y. Then we have the following Cauchy problem: find the solution of the wave equation (2.14) which satisfies the initial conditions (2.15).

Since the equation (2.14) is hyperbolic everywhere, a suitable change of independent variables will bring it into its canonical form. Taking the transformation (2.12) to be

$$\xi = x - cy, 0.5in\eta = x + cy \tag{2.16}$$

the partial differential equation (2.14) is then transformed into canonical form

$$u_{\xi\eta} = 0 \tag{2.17}$$

which is (1.41). From section 1.4. we have found that the general solution of the partial differential equation (2.17) is given by

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta) \tag{2.18}$$

where ϕ and ψ are arbitrary functions of ξ and η respectively.

Substituting the transformations (2.16) into (2.18) we have

$$u(x, y) = \phi(x - cy) + \psi(x + cy). \tag{2.19}$$

Now we have to determine the functions ϕ and ψ such that the initial conditions (2.15) are satisfied. By applying the first initial condition we have

$$\phi(x) + \psi(x) = r(x) \tag{2.20}$$

Differentiating (2.19) with respect to y, and applying the second initial condition, we obtain

$$-c\phi'(x) + c\psi'(x) = s(x).$$

Furthermore by integrating this from, say, x_0 to x , we have

$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} s(\xi)d\xi + K, \tag{2.21}$$

where K is a constant of integration. Solving equation (2.20) and (2.21) , we have

$$\phi(x) = \frac{1}{2}r(x) - \frac{1}{2c}\int_{x_0}^{x} s(\xi)d\xi + \frac{K}{c}$$

and

$$\psi(x) = \frac{1}{2}r(x) + \frac{1}{2c}\int_{x_0}^{x} s(\xi)d\xi - \frac{K}{c}.$$

Replacing x on ϕ and ψ by $x - cy$ and $x + cy$ respectively gives

$$\phi(x - cy) = \frac{1}{2}r(x - cy) - \frac{1}{2c}\int_{x_0}^{x-cy} s(\xi)d\xi - \frac{K}{c} \tag{2.22}$$

and

$$\psi(x + cy) = \frac{1}{2}r(x + cy) + \frac{1}{2c}\int_{x_0}^{x+cy} s(\xi)d\xi + \frac{K}{c} \tag{2.23}$$

Substituting these into (2.19) results in

$$u(x, y) = \frac{1}{2}(r(x - cy) + r(x + cy)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi. \tag{2.24}$$

This is the solution of the wave equation (2.14) with the initial conditions (2.15). The solution is well known as *the D'Alembert* solution of the one dimensional wave equation. The D'Alembert solution (2.24) is compatible with the solution (1.45) found in section 1.4. The solution (2.24) can be found by adopting directly the solution (1.45).

We know from section 1.4 that for a hyperbolic equation there are two characteristic curves, namely the α characteristic and the β characteristic through each point on the domain. We wish to examine here some significance influences the characteristic curves of the wave equation (2.14) into the D'Alembert solution (2.24).

Consider an arbitrary point $R = (x_0, y_0)$ in the xy-plane. Obviously the α characteristic line and the β characteristic line through the point R are

$$x - cy = x_0 - cy_0 \tag{2.25}$$

and

$$x + cy = x_0 + cy_0. \tag{2.26}$$

respectively. Substituting the coordinates of the point R to (2.24) we get

$$u(x_0, y_0) = \frac{1}{2}(r(x_0 - cy_0) + r(x_0 + cy_0)) + \frac{1}{2c}\int_{x_0-cy_0}^{x_0+cy_0} s(\xi)d\xi. \tag{2.27}$$

This result shows us that the solution at the point R is obtained as being the average of the values of the initial displacement $r(x)$ at points say $P = (x_0 - cy_0, 0)$ and $Q = (x_0 + cy_0, 0)$ and the initial velocity $s(x)$ along the segment PQ.

Geometrically the point P and Q are found by backtracking the characteristic lines (2.25) and (2.26). Referring to the discussion in section 1.4, the triangular PQR is called the domain of dependence of the solution u with respect to the point R, and denoted by Σ, while the interval PQ is called the interval of dependence of the point R. See figure 2.2.

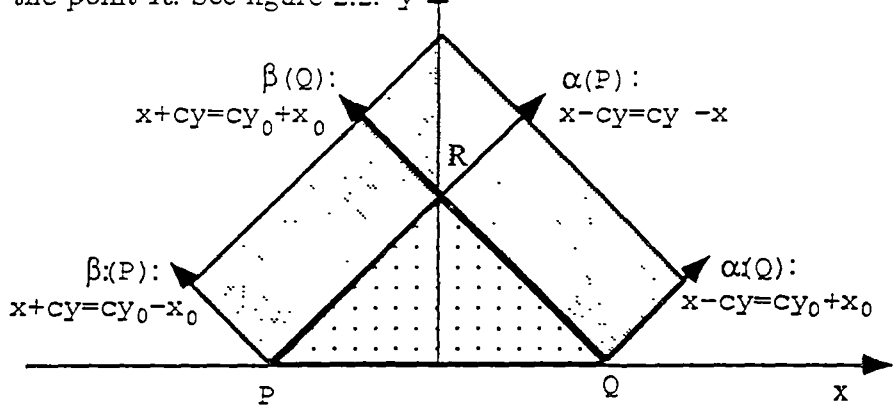


Fig.2.2. The characteristics of wave eqn.
PQ: the domain of determinacy wrt.R
: Σ ,the domain of dependent wrt.R
: $\Re$,the range of influence wrt.PQ

Figure 2.2:The domain of dependence and the range of infulence

However the initial values along the interval PQ can influence the solution only in the region bounded by the β characteristics line through the point P

$$x - cy = x_0 - cy_0$$

and the α characteristic line through the point $Q = (x_0 + y_0, 0)$

$$x + cy = x_0 + cy_0. \tag{2.28}$$

The region bounded by these characteristic lines is the range of influence of the initial data along the segment PQ. The shaded area in figure (2.2) indicates the range of influence and denoted by $\Re$. Hence the significance of the characteristic curves is that they form the boundaries of the domain of dependence and the range of influence.

Furthermore in order to give a *space-time* interpretation of the D'Alembert solution (2.24), consider two specific cases of the initial value problems; firstly when the string has a given non-zero initial displacement and *zero initial velocity*, secondly when the string has *zero initial displacement* and a given non zero initial velocity.

The first case the initial conditions (2.15) have the form

$$u(x,0) = r(x), \qquad u_t(x,0) = 0 \tag{2.29}$$

for $-\infty < x < \infty$ then the D'Alembert solution (2.24) is reduced to

$$u(x,y) = \frac{1}{2}(r(x-cy) + r(x+cy)). \tag{2.30}$$

The equation (2.28) shows that the solution at a point (x,t) may be interpreted as being the average of the initial displacement $r(x)$ at the point $(x-ct,0)$ and $(x+ct,0)$. The initial displacement $r(x)$ are propagated as time goes on. The solution represents as a superposing of two travelling waves which has the same profile and the same velocity c, but travelling in the opposite directions along the x-axis.

The second case when the string has initial displacement zero and a given initial velocity. The initial conditions (2.15) have the form

$$u(x,0) = 0, \qquad u_t(x,0) = s(x) \tag{2.31}$$

for $-\infty < x < \infty$. Then the D'Alembert solution (2.24) reduces to

$$u(x,t) = \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi. \tag{2.32}$$

Suppose $s^*(x)$ is the function such that $\frac{ds^*}{dx} = s$ then we have

$$u(x,t) = \frac{1}{2c}(s^*(x+cy) - s^*(x-cy)).$$

It represents a superimposing of travelling waves $\frac{1}{2c}s^*(x+cy)$ and $-\frac{1}{2c}s^*(x-cy)$ travelling in the opposite directions along the x-axis.

As a particular illustration of the first case, consider the example below.

Example 2.1.

Find the solution of wave equation

$$u_{yy} = u_{xx} \tag{2.33}$$

that satisfies the initial conditions

$$u(x,0) = r(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \tag{2.34}$$

and

$$u_y(x,0) = s(x) = 0 \tag{2.35}$$

where $-\infty < x < \infty$.

The characteristics lines at the points $(-1,0)$ and $(1,0)$ divide the xy-plane into six regions,

$$\begin{aligned}
R_1 &= \{(x,y); x-y > -1 \cap x+y < 1 \cap x > 0\}, \\
R_2 &= \{(x,y); x+y < -1\}, \\
R_3 &= \{(x,y); x+y > -1 \cap x+y < 1\}, \\
R_4 &= \{(x,y); x+y > 1 \cap x-y < -1\}, \\
R_5 &= \{(x,y); x-y > -1 \cap x-y < 1\} \\
R_6 &= \{(x,y); x-y > 1\}.
\end{aligned}$$

such as shown in figure 2.3.

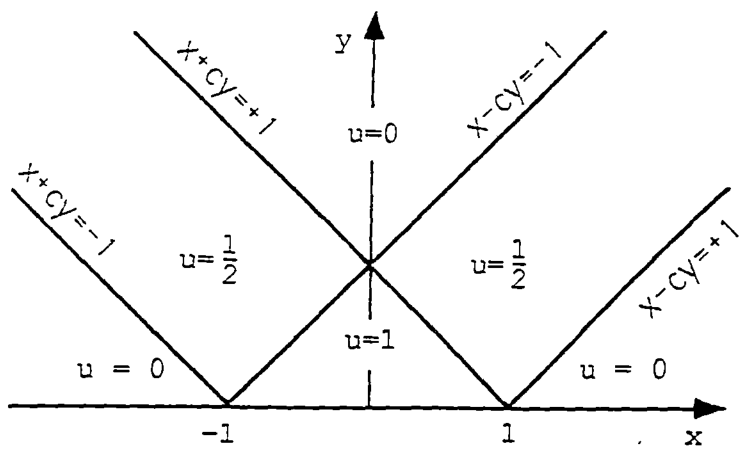


Figure 2.3: The regions of example 3.1

The initial values in (2.34) and (2.35) reduce the D'Alembert solution (2.24) to

$$u(x,y) = \frac{1}{2}(r(x-y) + r(x+y)), \tag{2.36}$$

that is

$$u(x,y) = \begin{cases} 1 & \forall (x,y) \in R_1 \\ 0 & \forall (x,y) \in R_2 \\ 0.5 & \forall (x,y) \in R_3 \\ 0 & \forall (x,y) \in R_4 \\ 0.5 & \forall (x,y) \in R_5 \\ 0 & \forall (x,y) \in R_6 \end{cases} \tag{2.37}$$

We may construct the surface of the solution depicting the motion of the wave as shown in figure. 2.4.

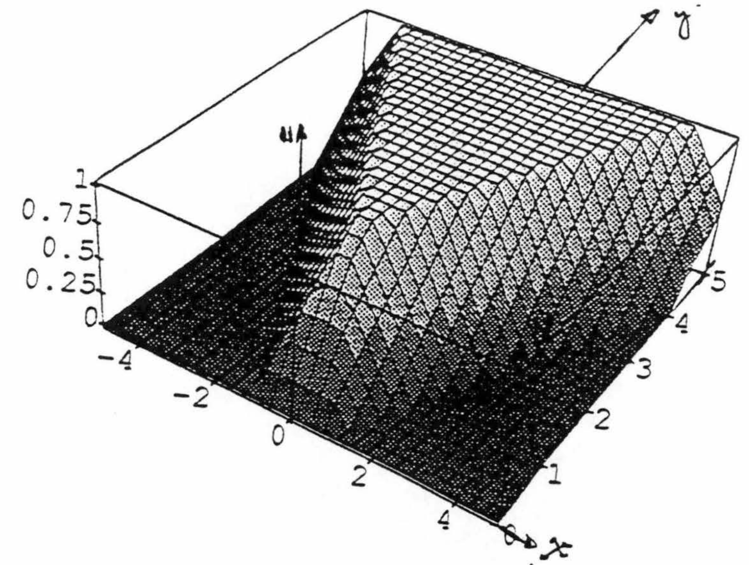


Figure 2.4: The solution of string with zero initial velocity

The slopes of the surface of motion are shown by its contour, the projection of the solution onto the xy-plane. (See figure 2.5)

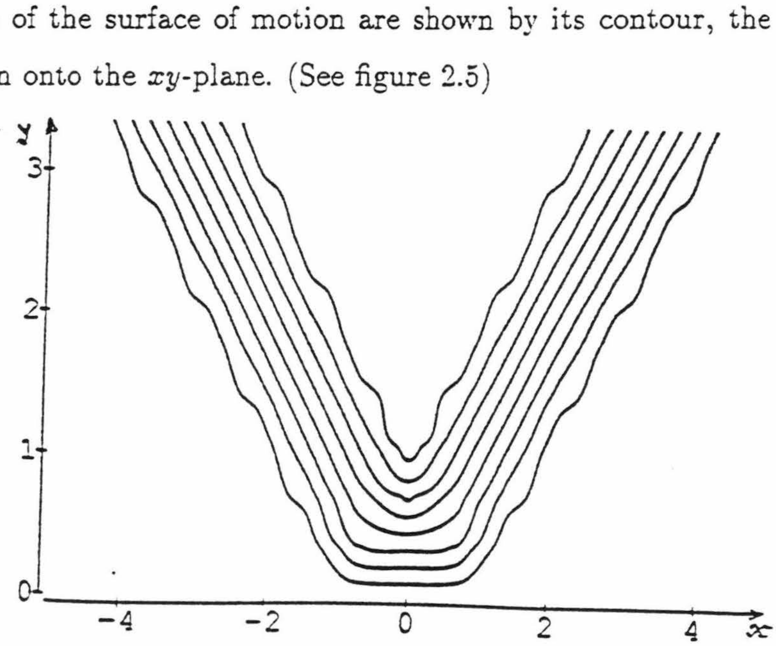


Figure 2.5:The contour of the solution of string with a zero initial velocity

Corresponding to various values of y, we may construct a series of graphs which represent the motion of the wave. (See figure.2.6)

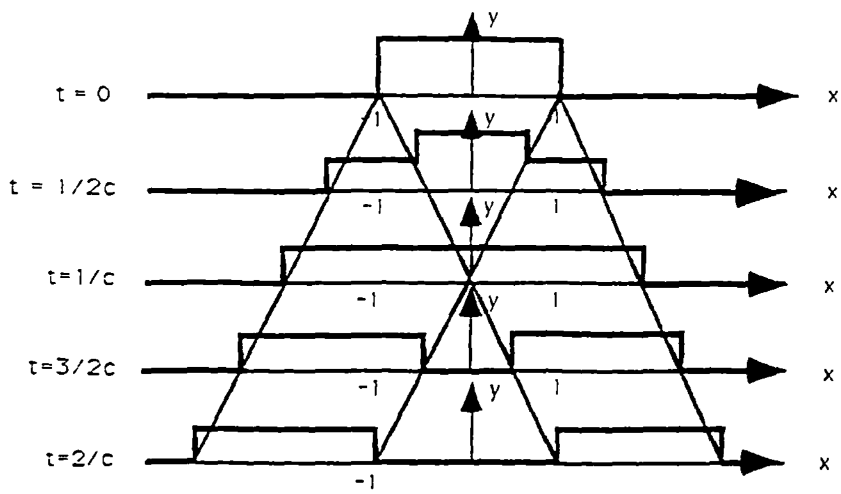


Figure 2.6:The wave motion of string with zero initial velocity

In the second case, the string has zero initial displacement and non zero initial velocity.

Example 2.2

Find the solution of the wave equation (1.21) satisfies the initial conditions

$$u(x,0) = r(x) = 0 \tag{2.38}$$

and

$$u_y(x,0) = s(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \tag{2.39}$$

As in the previous case, the characteristic lines in points $(-1,0)$ and $(1,0)$ divide the xy-plane into the regions R_1, R_2, R_3, R_4, R_5 and R_6, such as shown in figure 2.7.

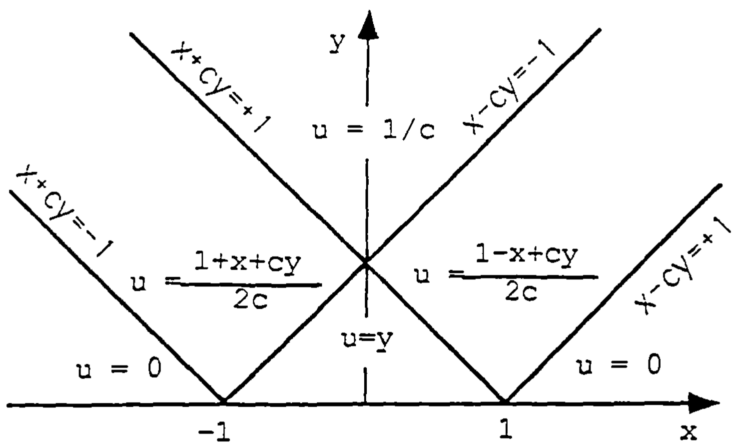


Figure 2.7:Regions example 2.2

Using the initial condition the D'Alembert solution is reduced to

$$u(x,y) = 0.5\int_{x-y}^{x+y} s(\xi)d\xi, \tag{2.40}$$

hence

$$u(x,y) = \begin{cases} 0.5\int_{x-y}^{x+y} d\xi = y, & \forall(x,y) \in R_1 \\ 0.5\int_{x-y}^{x+y} 0d\xi = 0, & \forall(x,y) \in R_2 \\ 0.5\int_{-1}^{x+y} d\xi = 0.5(x+y+1), & \forall(x,y) \in R_3 \\ 0.5\int_{-1}^{1} d\xi = 1, & \forall(x,y) \in R_4 \\ 0.5\int_{x-y}^{1} d\xi = 0.5(-x+y+1), & \forall(x,y) \in R_5 \\ 0.5\int_{x-y}^{x+y} 0d\xi = 0, & \forall(x,y) \in R_6 \end{cases} \tag{2.41}$$

This solution may be grawn as shown in figure 2.8.

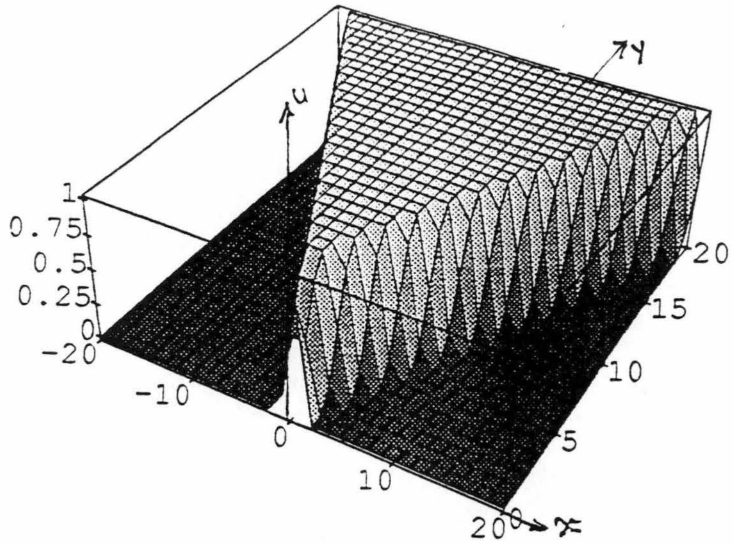


Figure 2.8: The solution of string with zero initial displacement

The slopes of the surface of motion are almost linear such as shown in the contours. (See figure 2.9)

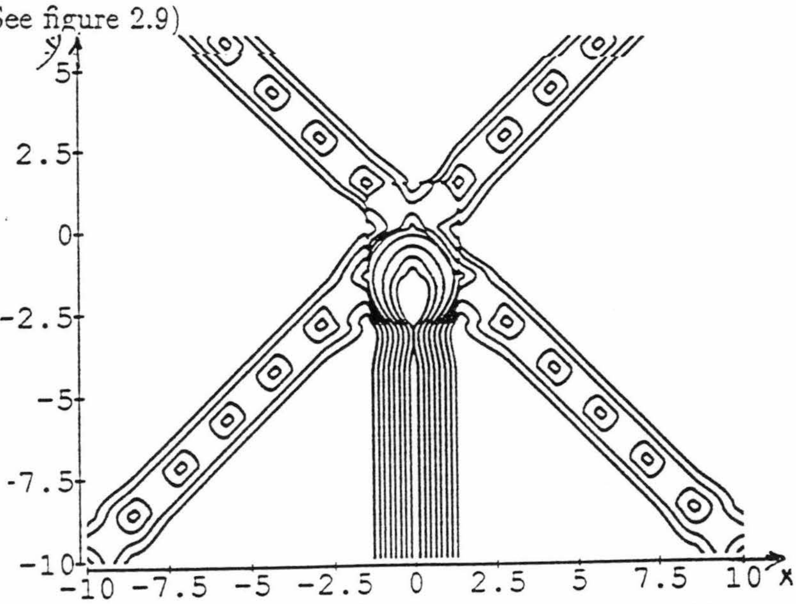


Figure 2.9:The contour of the solution of string with zero initial displacement.

And finally for various values of y, we may also construct a series of graphs which represent the motion of the wave. (See figure 2. 10)

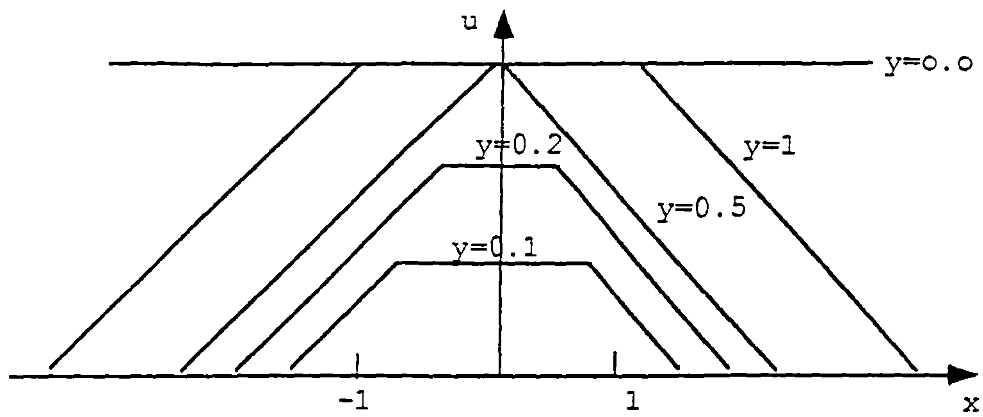


Figure 2.10The wave motion of string with zero initial displacement

2.5 Boundary Conditions Associated with the Wave Equations

So far the string has been supposed to be unbounded, that is the wave equation (2.14) is defined over infinite domain $-\infty < x < \infty$. The solution appears as travelling waves moving in opposite directions. Consider now, the portion of the string in the segment $0 < x < \infty$

$$u_{yy} = c^2 u_{xx} \tag{2.42}$$

for $0 < x < \infty$ and $0 < y < \infty$. Suppose that besides of the initial conditions (2.15) and (2.16), there are additional conditions which is applied at the end point of the string. Such conditions are called *boundary conditions.* Problems involving the initial and boundary conditions are called *boundary value problems.* By applying a particular boundary condition, the motion of the waves will be influenced at the boundary. Disturbance moving along the characteristics will then be reflected or transmitted at the boundary. In this section we wish to investigate the corresponding changes in the D'Alembert solution.

Assume that the end point of the string is attached by a linear spring such as shown in figure 2.11 below.

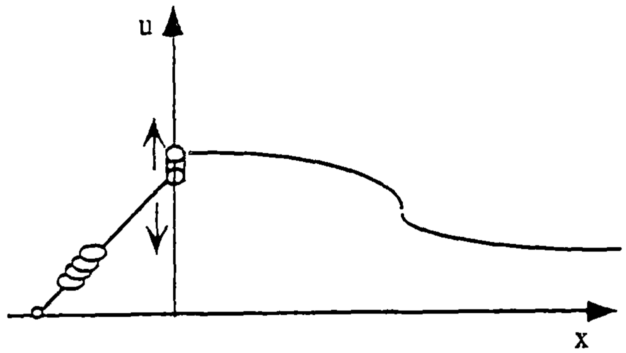


Figure 2.11:Elastic Attachment on the boundary

The string attachment give rise vertical force proportional to the displacement $u(0, y)$. Setting the tension force $Tu_x(0, y)$, we have

$$Tu_x(0, y) = ku(0, y)$$

where k is the spring constant and T is the string tension. Hence the condition should be satisfied at the boundary is

$$u_x(0, y) - \delta u(0, y) = 0, \qquad 0 < y < \infty. \tag{2.43}$$

with $\delta = \frac{k}{T}$. When $\delta \to \infty$ we have

$$u(0, y) = 0, \qquad 0 < y < \infty \tag{2.44}$$

the end point is fixed. When $\delta = 0$ we get

$$u_x(0, y) = 0, \qquad 0 < y < \infty, \tag{2.45}$$

physically, the string at the boundary has no resistance, free to move in the vertical direction. If the spring attachment is displaced according to the function $t^*(y)$, see figure 2.12, then the condition should be satisfied at the boundary is

$$Tu_x(0, y) = k[u(0, y) - t^*(y)].$$

hence we have a non-homogeneous boundary condition

$$u_x(0,y) - \delta u(0,y) = t(y), \qquad 0 < y < \infty. \tag{2.46}$$

with $\delta = \frac{k}{T}$ and $t(y) = \frac{t(y)^*}{T}$ is supposed to be first order continuous differentiable.

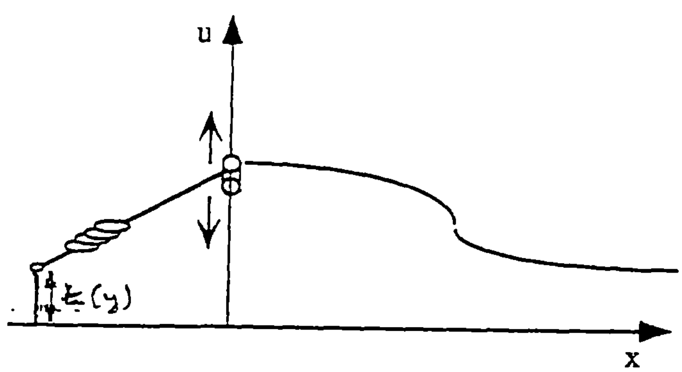


Figure 2.12:Non-homogeneous boundary condition

When δ in (2.46) is a function of y we have

$$u_x(0,y) - \delta(y)u(0,y) = t(y), \qquad 0 < y < \infty. \tag{2.47}$$

This is well known as *the boundary condition of third kind,* while the first and second kinds

$$u(0,y) = t(y), \qquad 0 < y < \infty, \tag{2.48}$$

and

$$u_x(0,y) = t(y), \qquad y \geq 0 \tag{2.49}$$

are well known as *the Dirichlet* and *the Neumann's* boundary conditions respectively. Physically, the Dirichlet boundary condition is due to *controlled end the point*, while the Neumann's boundary condition is due to *given force* at the end point, and the boundary condition of third kind is *elastic attachment* of the end point.

2.5.1 Controlled end points

Consider first the imposition of a condition such that the solution remains zero along the boundary. The problem can be formulated as follows: find the solution of the wave equation (2.42) satisfying the set of initial conditions (2.15), and the boundary condition (2.44). Notice that since the wave equation (2.42) is defined over the region $R = \{(x, y) | 0 < x < \infty, \ 0 < y < \infty\}$, then, in this case, the set of initial conditions (2.15) is defined over finite interval $0 < x < \infty$.

Using a method similar to the method in solving the infinite string problem, substituting the set of initial conditions, we have (2.22) and (2.23) which, in this case, hold everywhere in the first quadrant of the xy-plane. Hence we have to find

$$\phi(x - cy) \qquad \forall - \infty < x - cy < \infty$$

and

$$\psi(x + cy) \qquad \forall 0 < x + cy < \infty.$$

However since the initial data $r(x)$ and $s(x)$ are defined only for $x > 0$, (2.22) gives only the value of $\phi(x - cy)$ for $x - cy \geq 0$. Hence we still have to find the value of $\phi(x - cy)$ in the region $x - cy < 0$. Geometrically, the wave plane is divided into two regions, $x \geq cy$ and $x < cy$. The characteristic through $P = (x_0 - cy_0, 0)$ is reflected at the boundary. The reflection line intersect the characteristic through the point $Q = (x_0 + cy_0, 0)$ at the point $R = (x_0, y_0)$ in the region $x < cy$. Hence the solution at the point R is influenced by the boundary value. The situation is depicted by figure 2.13 below.

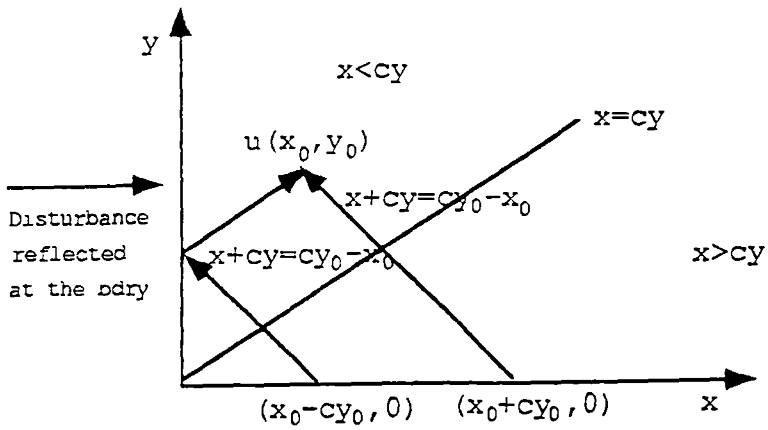


Fig.2.13: the semi-infinite string

For $x < cy$, substituting the boundary-condition (2.44) into (2.19), we have

$$\phi(-cy) + \psi(cy) = 0, \tag{2.50}$$

or

$$\phi(-cy) = -\psi(cy).$$

Letting $z = -cy$, the last equation becomes

$$\phi(z) = -\psi(-z),$$

and replacing z by $x - cy$ gives

$$\phi(x - cy) = -\psi(cy - x), \tag{2.51}$$

further by using (2.23) we have

$$\phi(x - cy) = -\frac{1}{2}r(cy - x) + \frac{1}{2c}\int_{x_0}^{x+cy} s(\xi)d\xi + \frac{K}{c}. \tag{2.52}$$

Finally substituting (2.23) and (2.52) into (2.19) we get

$$u(x, y) = \frac{1}{2}(r(x + cy) - r(cy - x)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi. \tag{2.53}$$

Hence the solution of the wave equation (2.42) satisfying the set of initial conditions (2.15), and the boundary condition (2.44) is

$$u(x, y) = \begin{cases} \frac{1}{2}(r(x - cy) + r(x + cy)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi, & x \geq cy \\ \frac{1}{2}(r(x + cy) - r(cy - x)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi, & x < cy \end{cases} \tag{2.54}$$

This equation show us that for $x \geq 0$ the solution is the same as the D'Alembert solution for the infinite wave, while for $x < 0$ the solution is modified due to the reflection of the wave at the boundary.

Secondly, when the boundary is moving in such a way that its displacement is $t(y)$, a function of y. The problem can be formulated as follows: find the solution of the wave equation (2.42) satisfying the set of initial conditions (2.15), and the boundary condition (2.48). However in order the boundary condition (2.48) is compatible with the set of initial conditions (2.15) it is necessary that

$$t(0) = r(0), \qquad t'(0) = s(0). \tag{2.55}$$

We solve the problem by using the same procedures such as in the first case. For $x < cy$, the relations (2.50) and (2.50) become

$$\phi(-cy) + \psi(cy) = t(y), \tag{2.56}$$

and

$$\phi(x - cy) = t\left(-\frac{x-c}{c}\right) - \psi(-(cy - x)), \tag{2.57}$$

respectivelly, therefore (2.53) will be

$$\phi(x - cy) = t\left(-\frac{x-c}{c}\right) - \frac{1}{2}r(cy - x) + \frac{1}{2c}\int_{x_0}^{x+cy} s(\xi)d\xi + \frac{K}{c}. \tag{2.58}$$

Substituting (2.23) and (2.58) into (2.19) we get

$$u(x,y) = t\left(-\frac{x-c}{c}\right) + \frac{1}{2}(r(x+cy) - r(cy-x)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi. \tag{2.59}$$

Hence the solution of the wave equation (2.42) satisfying the set of initial conditions (2.15), and the boundary condition (2.48) is

$$u(x,y) = \begin{cases} \frac{1}{2}(r(x-cy) + r(x+cy)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi, & x \geq cy \\ t\left(-\frac{x-c}{c}\right) + \frac{1}{2}(r(x+cy) - r(cy-x)) + \frac{1}{2c}\int_{x-cy}^{x+cy} s(\xi)d\xi, & x < cy \end{cases} \tag{2.60}$$

2.5.2 Elastic attachment of the end points

Suppose the end point of the string is attached to the origin by linear spring and the spring attachment is displaced according to the function of y we have the boundary condition of third kind (2.47). As a particular example consider the following example: find the solution of the wave equation (2.42) satisfying the set of initial conditions (2.15), and the boundary condition (2.43). However in order the boundary condition (2.43) is compatible with the set of initial conditions (2.15) it is necesary that

$$r'(0) - \delta r(0) = 0, \qquad s'(0) - \delta s(0) = 0. \tag{2.61}$$

Suppose $s(x) = 0$, then the equations (2.22) and (2.23) give us

$$\phi(x - cy) = \frac{1}{2} r(cy - x) \quad \forall x > cy$$

and

$$\psi(x + cy) = \frac{1}{2} r(x + cy) \quad \forall x > cy.$$

From subsection 2.5.1. we know that ψ is constant along the characteristic $x+cy = k$, for each constant k. Hence we have

$$\psi(x + cy) = \frac{1}{2} r(x + cy) \quad \forall x > cy \cup x < cy.$$

Using functional substitution such as in the subsection 2.5.1, we have

$$u(x,y) = \begin{cases} \frac{1}{2}[r(x - cy) + r(x + cy)], & x \geq cy \\ \frac{1}{2} r(x + cy) + \phi(x - cy) & \forall x < cy \end{cases}. \tag{2.62}$$

Suppose $\phi(x - cy) = \Phi\left(\frac{-(x-ct)}{c}\right)$ for some function Φ. Writing $\frac{-(x-cy)}{c} = y - \frac{x}{c}$ then $\phi(x - cy) = \Phi\left(y - \frac{x}{c}\right)$. Substituting this into (2.62) we have

$$u(x,y) = \frac{1}{2} r(x + cy) + \Phi\left(t - \frac{x}{c}\right) \quad \forall x < cy. \tag{2.63}$$

To determine the function $\Phi(x - cy)$ we may use the boundary condition (2.43). Applying the boundary condition (2.43) gives

$$\frac{1}{2} r'(cy) - \frac{1}{c}\Phi'(y) - \frac{\delta}{2} r(cy) - \delta\Phi(y) = 0.$$

Hence we have the first order linear ordinary differential equation

$$\Phi'(y) + c\delta\Phi(y) = \frac{c}{2}\left[r'(cy) - \delta r(cy)\right]$$

and the solution is

$$\Phi(y) = \frac{c}{2}e^{-\delta cy}\int_0^y \left[r^{'}(c\eta) - \delta r(c\eta)\right] e^{\delta c\eta} d\eta + ke^{-\delta cy} \tag{2.64}$$

where $k = Phi(0)$. Furthermore from (2.62) and (2.63) we have $Phi(0) = \frac{1}{2}r(0)$. Inserting this into (2.64) we have

$$\Phi(y) = \frac{c}{2}e^{-\delta cy}\int_0^y \left[r^{'}(c\eta) - \delta r(c\eta)\right] e^{\delta c\eta} d\eta + \frac{1}{2}r(0)e^{-\delta cy}. \tag{2.65}$$

Hence

$$\begin{aligned}\phi(x-cy) &= \Phi\left(\frac{-(x-cy)}{c}\right)\\ &= \frac{c}{2}e^{-\delta c(\frac{-(x-cy)}{c})}\int_0^{\frac{-(x-cy)}{c}} \left[r^{'}(c\eta) - \delta r(c\eta)\right] e^{\delta c\eta} d\eta + \frac{1}{2}r(0)e^{-\delta cy}.\end{aligned} \tag{2.66}$$

Substituting this result into (2.62) we get

$$u(x,y) = \frac{1}{2}[r(x-cy) + r(x+cy)], \;\; x \geq cy, \tag{2.67}$$

and

$$\begin{aligned}u(x,y) &= \frac{c}{2}e^{-\delta c(\frac{-(x-cy)}{c})}\int_0^{\frac{-(x-cy)}{c}} \left[r^{'}(c\eta) - \delta r(c\eta)\right] e^{\delta c\eta} d\eta\\ &\quad + \frac{1}{2}r(0)e^{-\delta cy}, \qquad\qquad \forall x < cy.\end{aligned} \tag{2.68}$$

2.6 Standing Waves

So far we have seen that the solution of the one dimensional wave equation for an unbounded domain appear to be travelling waves. For semi-infinite strings, by applying boundary at the end of the string the D'Alembert solution was modified. Now we study the motion of the string in a bounded region, say $0 \leq x \leq L$, with both end points $x = 0$ and $x = L$ are fixed. Due to repeated interactions with the boundaries, the waves are no longer moving, but appear to be what are known as *standing waves.*

Consider a vibrating string of finite length, L, which may be described by wave equation

$$u_{yy} = c^2 u_{xx} \tag{2.69}$$

for $0 < x < L$ and $0 < y < \infty$ with boundary conditions

$$u(0,y) = 0, \qquad u(L,y) = 0, \qquad 0 < y < \infty \tag{2.70}$$

and initial conditions

$$u(x,0) = r(x), \qquad u_y(x,0) = s(x), \qquad 0 \leq x \leq L. \tag{2.71}$$

The motion of the string is composed of two waves continually travelling along it in opposite directions, and the whole displacement being the resultant of the two waves and their reflections at the end points. To solve this problem we use *the Method of Separation Variable.* Assume that the solution has the form

$$u(x,y) = X(x).Y(y). \tag{2.72}$$

Differentiating this and substituting this into the equation (2.69) gives

$$\frac{Y''}{c^2 Y} = \frac{X''}{X}$$

The left-hand side does not vary with x, therefore the right-hand side cannot vary with x. Similarly the right-hand side does not vary with y, therefore the left-hand side cannot vary with y. The result is that both sides must be constant, say λ, hence we have two ordinary differential equations,

$$Y'' - c^2 \lambda Y = 0$$

and

$$X'' - \lambda X = 0$$

where $-\infty < \lambda < \infty$. When $\lambda \geq 0$, the boundary conditions (2.70) imply trivial solutions and only $\lambda < 0$ give feasible solutions. By rewriting $\lambda = -\mu^2$, the last two equations become

$$Y'' + (\mu c)^2 Y = 0 \tag{2.73}$$

$$X'' + \mu^2 X = 0. \tag{2.74}$$

The general solutions of these equations are

$$Y(y) = Asin(\mu cy) + Bcos(\mu cy) \tag{2.75}$$

and

$$X(x) = Csin(\mu x) + Dcos(\mu x) \tag{2.76}$$

respectively. Substituting these into (2.72) we have

$$u(x, y) = [Csin(\mu x) + Dcos(\mu x)][Asin(\mu cy) + Bcos(\mu cy)]. \tag{2.77}$$

Since according to the first boundary condition (2.70) $u(0, y) = X(0)Y(y) = 0$ and $Y(y) \neq 0$ in general then $X(0) = 0$ and hence (2.76) leads to $D = 0$. Furthermore the second boundary condition $u(L, y) = X(L)Y(y) = 0$ implies that $X(L) = 0$. Putting these into (2.76), we get $Csin(\mu L) = 0$. Since we are looking for a non-trivial solution, $C \neq 0$ and

$$sin(\mu L) = 0. \tag{2.78}$$

The solution of this equation are denoted by

$$\mu_n = \frac{n\pi}{L}, \qquad n = 0, 1, 2, ... \tag{2.79}$$

Substituting these into (2.77) we have an infinite number of solutions

$$u_n(x, y) = X_n(x)Y_n(y) = sin(\frac{n\pi}{L}x)[a_n sin(\frac{cn\pi}{L}y) + b_n cos(\frac{cn\pi}{L}y)], \quad n = 1, 2, 3......, \tag{2.80}$$

Since the partial differential equation and the boundary conditions are *linear* and *homogeneous* then any sum of the solutions (2.80) is also the solution, thus we have the general solution

$$u(x, y) = \sum_{n=1}^{\infty} sin(\frac{n\pi}{L}x)[a_n sin(\frac{cn\pi}{L}y) + b_n cos(\frac{cn\pi}{L}y)]. \tag{2.81}$$

Using the first equation in the initial condition (2.71) we have

$$u(x, 0) = r(x) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{L}x).$$

This series suggest that b_n should be chosen as the coefficient in the *half range Fourier sine series* of $r(x)$. Thus we have

$$b_n = \frac{2}{L}\int_0^L r(x)\sin\frac{n\pi x}{L}dx. \qquad (2.82)$$

Differentiating the equation (2.81) with respect to y and using the second equation in the initial condition (2.70) gives

$$u_y(x,0) = s(x) = \sum_{n=1}^{\infty} a_n(\frac{cn\pi}{L})sin(\frac{n\pi}{L}x).$$

Again a_n should be chosen as the coefficient in the Fourier's sine series of $s(x)$:

$$a_n = \frac{2}{n\pi c}\int_0^L s(x)\sin\frac{n\pi x}{L}dx. \qquad (2.83)$$

Hence equation (2.81) gives the solution of the boundary value problem governed by equations (2.69) ,(2.70) and (2.71). The coefficients of the solution (2.81) are given by (2.82) and (2.83).

In particular when *the initial velocity is zero*, that is $s(x) = 0$, implies $a_n = 0, \forall n$, then equation (2.80) is reduced to

$$u_n(x,y) = X_n(x)Y_n(y) = b_n sin(\frac{n\pi}{L}x)cos(\frac{cn\pi}{L}y),$$
$$n = 1,2,3....... \qquad (2.84)$$

For each n the solution $u_n(x,y)$ is a *standing wave* having the fixed shape $X(x) = sin(\frac{n\pi}{L}x)$ with varying amplitude $Y(y) = cos(\frac{cn\pi}{L}y)$. The zeros of $X(x)$ are called *nodes*.

Since

$$sin(\frac{n\pi}{L}x)cos(\frac{cn\pi}{L}y) = \frac{1}{2}[sin(\frac{n\pi}{L}(x-cy) + sin(\frac{n\pi}{L}(x+cy)],$$

then we have

$$u_n(x,y) = \frac{b_n}{2}[sin(\frac{n\pi}{L}(x-cy) + sin(\frac{n\pi}{L}(x+cy)]. \qquad (2.85)$$

The right hand side are two travelling waves of equal amplitude but going in the opposite direction. Hence we may consider the standing wave as a sum two travelling waves of equal amplitude but going in the opposite direction, see figure 2.12.

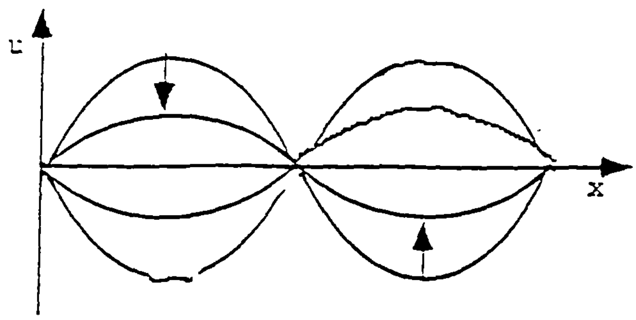


Fig.2.12: the standing wave

Again, since $s = 0$ then the solution (2.81) is reduced to

$$u(x,y) = \frac{1}{2}\sum_{n=1}^{\infty} b_n[sin(\frac{n\pi}{L}(x-cy) + \frac{1}{2}\sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{L}(x+cy). \tag{2.86}$$

Notice that these two series are the half range Fourier sine series of r when r is a function of $(x - cy)$ and $(x + cy)$ respectively. Hence the equation (2.86) can be written as

$$u(x,y) = \frac{1}{2}[r^*(x-cy) + r^*(x+cy)] \tag{2.87}$$

where r^* is the odd periodic extension of r with period $2L$. This result is known as the D'Alembert solution for the problem in the case 2 of section 2.3.

As a particular example consider initial value problem below.

Example 2.3. Suppose the boundary value problem govern by equation (2.69), (2.70) and (2.71) has zero initial velocity the triangular initial deflection

$$u(x,0) = r(x) = \begin{cases} \frac{2c}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2c}{L}(L-x), & |x| > 1 \end{cases} \tag{2.88}$$

Then the solution (2.86) gives

$$u(x,y) = \frac{1}{2}\sum_{n=1}^{\infty} b_n[sin(\frac{n\pi}{L}(x-cy) + \frac{1}{2}\sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{L}(x+cy). \tag{2.89}$$

which depicted by figure 1.13 below

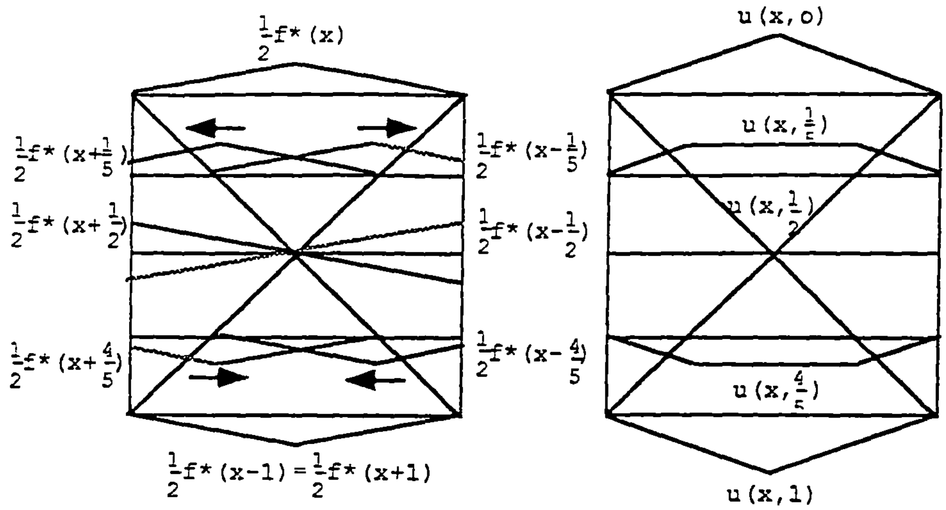


Figure 2.13 Standing wave as superimpose two travelling wave

2.7 The Riemann-Volterra Solution

So far the previous sections discussed the Cauchy problem for the wave equation (2.33), on which the Cauchy data (the initial conditions) given along the x-axis. In the present section we deal with the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 u}{\partial^2 y} \tag{2.90}$$

on which the Cauchy data given along an initial curve C with the parametric representation

$$x = x(\tau) \qquad\qquad y = y(\tau). \tag{2.91}$$

The initial curve C is assumed to be *strictly monotonic.* Then from section 1.3, the Cauchy problem can be described as follow: given continuously differentiable functions ϖ and ω, determine a solution of the equation (2.90) such that the initial conditions

$$u(x(\tau), y(\tau)) = \varpi(\tau),$$
$$u_n(x(\tau), y(\tau)) = \omega(\tau) \tag{2.92}$$

are satisfied along the initial curve (2.91).

Such a problem can be solve by using the Riemann method discussed in section 1.8. However the Riemann method gives the solution of the Cauchy problem for linear hyperbolic equation in the canonical form (1.66)

$$\frac{\partial^2 u}{\partial x \partial y} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g(x, y). \tag{2.93}$$

The method involves introducing the linear operator L containing the left-hand side terms of the equation (2.93) and the adjoint operator M such that $vL[u] - uM[v]$ becomes a divergence for some continuously differentiable function $v = v(x, y)$. Taking v as a Riemann-Green function, the method gives the solution in terms of prescribed values along the given curve and the values of the function g over the domain of dependence.

In this section we will generalize the Riemann method to determine the solution of the Cauchy problem for the linear equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial x \partial y} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g \tag{2.94}$$

where a, b, c, d, e, f and g are function of x and y. Without transforming this equation into the canonical form (2.93), the *Riemann-Voltera method* [17] defines the operator L by

$$L[u] = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial x \partial y} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu, \tag{2.95}$$

and the adjoint operator $M[v]$, by

$$M[v] = \frac{\partial^2 (av)}{\partial x^2} + \frac{\partial^2 (bv)}{\partial y^2} + \frac{\partial^2 (cv)}{\partial x \partial y} - \frac{\partial (dv)}{\partial x} - \frac{\partial (ev)}{\partial y} + fv, \tag{2.96}$$

for $v = v(x, y)$, an arbitrary continuously differentiable function, such that $vL[u] - uM[v]$ becomes a divergence of the vector (U, V) say, that is

$$vL[u] - uM[v] = U_x + V_y \tag{2.97}$$

Of special interest is the situation where the adjoint operator M, is identical with the operator L, $L = M$, the *self adjoint* operator.

Applying the Green's Theorem we have [1]

$$\begin{aligned} \iint\limits_{\Sigma} (vL[u] - uM[v])dxdy &= \iint\limits_{\Sigma} (U_x + V_y)dxdy \\ & \int\limits_{\Gamma} \mathbf{n}.(U, V)d\sigma \end{aligned} \tag{2.98}$$

where the line integration is evaluated in the counter-clockwise direction over the closed contour Γ bounding the region of integration Σ. The parameter σ is the arc length of the curve Γ, while $\mathbf{n}$ is the unit *normal* to the curve.

Consider now as an example, the wave equation (2.90). By defining the operator L

$$L[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \tag{2.99}$$

then the equation (2.90) may be written as

$$L[u] = 0. \tag{2.100}$$

Now we wish to find the solution at an arbitrary point $R = (x_0, y_0)$. In section 2.3, we have shown that the characteristics curves of the equation (2.90) through the point R have the forms

$$x + y = x_0 + y_0 \tag{2.101}$$

and

$$x - y = x_0 - y_0. \tag{2.102}$$

Denote by P and Q the intersection points between the curve C and these characteristic curves through the point R. (See figure. 2.14)

[1] the generalized form of Green's Theorem

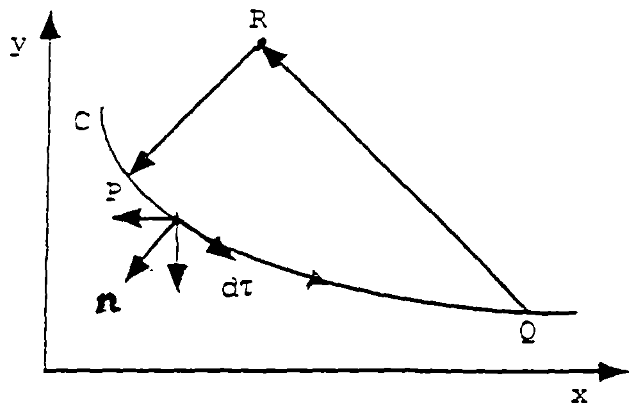


Figure 2.14:The Riemann Volterra's Method

Denote by Γ, the close curve involving the characteristic curves (2.101), (2.102) and the initial curve (2.91). Suppose Γ has the parametric representation

$$x = x(\sigma), \qquad\qquad y = y(\sigma). \tag{2.103}$$

Using the similar procedures as in the Riemann's method (see section 1.8), we have

$$U = v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \tag{2.104}$$

and

$$V = -v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}. \tag{2.105}$$

The Green's function $v(x, y)$ satisfies the conditions

$$M[v] = 0, \tag{2.106}$$

$$\frac{\partial v}{\partial n} = 0, \quad \forall (x, y) \in (PR \cup QR), \tag{2.107}$$

and

$$v(x_0, y_0) = 1. \tag{2.108}$$

If we take $v = 1$, the conditions (2.106), (2.107) and (2.108) are satisfied, then we have here

$$U = \frac{\partial u}{\partial x} \tag{2.109}$$

and

$$V = -\frac{\partial u}{\partial y}. \tag{2.110}$$

Since $L[u] = M[v] = 0$ then the equation (2.98) reduces to

$$\int_{\Gamma} \mathrm{n}.(U, V) d\sigma = 0.$$

Substituting (2.109) and (2.110) gives

$$\int_{\Gamma} \mathrm{n}. \left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y} \right) d\sigma = 0, \tag{2.111}$$

or

$$\left(\int_{RP} + \int_{PQ} + \int_{QR} \right) \left[\mathrm{n}. \left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y} \right) \right] d\sigma = 0. \tag{2.112}$$

Since the curve Γ is given parametrically by (2.103) then the unit normal $\mathbf{n}$ is given by

$$\mathrm{n} = \left(\frac{dy}{d\sigma}, -\frac{dx}{d\sigma} \right). \tag{2.113}$$

On RP, σ is an arc-length parameter on the characteristic (2.101), hence we have

$$\frac{dx}{d\sigma} = \frac{-1}{\sqrt{2}}, \quad \frac{dy}{d\sigma} = \frac{-1}{\sqrt{2}}. \tag{2.114}$$

Substituting these into (2.113) gives the unit normal on RP

$$\mathrm{n} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \tag{2.115}$$

On QR, σ is an arc-length parameter on the characteristic (2.102), hence we have

$$\frac{dx}{d\sigma} = \frac{-1}{\sqrt{2}}, \quad \frac{dy}{d\sigma} = \frac{1}{\sqrt{2}}. \tag{2.116}$$

Substituting these into (2.113) gives the unit normal on QR

$$\mathrm{n} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \tag{2.117}$$

On PQ, σ is an arc-length parameter on the initial curve (2.91), the parameter σ is replaced by τ. Hence the unit normal on PQ is

$$\mathrm{n} = \left(\frac{dy}{d\tau}, -\frac{dx}{d\tau} \right). \tag{2.118}$$

Using (2.115), integration along the characteristic RP gives

$$\begin{aligned}\int_{RP} \mathbf{n}.\left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\right) d\sigma &= \int_{RP} \left[-\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} - \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}\right] d\sigma \\ &= \int_R^P \left[\frac{\partial u}{\partial x}\frac{dx}{d\sigma} + \frac{\partial u}{\partial y}\frac{dy}{d\sigma}\right] d\sigma \\ &= \int_R^P \frac{du}{d\sigma} d\sigma \\ &= u(P) - u(R). \end{aligned} \tag{2.119}$$

Similarly, using (2.117), integration along the characteristic QR gives

$$\begin{aligned}\int_{QR} \mathbf{n}.\left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\right) d\sigma &= \int_{QR} \left[\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} - \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}\right] d\sigma \\ &= \int_Q^R \left[-\frac{\partial u}{\partial x}\frac{dx}{d\sigma} - \frac{\partial u}{\partial y}\frac{dy}{d\sigma}\right] d\sigma \\ &= -\int_Q^R \frac{du}{d\sigma} d\sigma \\ &= u(Q) - u(R). \end{aligned} \tag{2.120}$$

Since on PQ the parameter σ is replaced by τ, integration along PQ becomes

$$\begin{aligned}\int_{PQ} \mathbf{n}.\left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\right) d\sigma &= \int_{PQ} \left[\left(\frac{dy}{d\tau}, -\frac{dx}{d\tau}\right)\left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\right)\right] d\tau \\ &= \int_{PQ} \left(\frac{\partial u}{\partial x}\frac{dy}{d\tau} + \frac{\partial u}{\partial y}\frac{dx}{d\tau}\right) d\tau \\ &= \int_{PQ} \left(\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right) \end{aligned} \tag{2.121}$$

Substituting (2.119), (2.120) and (2.121) into (2.112) results

$$u(R) = \frac{1}{2}(u(P) + u(Q)) - \frac{1}{2}\int_{PQ} \left(\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right). \tag{2.122}$$

It is the solution of *the Cauchy problem* for the wave equation (2.90) satisfies initial conditions (2.92) along the initial curve (2.91).

For instance if the given curve (2.91) is the x-axes, i.e. $y = 0$ then the unit normal on PQ is $(0, 1)$, integration along interval PQ is

$$-\int_{PQ} \frac{\partial u}{\partial y} dx$$

such that (2.122) is reduced into

$$u(R) = \frac{1}{2}(u(P) + u(Q)) + \frac{1}{2}\int_{PQ} \frac{\partial u}{\partial y} dx. \tag{2.123}$$

It is the solution of *the Cauchy problem* for the wave equation (2.90) satisfies initial conditions (2.92) along the initial curve (2.91). For the point $R = (x_0, y_0)$, the characteristic curve (2.99) gives $P = (x_0 - y_0, 0)$ and the characteristic curve (2.100) gives $Q = (x_0 + y_0, 0)$. Hence we have

$$u(R) = \frac{1}{2}(u(x_0 - y_0, 0) + u(x_0 - y_0, 0)) + \frac{1}{2}\int_{x_0 - y_0}^{x_0 + y_0} \frac{\partial u}{\partial y} dx. \tag{2.124}$$

Since $y = 0$, the initial conditions (2.92) becomes

$$u(x, 0) = r(x), u_y(x, 0) = s(x). \tag{2.125}$$

The first initial condition gives

$$u(x_0 - y_0, 0) = r(x_0 - y_0), \quad u(x_0 - y_0, 0) = r(x_0 + y_0),$$

and the second gives

$$\int_{x_0 - y_0}^{x_0 + y_0} \frac{\partial u}{\partial y} dx = \int_{x_0 - y_0}^{x_0 + y_0} s(x) dx.$$

Substituting these into (2.122), gives

$$u(R) = \frac{1}{2}(r(x_0 - y_0) + r(x_0 + y_0)) + \frac{1}{2}\int_{x_0 - y_0}^{x_0 + y_0} s(x) dx \tag{2.126}$$

which is known as the D'Alembert solution (2.27) when $c = 1$.

2.8 Non-Homogeneous Wave Equations

We have mentioned in section 2.3 that by applying a driving force G, we have the non-homogeneous wave equations (2.10) or (2.11). In the case of (2.11) G is dependent on u and its derivatives. By transforming it into the canonical form (2.13), we may used the methods of successive approximation discussed in section 1.8.

Consider now the problem determining the solution of the non-homogeneous wave equation (2.10) defined on the finite interval $0 < x < L$. The problem can be described as follow: find the solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = G(x,y), \qquad -L < x < L, 0 < y < \infty \tag{2.127}$$

satisfying the initial conditions

$$u(x,0) = r(x), \quad u_y(x,0) = s(x), \quad 0 \leq x \leq L. \tag{2.128}$$

and the boundary conditions of the second kind

$$u(0,y) = t_1(y), \quad u(L,y) = t_2(y), \quad 0 < y < \infty. \tag{2.129}$$

Since the equation (2.127) involves the external driving force $G(x,y)$ then the method of separation variables cannot be used here. In the present section we are going to use the Cauchy Riemann's method, discussed in the previous section, to solve the problem.

From the previous sections we know that the characteristic curves of the wave equation (2.127) divide the domain into the sub-domains $I - IV$. See figure 2.15.

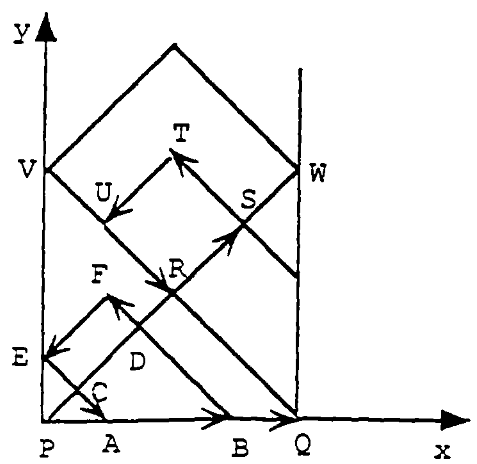


Figure 2.15 Sub-domains

Integrating the equation (2.127) over region I bounded by the triangular PQR, we have

$$\iint_{PQR} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} dx dy = \iint_{PQR} (G(\xi,\eta))\, d\xi d\eta. \tag{2.130}$$

From the previous section we know that by using the Cauchy Goursat method the left hand side integration in (2.130) is reduced to (2.112), the line integrations along initial line PQ, characteristic lines QR and RP. The solution is found to be (2.126). Hence the solution of the equation (2.127) at the sub-domain I is

$$u(x,y) = \frac{1}{2}(r(x-cy)+r(x+cy)) + \frac{1}{2}\int_{x-cy}^{x+cy} s(\xi)d\xi + \iint\limits_{\Sigma} (G(\xi,\eta))\, d\xi d\eta. \quad (2.131)$$

Furthermore we seek the solution of (2.127) at the point F lie on the sub-domain II. Integrating the equation (2.127) over the region bounded by rectangular $CDFE$ gives

$$\iint\limits_{CDFE} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} dxdy = \iint\limits_{CDFE} (G(\xi,\eta))\, d\xi d\eta. \quad (2.132)$$

The left hand side integration in (2.132) is reduced to the line integrations

$$\left(\int_{CD} + \int_{DF} + \int_{FE} + \int_{EC}\right)\left[\text{n.}\left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\right)\right] d\sigma = [u(D)-u(C)]$$
$$-[u(F)-u(D)] + [u(E)-u(F)] - [u(C)-u(E)]. \quad (2.133)$$

Hence the solution of the equation (2.127) at the point F is

$$u(F) = u(D) - u(C) + u(E) - \frac{1}{2}\iint\limits_{CDFE} (G(\xi,\eta))\, d\xi d\eta. \quad (2.134)$$

Furthermore integrating the equation (2.127) over the region bounded by rectangular $ABDC$ gives

$$u(D) - u(C) = \frac{1}{2}[u(C)+u(E)] + \frac{1}{2}\int_{AB} \frac{\partial u}{\partial y} dx - \frac{1}{2}\iint\limits_{ABDC} (G(\xi,\eta))\, d\xi d\eta. \quad (2.135)$$

Substituting this into (2.134) we get

$$u(F) = u(E)\frac{1}{2}[u(C)+u(E)] + \frac{1}{2}\int_{AB} \frac{\partial u}{\partial y} dx - \frac{1}{2}\iint\limits_{ABFE} (G(\xi,\eta))\, d\xi d\eta. \quad (2.136)$$

The equation (2.134) show us that at the sub-domain II the solution is dependent on the value of u on left boundary and the characteristic through CD. Hence the problem finding the solution of (2.127) in the sub-domain II is becomes problem finding the solution of (2.127) with the data are prescribed on the characteristic segment PR and the boundary segment PV which is other than a characteristic, such problem is called *Goursat problem,* see section 1.5 page 25. However the equation (2.135) tell us that the required values along CD can be replaced by the initial data along the along AB. The solution at the sub-domain III may be calculated by using a similar procedure such as for the sub-domain II.

Finally we seek the solution of (2.127) at the point T lie on the sub-domain IV. Integrating the equation (2.127) over the region bounded by rectangular $RSTU$ gives

$$\iint\limits_{RSTU} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} dx dy = \iint\limits_{RSTU} (G(\xi,\eta))\, d\xi d\eta. \tag{2.137}$$

The left hand side integration in (2.132) is reduced to the line integrations

$$\begin{aligned}\left(\int_{RS} + \int_{ST} + \int_{TU} + \int_{UR}\right) \left[\mathrm{n}.\left(\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\right)\right] d\sigma &= [u(S) - u(R)] \\ -[u(T) - u(S)] + [u(U) - u(T)] - [u(R) - u(U)]. \end{aligned} \tag{2.138}$$

Hence the solution of the equation (2.127) at the point T is

$$u(T) = u(S) - u(R) + u(U) - \frac{1}{2} \iint\limits_{RSTU} (G(\xi,\eta))\, d\xi d\eta. \tag{2.139}$$

This solution is dependent on the values of u at the points S, R and U which are lie on the two characteristic through R. Hence problem finding the solution (2.127) at the sub-domain IV is problem finding the solution (2.127) with data are given along the two characteristic through R. Such problem is called *characteristic Goursat problem*, see section 1.5 page 25.

2.9 Discussions and Conclusions

The D'Alembert solution of the one dimensional wave equation for an unbounded domain appear to be *travelling waves.* For the semi-infinite string, the motion of wave is influenced by the boundary and the D'Alembert solution must be modified. Further the motion of a finite string is composed of two waves continually travelling along it in opposite directions, the whole displacement being the resultant of the two waves and their reflections at the end points. The Method of Separation Variable gives a solution in term of a series of standing waves. Finally, the Riemann-Volterra solution, produces an explicit solution in which it is dependent on the prescribed boundary values. When initial conditions are given along x-axis, the Riemann-Volterra solution is reduced to the D'Alembert solution. When we deal with a boundary value problem, the domain is divided by the characteristic lines into four sub-domain. In the first sub-domain, below the characteristic lines, the problem is reduce into Cauchy problem. In the second and third sub-domains, in the left and right characteristic lines respectively, the problems becomes the Goursat problem. In the fourth sub-domain, above the characteristic lines, the problems becomes the characteristic Goursat problem.

Chapter 3

METHOD OF CHARACTERISTICS

3.1 Introduction

In the previous chapter we found that by an appropriate transformation a linear second order differential equation is reduced to one of three canonical forms classified as *parabolic*, *elliptic* and *hyperbolic*. In the case of *hyperbolic* equations, the discriminant of the principal part of the equation is positive and the *two characteristic directions* are real and distinct at all points in the domain of interest. The characteristics are independent of the solution u.

For the sake of variety, we generalize the concept of characteristics for a *quasi-linear equation*

$$a(x,y,u,u_x,u_y)u_{xx} + 2b(x,y,u,u_x,u_y)u_{xy} + c(x,y,u,u_x,u_y)u_{yy} = g(x,y,u,u_x,u_y). \tag{3.1}$$

From section 1.2, we know that the classification of the quasi linear equation (3.1) depends on the sign of the discriminant of the principal part of the equation, that is

$$\Delta(x,y,u,u_x,u_y) = b^2 - ac \tag{3.2}$$

However since the coefficient functions a, b, c involve x, y, u, u_x and u_y, then the discriminant depends on u as well as x and y. Hence, the type of equation depends on a particular solution considered, in a given problem.

Our first objective is to formulate a Cauchy problem to a quasi linear (3.1) equation which will be analogous to the one already discussed in chapter 1. That is the Cauchy problem reduces to finding the condition under which the prescribed values are sufficient to determine the solution along the characteristic curves. However since the coefficient functions of the second derivatives contain the dependent variable and its first derivatives, then the equation is non linear in the dependent variable as well as non linear in the first derivative of the dependent variable. In addition the equation cannot be transformed into one of the three canonical forms, such that the methods such as the Riemann's methods, discussed in section 1.8, is not applicable.

The *method of characteristics* involves expressing the partial differential equation in terms of its characteristic coordinates and integrating along the characteristic directions. The quasi linear hyperbolic equation will reduce to the *characteristic system.* The system involves non-linear ordinary differential equations which should be calculated simultaneously so that to solve the system we will need to discretize it. Discretization involves approximating the two families of the characteristics curves by characteristic grids and replacing the differential equations with appropriate finite difference equations.

In the last three sections we are going to explore some methods to approximate the characteristics curves, and then find the solutions of the partial differential equation at these characteristic grid points.

First of all, based on the Taylor's series of order one, the *straight line method* will be derived in section 3.4. This method leads to a discretization error $O(h)$. Geometrically the method approximates the characteristic curves by straight lines.

Secondly based on Taylor's series of order two, the *predictor-corrector methods* will be derived in section 3.5. Geometrically the method approximates the characteristic curves by parabolic arcs. The use of parabolic arcs give us a more accurate approximation, with discretization error $O(h^2)$.

Increasing the order of the Taylor's series may give a better approximation, however since we deal Cauchy problem of a second order of quasi linear equation, we have to avoid the evaluation derivatives of order higher then two. Hence we seek other method to improve the approximate values.

Since the characteristic curves are governed by first order differential equations, then we may integrate directly along the characteristic curves. Basically numerical procedures for solving differential equations can be used to find the characteristic grids. However since we only have one set of prescribed values as information, we have to use one step methods, such as the *Runge Kutta* method. However we need an interval integration. The approximate values found by the straight line method will be used to do so.

In section 3.6. we propose two alternative methods to improve the approximate values. We will use the approximate values calculated by the straight line approximations as the *initial guess.* Then the Runge Kutta method will be used to approximate the value on the characteristic curves. An improved approximate value will be found as the intersection point of the approximate curves. However if the approximate curves do not intersect, we may approximate the intersection point. In this section we approximate the intersection point by say, the method A and the method B. The method A approximates the intersection point by the intersection point of the arcs of approximate curves, while the method B approximates the intersection point by the intersection point of the tangents at the end point of the approximate curves. We will explore how better approximations can be achieved and compare the results.

3.2 Characteristics system

Suppose we are given a Cauchy data set along the initial curve C which is given by

$$x = x(\tau) \quad y = y(\tau), \qquad -\infty < \tau < \infty. \tag{3.3}$$

Suppose that the point $P(x,y)$ lies on the given curve C, then for some $\Delta x, \Delta y$, such that $(x+\Delta x, y+\Delta y) \in D$, the Taylor's series expansion gives

$$\begin{aligned} u(x+\Delta x, y+\Delta y) &= u(x,y) + u_x\Delta x + u_y\Delta y + \frac{1}{2}(u_{xx}(\Delta x)^2 \\ &\quad +2u_{xy}\Delta x\Delta y + u_{yy}(\Delta y)^2) + O[(\Delta x)^3, (\Delta x)^3], \quad (3.4) \end{aligned}$$

where $O[(\Delta x)^3, (\Delta x)^3]$ denotes higher order derivative terms.

By differentiating equation (3.1), all third and higher order derivatives at the point P can be found in terms of $u, u_x, u_y, u_{xx}, u_{xy}$ and u_{yy}, hence we need only find u_{xx}, u_{xy} and u_{yy}. However knowing the values of u, u_x and u_y is *not* sufficient to determine all the second derivatives u_{xx}, u_{xy} and u_{yy}. Therefore the problem reduces to finding the condition under which the known values of u, u_x, and u_y are sufficient for the determination of the unique values of u_{xx}, u_{xy} and u_{yy} along the given curve C that satisfy the equation (3.1).

Along the given curve C where u_x and u_y are prescribed, we know the values of differentials

$$\dot{u_x} = u_{xx}\,\dot{x} + u_{xy}\,\dot{y} \quad (3.5)$$

and

$$\dot{u_y} = u_{xy}\,\dot{x} + u_{yy}\,\dot{y}\,. \quad (3.6)$$

These rates of change can be considered as known quantities that are related to the values of u_{xx}, u_{xy} and u_{yy} along the given curve C, while $\dot{x}$ and $\dot{y}$ denoting $\frac{dx}{d\tau}$, and $\frac{dy}{d\tau}$ respectively.

Equations (3.1), (3.5) and (3.6) form a system of linear equations, in u_{xx}, u_{xy} and u_{yy}, which has the matrix form

$$\mathbf{A}\mathrm{x} = \mathrm{b} \quad (3.7)$$

where $\mathbf{A} = \begin{pmatrix} a & 2b & c \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{pmatrix}$, $\mathrm{x} = \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix}$, and $\mathrm{b} = \begin{pmatrix} g \\ \dot{u_x} \\ \dot{u_y} \end{pmatrix}$.

Using *Cramer's rule*, we have

$$\frac{u_{xx}}{\det(\mathbf{A_1})} = \frac{u_{xy}}{\det(\mathbf{A_2})} = \frac{u_{yy}}{\det(\mathbf{A_3})} = \frac{1}{\det(\mathbf{A})} \quad (3.8)$$

where $det(\mathbf{A}_i), i = 1(1)3$, are determinants of the matrixes $\mathbf{A}$ with its *ith* column replaced by $\mathbf{b}$. From these relations, the system of equations (3.7) will have a unique solution unless the determinant of matrix $\mathbf{A}$ vanishes.

When $det(\mathbf{A}) = \mathbf{0}$, the values of u_{xx}, u_{xy} and u_{yy} will usually be infinite. As a result the prescribed values, u, u_x and u_y, will not satisfy the partial differential equation. However if the compatibility conditions $det(\mathbf{A}_i) = \mathbf{0}$,$\forall i = 1(1)3$, are satisfied then u_{xx}, u_{xy} and u_{yy} can be finite and satisfy the partial differential equation (3.1) [10] . Further if any two $A_i = 0$, the determinants are zero.

Suppose that C is a curve on which $det(\mathbf{A}) = 0$, that is

$$a\left(\dot{y}\right)^2 - 2b\dot{y}\dot{x} + c\left(\dot{x}\right)^2 = 0.$$

Dividing by $\dot{x}$, we have the quadratic equation

$$a\left(\frac{\dot{y}}{\dot{x}}\right)^2 - 2b\frac{\dot{y}}{\dot{x}} + c = 0,$$

which is equivalent to the equation

$$a(\frac{dy}{dx})^2 - 2b(\frac{dy}{dx}) + c = 0. \tag{3.9}$$

Suppose that the curve C is chosen such that the slope of the tangent at every point on it are the roots of the equations (3.9). Then denoting the roots by α and β , we have

$$\frac{dy}{dx} = \alpha = \frac{1}{a}(b + \sqrt{b^2 - ac}) \tag{3.10}$$

and

$$\frac{dy}{dx} = \beta = \frac{1}{a}(b - \sqrt{b^2 - ac}), \tag{3.11}$$

which are equivalent to the equations (1.26) and (1.27) in section 1.2 when the coefficient functions a, b and c are independent of u, u_x and u_y. These equations give the directions of the characteristic curves at any point on C for the given values u,u_x, and u_y. A curve which at each of its points has the direction α is said to be an α *characteristic* curve, and a curve which at each of its points has the direction β is said to be a β *characteristic* curve. Then the solution of (3.10) and (3.11) define two families of characteristic curves of the quasi linear equation (3.1).

According to the values of a, b and c at that point, there are two *real characteristic directions* if the given partial differential is *hyperbolic*, one real characteristic direction if it is *parabolic*, and no real characteristic direction if it is *elliptic*. Notice that since a, b, and c are functions of x, y, u, u_x and u_y, then the type of the partial differential equation (3.1) may depend on the region in which the solution is to be found.

Furthermore from the equation (3.8) , we know that if $det(\mathbf{A}) = 0$ and $\exists k, \;\; k \in \{1, 2, 3\}$ such that $det(\mathbf{A_k}) = 0$ then the rest determinants are vanish. Suppose for example that

$$det(\mathbf{A_2}) = 0,$$

that is

$$a\,\dot{u}_x\,\dot{y} + c\,\dot{u}_y\,\dot{x} - g\,\dot{x}\,\dot{y} = 0.$$

Dividing by $\dot{x}$ we have

$$a\frac{\dot{y}}{\dot{x}}\,\dot{u}_x + c\,\dot{u}_y\,yg\,\dot{y} = 0,$$

which is equivalent to

$$a\frac{dy}{dx}d(u_x) + cd(u_y) - gdy = 0. \tag{3.12}$$

Assume the quasilinear (3.1) is hyperbolic throughout the domain; by substituting the characteristic directions from (3.10) and (3.11) into the equation (3.12), we see that the condition that should be satisfied along the α characteristic is,

$$a\alpha d(u_x) + cd(u_y) - gdy = 0, \tag{3.13}$$

and along the β characteristic is,

$$a\beta d(u_x) + cd(u_y) - gdy = 0. \tag{3.14}$$

These last two equations may be regarded as the representations of the partial differential equation in the characteristic directions.

When u does not appear explicitly as an argument of the coefficient functions a, b, c and d, then the system of equations (3.10), (3.11), (3.13) and (3.14) is sufficient to determine the solution of the Cauchy problem of (3.1). If a, b, c or d does depend on u, we can use the relation

$$du = u_x dx + u_y dy \tag{3.15}$$

to determine the solution.

Hence the system of equations (3.10), (3.11), (3.13), (3.14) and (3.15) provides five equations that govern the way in which the unknown function u and its first partial derivatives change with the independent variables x and y. The systm is called the *characteristic system.* Hence when the *quasi-linear equation* (3.1) remains hyperbolic throughout the domain, the Cauchy problem for it reduces to solving the characteristic system.

Suppose the given curve is other than the characteristic curves, then we have two families of characteristic curves over the domain D. From the discussion in section 1.4 we know that since the Cauchy data are propagated along the characteristic curves, then the solution of the Cauchy problem should be calculated at the intersection point of the characteristic curves. The intersection point are called *characteristic grid points.* The characteristic grid points are obtained by solving the ordinary differential equations (3.10) and (3.11). Furthermore the values of the first derivatives of u are calculated by (3.13) and (3.14). Finally the solution at each characteristic grid point is calculated by (3.15). However since a, b, c, and g are functions of x, y, u, u_x, and u_y then the equations (3.10) and (3.11) should be solved simultaneously with (3.13), (3.14) and (3.15). We will require *numerical processes* to do so.

3.3 Discretizations

To discretize the problem, let us consider the segment of the initial curves C

$$x = x(\tau_0) \quad y = y(\tau_0). \tag{3.16}$$

The subscript 0 on the parameter τ_0 indicates that the initial curve C is chosen to be the zero time level curve where the initial conditions are prescribed. Dividing the interval into a finite number of, say n, equally spaced sub intervals with the end points

$$\tau_{0_0} \leq \tau_{0_1} \leq \tau_{0_2} \leq ...... \leq \tau_{0_{n-1}} \leq \tau_{0_n},$$

the corresponding points on C are

$$(x_{0,0}, y_{0,0}), (x_{1,0}, y_{1,0}), \dots, (x_{n-1,0}, y_{n-1,0}), (x_{n,0}, y_{n,0})$$

where $(x_{i,0}, y_{i,0}) = (x(\tau_{0_i}), y(\tau_{0_i}))$ for $i = 0(1)n$.

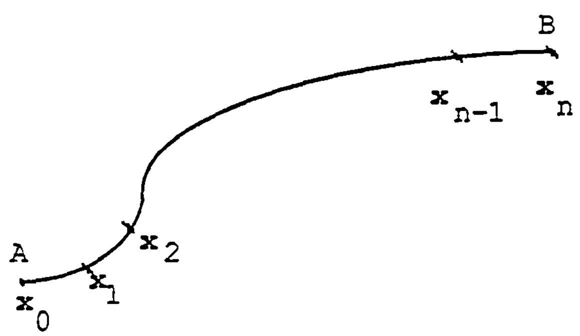


Figure 3.1:Initial curve

Using the given initial conditions we are able to calculate a set of values involving the exact values of the coordinate points, the solution, the first derivatives of the solution and the characteristic directions at all points in the initial curve. These values will be recognized as the given values.

Consider $P = (x_{0,0}, y_{0,0})$ and $Q = (x_{1,0}, y_{1,0})$ the first two adjacent points on the given initial curve (3.16) which is assumed to be not a characteristic curve. The $0's$ in the second subscripts of the coordinates points P an Q indicate that these points lie on the initial curve. Suppose the α characteristic curve through the point P and the β characteristic curve through the point Q intersect in an advance point

$R^* \in D$, the domain of the equation (3.1). Using the initial values at the points P and Q, we are able to calculate the solution at the point R^*. However since the characteristic curves are governed by ordinary differential equations (3.10) and (3.11) which are non linear in u and its first derivatives then we are going to approximate the intersection point R^* by $R = (x_{1,1}, y_{1,1})$. The $1's$ in the second subscripts of the coordinates of the point R indicates that the point R is located in the time level 1. In the next three sections we are going to approximate the characteristic curves (3.10) and (3.11). The process will result the point R. Some methods involve iterative procedures, so that the R will be denoted by R^n for non negative integer n.

Furthermore solving the difference representations of the equations (3.13) and (3.14) will give approximate values of the first derivatives of u at the point R. Finally the solution at the point R can be calculated by the difference representation of the equation (3.15).

This process is repeated for each adjacent pair of points on the initial curve C to obtain a set of grid points in the 1^{st} time level and the solution at these points. In addition, using the calculated values at grid points in the time level 1 we are able to calculate the solution at the grid points in the time level 2. These processes are repeated for further time levels.

Suppose that we know the solution at the points $(x_{i,j}, y_{i,j})$ and $(x_{i+1,j}, y_{i+1,j})$ at the time level j^{th}. The advance point $(x_{i+1,j+1}, y_{i+1,j+1})$ at the time level $(j+1)^{th}$ is found by means of the intersection of the intersection point of the α characteristic through the point $(x_{i,j}, y_{i,j})$ and the β characteristic through the point $(x_{i,j}, y_{i,j})$ which are governed by the equations (3.10) and (3.11) respectively. See figure 3.2.a.

When we deal with a boundary value problem the points at the boundaries are found by means of intersection of the characteristic curve and the boundaries. Using (3.10), the coordinates of the points $(x_{0,j+1} y_{0,j+1})$, at the left boundary, are found to be

$$x_{0,j+1} = x_{0,0}, \qquad y_{0,j+1} = y_{0,j} + \beta_{1,j}(x_{1,j} - x_{0,j}) \tag{3.17}$$

where $\beta_{1,j}$ is the β characteristic direction at the point $(x_{1,j}y_{1,j})$, see figure 3.2.b. Similarly using the equation (3.11), the coordinates of the points $(x_{n,j+1}y_{n,j+1})$, at the right boundary, are found to be

$$x_{n,j+1} = x_{n,0}, \qquad y_{n,j+1} = y_{n,j} + \alpha_{n-1,j}(x_{n,j} - x_{n-1,j}) \tag{3.18}$$

where $\alpha_{n-1,j}$ is the α characteristic direction at the point $(x_{n-1,j}y_{n-1,j})$. (See figure 3.2.c.)

Furthermore the solution and its first derivatives along the boundaries are calculated by using one of the three appropriate boundary conditions mentioned in section 2.5. In general, the discretization of boundary conditions may be written as

$$\lambda U_x(x_0, y_{j+1}) + \delta_1(j)U(x_0, yj+1) = t_1(j) \tag{3.19}$$

along $x = x_0$, and

$$\lambda u_x(x_n, y) + \delta_2(j)u(x_n, y) = t_2(j) \tag{3.20}$$

along $x = x_n$, where $y \geq 0$, and λ is a constant. $\lambda = 0$ gives a set of boundary conditions of the first kind (2.48). and $\delta_1(j) = \delta_2(j) = 0$, $\forall j \geq 0$ gives a set of boundary conditions of the second kind (2.49), while $\lambda = 1$ is associated with a set of boundary conditions of the third kind (2.47).

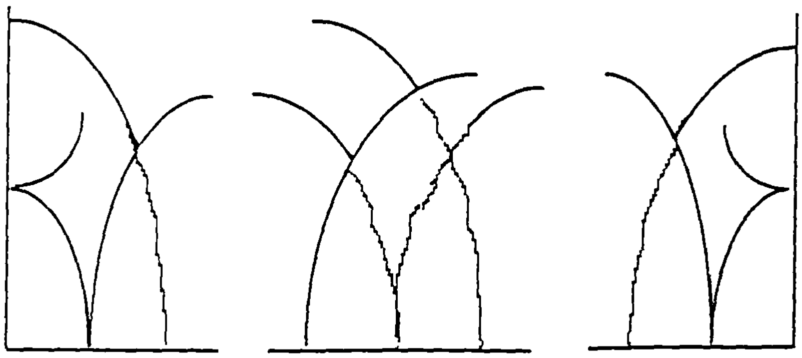


Figure 3.2.a Figure 3.2.b Figure 3.2.c

Since the characteristic curves from the same family do not intersect, we have characteristic grids over the domain D. (See Figure 3.3.)

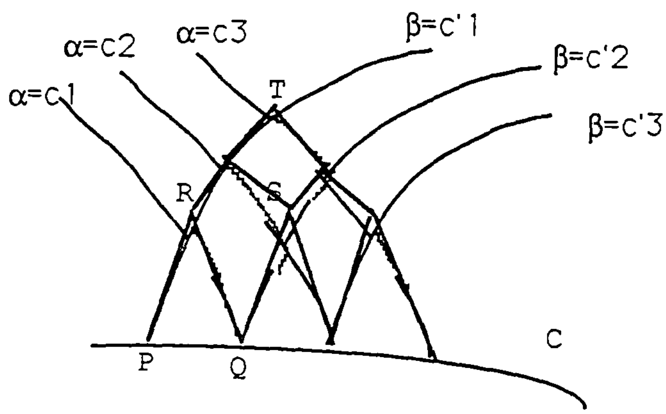


Fig.3.3: Characteristic grids

We deal with quasi linear equation ((3.1) on which the coefficient function a, b, c, and g are functions of x, y, u, u_x, and u_y, and hence the system of equations (3.10), (3.11). (3.13), (3.14) and (3.15) involve non linear equations and should be solved simultaneously. We will require *numerical processes* to do so. By substituting appropriate difference equations into the differential equations in the system of equations , we are going to derive some numerical procedures to solve the system.

3.4 Straight Line Approximations

Consider P and Q two adjacent points on the initial curve C, which is other than the characteristic curves. Suppose the α characteristic curve through P and the β characteristic curve through Q intersect at the point R^*. We are going to approximate the characteristic curves by straight lines, such that the point R^* will be approximated by R, the intersection point of the straight lines.

The *Taylor's series* expansion for $y(x_0 + \Delta x)$ about $x = x_0$ gives

$$y(x_0 + \Delta x) = y(x_0) + \Delta x \left.\frac{dy}{dx}\right|_{x=x_0} + O[\Delta x^2].$$

Corresponding to the Taylor's series above, set x_0 to be $x(P)$, the x-coordinate of the point P, and $x_0 + \Delta x = x(R^*)$, the x-coordinate of the point R^*. If the point P is close to the point Q , that the length of the arc PQ is small, we may expect that Δx small then the Taylor's series expansion for $y(R^*)$ about $x = x(P)$ gives

$$y(R^*) = y(P) + \Delta x \left.\frac{dy}{dx}\right|_{x=x(P)} + O[\Delta x^2],$$

which leads to

$$\left.\frac{dy}{dx}\right|_{x=x(P)} = \frac{y(R^*) - y(P)}{x(R^*) - x(P)} + O[\Delta x]. \tag{3.21}$$

This can be approximated by

$$\left.\frac{dy}{dx}\right|_{x=x(P)} = \frac{y(R) - y(P)}{x(R) - x(P)},$$

where R is the approximate value of the intersection point R^*. Hence (3.10), the direction of the α characteristic curve through the point P, can be approximated by

$$\frac{y(R) - y(P)}{x(R) - x(P)} = \alpha(P), \tag{3.22}$$

where the value of $\alpha(P)$ has been calculated,

$$\alpha(P) = \frac{1}{a}\left(b(P) + \sqrt{b^2(P) - a(P)c(P)}\right). \tag{3.23}$$

Similarly setting x_0 to be $x(Q)$, that is the x-coordinate of the point Q, and setting $x_0+\Delta x$ to be $x(R^*)$, the Taylor's series expansion for $y(R^*)$ about $x = x(Q)$ gives

$$\left.\frac{dy}{dx}\right|_{x=x(Q)} = \frac{y(R^*) - y(Q)}{x(R^*) - x(Q)} + O[\Delta x]. \tag{3.24}$$

Hence the approximate value to the direction of the β characteristic curve through the point Q is given by

$$\frac{y(R) - y(Q)}{x(R) - x(Q)} = \beta(Q), \tag{3.25}$$

where the value of $\beta(Q)$ has been calculated,

$$\beta(Q) = \frac{1}{a}\left(b(Q) - \sqrt{b^2(Q) - a(Q)c(Q)}\right). \tag{3.26}$$

Obviously, from (3.21) and (3.24) we know that the approximates values (3.22) and (3.25) have the discretization error $O[\Delta x]$.

Since we know the values of $x(P), y(P), x(Q), y(Q), \alpha(P)$ and $\beta(Q)$, then solving the equations (3.22) and (3.25) will give the coordinate point of R. Eliminating $y(R)$ from (3.22) and (3.25) gives

$$x(R) = \frac{y(Q) - y(P) + \alpha(P)x(P) - \beta(Q)x(Q)}{\alpha(P) - \beta(P)}. \tag{3.27}$$

Substituting (3.27) into the equation (3.22) gives

$$y(R) = y(P) + \alpha(P)(x(R) - x(P)), \tag{3.28}$$

and substituting (3.27) into the equation (3.25) gives

$$y(R) = y(Q) + \beta(Q)(x(R) - x(Q)). \tag{3.29}$$

Hence we have R, the approximate value to the point R^* with the discretization error $O[\Delta x]$.

Geometrically, the characteristic curves are approximated by their tangents. Hence intersection point of the curves is approximated by the intersection point of the tangents. See figure 3.4.

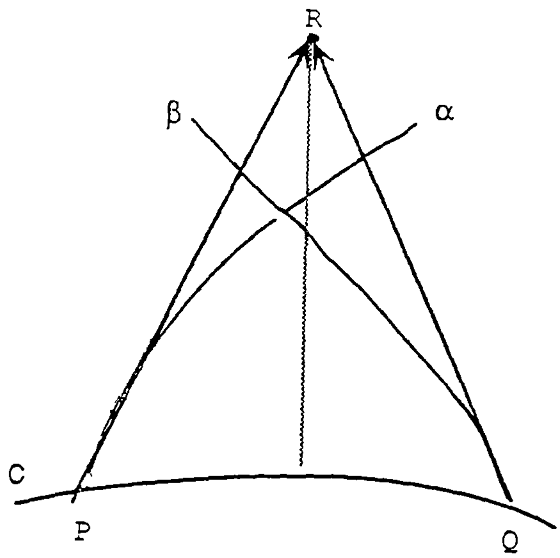


Figure 3.4:The straight line method

Furthermore by substituting finite differences for the differentials appearing in the differential equations (3.13) and (3.14) we have

$$a(P)\alpha(P)(U_x(R)-U_x(P))+c(P)(U_y(R)-U_y(P))-g(P)(y(R)-y(P))=0 \quad (3.30)$$

along the *characteristic segment* PR, and

$$a(Q)\beta(Q)(U_x(R)-U_x(Q))+c(Q)(U_y(R)-U_y(Q))-g(Q)(y(R)-y(Q))=0 \quad (3.31)$$

along the *characteristic segment* QR. Eliminating $U_y(R)$ from these equations give an approximate value of u_x,

$$\begin{aligned} U_x(R) \;=\; & [[a(P)c(Q)\alpha(P)U_x(P)-a(Q)c(P)\beta(Q)U_x(Q) \\ & +c(P)c(Q)[U_y(P)-U_y(Q)]+[c(Q)g(P)-c(P)g(Q)]y(R) \\ & +c(Q)g(P)y(P)-c(P)g(Q)y(Q)]] \\ & /[a(P)c(Q)\alpha(P)-a(Q)c(P)\beta(Q)]. \qquad (3.32) \end{aligned}$$

By substituting (3.32) into equation (3.30) give the approximate value of u_y,

$$U_y(R)=U_y(P)-\frac{1}{c(P)}[a(P)\alpha(P)(U_x(R)-U_x(P))-g(P)(y(R)-y(P))], \quad (3.33)$$

and substituting (3.32) into equation (3.31) we have

$$U_y(R)=U_y(Q)-\frac{1}{c(Q)}[a(Q)\beta(Q)(U_x(R)-U_x(Q))-g(P)(y(R)-y(Q))]. \quad (3.34)$$

In addition, by writing the total differential in a difference form and replacing the first partial differentials u_x and u_y by their average values, then along the characteristic segment PR, we may replace the equation (3.15) by

$$\begin{aligned} U(R)-U(P) &= \frac{1}{2}(U_x(R)+U_x(P))(x(R)-x(P)) \\ &+\frac{1}{2}(U_y(R)+U_y(P))(y(R)-y(P)). \end{aligned} \tag{3.35}$$

Similarly along the characteristic segment QR, the equation (3.15) may be replaced by

$$\begin{aligned} U(R)-U(Q) &= \frac{1}{2}(U_x(R)+U_x(Q))(x(R)-x(Q)) \\ &+\frac{1}{2}(U_y(R)+U_y(Q))(y(R)-y(Q)). \end{aligned} \tag{3.36}$$

Hence the first approximate value of U at the point R is given by

$$\begin{aligned} U(R) &= U(P)+\frac{1}{2}(U_x(R)+U_x(P))(x(R)-x(P)) \\ &+\frac{1}{2}(U_y(R)+U_y(P))(y(R)-y(P)), \end{aligned} \tag{3.37}$$

or by

$$\begin{aligned} U(R) &= U(Q)+\frac{1}{2}(U_x(R)+U_x(Q))(x(R)-x(Q)) \\ &+\frac{1}{2}(U_y(R)+U_y(Q))(y(R)-y(Q)). \end{aligned} \tag{3.38}$$

To construct a characteristic grid net and the solution of the Cauchy and boundary-value problems for the partial differential equation (3.1) we may use the procedures in the algorithms A.3.1 and A.3.2, respectively. The algorithms are given in Appendix .

As particular illustrations, consider Cauchy problem below.

Example 3.1.

Find the solution of

$$u_{xx}-(1-2x)u_{xy}+(x^2-x-2)u_{yy}=0 \tag{3.39}$$

satisfying

$$u(x,0)=x \qquad u_y(x,0)=0. \tag{3.40}$$

The straight line method produce the characteristic grids and the solutions such as shown in figure 3.5.

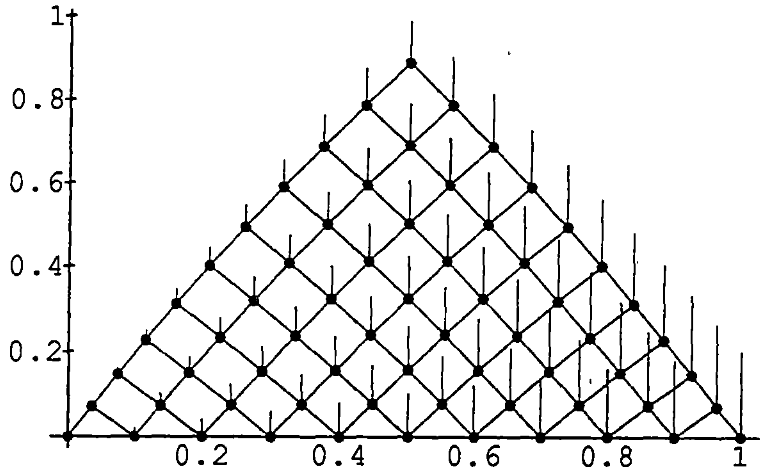


Figure 3.5 :The solution by straight line method
The vertical lines indicate the solutions at each intersection points of the characteriostc grids. They are $\frac{1}{20}$ of the actual size.

3.5 Predictor-Corrector Computations

The previous method, the *straight line method*, gives us the approximate values of x, y, U, U_x and U_y at the point R. Knowing these approximate values allows us to calculate the approximate values of a, b, c, and g at this point. Moreover we are able to calculate first approximate values of the α characteristic direction and β characteristic direction at the point R. These approximate values with discretization error $O(h)$ may be too rough because we have replaced the characteristic curves from the points P and Q by their tangents. Hence the point R found to be the intersection of the tangents whereas the 'true' point ought to be R^* the point of intersection between the characteristic curves. In addition, finite increments have been substituted throughout for the differentials.

In the present method we use *parabolic arcs* instead of tangents. The use of parabolic arcs gives us a better approximation with discretization error $O(h^2)$;

hence the approximate values are improved. In order to use the calculated values as a predicted values in the following calculation, the parabolic arc PR is chosen to be the *average value* of the tangent at the given points P and the tangent at the calculated point R. Similarly the parabolic arc QR is chosen to be the average value of the tangent at the given point Q and the tangent at the calculated point R. The intersection of these arcs gives us corrected values.

The Taylor's series expansion for $y(x+\Delta x)$ about $x = x_0$ gives

$$y(x_0+\Delta x) = y(x_0) + \Delta x \left.\frac{dy}{dx}\right|_{x=x_0} + (\Delta x)^2 \left.\frac{d^2y}{dx^2}\right|_{x=x_0} + O[\Delta x^3].$$

Set x_0 to be $x(P)$, the x-coordinate of the point P, and $x_0 + \Delta x = x(R^*)$, the x-coordinate of the intersection point R^*, the Taylor's series expansion for $y(R^*)$ about $x = x(P)$ gives

$$y(R^*) = y(P) + \Delta x \left.\frac{dy}{dx}\right|_{x=x(P)} + (\Delta x)^2 \left.\frac{d^2y}{dx^2}\right|_{x=x(P)} + O[\Delta x^3]. \tag{3.41}$$

On the other hand, the Taylor's series expansion for $y(x-\Delta x)$ about $x = x_0$ gives

$$y(x_0-\Delta x) = y(x_0) - \Delta x \left.\frac{dy}{dx}\right|_{x=x_0} + (\Delta x)^2 \left.\frac{d^2y}{dx^2}\right|_{x=x_0} - O[\Delta x^3].$$

Setting x_0 to be $x(R^*)$, and $x_0 + \Delta x = x(P)$, the Taylor's series expansion for $y(P)$ about $x = x(R^*)$ gives

$$y(P) = y(R^*) - \Delta x \left.\frac{dy}{dx}\right|_{x=x(R^*)} + \frac{(\Delta x)^2}{2} \left.\frac{d^2y}{dx^2}\right|_{x=x(R^*)} + O[\Delta x^3]. \tag{3.42}$$

By subtracting the equations(3.41) and (3.42) we have

$$y(R^*) - y(P) = \frac{\Delta x}{2}\left[\left.\frac{dy}{dx}\right|_{x=x(R^*)} + \left.\frac{dy}{dx}\right|_{x=x(P)}\right] + O[\Delta x^3].$$

Dividing it by Δx we obtain

$$\frac{y(R^*) - y(P)}{\Delta x} = \frac{1}{2}\left[\left.\frac{dy}{dx}\right|_{x=x(R^*)} + \left.\frac{dy}{dx}\right|_{x=x(P)}\right] + O[\Delta x^2].$$

Since $\Delta x = |x(R^*) - x(P)|$, we may write this as

$$\frac{y(R^*) - y(P)}{x(R^*) - x(P)} = \frac{1}{2}\left[\left.\frac{dy}{dx}\right|_{x=x(R^*)} + \left.\frac{dy}{dx}\right|_{x=x(P)}\right] + O[\Delta x^2]. \tag{3.43}$$

and its approximate value is

$$\frac{y(R)-y(P)}{x(R)-x(P)}=\frac{1}{2}\left[\left.\frac{dy}{dx}\right|_{x=x(R)}+\left.\frac{dy}{dx}\right|_{x=x(P)}\right]. \tag{3.44}$$

Denoted by

$$x(R^{(0)}), y(R^{(0)}), U(R^{(0)}), U_x(R^{(0)}), U_y(R^{(0)}),$$
$$a(R^{(0)}), b(R^{(0)}), c(R^{(0)}), e(R^{(0)}), \alpha(R^{(0)}), \beta(R^{(0)}),$$

the approximate values found by using the straight line method. The superscripts (0)'s indicate that these value will be used as the *first predicted values.* On the other hand, we use superscripts (1)'s for the *first corrected values.* Notice that the terms in the right hand side of (3.44) are the α characteristic directions at the predicted point R and the fixed point P respectively. Hence using the notation in equation (3.10), the equation (3.44) can be written as

$$\frac{y(R^{(1)})-y(P)}{x(R^{(1)})-x(P)}=\frac{1}{2}[\alpha(P)+\alpha(R^{(0)})]. \tag{3.45}$$

This equation approximates the direction of the α characteristic curve drawn from the fixed point P.

Similarly the direction of the β characteristic drawn from the fixed point Q can be approximated by

$$\frac{y(R^{(1)})-y(Q)}{x(R^{(1)})-x(Q)}=\frac{1}{2}[\beta(Q)+\beta(R^{(0)})]. \tag{3.46}$$

Since the values of $\alpha(P), \alpha(R^{(0)}), \beta(Q)$ and $\beta(R^{(0)})$ are known, then eliminating $y(R^{(1)})$ from the equations (3.45) and (3.46) gives

$$x(R^{(1)})=\frac{y(Q)-y(P)+\frac{1}{2}[\alpha(P)+\alpha(R^{(0)})]x(P)-\frac{1}{2}[\beta(Q)+\beta(R^{(0)})]x(Q)}{\frac{1}{2}[\alpha(P)+\alpha(R^{(0)})]-\frac{1}{2}[\beta(Q)+\beta(R^{(0)})]}. \tag{3.47}$$

Substituting (3.47) into the equation (3.45) gives

$$y(R^{(1)})=y(P)+\frac{1}{2}[\alpha(P)+\alpha(R^{(0)})][x(R^{(1)})-x(P)] \tag{3.48}$$

while substituting (3.47) into (3.46) gives

$$y(R^{(1)}) = y(Q) + \frac{1}{2}[\beta(Q) + \beta(R^{(0)})][x(R^{(1)}) - x(Q)]. \tag{3.49}$$

These equations gives the first corrected values to coordinates of the desired intersection point R^* with the discretization error $O[\Delta x^2]$.

Geometrically, a parabolic arc is drawn through the point P by its direction is chosen to be the average value of the direction of the α characteristic at the given point P and the direction of the α characteristic at the predicted point $R^{(0)}$. Another parabolic arc is drawn through the point Q in the direction of the β characteristic at the given point P and the direction of the β characteristic at the predicted point $R^{(0)}$. The corrected point $R^{(1)}$ is found to be the intersection point of these arcs.

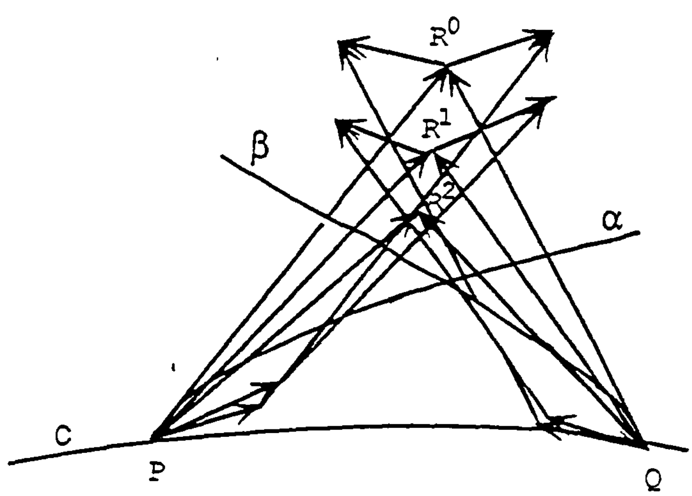


Figure 3.6: The predictor-corrector method

Substituting finite differences for the differentials appearing in the differential equations (3.13) and (3.14) we have

$$\frac{1}{2}[a(R^{(0)})\alpha(R^{(0)}) + a(P)\alpha(P)][U_x(R^{(1)}) - U_x(P)] + \frac{1}{2}[c(R^{(0)})$$
$$+c(P)][U_y(R^{(1)}) - U_y(P)] - \frac{1}{2}[g(R^{(0)}) + g(P)][y(R^{(1)}) - y(P)] = 0 \tag{3.50}$$

along the characteristic segment $PR^{(1)}$ and

$$\frac{1}{2}[a(R^{(0)})\beta(R^{(0)}) + a(Q)\beta(Q)][U_x(R^{(1)}) - U_x(Q)] + \frac{1}{2}[c(R^{(0)})$$
$$+c(Q)][U_y(R^{(1)}) - U_y(Q)] - \frac{1}{2}[g(R^{(0)}) + g(Q)][y(R^{(1)}) - y(Q)] = 0 \tag{3.51}$$

along the characteristic segment $QR^{(1)}$.

Hence the first corrected value of the first derivative u_x can be approximated by

$$\begin{aligned}
&U_x(R^{(1)}) = [[a(R^{(0)})\alpha(R^{(0)}) + a(P)\alpha(P)][c(R^{(0)}) + c(Q)]U_x(P)\\
&-[a(R^{(0)})\beta(R^{(0)}) + a(Q)\beta(Q)][c(R^{(0)}) + c(P)]U_x(Q)\\
&+[c(R^{(0)}) + c(P)][c(R^{(0)}) + c(Q)][U_y(P) - U_y(Q)]\\
&+[(c(R^{(0)}) + c(Q)][g(R^{(0)}) + g(P)] - [c(R^{(0)}) + c(P)][g(R^{(0)}) + g(Q)]y(R^{(1)})\\
&+[c(R^{(0)}) + c(Q)][g(R^{(0)}) + g(P)]y(P) - [c(R^{(0)}) + c(P)][g(R^{(0)}) + g(Q)]y(Q)]\\
&/[[a(R^{(0)})\alpha(R^{(0)}) + a(P)\alpha(P)][c(R^{(0)}) + c(Q)]\\
&-[a(R^{(0)})\beta(R^{(0)}) + a(Q)\beta(Q)][c(R^{(0)}) + c(P))],
\end{aligned} \tag{3.52}$$

while the first derivative u_y can be approximated by

$$\begin{aligned}
U_y(R^{(1)}) &= U_y(P) - \frac{1}{c(R^{(0)}) + c(P)}[[a(R^{(0)})\alpha(R^{(0)}) + a(P)\alpha(P)][U_x(R^{(1)})\\
&\quad -U_x(P)] - \frac{1}{2}[g(R^{(0)}) + g(P)][y(R^{(1)}) - y(P)],
\end{aligned} \tag{3.53}$$

or

$$\begin{aligned}
U_y(R^{(1)}) &= U_y(Q) - \frac{1}{(c(R^{(0)}) + c(Q))}[[a(R^{(0)})\beta(R^{(0)}) + a(Q)\beta(Q)][U_x(R^{(1)})\\
&\quad -U_x(Q)] - \frac{1}{2}[g(R^{(0)}) + g(P)][y(R^{(1)}) - y(Q)].
\end{aligned} \tag{3.54}$$

Finally using (3.15) the solution u at the point $R^{(1)}$ can be approximated by

$$\begin{aligned}
U(R^{(1)}) &= U(P) + \frac{1}{2}[U_x(R^{(1)}) + U_x(P)][x(R^{(1)}) - x(P)]\\
&\quad +\frac{1}{2}[U_y(R^{(1)}) + U_y(P)][y(R^{(1)}) - y(P)],
\end{aligned} \tag{3.55}$$

or by

$$
\begin{aligned}
U(R^{(1)}) &= U(Q)+\frac{1}{2}[U_x(R^{(1)})+U_x(Q)][x(R^{(1)})-x(Q)] \\
&\quad +\frac{1}{2}[U_y(R^{(1)})+U_y(Q)][y(R^{(1)})-y(Q)]. \qquad (3.56)
\end{aligned}
$$

We may improve the approximate values by means of iterations. Keeping the points P and Q fixed and using the corrected values in the previous iteration as the predicted values in the next iteration. However before we proceed to the next iteration we require updated approximate values of the given coefficient functions a, b, c, g and the characteristic directions. The iteration is terminated when two successive approximations agree with the same degree of accuracy.

To calculate the *2nd* corrected values for example; Substitute the calculated approximate values $x(R^{(1)}), y(R^{(1)}), U(R^{(1)}), U_x(R^{(1)}), U_y(R^{(1)})$ into the given coefficient functions

$$a(x,y,u,u_x,u_y), b(x,y,u,u_x,u_y), c(x,y,u,u_x,u_y), e(x,y,u,u_x,u_y).$$

Compute the α and β characteristic directions with

$$\alpha(R^{(1)})=\frac{1}{a(R^{(1)})}[b(R^{(1)})+\sqrt{b(R^{(1)})^2-a(R^{(1)})c(R^{(1)})}] \qquad (3.57)$$

and

$$\beta(R^{(1)})=\frac{1}{a(R^{(1)})}[b(R^{(1)})-\sqrt{b(R^{(1)})^2-a(R^{(1)})c(R^{(1)})}]. \qquad (3.58)$$

Let

$$
\begin{gathered}
x(R^{(1)}), y(R^{(1)}), U(R^{(1)}), U_x(R^{(1)}), U_y(R^{(1)}), \\
a(R^{(1)}), b(R^{(1)}), c(R^{(1)}), e(R^{(1)}), \alpha(R^{(1)}), \beta(R^{(1)})
\end{gathered}
$$

to be the predicted values for the 2nd iteration, and keep the values of the points P and Q fixed. Using the calculation procedure above we are able to calculate the 2nd corrected values. A complete procedure for solving Cauchy problems and boundary problem will be given below.

To solve the Cauchy problem of the equation (3.1) we have to construct the characteristic grid net and find the solution at these points. Similar to the straight line method we divide the initial curve into a finite number, say n, of equally spaced sub intervals. Using the given initial conditions we are able to calculate the exact values of the coordinates, the solution, the first derivatives of the solution and the characteristic directions at all points along the initial curve. These values will be recognized as prescribed values.

The procedures for solving the Cauchy and boundary-value problems for the partial differential equation (3.1), by using the predictor-corrector method , are given in the algorithms A.3.3 and A.3.4, in Appendix , respectively. The flowchart of the algorithms are depicted by figure A.3.1 in Appendix .

Using the predictor corrector method , the Cauchy problem in examples 3.1 has solution

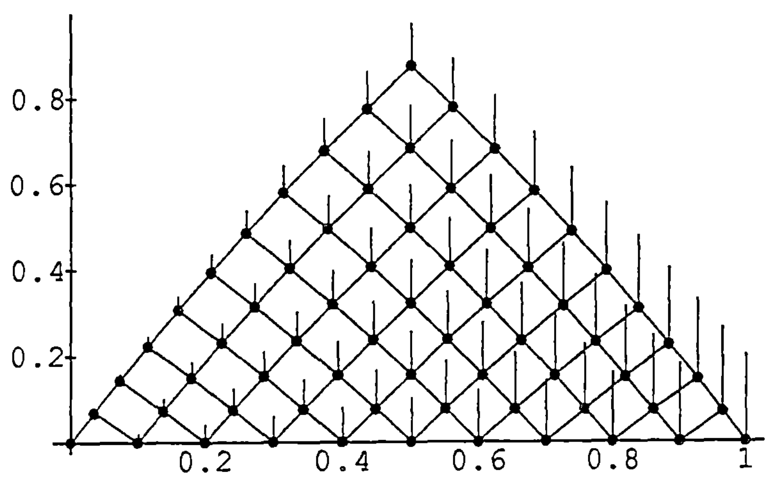


Figure 3.7 The solution by predictor-corrector method

3.6 Alternative methods

The straight line method discussed in the previous section uses *Taylor's series expansion of order one* and produces discretization errors of order one, while the predictor-corrector method uses *Taylor's series expansion of order two* produces discretization error of order two. We may improve the result by using higher order Taylor's series expansion. However we have to avoid the evaluation of derivatives of order higher that two.

Basically numerical procedures for solving differential equations can be used to find the characteristic grids. However since we only have one set of prescribed values as information, then we have to use one step methods, such as the *Runge Kutta* method.

In the present section, we use the *Runge Kutta method* to approximate the solution along the characteristic curves. The improved values will be found as the intersection of the approximate curve. In the next two subsections, we propose two methods to approximate the intersection point, namely the method A and the method B. The method A approximates the intersection point of the characteristic curves by intersection point of the arcs of approximate curves found by the Runge Kutta method. While the method B approximate the intersection point by intersection point of the tangents at the end point of the approximate curves.

Consider two adjacent points $P = (x_{0,0}, y_{0,0})$ and $Q = (x_{1,0}, y_{1,0})$ on the given curve C, which is assumed to be other than the characteristic curves. Denote by R^* the intersection point between the α characteristic through the point P and the β characteristic through the point Q. Suppose the intersection point R^* is approximated by $R = (x_{1,1}, y_{1,1})$. Using the straight line method we have already a first approximate values of R^*, say R^0. Hence at the point R^0 we have the values of

$$x(R^{(0)}), y(R^{(0)}), U(R^{(0)}), U_x(R^{(0)}), U_y(R^{(0)}),$$
$$a(R^{(0)}), b(R^{(0)}), c(R^{(0)}), e(R^{(0)}), \alpha(R^{(0)}), \beta(R^{(0)}).$$

These values are used as the first predictions.

Furthermore using the *Runge Kutta* method we integrate along the α characteristic curve (3.10) over the closed interval $x(P) \leq x \leq x(R^{(0)})$ to get y_α, the approximate value on the α characteristic through P. First of all divide the interval into finite number, say r_1, of equally spaced sub intervals, spacing h_1.

Taking $y_\alpha(x(P)) = y(P)$ as the initial value, generate the approximate value of the α characteristic at the end points of the sub intervals. Generate approximation y_{α_i} to $y_\alpha(x(P)+ih_1)$ for $i = 0,1,2,..r_1$ using the recursion formula

$$y_{\alpha_{i+1}} = y_{\alpha_i} + \frac{1}{6}(k_{12} + 2k_{12} + 2k_{13} + k_{14}) \tag{3.59}$$

where

$$\begin{aligned}
k_{11} &= h_1\alpha(x(P) + h_1 * i, y_{\alpha_i}) \\
k_{12} &= h_1\alpha(x(P) + h_1 * i + \frac{h_1}{2}, y_{\alpha_i} + \frac{k_{11}}{2}) \\
k_{13} &= h_1\alpha(x(P) + h_1 * i + \frac{h_1}{2}, y_{\alpha_i} + \frac{k_{12}}{2}) \\
k_{14} &= h_1\alpha(x(P) + h_1 * i + h_1, y_{\alpha_i} + k_{13}).
\end{aligned} \tag{3.60}$$

Denoted by $R_\alpha = (x(R_\alpha), y(R_\alpha))$, the point found by integrating along the α characteristic from the point P to the point $R^{(0)}$. Hence we have $x(R_\alpha) = x(R^{(0)})$ and $y(R_\alpha) = y_\alpha(x(R^{(0)})$. The later is found by (3.60).

Similarly in the opposite direction we integrate along the β characteristic curves in the closed interval $x(R^{(0)}) \leq x \leq x(Q)$ to get y_β, the approximate value on the β characteristic through Q. First of all divide the interval into finite number, say r_2, of equally spaced sub intervals , spacing h_2. Taking $y_\beta(x(Q)) = y(Q)$ as the initial value, generate the approximate value of the β characteristic at the end points of the sub intervals. Generate approximation y_{β_i} to $y_\beta(x(P)+ih_2)$ for $i = 0,1,2,..r_2$ using the recursion formula

$$y_{\beta_{i+1}} = y_{\beta_i} + \frac{1}{6}(k_{22} + 2k_{22} + 2k_{23} + k_{24}) \tag{3.61}$$

where

$$\begin{aligned}
k_{21} &= h_2\beta(x(Q) + h_2 * i, y_{\beta_i}) \\
k_{22} &= h_2\beta(x(Q) + h_2 * i + \frac{h_2}{2}, y_{\beta_i} + \frac{k_{21}}{2}) \\
k_{23} &= h_2\beta(x(Q) + h_2 * i + \frac{h_2}{2}, y_{\beta_i} + \frac{k_{22}}{2}) \\
k_{24} &= h_2\beta(x(Q) + h_2 * i + h_2, y_{\beta_i} + k_{23})
\end{aligned} \tag{3.62}$$

Denoted by $R_\beta = (x(R_\beta), y(R_\beta))$, the point found by integrating along the α characteristic from the point P to the point $R^{(0)}$. Hence we have $x(R_\beta) = x(R^{(0)})$ and $y(R_\beta) = y_\beta(x(R^{(0)})$. The later is found by (3.62).

The points R_α and R_β should be closed and give an improved approximation to R^*, the intersection point of the α characteristic curve through P and the β characteristic curve through Q. In other word, if

$$\left| y_\alpha(x(R^{(0)})) - y_\beta(x(R^{(0)})) \right| < \varepsilon, \tag{3.63}$$

where ε is the maximum allowed error, then the average value of $y(R_\alpha)$ and $y(R_\beta)$ is taken to be the first improved value of the y-coordinate of the intersection point. Hence the improved values of the coordinates of the intersection point is found to be

$$x(R^{(1)}) = x(R^{(0)}), \;\; y(R^{(1)}) = \frac{y_\alpha(x(R^{(0)})) + y_\beta(x(R^{(0)}))}{2}. \tag{3.64}$$

However if the points R_α and R_β are not closed enough, that is

$$\left| y_\alpha(x(R^{(0)})) - y_\beta(x(R^{(0)})) \right| > \varepsilon, \tag{3.65}$$

then we have to approximate further the intersection point.

3.6.1 The Method A

Suppose from the previous calculation, the situation (3.65) occur, that is

$$\left| y_\alpha(x(R^{(0)})) - y_\beta(x(R^{(0)})) \right| > \varepsilon,$$

for some maximum allowed error ε. Hence we cannot use equation (3.64) to calculate the first improved value of the point R. Then we have to approximate the intersection point.

Replace the direction of the α characteristic curve by the straight line which is governed by the equation

$$\frac{y - y_\alpha(x(R^{(0)})}{y(P) - y_\alpha(x(R^{(0)})} = \frac{x - x(R^{(0)})}{x(P) - x(R^{(0)})} \tag{3.66}$$

and replace the direction of the β characteristic curve by the straight line which is governed by the equation

$$\frac{y - y_\beta(x(R^{(0)})}{y(P) - y_\beta(x(R^{(0)})} = \frac{x - x(R^{(0)})}{x(P) - x(R^{(0)})}. \tag{3.67}$$

The solution of these equation, say R^1 is a new approximate value of the intersection point R^*. The points R^0 and R^1 should be closed and give an improved approximation to the intersection point R^*. In other word, if

$$\left|x(R^{(1)}) - x(R^{(0)})\right| < \varepsilon, \tag{3.68}$$

and

$$\left|y(R^{(1)}) - y(R^{(0)})\right| < \varepsilon, \tag{3.69}$$

then the improved value of the x-coordinate of the point R is taken to be the average value of $x(R^{(1)})$ and $x(R^{(0)})$, and the improved value of the y-coordinate of the point R is taken to be and the average value of $y(R^{(1)})$ and $y(R^{(0)})$. Hence we have

$$x(R) = \frac{1}{2}(x(R^{(1)}) + x(R^{(0)})), \quad y(R) = \frac{y(R^{(1)}) + y(R^{(0)})}{2}. \tag{3.70}$$

If the points $R^{(1)}$ and $R^{(0)}$ are not closed enough, that is the conditions (3.68) and (3.69) are not satisfied, we have to repeat the calculation process above. Taking the approximate values at the point R^1 as the predicted values and keeping the values at the points P and Q fixed, calculate the corrected values at the corrected point R^2, R^3, and soon. The calculation is terminated when at an iteration say $(k+1)^{th}$, where $k = 0, 1, 2, ...$; the Runge Kutta procedures give

$$\left|y_\alpha(x(R^{(k)})) - y_\beta(x(R^{(k)}))\right| < \varepsilon, \tag{3.71}$$

or R^k and R^{k+1} are closed enough, that is the set of conditions

$$\left|x(R^{(k+1)}) - x(R^{(k)})\right| < \varepsilon, \quad \left|y(R^{(k+1)}) - y(R^{(k)})\right| < \varepsilon \tag{3.72}$$

are satisfied, for some maximum allowed error ε. R^{k+1} is found solving the set of equations

$$\frac{y(R)^{k+1} - y_\alpha(x(R)^{(k)})}{y(P) - y_\alpha(x(R)^{(k)})} = \frac{x(R)^{k+1} - x(R)^{(k)}}{x(P) - x(R)^{(k)}} \tag{3.73}$$

and

$$\frac{y(R)^{k+1} - y_\beta(x(R)^{(k)})}{y(Q) - y_\beta(x(R)^{(k)})} = \frac{x(R)^{k+1} - x(R)^{(k)}}{x(Q) - x(R)^{(k)}}. \tag{3.74}$$

When the condition (3.71) is satisfied, the improved value of the coördinates of the point R is found to be

$$x(R) = x(R^{(k)}), \quad y(R) = \frac{y_\alpha(x(R^{(k)})) + y_\beta(x(R^{(k)}))}{2}, \tag{3.75}$$

while if the set of conditions (3.72) are satisfied then the improved value of the coordinates of the point R is taken to be

$$x(R) = \frac{1}{2}(x(R^{(k+1)}) + x(R^{(k)})), \quad y(R) = \frac{y(R^{(k+1)}) + y(R^{(k)})}{2}. \tag{3.76}$$

Geometrically, using Straight Line method, the first approximate value of the point R, say $R^{(0)}$ is found to be the intersection of two straight lines, one is drawn through the point P in the direction of the α characteristic curve and another is drawn through the point Q in the direction of the β characteristic curve. Furthermore by applying Runge Kutta method on the closed intervals $x(P) \leq x \leq x(R^{(k)})$, where k indicates that the point $R^{(k)}$ is calculated at the k^{th} iteration, we get the approximate value $y_\alpha(x(R^{(k)}))$. Also by applying Runge Kutta method on the closed intervals $x(Q) \leq x \leq x(R^{(k)})$ we get approximate value $y_\beta(x(R^{(k)})$. When the condition (3.71) is satisfied the improved value of the coordinates of the point R is calculated by (3.75), if it is not then $R^{(k+1)}$ is found to be the intersection of the straight line drawn from P to $R^{(k)}$, which is govern by the equation (3.73), and the straight line drawn from Q to $R^{(k)}$, which is govern by the equation (3.74). If $R^{(k+1)}$ and $R^{(k)}$ are closed, that is the set of conditions (3.72) are satisfied then the improved value of the coordinates of the point R is calculated by (3.76). The situation is described by figure 3.8 below

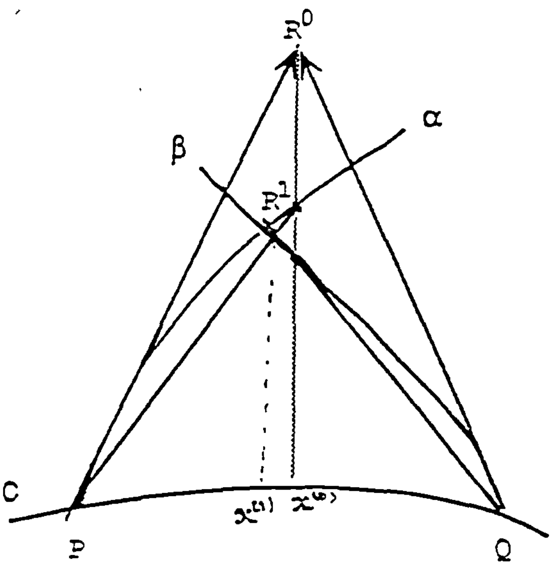


Figure 3.8 Method A

The procedures for solving the Cauchy and boundary-value problems of (3.1), using Method A, are given in the algorithms A.3.5 and A.3.6, in Appendix , respectively. The flowchart of the algorithms are depicted by figure A.3.2 in Appendix .

Using method A, the Cauchy problem in example 3.1 has the solution depicted in figure 3.9

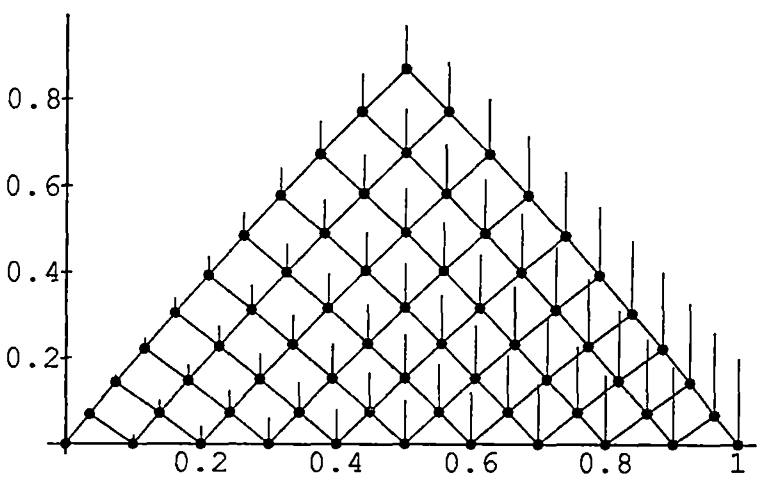


Figure 3.9 The solution by method A

3.6.2 The Method B

This method is quite similar to that in method A except that to obtain corrections to the approximate values of the intersection points, we do not use the straight lines but the tangent lines of the approximate curves. The improved value of the coordinates point may be found as the intersection point of these two tangents.

Suppose from the previous calculation, the situation (3.65) occur, that is

$$\left| y_\alpha(x(R^{(0)})) - y_\beta(x(R^{(0)})) \right| > \varepsilon,$$

for some maximum allowed error ε. Hence we cannot use equation (3.64) to calculate the first improved value of the point R. Then we have to approximate the intersection point.

Replace the direction of the α characteristic curve by the tangent line through the point $R_\alpha = (x(R)^{(0)}, y(R_\alpha))$ in the direction of α characteristic curve. This line is governed by

$$y - y_\alpha(x(R)^{(0)}) = \alpha(R_\alpha)[x - x(R)^{(0)}]. \tag{3.77}$$

Similarly replace the β characteristic curve by the tangent line through the point $R_\beta = (x(R)^{(0)}, y(R_\beta))$ which govern by equation

$$y - y_\beta(x(R)^{(0)}) = \beta(R_\beta)[x - x(R)^{(0)}]. \tag{3.78}$$

The solution of these equation, say R^1 is a new approximate value of the intersection point R^*. The points R^0 and R^1 should be closed and give an improved approximation to the intersection point R^*. In other word, if the conditions (3.68) and (3.69) are satisfied then the improved value of the x-coordinate of the intersection point is taken to be the average value of $x(R^{(1)})$ and $x(R^{(0)})$, and the improved value of the y-coordinate of the intersection point is taken to be and the average value of $y(R^{(1)})$ and $y(R^{(0)})$. That are (3.70).

If the points $R^{(1)}$ and $R^{(0)}$ are not closed enough, that is the conditions (3.68) and (3.69) are not satisfied, we have to repeat the calculation process above.

Taking the approximate values at the point R^1 as the predicted values and keeping the values at the points P and Q fixed, calculate the corrected values at the corrected point R^2, R^3, and soon. The calculation is terminated when at an iteration say $(k+1)^{th}$, where $k = 0, 1, 2, ...$; the Runge Kutta procedures give (3.71) or R^k and R^{k+1} are closed enough, that is the set of conditions (3.72) are satisfied, where R^{k+1} is found solving the set of equations (3.77) and (3.78).

When the condition (3.71) is satisfied, the improved value of the coordinates of the point R is found to be (3.75), while if the set of conditions (3.72) are satisfied then the improved value of the coordinates of the point R is taken to be (ref3.77.g). The calculation is terminated when two approximate points, say R^k and R^{k+1} for an integer k, are closed enough, that is the condition (3.71) or the set of conditions (3.72) is satisfied, for some maximum allowed error ε.

Geometrically, using Straight Line method, the first approximate value of the point R, say $R^{(0)}$ is found to be the intersection of two straight lines, one is drawn through the point P in the direction of the α characteristic curve and another is drawn through the point Q in the direction of the β characteristic curve. Furthermore by applying Runge Kutta method on the closed intervals $x(P) \leq x \leq x(R^{(k)})$, where k indicates that the point $R^{(k)}$ is calculated at the k^{th} iteration, we get the approximate value $y_\alpha(x(R^{(k)}))$. Also by applying Runge Kutta method on the closed intervals $x(Q) \leq x \leq x(R^{(k)})$ we get approximate value $y_\beta(x(R^{(k)})$. When the condition (3.71) is satisfied the improved value of the coordinates of the point R is calculated by (3.75), if it is not then $R^{(k+1)}$ is found to be the intersection of the tangent line drawn line through the point R_α in the direction of the α characteristic

$$y - y_\alpha(x(R)^{(k)}) = \alpha[R_\alpha)(x - x(R)^{(k)}]. \qquad (3.79)$$

and the tangent line drawn from the point R_β in the direction of the β characteristic

$$y - y_\beta(x(R)^{(k)}) = \beta[R_\beta)(x - x(R)^{(k)}]. \qquad (3.80)$$

If $R^{(k+1)}$ and $R^{(k)}$ are closed, that is the set of conditions (2.133) is satisfied then the improved value of the coordinates of the point R is calculated by (3.76). The situation is described by figure 3.10 below

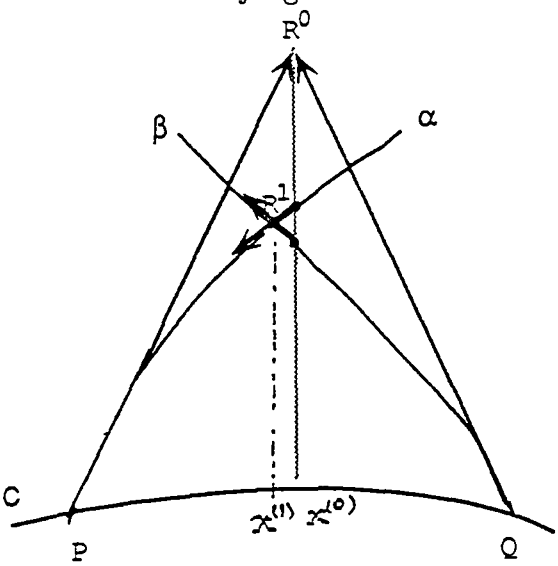


Figure 3.10The Method B

To construct a characteristic grid net and the solution of the Cauchy and boundary-value problems we may use the procedures in the algorithms in Appendix . The flowchart of the algorithms are depicted by figure A.3.3 in Appendix

.

Using method B, the Cauchy problem in example 3.1 has the solution depicted in figure 3.11.

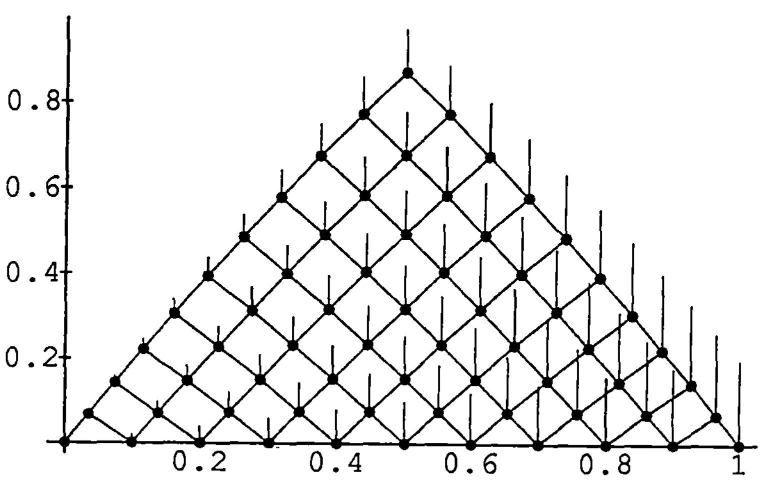


Figure 3.11The solution of Cauchy problem by method B

3.7 Comparisons

The predictor corrector method, method A and method B give better approximate values than of the straight line method, since the three former methods are using truncation error of order two, while the later is using truncation error of order two. However the methods A and method B give the same approximate values. For the Cauchy problem in example 3.1, the comparisons are depicted in figures 3.12, 3.13 and 3.14.

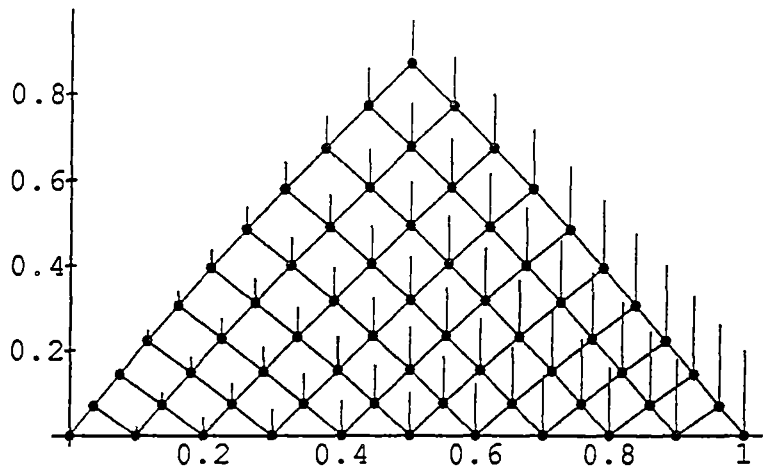


Figure 3.13 Comparison solutions : the methods A and B

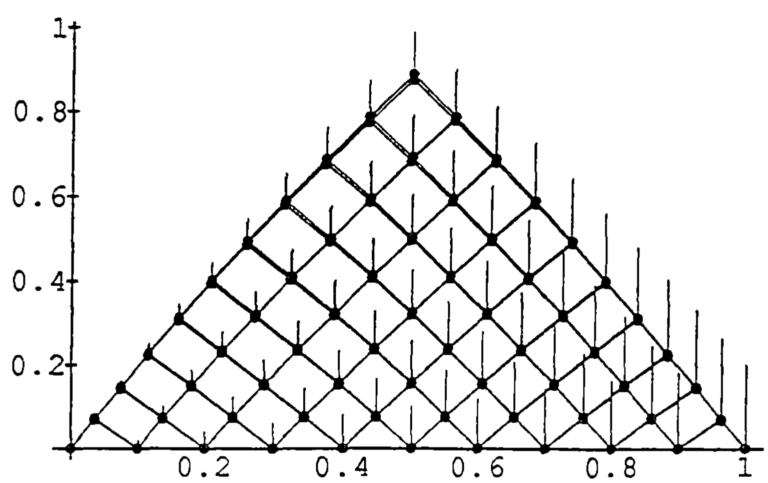


Figure 3.12 Comparison solutions : the straight line method and the predictor corrector method

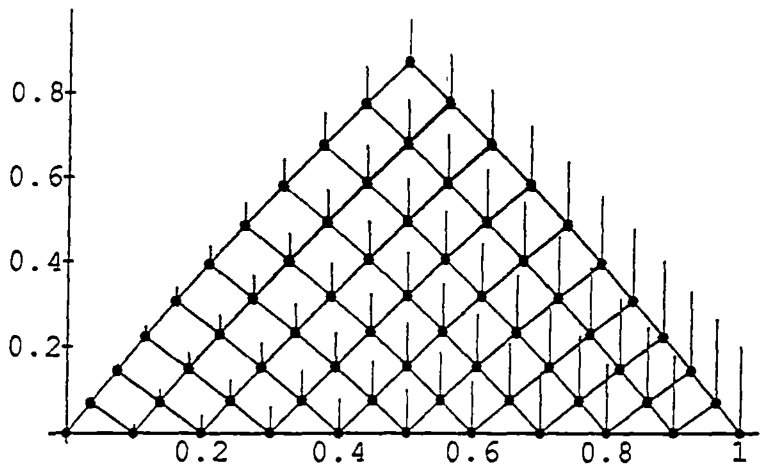


Figure 3.13:Comparison solutions : the straight line method
and methods A, B

The methods A and B faster and give better approximate values than of the predictor corrector method

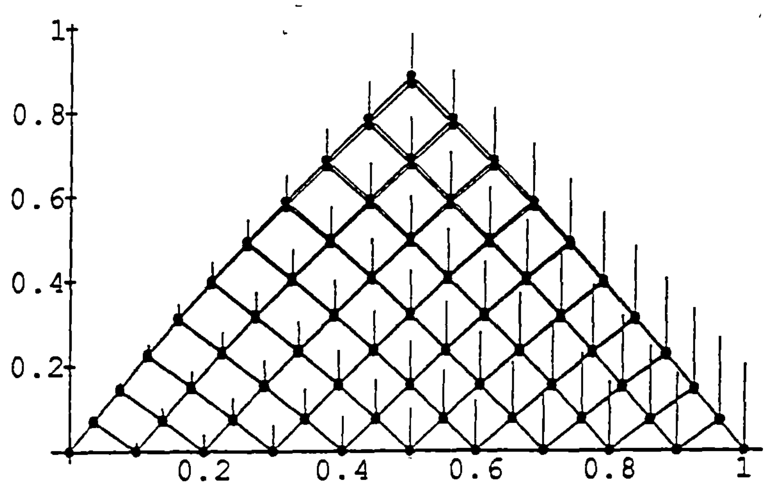


Figure 3.14:Comparison solutions : the predictor corrector method
and the methods A, B

since the two former methods use the Runge Kutta (of order 4) method while the later method use the trapezoidal rule which is equivalent to the Runge Kutta of order 2. However when the characteristic curves are straight lines, then the method A and method B are reduced into the straight line method.

3.8 Discussions and Conclusions

The method of characteristics involves expressing the partial differential equation in terms of its characteristic coordinates and integrating along the characteristic directions. Solving the quasi linear hyperbolic equation is reduced to solving the characteristic system. The system involves non-linear ordinary differential equations which should be calculated simultaneously. Discretization involves approximating the two families of the characteristics curves by characteristic grids and replacing the differential equations with appropriate finite difference equations.

Based on the Taylor's series of order one, the straight line method gives approximate solutions with discretization error of order one. Geometrically the method approximates the characteristic curves by straight lines.

Secondly based on Taylor's series of order two, the predictor-corrector methods produce approximate solutions with discretization error of order two. Geometrically the method approximates the characteristic curves by parabolic arcs. The parabolic arcs are chosen to be straight line through the given point and the calculated approximate point.

Since the methods of characteristic calculate the solution on the characteristic grid points, which approximate intersection points of the characteristic curve, then there is no restriction conditions to ensure the stability and convergence of the numerical methods above [4].

Basically step methods numerical integration, such as the Runge Kutta method, can be used to find the characteristic grids. Using the *Runge Kutta method*, the alternative methods, method A and method B, integrate the characteristic curves directly. The improved values are found to be the intersection of the approximate curves. When the approximate curves do dot intersect, the method A approximates the intersection point by the intersection point of their arcs, while the method B by the intersection point of their tangents.

From our experience, for the problems which have *semi-symmetric* characteristics, the predictor corrector method, the method A and method B are converge, since at each stage of calculation the characteristic curves remain inside the triangular formed by the tangents of the given points. Generally the predictor corrector method, the method A and method B give better approximate values than of the straight line method. However when the characteristic curves are straight lines, then the method A and method B are reduced into the straight line method. In calculation processes, the method A and method B are faster than the predictor corrector method, since the two former methods use the Runge Kutta (of order 4) method while the later method use the trapezoidal rule which is equivalent to the Runge Kutta of order 2.

Chapter 4

THE FINITE DIFFERENCE METHOD

4.1 Introduction

Many physical systems lead to complex systems of equations involving discontinuity, non-linearity, non homogeneous domains and irregular boundaries. One of the main advantages of the use of characteristics is the fact that the discontinuity in the prescribed initial values may be propagated along the characteristics. For systems which are governed by equations whose solutions are known to be well behaved, the propagation of discontinuity of the initial values can be handled accurately by the method of characteristics.

However, systems involving non linear equations and irregular boundaries we require alternative approximation methods. In this chapter we will use the *finite difference method* to solve such problems. *The finite difference method* involves discretization of the governing equations, the initial conditions and the boundary conditions of the continuous domain.

We will deal with the Cauchy problem of a second order linear equation of two variables. The discretization may be divided into two steps:
Firstly, replace the continuous domain by a mesh passing through discrete interior

points. In two dimensional problems, the continuous domain may be replaced by rectangular grids .

Secondly, the governing equations which are continuous formulations are replaced by *finite difference equations* as the approximations. The replacement of the governing equation by a finite difference equation is not unique. It depends on the configuration of the difference formulas we use.

Based on Taylor's series, the *forward*, *backward* and *central difference* formulas will be derived in section 4.2. Using these formulas we may construct some finite difference schemes that can be used to replace a given partial differential equation. The boundary conditions usually determine the scheme suitable for solving a particular problem. Basically the finite difference methods are classified into the *explicit* and *implicit* methods. The von Neumann stability condition will be used to examine stability of finite difference methods.

4.2 Difference Schemes

The derivative of a function is the limit of a difference quotient, and therefore the derivative can be approximated as close as desired by taking the points involved in the difference quotient close enough. Hence a partial differential equation may be replaced by a *finite difference equation.*

The finite difference equation as an approximate equation is dependent on the configuration of discretization of the continuous domain. In two-dimensional problems, the continuous domain in the xy-plane can be replaced by a rectangular grid with uniform spacing $h = \Delta x$ and $k = \Delta y$, as shown in figure 4.1. The space point $(i\Delta x, j\Delta y)$ which is also called the *grid point* (i, j) is surrounded by the neighbouring grid points. The grid points inside the domain are called *interior* points. Hereafter we will use the term grid points to mean interior points.

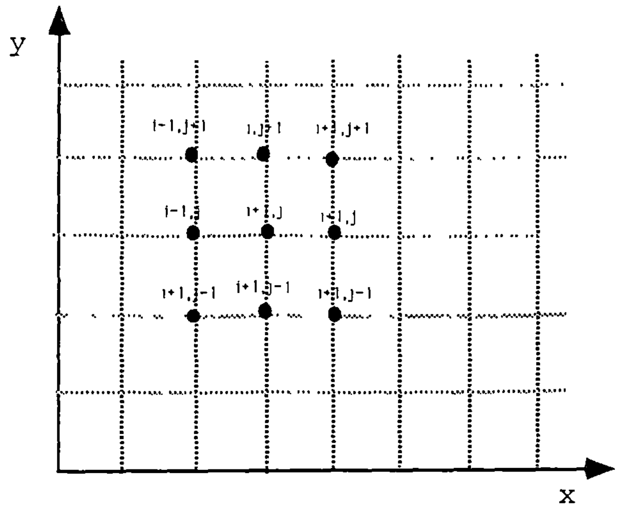


Figure 4.1: The rectangular grid

Suppose that the continuous function $u(x,y)$ posses a sufficient number of partial derivatives, then the *Taylor's series* expansion for $u(x+\Delta x, y+\Delta y)$ about (x,y), gives :

$$\begin{aligned} u(x+\Delta x, y+\Delta y) &= u(x,y) + (\Delta x \frac{\partial}{\partial x} + \Delta x \frac{\partial}{\partial y}) u(x,y) \\ &+ \frac{1}{2!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 u(x,y) \\ &+ \frac{1}{3!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^3 u(x,y) \\ &+ \frac{1}{(n-1)!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^{n-1} u(x,y) + R_n \end{aligned} \tag{4.1}$$

where the remainder term is given by

$$R_n = \frac{1}{n!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^n u(x + \xi \Delta x, y + \xi \Delta y) \tag{4.2}$$

for $0 < \xi < 1$. That is

$$R_n = O[(\|\Delta x\| + \|\Delta y\|)^n]. \tag{4.3}$$

There exists positive constant M say, such that

$$\|R_n\| \leq M(\|\Delta x\| + \|\Delta y\|)^n.$$

Taking $\Delta y = 0$ in the equation (4.1) and assuming that the second derivatives are bounded, the Taylor's series expansion for $u(x + \Delta x, y)$ about (x, y) gives

$$u(x + \Delta x, y) = u(x, y) + \Delta x \frac{\partial u(x, y)}{\partial x} + O[((\Delta x)^2].$$

Dividing it by Δx we have

$$\frac{\partial u(x, y)}{\partial x} = \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + O(\Delta x).$$

Similarly, the Taylor's series expansion for $u(x - \Delta x, y)$ about (x, y), gives

$$\frac{\partial u(x, y)}{\partial x} = \frac{u(x, y) - u(x - \Delta x, y)}{\Delta x} + O(\Delta x).$$

Now we wish to evaluate the first and second derivatives at the grids point (x_i, y_j). Suppose $x_i = x_0 + ih$ and $y_j = x_0 + jk$, where $h = \Delta x$ and $k = \Delta y$. Then using double subscript notation, the Taylor's series expansion for $u(x_i + h, y_i)$ about (x_i, y_j) with the second derivatives assumed to be bounded gives

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{h}(u_{i+1,j} - u_{i,j}) + O(h) \tag{4.4}$$

This equation indicates a formula to evaluate $\frac{\partial u}{\partial x}$ at (x_i, y_j) by using the values of u at the points (x_i, y_j) and $(x_i + h, y_j)$. Such formula is called a *forward* difference formula. Similarly the Taylor's series expansion for $u(x_i - h, y_i)$ about (x_i, y_j) with the second derivatives assumed to be bounded gives a *backward* difference formula

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{h}(u_{i,j} - u_{i-1,j}) + O(h). \tag{4.5}$$

which evaluates $\frac{\partial u}{\partial x}$ at (x_i, y_j) by using the values of u at the points (x_i, y_j) and $(x_i - h, y_j)$. In effect we may replace the value of $\frac{\partial u}{\partial x}$ at the point (x_i, y_j) by

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{h}(u_{i+1,j} - u_{i,j})$$

if we use forward difference formula, and by

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{h}(u_{i,j} - u_{i-1,j})$$

if we use backward difference formula. These are *first order* approximations of $\frac{\partial u}{\partial x}$ evaluated at (x_i, y_j) with a truncation error $O(h)$.

Increasing the order of the truncation error in the Taylor's series expansion improve the approximate value of $\frac{\partial u}{\partial x}$. Subtracting the Taylor's series expansions for $u(x + \Delta x, y)$ and $u(x - \Delta x, y)$ with fourth derivatives assumed to be bounded gives

$$u(x + \Delta x, y) - u(x - \Delta x, y) = 2\Delta x \frac{\partial u(x, y)}{\partial x} + O[((\Delta x)^3].$$

Note that all even-order terms cancel. Dividing by Δx and using double subscript notation leads to

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}) + O(h^2). \tag{4.6}$$

This equation provides another formula to evaluate $\frac{\partial u}{\partial x}$ at (x_i, y_j), in which we use the values of u at the points (x_i, y_j) and $(x_i + h, y_j)$. Such formula is called a *central* difference formula. In effect we may replace the value of $\frac{\partial u}{\partial x}$ at the point (x_i, y_j) by

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}).$$

This formula gives a *second order* approximate value of $\frac{\partial u}{\partial x}$ evaluated at (x_i, y_i) with a truncation error $O(h^2)$.

Similarly, by taking $\Delta x = 0$ at the equation (4.1) $\frac{\partial u}{\partial y}$ can be evaluated at grid pints (x_i, y_j), by using forward difference formula

$$\left.\frac{\partial u}{\partial y}\right|_{i,j} = \frac{1}{k}(u_{i,j+1} - u_{i,j}) + O(k), \tag{4.7}$$

or by backward difference formula

$$\left.\frac{\partial u}{\partial y}\right|_{i,j} = \frac{1}{k}(u_{i,j} - u_{i,j-1}) + O(k), \tag{4.8}$$

or by central difference formula

$$\left.\frac{\partial u}{\partial y}\right|_{i,j} = \frac{1}{2k}(u_{i,j+1} - u_{i,j-1}) + O(k^2). \tag{4.9}$$

Now we evaluate the second derivatives, adding the *Taylor series* expansions for $u(x+\Delta x, y)$ and $u(x-\Delta x, y)$ with fourth derivatives bounded leads to

$$u(x+\Delta x, y)+u(x-\Delta x, y)=2u(x,y)+(\Delta x)^2\frac{\partial^2 u(x,y)}{\partial x^2}+O[((\Delta x)^4].$$

Dividing by $(\Delta x)^2$ and using double subscript notation we have a central difference formula for $\frac{\partial^2 u(x,y)}{\partial x^2}$

$$\left.\frac{\partial^2 u(x,y)}{\partial x^2}\right|_{i,j}=\frac{1}{h^2}(u_{i-1,j}-2u_{i,j}+u_{i+1,j})+O(h^2). \tag{4.10}$$

Similarly we have a central difference formula for $\frac{\partial^2 u(x,y)}{\partial y^2}$

$$\left.\frac{\partial^2 u(x,y)}{\partial y^2}\right|_{i,j}=\frac{1}{k^2}(u_{i,j-1}-2u_{i,j}+u_{i,j+1})+O(k^2). \tag{4.11}$$

The mixed derivative $\frac{\partial^2 u(x,y)}{\partial x\partial y}$ can be obtained by mean of composite way. Using

$$\begin{aligned}\left.\frac{\partial^2 u}{\partial x\partial y}\right|_{i,j} &= \frac{\partial}{\partial x}\left[\left.\frac{\partial u}{\partial y}\right|_{i,j}\right]\\ &= \frac{1}{2h}\left[\left.\frac{\partial u}{\partial y}\right|_{i+1,j}-\left.\frac{\partial u}{\partial y}\right|_{i-1,j}\right]+O(h^2)\\ &= \frac{1}{2h}((\frac{1}{2k}(u_{i+1,j+1}-u_{i+1,j-1})+O(k^2))\\ &\quad -(\frac{1}{2k}(u_{i-1,j+1}-u_{i-1,j-1})+O(k^2)))+O(h^2).\end{aligned}$$

Hence we have a central difference formula for $\frac{\partial^2 u(x,y)}{\partial x\partial y}$

$$\begin{aligned}\left.\frac{\partial^2 u(x,y)}{\partial x\partial y}\right|_{i,j} &= \frac{1}{4hk}(u_{i-1,j-1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i+1,j+1})\\ &\quad +O(h^2)+O(k^2). \end{aligned}\tag{4.12}$$

In conclusion, at the point at point (x_i, y_j), the value of $\frac{\partial^2 u(x,y)}{\partial x^2}$ can be approximated by

$$\left.\frac{\partial^2 u(x,y)}{\partial x^2}\right|_{i,j}=\frac{1}{h^2}(u_{i-1,j}-2u_{i,j}+u_{i+1,j}),$$

the value of $\frac{\partial^2 u(x,y)}{\partial y^2}$ by

$$\left.\frac{\partial^2 u(x,y)}{\partial y^2}\right|_{i,j}=\frac{1}{k^2}(u_{i,j-1}-2u_{i,j}+u_{i,j+1}),$$

and the value of $\frac{\partial^2 u(x,y)}{\partial x \partial y}$ by

$$\left.\frac{\partial^2 u(x,y)}{\partial x \partial y}\right|_{i,j} = \frac{1}{4hk}(u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i+1,j+1}).$$

These three approximate values have the same truncation error, $O(h^2)$.

4.3 Explicit Difference Methods

In this chapter a study is made for the finite difference methods of solving the Cauchy problem and boundary value problem for linear hyperbolic equation

$$a(x,y)u_{xx} - c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \tag{4.13}$$

As we have done in the successive approximation method discussed in section 1.7, to set up *the Cauchy problem* and *the boundary value problem* of the equation (4.13) there is no lost of generality if we suppose that the initial curve (1.39), on which the Cauchy data are given, is the straight line (1.57). Furthermore by rotation, we may have x-axis as the initial curve such that the Cauchy problem can be described as follows: find the solution $u(x,y)$ of the equation (4.13) in the region

$$D = \{(x,y) | -\infty < x < \infty, y \geq 0\}$$

which satisfies the initial conditions:

$$u(x,0) = r(x),$$

$$u_y(x,0) = s(x) \tag{4.14}$$

where r and s are given functions.

The solution of the boundary value problem involves finding the solution $u(x,y)$ of the equation (4.13) in the region

$$D = \{(x,y) | x_0 \leq x \leq x_n, y \geq 0\}$$

which satisfies the initial conditions (4.14) in the interval $x_0 \leq x \leq x_n$ as well as certain conditions at the boundary which are straight lines $x = x_0$ and $x =$

x_n. Such as mentioned in section 3.3, in general, the discretization of boundary conditions may be written as

$$\lambda U_x(x_0, y_{j+1}) + \delta_1(j)U(x_0, yj+1) = t_1(j) \tag{4.15}$$

along $x = x_0$, and

$$\lambda u_x(x_n, y) + \delta_2(j)u(x_n, y) = t_2(j) \tag{4.16}$$

along $x = x_n$, where $y \geq 0$, and λ is a constant. $\lambda = 0$ gives a set of boundary conditions of the first kind (2.48) and $\delta_1(j) = \delta_2(j) = 0$, $\forall j \geq 0$ gives a set of boundary conditions of the second kind (2.49), while $\lambda = 1$ is associated with a set of boundary conditions of the third kind (2.47).

To find the approximate solution to the Cauchy problem and also the boundary value problem for the equation (4.13) using finite difference methods we need to discretize the equation (4.13) and the equations involved in the initial conditions (4.14) and boundaries conditions (4.15) and (4.16).

First of all cover the domain in the xy-plane by rectangular grid of mesh points with constant intervals $h = \Delta x$ and $k = \Delta y$. Instead of developing a solution defined everywhere in the domain we only calculate the approximation solution in these internal mesh points. Suppose at a point $(i\Delta x, j\Delta y)$ the exact solution is denoted by $u_{i,j} = u(i\Delta x, j\Delta y)$ and the approximation solution by $U_{i,j} = U(i\Delta x, j\Delta y)$ for $i = ... -3, -2, -1, 0, 1, 2, 3...$ and $j = 0, 1, 2, 3, ...$

Using the second order central difference formula ,the partial differential equation (4.13) may be replaced by *finite difference equation*

$$\begin{aligned} a_{i,j}\frac{1}{h^2}(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}) - c_{i,j}\frac{1}{k^2}(U_{i,j-1} - 2U_{i,j} + U_{i,j+1}) \\ +d_{i,j}\frac{1}{2h}(U_{i+1,j} - U_{i-1,j}) + e_{i,j}\frac{1}{2k}(U_{i,j+1} - U_{i,j-1}) + f_{i,j}U_{i,j} = g_{i,j} \end{aligned}$$

or it may be written as

$$A_{i,j}U_{i,j+1} + B_{i,j}U_{i,j-1} + C_{i,j}U_{i+1,j} + D_{i,j}U_{i-1,j} + E_{i,j}U_{i,j} = g_{i,j} \tag{4.17}$$

where

$$
\begin{aligned}
A_{i,j} &= -\frac{c_{i,j}}{k^2} + \frac{e_{i,j}}{2k}, \\
B_{i,j} &= -\frac{c_{i,j}}{k^2} - \frac{e_{i,j}}{2k}, \\
C_{i,j} &= \frac{a_{i,j}}{h^2} + \frac{d_{i,j}}{2h}, \\
D_{i,j} &= \frac{a_{i,j}}{h^2} - \frac{d_{i,j}}{2h}, \\
E_{i,j} &= -\frac{2a_{i,j}}{h^2} + \frac{2c_{i,j}}{k^2} + f_{i,j}.
\end{aligned} \tag{4.18}
$$

Provided h small, hence $A_{i,j} \neq 0$, the equation (4.17) can be written explicitly by

$$
U_{i,j+1} = \frac{g_{i,j}}{A_{i,j}} - \frac{B_{i,j}}{A_{i,j}} U_{i,j-1} - \frac{C_{i,j}}{A_{i,j}} U_{i+1,j} - \frac{D_{i,j}}{A_{i,j}} U_{i-1,j} - \frac{E_{i,j}}{A_{i,j}} U_{i,j}. \tag{4.19}
$$

If we know the value of the approximation solution $U_{i,j}$ at all mesh points in the $(j-1)^{th}$ and j^{th} horizontal lines, then we can find the solution at all mesh point in the advanced line $(j+1)^{th}$. To visualize the calculation it is useful to draw a schema such as shown in figure 4.2

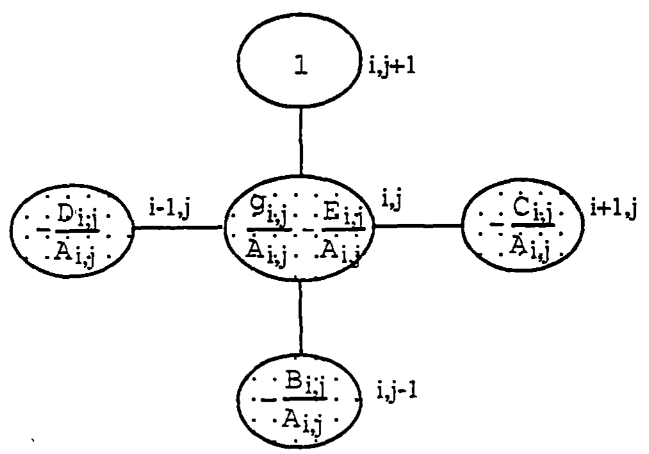


Figure 4.2. Explicit method

: computed values from previous calculations

: value to be computed

From the schema above, on each calculation process, the advanced line $(j+1)^{th}$ contains only one unknown, such computation is called an *explicit schema*. If there are two or more unknowns in the $(j+1)^{th}$ then it will be called an *implicit schema*. A finite difference method which has explicit schema is called the *explicit method*, and is called the *implicit method* if it has implicit schema.

4.3.1 Cauchy problems

To find an approximation solution of the Cauchy problem, replace the domain $D = \{(x, y)| -\infty < x < \infty \ \& \ y \geq 0\}$ by

$$D^* = \{(ih, jk)|i = 0, \pm 1, \pm 2, ...\&j = 0, 1, 2...\}.$$

In using the initial conditions we need to know the values of the solution in the first two lines at $j = 0$ and $j = 1$. To do this we can replace the first derivative in the initial condition by a forward difference formula or by a central difference formula . If we use the former we will have

$$U_{i,0} = r_i, \qquad U_{i,1} = r_i + ks_i, \tag{4.20}$$

while if we use central difference formula we have

$$U_{i,0} = r_i, \qquad \frac{U_{i,1} - U_{i,-1}}{2k} = s_i. \tag{4.21}$$

The central difference formula is sometimes preferable: the reasons will be discussed in the section concerning to error computations.

Furthermore, taking $j = 0$ in equation (4.19) we get

$$U_{i,1} = \frac{g_{i,0}}{A_{i,0}} - \frac{B_{i,0}}{A_{i,0}}U_{i,-1} - \frac{C_{i,0}}{A_{i,0}}U_{i+1,0} - \frac{D_{i,0}}{A_{i,0}}U_{i-1,0} - \frac{E_{i,0}}{A_{i,0}}U_{i,0}. \tag{4.22}$$

Eliminating $U_{i,-1}$ from the equation (4.22) and the second equation in (4.21) we get

$$U_{i,1} = \frac{g_{i,0}}{A_{i,0}} - \frac{B_{i,0}}{A_{i,0}}(U_{i,1} - 2ks_i) - \frac{C_{i,0}}{A_{i,0}}U_{i+1,0} - \frac{D_{i,0}}{A_{i,0}}U_{i-1,0} - \frac{E_{i,0}}{A_{i,0}}U_{i,0}$$

or

$$(A_{i,0} + B_{i,0})U_{i,1} = g_{i,0} + 2kB_{i,0}s_i - C_{i,0}U_{i+1,0} - D_{i,0}U_{i-1,0} - E_{i,0}U_{i,0}$$

Substituting the first initial condition we get

$$(A_{i,0} + B_{i,0})U_{i,1} = g_{i,0} + 2kB_{i,0}s_i - C_{i,0}r_{i+1} - D_{i,0}r_{i-1} - E_{i,0}r_i$$

Hence the second initial condition (4.21) may be replaced by

$$U_{i,1} = \frac{1}{A_{i,0} + B_{i,0}}(g_{i,0} + 2kB_{i,0}s_i - C_{i,0}r_{i+1} - D_{i,0}r_{i-1} - E_{i,0}r_i). \tag{4.23}$$

Hence using explicit method, the solution of Cauchy problem for the equation (4.13) can be calculated by following procedures. Calculate the solutions at each point in the first line by first equation in (4.20) for $i = 0, 1, 2, .., n$; and second lines by (4.23) for $i = 0, 1, 2, .., n$; and then the j^{th} line by (4.19) for $i = 0, 1, 2, ...j - 1$ & $j \geq 3$.

4.3.2 The Boundary Value problem

In solving a boundary value problem, we assume that the boundaries are $x = 0$ and $x = 1$, since it is always possible to transform the interval $[\alpha \leq x \leq \beta]$ into $[0, 1]$. Hence the domain becomes

$$D = \{(x, y)|0 \leq x \leq 1, y \geq 0\}$$

and then it is replaced by

$$D* = \{(ih, jk)|i = 0, 1, 2, .., n, j = 0, 1, 2....\},$$

where h and k are grid size in the x and y axes respectively.

Using the forward difference formula the boundary conditions (4.15) and (4.15) may be replaced by

$$\lambda\frac{U_{1,j} - U_{0,j}}{h} + \delta_{1_j}U_{0,j} = t_{1_j}, \tag{4.24}$$

and

$$\lambda\frac{U_{n,j} - U_{n-1,j}}{h} + \delta_{2_j}U_{n,j} = t_{2_j}. \tag{4.25}$$

Note that $\lambda = 0$, $\delta_{1_j} = \delta_{2_j} = 0$ and $\lambda = 1$ correspond to the difference formulas for the boundary condition of the first, second, and third kind respectively. The error in using the forward difference formula is $O(h)$.

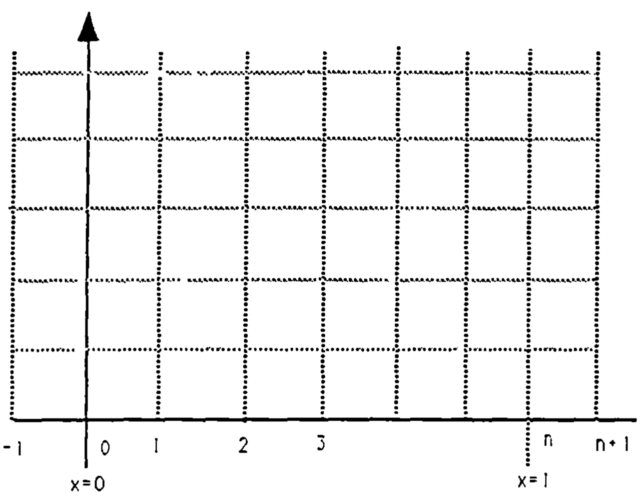
Hence by replacing the boundary conditions with the forward difference formulas, the solution of boundary value problem for the equation (4.13) can be calculated by following procedures: Calculate the solutions at each point in the first line $U_{i,0}$ for $i = 0, 1, 2, .., n$ by first equations in (4.20) Furthermore for the j^{th} line $j \geq 1$; calculate $U_{0,j}$, the solution at the left boundary by (4.24), and then calculate $U_{i,j}$, $i = 1, 2, ...(n-1)$, the solutions at the non boudary points, by (4.20) when $j = 1$ and by (4.19) when $j > 1$, finally calculate $U_{n,j}$, the solution at the right boundary by (4.25).

More accurate approximations can be made by replacing the boundary conditions with the central difference formulas. This can be done in two ways.

1st method: Expanding domain

We add two additional vertical lines $i = -1$ and $i = n + 1$ such as shown in figure 4.3 and approximate the boundary conditions of the third kind at the grid points $(0, j)$ and (n, j) by

$$\frac{U_{1,j} - U_{-1,j}}{2h} + \delta_{1_j} U_{0,j} = t_{1_j}, \quad \frac{U_{n+1,j} - U_{n-1,j}}{2h} + \delta_{2_j} U_{n,j} = t_{2_j}. \tag{4.26}$$



Taking $i = 0$ in the equation (4.17) we have

$$A_{0,j}U_{0,j+1} + B_{0,j}U_{0,j-1} + C_{0,j}U_{1,j} + D_{0,j}U_{-1,j} + E_{0,j}U_{0,j} = g_{0,j}, \tag{4.27}$$

and taking $i = n$ we have

$$A_{n,j}U_{n,j+i} + B_{n,j}U_{n,j-1} + C_{n,j}U_{n+1,j} + D_{n,j}U_{n-1,j} + E_{n,j}U_{n,j} = g_{n,j}. \tag{4.28}$$

Eliminating $U_{-1,j}$ from (4.27) and the first equation in (4.26) we get

$$A_{0,j}U_{0,j+1} + B_{0,j}U_{0,j-1} + (C_{0,j} + D_{0,j})U_{1,j} + (E_{0,j} + 2h\delta_{1_j}D_{0,j})U_{0,j}$$
$$= g_{0,j} + 2hD_{0,j}t_{1_j}, \tag{4.29}$$

and eliminating $U_{n+1,j}$ from (4.28) and the second equation in (4.26) we have

$$A_{n,j}U_{n,j+1} + B_{n,j}U_{n,j-1} + (C_{n,j} + D_{n,j})U_{n+1,j} + (E_{n,j} - 2h\delta_{2_j}C_{n,j})U_{n,j}$$
$$= g_{n,j} - 2hD_{n,j}t_{2_j} \tag{4.30}$$

Further by taking $j = 0$ in the last two equations gives

$$A_{0,0}U_{0,1} + B_{0,j}U_{0,-1} + (C_{0,0} + D_{0,0})U_{1,0} + (E_{0,0} + 2h\delta_{1_0}D_{0,0})U_{0,0}$$
$$= g_{0,0} + 2hD_{0,0}t_{1_0} \tag{4.31}$$

and

$$A_{n,0}U_{n,1} + B_{n,0}U_{n,-1} + (C_{n,0} + D_{n,0})U_{n+1,0} + (E_{n,0} - 2h\delta_{2_n}C_{n,0})U_{n,0}$$
$$= g_{n,0} - 2hD_{n,0}t_{2_0}. \tag{4.32}$$

Since we use the central difference formula for the boundary conditions, we apply the central difference formula to the second initial condition on the boundaries,

$$\frac{U_{0,1} - U_{0,-1}}{2k} = s_0, \tag{4.33}$$

and

$$\frac{U_{n,1} - U_{n,-1}}{2k} = s_n. \tag{4.34}$$

Eliminating $U_{0,-1}$ from (4.31) and (4.33) we get

$$U_{0,1} = \frac{g_{0,0} + 2kB_{0,0}s_0 + 2hD_{0,0}t_{1_0} - (C_{0,0} + D_{0,0})r_1 - (E_{0,0} + 2h\delta_{1_0}D_{0,0})r_0}{(A_{0,0} + B_{0,0})} \tag{4.35}$$

and eliminating $U_{n,-1}$ from (4.32) and (4.34) we get

$$U_{n,1} = \frac{g_{n,0} + 2hB_{n,0}s_n - 2hC_{n,0}t_{2_j} - (C_{n,0} + D_{n,0})r_{n-1} - (E_{n,0} - 2h\delta_{2_0}C_{n,0})r_n}{(A_{0,0} + B_{0,0})}. \tag{4.36}$$

Hence by expanding the domain we are able to replace the boundary conditions with the central difference formulas. The solution of boundary value problem for the equation (4.13) can be calculated by following procedures: Calculate the solutions at each point in the first line $U_{i,0}$ for $i = 0, 1, 2, .., n$ by first equations in

(4.20). When $j = 1$, calculate $U_{0,1}$, the solution at the left boundary by (4.35), and then calculate $U_{i,1}$ by (4.20), finally calculate $U_{0,1}$, the solution at the right boundary by (4.36). Furthermore for the j^{th} line $j > 1$; calculate $U_{0,j}$, the solution at the left boundary by (4.29), and then calculate $U_{i,j}$, $i = 1, 2, ...(n-1)$, the solutions at the non boundary points by (4.19), finally calculate $U_{n,j}$, the solution at the right boundary by (4.30).

2nd method : Reducing domain

We shift the boundaries by $\frac{h}{2}$ such as shown in figure 4.4

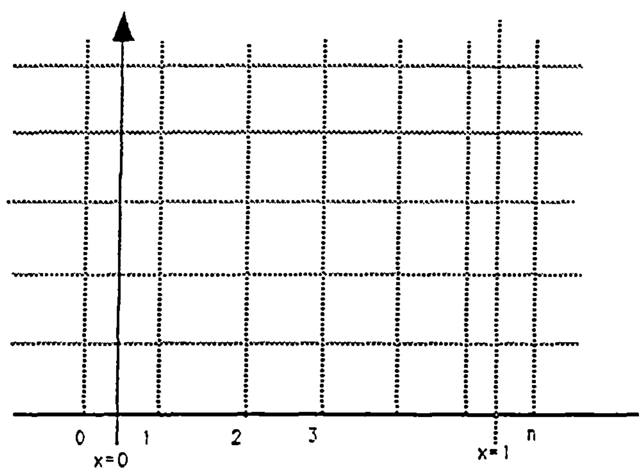


Figure 4.4: Reducing domain

The boundary conditions of the third kind are approximated by

$$\frac{U_{1,j} - U_{0,j}}{h} + \delta_{1_j} \frac{U_{1,j} + U_{0,j}}{2} = t_{1_j}, \tag{4.37}$$

and

$$\frac{U_{n,j} - U_{n-1,j}}{h} + \delta_{2_j} \frac{U_{n,j} + U_{n-1,j}}{2} = t_{2_j}. \tag{4.38}$$

From (4.37) and (4.38) we have

$$U_{0,j} = \frac{(2 + h\delta_{1_j})U_{1,j} - 2ht_{1_j}}{(2 - h\delta_{1_j})} \tag{4.39}$$

and

$$U_{n,j} = \frac{(2 + h\delta_{2_j})U_{n-1,j} - 2ht_{1j}}{(2 - h\delta_{2_j})} \tag{4.40}$$

Taking $j = 1$ in the last two equations we have

$$U_{0,1} = \frac{(2 + h\delta_{1_j})U_{1,1} - 2ht_{1_j}}{(2 - h\delta_{1_1})} \tag{4.41}$$

and

$$U_{n,1} = \frac{(2 + h\delta_{2_1})U_{n-1,1} - 2ht_{11}}{(2 - h\delta_{2_1})} \tag{4.42}$$

Hence by reducing the domain we are able to replace the boundary conditions with the central difference formulas. The solution of boundary value problem for the equation (4.13) can be calculated by following procedures: Calculate the solutions at each point in the first line $U_{i,0}$ for $i = 0, 1, 2, .., n$ by first equations in (4.20). The required values r_0, r_n, s_0 and s_n can be calculated by extrapolation. Hence we can find the values of the approximation solutions on the boundaries. When $j = 1$, calculate $U_{0,1}$, the solution at the left boundary by (4.41), and then calculate $U_{i,1}$ by (4.20), finally calculate $U_{0,1}$, the solution at the right boundary by (4.42). Furthermore for the j^{th} line $j > 1$; calculate $U_{0,j}$, the solution at the left boundary by (4.39), and then calculate $U_{i,j}$, $i = 1, 2, ...(n - 1)$, the solutions at the non boundary points by (4.19), finally calculate $U_{n,j}$, the solution at the right boundary by (4.40).

4.4 Domain of Dependencies

We mentioned in section 1.4 that the solution of a Cauchy problem for a hyperbolic differential equation at a particular point depends on the initial values along the segment of determination, the segment of the initial curve intercepted by the characteristic curves through the point, and the values over the domain of dependence of the differential equation with respect to the point. We wish to examine the domain of dependence of the finite difference solutions for the Cauchy problem for the homogeneous wave equation

$$u_{yy} = u_{xx} \tag{4.43}$$

with initial conditions (4.14). Using the procedures given in the section 4.3, the partial differential equation (4.43) may be replaced by the implicit difference equation

$$U_{i,j+1} = \rho^2(U_{i-1,j} + U_{i+1,j}) + 2(1-\rho^2)U_{i,j} - U_{i,j-1}, \tag{4.44}$$

where $\rho = \frac{k}{h}$. further the initial conditions are replaced by

$$U_{i,0} = r(ih) \tag{4.45}$$

and

$$U_{i,1} = \frac{1}{2}\rho^2(f_{i-1} + r_{i+1}) + (1-\rho^2)r_i + ks_i. \tag{4.46}$$

From section 2.4, the D'Alembert solution at a particular point (x_i, y_{j+1}) is

$$u(x_i, y_{j+1}) = \frac{1}{2}(r(x_i - y_{j+1}) + r(x_i + y_{j+1})) + \frac{1}{2}\int_{x_i - y_{j+1}}^{x_i + y_{j+1}} g(\xi)d\xi. \tag{4.47}$$

Geometrically this equation shows that the domain of dependence of the differential equation (4.43) with respect to the point (x_i, y_{j+1}) is a triangular bounded by characteristic lines $x - y = x_i - y_{j+1}$ and $x + y = x_i + y_{j+1}$ and the segment determination is interval $x_i - y_{j+1} \leq x_i + y_{j+1}$. However from (4.45), the domain of dependence of the difference equation (4.44) with respect to the mesh point $(i, j+1)$ is dependent on the ratio $\rho = \frac{k}{h}$ for a segment of determination $-(j+1)h \leq x \leq (j+1)h$.

When $\rho = 1$ the two domains of dependency coincide. However when $0 < \rho < 1$ the domain of dependency of the difference equation includes that for the differential equation, and the domain of dependency of the difference equation is inside that for the differential equation if $\rho > 1$; see figure 4.5.

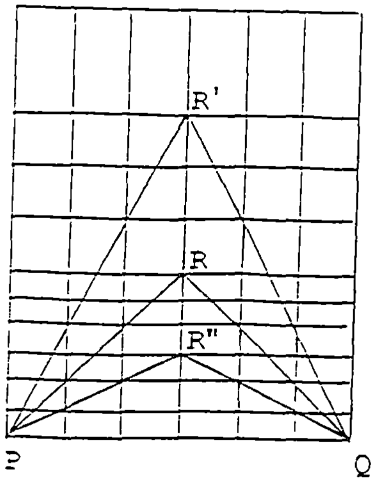


Figure 4.5: The domain of dependence

4.5 The Truncation Error and Stability

The explicit formula (4.45) has the *truncation error*

$$k^2h^2\left[\frac{1}{2}(\rho^2-1)\frac{\partial^4 u}{\partial x^4}+\frac{1}{360}(\rho^2-1)\frac{\partial^6 u}{\partial x^6}+..\right] \tag{4.48}$$

The truncation error vanishes completely when $\rho = 1$.

A truncation error or any other computational error may lead to numerical instability. A difference method is said to be *stable* if the calculation error does not increase by increasing calculation steps. However if the error is cumulative the method is said to be *unstable.* There are several methods for determining the stability criteria of a difference method. One of the most widely used is called *von Neumann stability.*

Von Neumann stability examines the propagation effect of errors along a single row, say the line j^{th}, on the next calculation processes. The errors are represented by finite Fourier series of the form

$$E(x)=\sum_{j}^{N_J}A_je^{\sqrt{-1}\mu_{1_j}x}$$

where N_J is the number of mesh points on the J^{th} line. Because of linearity, it is necessary to consider only a single term $e^{\sqrt{-1}\mu_1 x}$ where μ_1 is any real number and the coefficient A is constant and can be neglected. That is each term grows or decays independently to the others. Each single term is analysed separately and the complete effect is then obtained by linear superimposing.

To investigate the propagation of the error due to a single term $e^{\imath\mu_1 x}$ as y increases, we need to find a solution of the finite difference equation which reduces to $e^{\sqrt{-1}\mu_1 x}$ when $y = 0$. Let such a solution be

$$U(x, y) = e^{\sqrt{-1}\mu_1 x} e^{\mu_2 y} \tag{4.49}$$

where $\mu_2 = \mu_2(\mu_1)$ is, in general, complex. Then the condition to be satisfied such that the error introduced in $e^{\imath\alpha x}$ will not grow as y increases,

$$|e^{\mu_2}| \leq 1 \tag{4.50}$$

for all μ_2. This condition is well known as the *von Neumann condition.*

We will use the von Neumann method to examine the stability of various difference approximations. Let $\varepsilon_{i,j}$ be the difference between the exact solution $u_{i,j}$ and the approximate solution$U_{\imath,j}$ at a particular mess point (i, j), i.e. $\varepsilon_{i,j} = u_{i,j} - U_{i,j}$. Let us now examine the stability of the explicit scheme specified by equation (4.46). Employing the Taylor series into $\varepsilon_{i,j}$ we have

$$\frac{1}{k^2}(\varepsilon_{\imath,j-1} - 2\varepsilon_{i,j} + \varepsilon_{i,j+1}) + O[k^2] = \frac{1}{h^2}(\varepsilon_{\imath-1,j} - 2\varepsilon_{i,j} + \varepsilon_{\imath+1,j}) + O[h^2].$$

Hence the truncation error at a particular mesh point $(i, j+1)$ can be written explicitly as

$$\varepsilon_{\imath,j+1} = \rho^2(\varepsilon_{\imath-1,j} + \varepsilon_{\imath+1,j}) + 2(1-\rho^2)\varepsilon_{i,j} - \varepsilon_{i,j-1} + O[k^4] + O[k^2h^2]. \tag{4.51}$$

where the truncation error is given by (4.48).

Along the initial line $j = 0$, $\varepsilon_{\imath,0} = 0\forall i$, replacing the first derivative at the initial condition gives the truncation error at points along $1st$ line

$$\varepsilon_{\imath,1} = O[k^3]. \tag{4.52}$$

The von Neumann method will be used to investigate the stepwise stability of equation (4.51). According to the von Neumann method, we examine the propagation of a single term, $e^{\sqrt{-1}\mu_1 x}$, along the initial line, so that we have

$$\varepsilon_{i,0} = e^{\sqrt{-1}\mu_1 ih}, \tag{4.53}$$

and along the j^{th} line

$$\varepsilon_{i,j} = e^{\sqrt{-1}\mu_1 ih} e^{\mu_2 jk} \tag{4.54}$$

where $h = \Delta x$ and $k = \Delta y$. Using this, the equation (4.51) gives

$$e^{\mu_2 k} = \rho^2 (e^{\sqrt{-1}\mu_1 h} + e^{-\sqrt{-1}\mu_1 h}) + 2(1 - \rho^2) - e^{\mu_2 k}. \tag{4.55}$$

Since

$$e^{\sqrt{-1}\mu_1 ih} + e^{-\sqrt{-1}\mu_1 ih} = 2cos\mu_1 h \tag{4.56}$$

and

$$(1 - cos\mu_1 h = 2sin^2(\frac{\mu_1 h}{2}) \tag{4.57}$$

then (4.55) becomes a quadratic equation in $e^{\mu_2 k}$,

$$e^{2\mu_2 k} - 2(1 - 2\rho^2 sin^2(\frac{\mu_1 h}{2})e^{\mu_2 k} + 1 = 0 \tag{4.58}$$

which has the solution

$$e^{\mu_2 k} = (1 - 2\rho^2 sin^2(\frac{\mu_1 h}{2}) \pm \frac{1}{2}\sqrt{(2(1 - 2\rho^2 sin^2(\frac{\mu_1 h}{2})^2 - 4}.$$

In order to avoid an increasing error such as $j \rightarrow \infty$ it is necessary that $|e^{\mu_2 k}| \leq 1$ for all real values of μ_1. Hence we have

$$(2(1 - 2\rho^2 sin^2(\frac{\mu_1 h}{2})^2 - 4 \leq 0,$$

which gives

$$\rho^2 \leq \frac{1}{sin^2(\frac{\mu_1 h}{2})} \qquad \forall \mu_1.$$

Hence the condition to be satisfied such that the errors do not increase as j increases is

$$\rho \leq 1. \tag{4.59}$$

4.6 Implicit Methods

So far we have only considered the explicit method, the simplest finite difference methods. Now we will consider the *implicit method*, on which a line, say j^{th}, will contains more than one unknown. However for the sake of simplicity we will only deal with the wave equation (4.43). Replacing u_{yy} by the second order central difference approximation as before and u_{xx} by a linear combination of three second order central difference approximations each centred at $(i, j-1), (i, j)$ and $(i, j+1)$ then we have

$$\begin{aligned}\frac{(u_{i,j-1} - 2u_{i,j} + u_{i,j+1})}{k^2} &= \lambda\frac{(u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1})}{h^2} \\ &+(1-2\lambda)\frac{(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})}{h^2} \\ &+\lambda\frac{(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})}{h^2}\end{aligned} \tag{4.60}$$

for $0 \leq \lambda \leq 1$. Now we have three unknowns at $(j+1)^{th}$ line such as shown in figure 4.6.

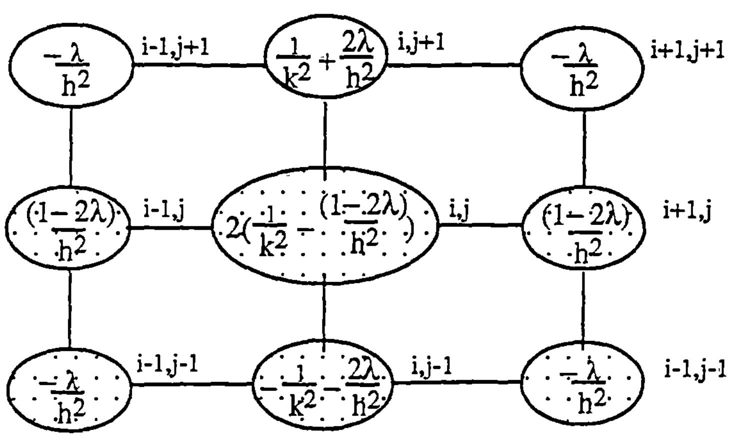


Figure 4.6. Implicit method
: computed values from previous calculations
: values to be computed

Taking $\rho = \frac{k}{h}$, the difference equation (4.61) can be written as

$$\begin{aligned}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) &= \rho^2[\lambda(u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1}) \\ &+(1-2\lambda)(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})\end{aligned}$$

$$+\lambda(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})]. \tag{4.61}$$

Again by employing the Taylor series into $z_{i,j} = U_{i,j} - u_{i,j}$ the truncation error for (4.61) is given by

$$k^2h^2[(\frac{1}{12}(\rho^2-1) - \lambda\rho^2)\frac{\partial^4 u}{\partial x^4} + \frac{1}{360}((\rho^2-1) - 30\lambda\rho^2(\rho^2+1))\frac{\partial^6 u}{\partial x^6} + ...] \tag{4.62}$$

Notice that when $\lambda = 0$, then (4.60) is reduced to the explicit approximation (4.44), while the truncation error (4.62) is reduced to the truncation error (4.48). Since $0 \leq \lambda \leq 1$ then the implicit finite difference approximation (4.60) gives a better approximate solution then that given by explicit finite difference approximation (4.44).

Using the von Neumann method such as for the explicit approximation (4.44), substituting (4.49) into the equation (4.61) gives

$$\begin{aligned} e^{\mu_2 k} - 2 + e^{-\mu_2 k} &= \rho^2[\lambda(e^{\sqrt{-1}\mu_1 h} - 2e^{\mu_2 k} + e^{-\sqrt{-1}\mu_1 h}) \\ &\quad +(1-2\lambda)(e^{\mu_2 k} - 2 + e^{-\mu_2 k}) \\ &\quad +\lambda(\frac{e^{\sqrt{-1}\mu_1 h}}{e^{\mu_2 k}} - \frac{2}{e^{\mu_2 k}} + \frac{e^{-\sqrt{-1}\mu_1 h}}{e^{\mu_2 k}}]. \end{aligned}$$

and using (4.56) and (4.57), we obtain

$$e^{2\mu_2 k} - 2\left(1 - \frac{2\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)}{1 + 4\lambda\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)}\right)e^{\mu_2 k} + 1 = 0 \tag{4.63}$$

or

$$\begin{aligned} e^{\mu_2 k} &= (1 - \frac{2\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)}{1 + 4\lambda\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)}) \\ &\quad \pm\frac{1}{2}\sqrt{2(1 - \frac{2\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)}{1 + 4\lambda\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)})^2 - 4}. \end{aligned}$$

Again in order to avoid an increasing error such as $j \to \infty$ it is necessary that $|e^{\mu_2 k}| \leq 1$ for all real value of μ_1. Hence we have

$$2(1 - 2\frac{2\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)}{1 + 4\lambda\rho^2\sin^2\left(\frac{\mu_1 h}{2}\right)})^2 - 4 \leq 0.$$

In the region of $0 \leq \lambda \leq 1$, μ_2 is real for all real values of μ_1 if

$$\lambda \geq \frac{1}{4}, \qquad \rho > 0 \tag{4.64}$$

or

$$0 < \lambda < \frac{1}{4}, \qquad 0 < \rho < \frac{1}{1-4\lambda}. \tag{4.65}$$

Then in order to prevent an increase in the error such as j, we have to choose λ and ρ such that they satisfy the *unconditional stability* (4.64) or *conditional stability* (4.65). Hence von Neumann stability gives unconditionally/conditionally stable to the implicit method when the central difference approximations in middle time level is greater/less than the sum of the above and lower time level, the conditionally stable satisfied when $0 < \rho < \frac{1}{1-4\lambda}$.

Since the implicit finite difference equation (4.61) contains nine variables with only six of them are known, then it is not suitable for solving the pure Cauchy problem with an unbounded domain. However by using the given values along the boundaries as additional known values, we will have a solvable system of equations.

A particular case when, $\lambda = \frac{1}{4}$, the lower bound for the unconditional stable approximation, the implicit difference equation (4.61) is reduced into

$$\begin{aligned}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) &= \frac{\rho^2}{4}[(u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1}) \\ &\quad +2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \\ &\quad +(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})]\end{aligned}$$

or may be written as

$$\begin{aligned}&-(u_{i+1,j+1} + (u_{i-1,j+1}) + 2(1+\frac{2}{\rho^2})u_{i,j+1} = 2[(u_{i+1,j} + u_{i-1,j}) \\ &-2(1-\frac{2}{\rho^2})u_{i,j}][(u_{i+1,j-1} + u_{i-1,j-1}) - 2(1+\frac{2}{\rho^2})u_{i,j-1}]\end{aligned} \tag{4.66}$$

This finite difference approximation is stable for all $\rho > 0$.

As in the explicit method, we assumed that the boundaries are $x = 0$ and $x = 1$, such that the continuous domain may be replaced by

$$D* = \{(ih, jk)|i = 0, 1, 2, .., n, j = 0, 1, 2....\},$$

where h and k are grid size in the x axis and y axis respectively. Denote by V_j the solution at the interior mesh points, i.e. mesh points other than boundary points, along the j^{th} line. For the sake of simplicity, we assume that the boundary values are zero. Applying the finite difference equation (4.66), a system of equations is generated . The system has the matrix form

$$AV_{j+1} = BV_j + CV_{j-1} \tag{4.67}$$

for $j = 1, 2, 3, ...$ where the matrices A, B and C are

$$A = -D + 4\rho^{-2}I, \tag{4.68}$$

$$B = 2D + 8\rho^{-2}I, \tag{4.69}$$

$$C = D - 4\rho^{-2}I, \tag{4.70}$$

where I is the identity matrix and D is the tridiagonal matrix,

$$\mathbf{D} = \begin{pmatrix} -2(1+\rho^2) & 1 & 0 & . & . & . & 0 \\ 1 & -2(1+\rho^2) & 1 & 0 & . & . & 0 \\ 0 & 1 & -2(1+\rho^2) & 1 & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & 1 & -2(1+\rho^2) \end{pmatrix} \tag{4.71}$$

Using the initial conditions (4.21) we know the solutions at all points along the initial line, $j = 0$, and the first line, $j = 1$, that is we have V_0 and V_1. For each $j = 1, 2, ..$ the right hand side of the equation (4.67) is a constant vector, hence we may rewrite the system of equations (4.67) as

$$-2(1 + \rho^2)U_{1,j} + U_{2,j} = d_{1,j}$$

$$U_{1,j} - 2(1+\rho^2)U_{2,j} + U_{3,j} = d_{2,j}$$

$$U_{2,j} - 2(1+\rho^2)U_{3,j} + U_{4,j} = d_{3,j}$$

....

$$U_{i-1,j} - 2(1+\rho^2)U_{i,j} + U_{i+1,j} = d_{i,j} \qquad i = 2,3,..,n-1$$

....

$$U_{n-3,j} - 2(1+\rho^2)U_{n-2,j} + U_{n-1,j} = d_{1,j}$$

$$-2(1+\rho^2)U_{n-1,j} + U_{n,j} = d_{1,j}. \tag{4.72}$$

To solve the system of equation (4.72) we may use *the tridiagonal algorithm.* Basically the tridiagonal algorithm, by Young [10], is based on the Gaussian elimination process. Transform the system (4.72) into *upper bidiagonal* form, and solve it for $U_{i,j} \qquad \forall i = 2,3,..,n-1$

As particular example, suppose the boundary value problem in example 2.3 has initial conditions

$$u(x,0) = \frac{1}{8}sin\pi x, \;\; u_y = 0, \;\; 0 \leq x \leq 1. \tag{4.73}$$

The method of separation variable gives the analytic solution

$$u(x,0) = \frac{1}{8}sin\pi x cos\pi y, \tag{4.74}$$

which may be depicted in figures 4.7 below

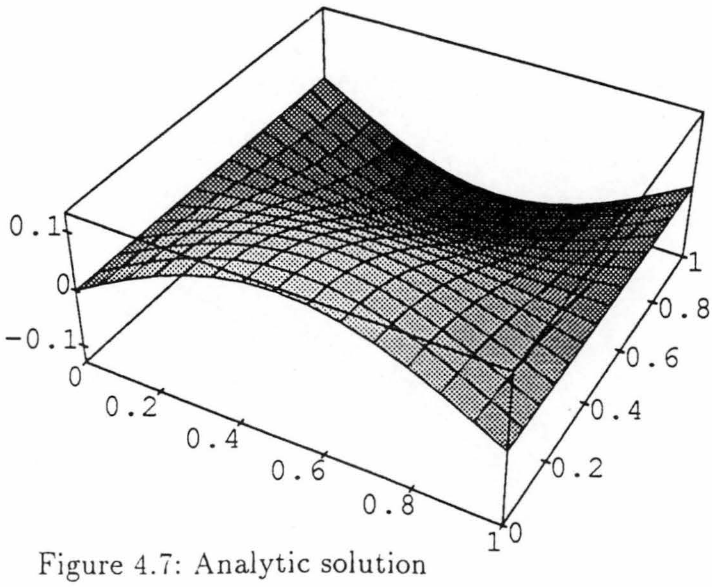


Figure 4.7: Analytic solution

The approximate solution by using the explicit method are depicted by figures 4.9.a and 4.9.b below

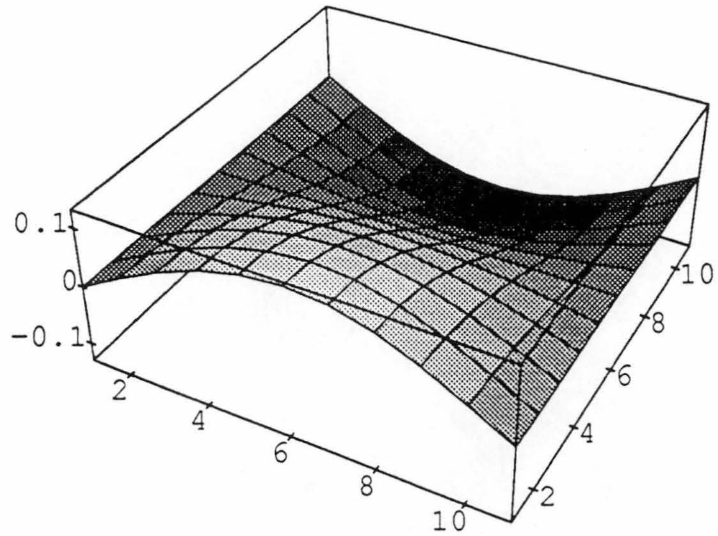


Figure 4.8.aThe solution by the explicit method

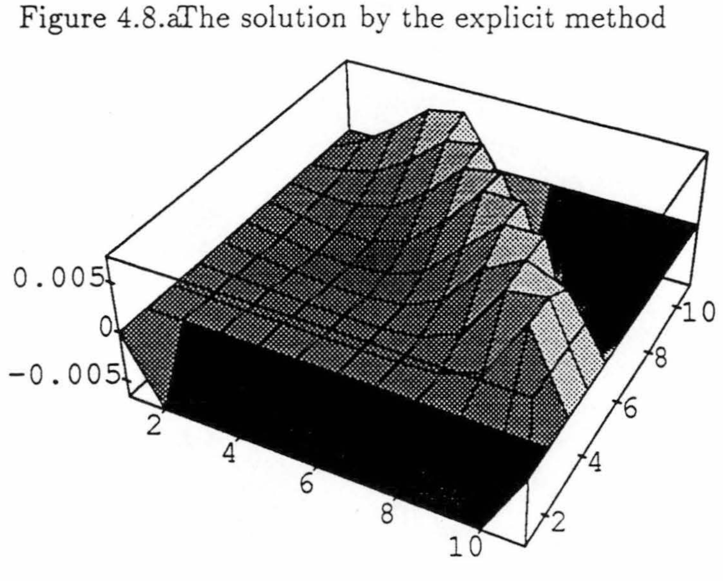


Figure 4.8.bThe error by the explicit method

while of the implicit method is depicted by figures 4.9.a and 4.9.b, respectively

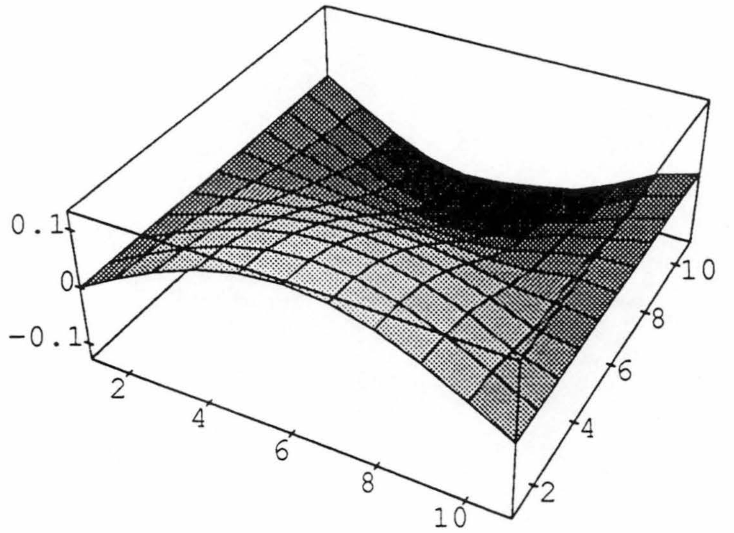
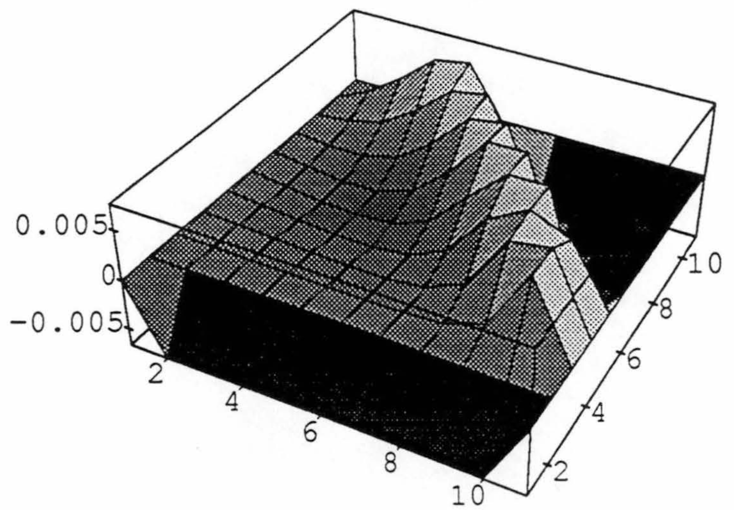


Figure 4.9.aThe solution by the implicit method



and Figure 4.9.bThe error by the implicit method

4.7 Discusion and Conclusion

The finite difference method involves discretization of the continuous domain by rectangular grids, and the governing equations, the initial conditions and the boundary conditions by finite difference equations. Using forward, backward and central difference formulas we may construct some finite difference schemes on which the partial differential equations are replaced by finite difference equations. The replacement of the governing equation by a finite difference equation is not unique. It depends on the configuration of the difference formulas we use.

We get explicit or implicit methods, depending on the choice of nodes. Replacing the second derivative with respect to x by a linear combination of three second order central difference approximations on three time levels will give an implicit. The use of a high-order difference approximation to a differential equation requires the use of an equally high-order approximation to the initial and boundary conditions. However when we deal with boundary value problems, replacing the first derivatives in the boundary conditions by the central difference formula we have to expand or reduce the domain.

The implicit method is not suitable for solving the pure Cauchy problem with an unbounded domain; for the initial value problem, the problem is reduced to solving a system of equations. For instant, when the central difference approximations in middle time level is equal to the sum of the upper and lower time level the problem is reduce to solving the tridiagonal system.

Examining the Cauchy problem for the homogeneous wave equation results that ρ, the proportion of grid sizes Δy to Δx have significance influence into the domains of dependency. When $\rho < 1$ domain of dependency of the difference equation includes that for the differential equation. However the two domains of dependency are coincide when $\rho < 1$,, and the domain of dependency of the difference equation is inside that for the differential equation if $\rho < 1$.

Moreover von Neumann stability gives the condition that the explicit method will stable, the calculation error does not increase by increasing calculation steps, if $\rho < 1$. The implicit finite difference approximation give a better approximate solution then that given by explicit one. Then von Neumann stability gives unconditionally/conditionally stable to the implicit method when the central difference approximations in middle time level is equal or greater/less than the sum of the above and lower time level, the conditionally stable satisfied when $0 < \rho < \frac{1}{1-4\lambda}$.

Chapter 5

CONCLUSION

5.1 Results

Examining the discriminant of a second order linear differential equation, we find that the number of real characteristic directions at a given point is zero, one or two according to whether the equation is elliptic, parabolic or hyperbolic. In the case of hyperbolic type there are two characteristic curves on which will becomes the coordinates system if the differential equation is in the canonical form . This fact suggest that we may integrate directly along the characteristic curves. It is found that Cauchy problem for the hyperbolic equation, the initial curve, on which the initial data are given, cannot be one of characteristic curves.

The values of the solution of the Cauchy problem for the hyperbolic equation at a particular point depend only on the segment of dependence, the segment of the curve intercepted by the characteristic lines through the point, and the value of the given function over the domain of dependence, bounded by the characteristic line and the segment of dependence. Further the solution of the Goursat problem, on which we have some values on a characteristic curve are known, depend on the data along the characteristic and initial curves.

Under the assumption that the hyperbolic equation satisfies the Lipschitz condition, the Cauchy problem for the hyperbolic equation in canonical form is a stable problem, hence methods of successive approximation generates a sequence of approximate solutions which is converges to the exact solution. By introducing the Riemann-Green function, the Riemann's method presents the solution of the Cauchy problem for linear hyperbolic equation, in a manner depending explicitly on prescribed initial conditions.

Using the displacement of a stretched string as a model, the D'Alembert solution appear to be travelling waves. When boundary conditions are taken into account, the solution is composed of two waves continually travelling along it in opposite directions, the whole displacement being the resultant of two waves and their reflections at the end points. The method of separation variable gives a solution in term of a series of standing waves. In the case moving boundary we have discussed only elastic attachment of the boundaries moving normal to the wave front.

By applying a driving forces, we have non-homogeneous wave equations; the Riemann-Volterra solution produces an explicit solution in which it is dependent on the prescribed initial values. When initial conditions are given along x-axis, the Riemann-Volterra solution is reduced to the D'Alembert solution. When we deal with the boundary value problem, we knew that the domain is divided by the characteristic lines into four sub-domain. In the first sub-domain, below the characteristic lines, the problem is reduce into Cauchy problem. In the second and third sub-domains, in the left and right characteristic lines respectively, the problems becomes the Goursat problem. In the fourth sub-domain, above the characteristic lines, the problems becomes the characteristic Goursat problem.

In the case of quasi linear, the Cauchy problem is reduced to the characteristic system. The system involves non-linear ordinary differential equations which should be discretized and calculated simultaneously. Based on the Taylor's series of order one, the straight line method gives approximate solutions with discretization error of order one. Geometrically the method approximates the characteristic curves by straight lines. The predictor-corrector methods produce approximate solutions with discretization error of order two. Geometrically the method approximates the characteristic curves by parabolic arcs. The parabolic arcs are chosen to be straight line through the given point and the calculated approximate point. These methods are well known.

We have proposed the alternative methods, method A and method B, on which we integrate the characteristic curves directly. The improved values are found to be the intersection of the approximate curves. When the approximate curves do dot intersect, the method A approximates the intersection point by the intersection point of their arcs, while the method B by the intersection point of their tangents.

From our experience, for the problems semi-symmetric characteristics, the predictor corrector method, the method A and method B are converge, since at each stage of calculation the characteristic curves remain inside the triangular formed by the tangents of the given points. Generally the predictor corrector method, the method A and method B give better approximate values than of the straight line method. However when the characteristic curves are straight lines, then the method A and method B are reduced into the straight line method. In calculation processes, the method A and method B are faster than the predictor corrector method, since the two former methods use the Runge Kutta (of order 4) method while the later method use the trapezoidal rule which is equivalent to the Runge Kutta of order 2.

The finite difference method involves discretization of the continuous domain by rectangular grids, and the governing equations, the initial conditions and the boundary conditions by finite difference equations. Using forward, backward and central difference formulas we may construct some finite difference schemes on which the partial differential equations are replaced by finite difference equations. The replacement of the governing equation by a finite difference equation is not unique. It depends on the configuration of the difference formulas we use.

On replacing the partial differential equations by finite difference equations, we get explicit or implicit finite difference methods, depending on the choice of nodes. The use of a high-order difference approximation to a differential equation requires the use of an equally high-order approximation to the initial and boundary conditions. However when we deal with boundary value problems, replacing the first derivatives in the boundary conditions by the central difference formula we have to expand or reduce the domain. The implicit methods contains more than one unknowns such that it is not suitable for solving the pure Cauchy problem with an unbounded domain, the boundary values are need. Solving boundary value problem of the homogeneous wave equation by explicit methods is reduced into solving tridiagonal system of equations.

Examining the Cauchy problem for the homogeneous wave equation results that ρ, the proportion of grid sizes Δy to Δx have significance influence into the domains of dependency. When $\rho < 1$ domain of dependency of the difference equation includes that for the differential equation. However the two domains of dependency are coincide when $\rho < 1$,, and the domain of dependency of the difference equation is inside that for the differential equation if $\rho < 1$.

Moreover von Neumann stability gives the condition that the explicit method will stable, the calculation error does not increase by increasing calculation steps, if $\rho < 1$. The implicit finite difference approximation give a better approximate solution then that given by explicit one. Then von Neumann stability gives unconditionally/conditionally stable to the implicit method when the central difference approximations in middle time level is equal or greater/less than the sum of the above and lower time level, the conditionally stable satisfied when $0 < \rho < \frac{1}{1-4\lambda}$.

5.2 Further Developments

In the discussion of moving boundaries we may apply elastic attachment boundaries, moving along the wave front. When velocity of the boundary is greater then the velocity of the string we will have a *supersonic* wave, and *sub-*

Chapter 6

APPENDICES

The algorithms below are derived for solving Cauchy /boundary-value problems of the quasi linear second order (3.1). Implementation in computer programs may vary. We have been used Think Pascal, to solve the second order quasi-linear partial differential equations. So far, it work well, as long as the equations are hyperbolic throughout the domain.Reader who interested in our software, may have it for free.

6.1 Algorithms of Straight Line Method

6.1.1 Cauchy problems

Algorithm 3.1: The straight line method for solving Cauchy problems.
Suppose we have already approximate values at the point

$$(x_{i,j}, y_{i,j}), \qquad (x_{i+1,j}, y_{i+1,j}).$$

To calculate the approximate values at the point $(x_{i+1,j+1}, y_{i+1,j+1})$, we carry out the following procedures:

1. Compute the x-coordinate of the point by

$$x_{i+1,j+1} = \frac{y_{i+1,j} - y_{i,j} + \alpha_{i,j} x_{i,j} - \beta_{i+1,j} x_{i+1,j}}{\alpha_{i,j} - \beta_{i,j}} \quad . \tag{6.1}$$

and the y-coordinate

$$y_{i+1,j+1} = y_{i,j} + \alpha_{i,j}(x_{i+1,j+1} - x_{i,j}). \tag{6.2}$$

The y-coordinate may also be calculated by

$$y_{i+1,j+1} = y_{i+1,j} + \beta_{i+1,j}(x_{i+1,j+1} - x_{i+1,j}). \tag{6.3}$$

2. Compute the first derivatives of u: u_x by

$$\begin{aligned} U_{x_{i+1,j+1}} &= [a_{i,j}c_{i+1,j}\alpha_{i,j}U_{x_{i,j}} - a_{i+1,j}c_{i,j}\beta_{i+1,j}U_{x_{i+1,j}} + c_{i,j}c_{i+1,j}(U_{y_{i,j}} - U_{y_{i+1,j}}) \\ &\quad +(c_{i+1,j}e_{i,j} - c_{i,j}e_{i+1,j})y_{i,j+1} + c_{i+1,j}e_{i,j}y_{i,j} - c_{i,j}e_{i+1,j}y_{i+1,j}] \\ &\quad /[a_{i,j}c_{i+1,j}\alpha_{i,j} - a_{i+1,j}c_{i,j}\beta_{i+1,j}], \end{aligned} \tag{6.4}$$

and u_y

$$U_{y_{i+1,j+1}} = U_{y_{i,j}} - \frac{1}{c_{i,j}}[a_{i,j}\alpha_{i,j}(U_{x_{i+1,j+1}} - U_{x_{i,j}}) - e_{i,j}(y_{i+1,j+1} - y_{i,j})], \tag{6.5}$$

or may also by

$$U_{y_{i+1,j+1}} = U_{y_{i+1,j}} - \frac{1}{c_{i+1,j}}[a_{i+1,j}\beta_{i+1,j}(U_{x_{i+1,j+1}} - U_{x_{i+1,j}}) - e_{i,j}(y_{i+1,j+1} - y_{i+1,j})] \tag{6.6}$$

3. Compute the first approximate value of u by

$$\begin{aligned} U_{i+1,j+1} &= U_{i,j} + \frac{1}{2}(U_{x_{i+1,j+1}} + U_{x_{i,j}})(x_{i+1,j+1} - x_{i,j}) \\ &\quad +\frac{1}{2}(U_{y_{i+1,j+1}} + U_{y_{i,j}})(y_{i+1,j+1} - y_{i,j}), \end{aligned} \tag{6.7}$$

or by

$$\begin{aligned} U_{i+1,j+1} &= U_{i+1,j} + \frac{1}{2}(U_{x_{i+1,j+1}} + U_{x_{i+1,j}})(x_{i+1,j+1} - x_{i+1,j}) \\ &\quad +\frac{1}{2}(U_{y_{i+1,j+1}} + U_{y_{i+1,j}})(y_{i+1,j+1} - y_{i+1,j}). \end{aligned} \tag{6.8}$$

4. Substitute the calculated approximate values

$$x_{i+1,j+1}, y_{i+1,j+1}, U_{i+1,j+1}, U_{x_{i+1,j+1}}, U_{x_{i+1,j+1}}$$

into the given coeficient functions

$$a(x, y, u, u_x, u_y), b(x, y, u, u_x, u_y), c(x, y, u, u_x, u_y), g(x, y, u, u_x, u_y).$$

5. Compute the α characteristic direction by

$$\alpha_{i+1,j+1} = \frac{1}{a_{i+1,j+1}}(b_{i+1,j+1} + \sqrt{b_{i+1,j+1}^2 - a_{i+1,j+1}c_{i+1,j+1}}) \tag{6.9}$$

and β characteristic direction by

$$\beta_{i+1,j+1} = \frac{1}{a_{i+1,j+1}}(b_{i+1,j+1} - \sqrt{b_{i+1,j+1}^2 - a_{i+1,j+1}c_{i+1,j+1}}). \tag{6.10}$$

6. Repeat the procedure 1, 2, 3 for i=j(1)n.
7. Repeat the procedure 1, 2, 3 for j=1(1)n-1.

6.1.2 Baundary Value problems

Algorithm 3.2.: The straight line method for solving boundary value problems. Using prescribed values at the point $(x_{i,j}, y_{i,j})$ and $(x_{i+1,j}, y_{i+1,j})$ for $i = 0(1)n$,
1. Say $(x_{0,j+1}, y_{0,j+1})$, for each j, is the coordinates of the point at the left boundary. $x_{0,j+1} = x_{0,0}$ for all j, and $y_{0,j+1}$ is computed by (3.17).
The first approximate values $U_{0,j+1}$ and $U_{x_{0,j+1}}$ are found by substituting $(x_{0,j+1}, y_{0,j+1})$ into (3.19), while $U_{y_{0,j+1}} = 0 \ \forall j$.
Substitute the calculated approximate values $x_{0,j+1}, y_{0,j+1}, U_{0,j+1}, U_{x_{0,j+1}}, U_{x_{0,j+1}}$ and $U_{x_{0,j+1}}$ into the given coefficient functions

$$a(x, y, u, u_x, u_y), b(x, y, u, u_x, u_y), c(x, y, u, u_x, u_y), g(x, y, u, u_x, u_y).$$

Compute the α characteristic direction by (6.9) when $i = -1$.

2. Compute the coordinates of the non boundary point $(x_{i+1,j+1}, y_{i+1,j+1})$ for $i = 0(1)n - 2$ by (6.1) and (6.2).
Compute the first derivatives of u a by (6.4) and (6.5).
Compute the approximate value of u by (6.7) or (6.8).

Substitute the calculated approximate values $x_{i+1,j+1}, y_{i+1,j+1}, U_{i+1,j+1}, U_{x_{i+1,j+1}}$ and $U_{x_{i+1,j+1}}$ into the given coefficient functions

$$a(x, y, u, u_x, u_y), b(x, y, u, u_x, u_y), c(x, y, u, u_x, u_y), g(x, y, u, u_x, u_y).$$

Compute the α and β characteristic directions by (6.9) and (6.10) respectively.

3. Say $(x_{n,j+1}, y_{n,j+1})$, for each j, is the coordinates of point at the right boundary. $x_{n,j+1} = x_{n,0}$ for all j, and $y_{n,j+1}$ is computed by (3.18).
The first approximate values $U_{n,j+1}$ and $U_{x_{n,j+1}}$ are found by substituting $(x_{n,j+1}, y_{n,j+1})$ into (3.20), while $U_{y_{n,j+1}} = 0 \forall j$.
Substitute the calculated approximate values $x_{n,j+1}, y_{n,j+1}, U_{n,j+1}, U_{x_{n,j+1}}$ and $U_{x_{n,j+1}}$ into the given coefficient functions

$$a(x, y, u, u_x, u_y), b(x, y, u, u_x, u_y), c(x, y, u, u_x, u_y), g(x, y, u, u_x, u_y).$$

Compute the β characteristic direction by (6.10) when $i = n - 1$.

4. Repeat the procedure 1,2,3,4 for j=1,2,3.... as desired.

6.2 Algorithms of Predictor Corrector Method

6.2.1 Cauchy problems

Algorithm 3.3: Predictor corrector method for solving Cauchy problems of the quasi linear second order (3.1).

1. Using straight line method we calculate the first approximate values of

$$x, y, u, U_x, U_y, a, b, c, g, \alpha, \beta$$

at the point $(x_{i+1,j+1}, y_{i+1,j+1})$. Use this set of values as the initial predictor, that we have

$$x^{(0)}_{i+1,j+1}, y^{(0)}_{i+1,j+1}, U^{(0)}_{i+1,j+1}, U^{(0)}_{x_{i+1,j+1}}, a^{(0)}_{i+1,j+1}, b^{(0)}_{i+1,j+1}, c^{(0)}_{i+1,j+1}, e^{(0)}_{i+1,j+1},$$

$\alpha^{(0)}_{i+1,j+1}$ and $\beta^{(0)}_{i+1,j+1}$.

2. A correction to the approximate value of coordinates of the point $(x_{i+1,j+1}, y_{i+1,j+1})$ may be calculated by

$$
\begin{aligned}
x^{(k+1)}_{i+1,j+1} &= [y_{i+1,j} - y_{i,j} + \frac{1}{2}(\alpha_{i,j} + \alpha^{(k)}_{i+1,j+1})x_{i,j} \\
&\quad -\frac{1}{2}(\beta_{i+1,j} + \beta^{(k)}_{i+1,j+1})x_{i+1,j}] \\
&\quad /[\frac{1}{2}(\alpha_{i,j} + \alpha^{(k)}_{i+1,j+1}) - \frac{1}{2}(\beta_{i+1,j} + \beta^{(k)}_{i+1,j+1})]
\end{aligned} \tag{6.11}
$$

and

$$
y^{(k+1)}_{i+1,j+1} = y_{i,j} + \frac{1}{2}(\alpha_{i,j} + \alpha^{(k)}_{i+1,j+1})(x^{(k+1)}_{i+1,j+1} - x_{i,j}) \tag{6.12}
$$

or

$$
y^{(k+1)}_{i+1,j+1} = y_{i+1,j} + \frac{1}{2}(\beta_{i+1,j} + \beta^{(k)}_{i+1,j+1})(x^{(k+1)}_{i+1,j+1} - x_{i+1,j}). \tag{6.13}
$$

3. Furthermore a correction to approximate values to the first partial derivatives, can be calculated by

$$
\begin{aligned}
U^{(k+1)}_{x_{i+1,j+1}} &= [(a^{(k)}_{i+1,j+1}\alpha^{(k)}_{i+1,j+1} + a_{i,j}\alpha_{i,j})(c^{(k)}_{i+1,j+1} + c_{i+1,j})U_{x_{i,j}} \\
&\quad -(a^{(k)}_{i+1,j+1}\beta^{(k)}_{i+1,j+1} + a_{i+1,j}\beta_{i+1,j})(c^{(k)}_{i+1,j+1} + c_{i,j})U_{x_{i+1,j}} \\
&\quad +(c^{(k)}_{i+1,j+1} + c_{i,j})(c^{(k)}_{i+1,j+1} + c_{i+1,j})(U_{y_{i,j}} - U_{y_{i+1,j}}) \\
&\quad +(c^{(k)}_{i+1,j+1} + c_{i+1,j})(e^{(k)}_{i+1,j+1} + e_{i,j})y^{(k+1)}_{i+1,j+1} \\
&\quad -(c^{(k)}_{i+1,j+1} + c_{i,j})(e^{(k)}_{i+1,j+1} + e_{i+1,j})y^{(k+1)}_{i+1,j+1} \\
&\quad +(c^{(k)}_{i+1,j+1} + c_{i+1,j})(e^{(k)}_{i+1,j+1} + e_{i,j})y_{i,j} \\
&\quad -(c^{(k)}_{i+1,j+1} + c_{i,j})(e^{(k)}_{i+1,j+1} + e_{i+1,j})y_{i+1,j}]/ \\
&\quad [(a^{(k)}_{i+1,j+1}\alpha^{(k)}_{i+1,j+1} + a_{i,j}\alpha_{i,j})(c^{(k)}_{i+1,j+1} + c_{i+1,j}) \\
&\quad -(a^{(k)}_{i+1,j+1}\beta^{(k)}_{i+1,j+1}) + a_{i+1,j}\beta_{i+1,j})(c^{(k)}_{i+1,j+1} + c_{i,j})],
\end{aligned} \tag{6.14}
$$

and

$$
\begin{aligned}
U^{(k+1)}_{y_{i+1,j+1}} &= U_{y_{i,j}} - \frac{1}{(c^{(k)}_{i+1,j+1} + c_{i,j})}[[(a^{(k)}_{i+1,j+1}\alpha^{(k)}_{i+1,j+1} + a_{i,j}\alpha_{i,j}) \\
&\quad (U^{(k+1)}_{x_{i+1,j+1}} - U_{x_{i,j}})] - \frac{1}{2}(e^{(k)}_{i+1,j+1} + e_{i,j})(y^{(k+1)}_{i+1,j+1} - y_{i,j})],
\end{aligned} \tag{6.15}
$$

or

$$U_{y_{i+1,j+1}}^{(k+1)} = U_{y_{i+1,j}} - \frac{1}{(c_{i+1,j+1}^{(k)} + c_{i+1,j})}[[(a_{i+1,j+1}^{(k)}\beta_{i+1,j+1}^{(k)} + a_{i+1,j}\beta_{i+1,j})$$
$$(U_{x_{i+1,j+1}}^{(k+1)} - U_{x_{i+1,j}})] - \frac{1}{2}(e_{i+1,j+1}^{(k)} + e_{i,j})(y_{i+1,j+1}^{(k+1)} - y_{i+1,j})]. \tag{6.16}$$

4. Substitute the calculated approximate values

$$x_{i+1,j+1}^{(k+1)}, y_{i+1,j+1}^{(k+1)}, U_{i+1,j+1}^{(k+1)}, U_{x_{i+1,j+1}}^{(k+1)}, U_{x_{i+1,j+1}}^{(k+1)}$$

into the given coefficient functions

$$a(x, y, u, u_x, u_y), b(x, y, u, u_x, u_y), c(x, y, u, u_x, u_y), e(x, y, u, u_x, u_y).$$

5. Compute the α characteristic direction at the point $(x_{i+1,j+1}, y_{i+1,j+1})$ by

$$\alpha_{i+1,j+1}^{(k+1)} = \frac{1}{a_{i+1,j+1}}^{(k+1)}(b_{i+1,j+1}^{(k+1)} + \sqrt{(b_{i+1,j+1}^{(k+1)})^2 - a_{i+1,j+1}^{(k+1)}c_{i+1,j+1}^{(k+1)}}) \tag{6.17}$$

and β characteristic direction at the point $(x_{i+1,j+1}, y_{i+1,j+1})$ by

$$\beta_{i+1,j+1}^{(k+1)} = \frac{1}{a_{i+1,j+1}^{(k+1)}}(b_{i+1,j+1}^{(k+1)} - \sqrt{(b_{i+1,j+1}^{(k+1)})^2 - a_{i+1,j+1}^{(k+1)}c_{i+1,j+1}^{(k+1)}}). \tag{6.18}$$

6. Repeat the procedure 1, 2, 3 ,4 and 5, and iterate on k until

$$\frac{|y_{i+1,j+1}^{(k+1)} - y_{i+1,j+1}^{(k)}|}{y_{i+1,j+1}^{(k)}} < \varepsilon, \tag{6.19}$$

and

$$\frac{|x_{i+1,j+1}^{(k+1)} - x_{i+1,j+1}^{(k)}|}{x_{i+1,j+1}^{(k)}} < \varepsilon \tag{6.20}$$

for a given error ε.

7. Repeat the procedure 1, 2, 3, 4, 5 and 6 for i=j(1)n.

8. Repeat the procedure 1, 2, 3, 4, 5, 6 and 7 for j=1(1)n-1.

6.2.2 Baundary Value problems

To construct a characteristic grid net and the solution of boundary value problem, we calculate the values at the non boundary points by the algorithm 3.3, and the values at the boundary points by procedures similar to those of the algorithm 3.2. The procedures are described by the flowchart in figure A1 next page.

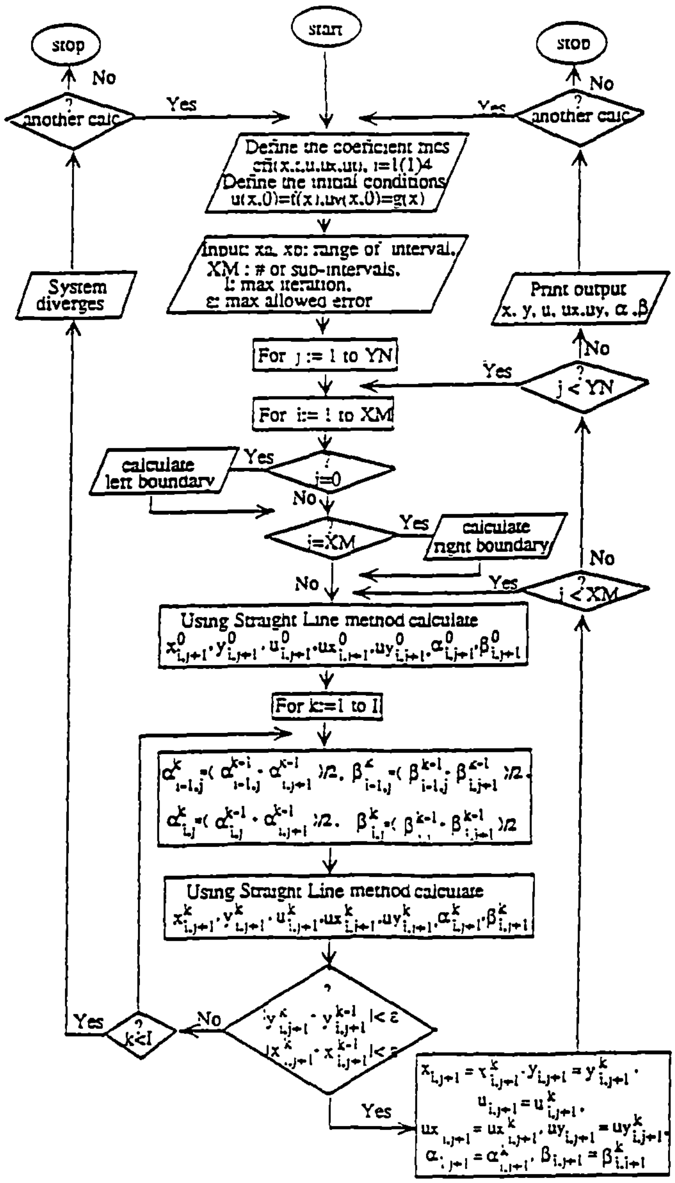


Figure A.1Predictor corrector method

6.3 Algorithms of Method A

6.3.1 Cauchy problems

Algorithm 3.4: Method of A for solving a Cauchy problem.

1.a Using straight line method we calculate the first approximate values of

$$x, y, u, U_x, U_y, a, b, c, g, \alpha, \beta$$

at the point $(x_{i+1,j+1}, y_{i+1,j+1})$. Using this set of approximate values as initial prediction, we have

$$x^{(0)}_{i+1,j+1}, y^{(0)}_{i+1,j+1}, U^{(0)}_{i+1,j+1}, U^{(0)}_{x,i+1,j+1}, a^{(0)}_{i+1,j+1}, b^{(0)}_{i+1,j+1}, c^{(0)}_{i+1,j+1}, e^{(0)}_{i+1,j+1},$$

$\alpha^{(0)}_{i+1,j+1}$ and $\beta^{(0)}_{i+1,j+1}$.

1.b Suppose for integers $k = 1, 2..$, the k^{th} and $(k-1)^{th}$ iterations give

$$\left|x^{(k)}_{i+1,j+1} - x^{(k+1)}_{i+1,j+1}\right| > \varepsilon, \tag{6.21}$$

or

$$\left|y^{(k)}_{i+1,j+1} - y^{(k+1)}_{i+1,j+1}\right| > \varepsilon, \tag{6.22}$$

and applying Runge Kutta integration gives

$$\left|y_\alpha(x^{(k)}_{i+1,j+1}) - y_\beta(x^{(k)}_{i+1,j+1})\right| > \varepsilon, \tag{6.23}$$

for an allowed error ε.

2.a Caclculate $y_\alpha(x^{(k+1)}_{i+1,j+1})$ and $y_\beta(x^{(k+1)}_{i+1,j+1}$ by (3.63) and (3.65) respectivelly.

2.b If

$$\left|y_\alpha(x^{(k+1)}_{i+1,j+1}) - y_\beta(x^{(k+1)}_{i+1,j+1})\right| < \varepsilon, \tag{6.24}$$

then a correction to the approximate values of coordinates of the point $(x_{i+1,j+1}, y_{i+1,j+1})$ may be calculated by

$$x_{i+1,j+1} = x^{(k)}_{i+1}, \quad y_{i+1,j+1} = \frac{1}{2}(y_\alpha(x^{(k+1)}_{i+1}) + y_\beta(x^{(k+1)}_{i+1})). \tag{6.25}$$

2.c If

$$\left|y_\alpha(x_{i+1,j+1}^{(k+1)}) - y_\beta(x_{i+1,j+1}^{(k+1)})\right| > \varepsilon, \tag{6.26}$$

then calculate the approximate value $(x_{i+1,j+1}^{k+1}, y_{i+1,j+1}^{k+1})$ by solving the equations

$$\frac{y_{i+1,j+1}^{k+1} - y_\alpha(x_{i+1,j+1}^{(k)})}{y_{i,j} - y_\alpha(x_{i+1,j+1}^{(k)})} = \frac{x_{i+1,j+1}^{k+1} - x_{i+1,j+1}^{(k)}}{x_{i,j} - x_{i+1,j+1}^{(k)}} \tag{6.27}$$

and

$$\frac{y_{i+1,j+1}^{k+1} - y_\beta(x_{i+1,j+1}^{(k)})}{y_{i+1,j} - y_\beta(x_{i+1,j+1}^{(k)})} = \frac{x_{i+1,j+1}^{k+1} - x_{i+1,j+1}^{(k)}}{x_{i+1,j} - x_{i+1,j+1}^{(k)}}. \tag{6.28}$$

2.d If the set of conditions

$$\left|x_{i+1,j+1}^{(k+1)} - x_{i+1,j+1}^{(k)}\right| < \varepsilon, \quad \left|y_{i+1,j+1}^{(k+1)} - y_{i+1,j+1}^{(k)}\right| < \varepsilon \tag{6.29}$$

are satisfied then a correction to the approximate values of coordinates of the point $(x_{i+1,j+1}, y_{i+1,j+1})$ is calculated by

$$x_{i+1,j+1} = \frac{1}{2}(x_{i+1,j+1}^{(k+1)} + x_{i+1,j+1}^{(k)}), \quad y_{i+1,j+1} = \frac{1}{2}(y_{i+1,j+1}^{(k+1)} + y_{i+1,j+1}^{(k)}). \tag{6.30}$$

3. Compute the first derivatives of u by (3.27) and (3.28).

Compute the first approximate value of u by (3.30) or (3.31).

Substitute the calculated approximate values $x_{n,j+1}, y_{n,j+1}, U_{n,j+1}, U_{x_{n,j+1}}$ and $U_{x_{n,j+1}}$ into the given coefficient functions

$$a(x, y, u, u_x, u_y), b(x, y, u, u_x, u_y), c(x, y, u, u_x, u_y), e(x, y, u, u_x, u_y).$$

4. Compute the α and β characteristic directions by (6.9) and (6.10) respectivelly.

5. Repeat the procedures 2.a, 2.b, 2.c, 3 and 4, iterating on k until the condition (6.24) or the set of conditions (6.29)is satisfied. 6. Repeat the procedure 5 for i=j(1)n.

7. Repeat the procedure 6 for j=1(1)n-1.

6.3.2 Baundary Value problems

To construct a characteristic grid net and the solution of boundary value problem, we calculate the values at the non boundary points by the algorithm 3.4, and the values at the boundary points by procedures similar to those of the algorithm 3.2. The procedures are described by the flowchart in figure A2 below

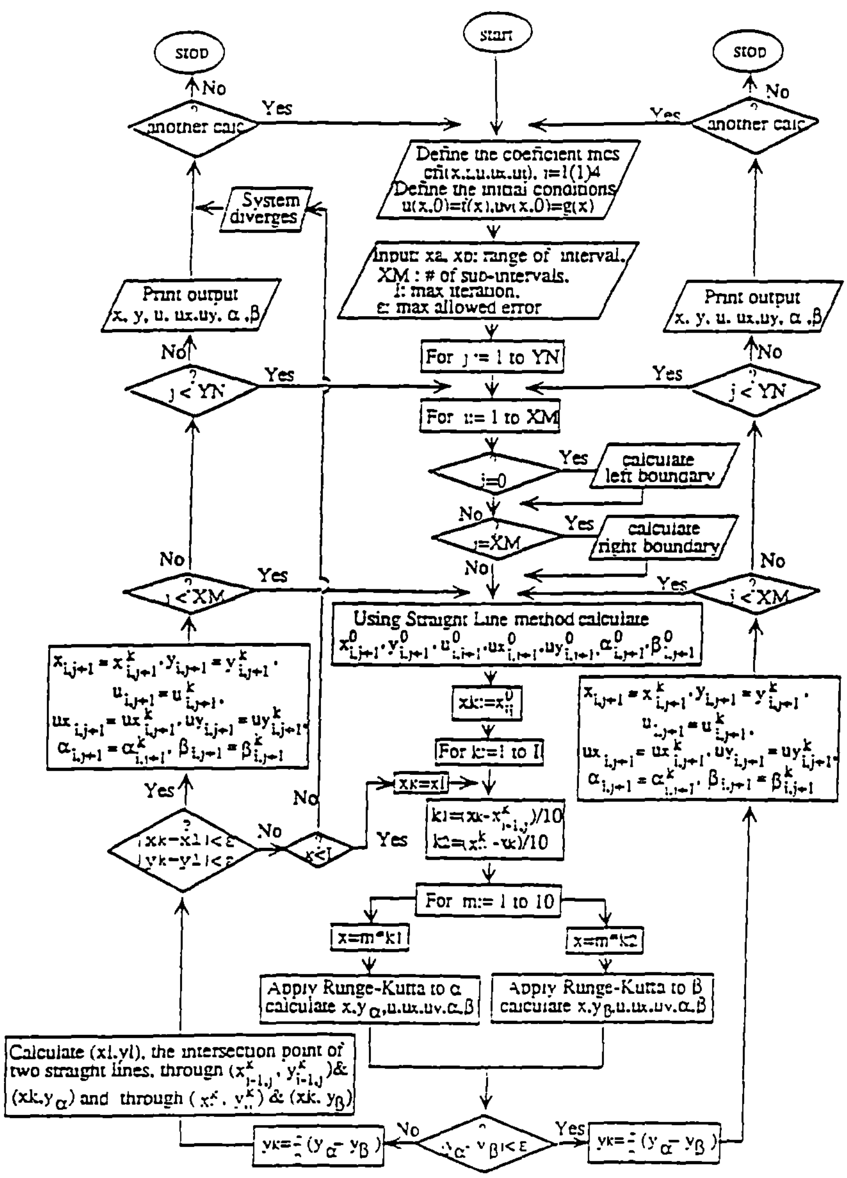


Figure A.2Method A

6.4 Algorithms of Method B

On replacing (6.27), (6.27) by

$$y^{k+1}_{i+1,j+1} - y_{\alpha}(x^{(k)}_{i+1,j+1}) = \alpha\left(R_{\beta})(x^{k+1}_{i+1,j+1} - x^{(k)}_{i+1,j+1}\right), \tag{6.31}$$

$$y^{k+1}_{i+1,j+1} - y_{\beta}(x^{(k)}_{i+1,j+1}) = \beta\left(R_{\beta})(x^{k+1}_{i+1,j+1} - x^{(k)}_{i+1,j+1}\right) \tag{6.32}$$

respectivel, we have procedures for method B which are described by the flowchart in figure A3 below

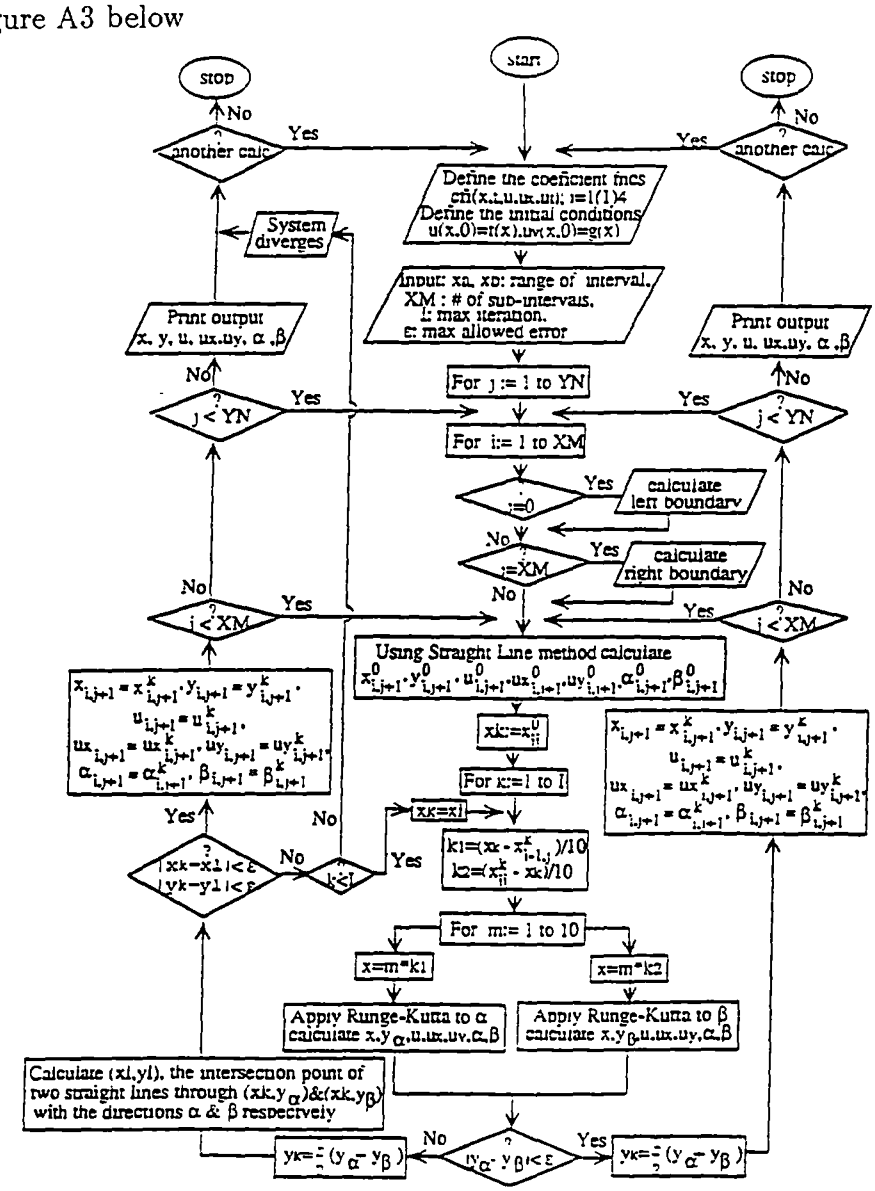


Fig.A.1: Method B

Bibliography

[1] Bleistein, Norman *Mathematical Methods for Wave Phenomena.* Academic Press, Inc. : Florida, 1984.

[2] Lapidus, Leon. *Numerical Solutions of Partial Differential Equations in Science and Engineering.* John Wiley and Sons,Inc.: New York, 1982.

[3] Lick, W.J. *Difference Equations from Differential Equations.* Sringer Verlag, 1989.

[4] Ames, William F. *Numerical Methods For Partial Differential Equation.* Academic Press,Inc.: New York, 1977.

[5] Farlow, S.J. *Partial Differential Equations for Scientists and Engineers.* John Wiley and Sons, Inc., 1982.

[6] Street, Robert L. *The Analysis and Solutions of Partial Differential Equations.* Wadsworth Publishing Company,Inc.: California, 1973.

[7] Chester, Clive R. *Technique in Partial Differential Equation and Boundary Value Problems.* McGraw-Hill,Inc.: New York, 1971.

[8] Dennemeyer, Rene. *Introduction to Partial Differential Equation and Boundary Value Problems.* McGraw-Hill,Inc.: New York, 1968.

[9] Garabedian, Paul R. *Partial Differential Equation.* John Wiley and Sons,Inc.: New York, 1967.

[10] Smith, G.D. *Numerical Solution of Partial Differential Equations : Finite Difference Methods.* Oxford University Press : Oxford, 1978.

[11] Laurie, D.P. *Numerical Solution of Partial Differential Equations: Theory, Tools and Case Studies.* Birkhauser Verlag : Basel. Boston. Stuttgart, 1983.

[12] Ferziger, Joel H. *Numerical Methods for Engineering Application.* John Wiley and Sons, Inc., 1981.

[13] Crank, John. *Free and Moving Boundary problems.* Clarendon Press : Oxford, 1984.

[14] Mitchell, A.R. *Computational Methods in Partial Differential Equations.* John Wiley and Sons. , 1969.

[15] Gladwell, I., Wait, R. *A Survey of Numerical Methods for Partial Differential Equations.* Clarendon Press : Oxford, 1979

[16] Crandal, Stephen H.,Ph.D. *Engineering Analysis : A Survey of Numerical Procedures.* McGraw-Hill Book Company, 1956.

[17] Sneddon, Ian N. *Elements of Partial Differential Equations.* J.McGraw-Hill Company, 1957.