ERRATA

Page:	line:	
(ii)	4	"Huat Lim" should be "Ming-Huat".
(iii)	8	"spaned" should be "spanned".
(iii)	8, 9	The first A should be $\overline{A}$.
(iv)	-3 & -4	There are commas missing after x_1 and before x_n.
1	4	A period is missing at the end of the line.
8	-7	Three dots are missing between "$x_1<$" and "$<x_n$".
19	6	"1-1" should be "i-1".
26	2	"semimodularity", should be "semimodular".
34	2	"$\dot{U}$" should be "U".
39	-2	An "F" is missing here.
45	3	"in the associated bijection $2^S \to 2^{S'}$" should be added .
46	6	Delete the period at the end of the line.
47	15	add ", and $\overline{a} = \{a\}$, $\forall\ a \in S$".
50	4	"and $\overline{\phi} = \phi$." should be added.
58	1	"contraction" should be "restriction".
72	3	The bracket should close after "finite".
83	9	"(X_I)" should be "$(X)_I$".
148	3	"." should be ",".
157	-10	"Although" should be "although".
169	11	"Murtey" should be "Murty".

Combinatorial Geometry

by

(Mrs)
Waree Karot, M.Sc. (Chula)

Submitted in fulfilment
of the requirements for the degree of
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Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any university, and to the best of the author's knowledge and belief, contains no copy or paraphrase of material previously published or written by another person, except when due reference is made in the text of the thesis.

Waree karot.

SUMMARY

We systematically give alternative characterisations of pregeometries, and examine their properties.

We examine well - known classes of pregeometries using the above characterisations.

In particular we (i) define "product" of pregoemetries, related to that given by Lim, and (ii) give some applications of this "product" .

CONTENTS

INTRODUCTION

Crapo and Rota [70] defined a pregeometry G(S) as a set S together with a closure function on its subsets. They gave various characterisations of pregeometries in terms of their ranks, independent sets, circuits and lattice of flats. Roberts [73] characterised any pregeometry in terms of its flats. A basis characterisation was given by Welsh [76] . He also gave a characterisation in terms of hyperplanes, for any pregeometry on a finite set S (we will use the term matroid for such a pregeometry). All these characterisations are derived in Chapter 1.

Basic properties of pregeometries are discussed in Chapter 2. Arising from a pregeometry G(S) a subpregeometry, $G_S(T)$ is induced on any subset T of S. Any pregeometry has special subpregeometries - called canonical geometries. They are isomorphic. Other pregeometries obtained from G(S) are contractions and duals (when G(S) = M(S) is a matroid).

In Chapter 3 we discuss the pregeometry G(S) obtainable from a family $(X)_I$ of subsets of sets - a transversal pregeometry in which the independent sets of G(S) are the partial transversals of $(X)_I$. Finally we investigate systems of distinct representatives giving the same transversal of such families.

Representable pregeometries, isomorphic to subpregeometries of finite dimensional vector spaces, are investigated in Chapter 4 .

In Chapter 5 we discuss the class of matroids arising from directed graphs - strict gammoids - together with their

SUMMARY

We systematically give alternative characterisations of pregeometries, and examine their properties.

We examine well - known classes of pregeometries using the above characterisations.

In particular we (i) define "product" of pregoemetries, related to that given by Lim, and (ii) give some applications of this "product" .

restrictions - gammoids. Also base orderable matroids are introduced and discussed.

In Chapter 6 we construct pregeometry products based on the work of Ming Huat Lim [77] and we apply these constructions to matroids defined on groups in which the geometric and algebraic structures are related. More precisely, group multiplication is a geometric automorphism.

The methods used in Chapter 1 unless otherwise stated in the text are based on lectures given to Honours students in Mathematics at the University of Tasmania. Similarly the methods of Chapter 2 (all but the last half of section 1), 3 section 1, 4 and 5 are based mainly on those of Welsh [76] , Crapo and Rota [70] , Mirsky [71] and Row [77] unless otherwise stated.

Some examples given in these chapters are original in particular those dealing with Steiner Triple Systems.

Section 3.2 dealing with multiplicity of system of transversals is new.

In Chapter 6, section 1, 3 and 4 are new - while section 2 comes from Lim [77] .

I would like to express my deep gratitude to Dr. D.H. Row for his assistance during the preparation of this thesis. I would like also thank Mrs. W. Gayong for her careful and patient typing.

NOTATION

A set $\{x,y,\ldots\}$ is often written $xy\ldots$. We use standard notation in set theory and algebra. Apart rom these we use :

A^C	complement of A (in the appropriate universal set).
$\bar{A}$	closure of A in a pregeometry
$A_f \subset\subset A$	A_f is a finite subset of A
$A \cup B$	union of disjoint sets A and B
A	subspace spaned by A
A	closure of A in a subpregeometry
$A_1 \Delta \ldots \Delta A_n$	symmetric difference of sets $A_1,\ldots,A_n$
$C(x, B)$	fundamental circuit of x in the basis B
$Cl(A)$	closure of A in a pregeometry
$G(S)$	pregeometry on S
$G_S(T)$	subpregeometry on T induced by G(S)
$G(S).T$	contraction of G(S) to a subset T of S
$G_1(S_1) \vee G_2(S_2)$	union of pregeometries $G_1(S_1)$ and $G_2(S_2)$
$G_1(S_1) \oplus G_2(S_2)$	direct sum of pregeometries $G_1(S_1)$ and $G_2(S_2)$
inf M	infimum of set M
L(G)	lattice of flats of a pregeometry G(S)
$M[\mathcal{A}]$	transversal matroid with a presentation $\mathcal{A}$

$M(S)$	matroid on S
$M(S)/T$	restriction of $M(S)$ to T
$M^*(S)$	dual matroid of $M(S)$
$r(A)$	rank of set A
$\mathcal{S}_n$	Steiner triple systems on a set of n elements
$\sup M$	supremum of set M
$U_{k,n}$	k - uniform geometry on n elements
(V,E)	directed graph with vertex set V and edge set E
$(v_o,v_1,\ldots,v_k)$	path in directed graph with initial vertex v_o and terminal vertex v_k
$x_1 v \ldots v\, x_n$	supremum of $x_1 \ldots x_n$
$x_1 \wedge \ldots \wedge x_n$	infimum of $x_1 \ldots x_n$
$x_o < x_1 < \ldots$	chain in a poset
$(X)_I, (X_i/i \in I)$	family of subsets of X with index I

1. EQUIVALENT CHARACTERISATIONS OF PREGEOMETRIES

1.1 CLOSURE

We begin with a definition of pregeometries in terms of closure

1.1.1 A *pregeometry*, $G(S)$, is a set S together with a closure $f : 2^S \to 2^S$ satisfying the following four conditions.

(C_1) For all $A \subsetneq S$, $A \subsetneq \bar{A}$; writing $\bar{A}$ for $f(A)$;

(C_2) If $A \subseteq \bar{B}$, then $\bar{A} \subseteq \bar{B}$, $\forall A,B \subsetneq S$.

(C_3) If $a \notin \bar{A}$ and $a \in \overline{A \cup b}$, then $b \in \overline{A \cup a}$, $\forall A \subsetneq S$, $a,b \in S$.

(C_4) For all $A \subseteq S$, $\exists\, A_f \subset\subset A$ with $\bar{A}_f = \bar{A}$.

(C_3) and (C_4) are the *exchange property* and *finite basis property* respectively.

1.1.2 LEMMA. If (C_1) is given, then (C_2) is equivalent to

(i) For all $A \subsetneq S$, $\bar{\bar{A}} = \bar{A}$.

(ii) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$, $\forall A,B \subseteq S$.

PROOF. Assume that (C_1) and (C_2) are given. Then $\bar{A} \subsetneq \bar{A} \Rightarrow \bar{\bar{A}} \subsetneq \bar{A}$ by (C_2). On the other hand $\bar{A} \subsetneq \bar{\bar{A}}$ by (C_1) so that $\bar{\bar{A}} = \bar{A}$. Now $A \subsetneq B \Rightarrow A \subsetneq B \subseteq \bar{B} \Rightarrow \bar{A} \subseteq \bar{B}$ by (C_2).

Assume that (C_1), and $\bar{\bar{A}} = \bar{A}$, $\forall A \subsetneq S$ and $A \subseteq B \Rightarrow \bar{A} \subsetneq \bar{B}$, $A,B \subseteq S$ are given. Then $A \subsetneq \bar{B} \Rightarrow \bar{A} \subsetneq \bar{\bar{B}} = \bar{B}$ which is as desired. //

1.1.3 A *geometry* is a pregeometry $G(S)$ satisfying the two additional properties:

(C_5) $\bar{\phi} = \phi$

(C_6) $\bar{\bar{a}} = a, \forall a \in S$.

1.1.4 EXAMPLE. The smallest coset of any subspace of a finite dimensional vector space V, containing a set A of vectors of V defines a closure $\bar{A}$ satisfying (C_1) - (C_4).

PROOF. The smallest coset of any subspace containing A is of the form $a_0 + W$, where W is the (unique) smallest subspace containing $A - a_0$, for any $a_0 \in A$.

We show that the closure defined satisfies (C_3) and (C_4).

Let $x \in \overline{A \cup y}$ and $x \notin \bar{A}$, $A \subseteq S$, $x,y \in S$. Then $x \in a + W$, where $W = [A \cup y - a]$ for some $a \in A$ so that $x - a = \sum_{i=1}^{n} c_i (a_i - a)$, where $a_i \in A \cup y$, $c_i \neq 0$. There exists j with $a_j = y$ (otherwise $x \in \bar{A}$). Putting $a_j - a = y - a$ on the left hand side we write

$$y - a = \sum_{i=1}^{m} d_i b_i, \text{ where } b_i \in (A \cup x) - a,\ d_i \neq 0.$$

Then $y \in \overline{A \cup x}$.

For any $A \subsetneq S$ consider a maximal independent subset A_{f_0} of $A - a_0$ for some $a_0 \in A$. Let $A_f = a_0 + A_{f_0}$. Then A_f is a finite subset of A and $\bar{A}_f = a_0 + [A_{f_0}] = \bar{A}$ as $[A_{f_0}]$ is the smallest subspace containing $A - a_0$ //

1.1.5 LEMMA. The conditions (C_1) - (C_6) are independent.

PROOF. We see this by examining the following six examples in each of which exactly one of (C_1) - (C_6) is not satisfied.

(i) Let $S = \{1,2,3\}$, $\bar{\phi} = \phi$, $\bar{1} = 1$, $\bar{2} = 2$, $\bar{3} = 3$, $\overline{12} = 1$, $\overline{13} = 13$, $\overline{23} = 23$, $\bar{S} = S$.

Then only (C_1)fails.

(ii) Let $S = \{1,2,3,4\}$, $\overline{12} = 1234$, $\overline{14} = 124$, $\overline{13} = 123$, $\overline{23} = 123$, $\overline{24} = 124$, $\bar{A} = A$ otherwise.

Then only (C_2) fails.

(iii) Let $S = \{1,2,3,4\}$. Define closure on S by

$$\bar{A} = \begin{cases} 123 & \text{if } A = 12 \\ A & \text{otherwise} \end{cases}$$

Then only (C_3) fails.

(iv) Let S be an infinite set and define $\bar{A} = A$, $\forall A \subsetneq S$.

Then only (C_4) fails.

(v) Let $S = \{1,2,3\}$ and define closure on S by

$$\bar{A} = \begin{cases} 1 & \text{if } A = \phi \\ A & \text{otherwise} \end{cases}$$

Then only (C_5) fails.

(vi) Let $S = \{1,2,3\}$ and define closure on S by

$$\bar{A} = \begin{cases} 12 & \text{if } A = 1 \\ A & \text{otherwise} \end{cases}$$

Then only (C_6) fails. //

1.1.6 A subset X in a pregeometry G(S) is *closed* or a *flat* if $X = \bar{B}$ for some $B \subsetneq S$.

1.1.7 LEMMA. In any pregeometry G(S). The following are true.

(i) A is a flat in G(S) if and only if $A = \bar{A}$.

(ii) Any intersection of flats is a flat.

(iii) $\bar{A}$ is the intersection of all flats containing A. That is $\bar{A}$ is the smallest flat containing A.

(iv) S is a flat in G(S).

(v) $B \subseteq \bar{A}$ if and only if $\bar{A} = \overline{A \cup B}$, $A, B \subseteq S$.

(vi) $\overline{\{a/a \in \bar{A}\}} = \bigcup_{a \in A} \bar{a}$, $\forall A \subseteq S$.

PROOF. (i) Let A be a flat in G(S). By definition there exists $B \subseteq S$ such that $A = \bar{B}$. Thus $\bar{A} = \bar{\bar{B}} = \bar{B} = A$. The converse is obvious.

(ii) Given any intersection, $\bigcap A_i$, of flats of G(S). Put $A = \bigcap A_i$. It suffices to show that $\bar{A} \subseteq A$. Since $A \subseteq A_i$ for all i which implies $\bar{A} \subseteq \bar{A}_i = A_i$ for all i, we have $\bar{A} \subseteq \bigcap A_i = A$.

(iii) Let $B = \bigcap A_i$, where A_i is a flat containing A. Then by (ii) B is a flat containing A. Therefore $\bar{A} \subseteq \bar{B} = B$. Since $\bar{A}$ is a flat containing A, $\bar{A} = A_i$ for some i and hence $B \subseteq \bar{A}$.

(iv) follows from (i).

(v) Assume that $B \subseteq \bar{A}$. By (C_1) $A \subseteq \bar{A}$ so that $A \cup B \subseteq \bar{A}$ and hence by (C_2) $\overline{A \cup B} \subseteq \bar{A}$. On the other hand $\bar{A} \subseteq \overline{A \cup B}$. Thus $\bar{A} = \overline{A \cup B}$.

Suppose that $\overline{A \cup B} = \bar{A}$. Let $x \in B$. Then $\overline{A \cup x} \subsetneq \overline{A \cup B} = \bar{A}$ and so $x \in \bar{A}$. Therefore $B \subseteq \bar{A}$.

(vi) follows from (v). //

1.1.8 A *Boolean geometry* is a pregeometry G(S) with $\bar{A} = A$, $\forall A \subseteq S$.

When $\bar{A} = A$ if $|A| < k$ and $\bar{A} = S$ otherwise provided $k \geq 1$

defines a k - *uniform geometry* on S.

1.2 LATTICES OF FLATS

We characterise any pregeometry in terms of a lattice of flats.

1.2.1 A *poset* is a set L together with a binary relation $\leq$ satisfying the following.

(i) For any $x \in L$, $x \leq x$, reflexive property;

(ii) If $x \leq y$ and $y \leq x$, then $x = y$, $\forall x, y \in L$, antisymmetric property;

(iii) If $x \leq y$ and $y \leq z$, then $x \leq z$, $\forall x,y,z \in L$, transitive property.

In a poset we write $x < y$ (or $y > x$) to mean $x \leq y$ and $x \neq y$.

A *greatest (least) element* of a subset M of L is an element x of M such that $x \geq (\leq)\ m$, $\forall m \in M$.

If x_1 and x_2 are greatest elements of M, then $x_1 \leq x_2$ and $x_2 \leq x_1$ so that by the antisymmetric property $x_1 = x_2$. Thus the greatest element is unique if existing. Also the least element is unique if existing. We denote the greatest (least) element of L if existing by 1 (0) .

A *lower (upper) bound* of M is an element y of L with $y \leq (\geq)$ m, $\forall m \in M$. The *infimum (supremum*) of M, written inf M (sup M), is the greatest (least) element of the set of lower (upper) bounds of M (if existing).

1.2.2 LEMMA. $\sup (M_1 \cup M_2) = \sup \{\sup M_1, \sup M_2\}$, provided the right hand side exists .

PROOF. Suppose that M_1, M_2 are subsets of a poset $(L, \leq)$, with $x_1 = \sup M_1$, $x_2 = \sup M_2$. If $x = \sup \{x_1, x_2\}$, then $x \geq x_1$ and $x \geq x_2$ so that x is an upper bound of $M_1 \cup M_2$.

For any upper bound x' of $M_1 \cup M_2$ we have $x' \geq x_1$ and $x' \geq x_2$ so that x' is an upper bound of $\{x_1, x_2\}$ and so $x' \geq x$. Thus the lemma is proved. //

1.2.3 A *lattice* is a poset $(L, \leq)$ with every pair of elements having a supremum and infimum.

For convenience in notation we write $x \wedge y$ and $x \vee y$ for $\inf \{x,y\}$ and $\sup \{x,y\}$ respectively, where $\wedge$ and $\vee$ are read "meet" and "join".

By an induction argument we see that the infimum and supremum of finite subsets exist in any lattice.

1.2.4 LEMMA. The set of flats, $L(G)$, of a pregeometry $G(S)$ is a lattice with respect to set inclusion. In this lattice

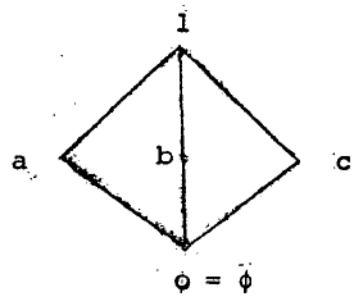
$$A \vee B = \overline{A \cup B} \quad , \quad A \wedge B = A \cap B \quad , \quad \forall A, B \in L(G).$$

As S is a flat and $\phi \subsetneq A$, $\forall A \subsetneq S$, S and $\bar{\phi}$ are the elements 1 and 0 respectively in $L(G)$.

1.2.5 We say y *covers* x in a lattice $(L, \leq)$ iff $x < y$ and there is no z in L with $x < z < y$.

A finite lattice can be conveniently represented by a Hasse diagram in which distinct elements are represented by distinct points so that x is above y iff $x > y$ and x, y are joined by a straight line whenever x covers y. We illustrate by

EXAMPLE 1.2.6



the lattice of flats of a 2 - uniform geometry on abc.

We characterise a lattice in terms of $\wedge$ and $\vee$.

1.2.7 THEOREM. A lattice $(L,\leq)$ is characterised by

(L_1) For every $x \in L$, $x \wedge x = x$ and $x \vee x = x$.

(L_2) For every $x,y \in L$, $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(L_3) For every $x,y,z \in L$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.

(L_4) For every $x,y \in L$, $x \wedge (y \vee x) = x$ and $x \vee (y \quad x) = x$.

PROOF. That a lattice $(L,\leq)$ satisfies (L_1) - (L_3) is immediate.

We show that $(L,\leq)$ satisfies (L_4). Let $x,y \in L$. Let $z = y \vee x$. Then $x \leq z$ and since $x \leq x$, x is a lower bound of $\{x,z\}$. For any lower bound a of $\{x,z\}$ we have $a \leq x$. Thus $x = \inf \{x,z\}$ as desired. Let $p = y \wedge x$. Then $x \geq p$ so that p is an upper bound of $\{x,p\}$. For any upper bound d of $\{x,p\}$ we have $d \geq x$. Hence $x = \sup \{x,p\}$ as desired.

We show that a given set L with $x \wedge y$, $x \quad y$ defined for every pair x,y in L satisfying (L_1) - (L_4) is a lattice.

We define $x \leq y$ when $x \vee y = y$. Then $x \leq y \Rightarrow x \vee y = y \Rightarrow x \wedge (x \vee y) = x \wedge y \Rightarrow x = x \wedge y$. Also $x = x \wedge y \Rightarrow x \vee y = (x \wedge y) \vee y \Rightarrow x \vee y = y \Rightarrow x \leq y$

(i) Since $x \vee x = x$, $x \leq x$, $\forall x \in L$.

(ii) Let $x \leq y$ and $y \leq x$. Then $x \vee y = y$ and $x = y \vee x$ so that $x = y$.

(iii) Let $x \leq y$ and $y \leq z$. Then $x \vee y = y$ and $y \vee z = z$ so that $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$. Hence $x \leq z$.

Then $(L, \leq)$ is a poset.

For any $x,y \in L$ we show that $\inf\{x,y\} = x \wedge y$ and $\sup\{x,y\} = x \quad y$.

Since $(x \wedge y) \vee x = x$ and $(x \wedge y) \vee y = y$, we have $x \wedge y \leq x$ and $x \wedge y \leq y$ and so $\{x,y\}$ has at least one lower bound. Let b be any lower bound of $\{x,y\}$. Then $b \wedge x = b$ and $b \wedge y = b$ so that $b \wedge (x \wedge y) = (b \wedge x) \wedge y = b \wedge y = b$. Thus $b \leq x \wedge y$ and so $x \wedge y = \inf\{x,y\}$.

Similarly we can show that $\sup\{x,y\} = x \vee y$. //

1.2.8 A *chain* in a poset is a subset with the induced order on it linear, it is *finite* if the subset is finite.

We write $x_0 < x_1 < \ldots$ to denote a chain. Given a finite chain $C : x_0 < x_1 < \quad < x_n = y$ we say that C is a *chain from x to y* with *length* $|C| - 1$.

1.2.9 LEMMA. Every chain in the lattice of flats of any pregeometry is finite.

PROOF. Suppose that C is an infinite chain in the lattice of flats of $G(S)$. Then C must contain an infinite ascending chain or an infinite descending chain.

First assume that there exists an infinite ascending chain. Choose a countably infinite subchain $C_1 : A_0 \subsetneqq A_1 \subsetneqq A_2 \subsetneqq \ldots$. Let $A = \bigcup_i A_i$. Then A is a flat and by (C_4) $\exists\, A_f \subset \not\subset A$ with $\bar{A}_f = \bar{A} = A$. Since A_f is finite and each $a_i \in A_f$ is contained in A_i, there exists n such that $A_f \subseteq A_n$. It then follows that $A = \bar{A} = \bar{A}_f \subseteq \bar{A}_n = A_n$. Thus $A_{n+m} \subseteq A \subseteq A_n$, $\forall_m$. This contradicts the fact that C_1 is infinite. Thus no ascending chain is infinite.

Next assume that C contains an infinite descending chain. Choose a countably infinite subchain C_2: $A_0 \supsetneqq A_1 \supsetneqq A_2 \supsetneqq \ldots$. For each i let $a_i \in A_{i-1} \setminus A_i$ and $T_{i+1} = \{a_{i+1}, a_{i+2}, \ldots\}$. Now $T_{i+1} \subsetneqq A_i$ and since $a_i \notin A_i$ and $\bar{T}_{i+1} \subseteq A_i$, we have $a_i \notin \bar{T}_{i+1}$. Consider for each i the set $B_i = \{a_j \,/\, j \in N\} \setminus a_i$. If there exists $a_i \in \bar{B}_i$ we then choose a maximum j such that $a_i \in \overline{\{a_j, \ldots, a_i, a_{i+1}, \ldots\}}$, where $1 \le j \le i - 1$ as $T_{i+1} \subseteq \{a_j, \ldots, a_{i-1}, a_{i+1}, \ldots\} \subseteq \bar{B}_i$. Put $B = \{a_{j+1}, \ldots, a_{i-1}, a_{i+1}, \ldots\}$. Then $a_i \notin \bar{B}$, but $a_i \in \overline{B \cup a_j}$ so that by (C_3) $a_j \in \overline{B \cup a_i} = \bar{T}_{j+1}$. A contradiction. Hence $a_i \notin \bar{B}_i$ for all i. This means that no proper subset of T_1 has closure $\bar{T}_1$ since $a_i \in \bar{T}_1$ for all i and any proper subset of T_1 which does not contain a_i is contained in B_i, contradicting the finite basis property for $\{ a_j / j \in j \in N \}$.

Thus the lemma is proved. //

1.2.10 LEMMA. For any flats A, B, C of G(S) we have

(i) A covers B <=> $A = \overline{B \cup a}$ for some $a \in A \setminus B$.

(ii) A covers B => $A \vee C$ covers $B \vee C$ or $A \vee C = B \vee C$.

PROOF. (i) Assume that A covers B. Then $\exists\ a \in A \setminus B$ and hence $B \subsetneqq \overline{B \cup a}$ so that $\overline{B \cup a} = A$.

Assume $A = \overline{B \cup a}$, for some $a \in A \setminus B$, where A, B are flats of G(S). Then $B \subsetneqq A$. Let X be any flat such that $B \subsetneqq X \subsetneqq \overline{B \cup a}$. We show that $\overline{B \cup a} \subseteq X$. Pick an element $b \in X \setminus B$. Then $b \notin \bar{B} = B$ and $b \in \overline{B \cup a}$ so that by (C_3), $a \in \overline{B \cup b} \subseteq X$. Thus $\overline{B \cup a} \subseteq X$. Therefore A covers B.

(ii) Let A cover B. Then $A = \overline{B \cup a}$ for some $a \in A \setminus B$. If $a \notin C$, then by (i) $\overline{A \cup C} = \overline{B \cup C \cup a}$ so that $A \vee C$ covers $B \vee C$. If $a \in C$ we have $\overline{A \cup C} = \overline{B \cup a \cup C} = \overline{B \cup C}$ and so $A \vee C = B \vee C$. //

1.2.11 A lattice $(L, \leq)$ is *semimodular* if it has no infinite chain and whenever x,y cover $x \wedge y$ we have $x \vee y$ covering x and y.

1.2.12 LEMMA. The lattice of flats of G(S) is semimodular.

PROOF. Follows from Lemma 1.2.10.

1.2.13 An *atom* is an element in any lattice that covers 0.

1.2.14 A *geometric lattice* is a semimodular lattice in which every element is a supremum of atoms.

1.2.15 LEMMA. Let $(L, \leq)$ be a geometric lattice. Then any x,y in L satisfy the following.

(i) Any two maximal chain from x to y have same length.

(ii) y covers x <=> $y = x \vee a$ for some atom $a \nleq x$.

PROOF. (i) We prove this by induction on chain length.

Let $x = s_0 < s_1 < \ldots < s_n = y$ be a maximal chain from x to y. Consider another maximal chain $x = t_0 < t_1 < \ldots < t_m = y$ from x to y. We can assume $s_1 \neq t_1$. Now s_1, t_1 both cover x and so $s_1 \wedge t_1 \geq x$. If $x < s_1 \wedge t_1$ and since either $s_1 \wedge t_1 < s_1$ or $s_1 \wedge t_1 < t_1$ is the case then either s_1 or t_1 does not cover x. Thus $x = s_1 \wedge t_1$. Semimodularity implies that $s_1 \vee t_1$ covers s_1 and t_1. As $s_1 \vee t_1 \leq y$ a chain from $s_1 \vee t_1$ to y exists. Let $s_1 \vee t_1 = u_1 < u_2 < \ldots < u_p = y$ be maximal. Then $s_1 < s_1 \vee t_1 < u_2 < \ldots < u_p = y$ and $t_1 < s_1 \vee t_1 < u_2 < \ldots < u_p = y$ are maximal chains from s_1 and t_1 respectively to y.

case 1. If $n = 2$, then since y covers s_1 we have $p = 0$ so that $s_1 \vee t_1 = y$. Thus $x < t_1 < s_1 \vee t_1 = y$ is maximal as required.

case 2. If $n > 2$. Assume that the lemma holds for any maximal chain of length $< n$. As $s_1 < s_2 < \ldots < s_n = y$ is a maximal chain from s_1 to y of length $n - 1$ the chain $s_1 < s_1 \vee t_1 < u_2 < \ldots < u_p = y$ has length $n - 1$. Since $t_1 < s_1 \vee t_1 < u_2 < \ldots < u_p = y$ is maximal, by the assumption it has length $n - 1$. Therefore the chain $t_0 < t_1 < \ldots < t_m = y$ has length n.

(ii) Assume y covers x. Since $(L, \leq)$ is geometric, $y = \sup A$ for some subset A of atoms of L. If $\forall a \in A$, $a \leq x$, then $y = \sup A \leq x$. A contradiction. Thus $\exists\, a \in A$ with $a \nleq x$. But $x \vee a > x$ (as $x \vee a = x \iff a \leq x$). Hence $x \vee a = y$.

Now let $y = x \vee a$, where a is an atom such that $a \nleq x$. We consider the following two cases.

case 1. If the maximal chain length from 0 to x is 1. Thus x is an atom so that by the above $x \wedge a = 0$. By semimodularity $x \vee a$

covers x and a as desired.

case 2. If the maximal chain length from 0 to x is $n > 1$. Assume that the lemma holds for any x with maximal chain length from 0 less than n. Let $0 < \ldots < x' < x$ be maximal. As x covers x' we have $x = x' \vee b$ for some atom $b \nleq x$. If $x' \vee b = x' \vee a$, then $x = x' \vee b = x' \vee a$ so that $a \leq x$. A contradiction. Thus $x' \vee b \neq x' \vee a$. and both cover x' and so $x' = (x' \vee a) \wedge (x' \vee b)$. By semimodularity $(x' \vee a) \vee (x' \vee b)$ covers both $x' \vee a$ and $x' \vee b = x$. Hence $y = x \vee a = (x \vee x') \vee a = x \vee (x' \vee a) = (x' \vee b) \vee (x' \vee a)$ which covers x. //

1.2.16 Two lattices $(L_1, \leq)$ and $(L_2, \leq)$ are *isomorphic* if there exists a bijection $f: L_1 \to L_2$ such that for every pair x,y in L_1 $f(x \wedge y)$ and $f(x \vee y)$ are meet and join of $f(x)$ and $f(y)$ respectively in L_2.

We call f an *isomorphism* from $(L_1, \leq)$ to $(L_2, \leq)$.

We now characterise pregeometries by these properties of their lattices of flats.

1.2.17 THEOREM. The lattice of flats of any pregeometry is geometric. Conversely any geometric lattice is isomorphic to the lattice of flats of some pregeometry.

PROOF. Let L (G) be the lattice of flats of G(S). We need to show that every flat in L(G) is a join of atoms. Let $B \in L(G)$. If B covers $\bar{\phi}$ there is nothing to prove. Consider a maximal chain $\phi < B_1 < \ldots < B_n = B$ of length $n > 1$ from $\bar{\phi}$ to B. Now as B_1 covers $\bar{\phi}$ we have $B_1 = \overline{\phi \cup a_1}$, where $a_1 \in B_1 \setminus \bar{\phi}$ so that $\bar{a}_1$ is an atom and $B_1 = \bar{\phi} \vee \bar{a}_1 = \bar{a}_1$. Inductively $B_i = B_{i-1} \vee \bar{a}_i$ for some $a_i \in B_i \setminus B_{i-1}$, where $i = 2, \ldots, n-1$. Thus $a_i \notin \bar{\phi}$ and $\bar{a}_i$ is an atom which implies

$B = B_n = B_{n-1} \vee \bar{a}_n = \ldots = \bar{a}_1 \vee \ldots \vee \bar{a}_n$, as required.

Conversely let $(L, \leq)$ be any geometric lattice. Consider the set S of atoms of $(L, \leq)$ and define closure on 2^S as follows.

$$\bar{A} = \{ a \in S \,/\, a \leq \sup A \}, \; A \subseteq S.$$

We show that the closure defined satisfies (C_1) - (C_4).

(C_1) : Let $A \subseteq S$. As $a \in A$, $a \leq \sup A$ so that $a \in \bar{A}$. Hence $A \subseteq \bar{A}$.

(C_2) : Let $A \subseteq \bar{B}$, where $A, B \subseteq S$. For every $x \in \bar{B}$ we have $x \leq \sup B$ so that $\sup \bar{B} \leq \sup B$. But $\sup B \leq \sup \bar{B}$. Thus $\sup B = \sup \bar{B}$. If $a \leq \sup A$, then as $\sup A \leq \sup \bar{B} = \sup B$ we have $a \leq \sup D$. That is $\bar{A} \subseteq \bar{B}$.

(C_3) : Let $a \in \overline{A \cup b}$ and $a \notin \bar{A}$, where $A \subseteq S$, $a, b \in S$. Then $a \in \overline{A \cup b} \Rightarrow a \leq \sup (A \cup b) \Rightarrow a \leq \sup \{\sup A, \sup b\} \Rightarrow a \leq \sup \{\sup A, b\} \Rightarrow a \leq \sup A \vee b \Rightarrow \sup A \vee a \leq \sup A \vee b \Rightarrow \sup A < \sup A \vee a \leq \sup A \vee b$ (as $a \notin \bar{A}$). Then $b \nleq \sup A$ (otherwise $a \leq \sup A \vee b = \sup A$) so that by Lemma 1.2.15 $\sup A \vee b$ covers $\sup A$ and hence $\sup A \vee a = \sup A \vee b$. It then follows that $\overline{A \cup a} = \overline{A \cup b}$ and $b \in \overline{A \cup a}$.

(C_4) : Let $A \subseteq S$. Well order A by $\{ a_1, a_2, \ldots \}$ and define inductively the set

$$b_r = \sup \{ a_1, a_2, \ldots , a_r \} .$$

Then the chain $b_1 \leq b_2 \leq b_3 \ldots$ is finite. Thus there exists n such that $b_{n+m} = b_n$ for all m. Therefore $\sup A = \sup \{ a_1, \ldots , a_n \}$ and hence $\bar{A} = \overline{\{ a_1, \ldots , a_n \}}$.

Thus the closure defines a pregeometry $G(S)$ on S.

To show that the lattice of flats of $G(S)$ is isomorphic to $(L, \leq)$ we define the function $f : L(G) \to L$ by $f(A) = \sup A$, $\forall A \in L(G)$.

Let $A_1 \neq A_2$ be flats of $G(S)$. If $\sup A_1 = \sup A_2$, then $A_1 = \bar{A}_1 = \bar{A}_2 = A_2$. Thus $f(A_1) = \sup A_1 \neq \sup A_2 = f(A_2)$ so that f is one to one.

Since $(L, \leq)$ is geometric, $x = \sup B$ for some $B \subseteq S$, where $x \in L$. Then $\bar{B} \in L(G)$ and $f(\bar{B}) = \sup \bar{B} = \sup B = x$. Hence f is onto.

To show that f preserves meet and join.

Let $A, B \in L(G)$. Then $A \cap B \subseteq A$, $A \cap B \subseteq B \Rightarrow \sup (A \cap B) \leq \sup A$ and $\sup (A \cap B) \leq \sup B \Rightarrow \sup (A \cap B) \leq (\sup A) \wedge (\sup B)$. Put $x = \sup (A \cap B)$ and $y = (\sup A) \wedge (\sup B)$. Let $Y = \{a \in S \,/\, a \leq y\}$. Then $y \geq \sup Y$. Since $(L, \leq)$ is geometric, $y = b_1 \vee b_2 \ldots \vee b_n$, where b_i's are atoms. Then $b_i \in Y$ for all i and so $y = \sup \{b_1, \ldots, b_n\} \leq \sup Y$. Therefore $y = \sup Y$. Now $a \in Y \Rightarrow a \leq \sup A \Rightarrow a \in \bar{A} = A$ so that $Y \subseteq A$. Similarly $Y \subseteq B$. Hence $Y \subseteq A \cup B$ and $y = \sup Y \leq \sup (A \cup B) = x$ and so $f\ (a \wedge B) = f(A) \wedge f(B)$.

Now $f(A \vee B) = \sup (\overline{A \cup B}) = \sup (A \cup B) = \sup \{\sup A, \sup B\} = (\sup A) \vee (\sup B) = f(A) \vee f(B)$ and the theorem is proved. //

1.2.18 A *hyperplane* in $G(S)$ is a flat covered by S in $L(G)$.

Thus no hyperplane properly contains another and so an intersection of distinct hyperplanes is not a hyperplane.

Before we close this section we prove a useful result.

1.2.19 LEMMA. Every flat is the intersection of all hyperplanes containing it.

PROOF. First we show that for any $A \subseteq T$ in $L(G)$ and for any X

with $A \subsetneq X \subseteq T$, $\exists\ Y \in L(G)$ such that $X \cap Y = A$ and $\overline{X \cup Y} = T$.

Since $\exists\ Y_i$ s.t. $A \subseteq Y_i \subsetneq T$ and $X \cap Y_i = A$, if $\overline{X \cup Y_i} \neq T$, let $b \in T \setminus \overline{X \cup Y_i}$ (as $\overline{X \cup Y_i}$ T). Let $Y_{i+1} = \overline{Y_i \cup b}$. Then $\overline{X \cup Y_{i+1}}$ $\subseteq T$ and $\overline{X \cup Y_{i+1}}$ covers $\overline{X \cup Y_i}$. It follows that $X \cap Y_{i+1} = A$ (otherwise $\exists\ c \in X \cap Y_{i+1} \setminus A$ so that $Y_i \subset \overline{Y_i \cup c} \subsetneq \overline{Y_{i+1} \cup c} = Y_{i+1}$. Since Y_{i+1} covers Y_i, $\overline{Y_i \cup c} = Y_{i+1}$. Then $\overline{X \cup Y_i} = \overline{\bar{c} \cup X \cup Y_i}$ = $\overline{X \cup (\bar{c} \cup Y_i)} = \overline{X \cup Y_{i+1}}$. A contradiction). If $\overline{X \cup Y_{i+1}} \neq T$ we then construct Y_{i+2} such that $X \cap Y_{i+2} = A$, $\overline{X \cup Y_{i+2}}$ T and $\overline{X \cup Y_{i+2}}$ covers $\overline{X \cup Y_{i+1}}$. As any chain in L(G) is finite, after finitely many steps we have Y_{i+n} satisfying $X \cap Y_{i+n} = A$ and $\overline{X \cup Y_{i+n}} = T$ as required.

Let $S \neq X$ be a flat of G(S). Put $Y = \bigcap H$, where H is a hyperplane containing X. (H exists as a maximal chain from X to S exists and is finite). Obviously $X \subseteq Y \subsetneq S$. By the above $\exists$ a flat Z with $Y \cap Z = X$ and $\overline{Y \cup Z} = S$. Suppose $X \neq Y$. Then $X = Y \cap Z \neq Y$ which implies $Z \neq S$. Hence there exists a hyperplane H_1 containing Z and so containing X as well. Thus $Y \subsetneq H_1$ and $\overline{Y \cup Z} = S \Rightarrow \overline{Y \cup H_1}$ $= S \Rightarrow Y \nsubseteq H_1$. A contradiction. Hence $X = Y$ and the theorem is proved. //

1.3 RANK

We characterise any pregeometry in terms of its rank.

1.3.1 The *rank* of any subset A of S in G(S), written r(A), is the maximal chain length from $\bar{\phi}$ to $\bar{A}$ in L(G).

r(S) is the *rank of the pregeometry* G(S). The *points* and *lines* are the rank 1 and rank 2 elements respectively, [illegible] [illegible].

1.3.2 LEMMA. In a geometric lattice the length of any chain from y to $x \vee y$ is not greater than that of any maximal chain from $x \wedge y$ to x.

PROOF. Let $x \wedge y = x_0 < x_1 < \ldots < x_n = x$ be a maximal chain from $x \wedge y$ to x. Put $y_i = y \vee x_i$. This gives

$$y_0 = (x \wedge y) \vee y = y \leq y_1 \leq \ldots \leq y_n = x \vee y \ .$$

Since x_{i+1} covers x_i, by Lemma 1.2.15 there exists an atom $a_i \nleq x_i$ with $x_{i+1} = x_i \vee a_i$ and hence $y_{i+1} = y \vee x_{i+1} = y \vee (x_i \vee a_i) = (y \vee x_i) \vee a_i = y_i \vee a_i$, $i = 0, 1, \ldots , n - 1$

If $a_i \nleq y_i$, then y_{i+1} covers y_i .

If $a_i \leq y_i$, then $y_{i+1} = y_i$.

Hence with possible repetition of some elements we have a maximal chain from y to $x \vee y$ of length $\leq n$. Thus the lemma is proved. //

1.3.3 LEMMA. The rank function r of any pregeometry $G(S)$ has the following properties.

(R_1) $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$, $\forall A, B \subseteq S$. (*semimodularity*)

(R_2) $r(\phi) = 0$ (*normalized*)

(R_3) r is increasing

(R_4) $r(A \cup a) = r(A) + \begin{cases} 0 \\ 1 \end{cases}$, $\forall A \subsetneq S$, $\forall a \in S$. (*unit increasing)*

(R_5) For all $A \subseteq S$, $\exists\, A_f \subset\subset A$ with $r(A_f) = r(A)$. (*finite basis property*)

PROOF. (R_1):Given $A, B \subseteq S$. We note that $r(A) = r(\bar{A})$ and $\overline{\bar{A} \cup \bar{B}} = \overline{A \cup B}$ by Lemma 1.1.7. Then

$$r(A\cup B) + r(A\cap B) = r(\overline{A\cup B}) + r(\overline{A\cap B})$$

$$= r(\overline{\bar{A}\cup\bar{B}}) + r(\overline{A\cap B})$$

$$\leq r(\bar{A} \vee \bar{B}) + r(\bar{A}\cap\bar{B}) \quad (\text{as } A\subseteq\bar{A},\ B\subseteq\bar{B} \Rightarrow \overline{A\cap B} \subseteq \overline{\bar{A}\cap\bar{\bar{B}}} = \bar{A}\cap\bar{B})$$

$$\leq r(\bar{A} \vee \bar{B}) + r(\bar{A}\wedge\bar{B})$$

Now by Lemma 1.3.2 we have $r(\bar{A} \vee \bar{B}) - r(\bar{B}) \leq r(\bar{A}) - r(\bar{A}\wedge\bar{B})$ so that $r(\bar{A}\vee\bar{B}) + r(\bar{A}\wedge\bar{B}) \leq r(\bar{A}) + r(\bar{B})$ which is as desired.

(R_2) follows from the definition of rank.

(R_3) follows from the fact that $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$.

(R_4) : Let $A \subseteq S$. We consider the two cases.
If $a \in \bar{A}$, then $\overline{A \cup a} = \bar{A}$ so that $r(A \cup a) = r(A)$.
If $a \notin \bar{A}$, then by Lemma 1.1.7 $\bar{A} \vee \bar{a}$ covers $\bar{A}$. Hence $r(A \cup a) = r(A)+1$.

(R_5) : Let $A \subseteq S$. By (C_4) $\exists\, A_f \subset\subset A$ with $\bar{A}_f = A$ and then $r(A) = r(A_f)$. //

We now link closure and rank function.

1.3.4 LEMMA. In any pregeometry $G(S)$ we have

$$\bar{A} = \{ a \in S \,/\, r(A \cup a) = r(A) \} \quad , \quad \forall A \subseteq S.$$

PROOF. Let $a \in \bar{A}$. Then by Lemma 1.1.7 $\bar{A} = \overline{A \cup a}$ so that $r(A) = r(A \cup a)$. Given any a with $r(A \cup a) = r(A)$. If there exists $b \in \overline{A \cup a} \setminus \bar{A}$, then by Lemma 1.2.10 $\overline{A \cup b}$ covers $\bar{A}$ and so $r(A \cup b) > r(A)$. But as $b \in \overline{A \cup a}$ we have $r(A \cup a) = r(A \cup a \cup b) \geq r(A \cup b) > r(A)$. A contradiction. Hence $\overline{A \cup a} = \bar{A}$ and $a \in \bar{A}$. //

The following theorem characterises any pregeometry in terms of its rank function.

1.3.5 THEOREM. Any function $r : 2^S \to Z$ which is semimodular, normalized, increasing, unit increasing and has the finite basis property is the rank function of a unique pregeometry on S, having closure given by

$$\bar{A} = \{a \in S \,/\, r(A \cup a) = r(A)\} \quad , \quad \forall A \subseteq S.$$

Conversely the rank function of any pregeometry G(S) is a function $r : 2^S \to Z$ which is semimodular, normalized, increasing, unit increasing, and has the finite basis property and the closure given by the above.

PROOF. Let $r : 2^S \to Z$ be semimodular, normalized, increasing, unit increasing and have finite basis property with the closure given by the above. We show that the closure as defined satisfies (C_1) - (C_4).

(C_1) is clear from definition.

(C_2) : Let $A \subseteq B$, $A,B \subseteq S$. For any $a \in \bar{A} \setminus B$ by semimodularity we have $r(A \cup a) + r(B) \geq r(B \cup a) + r(A)$. This implies $0 \leq r(B \cup a) - r(B) \leq r(A \cup a) - r(A) = r(A) - r(A) = 0$. Thus $a \in \bar{B}$ and so $\bar{A} \subsetneq \bar{B}$

To show that $\bar{\bar{A}} = \bar{A}$, $\forall A \subsetneq S$ we first show $r(\bar{A}) = r(A)$. Let $B_1 \subset\subset A$ with $r(B_1) = r(A)$ and let $C_1 \subset\subset \bar{A}$ with $r(C_1) = r(\bar{A})$. Consider $C = B_1 \cup C_1$ we have $C_1 \subseteq C \subsetneq \bar{A} \Rightarrow r(C) \leq r(\bar{A}) \Rightarrow r(C) = r(\bar{A})$ (as $r(C) \geq r(C_1) = r(\bar{A})$). Put $B = A \cap C$. Then $B_1 \subseteq B \subseteq A \Rightarrow r(B) = r(A)$.

If $B = C$, then $r(\bar{A}) = r(C) = r(B) = r(A)$.

If $\exists a \in C \setminus B$, then $r(B) \leq r(B \cup a) \leq r(A \cup a) = r(A) = r(B)$. Since $C \setminus B \subseteq \bar{A}$ and $C \setminus B$ is finite, suppose $C \setminus B = \{ a_1, \ldots , a_n \}$. Now by semimodularity we have

$$r(B \cup a_1) + r(B \cup a_2) \geq r(B) + r(B \cup a_1 \cup a_2)$$

$$\Rightarrow r(B) = r(B \cup a_1) \geq r(B \cup a_1 \cup a_2)$$

Also $r(B \cup a_1 \cup a_2) + r(B \cup a_3) \geq r(B) + r(B \cup a_1 \cup a_2 \cup a_3)$

and hence $r(B \cup a_1 \cup a_2) \geq r(B \cup a_1 \cup a_2 \cup a_3)$.

Inductively for $i = 1, \ldots, n$ we have

$$r(B \cup a_1 \cup \ldots \cup a_{i-1}) \geq r(B \cup a_1 \cup \ldots \cup a_i)$$

so that $r(B) \geq r(C)$. Hence $r(A) = r(B) \geq r(C) = r(\bar{A})$ and therefore $r(A) = r(\bar{A})$.

Now $r(A \cup a) \leq r(\bar{A} \cup a) \leq r(\overline{A \cup a})$ and since $r(A \cup a) = r(\overline{A \cup a})$ we have $r(A \cup a) = r(\bar{A} \cup a)$.

Thus $a \in \bar{A} \Leftrightarrow r(A) = r(A \cup a) \Leftrightarrow r(\bar{A}) = r(\bar{A} \cup a) \Leftrightarrow a \in \bar{\bar{A}}$.

(C_3) : Let $a \in \overline{A \cup b}$ and $a \notin \bar{A}$, $A \subsetneq S$, $a, b \in S$. Since $a \notin \bar{A}$ and hence $r(A \cup a) \neq r(A)$, we have $r(A \cup a) = r(A) + 1$. Now $r(A \cup a) \leq r(A \cup b \cup a) = r(A \cup b) \leq r(A) + 1$ so that $r(A \cup a) = r(A \cup a \cup b)$ and hence $b \in \overline{A \cup a}$.

(C_4) : Given $A \subsetneq S$. There exists $A_f \subset\subset A$ with $r(A_f) = r(A)$. For any $a \in \bar{A}$ we have $r(A_f) = r(A) = r(A \cup a) \geq r(A_f \cup a)$ so that $r(A_f) = r(A_f \cup a)$. Hence $a \in \bar{A}_f$, and $\bar{A}_f = \bar{A}$.

Then the closure defines a pregeometry $G(S)$ on S. We next show that r is rank function on $G(S)$. That is we have to show that $r(A)$ is the maximal chain length from $\bar{\phi}$ to $\bar{A}$. Given any $A \subseteq S$ and a maximal chain $\bar{\phi} < A_1 < \ldots < A_n = \bar{A}$ in $L(G)$. For each $i = 1, \ldots, n-1$, A_{i+1} covers A_i so that $A_{i+1} = \overline{A_i \cup a_{i+1}}$ for some $a_{i+1} \in A_{i+1} \setminus A_i$ and hence $r(A_{i+1}) = r(\overline{A_i \cup a_{i+1}}) = r(A_i \cup a_{i+1}) = r(A_i) + 1$ (as $a_{i+1} \notin A_i$). Inductively, $r(A) = r(A_n) = r(A_{n-1}) + 1 = r(A_1) + n - 2 + 1 = n$

The converse was proved in Lemma 1.3.3 and 1.3.4. //

1.3.6 COROLLARY. The pregeometry is a geometry if and only if all two element subsets have rank 2.

PROOF: If $G(S)$ is a geomety then $x \notin \overleftarrow{y} \Rightarrow r(xy) \neq r(y) = 1 \Rightarrow r(xy) = 2$.

Conversely, if $r(xy) = 2$, as r is normalized and unit increasing, $r(\phi) = 0$ and $r(x) = 1$. //

1.3.7 EXAMPLE. Recall that a projective plane is a set of points S and lines, where lines are specified sets of points, satisfying the following axioms.

Axiom 1. Every two points belong to exactly one line.

Axiom 2. Every two lines have exactly one point in common.

Axiom 3. There are four points no three of which are in any line.

Points on a line are collinear.

Define $r : 2^S \to Z$ as follows :

$$r(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \text{ is a singleton} \\ 2 & \text{if } |A| \geq 2 \text{ and } A \text{ is contained in a line} \\ 3 & \text{if } A \text{ contains 3 non-collinear points} \end{cases}$$

Then r is rank function on a pregeometry on S.

PROOF. To show that r is semimodular let $A, B \subseteq S$. We can assume that $A \neq B$.

case 1. A and B are singletons. Then $r(A) + r(B) = 1 + 1 = 2$. Now there is a line containing $A \cup B$ so that $r(A \cup B) = 2$. Hence $r(A \cup B) + r(A \cap B) = 2 + 0 = 2 = r(A) + r(B)$.

case 2. Both A and B are not singletons and contained in a line.

If A, B are contained in the same line, then $r(A \cup B) + r(A \cap B) \leq 2 + 2 = r(A) + r(B)$. In case A and B are contained in different lines we have $A \cap B$ is a singleton or the empty set so that $r(A \cap B) \leq 1$. Now $r(A \cup B) + r(A \cap B) \leq 3 + 1 = 4 = r(A) + r(B)$.

case 3. Both A and B contain 3 non-collinear points. Then $r(A) + r(B) = 3 + 3 = 6$. Since $r(X) + r(Y) \leq 6$, $\forall X, Y \subseteq S$, $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$.

case 4. A is a singleton x and B is contained in a line L, where $|B| \geq 2$. If $x \in L$, then $r(A \cup B) + r(A \cap B) = 2 + 1 = 3 = r(A) + r(B)$. If $x \notin L$, we have $r(A \cup B) + r(A \cap B) = 3 + 0 = 3 \leq r(A) + r(B)$.

case 5. A is a singleton x and B contains 3 non-collinear points. Then $r(A \cap B) \leq 1$ so that $r(A \cup B) + r(A \cap B) \leq 3 + 1 \leq 4 = r(A) + r(B)$.

case 6. A is contained in a line L, where $|A| \geq 2$ and B contains 3 non - collinear points. Then $r(A \cap B) \leq 2$ so that $r(A \cup B) + r(A \cap B) \leq 3 + 2 = r(B) + r(A)$.

That r satisfies (R_2) - (R_5) follows from the definition of r.

Thus r is rank function on a pregeometry on S. //

1.3.8 LEMMA. The conditions (R_1) - (R_5) are independent.

PROOF. We see this by examining five examples in each of which exactly one of (R_1) - (R_5) is not satisfied.

(i) Let $S = \{1,2,3\}$.

Define $r : 2^S \to z$ by

$$r(A) = \begin{cases} 0 & \text{if } A = \phi \, , \\ 2 & \text{if } A = S \, , \\ 1 & \text{otherwise.} \end{cases}$$

Then r does not satisfy (R_1) since

$$r(12) + r(23) = 1 + 1 < r(12 \cup 23) + r(2).$$

(ii) Let $S = \{1,2\}$.

Define $r : 2^S \to Z$ by

$$r(A) = 1 \quad , \quad \forall A \subseteq S.$$

Then r does not satisfy (R_2).

(iii) Let S be an infinite set.

Define $r : 2^S \to Z$ by

$$r(A) = \begin{cases} 0 & \text{if } A^C \text{ is finite,} \\ \min\{1, |A|\} & \text{if } A^C \text{ is infinite.} \end{cases}$$

That r satisfies (R_2) follows from the fact that S is infinite.

To show that r satisfies (R_1) let $A, B \subseteq S$. We consider three cases.

case 1. A^C and B^C are finite. Then $(A \cup B)^C = A^C \cap B^C$ is finite and $(A \cap B)^C = A^C \cup B^C$ is finite so that $r(A \cup B) + r(A \cap B) = 0 + 0 = r(A) + r(B)$.

case 2. A^C is finite and B^C is infinite. Then $(A \cup B)^C = A^C \cap B^C$ is finite and $(A \cap B)^C = A^C \cup B^C$ is infinite and so $r(A \cup B) + r(A \cap B) \leq 0 + 1 = r(A) + r(B)$.

case 3. A^C and B^C are infinite and $A, B \neq \phi$. Then $r(A) + r(B) = 1 + 1 = 2$.

As $r(X) + r(Y) \leq 2$, $\forall X,Y \subseteq S$ we have $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$.

To see that r is not increasing consider the set $S \setminus x$ which is infinite so that $r(S \setminus x) = 0$. Since $S \setminus x$ is infinite, there exists $x \neq y \in S \setminus x$. Now $r(y) = 1$ which is as desired.

That r is unit increasing follows from the definition of r.

To show that r has finite basis property let $A \subseteq S$. If A^c is finite, we have $r(A) = 0$. Then A is infinite and $\phi \subset\subset A$ with $r(\phi) = r(A)$. If A^c is infinite and A is infinite. Pick $a \in A$. Now $a \subset\subset A$ and $r(a) = 1 = r(A)$. If A^c is infinite and A is finite we are finished.

Therefore r satisfies (R_1) - (R_5) except (R_3).

(iv) Let $S = \{1,2\}$.
Define $r : 2^S \to Z$ by $r(\phi) = 0$, $r(1) = 1$, $r(2) = 2$, $r(S) = 3$.
Then r satisfies (R_1) - (R_5) except (R_4).

(v) Let S be an infinite set.
Define $r : 2^S \to Z$ by

$$r(A) = \begin{cases} 0 & \text{if } A = \phi , \\ 1 & \text{if } A \text{ is finite} , \\ 2 & \text{if } A \text{ is infinite} . \end{cases}$$

To show that r satisfies semimodularity let $A,B \subseteq S$. We consider the three cases and we may assume that $A,B \neq \phi$.

case 1. A and B are finite. Then $A \cup B$ is finite and $A \cap B$ is finite or empty so that $r(A) + r(B) = 1 + 1 \geq r(A \cup B) + r(A \cap B)$.

case 2. A is finite and B is infinite. Then $A \cup B$ is infinite and $A \cap B$ is finite or empty and so $r(A) + r(B) = 1 + 2 \geq$ [illegible]

$r(A \cup B) + r(A \cap B)$.

case 3. A and B are infinite. Then $r(A) + r(B) = 2 + 2 = 4$. As $r(X) + r(Y) \leq 4$, $\forall X, Y \subsetneq S$, we have $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$.

That r satisfies (R_2) - (R_4) is clear from the definition of r.

As $r(S) = 2$ and for every finite subset A of S we have $r(A) = 1 \neq 2$. Thus r does not satisfy (R_5). //

1.4 INDEPENDENT SETS

We characterise any pregeometry in terms of its independent sets.

1.4.1 An *independent set* of a pregeometry G(S) is a set with rank equal to its cardinality.

As every set has finite rank only finite sets can be independent.

Before we characterise any pregeometry in terms of independent sets we obtain some of their properties.

1.4.2 LEMMA. (i) Any subset of an independent set is independent.

(ii) All maximal independent subsets of any set A have same cardinality, i.e. $r(A)$.

(iii) If I_1, I_2 are independent sets in G(S) with $|I_1| < |I_2|$, then $\exists\, x \in I_2 \setminus I_1$ such that $I_1 \cup x$ is independent.

PROOF. (i) Let J be any subset of an independent set A. Then $r(A) = |A|$. By semimodularity $r(J) + r(A \setminus J) \geq r(A) + r(\phi) = |A| + 0 = |A| = |J| + |A \setminus J| \geq |J| + r(A \setminus J)$ we have $r(J) \geq |J|$. On the other hand $r(J) \leq |J|$. Thus $r(J) = |J|$ so that J is independent.

(ii) Let A be any subset of S. Suppose that I is a maximal independent subset of A with $|I| < r(A)$. Then $r(I) < r(A)$ and hence $\bar{I} \subsetneqq \bar{A}$. We observe that $\bar{I} \subsetneqq A$ (otherwise by Lemma 1.1.7 $\bar{I} = \bar{A}$) and so there exists $x \in \bar{A} \setminus \bar{I}$. Then $\overline{I \cup x}$ covers $\bar{I}$ and $r(I \cup x) = r(I) + 1 = |I| + 1 = |I \cup x|$. Thus $I \cup x$ is an independent subset of A containing I. This contradicts the maximality of I. Hence $r(I) = r(A)$.

(iii) Let I_1, I_2 be independent sets in G(S) with $|I_1| < |I_2|$. Since I_2 is an independent subset of $I_1 \cup I_2$, any maximal independent subset of $I_1 \cup I_2$ has size at least $|I_2|$. Let I be an independent subset of $I_1 \cup I_2$ containing I_1 (I exists as I_1 is independent). Then $I \setminus I_1 \neq \phi$ (otherwise $|I| = |I_1| < |I_2|$). Thus I contains an element of $I_2 \setminus I_1$ which is as desired. //

The following theorem characterises any pregeometry in terms of its independent sets.

1.4.3 THEOREM. Any nonempty family $\mathcal{J}$ of finite subsets of S satisfying:

(I_1) $\mathcal{J}$ is closed with respect to subsets.

(I_2) All elements of $\mathcal{J}$ contained in any subset A of S are contained in maximal elements of $\mathcal{J}$ having the same cardinality.

is the collection of independent sets of a (unique) pregeometry on S having closure given by

$$\bar{A} = \{ a \;/\; \exists\, A \supseteq I \in \mathcal{J} \text{ s.t } a \cup I \notin \mathcal{J}\} \overset{\circ}{\cup} A \;, \quad \forall A \subseteq S.$$

Conversely the independent sets of any pregeometry have the above properties.

PROOF. Let $\mathcal{J}$ be a subset of 2^S satisfying the above conditions. Define $r : 2^S \to Z$ as follows:

$$r(A) = \max \;\{|I| \;/\; A \supseteq I \in \mathcal{J}\}.$$

We shall show that r satisfies (R_1) - (R_5).

To show that r is semimodularity we note that for any subsets A, B of S a maximal independent set I_1 in $A \cap B$ can be extended to a maximal independent set I_2 in $A \cup B$. Thus

$$\begin{aligned} |I_2 \cap A| + |I_2 \cap B| &= |I_2 \cap A \cap I_2 \cap B| + |(I_2 \cap A) \cup (I_2 \cap B)| \\ &= |I_1| + |I_2| \\ &= r(A \cap B) + r(A \cup B) \end{aligned}$$

Now $r(A) + r(B) \geq |I_2 \cap A| + |I_2 \cap B|$ so that

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$

That r is increasing and unit increasing follows from the definition of r; and it is normalized as $\phi \in \mathcal{J}$.

To show that r has finite basis property let $A \subseteq S$. Pick a maximal element I of $\mathcal{J}$ which is contained in A. Then $r(A) = |I| = r(I)$.

Hence r satisfies (R_1) - (R_5) so that r is the rank function of a unique pregeometry G(S) on S.

To show that the closure of G(S) defined as above we observe that $r(I) = |I| \iff I \in \mathcal{J}$ and hence
$r(A \cup a) = r(A),\ a \in \bar{A} \setminus A \iff a \cup I \notin \mathcal{J},\ \exists A \supseteq I \in \mathcal{J}$.
which is as desired.

The converse follows from Lemma 1.4.2 and the uniqueness of the specification of a pregeometry from its rank function follows from the first half of this theorem. //

1.4.4 COROLLARY. A pregeometry is a geometry if and only if all 2 - point sets are independent.

PROOF. Follows from Corollary 1.3.6 and the definition of independent sets.

1.4.5 LEMMA. Conditions (I_1) and (I_2) are together equivalent to (I_1) and (I_2'), where (I_2') is as follows:
(I_2') If $A_1, A_2 \in \mathcal{J}$ and $|A_2| = |A_1| + 1$, then $\exists\, x \in A_2 \setminus A_1$ such that $A_1 \cup x \in \mathcal{J}$.

PROOF. Suppose that $A_1, A_2 \in \mathcal{J}$ are maximal subsets of $X \subsetneq S$ such that $|A_1| < |A_2|$. Then there exists $A_2' \subsetneq A_2$ with $|A_2'| = |A_1| + 1$ and so there exists $x \in A_2' \setminus A_1$ such that $A_1 \cup x \in \mathcal{J}$. This contradicts the maximality of A_1 in $\mathcal{J}$ contained in X. Hence $|A_1| \geq |A_2|$.

Similarly $|A_2| \geq |A_1|$ and therefore $|A_1| = |A_2|$. //

1.4.6 EXAMPLE. Let S be a finite dimensional vector space. If we define $\mathcal{J}$ to be the family of all linearly independent subsets A of S, then $\mathcal{J}$ is the family of independent sets of a pregeometry on S.

1.4.7 LEMMA. The conditions (I_1) and (I_2) are independent.

PROOF. We see this by examining the following two examples in each of which exactly one of (I_1) - (I_2) fails.

(i) Let $S = \{1,2\}$ and $\mathcal{J} = \{\phi, 12\}$. Then only (I_1) fails.

(ii) Let $S = \{1,2,3\}$. Let $\mathcal{J} = \{\phi, 1, 2, 3, 23\}$. Then only (I_2) fails. //

1.4.8 LEMMA. The following four conditions are equivalent.

(i) A is independent.

(ii) A is minimal among those sets having closure $\bar{A}$

(iii) In a giving listing $a_1, a_2, \ldots$ of elements of A we have

$a_i \notin \overline{a_1 \dots a_{i-1}}$, $\forall_i$.

(iv) There exists no $a \in A$ with $a \in \overline{A \setminus a}$.

PROOF. (iv) => (i).

In any G(S) listing elements of A as $a_1, a_2, \dots$

Put $X_i = \overline{a_1 \dots a_{i-1}}$.

Then $a_{i+1} \notin \overline{A \setminus a_{i+1}} \supseteq X_i$ so that $X_{i+1} \supsetneqq X_i$. Consider the chain $\phi \subsetneqq X_1 \subsetneqq X_2 \dots \subsetneqq \bar{A}$ in L(G) which is finite, of length n say. Thus $X_n = \bar{A}$ and $|A| = n$ so that A is independent.

(i) => (ii).

Suppose $\exists\, C \subsetneqq A$ with $\bar{C} = \bar{A}$. Let $a \in A \setminus C$. Then $a \in \bar{A}$ and so $a \in \bar{C} \setminus C$ so that there exists an independent set $I \subsetneqq C$ and $I \cup a$ is not independent. But $I \cup a \subsetneqq A$ and A is independent. A contradiction. Hence A is a minimal set having closure $\bar{A}$.

(ii) => (iii).

If $\exists\, a_i \in \overline{a_1 \dots a_{i-1}} \subseteq \overline{A \setminus a_i}$, then by Lemma 1.1.7 $\bar{A} = \overline{A \setminus a_i}$ so that A is not a minimal set having closure $\bar{A}$. Thus $a_i \notin \overline{a_1 \dots a_{i-1}}$, $\forall_i$.

(iii) => (iv).

Suppose that $\exists\, a \in A$ such that $a \in \overline{A \setminus a}$. Listing elements of A in a way that a is the i^{th} element, for some fixed integer i .

Now $a_i \in \overline{A \setminus a_i}$. By the finite basis property $\exists\, A_f \subset\subset A \setminus a_i$ with $\bar{A}_f = \overline{A \setminus a_i}$. Choose a minimal $B \subset\subset A \setminus a_i$ such that $\bar{B} = \overline{A \setminus a_i}$.

Let j be the maximal suffix such that $a_j \in B$. Then $j > i$ (otherwise $B \subsetneqq a_1 \dots a_{i-1} \Rightarrow a_i \in \overline{A \setminus a_i} = \bar{B} \subsetneqq \overline{a_1 \dots a_{i-1}}$ which contradicts the assumption). Now $a_i \notin \overline{B \setminus a_j}$ and $a_i \in \overline{(B \setminus a_j) \cup a_j}$ so that by (C_3), $a_j \in \overline{(B \setminus a_j) \cup a_i} \subseteq \overline{a_1 \dots a_{j-1}}$. A contradiction.

Thus there exists no $a \in \overline{A \setminus a}$.

1.5 BASES

We characterise any pregeometry in terms of its collection of bases.

1.5.1 Defining a *basis* of $G(S)$ as a minimal set having S as closure we have

1.5.2 LEMMA. The bases of $G(S)$ are exactly the maximal independent sets in $G(S)$.

PROOF. We first show that a basis B of $G(S)$ is a maximal independent set. Now B is a minimal set such that $\bar{B} = S$.
If $\exists x \in B$ such that $x \in \overline{B \setminus x}$, then $\bar{B} = \overline{B \setminus x}$ which is a contradiction.
Thus $\forall x \in B$, $x \notin \overline{B \setminus x}$ so that, by Lemma 1.4.8, B is independent.

For any $x \notin B$ we have $x \in S = \bar{B}$ and so $r(B \cup x) = r(B) < |B \cup x|$.
Therefore $B \cup x$ is not independent and hence B is a maximal independent set in $G(S)$.

Next suppose that B is a maximal independent set in $G(S)$.
Then $x \notin B \Rightarrow B \cup x$ is not independent, $\Rightarrow r(B \cup x) = r(B)$, $\Rightarrow x \in \bar{B}$, $\Rightarrow \bar{B} = S$.

If $\exists x \in B$ such that $\overline{B \setminus x} = S = \bar{B}$, then $r(B) = r(\bar{B}) = r(B \setminus x) < |B|$, a contradiction. Thus B is a minimal set having S as closure; which is as required. //

1.5.3 A subset A *spans* (*generates*) B in $G(S)$ if $B = \bar{A}$.

Thus every basis spans S.

We say that a *depends* on A if $a \in \bar{A}$. Then every element depends on any basis.

1.5.4 LEMMA. (i) Every independent set extends to a basis and this property characterises independent sets.

(ii) If A is an independent subset of a spaning set C, then there exists a basis B such that $A \subseteq B \subsetneqq C$.

PROOF. (i) Given an independent set I in G (S). If $\bar{I} \subsetneqq S$ consider a maximal chain of length n from $\bar{I}$ to S :
$\bar{I} \subsetneqq \overline{I \cup x_1} \subsetneqq \overline{I \cup x_1 x_2} \ \ldots \subsetneqq \overline{I \cup x_1 \ldots x_n} = S$. Then $I \cup x_1 \ldots x_n$ is a basis, having rank equal to its size.

(ii) As C is spaning we have $r(C) = r(S)$ so that there exists an independent subset of C of size $r(S)$. Let B be a maximal independent subset of C containing A. Then $r(B) = r(S)$ so that by Lemma 1.5.2 B is a basis. //

We characterise any pregeometry in terms of its bases.

1.5.5 THEOREM. A nonempty family $\mathcal{B}$ of finite subsets of S, each of the same size, is the collection of bases of a pregeometry on S if and only if it satisfies the following:
(B) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \smallsetminus B_2$, then $\exists\, y \in B_2 \smallsetminus B_1$ such that $(B_1 \cup y) \smallsetminus x \in \mathcal{B}$.

PROOF. That the family $\mathcal{B}$ of base of G(S) satisfies (B) follows from Lemma 1.4.2.

Let $\mathcal{B}$ be a nonempty family of finite subsets of S of the same size satisfying (B).

Put $\mathcal{J} = \{ I \mid I \subseteq A \in \mathcal{B} \}$

Then $\mathcal{J} \neq \phi$ as $\mathcal{B} \neq \phi$ and (I_1) is satisfied from the definition of $\mathcal{J}$.

To show that $\mathcal{J}$ satisfies (I_2') let $A_1, A_2 \in \mathcal{J}$ with $|A_2| = |A_1| + 1$. Then there exist $B_1, B_2 \in \mathcal{B}$ such that $A_1 \subseteq B_1$, $A_2 \subseteq B_2$. Let

$$\begin{aligned} A_1 &= \{x_1, \ldots, x_n\}, \\ B_1 &= \{x_1, \ldots, x_n, b_1, \ldots, b_r\}, \\ A_2 &= \{y_1, \ldots, y_n, y_{n+1}\}, \\ B_2 &= \{y_1, \ldots, y_n, y_{n+1}, c_1 \ldots, c_{r-1}\}. \end{aligned}$$

Consider B_1, B_2. By (B), there exists $z_1 \in B_2$ such that $B_1' = (B_1 \setminus b_1) \cup z_1 \in \mathcal{B}$.

If $z_1 \in A_2$, then $A_1 \cup z_1 \in \mathcal{J}$ and (I_2') is satisfied.

If $z_1 \notin A_2$ consider B_1' and B_2. By (B) there exists $z_2 \in B_2$ such that $B_2' = (B_1' \setminus b_2) \cup z_2 \in \mathcal{B}$. If $z_2 \in A_2$ we are finished, if not remove b_3 from B_2' and so on. Since $|\{b_1, \ldots, b_r\}| > |\{c_1, \ldots, c_{r-1}\}|$, we reach step $k (k \leq r)$, where $B_k' = (B_{k-1}' \setminus b_k) \cup z_k \in \mathcal{B}$ and $z_k \in A_2$. Thus $B_k' \supseteq A_1 \cup z_k \in \mathcal{J}$.

Therefore by Lemma 1.4.5 $\mathcal{J}$ is the family of independent sets of a pregeometry on S with $\mathcal{B}$ its family of bases //

1.5.6 LEMMA. In any G(S) the following statements are equivalent.

(i) H is a hyperplane of G(S).

(ii) $\bar{H} \neq S$ but $\overline{H \cup x} = S$, $\forall x \in S \setminus H$.

(iii) No basis B is contained in H but if $x \in S \setminus H$, $\exists$ a basis B' such that $x \in B \subseteq H \cup x$.

(iv) H is a maximal subset of S which is not spanning.

(v) H is a maximal set of rank $r(S) - 1$.

PROOF. (i) => (ii) follows from the definition of H.

(ii) => (iii) : Suppose that H contains a basis B. The $S = \bar{B} \subseteq \bar{H}$ so that $\bar{H} = S$. Thus H does not contain any basis. Let $x \in S$ H. Then $\overline{H \cup x} = S$ so that $r(H \cup x) = r(S)$ and so $H \cup x$ contains a basis B and $x \in B$ (otherwise H contains a basis).

(iii) => (iv) is obvious.

(iv) => (v) : By the assumption $r(H) < r(S)$. If $r(H) < r(S) - 1$, then H is not maximal non - spanning set. Thus $r(H) = r(S) - 1$. For any $x \in S$ H, $H \cup x$ is a spanning set and so it contains a basis. Hence $r(H \cup x) = r(S)$. Therefore H is a maximal set of rank $r(S) - 1$.

(v) => (i) : It suffices to show that $\bar{H} = H$. Suppose $\exists x \in \bar{H} \setminus H$. Then $r(H \cup x) = r(\overline{H \cup x}) = r(\bar{H}) = r(H) = r(S) - 1$, contradicting the maximality of H of rank $r(S) - 1$. Thus $\bar{H} = H$. //

1.6 CIRCUITS

We characterise any pregeometry in terms of its circuits.

1.6.1. A subset A of S is *dependent* in G(S) if it is not independent.

A *circuit* of $G(S)$ is a minimal dependent subset of S.

1.6.2 LEMMA. The collection $\mathcal{K}$ of circuits of $G(S)$ has the following properties.

(K_0) $C \in \mathcal{K} \iff r(C) = |C| - 1 = r(C \setminus a), \forall a \in C$.

(K_1) Any circuit is nonempty and finite.

(K_2) No circuit properly contains another.

(K_3) Every infinite subset of S contains a circuit.

(K_4) If $C_1, C_1 \neq C_2 \in \mathcal{K}$ and $a \in C_1 \cap C_2$, then $\exists C \in \mathcal{K}$ s.t. $C \subseteq C_1 \cup C_2 \setminus a$.

PROOF. (K_0) is clear from the definition of circuits.

(K_1) : Since ϕ is independent, any circuit is nonempty. As $r(C) = |C| - 1$ any circuit is finite.

(K_2) : Since any circuit is a minimal depependent set, any circuit properly contains no other circuit.

(K_3) : If A is an infinite subset of S in $G(S)$, then A is dependent. Thus it contains a minimal dependent set which is a circuit.

(K_4) : By semimodularity we have

$r(C_1 \cup C_2) \leq r(C_1) + r(C_2) - r(C_1 \cap C_2) = r(C_1) + r(C_2) + |C_1 \cap C_2|$

(as $C_1 \cap C_2$ is independent.)

and so

$$\begin{aligned} r(C_1 \cup C_2 \setminus a) &\leq r(C_1 \cup C_2) \\ &\leq |C_1| - 1 + |C_2| - 1 - |C_1 \cap C_2| \\ &< |C_1 \cup C_2 \setminus a| \end{aligned}$$

Thus $C_1 \cup C_2 \setminus a$ is dependent and hence contains a circuit which is as desired. //

We link closure and circuits.

1.6.3 LEMMA. $\bar{A} = \{a \,/\, a \in C \subseteq A \cup a, \text{ some circuit } C\} \overset{\circ}{\cup} A$.

PROOF. Let $a \in C \subsetneq A \cup a$, for some circuit C. Then $C \setminus a$ is independent so that by Theorem 1.4.3 $a \in \bar{A}$.

Conversely if $a \in \bar{A} \setminus A$, then there exists an independent $I \subsetneq A$ such that $I \cup a$ is dependent. Pick a circuit $C \subsetneq a \cup I$. Then $a \in C \subseteq I \cup a$ (otherwise C is independent). //

1.6.4 THEOREM. Any subset $\mathcal{C}$ of 2^S which satisfies (K_1) - (K_4) is the collection of circuits of a unique pregeometry on S, having closure given by

$$\bar{A} = \{a \,/\, a \in C \subseteq A \cup a, \text{ some } C \in \mathcal{C}\} \overset{\circ}{\cup} A, \ \forall A \subseteq S.$$

Conversely the circuits of any pregeometry on S have the above properties.

PROOF. Assume that $\mathcal{C}$ is a subset of 2^S satisfying (K_1) - (K_4). We first show that the closure defined as above satisfies (C_1) - (C_4).

(C_1) is trivial from the definition of closure.

(C_2): Let $A \subseteq B$. For any $a \in \bar{A} \setminus B$, $a \in C \subseteq A \cup a$ for some $C \in \mathcal{C}$ we have $a \in C \subseteq B \cup a$ so that $a \in \bar{B}$. Thus $\bar{A} \subseteq \bar{B}$.

To show that $\bar{\bar{A}} = \bar{A}$, $\forall A \subseteq S$ we first show that C in (K_4) can be chosen to contain any given $b \in C_2 \setminus C_1$, i.e. we shall show

(K_4') $C_1, C_2 \in \mathcal{C}$, $a \in C_1 \cap C_2$, $b \in C_2 \setminus C_1 \Rightarrow \exists\, C \in \mathcal{C}$ s.t. $b \in C \subseteq C_1 \cup C_2 \setminus a$.

If not, there exist C_1, C_2, a, b, $a \in C_1 \cap C_2$, $b \in C_2 \setminus C_1$ such that for any $C \in \mathcal{C}$ and $a \notin C \subsetneq C_1 \cup C_2$ we have $b \notin C$. Choose one such with $|C_1 \cup C_2|$ minimal. Let $C \in \mathcal{C}$ be such that $a \notin C \subseteq C_1 \cup C_2$. By (K_2) $C \subsetneqq C_2$ and so $\exists\, b \neq c \in C \setminus C_2$. Since $C \subseteq C_1 \cup C_2$ and $c \notin C_2$, $c \in C_1$. Now $C_1 \cup C \subsetneq C_1 \cup C_2$ and $b \in (C_1 \cup C_2) \setminus (C_1 \cup C)$ so that $|C_1 \cup C| < |C_1 \cup C_2|$. Therefore $\exists\, C_3 \in \mathcal{C}$, $C_3 \subseteq C_1 \cup C$, containing a but not c (as $|C_1 \cup C_2|$ is minimal such that (K_4') fails).

Observe that $C_3 \cup C_2 \subsetneq C_1 \cup C \cup C_2 = C_1 \cup C_2$ and $b \notin C_3$ as $b \notin C_1$, C and so $b \in C_2 \setminus C_3$. Since $c \notin C_3, C_2$ but $c \in C_1, C_2$ we have $c \in (C_1 \cup C_2) \setminus (C_3 \cup C_2)$ and hence $|C_3 \cup C_2| < |C_1 \cup C_2|$ As $|C_1 \cup C_2|$ is minimal there exists an element C' of $\mathcal{C}$ contained in $C_3 \cup C_2$ containing b and not containing $a \in C_3 \cap C_2$. Thus $\exists\, C' \in \mathcal{C}$ such that $a \notin C' \subseteq C_1 \cup C_2$ and $b \in C'$. This contradicts the assumption of C_1, C_2. Therefore (K_4') is obtained.

Given $A \subseteq S$. If $b \in \bar{\bar{A}} \setminus \bar{A}$, then $\exists\, C_2 \in \mathcal{C}$ s.t. $b \in C_2 \subsetneqq \bar{A} \cup b$ and $C_2 \subsetneqq A \cup b$ (as $b \notin \bar{A}$). Hence $C_2 \subseteq \bar{A} \cup b = (\bar{A} \setminus A) \cup (A \cup b)$ so that we can choose $a \in C_2 \cap (\bar{A} \setminus A)$. Thus $a \in \bar{A} \Rightarrow \exists\, C_1 \in \mathcal{C}$ s.t $a \in C_1 \subseteq A \cup a$. Hence $a \in C_1 \cap C_2$ and $b \in C_2 \setminus C_1$ (as $b \notin \bar{A} \Rightarrow b \notin C_1 \subseteq A \cup a \subseteq \bar{A}$) so that by (K_4') $\exists\, C_3 \in \mathcal{C}$ s.t $b \in C_3 \subseteq C_1 \cup C_2 \subseteq \bar{A} \cup b$ and $a \notin C_3$. Now the finite set $(\bar{A} \setminus A) \cap C_3 \subsetneqq (\bar{A} \setminus A) \cap C_2$ (as $(\bar{A} \setminus A) \cap C_3 \subseteq \left[(\bar{A} \setminus A) \cap C_1\right] \cup \left[(\bar{A} \setminus A) \cap C_2\right]$ and $(\bar{A} \setminus A) \cap C_1 = a$, $a \in (\bar{A} \setminus A) \cap C_2 \setminus (\bar{A} \setminus A) \cap C_3$).

If $(\bar{A} \setminus A) \cap C_3 \neq \phi$ consider C_3 and C_1 instead of C_2 and C_1 and obtain $C_4 \in \mathcal{C}$ s.t $(\bar{A} \setminus A) \cap C_4 \subsetneqq (\bar{A} \setminus A) \cap C_3$ and $b \in C_4$. Eventually in finitely many steps we obtain $C \in \mathcal{C}$ such that $(\bar{A} \setminus A) \cap C = \phi$ and $b \in C$. Then $b \in C \subseteq A \cup b \Rightarrow b \in \bar{A}$. A contradiction. Therefore $\bar{\bar{A}} = \bar{A}$.

(C_3) : Let $a \in \overrightarrow{A \cup b}$ and $a \notin \bar{A}$, $A \subseteq S$, $a,b \in S$. There exists

$C \in \mathcal{C}$ s.t $a \in C \subseteq A \cup b \cup a$. Now $C \subsetneq A \cup a$ since $a \notin \bar{A}$. Hence $C \subseteq A \cup b$ and $b \in C$ so that $C \subseteq A \cup a$. Thus $b \in \overline{A \cup a}$.

(C_4) : Let $A \subsetneq S$. Consider the family of subsets of A which do not contain an element of $\mathcal{C}$ and partially order them by set inclusion. This family is not empty as it contains ϕ . It contains only finite sets. If the upper bound of any chain contains an element of $\mathcal{C}$ then some member of the chain will also contain this finite set. This contradicts the choice of members of the set, so the set contains an upper bound of each chain. By Zorn's Lemma there exists a maximal element A_f in this family. Obviously A_f is finite. If $a \in A \setminus A_f$, then $\exists\, C \in \mathcal{C}$ s.t. $a \in C \subseteq A_f \cup a$ (as A_f is maximal). Thus $\bar{A}_f \supseteq A$ and hence $\bar{\bar{A}}_f = \bar{A}_f \supseteq \bar{A}$. Since $A_f \subseteq A$, $\bar{A}_f \subsetneq \bar{A}$ and therefore $\bar{A}_f = \bar{A}$.

Hence the closure defines a pregeometry G(S) on S. To show that $\mathcal{C}$ is the family of circuits we first show that every element $C \in \mathcal{C}$ is a circuit.

$C \in \mathcal{C} \Rightarrow c \in C \subsetneq (C \setminus c) \cup c \Rightarrow c \in \overline{C \setminus c} \Rightarrow \exists\, C_0 \in \mathcal{K}$ s.t $c \in C_0 \subsetneq (C \setminus c) \cup c = C \Rightarrow C_0 \subsetneq C$

$x \in C_0 \subseteq (C_0 \setminus x) \cup x \Rightarrow x \in \overline{C_0 \setminus x} \Rightarrow \exists\, C' \in \mathcal{C}$ s.t $x \in C' \subseteq (C_0 \setminus x) \cup x \Rightarrow C' \subseteq C_0 \subseteq C \Rightarrow C' = C = C_0$

Thus $C \in \mathcal{K}$.

By interchanging the roles of $\mathcal{K}$ and $\mathcal{C}$ in the above we show that every circuit is in $\mathcal{C}$.

The uniqueness of the pregeometry follows from the definition of closure in terms of circuits.

The converse has already been proved. //

1.6.5 COROLLARY. The pregeometry is a geometry if and only if all circuits have cardinality at least 3.

PROOF. Follows from Corollary 1.4.4 .

We recall that a graph (V,E) is a set V of *vertices* and a family E of unordered pairs of vertices, called *edges* , and a *polygon* is a finite set of edges $\{(v_1,v_2), (v_2,v_3), \dots (v_{n-1}, v_n), (v_n, v_1)\}$ with $i \neq j \Rightarrow v_i \neq v_j$.

1.6.6. EXAMPLE. The polygons of any graph, in which any infinite collection of edges contains a polygon, are the circuits of a pregeometry on the set of edges of the graph.

PROOF. We need only show that the collection of polygons satisfy (K_4).

Consider two polygons,

$$C_1 : (v_1,v_2) , \dots , (v_{n-1},v_n), (v_n,v_1)$$
$$C_2 : (w_1,w_2) , \dots , (w_{m-1},w_m), (w_m, w_1)$$

such that $C_1 \cap C_2 \neq \phi$. Without loss of generality we assume $(v_1,v_2) = (w_1,w_2)$ and $v_1 = w_1$, $v_2 = w_2$.

Consider the collection $(v_2,v_3) \dots , (v_{n-1}, v_n), (v_n, v_1)$, $(w_1, w_m), (w_m, w_{m-1}), \dots , (w_3, w_2)$. This is a finite closed path. Hence it contains a minimal closed path - or polygon - which does not contain (v_1, v_2). //

1.6.7 LEMMA. The conditions (K_1) - (K_4) are independent.

PROOF. We see this by examining the following four examples in each of which exactly one of (K_1) - (K_4) is not satisfied.

(i) Let S be any set and $\mathcal{K} = \{\phi\}$. Then only (K_1) fails.

(ii) Let $S = \{1,2\}$ and $\mathcal{K} = \{1,2,12\}$. Then only (K_2) fails.

(iii) Let S be an infinite set. Let $x \neq y \in S$. Put $\mathcal{K} = \{x,y\}$. Then only (K_3) fails.

(iv) Let $S = \{1,2,3\}$ and $\mathcal{K} = \{12,23\}$. Then only (K_4) fails. //

We link bases and circuits of pregeometries.

1.6.8 LEMMA. If B is a basis of $G(S)$ and $x \in S \setminus B$, then there exists a unique circuit $C = C(x, B)$ such that $x \in C \subseteq B \cup x$.

This circuit is the *fundamental circuit* of x with respect to the basis B.

PROOF. First we show that $I \cup x$ contains at most one circuit if I is independent in $G(S)$ and $x \in S$. Suppose that $I \cup x$ contains two distinct circuits C_1, C_2. As I is independent C_1 and C_2 both contain x. Now $x \in C_1 \cap C_2$ and hence by (K_4) $\exists$ a circuit C_3 of $G(S)$ such that $C_3 \subseteq C_1 \cup C_2 \setminus x$. But $C_1 \cup C_2 \setminus x \subseteq I$. Thus I contains a circuit C_3. A contradiction.

Let B be a basis of $G(S)$ and $x \in S \setminus B$. Then $B \cup x$ is dependent and it contains a circuit C. Since B is independent, $C \subsetneq B$ and so $x \in C \subseteq B \cup x$. As $B \cup x$ contains at most one circuit, C is unique. //

As a consequence of Lemma 1.6.8 we have the following stronger result, writing $C(x, B)$ for the unique circuit C such that $x \in C \subseteq B \cup x$.

1.6.9 THEOREM. Consider any basis B of $G(S)$ and any $x \in S \setminus B$. Then $(B \setminus y) \cup x$ is a basis of $G(S)$ if and only if $y \in C(x, B)$.

PROOF. Let $(B \setminus y) \cup x$ be a basis of $G(S)$. Then $C(x, B) \subseteq B \cup x$. Suppose $y \notin C(x, B)$. Then $C(x, B) \subseteq (B \setminus y) \cup x$ and so $(B \setminus y) \cup x$ can not be a basis. Hence $y \in C(x, B)$.

Assume that $y \in C(x, B)$. If $(B \setminus y) \cup x$ is not a basis, Then $(B \setminus y) \cup x$ contains a circuit C'. Hence $B \cup x$ contains a circuit $C' \neq C(x, B)$. A contradiction. Thus $(B \setminus y) \cup x$ is a basis. //

1.7 FLATS

We present a characterisation of any pregeometry in terms of flats, which is due to Roberts [73].

1.7.1 THEOREM. Let $\mathcal{F}_r$, $r = 0,1,2, \ldots , n$, be disjoint families of subsets of an arbitrary set S, with $\mathcal{F}_n$ consisting of S alone. A subset A of S is $\mathcal{F}$-dependent iff A is contained in some member of $\mathcal{F}_r$, for some $r < |A|$; otherwise A is $\mathcal{F}$-independent. Suppose

(1) Each $\mathcal{F}$-independent r-element subset R of S is contained in exactly one member of $\mathcal{F}_r$, denoted by $M(R)$, for $r = 0, 1, \ldots, n$; denoting a typical member of $\mathcal{F}_r$ by F_r,

(2) If F_r contains an $\mathcal{F}$-independent $(r-1)$-element subset R of S, then $F_r \supsetneqq M(R)$, for $r = 1,2, \ldots , n$;

(3) $F_r \supsetneqq F_s \Rightarrow r > s$.

For $A \subseteq S$, we define $J(A)$ to be the intersection of all members of $\bigcup_{r=0}^{n} \mathcal{F}_r$ containing A.

The above conditions define a pregeometry on S, with J the closure, $\mathcal{F}_r$ the set of flats of rank r, and the $\mathcal{F}$-independent sets being exactly the independent sets. Conversely, given a pregeometry on S with $\mathcal{F}_r$ its set of flats of rank r, the above conditions are fulfilled, with the independent sets being exactly the $\mathcal{F}$-independent sets, and the closure being J.

The pregeometry is a geometry iff $\mathcal{F}_0$ consists of the empty set alone and $\mathcal{F}_1$ consists of all singleton subsets of S.

PROOF. We require several preliminary lemmas.

1.7.2 LEMMA. If R is an $\mathcal{F}$-independent r-element set and $a \not\in M(R)$, then $R \cup a$ is an $\mathcal{F}$-independent $(r + 1)$-element set.

PROOF. By the definition of $\mathcal{F}$-independence, R, and hence $R \cup a$, is contained in no F_t, for $t < r$. Suppose $R \cup a \subseteq F_r$. Then by (1), $F_r = M(R)$, contradicting the choice of a. //

1.7.3 LEMMA. If $t < r$, and F_r contains an $\mathcal{F}$-independent t-element set T, then $F_r \supsetneqq M(T)$.

PROOF. If $t = r-1$, the above is true by (2). If $t < r-1$, $M(T) \not\supseteq F_r$ by (3), and $M(T) \neq F_r$ since the families $\mathcal{F}_r$ are disjoint. Thus there is an $a_1 \in F_r \setminus M(T)$. By Lemma 1.7.2, $T_1 = T \cup a_1$ is an $\mathcal{F}$-independent (t+1)-element set. Again, if $t+1 < r-1$, there is an $a_2 \in F_r \setminus M(T_1)$ such that $T_2 = T_1 \cup a_2$ is an $\mathcal{F}$-independent (t+2)-element set. We continue thus until we have an $\mathcal{F}$-independent (r-1)-element set T_{r-t-1}. Then, by repeated use of (2),

$$M(T) \subsetneqq M(T_1) \subsetneqq \cdots \subsetneqq M(T_{r-t-1}) \subsetneqq F_r \qquad //$$

1.7.4 LEMMA. If $A \subseteq S$, there exists a maximal $\mathcal{F}$-independent subset of A. If R is any maximal $\mathcal{F}$-independent subset of A, then

$J(A) = M(R)$.

PROOF. Every subset of S of cardinality exceeding n is $\mathcal{F}$-dependent, while the empty set is $\mathcal{F}$-independent. Thus there exists at least one maximal $\mathcal{F}$-independent subset of A, say R; let R have cardinality r. Then

$$F_t \supseteq A \Rightarrow F_t \supseteq R,$$
$$\Rightarrow t \geq r,$$

by the definition of $\mathcal{F}$-independence. If $t > r$, $F_t \supsetneq M(R)$ by Lemma 1.7.3; if $t = r$, $F_t = M(R)$ by (1). Thus $F_t \supseteq A \Rightarrow F_t \supseteq M(R)$. If $M(R) \not\supseteq A$, let $x \in A \setminus M(R)$. Then $R \cup x$ is $\mathcal{F}$-independent by Lemma 1.7.2, contradicting our choice of R. Thus $J(A) = M(R)$. //

1.7.5 LEMMA. Any subset of an $\mathcal{F}$-independent set is $\mathcal{F}$-independent.

PROOF. We show that any superset of an $\mathcal{F}$-dependent subset T of S is $\mathcal{F}$-dependent. Let R be a maximal $\mathcal{F}$-independent subset of T, and let $a \in S \setminus T$. Then if $a \in M(R)$, $T \cup a \subseteq M(R) = J(T)$ by Lemma 1.7.4, and $T \cup a$ is $\mathcal{F}$-dependent. If $a \notin M(R)$, $R \cup a$ is $\mathcal{F}$-independent by Lemma 1.7.2. Thus, using (2),

$$T \subseteq J(T) = M(R) \subsetneq M(R \cup a) \quad ,$$

and $T \cup a \subseteq M(R \cup a)$; hence $T \cup a$ is again $\mathcal{F}$-dependent, since $R \cup a$ has lesser cardinality than $T \cup a$. Since any infinite subset of S is $\mathcal{F}$-dependent, any superset of an $\mathcal{F}$-dependent subset of S is $\mathcal{F}$-dependent. //

1.7.6 LEMMA. Let R be a maximal $\mathcal{F}$-independent subset of $A \subsetneq S$, and let $R \cup x$ be $\mathcal{F}$-independent for some $x \in S \setminus A$. Then $R \cup x$ is a maximal $\mathcal{F}$-independent subset of $A \cup x$.

PROOF. Suppose $R \cup x \cup y$ is an $\mathcal{F}$-independent subset of $A \cup x$, where $y \in A \setminus R$. Then by Lemma 1.7.5, $R \cup y$ is $\mathcal{F}$-independent, contradicting our choice of R. //

PROOF OF THEOREM 1.7.1 Suppose the families $\mathcal{F}_r$ of subsets of S satisfy the conditions of the theorem. Since $S \in \mathcal{F}_n$, if $A \subsetneq S$, the intersection in the evaluation of J(A) is not vacuous. Thus, from the definition of J, $A \subseteq J(A)$, and if $A \subsetneq B$, $J(A) \subseteq J(B)$. If R is a maximal $\mathcal{F}$-independent subset of A, $J(A) = M(R)$ by Lemma 1.7.4, giving immediately the finite basis property for J (since $J(A) = J(R) = M(R)$) and the idempotency of J (since $J(M(R) = M(R))$.

To show that S is a pregeometry with J as its closure, we still need to show that J satisfies the exchange property. Suppose $a,b \in S$ such that $b \notin J(A) = M(R)$, but $b \in J(A \cup a)$, with a and R as above. If $R \cup a$ is $\mathcal{F}$-dependent, R is a maximal $\mathcal{F}$-independent subset of $A \cup a$, so that $J(A \cup a) = M(R) = J(A)$, contradiction. Thus $R \cup a$ is $\mathcal{F}$-independent, and, since $a \notin A$, $R \cup a$ is a maximal $\mathcal{F}$-independent subset of $A \cup a$ by Lemma 1.7.6. Then $J(A \cup a) = M(R \cup a)$ by Lemma 1.7.4, and $b \in M(R \cup a)$. Now $b \notin M(R)$, so that $R \cup b$ is $\mathcal{F}$-independent by Lemma 1.7.2, and $M(R \cup b) = M(R \cup a)$. From Lemma 1.7.6 and 1.7.4, since $b \notin A$ and $R \cup b$ is $\mathcal{F}$-independent, $J(A \cup b) = M(R \cup b)$. Thus

$$a \in M(R \cup a) = M(R \cup b) = J(A \cup b),$$

and the exchange property for J is verified. Thus the closure J difines a pregeometry G(S) on S.

Let $A \subsetneq S$, and let R be a maximal $\mathcal{F}$-independent subset of A. Then $J(A) = M(R) \in \bigcup_{r=0}^{n} \mathcal{F}_r$ by Lemma 1.7.4; on the other hand

$F \in \bigcup_{r=0}^{n} \mathcal{F}_r \Rightarrow J(F) = F$, by the definition of J, and the flats of G(S) are exactly the members of $\bigcup_{r=0}^{n} \mathcal{F}_r$.

If $T \in \mathcal{F}_t$, let R be a maximal $\mathcal{F}$-independent subset of T. If $|R| < t$, $T \supsetneqq M(R)$ by Lemma 1.7.3; then if $x \in T \setminus M(R)$, $R \cup x$ is $\mathcal{F}$-independent by Lemma 1.7.2, contradicting our choice of R. Thus $|R| = t$; let $R = r_1 \dots r_t$. If $1 \leq i \leq t$,

$$r_1 \dots r_{i-1} \subseteq r_1 \dots r_i$$

$$\Rightarrow J(r_1 \dots r_{i-1}) = M(r_1 \dots r_{i-1}) \subsetneqq M(r_1 \dots r_i) = J(r_1 \dots r_i).$$

Then
$$J(\phi) \subsetneqq J(R_1) \subsetneqq J(r_1, r_2) \subsetneqq \dots\dots \subsetneqq J(R) = T$$

is a maximal chain from $J(\phi)$ to T in L(G), and $\mathcal{F}_t$ is the set of flats of rank t in G(S), for each t.

Let A be a subset of S, and R a maximal $\mathcal{F}$-independent subset of A. Then the rank of A in G(S) is the (finite) cardinality of R, and A is $\mathcal{F}$-independent iff R = A, iff the rank and cardinality of A are the same. Thus the $\mathcal{F}$-independent sets are precisely the independent sets of G(S), completing the proof of the first part of the theorem.

Conversely, let G(S) be a pregeometry of rank n on a set S, and L(G) its lattice of flats. Let $\mathcal{F}_r$ be the family of flats of rank r in G(S), for r = 0,1, ... ,n then the families $\mathcal{F}_r$ are disjoint, and $\mathcal{F}_n$ consists of S alone. Now any infinite subset of S is both dependent and $\mathcal{F}$-dependent; hence let A be a finite subset of S, of cardinality t. If A is dependent, $r(A) < t$, and A is F-dependent. If A is $\mathcal{F}$-dependent, $A \subseteq F_r$ for some $r < t$; then $\bar{A} \subseteq F_r$ by the defintion of closure and $r(A) \leq r < t$, so that A is dependent. Thus the independent sets of G(S) are exactly the $\mathcal{F}$-independent sets.

Any independent r-element subset R of S is contained in one member of $\mathcal{F}_r$, namely $\bar{R}$. If $R \subseteq F_r \neq \bar{R}$, $R \subseteq F_r \cap \bar{R} \in \mathcal{F}_t$, for some $t < r$, contradicting the fact that $r(R) = r$. Thus (1) is true. If $\mathcal{F}_r$ contains an independent (r-1)-element subset R of S, $F_r \supseteq \bar{R}$ by the definition of closure; $F_r \neq \bar{R}$ because their ranks in G(S) differ, and so (2) is true. (3) is true by the definition of rank in G(S).

The last statement of the theorem is immediate from (C_5) and (C_6), and the proof of the theorem is complete. //

2 BASIC PROPERTIES OF PREGEOMETRIES

2.1 ISOMORPHISMS

2.1.1 Two pregeometries $G(S)$ and $G(S')$ are *isomorphic* if there is a bijection $i: S \to S'$ such that $i(\bar{A}) = \overline{i(A)}$, $\forall A \subseteq S$.

We write $G(S) \cong G(S')$ and call i an *isomorphism* from $G(S)$ to $G(S')$.

Now we examine the relations between isomorphism and the various characterisations of pregeometries.

2.1.2 THEOREM. Two pregeometries $G(S)$ and $G(S')$ with rank functions r and r' respectively are isomorphic if and only if there exists a bijection $i: S \to S'$ satisfying $r(A) = r'(iA)$, $\forall A \subseteq S$.

PROOF. First assume that $G(S) \cong G(S')$. Then there exists an isomorphism $i : S \to S'$. Let $A \subseteq S$. Consider any maximal chain $\bar{\phi} < \bar{A}_1 <\ldots< \bar{A}_n = \bar{A}$ in the lattice of flats of $G(S)$. Then $\bar{\phi} < \overline{iA_1} <\ldots< \overline{iA_n} = \overline{iA}$ is a chain in the lattice of flats in $G(S')$. Suppose $\exists\, j$ such that $\overline{iA_j} < Y < \overline{iA_{j+1}}$ for some flat Y of $G(S')$. Hence $\bar{A}_j < X < \bar{A}_{j+1}$, where $i(X) = Y$. Now $i(\bar{X}) = \overline{iX} = \bar{Y} = Y = iX$ so that $X = \bar{X}$ and hence X is a flat in $G(S)$. A contradiction. Thus $r'(iA) = r(A)$.

Conversely let $i : S \to S'$ be a bijection satisfying $r(A) = r'(iA)$, $\forall A \subseteq S$. Let $A \subseteq S$. Then

$x \in i(\bar{A}) \Rightarrow x = i(y)$ for some $y \in \bar{A}$, $\Rightarrow r(A \cup y) = r(A)$, $\Rightarrow r'(i(A \cup y)) = r'(iA)$, $\Rightarrow r'(iA \cup iy) = r'(iA) \Rightarrow r'(iA \cup x) = r'(iA)$, $\Rightarrow x \in \overline{iA}$.

Thus $i(\bar{A}) \subseteq \overline{iA}$.

$y \in \overline{iA} \Rightarrow r'(iA \cup y) = r'(iA), \Rightarrow r'(i(A \cup x)) = r'(iA),$

where $y = ix$ for some $x \in S$, $\Rightarrow r(A \cup x) = r(A) \Rightarrow x \in \bar{A}$,

$\Rightarrow y \in i(\bar{A})$.

Therefore $\overline{iA} \subseteq i(\bar{A})$ and so $i(\bar{A}) = \overline{iA}$. //

In fact an isomorphism between two pregeometries preserves. rank, independence, bases and circuits and vice versa.

2.1.3 THEOREM. A bijection $i : S \to S'$ is an isomorphism from $G(S)$ to $G(S')$ exactly when any one of the following three conditions holds,

(i) I is independent in $G(S)$ $\Leftrightarrow$ $i(I)$ is independent in $G(S')$,

(ii) B is a basis in $G(S)$ $\Leftrightarrow$ $i(B)$ is a basis in $G(S')$,

(iii) C is a circuit in $G(S)$ $\Leftrightarrow$ $i(C)$ is a circuit in $G(S')$.

If i is an isomorphism then the induced map on flats is a lattice isomorphism.

Conversely, if $G(S)$ and $G(S')$ are geometries the existence of the lattice isomorphism induces a geometric isomorphism.

PROOF. First; i is an isomorphism from $G(S)$ to $G(S')$

$\Leftrightarrow$ $r(A) = r'(iA)$, $\forall A \subseteq S$,

$\Leftrightarrow$ $|A| = r(A)$ exactly when $|iA| = r'(iA)$,

$\Leftrightarrow$ Condition (i) holds.

Secondly; condition (ii) is equivalent to (i).

Thirdly; the equivalence of (iii) follows from (i) and (K_0).

Let i be an isomorphism from $G(S)$ to $G(S')$. Let $L(G)$ and $L(G')$

be the lattices of flats of G(S) and G(S') respectively.

Define $\psi : L(G) \to L(G')$ by

$$\psi\ (\bar{A}) = i(\bar{A})\ ,\ \forall \bar{A} \in L(G).$$

Let $\bar{A}, \bar{B} \in L(G)$. Then $\phi(\bar{A})\ v\ \phi(\bar{B}) = i(\bar{A})\ v\ i(\bar{B}) = \overline{i(A)}\ \ v\ \overline{i(B)}$

$= \overline{\overline{i(A)} \cup \overline{i(B)}} = \overline{i(\bar{A}) \cup i(\bar{B})} = i(\overline{\bar{A} \cup \bar{B}}) = i(\overline{\bar{A} \cup \bar{B}}) = \phi(\overline{\bar{A} \cup \bar{B}}) = \phi(\bar{A}\ v\ \bar{B})$

and $\phi(\bar{A} \wedge \bar{B}) = \phi(\bar{A} \cap \bar{B}) = \phi(\overline{\bar{A} \cap \bar{B}}) = i(\overline{\bar{A} \cap \bar{B}}) = i(\bar{A} \cap \bar{B}) = i(\bar{A}) \cap i(\bar{B})$

$= \overline{i(A)} \cap \overline{i(B)} = \overline{i(A)} \wedge \overline{i(B)} = i(\bar{A}) \wedge i(\bar{B}) = \phi(\bar{A}) \wedge \phi(\bar{B})$.

Thus L(G) and L(G') are isomorphic.

Let G(S) and G(S') be geometries such that $L \cong L'$, where L, L' are lattices of flats of G(S), G(S') respectively. Let $\psi : L \to L'$ be lattice isomorphism. Since G(S), G(S') are geometries, the atoms in L, L' are $\{ \bar{a} / a \in S \}$, $\{ \bar{b} / b \in S' \}$ respectively.

Define $i : S \to S'$ by $i = \psi$ / atoms .

Then i is one to one and onto as ψ / atoms is one to one and $\psi\ \{\bar{a} / a \in S\} = \{ \bar{b} / b \in S'\}$.

Let $A \subseteq S$. Then $i(\bar{A}) = i(\sup A) = \psi(\sup A) = \sup\ (\psi A)$ $= \overline{\psi A} = \overline{iA}$.

2.2 SUBPREGEOMETRIES

We show that in a natural way any pregeometry on S induces a pregeometry on any subset of S.

2.2.1 THEOREM. Any pregeometry $G(S)$ induces a pregeometry $G_S(T)$, on any subset T of S, called the subpregeometry on T induced by $G(S)$ with closure $\tilde{A}$ defined by $\tilde{A} = \bar{A} \cap T$, $\forall A \subseteq T$.

PROOF. It is obvious that $A \subsetneq \tilde{A}$, $\forall A \subsetneq T$ so that (C_1) is satisfied.

Let $A \subseteq \tilde{B}$. Then $A \subseteq \bar{B} \cap T$ so that $A \subseteq \bar{B}$ and hence $\bar{A} \subseteq \bar{B}$. Thus $\tilde{A} = \bar{A} \cap T \subseteq \bar{B} \cap T = \tilde{B}$.

Given $a \in \widetilde{A \cup b}$ and $a \notin \tilde{A}$, where $A \subseteq T$, $a,b \in T$. Now $a \notin \bar{A}$ and $a \in \overline{A \cup b}$ as $a \in T$. By the exchange property in $G(S)$ we have $b \in \overline{A \cup a}$. Therefore $b \in \widetilde{A \cup a}$.

Let $A \subseteq T$. Since $A \subseteq S$, $\exists A_f \subset\subset A$ with $\bar{A}_f = \bar{A}$. Now $\tilde{A}_f = \bar{A}_f \cap T = \bar{A} \cap T = \tilde{A}$. //

Any pregeometry and its subpregeometries have structures related as in the following lemma.

2.2.2. LEMMA. (i) In any subpregeometry $G_S(T)$, $\tilde{A} = \bar{A}$, $\forall A \subseteq T$, if and only if T is a flat in $G(S)$.

(ii) The independent sets in $G_S(T)$ induced on $T \subseteq S$ by $G(S)$ are exactly the subsets of T which are independent in $G(S)$.

(iii) The rank of $A \subseteq T$ in $G_S(T)$ is its rank in $G(S)$.

(iv) The circuits of $G_S(T)$ are exactly the subsets of T which are the circuits of $G(S)$.

PROOF. (i) Assume that $\tilde{A} = A$, $\forall A \subseteq T$. Thus $\bar{T} = \tilde{T} = \bar{T} \cap T = T$ so that T is a flat in $G(S)$.

Assume that T is a flat in $G(S)$. Let $A \subseteq T$. Then $\bar{A} \subseteq \bar{T} = T$ so that $\tilde{A} = \bar{A} \cap T = \bar{A}$.

(ii) Let A be independent in $G_S(T)$. By Lemma 1.4.8 there is no $a \in A$ such that $a \in \widetilde{A \setminus a} = \overline{A \setminus a} \cap T$. If there is $a \in A$ such that $a \in \overline{A \setminus a}$, then $a \in \overline{A \setminus a} \setminus T$. But $A \subseteq T$. Therefore there is no $a \in A$ such that $a \in \overline{A \setminus a}$ and hence A is independent in $G(S)$.

Assume that A is independent in $G(S)$ and $A \subseteq T$. Now there is no $a \in A$ such that $a \in \overline{A \setminus a} \supseteq \widetilde{A \setminus a}$ so that there is no $a \in A$ such that $a \in \widetilde{A \setminus a}$. Hence A is independent in $G_S(T)$

(iii) follows from (ii) and the fact that $r_T(A)$ is the cardinality of a maximal independent set of $G_S(T)$ contained in $A \subseteq T$.

(iv) follows from (K_0) and (iii). //

2.3 CANONICAL GEOMETRIES

We examine particular subpregeometries.

2.3.1 A subpregeometry $G_S(T)$ is a *canonical geometry* of $G(S)$ if it satisfies the following.

(CG 1) $T \cap \bar{\phi} = \phi$

(CG 2) $|T \cap (\bar{a} \setminus \bar{\phi})| = 1 \quad , \quad \forall a \not\in \bar{\phi}$.

Obviously a canonical geometry is a geometry, as the induced closure of a singleton is the singleton.

The existence of a canonical geometry of any pregeometry is guaranteed by

2.3.2 THEOREM. A canonical geometry of any pregeometry G(S) exists and all canonical geometries of G(S) are isomorphic.

PROOF. In G(S) consider the equivalence relation $\equiv$ on $S \setminus \bar{\phi}$ defined by $a \equiv b$ iff $\bar{a} = \bar{b}$. Let T be a set of elements each from one equivalence class, no two elements of T from the same class, then T satisfies (CG 1) and (CG 2) so that $G_S(T)$ is a canonical geometry of G(S).

Let $G_S(T_1)$ and $G_S(T_2)$ be canonical geometries of G(S). Define a bijection $f: T_1 \to T_2$ by $f : (T_1 \cap (\bar{a} \setminus \bar{\phi})) \mapsto T_2 \cap (\bar{a} \setminus \bar{\phi})$. To show that f is an isomorphism let $A \subseteq T$. First notice that $\bar{t} = \overline{f(t)}$, $\forall t \in T_1$ since t and f(t) are in the same equivalence class. Now

$$\bar{A} = \overline{\{ a / a \in A \}} = \overline{\{ \bar{a} / a \in A \}} = \overline{\{ \overline{f(a)} / a \in A \}}$$
$$= \overline{\{ f(a) / a \in A \}} = \overline{f(A)} \quad .$$

Thus since $x \in \bar{A} \cap T_1 \iff f(x) \in \bar{A} \cap T_2$ we have

$$f(\mathcal{C}\ell_1 (A) = f \{x / x \in \mathcal{C}\ell_1 (A)\},$$
$$= f \{x / x \in \bar{A} \cap T_1\} ,$$
$$= \{ f(x) / f(x) \in \bar{A} \cap T_2\} ,$$

$$= \bar{A} \cap T_2 ,$$
$$= \overline{f(A)} \cap T_2 ,$$
$$= \mathcal{C}\ell_2(f(A)) . \qquad //$$

In a geometry $G(S)$ if $G_S(T)$ is a canonical geometry we have $T = S$ exactly when $a \in S$ implies $\bar{a} \cap T = a \cap T = a$. We then have proved.

2.3.3 COROLLARY. A pregeometry is a geometry if and only if it is a canonical geometry of itself.

2.3.4 THEOREM. Any pregeometry $G(S)$ canonically determines a geometry on the equivalence classes of $S \setminus \bar{\phi}$.

PROOF. As the equivalence relation $\equiv$ in the proof of Theorem 2.3.2 partitions $S \setminus \bar{\phi}$ into equivalence classes S', where every element of S' is of the form $\bar{x}$ for some $x \in S \setminus \bar{\phi}$ and $x \in \bar{y} \iff \bar{x} = \bar{y}$, $\forall \bar{x}, \bar{y} \in S'$. For any $A' \subseteq S'$ define $\mathcal{C}\ell(A')$ as follows :

$$\mathcal{C}\ell(A') = \{ \bar{b} \in S' \,/\, b \in \overline{\bigcup_{a' \in A'} a'} \} .$$

We show that $\mathcal{C}\ell$ satisfies (C_1) - (C_6).

(C_1) : Given $A' \subseteq S'$. Every element of A' is of the form $\bar{x}$ for some $x \in S \setminus \bar{\phi}$ and $x \in \overline{\bigcup_{a' \in A'} a'}$. Hence $\bar{x} \in \mathcal{C}\ell(A')$.

(C_2) : Let $A' \subseteq \mathcal{C}\ell(B')$, where $A', B' \in S'$. Then

$$A' \subseteq \mathcal{C}\ell(B') \Rightarrow \bigcup_{a' \in A'} a' \subseteq \bigcup_{x' \in \mathcal{C}\ell(B')} x' ,$$

$$\Rightarrow \overline{\bigcup_{a' \in A'} a'} \subseteq \overline{\bigcup_{x' \in \mathcal{C}\ell(B')} x'} = \overline{\bigcup_{x' \in B'} x' \cup \{\bar{a}/\bar{a} \notin B', \bar{a} \in \overline{\bigcup_{x' \in B'} x'}\}},$$

$$\Rightarrow \overline{\bigcup_{a' \in A} a'} \subseteq \overline{\bigcup_{x' \in \mathcal{Cl}(B')} x'} = \overline{\bigcup \{x'/\ x' \in B'\}}\ , \text{ (by Lemma 1.1.7)}$$

$$\Rightarrow \mathcal{Cl}(A') \subseteq \mathcal{Cl}(B').$$

(C_3) : Let $\bar{a} \in \mathcal{Cl}(A' \cup \bar{b})$ and $\bar{a} \notin \mathcal{Cl}(A')$, where $A' \in S'$, $\bar{a}, \bar{b} \in S'$. Then $a \in \overline{\{\bigcup a'/a' \in A' \cup \bar{b}\}}$ and $a \notin \overline{\{\bigcup a'/a' \in A'\}}$. Put $A = \bigcup_{a' \in A'} a'$. Now $a \notin \bar{A}$ and $a \in \overline{A \cup b}$ so that by the exchange property in G(S) we have $b \in \overline{A \cup a} = \overline{\bigcup_{a' \in A'} a' \cup a} = \overline{\bigcup_{a' \in A'} a' \cup \bar{a}}$. Hence $b \in \mathcal{Cl}(A' \cup \bar{a})$.

(C_4) : Given $A' \subseteq S'$. Let $A = \bigcup_{a' \in A'} a'$. Then $A \subsetneq S$ so that $\exists\ A_f \subset\subset A$ with $\bar{A}_f = \bar{A}$. Consider $A'_f = \bigcup_{a \in A_f} \bar{a}$ which is finite. We show that $A'_f \subset\subset A'$ and $\mathcal{Cl}(A'_f) = \mathcal{Cl}(A')$.

$\bar{a} \in A'_f \Rightarrow \exists\ b \in A_f$ and $b \in \bar{a} \setminus \bar{\phi} \Rightarrow b \in A \Rightarrow b \in \bar{c}$ for some $\bar{c} \in A'$ $\Rightarrow \bar{a} = \bar{b} = \bar{c} \in A'$.

That is $A'_f \subset\subset A'$. Now as $\bar{b} \in A'_f \Leftrightarrow \exists\ c \in \bar{b}$ s.t $c \in A_f$ and $\bar{c} = \bar{b}$ we have

$$\mathcal{Cl}(A'_f) = \{\bar{a}\ /\ a \in \overline{\bigcup_{\bar{b} \in A'_f} \bar{b}}\} = \{\bar{a}\ /\ a \in \overline{\bigcup_{c \in A_f} c}\} = \{\bar{a}\ /\ a \in \bar{A}_f\}$$

$$= \{\bar{a}\ /\ a \in \bar{A}\} = \{\bar{a}\ /\ a \in \overline{\bigcup_{a \in A'} a}\} = \mathcal{Cl}(A').$$

(C_5) and (C_6) follow at once from the definition of $\mathcal{Cl}$ and the property that $a \in \bar{b} \Leftrightarrow \bar{a} = \bar{b}$. //

Notice that the geometry obtained in Theorem 2.3.4 is isormorphic to any canonical geometry of G(S). Rado [57] defined the canonical geometry of G(S) to be the geometry obtained as in Theorem 2.3.4.

2.3.5 THEOREM. For any geometry G(S) a partition of a super set

V of S in such a way that no two elements of S are in the same equivalence class of V determines a pregeometry on V having $G_V(S) = G(S)$ as a canonical geometry of G(V).

PROOF. For any subset A of V, define $\mathcal{Cl}(A)$ as follows :

$$\mathcal{Cl}(A) = \bigcup_{b \in \bar{S}_A} \{\text{ equivalence class of V containing } b\} \cup \{\text{equivalence class of V containing no element of S }\}, \text{ where}$$

$$S_A = S \cap \{ \bigcup_{a \in A} \text{ equivalence class of V containing a } \}.$$

We show that $\mathcal{Cl}$ satisfies $(C_1) - (C_4)$.

(C_1) : Given $A \subseteq V$. For any $a \in A$ and a is not in an equivalence class of V containing an element of S we have $a \in \mathcal{Cl}(X)$, $\forall X \subseteq V$. If $a \in$ equivalence class containing an element b of S, then $b \in \bar{S}_A$ so that $a \in \mathcal{Cl}(A)$.

(C_2) : Let $A \subseteq B \subseteq V$. It is clear from the definition of $\mathcal{Cl}$ that $\mathcal{Cl}(A) \subseteq \mathcal{Cl}(B)$. To show that $\mathcal{Cl}(\mathcal{Cl}(A)) = \mathcal{Cl}(A)$ for every $A \subseteq V$ we observe that

$$B' = \overline{S \cap \{ \bigcup_{a \in \mathcal{Cl}(A)} \text{ equivalence class containing a } \}},$$

$$= \overline{S \cap \{ \bigcup_{a \in A} \text{ equivalence class containing a } \}}.$$

For any $x \in \mathcal{Cl}(\mathcal{Cl}(A))$ if $x \in$ equivalence class containing b, where $b \in \bar{B}'$, then $x \in \mathcal{Cl}(A)$. Thus $\mathcal{Cl}(\mathcal{Cl}(A)) = \mathcal{Cl}(A)$.

(C_3) : Let $a \in \mathcal{Cl}(A \cup b)$ and $a \notin \mathcal{Cl}(A)$, where $A \subseteq V$, $a,b \in V$. We consider the class containing a. If this class contains no element of S we have $a \in \mathcal{Cl}(A)$. A contradiction. Hence there exists an element a' of S in this equivalence class. If the equivalence class containing b contains no element of S , then $b \in \mathcal{C}\ell(A \cup a)$. Assume that b, b' are in the same class, where $b' \in S$.

Now $a \notin \mathcal{C}\ell(A) \Rightarrow a \notin$ equivalence class containing x, $\forall x \in \bar{S}_A$. As the union of equivalence classes containing an element of $A \cup b$ is the union of equivalence classes containing an element of $A \cup b'$ we have $\bar{S}_{A \cup b} = \bar{S}_{A \cup b'}$ and $\mathcal{C}\ell(A \cup b) = \mathcal{C}\ell(A \cup b')$ Therefore $a \in \mathcal{C}\ell(A \cup b) \Rightarrow a' \in \mathcal{C}\ell(A \cup b) \Rightarrow a' \in \mathcal{C}\ell(A \cup b') \Rightarrow a' \in \bar{S}_{A \cup b'}$. By the exchange property in G(S) we have $b' \in \bar{S}_{A \cup a'} \subsetneq \mathcal{C}\ell(A \cup a') = \mathcal{C}\ell(A \cup a)$ so that $b' \in \mathcal{C}\ell(A \cup a)$. Hence $b \in \mathcal{C}\ell(A \cup a)$.

(C_4) : Given $A' \subseteq V$. Let

$$A'' = \bigcup_{a' \in A'} \{\text{ equivalence class containing } a' \} \text{ and}$$

$A = A'' \cap S$. Then $A \subseteq S$ and by the finite basis property in G(S) $\exists A_f \subset\subset A$ with $\bar{A}_f = \bar{A}$. For each $a \in A_f$, pick one element $x_a \in$ equivalence class containing a and $x_a \in A'$ (x_a exists as every element in A is in a class containing an element of A'). Let $A'_f = \{ x_a / a \in A_f \}$. Then $A'_f \subset\subset A'$. Now

$$\begin{aligned}\mathcal{C}\ell(A'_f) &= \bigcup_{a \in \bar{A}_f} \{\text{ equivalence class of } a \} \cup \{\text{ equivalence class containing no element of } S \}, \\ &= \bigcup_{a \in \bar{A}} \{\text{ equivalence class of } a \} \cup \{\text{equivalence class containing no element of } S \}, \\ &= \mathcal{C}\ell(A').\end{aligned}$$

We see from the definition of $\mathcal{C}\ell$ that $G_V(S) = G(S)$. To show that G(S) is a canonical geometry we note that $\mathcal{C}\ell(\phi) = \cup$ {equivalence class containing no element of S }. Therefore $S \cap \mathcal{C}\ell(\phi) = \phi$.

For any $a \notin \mathcal{Cl}(\phi) \Rightarrow a \in$ equivalence class containing an element s of $S \Rightarrow \bar{s}_a = s \Rightarrow \mathcal{Cl}(a) \setminus \mathcal{Cl}(\bar{\phi})$ = equivalence class containing $s \Rightarrow S \cap (\mathcal{Cl}(a) \setminus \mathcal{Cl}(\phi)) = s$. //

2.3.6 THEOREM. Two pregeometries with isomorphic canonical geometries have isomorphic lattices of flats.

PROOF. We first show that any pregeometry G(S) and its canonical geometry $G_S(T)$ have isomorphic lattices of flats. Let L(S) and L(T) be lattices of flats of G(S) and $G_S(T)$ respectively. Consider the function $f : L(S) \to L(T)$ defined by $f(A) = A \cap T$, $\forall A \in L(S)$. Obviously f is one to one. To see that f is onto we observe that $B \in L(T) \Leftrightarrow B = \bar{B} \cap T$, $B \subseteq T$ so that $\bar{B} \in L(S)$ and $f(\bar{B}) = B$.

We show that f preserves meet and join. Let $A, B \in L(S)$. Then $f(A \cap B) = (A \cap B) \cap T = (A \cap T) \cap (B \cap T) = f(A) \cap f(B)$ and $f(A) \vee f(B) = (A \cap T) \vee (B \cap T) = (\bar{A} \cap T) \vee (\bar{B} \cap T) = \tilde{A} \vee \tilde{B} = \widetilde{A \cup B}$ $= \overline{A \cup B} \cap T = f(\overline{A \cup B}) = f(A \vee B)$.

Let $G_S'(T')$ be a canonical geometry of G(S'), where $G_S(T) \cong G_S'(T')$. The theorem is proved if we can show that $G_S(T)$ and $G_{S'}(T')$ have isomorphic lattices of flats. Let i be an isomorphism from $G_S(T)$ onto $G_S'(T')$. Denote by L(T) and L(T') the lattices of flats of $G_S(T)$ and $G_S'(T')$ respectively.

Define $\phi : L(T) \to L(T')$ by $\phi(A) = i(A)$, $\forall A \in L(T)$. Then ϕ is one to one and onto. Let $A, B \in L(T)$. Then $\phi(A \cap B) = i(A \cap B) = i(A) \cap i(B) = \phi(A) \cap \phi(B)$ and $\phi(\overline{A \cup B}) = i(\overline{A \cup B}) = \overline{i(A \cup B)}$ $= \overline{i(A) \cup i(B)} = \overline{\phi(A) \cup \phi(B)}$.

Thus L(T) and L(T') are isomorphic and the theorem is proved. //

2.4 TRUNCATION

We define the *truncation* of any pregeometry.

2.4.1 THEOREM. Let $\mathcal{J}$ be the family of independent sets of G(S). Then

$$\mathcal{J}_k = \{ I \in \mathcal{J} / |I| \leq k \}, \text{ for some positive integer } k \leq r(S),$$

is the family of independent sets of a pregeometry on S - the truncation of G(S) at k.

PROOF. It is clear that $\mathcal{J}_k$ satisfies (I_1).

We show that $\mathcal{J}_k$ satisfies (I_2). Let $A \subseteq S$. If I_1, I_2 are maximal elements of $\mathcal{J}_k$ contained in A and $|I_1| < |I_2|$. By Lemma 1.4.2 $\exists\, x \in I_2 \setminus I_1$ such that $I_1 \cup x \in \mathcal{J}$. As $I_1, I_2 \in \mathcal{J}_k$, $|I_1| < k$ and so $|I_1 \cup x| \leq k$. Thus $I_1 \cup x \in \mathcal{J}_k$. A contradiction. Hence $|I_1| = |I_2|$.

Therefore $\mathcal{J}_k$ is the the family of independent sets of a pregeometry on S. //

We note that the k - uniform geometry on a set S is the truncation at k of the Boolean geometry on S. //

2.5 CONTRACTION

We define the *contraction* of any pregeometry.

2.5.1 THEOREM. Let $\mathcal{J}$ be the family of independent sets of G(S). Let $T \subseteq S$ and define $\mathcal{J}(T)$ to be the family of subsets X of T such that there exists a maximal independent subset Y of $S \setminus T$ with $X \cup Y \in \mathcal{J}$. Then $\mathcal{J}(T)$ is the family of independent sets of a pregeometry G(S).T on S - the contraction of G(S) to T.

PROOF. We see that $\mathcal{J}(T)$ is a family of finite subsets of T if $\mathcal{J}(T) \neq \phi$. Since $S \setminus T$ contains a maximal independent subsets of G(S) and $\phi \subseteq T$, $\phi \in \mathcal{J}(T)$ so that $\mathcal{J}(T) \neq \phi$. The theorem is proved if we can show that $\mathcal{J}(T)$ satisfies (I_1) and (I_2).

(I_1) : Let $B \in \mathcal{J}(T)$ and $A \subsetneq B$. Then there exists a maximal independent subset Y of $S \setminus T$ with $B \cup Y \in \mathcal{J}$. Now $A \cup Y \subseteq B \cup Y$ and so $A \cup Y \in \mathcal{J}$. Thus $A \in \mathcal{J}(T)$.

(I_2) : Let $A \subseteq T$. Let X_1, X_2 be maximal sets in $\mathcal{J}(T)$ contained in A. Then there exist maximal independent subsets Y_1, Y_2 of $S \setminus T$ with $X_1 \cup Y_1 \in \mathcal{J}$ and $X_2 \cup Y_2 \in \mathcal{J}$. Put $T_1 = (S \setminus T) \cup A$. Then $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are maximal independent **sets** of $G_S(T_1)$ and so $|X_1 \cup Y_1| = |X_2 \cup Y_2|$. But $|Y_1| = |Y_2|$ and $X_1 \cap Y_1 = X_2 \cap Y_2 = \phi$. Thus $|X_1| = |X_2|$. //

2.5.2 LEMMA. Let r^T be the rank function of G(S).T. Then

$$r^T(A) = r(A \cup (S \setminus T)) - r(S \setminus T), \quad \forall A \subseteq T.$$

In particular $r^T(T) = r(S) - r(S \setminus T)$.

PROOF. Let $A \subseteq T$. As in the proof of Theorem 2.5.1, $X \cup Y$ is a maximal independent set of $G_S(A \cup (S \setminus T))$ if X is a maximal independent subset of A in G(S). T and Y is a maximal independent subset of $S \setminus T$ in G(S). Thus $r(A \cup (S \setminus T)) = |X \cup Y| = |X|+|Y| = r^T(A) + r(S \setminus T)$ as desired. //

2.5.3 EXAMPLE. Let M(G) be a pregeometry derived from a finite graph $G = (V, E)$ as in Example 1.6.6. For any $T \subsetneq E$, let G_T be a subgraph of G obtained from G by deleting all edges not in T. Then $M(G_T)$ is the contraction of $M(G)$ to T.

PROOF. Let $\mathcal{J}$, $\mathcal{J}(T)$ be the families of independent sets of $M(G)$, $M(G_T)$ respectively. Observe that I is independent in $M(G) \Leftrightarrow I$ does not contain a polygon of G.

Let $I \in \mathcal{J}(T)$. Then I does not contain a polygon of G_T. There exists a maximal subset X of $S \setminus T$ such that $I \cup X$ is not a polygon of G and so $X \in \mathcal{J}$ and $I \cup X \in \mathcal{J}$.

Let $I \cup X \in \mathcal{J}$, where X is a maximal independent subset of $S \setminus T$ and $I \subseteq T$. Thus $I \cup X$ does not contain a polygon of G and hence I does not contain a polygon of G_T so that $I \in \mathcal{J}(T)$.

Hence $M(G_T)$ is the contraction of $M(G)$ to T. //

2.6 UNION AND DIRECT SUM OF PREGEOMETRIES

We discuss the *union* of two pregeometries.

2.6.1 THEOREM. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the collections of independent sets of $G_1(S_1)$ and $G_2(S_2)$ respectively. Then the collection

$$\mathcal{J} = \{ I_1 \cup I_2 \;/\; I_i \in \mathcal{J}_i \,,\; i = 1,2 \}$$

is the collection of independent sets of a pregeometry on $S_1 \cup S_2$ - the union of $G_1(S_1)$ and $G_2(S_2)$ - denoted by $G_1(S_1) \vee G_2(S_2)$.

PROOF. Since $\mathcal{J}_1 \neq \phi$, $\mathcal{J}_2 \neq \phi$, we have $\mathcal{J} \neq \phi$. We see that any set in $\mathcal{J}$ is a finite subset of $S_1 \cup S_2$.

We show that $\mathcal{J}$ satisfies (I_1). Let $I \in \mathcal{J}$ and $J \subsetneq I$. Then $I = I_1 \cup I_2$, where $I_i \in \mathcal{J}_i$, $i = 1,2$. Put $J_i = J \cap I_i$, $i = 1,2$. Then $J_i \in \mathcal{J}_i$ and $J = J_1 \cup J_2 \in \mathcal{J}$.

We next show that $\mathcal{J}$ satisfies (I_2). Let $A \subseteq S_1 \cup S_2$. Let I, J be maximal sets in $\mathcal{J}$ contained in A. Suppose that $|I| < |J|$. There exists a presentation $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ with $I_1 \cap I_2 = J_1 \cap J_2 = \phi$. Choose one of these such that $|I_1 \cap J_2| + |I_2 \cap J_1|$ is minimum. Now $|I_1| + |I_2| = |I| < |J| = |J_1| + |J_2|$ and so $|I_1| < |J_1|$ or $|I_2| < |J_2|$. For definiteness assume $|I_1| < |J_1|$. Since J_1 and I_1 are independent in $G_1(S_1)$, $\exists\, y \in J_1 \setminus I_1$ with $I_1' = I_1 \cup y \in J_1$.

If $y \in I$, then as $y \notin I_1$, $y \in I_2$. Put $I_1^* = I_1 \cup y \in {}_1$ and $I_2^* = I_2 \setminus y \in {}_2$. Then $I = I_1^* \cup I_2^*$ and $I_1^* \cap I_2^* = \phi$. But $|I_1^* \cap J_2| = |(I_1 \cup y) \cap J_2| = |I_1 \cap J_2|$ (as $y \notin J_2$) and $|I_2^* \cap J_1| = |(I_2 \setminus y) \cap J_1| = |I_2 \quad J_1| - 1$ (as $y \in J_1 \cap I_2$), contradicting the minimality of $|I_1 \cap J_2| + |I_2 \cap J_1|$. Hence $y \notin I$ and $I \cup y = (I_1 \cup y) \cup I_2 \in \mathcal{J}$. This contradicts the maximality of I.

Thus $|I| \geq |J|$.

Similarly $|J| \geq |I|$ so that $|I| = |J|$ and the theorem is proved. //

Inductively we have

2.6.2 THEOREM. The union of any finite collection of pregeometries exists and is a pregeometry.

PROOF. Let $G_1(S_1), \ldots, G_n(S_n)$ be pregeometries. The theorem is true when $n = 2$. Assume that the theorem is true for any union of k pregeometries, when $k < n$. Let $\mathcal{J}_i$ be the collection of independent sets of $G_i(S_i)$, $i = 1, 2, \ldots, n$. By the assumption

$$\mathcal{J} = \{ I_1 \cup \ldots \cup I_{n-1} \,/\, I_i \in \mathcal{J}_i,\ i = 1, \ldots n-1 \}$$

is the collection of independent sets of a pregeometry $G(S_1 \cup \ldots \cup S_{n-1})$. Thus by Theorem 2.6.1 the union of $G(S_1 \cup \ldots \cup S_{n-1})$ and $G_n(S_n)$ is a pregeometry on $S_1 \cup \ldots \cup S_n$ as required. //

2.6.3 EXAMPLE. (i) Union of a k_1 - uniform geometry on S and a k_2 - uniform geometry on S is a $(k_1 + k_2)$ - uniform geometry on S. provided $|S| \geq k_1 + k_2$. It is Boolean if $k_1 + k_2 \geq |S|$.

(ii) Any k - uniform geometry on S is the union of k 1 - uniform geometries on S.

2.6.4 COROLLARY. The union of geometries is a geometry.

PROOF. It suffices to show that $G_1(S_1) \vee G_2(S_2)$ is a geometry if $G_1(S_1)$ and $G_2(S_2)$ are geometries. Let $A = \{x,y\}$ be any 2 - point subset of $S_1 \cup S_2$. Thus $A = A_1 \cup A_2$, where $A_1 \subseteq S_1$, $A_2 \subseteq S_2$. Thus $|A_i| \leq 2$ and since $G_i(S_i)$ is a geometry, by Corollary 1.4.4 A_i is independent in $G_i(S_i)$, $i = 1,2$. Thus A is independent in $G_1(S_1) \vee G_2(S_2)$.

Hence $G_1(S_1) \vee G_2(S_2)$ is a geometry by Corollary 1.4.4. //

The following example shows that the converse is not true.

Let $S_1 = \{1,2,3\}$, $S_2 = \{3,4,5\}$, $\mathcal{J}_1 = \{\phi,1,2,3, 12,13,23\}$

$\mathcal{J}_2 = \{\phi,4,5,45\}$, $\mathcal{J} = 2^S$, where $S = S_1 \cup S_2$

Then $\mathcal{J}$ is the family of independent set of the geometry $G_1(S_1) \vee G_2(S_2)$. By Corollary 1.6.5 $G_2(S_2)$ is not a geometry since $\mathcal{C}_2 = \{3\}$. //

2.6.5 A pregeometry $G(S_1 \cup S_2)$ is the *direct sum* of $G_1(S_1)$ and $G_2(S_2)$ if $G_{S_1 \cup S_2}(S_i) = G_i(S_i)$, $i = 1,2$, and each independent set I in

$G(S_1 \cup S_2)$ can be written uniquely $I = I_1 \overset{\circ}{\cup} I_2$, where I_j is unique and independent in $G_j(S_j)$ for $j = 1, 2$.

We denote the direct sum of $G_1(S_1)$ and $G_2(S_2)$ by $G_1(S_1) \oplus G_2(S_2)$.

Thus the union of $G_1(S_1)$ and $G_2(S_2)$ is certainly a direct sum if $S_1 \cap S_2 = \phi$.

2.6.6 THEOREM If $S_1 \cap S_2 = \phi$ we have the following.

(i) The independent sets in $G_1(S_1) \oplus G_2(S_2)$ are the disjoint union of independent sets, one from $G_1(S_1)$, one from $G_2(S_2)$.

(ii) The rank function of $G_1(S_1) \oplus G_2(S_2)$ is given by

$$r(A) = r_1(A \cap S_1) + r_2(A \cap S_2) \;, \; \forall A \subseteq S_1 \cup S_2 \;,$$

where r_1, r_2 are the rank functions of $G_1(S_1)$, $G_2(S_2)$ respectively.

(iii) The closure $\bar{A}$ of A in $G_1(S_1) \oplus G_2(S_2)$ is given by

$\bar{A}$ = closure of $(A \cap S_1)$ in $G_1(S_1) \overset{\circ}{\cup}$ closure of $(A \cap S_2)$ in $G_2(S_2)$.

Conversely if the rank, closure or independent structure of $G(S_1 \overset{\circ}{\cup} S_2)$ is given in the above way with respect to $G_{S_1 \cup S_2}(S_1)$ and $G_{S_1 \cup S_2}(S_2)$, then $G(S_1 \overset{\circ}{\cup} S_2) = G_{S_1 \cup S_2}(S_1) \oplus G_{S_1 \cup S_2}(S_2)$.

PROOF. (i) follows directly from the definition of direct sum

(ii) Given $A \subsetneq S_1 \cup S_2$. Let $r(A) = |I|$, where I is a maximal independent subset in $G_1(S_1) \oplus G_2(S_2)$ contained in A. Then $I = I_1 \overset{\circ}{\cup} I_2$, where I_j is independent in $G_j(S_j)$, $j = 1, 2$. Thus I_j is a maximal independent set contained in $A \cap S_j$ (otherwise I is not maximal). Hence $r_j(A \cap S_j) = |I_j|$ and so $r(A) = |I| = |I_1| + |I_2| = r_1(A \cap S_1) + r_2(A \cap S_2)$ as required.

(iii) Let $A \subseteq S_1 \cup S_2$. We first show that for $j = 1,2$, $r_j((A \cup a) \cap S_j) = r_j(A \cap S_j) \Leftrightarrow a \in \bar{A}$. Let I be a maximal independent subset contained in A. Then $I = I_1 \overset{o}{\cup} I_2$, where I_j is a maximal independent subset contained in $A \cap S_j$, $j = 1,2$. Thus $a \in \bar{A} \Leftrightarrow r(A \cup a) = r(A) \Leftrightarrow I$ is maximal in $A \cup a \Leftrightarrow I_j$ is maximal in $(A \cup a) \cap S_j$, $j = 1,2 \Leftrightarrow r_j(A \cap S_j) = r_j((A \cup a) \cap S_j)$, $j = 1,2$.

As $a \in \bar{A}$ only one of $a \in S_1$ or $a \in S_2$ occurs, for definiteness suppose $a \in S_1$ and hence $(A \cup a) \cap S_1 = (A \cap S_1) \cup a$. Thus $a \in \bar{A} \Rightarrow r_1((A \cup a) \cap S_1) = r_1(A \cap S_1) \Rightarrow r_1((A \cap S_1) \cup a) = r_1(A \cap S_1) \Rightarrow a \in$ closure of $A \cap S_1$ in $G_1(S_1)$. Clearly the closure of $A \cap S_j$ in $G_j(S_j) \subsetneq \bar{A}$, $j = 1,2$, so that (iii) is proved.

Now conversely we show that any independent subset I in $G(S_1 \overset{o}{\cup} S_2)$ can be written uniquely as $I_1 \overset{o}{\cup} I_2$, where I_j is independent in $G_{S_1 \overset{o}{\cup} S_2}(S_j)$, $j = 1,2$. Let I be independent in $G(S_1 \overset{o}{\cup} S_2)$. Observe that the rank of $A \cap S_j$ in $G_{S_1 \overset{o}{\cup} S_2}(S_j)$ is the rank of $A \cap S_j$ in $G(S_1 \overset{o}{\cup} S_2)$. Thus $|I| = r(I) = r(I \cap S_1) + r(I \cap S_2)$. But $|I| = |I \cap S_1| + |I \cap S_2|$. Since $r(I \cap S_j) \leq |I \cap S_j|$, $r(I \cap S_j) = |I \cap S_j|$, $j = 1,2$. Put $I_j = I \cap S_j$, $j = 1,2$. Then I_j is independent in $G_{S_1 \overset{o}{\cup} S_2}(S_j)$ and $I = I_1 \overset{o}{\cup} I_2$. Suppose $I = I_1' \overset{o}{\cup} I_2'$, where I_j' is independent in $G_{S_1 \overset{o}{\cup} S_2}(S_j)$, $j = 1,2$. Then $I_j' \subsetneq I \cap S_j = I_j$. But $|I_1'| + |I_2'| = |I \cap S_1| + |I \cap S_2|$. Thus $I_1' = I \cap S_1 = I_1$ and $I_2' = I \cap S_2 = I_2$. //

2.6.7 S_1 is a *separator* of G(S) if $G(S) = G_S(S_1) \ast G_S(S \setminus S_1)$.

Observe that S_1 is a separator if and only if $S \setminus S_1$ is a separator.

We have one further characterisation of direct sums in terms of circuits.

2.6.8 LEMMA. S_1 is a separator of $G(S)$ if and only if every circuit of $G(S)$ is contained in either S_1 or $S \setminus S_1$.

PROOF. Assume that every circuit of $G(S)$ is contained in either S_1 or $S \setminus S_1 = S_2$. Let I be any independent set of $G(S)$. Consider $I \cap S_j$, $j = 1,2$. If $I \cap S_j$ is dependent in $G_S(S_j)$, then it contains a circuit of $G_S(S_j)$ which is also a circuit of $G(S)$. Thus $I \cap S_j$ is independent in $G_S(S_j)$, $j = 1,2$. Also $I = (I \cap S_1) \overset{\circ}{\cup} (I \cap S_2)$. Let I_1 and I_2 be independent in $G_S(S_1)$ and $G_S(S_2)$ respectively. If $I = I_1 \overset{\circ}{\cup} I_2$ is not independent in $G(S)$, then it contains a circuit C. We can assume that $C \subseteq S_1$. Thus $C \subseteq I_1$ which is imposible. Hence I is independent in $G(S)$.

Let $A \subseteq S$. Then by the above $r(A) = r(A \cap S_1) + r(A \cap S_2)$. We show that $\bar{A}$ = closure of $(A \cap S_1)$ in $G_S(S_1) \overset{\circ}{\cup}$ closure of $(A \cap S_2)$ in $G_S(S_2)$. Let $\bar{A}_j$ be closure of $(A \cap S_j)$ in $G_S(S_j)$, $j = 1,2$. We see that $\bar{A}_1 \cap \bar{A}_2 = \phi$ and $\bar{A}_1 \cup \bar{A}_2 \subseteq \bar{A}$. Let $a \in \bar{A} \setminus A$. Then there exists a circuit C of $G(S)$ with $a \in C \subseteq A \cup a$. If $C \subseteq S_1$, then $a \in \bar{A}_1$. In case $C \subsetneq S_2$ we have $a \in \bar{A}_2$.

By theorem 2.6.6, $G(S) = G_S(S_1) \oplus G_S(S \setminus S_1)$ so that S_1 is a separator of $G(S)$.

Let S_1 be a separator of $G(S)$. If there is a circuit C of $G(S)$ with $C \cap S_1 \neq \phi$ and $C \cap (S \setminus S_1) \neq \phi$ .. Let $S_2 = S \setminus S_1$. Then $C \cap S_j$ is independent in $G_S(S_j)$, $j = 1,2$ and $r(C) = r(C \cap S_1) + r(C \cap S_2)$ $= |C \cap S_1| + |C \cap S_2| = |C|$. A contradiction. Thus every circuit is contained in either S_1 or $S \setminus S_1$. //

2.6.9 LEMMA. In any $G(S)$ with $\bar{\phi} = \phi$ if $r(S_1) + r(S \setminus S_1) \leq r(S)$ then any hyperplane contains either S_1 or $S \setminus S_1$.

PROOF Given any $G(S)$ with $\bar{\phi} = \phi$ and $r(S_1) + r(S \setminus S_1) \leq r(S)$ for some $S_1 \subseteq S$. Put $S_2 = S \setminus S_1$. We first show that S_1 and S_2 are flats.

$$\begin{aligned}
b \in \bar{S}_1 \setminus S_1 &\Rightarrow r(S_1 \cup b) + r(S_2) \geq r((S_1 \cup b) \cup S_2) + r((S_1 \cup b) \cap S_2), \\
&\Rightarrow r(S_1 \cup b) + r(S_2) \geq r(S) + r(b), \\
&\Rightarrow r(b) \leq r(S_1 \cup b) + r(S_2) - r(S), \\
&\Rightarrow r(b) \leq r(\bar{S}_1) + r(S_2) - r(S), \\
&\Rightarrow r(b) \leq r(S_1) + r(S_2) - r(S), \\
&\Rightarrow r(b) \leq 0, \\
&\Rightarrow b \in \bar{\phi} .
\end{aligned}$$

A contradiction. Hence $\bar{S}_1 = S_1$ and similarly $\bar{S}_2 = S_2$.

Suppose that H is a hyperplane of $G(S)$ such that $H \not\supseteq S_1$, $H \not\supseteq S_2$. Then as $H \cap S_j$ is a flat we have $r(H \cap S_j) \leq r(S_j)$, $j = 1,2$ and so $r(H \cap S_1) + r(H \cap S_2) \leq r(S_1) + r(S_2) - 2$. By (R_1) we have $r(H \cap S_1) + r(H \cap S_2) \geq r(H) + r(\phi)$ so that $r(H) \leq r(S_1) + r(S_2) - 2$ $\leq r(S) - 2$. A contradiction. Thus either $H \supseteq S_1$ or $H \supseteq S_2$ and the lemma is proved. //

We now characterise separators.

2.6.10 THEOREM. In any $G(S)$ with $\bar{\phi} = \phi$, $r(S_1) + r(S \setminus S_1) \leq r(S) \Leftrightarrow S_1$ is a separator of $G(S)$.

PROOF. Assume $r(S_1) + r(S \setminus S_1) \leq r(S)$, where $S_1 \subsetneq S$. Let $S_2 = S \setminus S_1$. By theorem 2.6.6 and Lemma 2.2.2 it suffices to show that $\bar{A} = \overline{(A \cap S_1)} \overset{o}{\cup} \overline{(A \cap S_2)}$, $\forall A \subsetneq S$. Let $A \subseteq S$. Since any hyperplane contains either S_1 or S_2, any hyperplane contains $A \cap S_1$ either contains $A \cap S_1$ and $A \cap S_2$ or contains $A \cap S_1$ but not $A \cap S_2$. Let $\mathcal{H}$ be the family of hyperplanes of $G(S)$.

Now $\overline{A \cap S_1} = \cap \{H/H \in \mathcal{H}$ and $H \supseteq A \cap S_1\}$,

$$= (\cap\{H/H \in \mathcal{H},\ H \supseteq A \cap S_1, H \supseteq A \cap S_2\}) \cap (\cap\{H/H \in \mathcal{H}, H \supseteq A \cap S_1, H \not\supseteq A \cap S_2\}),$$

and $\overline{A \cap S_2} = (\cap\{H/H \in \mathcal{H},\ H \supseteq A \cap S_1, H \supseteq A \cap S_2\}) \cap (\cap\{H/H \in \mathcal{H}, H \supseteq A \cap S_2, H \not\supseteq A \cap S_1\})$.

If we let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ be the families of hyperplanes of $G(S)$ containing $(A \cap S_1) \cup (A \cap S_2)$, $A \cap S_1$ but not $A \cap S_2$, $A \cap S_2$ but not $A \cap S_1$ respectively, then by distributive law of sets we have

$$\begin{aligned}(\overline{A \cap S_1}) \mathring{\cup} (\overline{A \cap S_2}) &= (\bigcap_{H \in \mathcal{H}_1} H) \cap \Big[(\bigcap_{H \in \mathcal{H}_2} H) \cup (\bigcap_{H \in \mathcal{H}_3} H) \Big], \\ &= \bar{A} \cap \Big[(\bigcap_{H \in \mathcal{H}_2,\text{some } H \supsetneq S_1} H) \cup (\bigcap_{H \in \mathcal{H}_3,\text{some } H \supsetneq S_2} H) \Big], \\ &= \bar{A} \cap [S] \\ &= \bar{A}.\end{aligned}$$

The converse follows from Theorem 2.6.6. //

2.6.11 LEMMA. Let S_1, S_2 be disjoint separators of $G(S)$. Then $S_1 \cup S_2$ and $S_1 \cap S_2$ are also separators of $G(S)$.

Furthermore S_1 is a separator of $G_S(T)$, $\forall T \supseteq S_1$.

PROOF. By semimodularity $r((S \setminus S_1) \cup (S \setminus S_2)) + r((S \setminus S_1) \cap (S \setminus S_2)) \leq r(S \setminus S_1) + r(S \setminus S_2)$ and $r(S_1 \cup S_2) + r(S_1 \cap S_2) \leq r(S_1) + r(S_2)$ so that adding two inequalities yields $r((S \setminus S_1) \cup (S \setminus S_2)) + r(S_1 \cap S_2) + r((S \setminus S_1) \cap (S \setminus S_2)) + r(S_1 \cup S_2) \leq r(S \setminus S_1) + r(S_1) + r(S \setminus S_2) + r(S_2) \leq r(S) + r(S)$ which gives $r(S) + r(\phi) + r(S \setminus S_1 \cup S_2) + r(S_1 \cup S_2) \leq 2r(S)$. Thus $r(S \setminus S_1 \cup S_2) + r(S_1 \cup S_2) \leq r(S)$ and so $S_1 \cup S_2$ is a separator of $G(S)$.

Since S_1^C and S_2^C are also disjoint separators of G(S), $S_1 \cap S_2 = (S_1^C \cup S_2^C)$ is a separator of G(S).

Let r_T be the rank function of $G_S(T)$. Then $r_T(T) = r(T) = r(T \cap S_1) + r(T \cap (S \setminus S_1)) = r(S_1) + r(T \setminus S_1) = r_T(S_1) + r_T(T \setminus S_1)$ so that by Theorem 2.6.10 T is a separator of $G_S(T)$. //

2.6.12 LEMMA. The family $\{S_i\}$ of minimal nontrivial separators of G(S) is finite and $S = S_1 \mathring{\cup} \dots \mathring{\cup} S_m$, where S_i's are the minimal nontrivial separators of G(S).

PROOF. Let $\{S_i\}$ be the family of minimal nontrivial separators of G(S). If $S_i \cap S_j \neq \phi$ for some $i \neq j$, then $S_i \cap S_j$ is a separator and $S_i \cap S_j \subseteq S_i$, contradicting minimality of S_i. Thus $S_i \cap S_j = \phi$, $\forall i \neq j$. Now $r(S) \geq r(S_1 \mathring{\cup} S_2 \mathring{\cup} \dots)$. Suppose that there exists $k (> r(S))$ elements in $\{S_i\}$. Then $r(S_1) + \dots + r(S_k) \geq k > r(S)$. This implies $r(S_1 \mathring{\cup} S_2 \mathring{\cup} \dots) \geq r(S_1 \mathring{\cup} \dots \mathring{\cup} S_k) = r(S_1) + \dots + r(S_k) \geq k > r(S)$. A contradiction. Thus $\{S_i\}$ consists of m elements, where $m \leq r(S)$. By Lemma 2.6.11 and the finite induction $S_1 \mathring{\cup} \dots \mathring{\cup} S_m$ is a separator of G(S).

If $S \setminus S_1 \mathring{\cup} \dots \mathring{\cup} S_m \neq \phi$, then it is a separator of G(S) and contains a minimal nontrivial separator, S_j say. Thus $S_j \subsetneq S \setminus S_1 \mathring{\cup} \dots \mathring{\cup} S_m$ which is imposible. Therefore $S = S_1 \mathring{\cup} \dots \mathring{\cup} S_m$. //

As a direct consequence we have

2.6.13 THEOREM. Every G(S) has a unique decomposition into a direct sum of irreducible direct summands.

That is $G(S) = G(S_1) \oplus \dots \oplus G(S_m)$, where $S_1, \dots, S_m$ are the minimal nontrivial separators of G(S).

2.7 CONNECTED PREGEOMETRIES

We give necessary and sufficient conditions for a pregeometry to be connected.

2.7.1 A pregeometry $G(S)$ is *connected* if the only separators of $G(S)$ are ϕ and S, thus $G(S)$ with $\bar{\phi} = \phi$ is connected iff $r(A) + r(S \setminus A) > r(S)$, $\forall A \subsetneqq S$. A pregeometry is *disconnected* if it is not connected.

2.7.2 LEMMA. $G(S)$ is connected if and only if $\forall \phi \neq A \subsetneqq S$, there exists a circuit containing elements of both A and $S \setminus A$.

PROOF. Follows from Lemma 2.6.8. //

The following useful necessary and sufficient condition for connectivity is due to Whitney [35].

2.7.3 THEOREM. $G(S)$ is connected if and only if every two distinct elements are contained in a circuit of $G(S)$.

PROOF. Assume that $G(S)$ is connected. Let x_1, x_2 be distinct elements in S. By Lemma 2.7.2 there exists a circuit containing x_1 and some elements of $S \setminus x_1$. Suppose that there exists no circuit containing both x_1 and x_2. Let $S_1 = x_1 \cup$ all circuits containing x_1. Then $\phi \neq S_1 \subsetneqq S$. Again by Lemma 2.7.2 there exists a circuit P_3 containing elements of both S_1 and $S \setminus S_1$. Pick an element $x_4 \in P_3 \cap S_1$. Since $x_1 \notin P_3$, $x_4 \neq x_1$ so that by the definition of S_1 there is a circuit P_1 containing x_1 and x_4. Let $S_2 = P_3 \cap (S \setminus S_1)$ and choose $x_3 \in S_2$. Now $S_1 \cup S_2$ is a subset of S such that it contains circuits P_1 and P_3 containing x_1 and x_3 respectively and P_1, P_3 have a common element.

We choose a smallest subset S' of S with such property. Then $S' = P_1 \cup P'_3$, where P'_1 and P'_3 are circuits containing x_1 and an element x_3 of $S \setminus S_1$ respectively and P'_1, P'_3 have a common element (otherwise S' is not smallest with the specified property). Let x'_4 be a common element of P'_1 and P'_3. By (K'_4) there exist circuits P_4 and P_5 both containing x_1 and x_3 respectively but not x'_4. Thus $P_4 \cup P_5 \subsetneq P'_1 \cup P'_3$ and so P_4, P_5 have no common element (as $P'_1 \cup P'_3$ is smallest with the specified property). Since $P_4 \setminus P'_3 \not\subseteq P'_1$, P_4 contains an element x_5 of $P'_3 \setminus P'_1$. Also P_5 contains an element x_6 of $P'_1 \setminus P'_3$. Consider the circuits P'_1 and P_5. Now P'_1 contains x_1 and P_5 contains x_3 and they have a common element x_6. But $x_5 \notin P_5$ and so $P'_1 \cup P_5 \subsetneq P'_1 \cup P'_3$. A contradiction. Thus there exists a circuit containing both x_1 and x_2.

Let $G(S)$ be a pregeometry such that every two distinct elements are contained in a circuit of $G(S)$. If $\phi \neq S_1 \subseteq S$ is a separator of $G(S)$, let $x_1 \in S_1$. By the assumption $\forall x_1 \neq x \in S$ there exists a circuit C_x containing both x and x_1. By Lemma 2.6.8 $C_x \subseteq S_1$ and so $x \in S_1$. Thus $S_1 = S$. Therefore $G(S)$ is connected. //

2.7.4 EXAMPLE. Any k - uniform pregeometry on a set of size $> k$ is connected.

2.7.5 A subset T of S is *connected* in $G(S)$ if $G_S(T)$ is connected.

It then follows that any minimal separator of $G(S)$ is connnected.

2.7.6 LEMMA. If C_1 and C_2 are circuits of $G(S)$ containing x, y and

x, z respectively, then there exists a circuit C of $G(S)$ containing y, z and $C \subsetneq C_1 \cup C_2$.

PROOF. We proceed first to prove this for finite S by induction on $|S|$. It is true for $|S| \leq 3$. Assume that it is true for any $G(T)$ with $|T| < n$. Let $G(S)$ be a pregeometry on a set S of n elements. Let C_1 and C_2 be circuits of $G(S)$ containing x, y and x, z respectively.

If $C_1 \cup C_2 \neq S$, let $T = S \setminus a$ for some $a \in S \setminus C_1 \cup C_2$. Then C_1 and C_2 are circuits of $G_S(T)$ containing x, y and x, z respectively and so by induction hypothesis there exists a circuit C of $G_S(T)$ containing y, z as required.

If $C_1 \cup C_2 = S$. By (K_4') there exist circuits C_3, C_4 with $y \in C_3 \subseteq C_1 \cup C_2 \setminus x$, $z \in C_4 \subseteq C_1 \cup C_2 \setminus x$. Obviously $C_3 \cap C_1 \subseteq C_1 \setminus C_2$ and $C_3 \cap C_1 \neq \phi$. If $C_3 \cap C_1 \subsetneqq C_1 \setminus C_2$, then $C_3 \cup C_2 \neq S$ and $C_3 \cap C_2 \neq \phi$ so that by the induction hypothesis there exists a circuit C_3 of $G_S(C_3 \cup C_2)$ containing y and z. Thus we have the result if $C_3 \cap C_1 \subsetneqq C_1 \setminus C_2$ or $C_4 \cap C_2 \subsetneqq C_2 \setminus C_1$. Suppose $C_3 \cap C_1 = C_1 \setminus C_2$ and $C_4 \cap C_2 = C_2 \setminus C_1$. Now $C_3 \cup C_4 \subseteq C_1 \cup C_2 \setminus x$ and as $C_3 \cap (C_2 \setminus C_1) \neq \phi$ we have $C_3 \cap C_4 \neq \phi$. By the induction hypothesis there exists a circuit C of $G_S(C_3 \cup C_4)$ containing y, z. Hence we have the result for finite S.

In case S is infinite we apply the above for $G_S(C_1 \cup C_2)$. //

As a consequence of Lemma 2.7.6 we note that $G_S(C_1 \cup C_2)$ is connected if C_1, C_2 are circuits of $G(S)$ having a common element.

2.7.7. THEOREM. Let A, B be connected in G(S). If $A \cap B \neq \phi$, the $A \cup B$ is connected.

(In case $A \cap B = \phi$ this is not necessarily true. For example, the union of two disjoint polygons of a graph is not connected in the polygon pregeometry of that graph but both of the two polygons are connected).

PROOF. Pick an element $x \in A \cap B$. Let y, z be distinct elements in $A \cup B$. We show that there is a circuit of G(S) containing y and z. If both y and z are in A or B, then y, z are contained in a circuit of G(S) as A and B are connected. Suppose that $y \in A \setminus B$, $z \in B \setminus A$. Then by Theorem 2.7.3 there exist circuits C_1 or $G_S(A)$ and C_2 of $G_S(B)$ containing x, y and x, z respectively. By Lemma 2.7.6 there exists a circuit $C \subseteq C_1 \cup C_2$ containing both y and z. The theorem is proved. //

The following theorem shows that any connected pregeometry contains subpregeometry or contraction which is connected. The proof is due to Murty [66] .

2.7.8 THEOREM. If G(S) is connected, then for every $x \in S$ at least one of $G_S(S \setminus x)$ and $G(S).(S \setminus x)$ is connected.

PROOF. The theorem is true when $|S| = 1$. Let $G(S)$ be connected and $|S| \geq 2$. Let $x \in S$. Suppose that $G(S).(S \setminus x)$ is not connected and so it has a separator $S_1 \neq \phi$, $S \setminus x$. Then $S \setminus x \setminus S_1 \neq \phi$, $S \setminus x$ and $S \setminus x \setminus S_1$ is also a separator of $G(S).(S \setminus x)$. Let $S_1, S_2, \ldots, S_t$ be all minimal nontrivial separators of $G(S).(S \setminus x)$. Thus $t \geq 2$. We show that $G_S(S \setminus x)$ is connected. Let y, z be distinct elements of $S \setminus x$.

case 1. If $y \in S_i$, $z \in S_j$, where $i \neq j$. Since $G(S)$ is connected, y and z are contained in a circuit C of $G(S)$. Suppose, that $x \in C$. Then x is maximally independent in $S \setminus (S \setminus x)$ so that $C \setminus x$ is dependent in $G(S).(S \setminus x)$. For any $y \in C \setminus x$, $C \setminus y = ((C \setminus x) \setminus y) \cup x$ is independent in $G(S)$ and hence $(C \setminus x) \setminus y$ is independent in $G(S).(S \setminus x)$. Thus $C \setminus x$ is a circuit of $G(S).(S \setminus x)$. By Lemma 2.6.12, $S_i \cap S_j = \phi$. Therefore $C \setminus x$ is a circuit of $G(S).(S \setminus x)$ which is not contained in S_i or $(S \setminus x) \setminus S_i$. This contradicts the separability of S_i. Hence $x \notin C$ and so C is a circuit of $G_S(S \setminus x)$ containing y and z.

case 2. If $y, z \in S_i$. Pick $a \in S_j$ for some $j \neq i$. Then there exist circuits C_1 and C_2 of $G(S)$ containing a, y and a, z respectively. By case 1, $x \notin C_1$ and $x \notin C_2$. Now $C_1 \cap C_2 \neq \phi$ and so there exists a circuit C_3 of $G_S(C_1 \cup C_2)$ containing y and z. Since $C_3 \subseteq C_1 \cup C_2$, $x \notin C_3$ and hence C_3 is a circuit of $G_S(S \setminus x)$ as desired. //

2.8 DUALITY

We define the *dual* of any matroid. (Remembering a matroid $M(S)$ is any pregeometry $G(S)$, where S is finite and use this concept for a hyperplane characterisation.)

2.8.1 THEOREM. In $M(S)$ with rank function r the function $r^* : A \longmapsto |A| + r(S \setminus A) - r(S)$ is the rank function of a matroid $M^*(S)$ on S – the dual matroid of $M(S)$. Moreover the dual matroid has rank $|S| - r(S)$.

PROOF. We first show that r^* is unit increasing. Let $A \subseteq S$, $a \notin A$. Put $B = A \cup a$. Then $r^*(B) = |B| + r(S \setminus B) - r(S) = |A| + 1 + r((S\setminus A)\setminus a) - r(S) = |A| + 1 + r(S\setminus A) - \{ {0 \atop 1} - r(S) = |A| + r(S\setminus A) - r(S) + (1 - \{ {0 \atop 1}) = r^*(A) + \{ {0 \atop 1}$.

Since S is finite, for a given $A \subseteq B$ we have $B = A \overset{\circ}{\cup} a_1 \ldots a_n$ for some $n \geq 0$ so that $r^*(A \cup a_1) \geq r^*(A)$ by the above and inductively $r^*(B) \geq r^*(A)$. Thus r^* is increasing.

We show that r^* is semimodular. Let $A,B \subseteq S$. Then

$$\begin{aligned} r^*(A) + r^*(B) &= |A| + |B| + r(S\setminus A) + r(S\setminus B) - 2r(S), \\ &\geq |A| + |B| + r\left[(S\setminus A) \cup (S\setminus B)\right] + r\left[(S\setminus A) \cap (S\setminus B)\right] - 2r(S), \\ &\geq |A \cup B| + |A \cap B| + r(S\setminus A \cap B) + r(S\setminus A \cup B) - 2r(S), \\ &\geq r^*(A \cup B) + r^*(A \cap B). \end{aligned}$$

Obviously r^* has finite basis property and $r^*(\phi) = |\phi| + r(S\setminus\phi) - r(S) = 0$.

Hence r^* is the rank function of a unique matroid on S. //

Now $r^{**}(A) = |A| + r^{*}(S \setminus A) - r^{*}(S) = |A| + |S \setminus A| + r(A)$ $- r(S) - |S| + r(S) = r(A)$. Thus we have

2.8.2 LEMMA. The dual of the dual of M(S) is the matroid M(S) itself.

We next link bases, circuits of M(S) and $M^{*}(S)$. The word *cobases* and *cocircuits* are used for bases and circuits of $M^{*}(S)$ respectively.

2.8.3 THEOREM. In any matroid M(S) the following are true.

(i) The cobases are exactly the complement of bases of M(S).

(ii) The cocircuits are exactly the complement of hyperplanes of M(S).

(iii) The cocircuits are exactly the subsets of S which minimally intersect all bases of M(S).

PROOF. (i) For any subset A of S we have $r^{*}(S \setminus A) = |S \setminus A| + r(A) - r(S)$ so that $r^{*}(S) - r^{*}(S \setminus A) = r^{*}(S) + r(S) - |S \setminus A| - r(A) = |S| - |S \setminus A| - r(A) = |A| - r(A)$. Thus A is independent in M(S) iff $S \setminus A$ is a spanning set in $M^{*}(S)$. If we replace A by $S \setminus A$ in $M^{*}(S)$ we see that $S \setminus A$ is independent in $M^{*}(S)$ iff A is a spanning set in M(S). Now

$$\begin{aligned}
A \text{ is spanning in } M(S) &\Leftrightarrow S \setminus A \text{ is independent in } M^{*}(S),\\
&\Leftrightarrow r^{*}(S \setminus A) = |S \setminus A| + |A| - |A| \quad ,\\
&\Leftrightarrow r^{*}(S \setminus A) = |S| - r(S) \quad (\text{as } r(S) = |A|\ ),\\
&\Leftrightarrow r^{*}(S \setminus A) = r^{*}(S).
\end{aligned}$$

(ii) C is a circuit of $M^{*}(S)$ <=> C is minimal dependent in $M^{*}(S)$ <=> $S \setminus C$ is maximal non - spanning subset in M(S) <=> $S \setminus C$ is a hyperplane of M(S).

(iii) C is a cocircuit of $M(S)$ $\Leftrightarrow$ $S \setminus C$ is a hyperplane of $M(S)$ $\Leftrightarrow$ $S \setminus C$ is a maximal non-spanning set in $M(S)$ $\Leftrightarrow$ $S \setminus C$ is a maximal set not containing any basis.

If $S \setminus C$ does not contain any basis, then C must intersect every basis. Suppose $\exists\, C' \subsetneqq C$ such that C' intersects every basis. Let $a \in C \setminus C'$. Then $(S \setminus C) \cup a$ contains a basis B which contains a and so $B \cap C' = \phi$. A contradiction. Therefore C is a minimal set intersecting every basis.

Conversely let C be a minimal set with this property. Then $S \setminus C$ does not contain any basis. If there exists $x \in C$ such that $(S \setminus C) \cup x$ does not contain any basis. Then $C' = C \setminus x$ intersects every basis, contradicting the minimality of C. Thus $S \setminus C$ is a maximal set not containing any basis and the theorem is proved. //

2.8.4 LEMMA. Let A, A^* be independent in $M(S)$ and $M^*(S)$ respectively with $A \cap A^* = \phi$. Then there exists a basis B such that $A \subseteq B$, $A^* \subseteq S \setminus B$.

PROOF. Let r, r^* be rank functions of $M(S)$ and $M^*(S)$ respectively. Thus $r(S \setminus A^*) = |S| - |A^*| - r^*(S) + r^*(A^*) = |S| - r^*(S) = r(S)$. Extend A to a basis $B \subseteq S \setminus A^*$. Then $A^* \subseteq S \setminus B$ as required. //

2.8.5 LEMMA. For any circuit C and any cocircuit C^* of $M(S)$ we have $|C \cap C^*| \neq 1$.

PROOF. Let C and C' be any circuit and cocircuit of $M(S)$ respectively. We may assume that $C \cap C^* \neq \phi$. Consider $C \setminus C \cap C^*$ and $C^* \setminus C \cap C^*$ which are independent in $M(S)$ and $M^*(S)$ respectively.

By Lemma 2.8.4 there exists a basis B with $C \setminus C \cap C^* \subseteq B$ and $C^* \setminus C \cap C^* \subsetneq S \setminus B$. Since $C^* \cap B \neq \phi$ and $(C^* \setminus C \cap C^*) \cap B = \phi$, there exists $y \in (C \cap C^*) \cap B$. If $|C \cap C^*| = 1$, then $C \subseteq B$ which is a contradiction. Thus $|C \cap C^*| \neq 1$. //

2.8.6 LEMMA. Let B be a basis of M(S). For any $e \in B$ there is a unique cocircuit C^* of M(S) with $(B \setminus e) \cap C^* = \phi$.

PROOF. Since $(S \setminus B)$ is a basis of $M^*(S)$ and $e \notin S \setminus B$, by Theorem 1.6.8 there exists a unique circuit C^* of $M^*(S)$ with $e \in C^* \subsetneq S \setminus B$. Thus $(B \setminus e) \cap C^* = \phi$. Let C_1^* be a cocircuit of M(S) with $(B \setminus e) \cap C_1^* = \phi$. Then $C_1^* \subseteq (S \setminus B) \cup e$ and $e \in C_1^*$. By the uniqueness of C^* we have $C^* = C_1^*$ and the lemma is proved. //

2.8.7 LEMMA. Let a, b be distinct elements of a circuit C of M(S). Then there exists a cocircuit C^* of M (S) with $C \cap C^* = a\,b$.

PROOF. Extend $C \setminus a$ to a basis B of M(S). Then $B^* = S \setminus B$ is a basis of $M^*(S)$ and $a \in B^*$. Now $b \notin B^*$. Consider the fundamental circuit C^* (in $M^*(S)$) of b in B^*. If $a \notin C^*$, then $C \cap C^* = b$ and so $|C \cap C^*| = 1$ which is impossible. Thus $C \cap C^* = ab$. //

Duality helps characterise a matroid in terms of its hyperplanes.

2.8.8 THEOREM. A collection $\mathcal{H}$ of nonempty proper subsets of S is the set of hyperplanes of M(S) if and only if it satisfies the following.

(H_1) For any H_1, H_2 in $\mathcal{H}$, $H_1 \subsetneq H_2$.

(H_2) If $H_1, H_2 \in \mathcal{H}$ and $x \notin H_1 \cup H_2$, then $\exists\, H_3 \in \mathcal{H}$ such that $H_3 \supseteq (H_1 \cap H_2) \cup x$.

PROOF. Let $\mathcal{H}$ be the family of hyperplanes of $M(S)$. Then (H_1) follows directly from Lemma 1.5.6 . Observe that (H_2) is equivalent to

(H_2') If $H_1, H_2 \in \mathcal{H}$ and $x \in (S\setminus H_1) \cap (S\setminus H_2)$ then $\exists H_3 \in \mathcal{H}$ such that $x \notin (S\setminus H_3) \subseteq (S\setminus H_1) \cup (S\setminus H_2)$ which is (K_4) in $M^*(S)$. Thus (H_2) follows.

Conversely let $\mathcal{H}$ be a collection of nonempty proper subsets of S satisfying (H_1) and (H_2). We show that $\mathcal{C} = \{ S\setminus H \ / \ H \in \mathcal{H} \}$ is the family of circuits of $M^*(S)$. Obviously $\mathcal{C}$ satisfies (K_1) and (K_3) and by (H_1) $\mathcal{C}$ satisfies (K_2). That $\mathcal{C}$ satisfies (K_4) follows from the fact that $(H_2) \Leftrightarrow (H_2')$.

Thus $\mathcal{C}$ is the family of circuits of $M^*(S)$ and hence by Theorem 2.8.3 $\mathcal{H}$ is the family of hyperplanes of $M(S)$. //

2.8.9 A *Steiner triple system* on a set S_n of n elements is a collection $\mathcal{S}_n$ of 3 - element subsets of S_n, called triples, having any two distinct elements of S_n in a unique triple. (cf Hall [67], p236)

We note some properties of any Steiner triple system $\mathcal{S}_n$ which are needed later.

(i) A necessary and sufficient condition for the existence of some $\mathcal{S}_n$ on a set of size n is that $n \equiv 1$ or $3 \pmod 6$

(ii) Any element of S_n occurs in exactly $\frac{n-1}{2}$ triples of $\mathcal{S}_n$

(iii) The number of triples in $\mathcal{S}_n$ is $\frac{n(n-1)}{6}$.

2.8.10 EXAMPLE. $\mathcal{S}_n$ is the collection of hyperplanes of a matroid $M(\mathcal{S}_n)$ on S_n with rank 3. The bases of $M(\mathcal{S}_n)$ are all 3 - element subsets of S_n which are not in $\mathcal{S}_n$.

For n = 7 $M(\mathcal{S}_n)$ is the well known Fano matroid.

PROOF. We first show that $\mathcal{S}_n$ satisfies (H_1) and (H_2). That $\mathcal{S}_n$ satisfies (H_1) is clear from its definition. Let A, B be triples of $\mathcal{S}_n$ and $x \notin A \cup B$. Since any two distinct triples intersect in one element or no element, $(A \cap B) \cup x$ is contained in one triple of $\mathcal{S}_n$. Thus $\mathcal{S}_n$ satisfies (H_2) and so it is the collection of hyperplanes of a matroid $M(\mathcal{S}_n)$ on S_n.

Since any element of S_n occurs in $\frac{n-1}{2}$ triples and the number of triples is $\frac{n(n-1)}{6} \neq \frac{n-1}{2}$ if $n > 3$, the intersection of all triples is empty and thus $\bar{\phi} = \phi$. For any $x \in S_n$ the intersection of all triples containing x is x and so $\bar{x} = x$. Thus $M(\mathcal{S}_n)$ is a geometry and hence every 2 - element subset of S_n is independent. We show that if $X = \{x,y,z\} \notin \mathcal{S}_n$, then X is independent. Let $A = \{x,y,a\}$ be the triple containing x,y. Then $a \neq z$. Now $S_n \setminus A$ is a cocircuit of $M(\mathcal{S}_n)$. Suppose that X is not independent. Since any proper subset of X is independent, X is a circuit of $M(\mathcal{S}_n)$. But $X \cap (S_n \setminus A) = z$, contradicting Lemma 2.8.5. Hence X is independent as required. //

We link submatroids, contractions and duals.

2.8.11 THEOREM. In any matroid M(S) for $T \subseteq S$, we have the following.

(i) $(M_S(T))^* = M^*(S).T$

(ii) $(M(S).T)^* = M_S^*(T)$

PROOF. Let r, r^*, $(r^*)^T$, ρ and ρ^* be rank functions of M(S), $M^*(S)$, $M^*(S).T$, $M_S(T)$ and $(M_S(T))^*$ respectively.

(i) For any subset A of T we have $\rho^*(T \setminus A) = |T \setminus A| + \rho(A) - \rho(T)$ $= |T| - \rho(T) - |A| + \rho(A)$. By Lemma 2.5.2 we have $(r^*)^T(T \setminus A) =$ $r^*((T \setminus A) \cup (S \setminus T)) - r^*(S \setminus T) = r^*(S \setminus A) - r^*(S \setminus T) = |S| - r(S)$

$- |A| + r(A) - (|S| - r(S) - |T| + r(T)) = |T| - r(T) - |A| + r(A) = |T| - \rho(T) - |A| + \rho(A) = \rho^*(T \setminus A)$. Since A is arbitrary, $\rho^* = (r^*)^T$ so that (i) is proved.

(ii) By (i) we have $(M_S^*(T))^* = (M^*(S))^*.T = M(S).T$. Taking dual both sides we obtain $M_S^*(T) = (M(S).T)^*$ //

We also have information about connectedness of duals

2.8.12 LEMMA. S_1 is a separator of M(S) if and only if S_1 is a separator of $M^*(S)$.

PROOF. Since $(M^*(S))^* = M(S)$, it suffices to show that a separator S_1 of M(S) is a separator of $M^*(S)$. Now $r^*(S_1) + r^*(S \setminus S_1) = |S_1| + r(S \setminus S_1) - r(S) + |S| - |S_1| + r(S_1) - r(S) = r(S_1) + r(S \setminus S_1) - r(S) - r(S) + |S| \leq r(S) - r(S) - r(S) + |S| \leq r^*(S)$. Thus by Theorem 2.6.10 S_1 is a separator of $M^*(S)$ and the lemma is proved. //

As a consequence of Lemma 2.8.12 we have

2.8.13 LEMMA. M(S) is connected if and only if $M^*(S)$ is connected.

2.8.14 A *loop* of M(S) is an element which is a circuit of M(S) and a *coloop* of M(S) is a loop of $M^*(S)$.

We now obtain a lower bound for the number of bases of a connected matroid. We first note the following.

2.8.15 LEMMA. (i) x is a loop of M(S) if and only if x is not contained in any basis of M(S).

(ii) x is a coloop of M(S) if and only if x is contained in every basis of M(S).

(iii) x is a coloop of $M(S)$ if and only if x is not contained in any circuit of $M(S)$.

2.8.16 THEOREM. Let $\mathcal{B}$ be the family of bases of a connected matroid. Then $|\mathcal{B}| \geq |S|$.

PROOF. We prove the theorem by induction on $|S|$. The Theorem is true when $|S| = 1$. Assume that the theorem is true for any connected matroid $M(T)$, where $1 \leq |T| < n$. Let $M(S)$ be a connected matroid on a set S of n elements.

For any $x \in S$ there exists $x \neq y \in S$ and thus x,y are contained in a circuit C of $M(S)$. Hence x is not a loop of $M(S)$. Now $C \setminus x$ is independent and $C \setminus x \subseteq$ a basis $B \in \mathcal{B}$. As C is dependent, $x \notin B$. Therefore x is not a coloop of $M(S)$. That is $M(S)$ has no loops and coloops.

Let $x \in S$ and let n_1, n_2 be the number of bases of $M_S(S \setminus x)$, $M(S).(S \setminus x)$ respectively. Observe that $M_S(S \setminus x)$ and $M(S).(S \setminus x)$ have no common basis since the bases of $M_S(S \setminus x)$ are the bases of $M(S)$ not containing x and the bases of $M(S).(S \setminus x)$ are the bases of $M(S)$ containing x. As any basis of $M(S)$ either contains x or does not contain x we have $|\mathcal{B}| = n_1 + n_2$. By Theorem 2.7.9 at least one of $M_S(S \setminus x)$ and $M(S).(S \setminus x)$ is connected.

If $M_S(S \setminus x)$ is connected, then $n_1 \geq |S| - 1$. Since x is not a loop of $M(S)$, x is contained in a basis of $M(S)$ so that $n_2 \geq 1$. Thus $|\mathcal{B}| \geq |S| - 1 + 1 = |S|$ ..

If $M(S).(S \setminus x)$ is connected we have $n_2 \geq |S| - 1$. Also x is not a coloop and hence is not contained in a basis of $M(S)$ so that

$n_1 \geq 1$. Thus $|\mathcal{B}| \geq 1 + |S| - 1 = |S|$ and the theorem is proved. //

In fact Murty [66] showed that $|\mathcal{B}| \geq r(n - r) + 1$ if $M(S)$ is a connected matroid of rank r on a set S of n elements.

2.8.17 LEMMA. A nonempty separator of a matroid without loops and coloops has cardinality at least 2.

PROOF. Suppose that x is a separator of $M(S)$ which has no loops and coloops. Thus x is not a loop and so x is not a circuit. Hence all circuits of $M(S)$ are contained in $S \setminus x$. Therefore x is not contained in any circuit of $M(S)$ so that by Lemma 2.8.9 x is a coloop of $M(S)$. This contradicts the assumption. So any separator of $M(S)$ has cardinality at least 2. //

2.8.18 LEMMA. Let S_1 be a separator of $M(S)$. If $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are the families of bases of $M(S)$, $M_S(S_1)$ and $M_S(S \setminus S_1)$ respectively. Then $|\mathcal{B}| = |\mathcal{B}_1|\,|\mathcal{B}_2|$.

PROOF. We first show that $|B_1 \cap S_1| = |B_2 \cap S_1|$ for every two bases B_1, B_2 of $M(S)$. Suppose that B_1, B_2 are bases of $M(S)$ with $|B_1 \cap S_1| < |B_2 \cap S_1|$. Let $I_1 = B_1 \cap S_1$ and $I_2 = B_2 \cap S_1$. By Lemma 1.4.2 and finite induction there exists a nonempty subset I of $I_2 \setminus I_1$ such that $I_1 \cup I$ is independent and $|I_1 \cup I| = |I_2|$. Observe that $B_1 \cap S_1$ is a maximal subset of B_1 contained in S_1. Consider any x in I we see that $x \notin B_1$. Then $C(x, B_1)$ is contained in S_1 or $S \setminus S_1$. But $x \in S_1$, hence $C(x, B_1) \subsetneq S_1$. Now $C(x, B_1) \subsetneq (B_1 \cap S_1) \cup x = I_1 \cup x$. A contradiction. Thus $|I_1| \geq |I_2|$. Similarly $|I_2| \geq |I_1|$ and so $|I_1| = |I_2|$.

Let $B \in \mathcal{B}$. By the above $B \cap S_1$ is a basis of $M_S(S_1)$ and

$B \cap (S \setminus S_1)$ is a basis of $M_S(S \setminus S_1)$. Conversely if $B_1 \in \mathcal{B}_1$ we can extend B_1 to a basis B of M(S). Then $B \cap (S \setminus S_1)$ is a basis of $M_S(S \setminus S_1)$. Thus a subset B of S is a basis of M(S) if and only if $B \cap S_1$ is a basis of $M_S(S_1)$ and $B \cap (S \setminus S_1)$ is a basis of $M_S(S \setminus S_1)$. Therefore $|\mathcal{B}| = |\mathcal{B}_1||\mathcal{B}_2|$. //

2.8.19 THEOREM. Let $\mathcal{B}$ be the family of bases of a disconnected matroid which has no loops and coloops. Then $|\mathcal{B}| \geq |S|$.

PROOF. We first show that the theorem is true for $|S| \leq 4$. The theorem is obviously true for $|S| = 1$.

For $|S| = 2$. Let $x_1 \neq x_2 \in S$. Since x_1 and x_2 are not coloops, $x_1 \notin B_1$ and $x_2 \notin B_2$ for some bases B_1, B_2 of M(S). If $B_1 = B_2$ and no other bases, then x_1 and x_2 are coloops which is not so. Thus M(S) has at least two bases.

For $|S| = 3$. Let $S = \{x_1, x_2, x_3\}$. As each of x_1, x_2, x_3 is contained in a basis every basis has at least one element. If M(S) has only one basis, then every element is a coloop. Thus M(S) has at least 2 bases. Suppose that there are only 2 bases. Then every basis consists of exactly 2 elements and the two bases have a common element which is a coloop. A contradiction. Thus M(S) has at least 3 bases.

For $|S| = 4$. We can show that $|\mathcal{B}| \geq 4$ by using the same argument as the case $|S| = 3$.

Assume that the theorem is true for all M(T) with $4 \leq |T| < n$. Let M(S) be a disconnected matroid on a set S of n elements which has no loops and coloops. Then there is a separator S_1 of S with $S_1 \neq \phi$, S.

Also $S \setminus S_1$ is a separator of $M(S)$ with $S \setminus S_1 \neq \phi$, S. Let $\mathcal{B}_1$, and $\mathcal{B}_2$ be the families of bases of $M_S(S_1)$ and $M_S(S \setminus S_1)$ respectively. Then by Lemma 2.8.12 we have $|\mathcal{B}| = |\mathcal{B}_1|\ |\mathcal{B}_2|$. By the induction hypothesis $|\mathcal{B}_1| \geq |S_1|$ and $|\mathcal{B}_2| \geq |S \setminus S_1|$ so that $|\mathcal{B}| \geq |S_1|\,|S \setminus S_1|$. Now $|S_1| \geq 2$ by Lemma 2.8.11 and so $|\mathcal{B}| \geq 2n - 4 \geq n$ as required. //

3. TRANSVERSAL PREGEOMETRIES

We define and obtain simple properties of the important class of transversal pregeometries.

3.1 REPRESENTATIONS

Here we define, and discuss various representations, of transversal pregeometries.

3.1.1 A *family* (or *listing*) of subsets of a set X is a function $f : I \to 2^X$ with I well-ordered.

We usually denote it by (X_I) or $(X_i, i \in I)$; I being the *index set* of the family.

3.1.2 Given a family (X_I), $X \subsetneq S$. We define as a *system of representatives of* $(X)_I$ (or *choice function*), denoted by SR any function $\phi : I \to S$ satisfying $\phi(i) \in X_i$, $\forall i \in I$.

If ϕ is injective, it is a *system of distinct representatives of* $(X)_I$, denoted by SDR, and its image $\phi(I)$ is a *transversal of* $(X)_I$.

In general a family $(X)_I$ of nonempty sets may not have an SDR. For example if $X_1 = \{a,b\}$, $X_2 = \{a,c\}$, $X_3 = \{b,c\}$, $X_4 = \{a,b,c\}$, then $\phi : I = \{1,2,3,4\} \to \{a,b,c\}$ defined by $\phi(1) = a$, $\phi(2) = c$, $\phi(3) = b$, $\phi(4) = a$ is an SR of $(X)_I$ but $(X)_I$ has no SDR.

3.1.4 A *subfamily* $(X)_J$ *of a family* $(X)_I$ is a restriction of $f : I \to 2^S$ to $J \subsetneq I$.

We write $\bigcup_{J} X$ or $\bigcap_{J} X$ to denote the union or intersection of sets in f(J) and we write $(X)_J$ to denote f(J).

3.1.5 A *partial system of (distinct) representatives of a family* $(X)_I$, denoted by PSR(PSDR), is an SR(SDR) of some subfamily of $(X)_I$.

A *partial transversal of* $(X)_I$ is a transversal of some subfamily of $(X)_I$.

3.1.6 THEOREM. Let $(X)_I$ be a finite family of subsets of a set S. Then the collection $\mathcal{J}$ of all partial transversals of $(X)_I$ is the family of independent sets of a pregeometry on S.

PROOF. For each $i \in I$, let $\mathcal{J}_i$ be the collection of empty set and all singletons of X_i. Then $\mathcal{J}_i$ is the collection of independent sets of a pregeometry $G_i(X_i)$ on X_i. Let $\mathcal{J}'$ be the collection of independent sets of the union of $G_i(X_i)$, where $i \in I$. We show that $\mathcal{J} = \mathcal{J}'$.

For each PT $E = \{x_1, \ldots , x_r\}$ of $(X)_I$ there exists a subset J of I such that E is a transversal of $(X)_J$. We can assume that $J = \{1, \ldots, r\}$ and $x_j \in X_j$, $\forall_j \in J$. Thus $x_j \in \mathcal{J}_j$, $\forall_j \in J$ and so $E = \bigcup_{j \in J} x_j \in \mathcal{J}'$.

Let $A \in \mathcal{J}'$. Then $A = \bigcup_{r \in R} x_r$ for some $R \subsetneq I$ and $r \neq s$ $\Rightarrow x_r \neq x_s$. Define $\phi\colon R \to A$ by $\phi(r) = x_r$. We see that ϕ is bijective and $\phi(R) = A$. Hence A is a PT of $(X)_I$. Thus $A \in \mathcal{J}$ and the theorem is proved. //

3.1.7 LEMMA. Let $(X)_I$ be a finite family of subsets of a set S.

Let $\mathcal{F}$ be the collection of subsets of I satisfying : $J \in \mathcal{F}$ if and only if $(X)_J$ has a transversal. Then $\mathcal{F}$ is the collection of independent sets of a matroid on I.

PROOF. Since ϕ is a PT of $(X)_I$ and any subset of a PT of $(X)_I$ is also a PT of $(X)_I$, we need to show that $\mathcal{F}$ satisfies (I_2). Let J_1, J_2 be subsets of I with $|J_1| < |J_2|$. Then $(X)_{J_1}$ and $(X)_{J_2}$ have transversals E_1 and E_2 respectively. Now E_1 and E_2 are PT of $(X)_I$ and $|E_1| < |E_2|$. Thus there exists $x \in E_2 \setminus E_1$ such that $E_1 \cup x$ is a PT of $(X)_I$. Since $x \in E_2$, $x \in X_j$ for some $j \in J_2$. As $E_1 \cup x$ is a PT, $j \notin J_1$ and the lemma is proved. //

3.1.8 A pregeometry G(S) is *transversal* if there exists a finite family $\mathcal{A} = (X)_I$ of subsets of S such that the collection of all PT of $(X)_I$ is the collection of independent sets of G(S).

We denote G(S) by $M[\mathcal{A}]$ or $M[X_1, \dots, X_n]$, where $I = \{1, \dots, n\}$ and call $\mathcal{A}$ a *presentation* of G(S).

Indeed a presentation of a transversal pregeometry need not be unique. As an easy example consider the matroid M(S) = M[14,234,13] on the set S ={1,2,3,4} . Another presentation of M(S) is [123, 12, 24] .

3.1.9 LEMMA. Any subpregeometry of a transversal pregeometry is transversal.

PROOF. Let $G_S(T)$ be any subpregeometry of a transversal $G(S) = M[X_1, \dots, X_n]$. Put $I = \{1, \dots, n\}$ and let $(Y)_I$ be the family of subsets of T defined by $Y_i = X_i \cap T$, $\forall i \in I$. Let $J = \{i \in I / Y_i \neq \phi\}$. Then it is clear that $G_S(T) = M[(Y_j / j \in J)]$

and so the lemma is proved. //

3.1.10 LEMMA. If $G(S)$ is a transversal pregeometry of rank r, then $G(S)$ has a presentation consisting of r sets.

PROOF. We first show that if $G(S) = G_1(S_1) \vee G_2(S_2)$ and $r(G(S)) = r(G_1(S_1))$, then $\mathcal{J} = \mathcal{J}_1$, where $\mathcal{J}$ and $\mathcal{J}_1$ are the collections of independent sets of $G(S)$ and $G_1(S_1)$ respectively. Let $\mathcal{J}_2$ be the collection of independent sets of $G_2(S_2)$. Clearly $\mathcal{J} \supseteq \mathcal{J}_1$. Let $I \in \mathcal{J}$. Then $I = I_1 \cup I_2$ for some $I_1 \in \mathcal{J}_1$ and $I_2 \in \mathcal{J}_2$. Extend I_1 to a basis B_1 of $G_1(S_1)$. Since $r(G(S)) = r(G_1(S_1))$, $|B_1 \cup I_2| \leq |B_1|$ and so $I_2 \subseteq B_1$. Thus $I_1 \cup I_2 \in \mathcal{J}_1$. Therefore $\mathcal{J} = \mathcal{J}_1$.

Let $G(S) = M[X_1, \ldots, X_n]$ and $I = \{1, \ldots, n\}$. Pick a maximal PT E of $(X)_I$. Then $|E| = r$. Suppose that E is a transversal of $(X)_R$, where $R = \{1, \ldots, r\}$. For each $i \in I$ let $\mathcal{J}_i$ and $G_i(X_i)$ be defined as in the proof of Theorem 3.1.6. Put $S_1 = \bigcup_R X$ and $S_2 = \bigcup_{I \setminus R} X$. Let $G'(S_1) = G_1(X_1) \vee \ldots \vee G_r(X_r)$ and $G''(S_2) = G_{r+1}(X_{r+1}) \vee \ldots \vee G_n(X_n)$. By the above $\mathcal{J} = \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_r$ so that $G(S) = M[X_1, \ldots, X_r]$ as required. //

Bondy and Welsh [71] showed that there exists a presentation of a transversal matroid in which each of the sets of the presentation is a cocircuit of the matroid. We now obtain this result.

3.1.11 LEMMA Let $M(S) = M[X_1, \ldots, X_n]$ be a transversal matroid of rank r. If E is a transversal of $(X_2, \ldots, X_r)$ such that $A = E \cap X_1$ has minimum cardinality. Then $M(S) = M[X_1 \setminus A, X_2, \ldots, X_r]$.

PROOF. Clearly any PT of $(X_1 \setminus A, X_2, \ldots, X_r)$ is a PT of $(X_1, \ldots, X_r)$. We show that any basis B of M(S) is a transversal of $(X_1 \setminus A, X_2, \ldots, X_r)$. Let $B = \{b_1, \ldots, b_r\}$ be any basis of M(S) and $b_i \in X_i$ where $1 \leq i \leq r$. Suppose $E = \{e_2, \ldots, e_r\}$, where $e_i \in X_i$, $2 \leq i \leq r$. The theorem is proved if $b_1 \in X_1 \setminus A$. Assume that $b_1 \in A$. Then $b_1 \in E$ and $b_1 = e_2$, say. Consider the two possibilities for b_2.

case 1. If $b_2 \in X_1 \setminus A$, then B is a transversal of $(X_1 \setminus A, X_2, \ldots, X_r)$.

case 2. If $b_2 \notin X_1 \setminus A$. We show that $b_2 \in E$. Suppose that $b_2 \notin X_1 \cup E$. Then $E' = \{b_2, e_3, \ldots, e_r\}$ is a transversal of $(X_2, \ldots, X_r)$ with $|E' \cap X_1| < |E \cap X_1|$ which is a contradiction. Thus $b_2 \in X_1 \cup E$. If $b_2 \in X_1 \setminus E$, then $b_2 \in X_1 \setminus A$ which is not so. Thus $b_2 \in E$.

Now $b_2 \neq e_2$ since $e_2 = b_1 \neq b_2$. Let $b_2 = e_3$ and repeat the same argument as above for b_3 and we shall have $b_3 = e_4 \in E$. Carrying on in this way we see that there exists i such that $b_i \in X_1 \setminus A$ and $b_j \in X_{j+1}$, $1 \leq j < i$. (otherwise we get a contradiction at the final step and so $b_1 \in X_1 \setminus A$). Thus $B = \{b_i, b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots b_r\}$ is a transversal of $(X_1 \setminus A, X_2, \ldots, X_r)$ and the lemma is proved. //

In fact we have the following stronger result

3.1.12 LEMMA. If $M(S) = M[X_1, \ldots, X_n]$ and E is a maximal partial transversal of $(X_2, \ldots, X_n)$ with $|E \cap X_1|$ minimal, then $M(S) = M[X_1 \setminus (E \cap X_1), X_2, \ldots, X_n]$.

PROOF. We can assume that $X_1 \neq \phi$. Let $x \in X_1$, Extend x to a basis B of $M(S)$. Then $B \setminus x$ is a maximal PT of $(X_2, \ldots, X_n)$. Without loss of generality assume that $B \setminus x$ is a transversal of $(X_2, \ldots, X_r)$. Observe that $M[X_2, \ldots, X_n]$ has rank $r - 1$. As in the proof of Lemma 3.1.10 $M[X_2, \ldots, X_n] = M[X_2, \ldots, X_r]$ so that $M(S) = M[X_1, \ldots, X_r]$. Since E is a transversal of $(X_2, \ldots, X_r)$ with $|E \cap X_1|$ minimal, by Theorem 3.1.11 we have $M[X_1 \setminus (E \cap X_1), X_2, \ldots X_r] = M[X_1, \ldots, X_r]$ so that $M[X_1 \setminus (E \cap X_1), X_2, \ldots, X_n] = M[X_1, \ldots, X_n]$. The lemma is proved. //

3.1.13 THEOREM. Let $M(S) = M[X_1, \ldots, X_r]$ be a transversal matroid of rank r. Then there exist distinct cocircuits $C_1^*, \ldots, C_r^*$ of $M(S)$ such that $M(S) = M[C_1^*, \ldots, C_r^*]$ and for some distinct integers $i_1, \ldots, i_r$, $C_j \subsetneq X_{i_j}$, where $1 \le j \le r$.

This presentation is *minimal* in the sense that for any i, $1 \le i \le r$ and for any $x \in C_i^*$,

$$M(S) \ne M[C_1^*, \ldots, C_{i-1}^*, C_i^* \setminus x, C_{i+1}^*, \ldots, C_r^*]$$

PROOF. Let E be a transversal of $(X_2, \ldots, X_r)$ such that $E \cap X_1$ has minimum cardinality. Put $A = E \cap X_1$. Then $(X_1 \setminus A) \cap E = \phi$ Since E is a transversal of $(X_2, \ldots, X_r)$, for any $x \in X_1 \setminus A$, $E \cup x$ is a transversal of $(X_1, \ldots, X_r)$ and hence is a basis of $M(S)$. By Lemma 3.1.11 for any basis B of $M(S)$ we have

$$B = \{x_1, \ldots, x_r\} \quad \text{where } x_1 \in X_1 \setminus A \text{ and } x_j \in X_j,\ j = 2, \ldots, r.$$

That is $X_1 \setminus A$ is a set intersecting every basis of $M(S)$. If $y \in X_1 \setminus A$ then $E \cup y$ is a basis of $M(S)$ and $(X_1 \setminus A \setminus y) \cap (E \cup y) = \phi$. Therefore $X_1 \setminus A$ is a minimal set intersecting every basis of $M(S)$. Hence $X_1 \setminus A$ is a cocircuit of $M(S)$ and $M(S) = M[X_2, X_1 \setminus A, \ldots, X_r]$. Apply the same procedure to $(X_2, X_1 \setminus A_1, \ldots, X_r)$ and so on until we obtain $M(S) = M[X_r \setminus A_r, \ldots, X_1 \setminus A_1]$ and $X_j \setminus A_j = C_j^*$ is a cocircuit of $M(S)$, $1 \leq j \leq r$. Observe that for any $x \in C_i^*$ we have $M(S) \neq M[C_1^*, \ldots, C_{i-1}^*, C_i^* \setminus x, C_{i+1}^*, \ldots, C_r^*]$ as $C_i^* \setminus x$ has empty intersection with some basis of $M(S)$ which is a transversal of $M[C_1^*, \ldots, C_r^*]$.

To see that $C_i^* \neq C_j^*$ if $i \neq j$. Suppose that there exist $i \neq j$ with $C_i^* = C_j^*$. We show that $M[C_1^*, \ldots, C_r^*]$ $= M[C_1^*, \ldots, C_i^* \setminus x, \ldots, C_r^*]$ for any $x \in C_i^*$. For any PT of M(S) such that x represents C_i^* and y represents C_j^* we obtain the same PT by representing C_i^* by y and C_j^* by x. This contradicts the minimality of $(C_1^*, \ldots, C_r^*)$. Hence all circuits are distinct and the theorem is proved. //

Moreover Theorem 3.1.13 gives an algorithm for testing whether or not a matroid is transversal.

As an example we show that the Fano matroid is not transversal.

PROOF. Suppose that $M(\mathcal{S}_7)$ is transversal. As $\mathcal{S}_7$ is the set of hyperplanes, $M(\mathcal{S}_7) = M[S_7 \setminus A_1, S_7 \setminus A_2, S_7 \setminus A_3]$, for some A_1, A_2, A_3 in $\mathcal{S}_7$. We consider the two possibilities of A_1, A_2, A_3 : (i) $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$, (ii) $A_1 \cap A_2 \neq A_1 \cap A_3 \neq A_2 \cap A_3$.

case 1. $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$

Without loss of generality assume that $A_1 = \{x_1, x_2, x_3\}$, $A_2 = \{x_1, x_4, x_5\}$, $A_3 = \{x_1, x_6, x_7\}$. Then $A_1 \cup A_2 \cup A_3 = S_7$ and $x_1 \notin S_7 \setminus A_1, S_7 \setminus A_2, S_7 \setminus A_3$. Hence $A = \{x_1, x_2, x_4\} \notin \mathcal{S}_7$. But $A \notin M[S_7 \setminus A_1, S_7 \setminus A_2, S_7 \setminus A_3]$. A contradiction.

case 2. $A_1 \cap A_2 \neq A_1 \cap A_3 \neq A_2 \cap A_3$

We note that any two distinct triples of $\mathcal{S}_7$ intersect in one element. Without loss of generality assume that $A_1 = \{x_1, x_2, x_4\}$, $A_2 = \{x_1, x_3, x_5\}$, $A_3 = \{x_2, x_3, x_6\}$. Then $S_7 \setminus A_1 = \{x_3, x_5, x_6, x_7\}$ $S_7 \setminus A_2 = \{x_2, x_4, x_6, x_7\}$, $S_7 \setminus A_3 = \{x_1, x_4, x_5, x_7\}$. Consider the tripte $A = \{x, x_1, x_7\}$ containing x_1, x_7. Then $x = x_2$ or x_3 or x_4 or x_5 or x_6. As $S_7 \setminus A_1 \setminus x_7 = \{x_3, x_5, x_6\}$ and $S_7 \setminus A_2 \setminus x_7 = \{x_2, x_4, x_6\}$ and $x_7 \in S_7 \setminus A_1, S_7 \setminus A_2$. Since $x_1 \in S_7 \setminus A_3$, it follows that $A \in M[S_7 \setminus A_1, S_7 \setminus A_2, S_7 \setminus A_3]$. A contradiction.

Thus $M(\mathcal{S}_7)$ is not transversal. //

By Hall [67] for any $n \equiv 1$ or $3 \bmod 6$ and $n \notin N_0 = \{9, 13, 25, 27, 33, 37, 67, 69, 75, 81, 97, 109, 201, 289, 321\}$, a Steiner triple system $\mathcal{S}_n$ containing $\mathcal{S}_7$ exists and since $\mathcal{S}_7$ is unique we note from Lemma 3.1.9 that a non - transversal matroid $M(\mathcal{S}_n)$ exists.

As a consequence of Theorem 3.1.13 we have

3.1.14 LEMMA. Each C_i^* in any minimal presentation $(C_1^*, \ldots, C_r^*)$ of a transversal matroid $M(S)$ is a PT of the family of bases of $M(S)$.

PROOF. As in the proof of Theorem 3.1.13 for any $y \in C_i^*$, $D \cup y$ is a basis of $M(S)$ for some PT D of $(X)_I$. Since $y_1 \neq y_2 \in C_i^*$, $D \cup y_1$ and $D \cup y_2$ are distinct bases of $M(S)$. Thus C_i^* is a PT of the family of bases of $M(S)$ as required. //

3.1.15 THEOREM. $M(S) = M[X_1, \ldots, X_n] = M[X_1 \cup A, X_2, \ldots, X_n]$ if and only if every element of $A \setminus X_1$ is a coloop of

$M[X_2 \setminus X_1, X_3 \setminus X_1, \ldots, X_n \setminus X_1] = M_S(S \setminus X_1)$.

PROOF. Observe that if $a,b \in S \setminus X_1$ with $M(S) = M[X_1 \cup a, X_2,\dots,X_n] = M[X_1 \cup b, X_2,\dots,X_n]$, then $M(S) = M[X_1 \cup ab, X_2,\dots,X_n]$. Thus to prove the theorem it suffices to show that if $a \notin X_1$, then $M[X_1 \cup a, X_2,\dots,X_n] = M[X_1, X_2,\dots,X_n]$ if and only if a is a coloop of $M[X_2 \setminus X_1, X_3 \setminus X_1,\dots,X_n \setminus X_1]$.

Let a be a coloop of $M[X_2 \setminus X_1,\dots,X_n \setminus X_1]$. Then every maximal PT of $(X_2,\dots,X_n)$ intersects $X_1 \cup a$. Choose a maximal PT B_1 of $(X_2,\dots,X_n)$ with $|B_1 \cap (X_1 \cup a)|$ minimal. We consider the two possibilities for a.

case 1. If $a \in B_1$. Then by Lemma 3.1.12 we have $M[X_1 \cup a, X_2,\dots,X_n] = M[(X_1 \cup a) \setminus (B_1 \cap (X_1 \cup a)), X_2,\dots,X_n] = M[X_1 \setminus B_1, X_2,\dots,X_n]$. But every maximal PT of $(X_1 \setminus B_1,\dots,X_n)$ is a maximal PT of $(X_1,\dots,X_n)$. Thus $M[X_1 \cup a,\dots,X_n] = M[X_1,\dots,X_n]$.

case 2. If $a \notin B_1$, then since a belongs to every maximal PT of $(X_2 \setminus X_1,\dots,X_n \setminus X_1)$, $B' = B_1 \setminus (X_1 \cup a)$ is a PT of $(X_2 \setminus X_1,\dots,X_n \setminus X_1)$. We see from the choice of B_1 that B' is a maximal PT of $(X_2 \setminus X_1,\dots,X_n \setminus X_1)$.
Extend B' to a basis B_2 of $M[X_2,\dots,X_n]$. Since $a \in B'$, $a \in B_2$. Now $|B_2 \cap (X_1 \cup a)| = |B_1 \cap (X_1 \cup a)|$ and we apply case 1 to B_2

Conversely suppose $M[X_1 \cup a,\dots,X_n] = M[X_1,\dots,X_n]$. Consider any maximal PT E of $(X_2 \setminus X_1,\dots,X_n \setminus X_1)$ which is a PT of $(X_2,\dots,X_n)$ and hence $E \cup a$ is a PT of $(X_1 \cup a, X_2,\dots,X_n)$. Then $E \cup a$ is a PT of $(X_1,\dots,X_n)$ (as $a \notin X_1$). Since $(E \cup a) \cap X_1 = \phi$, $E \cup a$ is a PT of $(X_2 \setminus X_1,\dots,X_n \setminus X_1)$. But E is a maximal PT of $(X_2 \setminus X_1,\dots,X_n \setminus X_1)$ and so $E = E \cup a$. Thus $a \in E$. Therefore a is a coloop of $M[X_2 \setminus X_1,\dots,X_n \setminus X_1]$ and the theorem is proved //

3.1.16 A *maximal presentation* $[X_1, \ldots, X_r]$ of a transversal matroid $M(S)$ of rank r is a presentation of $M(S)$ such that for any $i = 1, \ldots, r$ and each $x \notin X_i$, $M(S) \neq M[X_1, \ldots, X_{i-1}, X_i \cup x, \ldots, X_r]$.

Bondy [72] showed that a maximal presentation of any transversal matroid exists and is unique.

3.1.17 THEOREM. A maximal presentation of a transversal matroid $M(S) = M[X_1, \ldots, X_r]$ of rank r is unique.

PROOF. We first show that a maximal presentation of $M(S)$ exists. Let A_1 be the set of coloops of $M[X_2 \setminus X_1, \ldots, X_r \setminus X_1]$. Then by Theorem 3.1.15 $M(S) = M[X_1 \cup A_1, X_2, \ldots, X_r]$. Inductively for each i, $2 \leq i \leq r$, having A_{i-1} we let A_i be the set of coloops of $M[X_1 \cup A_1 \setminus X_i, X_2 \cup A_2 \setminus X_i, \ldots, X_{i-1} \cup A_{i-1} \setminus X_i, X_{i+1}, \ldots, X_r] = M_S(S \setminus X_i)$ so that by Theorem 3.1.15 $M[X_1 \cup A_1, \ldots, X_i \cup A_i, X_{i+1}, \ldots, X_r] = M[X_1 \cup A_1, \ldots, X_{i-1} \cup A_{i-1}, X_i, \ldots, X_r] = M(S)$. We claim that $M[X_1 \cup A_1, X_2 \cup A_2, \ldots, X_r \cup A_r]$ is a maximal presentation of $M(S)$. Suppose $M[X_1 \cup A_1, \ldots, X_i \cup A_i \cup x, \ldots\ldots\ldots, X_r \cup A_r] = M[X_1 \cup A_1, \ldots, X_r \cup A_r]$. Then by Theorem 3.1.15 x is a coloop of $M_S(S \setminus X_i)$ so that $x \in A_i$

To show the uniqueness we suppose that $\mathcal{A} = (A_1, \ldots, A_r)$ and $\mathcal{B} = (B_1, \ldots, B_r)$ are distinct maximal presentations of $M(S)$. Thus there exists a subset X of S such that ℓ of the sets in $\mathcal{A}$ and m of the sets in $\mathcal{B}$ are equal to X, with $\ell \neq m$. Choose such an X with $|X|$ minimal. We may assume that $\ell > m$. Let k be such that k of the sets in $\mathcal{A}$ are properly contained in X . Since $|X|$ is minimal, k of the sets in $\mathcal{B}$ are properly contained in X. Put $T = S \setminus X$. Order the sets in $\mathcal{A}$ and $\mathcal{B}$ so that

$$A_i \cap T = \phi \iff 1 \le i \le k + \ell ,$$

$$B_i \cap T = \phi \iff 1 \le i \le k + m .$$

As in the proof of Lemma 3.1.9, $(A_1 \cap T, \ldots, A_r \cap T)$ and $(B_1 \cap T, \ldots, B_r \cap T)$ are presentations of $M_S(T)$ and so $\mathcal{A}' = (A_{k+\ell+1} \cap T, \ldots, A_r \cap T)$ and $\mathcal{B}' = (B_{k+m+1} \cap T, \ldots, B_r \cap T)$ are presentations of $M_S(T)$. Now $r(M_S(T) \le r - k - \ell$. By Lemma 3.1.10 there exists a subfamily $\mathcal{B}''$ of $\mathcal{B}$ with $|\mathcal{B}''| < |\mathcal{B}'|$ and $\mathcal{B}''$ is a presentation of $M_S(T)$. Let $B_j \cap T \in \mathcal{B}' \setminus \mathcal{B}''$. Since $B_j \cap T \neq \phi$, there exists $y \in B_j \cap T$. Thus every maximal PT of $\mathcal{B}'$ contains y so that y is a coloop of $M_S(T)$. Thus by Theorem 3.1.15 $M(S) = M\left[A_1, \ldots, A_k, A_{k+1} \cup y, \ldots, A_r\right]$, contradicting the maximality of $\mathcal{A}$. Therefore the theorem is proved. //

We note that every presentation $(X)_I$ of a transversal matroid lies between a minimal presentation $(m)_I$ and the maximal presentation $(M)_I$ in the sense that for all $i \in I$, $m_i \subsetneq X_i \subsetneq M_i$.

We close this section by the following theorem due to Bondy [72] .

3.1.18 THEOREM. Let $(M_1, \ldots, M_r)$ be the maximal presentation of a transversal matroid $M(S)$ of rank r. If $(C_1, \ldots, C_r)$ and $(D_1, \ldots, D_r)$ are cocircuit presentations of $M(S)$ with $C_i \cup D_i \subsetneq M_i$, $1 \le i \le r$. Then $|C_i| = |D_i|$, $1 \le i \le r$.

PROOF. Let $|C_i \cap D_i| = k_i$, $|C_i \setminus D_i| = \ell_i$, $|D_i \setminus C_i| = m_i$. Pick $x \in C_i$. Now C_i is a circuit of $M^*(S)$ so that $C_i \setminus x \subsetneq$ a basis B^* of $M^*(S)$. Then C_i is the fundamental circuit of B^* in x so that $C_i \cap (S \setminus B^*) = x$. Hence $S \setminus B^*$ is a basis of $M(S)$ which intersects C_i

in one element and so $(M_1 \setminus C_i, \ldots, M_{i-1} \setminus C_i, M_{i+1} \setminus C_i, \ldots, M_r \setminus C_i)$ has at least one transversal. Observe that $(M_1, \ldots, M_{i-1}, C_i, M_{i+1}, \ldots M_r)$ and $(M_1, \ldots, M_{i-1}, C_i \cup D_i, \ldots, M_r)$ are presentations of $M(S)$. By Theorem 3.1.15 every element in $D_i \setminus C_i$ is a coloop of $M_S(S \setminus C_i)$. That is $D_i \setminus C_i$ is contained in every transversal of $(M_1 \setminus C_i, \ldots, M_{i-1} \setminus C_i, M_{i+1} \setminus C_i, \ldots, M_r \setminus C_i)$. Similarly $C_i \setminus D_i$ is contained in every transversal of $(M_1 \setminus D_i, \ldots, M_{i-1} \setminus D_i, M_{i+1} \setminus D_i, \ldots, M_r \setminus D_i)$ But $(S \setminus C_i)$ and $(S \setminus D_i)$ and $(S \setminus D_i)$ are hyperplanes of $M(S)$. Hence $r(M_S(S \setminus C_i) = r(M_S(S \setminus D_i)) = r - 1$. Thus every transversal of $(M_1 \setminus D_i, \ldots, M_{i-1} \setminus D_i, M_{i+1} \setminus D_i, \ldots, M_r \setminus D_r)$ contains at least $|D_i \setminus C_i|$ elements of $C_i \setminus D_i$ and so

$$\ell_i = |C_i \setminus D_i| \geq |D_i \setminus C_i| = m_i$$

Similarly we can show that $m_i \geq \ell_i$ and hence $\ell_i = m_i$. Thus $|C_i| = k_i + \ell_i = k_i + m_i = |D_i|$ and this is true for every i, $1 \leq i \leq r$. The theorem is proved. //

3.2 MULTIPLICITY

As every transversal pregeometry has a presentation with a transversal it is interesting to find criteria for the existence of transversals of families. Throughout this section the families discussed are finite.

3.2.1 THEOREM. (Hall's Criterion).

Given a finite family $(X)_I$ with each X finite. Then $(X)_I$ has a transversal if and only if

$$\left|\bigcup_J X\right| \geq J \quad , \quad \forall J \subseteq I. \tag{H}$$

PROOF. Let $\phi(I)$ be a transversal of $(X)_I$. Suppose that $J \subsetneq I$ with $|\bigcup_J X| < |J|$. Then ϕ is not injective on J so that ϕ is not injective on I. A contradiction. Hence $|\bigcup_J X| \geq |J|$, $\forall J \subseteq I$.

Assume that $|\bigcup_J X| \geq |J|$, $\forall J \subsetneq I$. If all X are singletons the theorem is proved. We may assume that $I = \{1,2,\ldots, n\}$ and X_1 is not singleton. We shall show that $\exists\, a \in X_1$ such that $(X')_I$ satisfies (H), where $X_1' = X_1 \setminus a$, $X_i' = X_i$, $2 \leq i \leq n$.

suppose not. Let $a_1 \neq a_2 \in X_1$. Then there exists $J_1' \subseteq \{1,\ldots,n\}$ such that $|\bigcup_{J_1'} X'| < |J_1'|$. That is

$$|(X_1 \setminus a_1) \cup (\bigcup_{J_1} X)| < |J_1| + 1 \text{ , where } J_1 = J_1' \setminus 1$$

Also there exists $J_2 \subseteq \{2,\ldots, n\}$ such that

$$|(X_1 \setminus a_2) \cup (\bigcup_{J_2} X)| < |J_2| + 1$$

Let $A = (X_1 \setminus a_1) \cup (\bigcup_{J_1} X)$

$B = (X_1 \setminus a_2) \cup (\bigcup_{J_2} X)$

Then $A \cup B = X_1 \cup (\bigcup_{J_1 \cup J_2} X)$ and

$$A \cap B = (X_1 \setminus a_1\, a_2) \cup (\bigcup_{J_1 \cap J_2} X)$$

Now $|J_1| + 1 + |J_2| + 1 > |A| + |B| + 1 = |A \cup B| + |A \cap B| + 1$

$$\begin{aligned}|A \cup B| + |A \cap B| + 1 &= |X_1 \cup (\bigcup_{J_1 \cup J_2} X)| + |(X_1 \setminus a_1\, a_2) \cup (\bigcup_{J_1 \cup J_2} X)| + 1 \\ &\geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| + 1 \\ &\geq |J_1| + |J_2| - |J_1 \cap J_2| + 1 + |J_1 \cap J_2| + 1\end{aligned}$$

$$\geq \; |J_1| + |J_2| + 1 + 1$$

Hence $|J_1| + 1 + |J_2| + 1 \; > \; |J_1| + |J_2| + 1 + 1$ which is a contradiction. Thus $(X')_I$ satisfies (H) for some $a \in X_1$. Therefore after finitely many steps we can reduce the family $(X)_I$ to a family $(Y)_I$ of singletons and $(Y)_I$ still satisfies (H) so that $\bigcup_I Y$ is a transversal of $(X)_I$. //

Even if we know that a given family $(X)_I$ has a transversal E we may ask how many distinct SDR's give rise to the transversal E.

The next theorem gives a necessary and sufficient condition for uniqueness.

3.2.2 THEOREM. Let E be a transversal of a family $(X)_I$. Then a necessary and sufficient condition for the uniqueness of SDR giving E is the following.

If $(Y)_I$ is a family satisfying the two conditions

$$\left.\begin{array}{l} \text{(i) There exists } J \subseteq I \text{ and } x_j \in X_j \text{ with } Y_j = x_j \text{ for all } j \in J \text{ and} \\ Y_j = X_j \setminus x_j' \text{, for some } x_j' \in X_j,\ j \notin J. \\ \text{(ii) } (\bigcup_J Y) \cup (\bigcup_{i \notin J} x_i') = E, \end{array}\right\} (T_1)$$

then E is not a transversal of $(Y)_I$.

PROOF. Necessity : Let $(Y)_I$ be a family satisfying (T_1). Define $\phi_1 : I \to \bigcup_I Y$ by

$$\phi_1(i) \begin{cases} x_i & \text{if } i \in J, \\ x_i' & \text{if } i \notin J. \end{cases}$$

Then ϕ_1 is injective and $\phi_1(I) = E$. Suppose that $(Y)_I$ has a transversal E. Let ϕ_2 be an SDR of $(Y)_I$. Since $Y_i \subseteq X_i$, ϕ_2 is also an SDR of $(X)_I$. Thus $\phi_1(I) = \phi_2(I) = E$. Now for $i \notin J$ we have $\phi_2(i) \neq x_i' = \phi_1(i)$ so that $\phi_1 \neq \phi_2$ which is a contradiction. Thus E is not a transversal of $(Y)_I$.

Sufficiency : Given $\phi(I) = \phi'(I) = E$ and suppose that $\phi \neq \phi'$. Then $\exists$ nonempty set $I_1 \subseteq I$ such that $\phi(i) \neq \phi'(i)$ if and only if $i \in I_1$. We shall show that $\phi(I_1) = \phi'(I_1)$. For each $i \in I_1$ there exists $i_r \neq i$ such that $\phi'(i_r) = \phi(i)$ and hence $i_r \in I_1$. Similarly for $j \in I_1$ there exists $j_s \neq j$ such that $\phi(j_s) = \phi'(j)$. Thus $\phi(I_1) = \phi'(I_1)$. Put $J = I \setminus I_1$

$$\text{Define } Y_i = \begin{cases} \phi(i) & i \in J , \\ X_i \setminus \phi(i) & i \notin J . \end{cases}$$

Then $(Y)_I$ satisfies the condition (T_1) and so E is not a transversal of $(Y)_I$. Now for $i \notin J$ we have $\phi(i) \neq \phi'(i)$ so that $\phi'(i) \in Y_i$. Since $\phi(I_1) = \phi'(I_1) \subseteq \bigcup_{i \notin J} Y_i$ and $\phi'(J) = \phi(J) \subseteq \bigcup_{i \in J} Y_i$, it follows that $\phi'(I) \subseteq \bigcup_I Y$ so that $(Y)_I$ has a transversal $\phi'(I) = E$. A contradiction and then the sufficiency is proved. //

3.2.3 A family $(X)_I$ has a *transversal E of multiplicity k* if every element in E occurs in exactly k sets of $(X)_I$.

We have another sufficient condition for the uniqueness of the SDR giving a particular transversal.

3.2.4 THEOREM. Let $(X)_I$ be a family with a transversal E of multiplicity 2. If there exists no subset $\{ x_{i_1}, \ldots, x_{i_r} \}$ of E of cardinality $r \geq 2$ such that

$$\{ x_{i_j} , x_{i_{j+1}} \} \subseteq X_{i_j} \qquad j = 1, \ldots , r \qquad (T_2)$$

where the addition of the subscript is modulo r. Then $(X)_I$ has unique SDR giving E.

PROOF. Let $\phi(I) = \phi'(I) = E$. Suppose $\phi \neq \phi'$. Then there exists $I_1 \subseteq I$ such that $\phi(I_1) = \phi'(I_1)$ and $\phi(i) \neq \phi'(i) \iff i \in I_1$

Define $f : I_1 \rightarrow I_1$ by

$$f(i_j) = i_k \text{ , where } \phi'(i_j) = \phi(i_k) .$$

Then f is a permutation on I_1. Since f is not the identity permutation, it can be written as a product of disjoint cycles $C_1, C_2, \ldots, C_k$, where at least one cycle, C_j say, has length ≥ 2.

Let $C_j = (i_1, i_2, \ldots, i_s)$, $s \geq 2$.

Suppose $\phi(i) = x_i$, $\forall i \in I$. Consider $1 \leq j \leq s - 1$, we have $f(i_j) = i_{j+1}$ so that $\phi'(i_j) = \phi(i_{j+1}) = x_{i_{j+1}}$. Thus $x_{i_j}, x_{i_{j+1}} \in X_{i_j}$. Now $f(i_s) = i_1$, so that $\phi'(i_s) = \phi(i_1)$ and hence $x_{i_1}, x_{i_s} \in X_{i_s}$. Therefore $\{x_{i_1}, \ldots, x_{i_s}\}$ satisfies the condition (T_2). A contradiction. Hence $\phi = \phi'$. //

3.2.5 REMARK. The condition (T_2) is not necessary for the uniqueness of SDR

As an example consider $X_1 = 1$, $X_2 = 24$, $X_3 = 345$, $X_4 = 4$ $X_5 = 56$, $X_6 = 64$. $(X)_I$ has unique SDR ϕ giving the transversal $\{1,2,3,4,5,6\}$, namely, $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 3$, $\phi(4) = 4$, $\phi(5) = 5$, $\phi(6) = 6$. The set $\{4,5,6\}$ is such that $4,5 \in X_3$, $5,6 \in X_5$, $6,4 \in X_6$ and so it satisfies (T_2).

3.2.6 Any subset $C = (x_{i_1}, \ldots, x_{i_r})$ of a transversal E of a family $(X)_I$ satisfying the condition (T_2) is a *cycle* of length r with *index set* $\{i_1, \ldots, i_r\}$ and *associate sets* $X_{i_1}, \ldots, X_{i_r}$.

The exact number of SDR's giving a transversal E of $(X)_I$ is known if E is a transversal of multiplicity 2. To prove this we need the following ten lemmas.

3.2.7 LEMMA. Given a transversal E of $(X)_I$ of multiplicity 2. Let $I_1 \subseteq I$. Then there exists at most one cycle of E with index set I_1.

PROOF. Let C_1, C_2 be cycles of E with the same index set I_1

Case 1 $I_1 = I$.

Then $|C_1| = |I_1| = |I| = |E|$ and also $|C_2| = |E|$ so that $C_1 = C_2$.

Case 2 $I_1 \subsetneqq I$.

Suppose $C_1 \neq C_2$. Since $|C_1| = |C_2|$, $|C_2 \setminus C_1| > 0$

so that $\left|\left(\bigcup_{I_1} X\right) \cap (C_1 \cup C_2)\right| = |C_1| + |C_2 \setminus C_1| > |I_1|$

Now $x \in \left(\bigcup_{I_1} X\right) \cap (C_1 \cup C_2) \Rightarrow x \notin \left(\bigcup_{I \setminus I_1} X\right) \cap E$ and so

$$\left|\left(\bigcup_{I \setminus I_1} X\right) \cap E\right| \leq |I| - \left|\left(\bigcup_{I_1} X\right) \cap (C_1 \cup C_2)\right|$$

$$< |I| - |I_1|$$

Hence $\phi(I \setminus I_1) < |I| - |I_1|$. A contradiction. //

3.2.8 LEMMA. Any proper subset of a cycle of a transveral of multiplicity 2 is not a cycle of that transversal.

PROOF. Let C' be a proper subset of a cycle $C = (x_{i_1}, \ldots, x_{i_r})$ of a transversal E of a family $(X)_I$. Let $C' = (x_{i_{j_1}}, \ldots, x_{i_{j_k}})$, $k < r$.

As E is of multiplicity 2, $\{ i_{j_1}, \ldots, i_{j_k} \} = \{ i_s, i_{s+1}, \ldots, i_p \}$ for some s, $1 \leq s \leq r$ and $p = s + (k - 1)$. We can assume that $s > 1$. Now x_{i_s} must occur in exactly 2 associate sets of C'. Also $x_{i_s} \in X_{i_{s-1}}$ which is not an associate set of C' so that x_{i_s} belongs to 3 sets of $(X)_I$.

A contraction. //

3.2.9 LEMMA. Let C_1, C_2 be cycles of a transversal E of multiplicity 2 of a family $(X)_I$ with index sets I_1, I_2 respectively. Then $I_1 \cap I_2 = \phi$ if and only if $C_1 \cap C_2 = \phi$.

PROOF. Assume that $I_1 \cap I_2 = \phi$. Then $(X)_I = (X)_{I_1 \mathring{\cup} I_2 \mathring{\cup} (I \setminus I_1 \cup I_2)}$. Let $x \in C_1$. Then x belongs to exactly 2 sets of $(X)_{I_1}$. Since x belongs to exactly 2 sets of $(X)_I$, x does net belong to any set of $(X)_{I_2}$. But $C_2 \subseteq \bigcup_{I_2} X$. Thus $x \notin C_2$.

Next we assume that $C_1 \cap C_2 = \phi$. Suppose $I_1 \cap I_2 \neq \phi$. Then $|I \setminus I_1 \cup I_2| > |I| - |I_1| - |I_2|$ and since $x \in (\bigcup_{I_1 \cup I_2} X) \cap (C_1 \cup C_2) \Rightarrow x \notin (\bigcup_{I \setminus I_1 \cup I_2} X) \cap E$ we have

$$\begin{aligned} \left|\left(\bigcup_{I \setminus I_1 \cup I_2} X\right) \cap E\right| &\leq |E| - \left|\left(\bigcup_{I_1 \cup I_2} X\right) \cap (C_1 \cup C_2)\right| \\ &\leq |I| - |I_1| - |I_2| \\ &< |I \setminus I_1 \cup I_2| \end{aligned}$$

Thus $|\phi(I \setminus I_1 \cup I_2)| < |I \setminus I_1 \cup I_2|$ and so ϕ is not injective. A contradiction. //

3.2.10 LEMMA. Let C_1 be a cycle of a transversal E of multiplicity 2 of a family $(X)_I$ with index set I_1. Let C' be a proper subset of C_1.

Then the following are true.

(i) For any $\phi \neq C_2 \subseteq (\bigcup_{I \setminus I_1} X) \cap E$ with $C_2 \cap C' = \phi$, $C' \cup C_2$ is not a cycle of E.

(ii) For any $C_2 \subseteq (\bigcup_{I_1} X) \cap E$ with $C_2 \cap C' = \phi$, $C' \cup C_2$ is not a cycle of E.

PROOF. Let $C_1 = (x_{i_1}, \ldots, x_{i_r})$ and $I_1 = \{i_1, \ldots, i_r\}$.
(i) Let $I = \{i_j \in I_1 \,/\, x_{i_j} \in C'\}$. Then $I' \subsetneq I_1$. Without loss of generality assume that $I' = \{i_{j_1}, i_{j_2}, \ldots, i_{j_k}\}$, where $1 \leq j_1 < j_2 < \ldots < j_k < r$. Suppose that $C' \cup C_2$ is a cycle of E. Let $I'' = \{i_{j_1}, \ldots i_{j_k}, i'_{j_1}, \ldots, i'_{j_e}\}$ be the index set of $C' \cup C_2$ where $e \geq 1$, $i'_{j_m} \notin I'$, $m = 1, \ldots, e$.

If there exists $i_{j_r} \in I'$ with $i_{j_r-1} \notin I'$, where $j_1 < j_r \leq j_k$. Then $X_{i_{j_r-1}}$ is not an associate set of the cycle $C' \cup C_2$ and since $x_{i_{j_r}}$ must belong to exactly 2 associate sets of $C' \cup C_2$, there exists i'_p such that $x_{i_{j_r}} \in X_{i'_p}$ so that $x_{i_{j_r}}$ belongs to at least 3 sets of $(X)_{I''}$ namely $X_{i_{j_r}}, X_{i_{j_r-1}}, X_{i'_p}$ which is a contradiction. Hence we can assume that $I'' = \{i_1, \ldots, i_k, i'_1, \ldots, i'_e\}$, where $k < r$ and $e \geq 1$. A cycle form of $C' \cup C_2$ can not have x_{i_m} in between x_{i_r} and $x_{i_{r-1}}$ (otherwise x_{i_r} belongs to 3 sets of $(X)_{I''}$). Then $C' \cup C_2$ $= (x_{i_1}, \ldots, x_{i_k}, x_{i'_1}, \ldots, x_{i'_e})$ and so x_{i_k} belongs to at least 3 sets of $(X)_{I'}$, namely $X_{i_{k-1}}, X_{i_k}, X_{i'_1}$. A contradiction. Hence (i) is proved.

(ii) Let $|C'| = k$, $k < r$ and $|C_2| = s > 0$. Suppose that $C' \cup C_2$ is a cycle of E. Then it has length $k + s \leq r$ and we have

$|\phi(I \setminus I_1)| = |(\bigcup_{I \setminus I_1} X) \cap E| \leq |E| - |(\bigcup_{I_1} X) \cap (C_1' \cup C_2)| \leq |E| - (|C_1'| + |C_2|) \leq |I| - (|I_1| + |C_2|) < |I| \; |I_1|$ which is a contradiction. //

3.2.11 LEMMA. It follows from Lemma 3.2.7 - 3.2.10 that two cycles of a transversal of multiplicity 2 are either disjoint (with disjoint index sets) or identical.

3.2.12 LEMMA. Let $C = (x_{i_1}, \ldots, x_{i_r})$ be a cycle of a transversal $E = \phi(I)$ of multiplicity 2 of a family $(X)_I$. Then $\phi(i_j) = x_{i_j}$ or $x_{i_{j+1}}$, where $i_j \in \{i_1, \ldots, i_r\}$ and the addition of the subscript of x_{i_j} is modulo r.

PROOF. Let $I' = \{i_1, \ldots, i_r\}$. It is obvious if $\phi(I') = C$. Suppose that $\phi(I') \neq C$. Since ϕ is one to one and onto and $|I'| = |C|$, there exists $x_{i_k} \in C$ such that $\forall i_j \in I'$, $\phi(i_j) \neq x_{i_k}$ and so there exists $j \in I \setminus I'$ such that $\phi(j) = x_{i_k}$. Then x_{i_k} occurs in 3 sets of $(X)_I$ which is impossible. Therefore $\phi(i_j) = x_{i_j}$ or $x_{i_{j+1}}$. //

We define *the SDR induced by the cycle C* in the following lemma.

3.2.13 LEMMA. Let $C = (x_{i_1}, \ldots, x_{i_r})$ be a cycle of a transversal $E = \phi(I)$ of multiplicity 2 of a family $(X)_I$ with index set I'.

Define $\phi_C : I \to E$ by

$$\phi_C(i_j) = \begin{cases} x_{i_j+1} & \text{if } \phi(i_j) = x_{i_j} \text{ and } i_j \in I', \\ x_{i_j} & \text{if } \phi(i_j) = x_{i_j+1} \text{ and } i_j \in I', \\ \phi(i_j) & i_j \in I \setminus I' . \end{cases}$$

Then ϕ_C is an SDR of $(X)_I$—the SDR induced by the cycle C—which is different from ϕ and $\phi_C(I) = E$.

PROOF. That $\phi_C \neq \phi$ is clear from the definition of ϕ_C and $\phi_C(x_i) \in X_i$, $\forall_i \in I$. To show that ϕ_C is an SDR, let $\phi_C(i_j) = \phi_C(i_k)$. If one of i_j, i_k, i_j say belongs to I' and $i_k \in I \setminus I'$, then $\phi_C(i_j) \in (\bigcup_{I'} X) \cap E$. Since $\forall x \in (\bigcup_{I'} X) \cap E$ we have $x \not\in (\bigcup_{I \setminus I'} X) \cap E$, it follows that $\phi_C(i_k) = \phi_C(i_j) \not\in (\bigcup_{I \setminus I'} X_\iota) \cap E$. Thus

$\phi_C(i_k) \in (\bigcup_{I'} X) \cap E$ so that $i_k \in I'$ which is a contradiction. Hence both i_j, i_k must belong to either I' or $I \setminus I'$. In either case we have $i_j = i_k$. Now $\phi_C(I) = \phi(I') \cup \phi(I \setminus I') = C \cup (E \setminus C) = E$. //

3.2.14 LEMMA. Disjoint cycles of a transversal give rise to different induced SDR's giving that transversal.

PROOF. Let C_1, C_2 be disjoint cycles of a transversal $\phi(I) = E$ with index sets I_1, I_2 respectively. For each $i \in I_1$ we have $\phi_{C_1}(i) \neq \phi(i)$. But $\phi_{C_2}(i) = \phi(i)$ whenever $i \in I_1$. Thus $\phi_{C_1} \neq \phi_{C_2}$. //

We define *the SDR induced by the disjoint cycles $C_1, \ldots, C_k$* in the following lemma.

3.2.15 LEMMA. Let $C_1, \ldots, C_k$ be disjoint cycles of a transversal

$E = \phi(I)$ of (X_I) of multiplicity 2 with index sets $I_1,\ldots,I_k$ respectively.

Define $\phi_{C_1\ldots C_k} : I \to E$ by

$$\phi_{C_1\ldots C_k}(i) = \begin{cases} \phi_{C_j}(i) & \text{if } i \in I_j \,, \\ \phi(i) & \text{if } i \in I \setminus I_1 \cup \ldots \cup I_k \,. \end{cases}$$

Then $\phi_{C_1\ldots C_k}$ is an SDR of $(X)_I$ — the SDR induced by the cycles $C_1,\ldots,C_k$, which is different from each of $\phi_{C_1},\ldots,\phi_{C_k}$, ϕ, and $\phi_{C_1,\ldots,C_k}(I) = E$.

PROOF. By induction on the number of the cycles $C_1,\ldots,C_k$. //

3.2.16 LEMMA. Let ϕ be an SDR of multiplicity 2 of a family $(X)_I$. For another SDR $\phi^1 \neq \phi$ such that $\phi(I) = \phi'(I)$, the set

$$\{ \phi'(i) \;/\; \phi'(i) \neq \phi(i) \}$$

determines disjoint cycles of $E = \phi(I)$.

PROOF. Follows from the proof of Theorem 3.2.4. //

We are now ready to find the number of different SDR's giving the same transversal.

3.2.17 THEOREM. Let $E = \phi(I)$ be a transversal of $(X)_I$ of multiplicity 2 . If E has r disjoint cycles, then the number, n(E) , of distinct SDR's giving the transversal E is

$$n(E) = 1 + {}^rC_1 + {}^rC_2 + \ldots + {}^rC_r$$

PROOF. From r disjoint cycles of E we can form

${}^{r}C_1 + \ldots + {}^{r}C_r = k$ combinations of these cycles so that they induce the k different SDR's and each of these SDR's is different from ϕ. Thus

$$n(E) \geq 1 + k$$

By Lemma 3.2.16 for a given SDR $\phi \neq \phi'$ of $(X)_I$ we have that $\{ \phi'(i) / \phi'(i) \neq \phi(i) \}$ determines disjoint cycles of E so that ϕ' is one of the above k SDR's and the theorem is proved. //

If $(X)_I$ has a transversal of multiplicity 2, it might have only one SDR giving the transversal if it has no cycles. We shall show that $(X)_I$ has at least 2 SDR's. That is a cycle of a transversal of multiplicity 2 must exist.

3.2.18 THEOREM. Let $E = \phi(I)$ be a transversal of multiplicity 2 of $(X)_I$ which contains a singleton. Then there exists an SDR $\phi \neq \phi'$ of $(X)_I$ giving E .

We need the following lemma to prove the theorem.

3.2.19 LEMMA. Let $E = \phi(I)$ be a transversal of multiplicity 2 of $(X)_I$ and $\bigcup_I X = E$. Suppose that $(X)_I$ contains a singleton. Let $I = \{1, \ldots, n\}$ and $\phi(i) = x_i \in X_i$, $\forall i \in I$. Define the subfamily $\mathcal{A}_1$ of $(X)_I$ by

$$\mathcal{A}_1 = \{ A_1 / A_1 \in (X)_I \text{ and } |A_1| = 1 \}$$

For any positive integer k, $2 \leq k < n$, if $\mathcal{A}_{k-1} \neq \phi$ we construct $\mathcal{A}_k$ as follows

$$\mathcal{A}_k = \{ A_k \in (X)_I / A_k = x \cup A_{k-1} \text{ for some } x \notin \bigcup (\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{k-1})$$

$$\text{and } A_{k-1} \subseteq \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k-1})\}$$

After finite steps of construction, m say, the process terminates when $\mathcal{A}_{m+1} = \phi$ and $\mathcal{A}_k \neq \phi$, $k \leq m$. Then

(i) For each $k = 1,2,\ldots, m$ and each $x \in \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ we have $\phi(i) \neq x$, $\forall X_i \notin \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$

(ii) $X_r \in \mathcal{A}_{k+1} \Rightarrow \phi(r) \not\in \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ $\forall k$, $r = 1,\ldots, m$. In fact $X_r = x_r \cup X_k'$ for some $X_k' \subseteq \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ and $x_r \not\in \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$.

(iii) $|(X)_I \setminus \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m| > 1$.

PROOF. (i) For $k = 1$, let $x \in (\cup \mathcal{A}_1)$. Hence $\{x\} \in \mathcal{A}_1$. Without loss of generality assume that $\phi(1) = x$. Consider $X_i \notin \cup \mathcal{A}_1$ and so $i \neq 1$. By definition of SDR $\phi(i) \neq \phi(1) = x$. Assume that the hypothesis is true for $k - 1$. Let $x \in \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$. Suppose that there exists $X_i \notin \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ such that $\phi(i) = x$. Hence $x \in \cup \mathcal{A}_k \setminus \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k-1})$ [otherwise the hypothesis is not true for $k - 1$]. Thus there exists $X_k \in \mathcal{A}_k$ such that $X_k = x \cup X_k'$ for some $X_k' \subseteq \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k-1})$ [as $x \in X_k \in \mathcal{A}_k$ and $X_k = x' \cup X_k'$, for some $x' \not\in \cup(\mathcal{A}_1 \ldots \cup \mathcal{A}_{k-1})$ and $X_k' \subseteq \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k-1})$ then $x = x'$; otherwise $x \in \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k-1})$]. By the assumption $\phi(k) \not\in X_k'$ (as $X_k \notin \cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k-1})$ and hence $\phi(k) = x = \phi(i)$ so that $k = i$ and $X_i \subseteq \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$. A contradiction.

(ii) Since $X_r \in \mathcal{A}_{k+1}$, $X_r \not\subseteq \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ and so by (i) $\phi(r) \notin \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$. For $X_r \in \mathcal{A}_{k+1}$ we have $X_r = x \cup X_k'$ for some $X_k' \subseteq \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ and $x \notin \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$. If $x_r \neq x$, then $x_r \in X_k' \subseteq \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)$ which is not so. Thus $x_r = x$.

(iii) We first show that $(X)_I \setminus \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m \neq \phi$.

Suppose the contrary. Without loss of generality let

$$\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{m-1} = \{ X_1, \ldots, X_{r-1} \},$$

$$\mathcal{A}_m = \{ X_r, \ldots, X_n \},$$

where for each $i = r, \ldots, n$, there exists i_j, $1 \leq i_j \leq r-1$ such that $X_i = x_i \cup X_{i_j}$, $X_{i_j} \subseteq \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{m-1})$ and $x_i \notin \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{m-1})$. If there exist i, j, $r \leq i \neq j \leq n$ such that $a \in X_i \cap X_j$. Then since $a \neq$ one of x_i, x_j, there exists $X_t \in \mathcal{A}_t$, $1 \leq t \leq m-1$ such that $a \in X_t$ which is a contradiction. Hence $X_i \cap X_j = \phi$ $\forall i \neq j \in \{r, \ldots, n\}$. Consider x_i, where $r \leq i \leq n$ we see that $x_i \notin X_j$, $\forall_j \in \{1, \ldots, r-1\}$ and thus x_i belongs to exactly one set of $(X)_I$. A contradiction. Thus $(X)_I \setminus \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m \neq \phi$.

Suppose that there exists only one $X_i \in (X)_I \setminus \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m$. Then $|X_i \setminus \bigcup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m)| \geq 1$ (otherwise

$x_i \in \bigcup(\mathscr{A}_1 \cup \ldots \cup \mathscr{A}_m)$ and by (i) $\phi(i) \neq x_i$). Let $x \in X_i \setminus \bigcup(\mathscr{A}_1 \cup \ldots \cup \mathscr{A}_m)$. Then x belongs to exactly one set of $(X)_I$ namely X_i. A contaradiction. Thus $|(X)_I \setminus \mathscr{A}_1 \cup \ldots \cup \mathscr{A}_m| > 1$. //

PROOF OF THEOREM 3.2.18. It suffices to prove the theorem for the case $\bigcup_I X = E$.

If $\bigcup_I X \supsetneq E$, let $Y_i = X_i \cap E$, $i \in I$. Then $\bigcup_I Y = E$. If there exist $i \neq j$ such that $Y_i = y_j$. For each i there is at most one integer $j \neq i$ such that $Y_i = y_j$ and if so $|Y_i| = |Y_j| = 2$. (otherwise $\exists$ $x \in E$ such that $\phi(i) \neq x \forall i \in I$).

Put $I_1 = \{ i / \exists j \neq i$ such that $Y_i = Y_j \}$ and $I' = I \setminus I_1$. If $I' \neq \phi$, let $E' = E \setminus \bigcup_{I_1} Y$. We claim that $(Y)_I'$ has a transversal E' of multiplicity 2 and $\bigcup_{I'} Y = E'$. Let $x \in E'$. Then $x \in E \setminus \bigcup_{I_1} Y = (\bigcup_{I'} Y) \cup (\bigcup_{I_1} Y) \setminus \bigcup_{I_1} Y$ so that $x \in (\bigcup_{I'} Y)$. Since x belongs to exactly 2 sets of $(X)_I$ and $x \notin$ any set of $(Y)_{I_1}$, x belongs to exactly 2 sets of $(Y)_I'$. As $i \neq j$ we have $Y_i \neq Y_j$ $\forall_{i,j} \in I'$. Hence E' is a transversal of $(Y)_I'$. Suppose that $\phi(i) = x_i \in X_i$ for every $i \in I$ and $\phi(I) = E$. Then $\phi / I'(I') = E'$. Put $\phi_1 = \phi/ I'$. If $\phi_1 \neq \phi'$ is another SDR of $(Y)_I'$ giving the transversal E', we can define an SDR $\phi \neq \phi''$ of $(X)_I$ such that $\phi''(I) = E$ as follows.

$$\phi''(i) = \begin{cases} \phi'(i) & \text{if } i \in I' \\ \phi'(i) & \text{if } i \notin I' \end{cases}$$

If $I' = \phi$. Without loss of generality assume that $Y_i = Y_{i+1}$,

where $i = 1, 3, \ldots, n-1$. Let $\phi(i) = x_i \in Y_i = \{x_i, y_i\}$.
Define $\phi' : I \to E$ by $\phi'(i) = y_i$, $\forall i \in I$. Then ϕ' is an SDR of $(X)_I$ giving E and $\phi' \neq \phi$.

Hence to prove the theorem we can assume that $\bigcup_I X = E$ and $i \neq j \Rightarrow X_i \neq X_j$, $\forall i \in I$. Let $I = \{1, \ldots, n\}$ and $\phi(i) = x_i \in X_i$. Define subfamilies $\mathcal{A}_1, \ldots, \mathcal{A}_m$ as in Lemma 3.2.19. Without loss of generality assume that $(X)_I \setminus \mathcal{A}_1 \cup \ldots\ldots \cup \mathcal{A}_m = \{X_{i_1}, \ldots, X_{i_k}\}$ and $\cup(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m) = X'$. Observe that $x_{i_j} \not\in X'$, $1 \leq j \leq k$.

We show that for each j, $1 \leq j \leq k$ we have $|X_{i_j} \setminus X'| \geq 2$. If there exists no $x \in X_{i_j} \setminus X' \cup x_{i_j}$, then $X_{i_j} = X'' \cup x_{i_j}$, where $X'' \subseteq X'$. Now $x_{i_j} \not\in X''$ and so $x_{i_j} \in \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{m+1}$. A contradiction. Thus $|X_{i_j} \setminus X'| \geq 2$, $1 \leq j \leq k$. Hence there exists $x_{i_j} \in X_{i_1} \setminus X' \cup x_{i_1}$ and $i_j \neq i_1$. Without loss of generality assume $i_j = i_2$. We first assume that $k \geq 4$.

case 1. $x_{i_1} \in X_{i_2}$

Define $\phi' : I \to E$ by $\phi'(i_1) = x_{i_2}$, $\phi'(i_2) = x_{i_1}$, $\phi'(j) = \phi(j)$ otherwise. Then $\phi' \neq \phi$ and ϕ' is an SDR of $(X)_I$ giving the transversal E.

case 2. $x_{i_1} \not\in X_{i_2}$

By the same argument as above there exists $x_{i_3} \in X_{i_2} \setminus X' \cup x_{i_1} x_{i_2}$ where $X_{i_3} \in \{X_{i_1}, \ldots, X_{i_k}\}$ and $i_3 \neq i_2$.

If $x_{i_1} \in X_{i_3}$ we can define $\phi' : I \to E$ by $\phi'(i_1) = x_{i_2}$, $\phi'(i_2) = x_{i_3}$ $\phi'(i_3) = x_{i_1}$ and $\phi'(j) = \phi(j)$ otherwise so that $\phi \neq \phi'$ is an SDR of $(X)_I$ and $\phi'(I) = E$.

If $x_{i_1} \notin X_{i_3}$, then there exists $x_{i_4} \in X_{i_3} \setminus X' \cup x_{i_3}$, where $x_{i_4} \in \{X_{i_1}, \ldots, X_{i_k}\}$ and $i_4 \neq i_3$. For $3 \leq j \leq k - 1$, if $x_{i_1} \notin X_{i_{j-1}}$ we can choose $x_{i_j} \in X_{i_{j-1}} \setminus X' \cup x_{i_1} \ldots x_{i_{j-1}}$ (See Lemma 3.2.20 below). The process must terminate after finite steps since there exists exactly one set from $X_{i_2}, \ldots, X_{i_k}$ containing x_{i_1}. Assume that the process stops after q steps; that is $x_{i_1} \in X_{i_q} \setminus X' \cup x_{i_1} \ldots x_{i_{q-1}}$ and $x_{i_1} \notin X_{i_j}$, $2 \leq j \leq q - 1$

Define $\phi' : I \to E$ as follows

$$\phi'(i_1) = x_{i_2},$$

$$\phi'(i_j) = x_{i_{j+1}}, \quad 2 \leq j \leq q - 1,$$

$$\phi'(i_q) = x_{i_1},$$

$$\phi'(j) = \phi(j) \quad \text{otherwise.}$$

Then ϕ' is an SDR of $(X)_I$ such that $\phi'(I) = E$ and $\phi' \neq \phi$.

By Lemma 3.2.19 $k > 1$. For $k = 2$ we have $x_{i_1} \in X_{i_2} \setminus X' \cup x_{i_1}$ and $x_{i_2} \in X_{i_1} \setminus X' \cup x_{i_1}$. Thus a function $\phi' : I \to E$ defined by $\phi'(i_1) = x_{i_2}$, $\phi'(i_2) = x_{i_1}$ and $\phi'(i) = \phi(i)$; otherwise, is an SDR of $(X)_I$ such that $\phi' \neq \phi$ and $\phi'(I) = E$.

For $k = 3$. The theorem is obvious when $x_{i_1} \in X_{i_2}$. If $x_{i_1} \notin X_{i_2}$. Then $x_{i_3} \in X_{i_2} \setminus X' \cup x_{i_2}$ and $x_{i_1} \in X_{i_3} \setminus X' \cup x_{i_3}$.

Define $\phi' : I \to E$ by $\phi'(i_1) = x_{i_2}$, $\phi'(i_2) = x_{i_3}$, $\phi'(i_3) = x_{i_1}$, $\phi'(i) = \phi(i)$ otherwise. Hence ϕ' is an SDR of $(X)_I$ such that $\phi(I) = E$ and $\phi' \neq \phi$.

The theorem is then proved. //

3.2.20 LEMMA. Assume the hypothesis and notation as in Theorem 3.2.18 Then for $j = 3,\ldots, k-1$, if $x_{i_1} \notin X_{i_{j-1}}$ We can choose

$x_{i_j} \in X_{i_{j-1}} \setminus X' \cup x_{i_1} \ldots\ldots x_{i_{j-1}}$.

PROOF. True for $j = 3$. Assume the lemma is true for $j \leq r$. Hence $x_{i_j} \in X_{i_{j-1}}$ $X' \cup x_{i_1} \ldots\ldots x_{i_{j-1}}$ $3 \leq j \leq r$. That is

$$x_{i_j} \in X_{i_j} \cap X_{i_{j-1}}, \; 3 \leq j \leq r$$

Assume $x_{i_1} \notin X_{i_r}$ and suppose that there is no $x_{i_{r+1}} \in X_{i_r} \setminus X' \cup x_{i_1},\ldots,x_{i_r}$. Thus $X_{i_r} \subseteq X' \cup x_{i_1} \ldots\ldots x_{i_r}$. Since $|X_{i_r} \setminus X'| \geq 2$, there exists $x \in X_{i_r} \setminus X' \cup x_{i_r}$ and so as $x_{i_1} \notin X_{i_r}$, $x \in \{x_{i_2}, \ldots, x_{i_{r-1}}\}$. If $x = x_{i_2}$, then $x \in X_{i_r} \cap X_{i_2} \cap x_{i_1}$ which is impossible. Thus there exists s, $3 \leq s \leq r-1$ such that $x = x_s$. Now $x \in X_{i_s} \cap x_{i_{s-1}} \cap X_r$ which is a contradiction. Hence the lemma is proved. //

3.2.21 THEOREM. The conclusion of Theorem 3.2.18 holds even though $(X)_I$ does not contain a singleton.

PROOF. We first show that if $(X)_I$ does not contain a singleton then $|X_i| = 2$, $\forall_i \in I$. Let $I = \{1,\ldots n\}$ and $\phi(i) = x_i \in X_i$, $i \in I$.

Put $Y = \{(x, i) \,/\, x \in E, x \in X_i\}$.

Then 2 distinct elements in E give rise to four different elements in Y.

Foreach $i = 1,\ldots, n$, let

$$Y_i = \{(x_i, j) / x_i \in X_j\}$$

Then $Y_i \subseteq Y$, $Y_i \cap Y_j = \phi$ if $i \neq j$ and $Y = \bigcup_{i=1}^{n} Y_i$

Now $|Y| = |\bigcup_{i=1}^{n} Y_i| = \sum_{i=1}^{n} |Y_i| = 2n$

We can write $Y = \bigcup_{i=1}^{n} Z_i$, where $Z_i = \{(x, i) / x \in X_i\}$

Thus $|Z_i| = |X_i|$ and $Z_i \cap Z_j = \phi$ if $i \neq j$.

Since $|X_i| \geq 2$, $|Z_i| \geq 2$. Suppose that there exists k such that $|X_k| > 2$. Then

$$|Y| = \sum_{i=1}^{n} |Z_i| \geq 2(n-1) + |Z_k| > 2(n-1) + 2 = 2n$$

A contradiction. Hence $|X_i| = 2 \quad \forall_i \in I$.

Now we write $X_i = \{x_i, y_i\}$, where $x_i \neq y_i$. Observe that $y_i \neq y_j$ if $i \neq j$ (otherwise there exists $k \neq i, j$ such that $\phi(k) = y_i$ so that y_i occurs in 3 sets of $(X)_I$)

We define $\phi' : I \to E$ by

$$\phi'(i) = y_i \quad \forall i \in I$$

Then ϕ' is an SDR of $(X)_I$ and $\phi'(I) = E$. //

In general a transversal E of multiplicity m is determined by at least m SDR's .

3.2.22 THEOREM. Let $E = \phi(I)$ be a transversal of multiplicity m of a family $(X)_I$. Then $(X)_I$ has at least $m - 1$ distinct SDR's each of which is different from ϕ and gives the transversal E.

PROOF. We prove the theorem by induction on m. The theorem is true for $m = 2$. Assume the theorem is true for any transversal E of multiplicity $k < m$. Let $\phi(I) = E$ be a transversal of multiplicity m of a family $(X)_I$. Let $I = \{1,2,\ldots, n\}$ and $\phi(i) = x_i \in X_i$, $i \in I$. We can assume that $\bigcup_I X = E$. For each $i \in I$ construct inductively the subset E_i of X_i as follows.

$$E_1 = X_1 \setminus \phi(1) ,$$

$$E_i = X_i \setminus \phi(i) \setminus \bigcup_{r < i} E_r \quad , \quad 2 \leq i \leq n.$$

Put $X_i' = X_i \setminus E_i$, $\forall i \in I$. Then we have

(i) $E_i \cap E_j = \phi$ if $i \neq j$, (ii) $\phi(i) \notin E_i$, (iii) $\bigcup_{i=1}^{n} E_i = E$

(iv) $(X')_I$ has a transversal $E = \phi(I)$ of multiplicity $m - 1$

To show (iii) let $a \in E$. There exists $i_1 \in I$ such that $\phi(i_1) = a$. Since a occurs in exactly m sets of $(X)_I$, $\exists\, i_2, \ldots, i_m$ such that $a \in X_{i_j}$, $j = 2,\ldots, m$. Then $\phi(i_j) \neq a$ $\forall_j = 2,\ldots, m$. Without loss of generality assume that $i_2 < i_3 < \ldots < i_m$.

case 1. $i_1 < i_2 < \ldots < i_m$

If $i_1 = 1$, $i_2 = 2$, then $a \notin E_1$ so that $a \in E_2$. Suppose that at least one of $i_1 \neq 1$ and $i_2 \neq 2$ holds. Then for each r, $1 \leq r < i_2$ we have $a \notin E_r$ since $a \notin X_r$ or $a = \phi(i_1)$ if $r = i_1$. Now $a \notin \bigcup_{i < i_2} E_i$ but $a \in X_{i_2}$ and $a \neq \phi(i_2)$. Thus $a \in E_{i_2}$.

case 2. i_1 is in between i_j and i_k for some $j, k \in \{2,\ldots, m\}$. We may assume that $i_2 < i_1 < i_3 < \ldots < i_m$. If $a \notin E_{i_3}$, then since $a \in X_{i_3}$ and $a \neq \phi(i_3)$ this implies that $a \in \bigcup_{i < i_3} E_i$. Thus $a \in E_i$

for some $i < i_3$

case 3 $i_2 < i_3 < \ldots < i_m < i_1$

If $a \notin E_{i_m}$, then by the same argument as in case 2 we have $a \in E_i$ for some $i < i_m$.

To show (iv) we observe that $\phi(i) \in X_i'$ for every $i \in I$ so that ϕ is also an SDR of $(X')_I$. Let $a \in E$. Then there exist $i_1 \neq i_2 \neq \ldots \neq i_m$ such that $a \in X_{i_j} \iff i_j \in \{i_1, \ldots, i_m\}$. Since $\overset{\circ}{\bigcup\limits_{i \in I}} E_i = E$, $a \in E_{i_j}$ for a unique $i_j \in \{i_1, \ldots, i_m\}$. Thus $a \notin X'_{i_j}$ and $a \in X'_{i_k}$ $\forall k \neq j$, $1 \leq k \leq m$. If a is in more than $m - 1$ sets of $(X')_I$, then a belongs to more than m sets of $(X)_I$. Thus (iv) is proved.

By induction hypothesis there exist $m - 2$ distinct SDR's $\phi_1', \ldots, \phi_{m-1}'$ each of which is different from ϕ, giving the transversal $E = \phi(I)$ of $(X')_I$.

For any $i \in I$, let

$$E_i' = X_i' \setminus \phi_1'(i)$$

and
$$X_i'' = (X_i' \setminus E_i') \cup E_i$$

Then $\bigcup\limits_{i=1}^{n} E_1' = E$ and $\phi_1'(i) \in X_i''$ $\forall_i \in I$. Now as $X_i'' = \phi_1'(i) \cup E_i$ and $\forall x \in E$, $x = \phi_1'(i)$ belongs to $m - 1$ sets of $(X_I'), (X_I'')$ has a transversal $\phi_1'(I) = E$ of multiplicity $m - 1$ and so by induction hypothesis $(X'')_I$ has at least $m - 2$ distinct SDR $\phi_1, \phi_2, \ldots, \phi_{m-2}$ giving the transversal and each of them is different from ϕ_1'.

We show that $\phi_i \neq \phi$ $\forall_i = 1, \ldots, m - 2$. Observe that $X_i = \phi_1'(i) \cup E_i \cup E_i'$ (as $X_i = X_i' \cup E_i$). $\phi_i \neq \phi_1' \Rightarrow \exists\, r_i$ such that

$\phi_i(r_i) \neq \phi_1'(r_i)$. Since $\phi_i(r_i) \in X_{r_i}$ and $\phi_i(r_i) \notin E'_{r_i}$, $\phi_i(r_i) \in E_{r_i}$

But $\phi(r_i) \notin E_{r_i}$. Thus $\phi_i \neq \phi$.

Hence $\phi_1, \ldots, \phi_{m-2}, \phi_1', \phi$ are different SDR's of $(X)_I$ giving the transversal E and the theorem is proved. //

3.2.23 Theorem. Let E be a transversal of a family $(X)_I$. If $a \in E$, then a necessary and sufficient condition for

$\phi_1(i) = \phi_2(j) = a \Rightarrow i = j$, where ϕ_1, ϕ_2 are SDR's of $(X)_I$ and $\phi_1(I) = \phi_2(I) = E$ is that $a \notin \bigcup_{I \setminus J} Y$ for every family $(Y)_I$ satisfying the condition (T_1) corresponding to E and with a transversal E.

PROOF. Necessity : Let $(Y)_I$ be any family satisfying the condition (T_1) corresponding to E and $(Y)_I$ has a transversal E. Thus there exists $J \subseteq I$ and $x_j \in X_j$ with $Y_i = x_i$, $i \in J$, and $Y_i = X_i \setminus x_i'$, $i \in I \setminus J$ and $(\bigcup_{i \in J} x_i) \cup (\bigcup_{i \in I \setminus J} x_i') = E$. It is obvious that $a \notin \bigcup_{I \setminus J} Y$ if $J = I$. Thus we may assume that $J \subsetneq I$ and so $I \setminus J \neq \phi$.

Define $\phi_1 : I \to E$ by

$$\phi_1(i) = \begin{cases} x_i & i \in J, \\ x_i', & i \in I \setminus J. \end{cases}$$

Since $(\bigcup_J Y) \cup (\bigcup_{i \in I \setminus J} x_i') = E$, ϕ_1 is an SDR of $(X)_I$ with $\phi_1(I) = E$.

As $(Y)_I$ has a transversal E, there exists an SDR ϕ_2 of $(Y)_I$ giving E. Also ϕ_2 is an SDR of $(X)_I$. Now for every $i \in J$, $\phi_1(i) = \phi_2(j)$ but for every $i \in I \setminus J$, $\phi_1(i) \notin Y_i$ and hence $\phi_1(i) \neq \phi_2(i)$. Suppose $a \in \bigcup_{I \setminus J} Y$. Since $\phi_2(I \setminus J) = (\bigcup_{I \setminus J} Y) \cap E$, $a = \phi_2(j)$ for some

$j \in I \setminus J$. But $a \in E = \phi_1(I)$. Thus $a = \phi_1(i)$ for some $i \in I$. Now $\phi_1(i) = \phi_2(j) = a$ so that by the assumption $i = j \in I \setminus J$ which is a contradiction. Hence $a \notin \bigcup_{I \setminus J} Y$.

Sufficiency : Assume that ϕ_1, ϕ_2 are SDR's of $(X)_I$ giving the same transversal E. Let $\phi_1(i) = \phi_2(j) = a$. We may assume that $\phi_1 \neq \phi_2$. Let $\phi \neq I_1 \subseteq I$ be such that $\phi_1(i) \neq \phi_2(i) \Leftrightarrow i \in I_1$. Define the family $(Y)_I$ by

$$Y_i = \begin{cases} \phi_1(i) & i \in I \setminus I_1, \\ X_i \setminus \phi_1(i) & i \in I_1 \end{cases} .$$

Then $(Y)_I$ satisfies (T_1) corresponding to E. By the same argument as in the proof of Theorem 3.2.4, $\phi_1(I) = \phi_2(I)$ so that $(Y)_I$ has a transversal $E = \phi_2(I)$. Thus by the assumption $a \notin \bigcup_{I_1} Y$ and hence $\phi_2(j) = a = \phi_1(i) = \phi_2(i)$. Therefore $i = j$ as required. //

4 REPRESENTABLE AND BINARY PREGEOMETRIES

In this chapter we examine the class of pregeometries isomorphic to subpregeometries of finite dimensional vector spaces.

4.1 REPRESENTABLE PREGEOMETRIES

4.1.1 A pregeometry $G(S)$ is *representable over the field F* if there exists a vector space V over F and a function $f : S \to V$ whose natural extension to $2^S \to 2^V$ preserves rank.

The function f is a *representation* of $G(S)$.

As rank of any set in a subpregeometry of $G(S)$ is equal to its rank in $G(S)$ we have

4.1.2 LEMMA. If $G(S)$ is representable over F, then any subpregeomtry of $G(S)$ is also representable over F.

From Mirsky [71],

4.1.3 THEOREM. Any transversal pregeometry is representable.

PROOF. Let $G(S)$ be any transversal pregeometry of rank r with a presentation $(X)_I$, where $|I| = r$. Let $Z = \{ Z_{\epsilon i} / i \in I, e \in X_i\}$, where the Z's are independent indeterminates over the field of rational numbers. Let F be the field of rational functions in the Z's (each function involving only a finite number of indeterminates). For each $e \in S$ define the mapping $\psi_e : I \to F$ as follows.

$$\psi_e(i) = \begin{cases} Z_{ei} & \text{if } e \in X_i , \\ 0 & \text{otherwise .} \end{cases}$$

For $\alpha_1, \alpha_2 \in F$ and $e_1, e_2 \in S$, let the mapping

$\alpha_1 \psi_{e_1} + \alpha_2 \psi_{e_2} : I \to F$ be defined by the equation $(\alpha_1 \psi_{e_1} + \alpha_2 \psi_{e_2})(i) = \alpha_1 \psi_{e_1}(i) + \alpha_2 \psi_{e_2}(i)$, $\forall i \in I$. Let V be the set of all linear combinations, with coefficients in F, of the mapping ψ_e, $e \in S$. Then V is a vector space over F. Consider $f : S \to V$ defined by $f(e) = \psi_e$ if $e \in \underset{I}{\cup} X$ and $f(e) = 0$ otherwise. Then f is injective. We show that f is a representation of G(S).

Let $E = \{e_1, \ldots, e_k\}$ be a PT of $(X)_I$. Then there exist $i_1 \neq i_2 \neq \ldots \neq i_k$ with $e_j \in A_{i_j}$, $1 \leq j \leq k$. Consider the $k \times k$ matrix M whose (r, s) element is $\psi_{e_r}(i_s)$, where $1 \leq r, s \leq k$. All elements on the main diagonal of M are independent indeterminates and any other element of M is either o or an indeterminate. But all indeterminates occuring in the entries of M are different. Thus M is non-singular. Suppose that $f(E) = \{\psi_{e_1}, \ldots, \psi_{e_k}\}$ is linearly dependent in V. Hence $\alpha_1 \psi_{e_1} + \ldots + \alpha_k \psi_{e_k} = 0$ for some $\alpha_1, \ldots, \alpha_k$ in F and all $\alpha_1, \ldots, \alpha_k$ are not zero. Therefore $\alpha_1 \psi_{e_1}(i_s) + \ldots + \alpha_k \psi_{e_k}(i_s) = 0$, $1 \leq s \leq k$, and so the rows of M are linearly dependent over F which is a contradiction. Thus f(E) is linearly independent in V.

Suppose that $G = \{e_1, \ldots, e_k\}$ is not a PT of $(X)_I$. We show that f(G) is linearly dependent in V. Since G is not a PT of $(X)_I$, G contains a maximal PT E.

If $|E| < r$, there exists a non-empty subset $J = \{i_1, \ldots, i_p\}$ of I with $e_j \notin A_i$, $1 \leq j \leq k$; $i \in I \setminus J$ (otherwise E is not a maximal PT contained in G). Hence $\psi_{e_j}(i) = 0$, $i \leq j \leq k$; $i \in I \setminus J$. Consider the $k \times p$ matrix N whose (j, s) element is $\psi_{e_j}(i_s)$,

$1 \leq j \leq k$, $1 \leq s \leq p$. Suppose that the rows of N are linearly independent over F. Then $k \leq p$ so that N has a non-singular matrix, say $N' = (\psi_{e_j}(i_s))$, $1 \leq j, s \leq k$. Thus all elements on the main diagonal of N' are indeterminates and hence G is a PT of $(X)_I$ which is not so. Hence the rows of N are linearly dependent. Then $\exists\ \alpha_1, \ldots, \alpha_k$ of F, not all zero such that

$$\alpha_1 \psi_{e_1}(i) + \ldots + \alpha_k \psi_{e_k}(i) = 0 \quad , \quad i \in J .$$

But $\psi_{e_j}(i) = 0$, $1 \leq j \leq k$; $i \in I \setminus J$ and therefore

$$\alpha_1 \psi_{e_1} + \ldots + \alpha_k \psi_{e_k} = 0$$

Thus f(G) is linearly dependent in V.

If $|E| = r$, we consider the $k \times r$ matrix $N = (\psi_{e_j}(i))$, $1 \leq j \leq k$; $i \in I$. Then the rows of N are linearly dependent (as $k > r$) so that by the above f(G) is linearly dependent in V. Thus the theorem is proved. //

We note that a matroid M(S) on $S = \{x_1, \ldots, x_n\}$ is representable over F if and only if there exists a matrix A of n columns with elements in F such that the function f on S defined by

$f(x_i)$ = the ith column vector of A ,

in the vector space of columns of A is a representation of M(S) over F.

Thus if any matroid M(S) of rank r is representable over F, then for any given basis $B = \{b_i, \ldots, b_r\} = S \setminus \{b_{r+1}, \ldots, b_n\}$ of M(S) there exists a *standard matrix representation* $A = [\, I_r, D \,]$, where I_r is the $r \times r$ identity matrix and D is an $r \times (n - r)$ matrix with entries in F.

4.1.4 EXAMPLE. The 2 uniform matroid $U_{2,4}$ on 4 elements is representable over every field except GF(2).

PROOF. We first show that $U_{2,4}$ is not representable over GF(2). Suppose the contrary. Then there exists a matrix

$A = \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix}$ with elements in GF(2) such that any two columns are independent. Now the column vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ are independent and so $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since the two elements in $U_{2,4}$ represented by the second and third columns of A are independent, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now $\begin{bmatrix} c \\ d \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are independent so that $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is impossible. Thus $U_{2,4}$ is not representable over GF(2).

For any field F of more than 2 elements consider the matrix

$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ with elements in F. We see that any two columns of A are independent while any three columns are dependent. Thus $U_{2,4}$ is representable over F with a standard matrix A. //

4.1.5 THEOREM. If M(S) is representable over F, then $M^*(S)$ is also representable over F.

PROOF. Let M(S) be a matroid of rank r on the set S of n elements which is representable over a field F.

Let the $r \times n$ matrix $A = (a_{ij})$, $a_{ij} \in F$ be a matrix representation of M(S). Consider the linear transformation ψ from the vector space V(n, F) of n-tuples of elements in F to the vector space $V(r, F)$ defined by $\psi(x) = Ax'$.

Now $\text{Ker } \psi = \{ x \in V(n, F) / Ax' = 0 \}$ and $\dim(\text{Ker } \psi) = n - r$. Choose $n \times (n - r)$ matrix B with entries from F such that the columns of B span $\ker \psi$. Thus

(1) $\quad Ax' = 0 \iff x' = By'$ for some $y \in V(n - r, F)$

We show that B' is a matrix representation of the dual $M^*(S)$

To prove this we need to show that r columns of A are linearly independent over F if and only if the complementary set of $n - r$ columns of B' are linearly independent over F. Also by reordering the columns of A (and B) it is sufficient to show that the first r columms of A are linearly independent if and only if the last $n - r$ columns of B' are linearly independent. Again it is sufficient to show that the first r columns of A are linearly dependent if and only if the last $n - r$ columns of B' are linearly dependent. We shall show this. By (1) there exists $0 \neq y = (y_1, y_2, \ldots, y_r, 0, \ldots, 0) \in V(n, F)$ with $Ay' = 0$ if and only if there exists $0 \neq z \in V(n - r, F)$ such that $y' = Bz'$.

We can write B in the form $B = (B_1, B_2)'$, where B_1 is $(n - r) \times r$ and B_2 is $(n - r) \times (n - r)$. It then follows that $B_2' z' = 0$. But $z \neq 0$. Hence B_2' and so B_2 is singular so that its columns are linearly dependent and the theorem is proved. //

4.1.6 LEMMA If a matroid $M(S)$ of rank r on $S = \{ x_1, \ldots, x_n \}$ has the standard representation

$$\begin{array}{c} x_1, \ldots, x_r, x_{r+1}, \ldots, x_n \\ \hline \left| \begin{array}{cc} I_r\,, & A \end{array} \right. \end{array}$$

then $M^*(S)$ has the standard representation

$$\begin{array}{c} x_1, \ldots, x_r, x_{r+1}, \ldots, x_n \\ \hline \left[-A', \quad I_{n-r} \right] \end{array}$$

PROOF It suffices to show that the columns of $\left[-A', I_{n-r} \right]'$ span the space of solutions of $\left[I_r, A \right] x' = 0$.

Let $x = \left[-A', I_{n-r} \right]' y$, where $y \in V(n, F)$. Then

$$\left[I_r, A \right] x' = \left[I_r, A \right] \begin{bmatrix} -A \\ I'_{n-r} \end{bmatrix} y' = \left[-I_r A + A I_r \right] y' = 0$$

Thus x is a solution of $\left[I_r, A \right] x' = 0$

Let $x = (x_1, \ldots, x_n)$ be a solution of $\left[I_r, A \right] x' = 0$.

Put $y = (x_{r+1}, \ldots, x_n) \in V(n - r, F)$. Then

$$x = \begin{bmatrix} -A \\ I_{n-r} \end{bmatrix} y = \left[-A', I_{n-r} \right]' y$$

and the lemma is proved. //

4.2 BINARY MATROIDS

4.2.1 A matroid $M(S)$ is *binary* if it is representable over $GF(2)$.

It follows easily from Theorem 4.1.6 that $M(S)$ is binary if an only if $M^*(S)$ is binary.

Welsh [76] gave necessary and sufficient conditions for a matroid M(S) to be binary. The following definition and lemma are needed for these conditions.

4.2.2 The *symmetric difference* of sets $C_1, \ldots, C_n$, written $C_1 \Delta \ldots \Delta C_n$, is the set $\bigcup_{i=1}^{n} (C_i \setminus \bigcup_{C \neq C_i} C)$.

Notice that $x \in C_1 \Delta \ldots \Delta C_n$ if and only if x belongs to exactly one of $C_1, \ldots, C_n$. Thus $C_1 \Delta \ldots \Delta C_n = (C_1 \Delta \ldots \Delta C_{n-1}) \Delta C_n$.

4.2.3 LEMMA. If $C_1, \ldots, C_n$ are sets such that $|C \cap C_i|$ is even, $1 \leq i \leq n$. Then $|(C_1 \Delta \ldots \Delta C_n) \cap C|$ is even.

PROOF. We first show that $(C_1 \Delta \ldots \Delta C_n) \cap C$ $= (C_1 \cap C) \Delta \ldots \Delta (C_n \cap C)$. For n = 2 we have $(C_1 \Delta C_2) \cap C$ $= ((C_1 \setminus C_2) \mathring{\cup} (C_2 \setminus C_1)) \cap C = ((C_1 \setminus C_2) \cap C) \mathring{\cup} ((C_2 \setminus C_1) \cap C)$ $= (C_1 \cap C \setminus C_2 \cap C) \mathring{\cup} (C_2 \cap C \setminus C_1 \cap C) = (C_1 \cap C) \Delta (C_2 \cap C)$. Assume that $(C_1 \Delta \ldots \Delta C_k) \cap C = (C_1 \cap C) \Delta \ldots \Delta (C_k \cap C)$, where $2 \leq k < n$. Then $(C_1 \Delta \ldots \Delta C_{k+1}) \cap C = ((C_1 \Delta \ldots \Delta C_k) \Delta C_{k+1}) \cap C$ $= ((C_1 \Delta \ldots \Delta C_k) \cap C) \Delta (C_{k+1} \cap C) = ((C_1 \cap C) \Delta \ldots \Delta (C_k \cap C)) \Delta$ $(C_{k+1} \cap C) = (C_1 \cap C) \Delta \ldots \Delta (C_{k+1} \cap C)$. Hence $(C_1 \Delta \ldots \Delta C_n) \cap C$ $= (C_1 \cap C) \Delta \ldots \Delta (C_n \cap C)$.

We next show that for any sets A, B if $|A|$, $|B|$ are even, then $|A \Delta B|$ is even. Observe that $A \Delta B = (A \setminus A \cap B) \mathring{\cup} (B \setminus A \cap B)$. If $|A \cap B|$ is odd, then since $A = (A \setminus A \cap B) \mathring{\cup} (A \cap B)$, $|A \setminus A \cap B|$ is odd. Also $|B \setminus A \cap B|$ is odd and hence $|A \Delta B| = |A \setminus A \cap B|$ $+ |B \setminus A \cap B|$ is even. If $|A \cap B|$ is even, by the same argument we obtain $|A \Delta B|$ even.

Thus $|(C_1 \Delta C_2) \cap C| = |(C_1 \cap C) \Delta (C_2 \cap C)|$ is even by the above.

Assume that $|(C_1 \Delta \ldots \Delta C_k) \cap A|$ is even, where $2 \leq k < n$. Thus $r = |(C_1 \Delta \ldots \Delta C_{k+1}) \cap C| = |(C_1 \cap C) \Delta \ldots \Delta (C_{k+1} \cap C)| = |((C_1 \cap C) \Delta \ldots \Delta (C_k \cap C)) \Delta (C_{k+1} \cap C)|$. Let $C' = (C_1 \cap C) \Delta \ldots \Delta (C_k \cap C)$. By the assumption $|C'|$ is even and so by the above $r = |C' \Delta ((C_{K+1} \cap C))|$ is even //

4.2.4 THEOREM. The following statements about M(S) are equivalent.

(i) For any circuit C and any cocircuit C^* of M(S), $|C \cap C^*|$ is even.

(ii) The symmetric difference of any finite collection of distinct circuits of M(S), if not empty, is the union of disjoint circuits of M(S).

(iii) The symmetric difference of any distinct circuits C_1, C_2 of M(S) contains a circuit of M(S).

(iv) If $C \setminus B = \{ x_1, \ldots, x_q \}$, where C is a circuit of M(S) and B is a basis of M(S). Then

$$C = C(x_1, B) \Delta \ldots \Delta C(x_q, B)$$

(v) M(S) is binary.

PROOF. We prove the theorem in 3 steps. Firstly we show that (i), (ii) and (iii) are equivalent. Secondly we show (i) <=> (iv) and finally we show (iv) <=> (v).

(i) => (ii) :

Let $C_1, \ldots, C_k$ be distinct circuits of M(S). Put $A = C_1 \Delta \ldots \Delta C_k$. Suppose that A is independent and non-empty. Extend A to a basis B. Let $x \in A$. By Lemma 2.8.6 there is a cocircuit C^* of M(S) with $(B \setminus x) \cap C^* = \phi$ and $x \in C^*$. Then $|C^* \cap A| \leq |C^* \cap B| = 1$. By Lemma 4.2.3 $|C^* \cap A|$ is even. Thus we have a contradiction.

Hence A is dependent in M(S) and so it contains a circuit C.

If $A = C$ we are finished, if $A \neq C$ we consider $A_1 = C \Delta C_1 \Delta \ldots \Delta C_k$ and apply the above argument with $A = A_1$. Since A is finite and $A_1 = A \setminus C$, this process eventually terminates giving a finite collection of disjoint circuits whose union is A and (ii) is proved.

(ii) => (iii) is clear.

(iii) => (i) :

Suppose that M(S) satisfies (iii) but not (i). Then there exist a circuit C and a cocircuit C^* of M(S) such that $|C \cap C^*|$ is not even. Choose such C and C^* with $|C \cap C^*|$ minimum. By Lemma 2.8.5 $|C \cap C^*| \neq 1$ and so $|C \cap C^*| \geq 3$. Let a, b, c be distinct elements of $C \cap C^*$. By Lemma 2.8.7 there exists a circuit C_1 with $C_1 \cap C^* = ac$. By (K_4') there exists a circuit $C_2 \subsetneq (C \cup C_1) \setminus a$ and $b \in C_2$. Choose C_2 so that $C \cup C_2$ is minimal. Also by (K_4') there exists a circuit $C_3 \subsetneq (C \cup C_2) \setminus b$ with $a \in C_3$ and so there exists a circuit $C_4 \subsetneq (C \cup C_3) \setminus a$ with $b \in C_4$. Now $C \cup C_4 \subseteq C \cup C_3 \subsetneq C \cup C_2 \subseteq C \cup C_1$ and $b \in C_4$. Thus $C_4 \subsetneq (C \cup C_1) \setminus a$. Since $C \cup C_2$ is minimal, $C \cup C_2 \subseteq C \cup C_4$ so that $C \cup C_2 \subsetneq C \cup C_4 \subsetneq C \cup C_3 \subseteq C \cup C_2$. Thus $C \cup C_3 = C \cup C_2$ and so $C_3 \setminus C = C_2 \setminus C$. Hence $C_2 \Delta C_3 = (C_2 \setminus C_3) \mathbin{\dot{\cup}} (C_3 \setminus C_2) \subseteq C$. By the assumption $C_2 \Delta C_3$ contains a circuit. Thus $C_2 \Delta C_3 = C$. Observe that $|C_3 \cap C^*|$ is positive (as $a \in C_3 \cap C^*$). If $|C_3 \cap C^*|$ is even, then as $|C \cap C^*| = |(C_2 \Delta C_3) \cap C^*| = |C_2 \cap C^*| + |C_3 \cap C^*| - 2\,|C_2 \cap C_3 \cap C^*|$, $|C_2 \cap C^*|$ is odd (otherwise $|C \cap C^*|$ is even). But $|C_2 \cap C^*| < |C \cap C^*|$, contradicting minimality of $|C \cap C^*|$. Thus $|C_3 \cap C^*|$ is odd and this also contradicts the minimality of $|C \cap C^*|$. Therefore (i) is proved.

(i) => (iv) :

Let $M(S)$ satisfy (i). Let $C \setminus B = \{e_1, \ldots, e_t\}$, where C is a circuit of $M(S)$ and B is a basis of $M(S)$. Put $Z = C(e_1, B) \Delta \ldots \Delta C(e_t, B)$. Then $\{e_1, \ldots, e_t\} \subseteq Z$. Now $C \Delta Z \subseteq B$. Since $M(S)$ also satisfies (ii), if $C \Delta Z \neq \phi$, it is the union of disjoint circuits which is impossible. Thus $C \Delta Z = \phi$ and so $C = Z$ as required.

(iv) => (i) :

Let $M(S)$ satisfy (iv). Since (i) <=> (iii), it suffices to show that $M(S)$ satisfies (iii). Let D_1, D_2 be distinct circuits of $M(S)$. We show that $D_1 \Delta D_2$ is dependent. Suppose not, and let $D_1 \cap D_2 = \{x_1, x_2, \ldots, x_k\}$. Then $D_1 \Delta D_2 = (D_1 \cup D_2) \setminus x_1 \ldots x_k$ is independent. Extend $D_1 \Delta D_2$ to a basis B of $M(S)$. Thus $D_1 \setminus B = D_2 \setminus B = \{x_1, \ldots, x_q\}$ and so $D_1 = C(x_1, B) \Delta \ldots \Delta C(x_q, B) = D_2$ by the assumption which is a contradiction. Hence $D_1 \Delta D_2$ is dependent and so it contains a circuit.

(v) => (iv) :

Let $M(S)$ be binary and let $B = \{b_1, \ldots, b_r\} = S \setminus \{e_1, \ldots, e_q\}$ be a basis of $M(S)$. Then there exists a standard matrix representation of $M(S)$ over $GF(2)$ of the form

$$\begin{array}{c|c} b_1 \ldots\ldots b_r & e_1 \ldots\ldots e_q \\ \hline I_r & A \end{array}$$

The elements of A are in $GF(2)$. Let C be a circuit of $M(S)$. We may assume that $C = \{b_1, \ldots, b_t, e_1, \ldots, e_p\}$. Then for each j, $1 \leq j \leq p$ we have

$$C(e_j, B) \setminus e_j = \{ b_i \,/\, a_{ij} = 1 \} .$$

We show that for any $b_i \in B \cap C$, $b_i \in C(e_j, B)$ for a unique j. Suppose that there exists $b_i \in B \cap C$ with $b_i \notin C(e_j, B)$, $j = 1,\ldots, p$. Then $a_{ij} = 0$, $j = 1,\ldots,p$. Now $C_1 = C \setminus b_i$ is independent. Let f be the representation given by the columms of the above matrix. Consider $\sum_{c_i \in C_i} x_i f(c_i) + yf(b_i) = 0$. We see that y must be zero and since C_1 is independent, $x_i = 0$, $\forall i$. Hence $f(C)$ is linearly independent over GF(2). This is a contradiction. Thus $b_i \in C(e_j, B)$.

Since C is a circuit, f(c) is linearly dependent over GF(2) so that $f(C) = \sum_{i=1}^{t} f(b_i) + \sum_{j=1}^{p} f(e_i)$ is the zero vector. But each b_i in $B \cap C$ is in $C(e_j, B)$ for some j, thus each row of the matrix $[f(C)]$ is occupied by 1 in even number of times. Hence each b_i in $B \cap C$, b_i is in odd number of $C(e_1, B),\ldots, C(e_p, B)$. Suppose that there exists, b_t say, so that b_t is in at least 3 sets of $C(e_1, B),\ldots, C(e_p, B)$. We may assume that b_t is in $C(e_1, B)$, $C(e_2, B)$ and $C(e_p, B)$ Consider the vector $f(e_p)$. We choose $x_i = 0$ if the i th component of $f(e_p)$ is 1 and $x_i = 1$ otherwise. Thus $\sum_{i=1}^{t-1} x_i(f(b_i)) + (f(b_t) + f(e_1) + f(e_2) +\ldots+ f(e_{p-1})$ is the zero vector, contradicting the fact that $\{f(b_1),\ldots, f(b_t), f(e_1),\ldots,f(e_{p-1})\}$ is linearly independent over GE(2). Thus any $b_i \in B \cap C$ is in exactly one of $C(e_1, B),\ldots, C(e_p, B)$.

We show that $C = C(e_1, B)\Delta \ldots \Delta C(e_p, B)$. Since $f(C(e_i, B)) = 0$, $i = 1,\ldots, p$, we have $\sum_{i=1}^{p} f(C(e_i, B)) = 0$. Let $C' = (\bigcup_{i=1}^{p} C(e_i, B)) \cap (B \setminus C)$. We write

$\sum_{i=1}^{p} f(C(e_i, B)) = \sum_{b_i \in C'} f(b_i) + f(C) = 0$. But $f(C) = 0$ and so

$\sum_{b_i \in C'} f(b_i) = 0$. We see that for any $b_i \in C'$, b_i must occur in even number of $C(e_1, B), \ldots, C(e_p, B)$. Therefore $b_i \notin C(e_1, B) \Delta \ldots \Delta C(e_p, B)$ and hence $C = C(e_1, B) \Delta \ldots \Delta C(e_p, B)$.

(iv) => (v) :

Let $M(S)$ satisfy (iv). Let $B = \{b_1, \ldots, b_r\}$ be a basis of $M(S)$ and $S \setminus B = \{e_1, \ldots, e_q\}$. Define a matrix A by

$$a_{ij} = \begin{cases} 1 & \text{if } b_i \in C(e_j, B), \ 1 \le i \le p, \ 1 \le j \le q, \\ 0 & \text{if } b_i \notin C(e_j, B), \ 1 \le i \le p, \ 1 \le j \le q. \end{cases}$$

Put $B = [\, I_r, A \,]$. We show that the function f on S defined by $f(b_i)$ = the i th column vector of I_r and $f(e_j)$ = the j th column vector of A is a representation of $M(S)$ over $GF(2)$.

Let C be a circuit of $M(S)$. We show that $f(C)$ is linearly dependent over $GF(2)$. Let $C \setminus B = \{e_{i_1}, \ldots, e_{i_k}\}$. By the assumption $C = C(e_{i_1}, B) \Delta \ldots \Delta C(e_{i_k}, B)$. If $B \cap C \neq \phi$, then for any $b_k \in B \cap C$, there is unique j with $b_k \in C(e_j, B)$. Thus $(B \cap C) \cap C(e_j, B) \neq \phi$ for some j. We may assume without loss of generality that $(B \cap C) \cap (C(e_{i_1}, B)) \neq \phi$. Suppose $(B \cap C) \cap C(e_{i_1}, B) = \{b_1, \ldots, b_s\}$. Thus $a_{mi_1} = 1$, $1 \le m \le s$, and $a_{mi_1} = 0$, $s < m \le r$, and so $f(e_{i_1}) = f(b_1) + \ldots + f(b_s)$. Hence $f(C)$ is linearly dependent over $GF(2)$. If $B \cap C = \phi$, then any $b_i \in \bigcup_{j=1}^{k} C(e_{i_j}, B)$ occurs in even number of $C(e_{i_1}, B), \ldots, C(e_{i_k}, B)$. Thus $f(C) = \{f(e_{i_1}), \ldots, f(e_{i_k})\}$ is linearly dependent over $GF(2)$ (as $\sum_{j=1}^{k} f(e_{i_j}) = 0$).

Let $f(U)$ be linearly dependent over GF(2) and such that every proper subset of $f(U)$ is linearly indendent over GF(2). Suppose that $U = \{b_1, \ldots, b_t, e_1, \ldots, e_p\}$. We show that U is a circuit of $M(S)$. Firstly we show that $U = C(e_1, B) \Delta \ldots \Delta C(e_p, B)$. By the same argument as above for any $b_i \in B \cap U$, $b_i \in C(e_j, B)$ for a unique j. Thus $U \subseteq C(e_1, B) \Delta \ldots \Delta C(e_p, B)$. We are left to show that each b_k in $(B \setminus U) \cap (\bigcup_{j=1}^{p} C(e_j, B)) = C'$, b_k occurs in even number of $C(e_1, B), \ldots, C(e_p, B)$. Consider $\sum_{j=1}^{p} f(C(e_j, B)) = 0$ we can write $\sum_{j=1}^{p} f(C(e_j, B)) = \sum_{b_k \in C'} f(b_k) + f(U)$. But $f(U) = 0$ so that $\sum_{b_k \in C'} f(b_k) = 0$. Thus each $b_k \in C'$ occurs in even number of $C(e_1, B), \ldots, C(e_p, B)$.

Since (iv) <=> (iii), $C(e_1, B) \Delta C(e_2, B)$ contains a circuit C_1 and so $C_1 \Delta C(e_3, B) \Delta \ldots \Delta C(e_p, B) \subseteq U$. Consider $C_1 \Delta C(e_3, B)$. Then there exists a circuit C_2 with $C_2 \Delta C(e_4, B) \Delta \ldots \Delta C(e_p, B) \subseteq U$. Carry on in this way we reach the step $C_{p-2} \Delta C(e_p, B) \subseteq U$, where C_{p-2} is a circuit of $M(S)$, and so there is a circuit $C_{p-1} \subseteq U$. Thus U is dependent. Observe that $f(A)$ is linearly independent implies that A is independent in $M(S)$. Thus every proper subset of U is independent in $M(S)$ and so U is a circuit. Thus (v) is proved.

Hence the theorem is proved. //

4.2.5 EXAMPLE. $M(\mathcal{S}_n)$ is binary if and only if $n = 7$.

PROOF. We first show that $M(\mathcal{S}_n)$ is not binary when $n \neq 7$.

First we show that $n^2 - 10n + 21 > 0$ if $n > 7$. For $n = 8$

we have $n^2 - 10n + 21 = 64 - 80 + 21 > 0$. Assume $k^2 - 10k + 21 > 0$ and $k > 7$. Then $(k + 1)^2 - 10(k + 1) + 21 = (k^2 - 10k + 21) + (2k - 9) > 0$ (as $2k - 9 > 0$).

Secondly for, any $n \equiv 1$ or $3 \pmod 6$, $n > 7$ we claim that there exist two disjoint triples of $\mathcal{S}_n$. If not, suppose $123 \in \mathcal{S}_n$. Then any triple intercects 123. As each of 1,2,3 occurs in $\frac{n-1}{2}$ triples and the number of triples in $\mathcal{S}_n$ is $\frac{n(n-1)}{6}$, we have

$$3 \frac{(n-1)}{2} - 2 = \frac{n(n-1)}{6}$$

which implies $n^2 - 10n + 21 = 0$. This is not so. Hence there exist two disjoint triples. Let A_1, A_2 be disjoint triples in $\mathcal{S}_n$. As shown in Chapter 2 , A_1 is a hyperplane and hence $S_n \setminus A_1$ is a cocircuit. Now A_2 is a circuit of $M(\mathcal{S}_n)$ and $|(S_n \setminus A_1) \cap A_2| = 3$ which is odd. By Theorem 4.2.4 $M(\mathcal{S}_n)$ is not binary.

To show that $M(\mathcal{S}_7)$ is binary let C_1, C_2 be distinct circuits of $M(\mathcal{S}_7)$ and $C_1 \cap C_2 \neq \phi$. We shall show by exhaustion that $C_1 \Delta C_2$ contains a circuit. Observe that the set of circuits of $M(\mathcal{S}_n)$ is the union of $\mathcal{S}_n$ and the family $\mathcal{C}_n$ of sets $A \subsetneq S_n$ with $|A| = 4$ and such that $A \setminus x \notin \mathcal{S}_n$, $\forall x \in A$.

case 1. $C_1, C_2 \in \mathcal{S}_7$.

Then $|C_1 \Delta C_2| = 4$ and any 3 - subset of $C_1 \Delta C_2$ can not be a triple. Thus $C_1 \Delta C_2$ is a circuit.

case 2. $C_1 \in \mathcal{S}_7$, $C_2 \in \mathcal{C}_7$, $|C_1 \cap C_2| = 1$

Without loss of generality let $C_1 = \{x_1, x_2, x_3\}$, $C_2 = \{x_1, x_4, x_5, x_6\}$

If $\{x_4, x_5, x_7\}$ is a triple, let C be the triple containing x_4, x_6.

Then $C \neq \{x_4, x_6, x_7\}$. Thus C contains an element of $C_1 \setminus C_2$. That is $C \subseteq C_1 \Delta C_2$ . .

If $\{x_4, x_5, x_7\}$ is not a triple, consider the triple C' containing x_4, x_5. Then we have $C' \subseteq C_1 \Delta C_2$

Case 3. $C_1 \varepsilon \mathscr{S}_7$, $C_2 \varepsilon \mathscr{C}_7$, $|C_1 \cap C_2| = 2$

Again we can assume that $C_1 = \{x_1, x_2, x_3\}$, $C_2 = \{x_1, x_2, x_4, x_5\}$. Since the triple C containing x_4, x_5 must intersect C_1 in exactly one element, $C \cap C_1 = x_3$. Hence $C \subseteq C_1 \Delta C_2$.

case 4. $C_1, C_2 \varepsilon \mathscr{C}_7$.

We shall first show that $|C_1 \cap C_2| \neq 1$, if not so let $C_1 = \{a,b,c,d\}$ $A_2 = \{a, p, q, r\}$. Then $C_1 \cup C_2 = S_7$. Consider the triples containing a and b, a and c, a and d, we see that each of these triples must have exactly two elements in A_2. We can assume these triples to be $\{a, b, p\}$, $\{a, c, q\}$, $\{a, d, r\}$. Then the triples containing b and c, b and d, c and d must be $\{b, c, r\}$, $\{b, d, q\}$, $\{c, d, p\}$. Now consider the other element x in the triple containing p and q. We see that $x \neq a, p, q, r$. But $x \neq b$ (othewise b, p are in the two triples). Also $x \neq c, d$. Hence no triple contains p and q which is a contradiction. Thus $|C_1 \cap C_2| \neq 1$. We shall show that $|C_1 \cap C_2| = 2$. Suppose that $|C_1 \cap C_2| = 3$, let $C_1 \cap C_2 = \{x_1, x_2, x_3\}$ and $S_7 \setminus C_1 \cup C_2 = \{x_4, x_5\}$. We can form three distinct 2-subsets from $C_1 \cap C_2$ and since any two elements are contained in exactly one triple, there is one 2-subset from $C_1 \cap C_2$, $\{x_1, x_2\}$ say, which does not form a triple with either x_4 or x_5. Thus another element in the triple containing x_1, x_2 is in $C_1 \Delta C_2$, contradicting the assumption that $C_1, C_2 \varepsilon \mathscr{C}_7$. Hence $|C_1 \Delta C_2| = 2$ and so $|C_1 \Delta C_2| = 4$. Thus $C_1 \Delta C_2$ must contain a circuit.

In fact the Fano matroid is only representable over GF(2) (Rado [57]). The standard representation of the Fano matroid over GF(2) is given by the following.

$$\begin{array}{c|ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}$$

By Lemma 4.1.6 the dual of the Fano matroid has the following standard representation

$$\begin{array}{c|ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}$$

4.2.6 EXAMPLE. Let $S_n = \{1,2,\ldots, n = 2m\}$, $n \geq 6$. Let $\mathcal{B}_n$ be the family of 2-subsets of S excluding 2-subsets of the form $\{i, i+1\}$, where $i = 1,3,\ldots, 2m-1$.

Then $\mathcal{B}_n$ is the family of bases of the matroid $M(\mathcal{B}_n)$ on S_n and $M(\mathcal{B}_n)$ is binary if and only if $n = 6$.

PROOF. To prove that $M(\mathcal{B}_n)$ is a matroid we only need to show that for any $i \neq j \neq k$ and $\{i, j\} \in \mathcal{B}_n$, at least one of $\{i, k\}$ or $\{j, k\}$ is in $\mathcal{B}_n$. Let $\{i, j\} \in \mathcal{B}_n$ and $k \neq i, j$.

case 1. k is odd.

If $k = i - 1$ or $j - 1$, without loss of generality assume that $k = i - 1$, hence $k \neq j - 1$ so that $\{j, k\} \in \mathcal{B}_n$. If $k \neq i - 1, j - 1$, then $\{i, k\}$; $\{j, k\} \in \mathcal{B}_n$.

case 2. k is even.

If $k < i$, then $\{i, k\} \in \mathcal{B}_n$. If $k > i$, then $\{j, k\} \in \mathcal{B}_n$ in case $i + 1 = k$ and $\{i, k\} \in \mathcal{B}_n$ otherwise.

Hence $M(\mathcal{B}_n)$ is a matroid on S.

For each $i \in I = \{1,3,\ldots, 2m - 1\}$, let $F_i = \{i, i + 1\}$ then F_i is a circuit of $M(\mathcal{B}_n)$. For distinct integers i, j, k in I the set $\{x_i, x_j, x_k\}$, where $x_r \in F_r$ $\forall r = i, j, k$ is a circuit of $M(\mathcal{B}_n)$. Let $\mathcal{C}(n)$ be the family of $\{x_i, x_j, x_k\}$ defined as above. Then the circuits of $M(\mathcal{B}_n)$ are $(\bigcup_{i \in I} F_i) \cup \mathcal{C}(n)$.

To show that $M(\mathcal{B}_6)$ is binary let C_1, C_2 be distinct circuits of $M(\mathcal{B}_6)$ such that $C_1 \cap C_2 \neq \phi$.

case 1. $C_1 = F_i$, $C_2 \in \mathcal{C}(n)$

Let $C_1 = \{x_i, y_i\}$ and $C_2 = \{x_i, x_j, x_k\}$. Then $i \neq j \neq k$ so that $C_1 \Delta C_2 = \{y_i, x_j, x_k\} \in \mathcal{C}(n)$.

case 2. $C_1, C_2 \in \mathcal{C}(n)$ and $|C_1 \cap C_2| = 1$

Then $C_1 = \{x_i, x_j, x_k\}$ and $C_2 = \{x_i, y_j, y_k\}$ where $F_j = \{x_j, y_j\}$ and $\{x_k , y_k\} = F_k$ and $i \neq j \neq k$. Hence $C_1 \Delta C_2$ contains a circuit F_j.

case 3. $C_1, C_2 \in \mathcal{C}(n)$ and $|C_1 \cap C_2| = 2$

Let $C_1 = \{x_i, x_j, x_k\}$ and $C_2 = \{x_i, x_j, y_k\}$.

Then $C_1 \Delta C_2 = \{x_k, y_k\} = F_k$ which is a circuit of $M(\mathcal{B}_6)$.

Thus M($_6$) is binary.

For $n > 6$ we can choose $C_1 = \{x_i, x_j, x_k\}$ and C_2 $\{x_i, x_j, x_r\}$ where $i \neq j \neq k \neq r$. Then $C_1 \Delta C_2 = \{x_k, x_r\}$ is not a circuit and so M($_n$) is not binary if $n > 6$.

The next theorem due to Tutte [65] gives a necessary and sufficient condition for a matroid to be binary. The proof is drawn from Welsh [76] . The following two lemmas are needed in the proof.

4.2.7 LEMMA. Let C be a circuit of $G(S)$. If $z \in C$, then $C \setminus z$ is a circuit of $G(S).(S \setminus z)$. If $z \notin C$, then either C is a circuit of $G(S).(S \setminus z)$ or C is the disjoint union of two circuits of $G(S).(S \setminus z)$.

PROOF. We first assume that $z \in C$. We showed in the proof of Theorem 2.7.8 that $C \setminus z$ is a circuit of $G(S).(S \setminus z)$.

We next assume that $z \notin C$. Suppose that C is not a circuit of $G(S).(S \setminus z)$. But C is dependent in $G(S).(S \setminus z)$. Thus there exists a proper subset D of C such that D is a circuit of $G(S).(S \setminus z)$. If $D \cup z$ is independent in $G(S)$, then D is independent in $G(S).(S \setminus z)$ which is not so. Hence $D \cup z$ is dependent in $G(S)$ so that $D \cup z$ is a circuit of $G(S)$. We show that $(C \setminus D) \cup z$ is a circuit of $G(S)$.
There exists a circuit $C_1 \subseteq (C \setminus D) \cup z) \setminus x_1$ with $z \in C_1$.
If there exists a in $(C \setminus D) \setminus C_1$. Pick $x_1 \in D$. Then $a \neq x_1$ and so there exists a circuit $C' \subsetneq (C \cup D \cup z) \setminus x_1$ with $a \in C'$. Since $C \setminus x_1$ is independent, $(C \setminus x_1) \cup z$ contains at most one circuit. Hence $C' = C_1$ and so $a \in C_1$ which is a contradiction. Thus $C_1 = (C \setminus D) \cup z$. Hence $C \setminus D$ is a circuit of $G(S).(S \setminus z)$ as required //

4.2.8 LEMMA. Let C' be a circuit of $G(S).(S \setminus z)$. Then C' is a circuit of $G(S)$ or $C' \cup z$ is a circuit of $G(S)$.

PROOF. Suppose that C' is not a circuit of $G(S)$. But every proper subset of C' is independent in $G(S).(S \setminus z)$ and so in $G(S)$. Thus C' is independent in $G(S)$. If z is dependent in $G(S)$, then C' must be independent in $G(S).(S \setminus z)$ which is not so. Thus z is independent in $G(S)$. Since C' is dependent in $G(S).(S \setminus z)$, $C' \cup z$ is dependent in $G(S)$. As $(C' \setminus x)$ is independent in $G(S).(S \setminus z)$, $(C' \setminus x) \cup z$ is independent in $G(S)$ and hence $C' \cup z$ is a circuit of $G(S)$. //

4.2.9 A *minor* of a matroid $M(S)$ is a matroid on a subset of S obtained by any combination of submatroids and contractions of $M(S)$.

4.2.10 THEOREM. A matroid $M(S)$ is binary if and only if it has no minor isomorphic to $U_{2,4}$.

PROOF. Let $M(S)$ be binary. If there exists a minor of $M(S)$ which is isomorphic to $U_{2,4}$, then since the minor is also binary, $U_{2,4}$ is also binary. This is a contradiction. Hence all minors of $M(S)$ are not isomorphic to $U_{2,4}$.

Let $M(S)$ be a matroid which has no minor isormorphic to $U_{2,4}$. We prove the theorem by induction on $|S|$. Assume the theorem is true for any matroid $M(T)$ which has no minor isomorphic to $U_{2,4}$ and $|T| < |S|$. Let C_1, C_2 be disjoint circuits of $M(S)$, where $C_1 \cap C_2 \neq \phi$. We shall show that $C_1 \Delta C_2$ is a disjoint union of circuits, that is $C_1 \Delta C_2$ contains a circuit.

We may assume that $S = C_1 \cup C_2$ (otherwise consider the matroid $M_S(C_1 \cup C_2)$. Let $X = C_1 \cap C_2$, $Y_1 = C_1 \setminus C_2$, $Y_2 = C_2 \setminus C_1$ and $Y = Y_1 \cup Y_2$. We show that Y is a union of disjoint circuits of $M(S)$ by considering all possibilities.

case 1. $|Y_1| = |Y_2| = 1$

If $|X| = 1$, then by (K_4) there exists a circuit $C \subseteq Y_1 \cup Y_2 = C_1 \Delta C_2$ and we are finished. The result also follows if $|X| > 1$ and $Y_1 \cup Y_2$ is dependent. If $Y_1 \cup Y_2$ is independent, then $|X| > 1$ (otherwise $Y_1 \cup Y_2$ is dependent). Extend $Y_1 \cup Y_2$ to a basis $Y_1 \cup Y_2 \cup I = B$. Then $I \subsetneq X$. If $X \setminus I = a$, then $B \cup a$ contains 2 circuits C_1, C_2 which is not so. Hence $|X \setminus I| \geq 2$. In case $|X \setminus I| > 2$ we have $r(B) = 2 + |I| < 2 + |X| - 2 = |X|$ so that $r(M) + 1 = r(B) + 1 < |X| + 1 = |C_1|$. A contradiction. Thus $|X \setminus I| = 2$ and so $Y \cup (X \setminus x_1\ x_2)$ is a basis of $M(S)$. Let $T = \{x_1, x_2, y_1, y_2\}$. Consider $M(S).T$. We see that any 3-subset of T is a circuit of $M(S).T$ so that $M(S).T$ is $U_{2,4}$ which is a contradiction.

case 2. $|Y_1| > 1$.

Let $Y_1 = \{y, z, \ldots\}$. By Lemma 4.2.7 $C_1 \setminus y$ is a circuit of $M(S).(S \setminus y)$. Also by Lamma 4.2.7 C_2 is either a circuit of $M(S).(S \setminus y)$ or C_2 is the disjoint union of two circuits of $M(S).(S \setminus y)$. By the induction hypothesis and by Theorem 4.2.4 the symmetric difference of $C_1 \setminus y$ and C_2 is a disjoint union of circuits of $M(S).(S \setminus y)$. By Lemma 4.2.8 we then can write

$$Y = S_1 \cup \ldots \cup S_r \cup \ldots \cup S_t$$

where each S_i is a circuit of $M(S)$ and

$$S_i \cap S_j = \begin{cases} \{y\} & 1 \leq i \neq j \leq r, \\ \phi & \text{otherwise}. \end{cases}$$

We show that r is odd. Suppose that r is even. Then we pair S_i and S_{i+1} for $i = 1, 3, \ldots, r-1$. By the induction hypothesis $S_i \,\Delta\, S_{i+1}$ is a union of disjoint circuits of $M_S(S_i \cup S_{i+1})$. As any circuit of $M_S(S_i \cup S_{i+1})$ is also a circuit of $M(S)$ it follows that

$$Y = T_1 \cup \ldots \cup T_h \cup y ,$$

where $T_1, \ldots, T_h$ are disjoint circuits of $M(S)$ which do not contain y. Now for each $i = 1, \ldots, h$, T_i is a circuit of $M_S(S \setminus y)$ and so by Lemma 4.2.7 if $z \notin T_i$, $T_i \setminus z$ is a disjoint union of at most two circuits of $M_S(S \setminus y).(S \setminus yz)$ and hence of $M(S).(S \setminus z)$ and T_i is a circuit of $M(S).(S \setminus z)$ if $z \in T_i$. Thus

$$Y \setminus z = R_1 \cup \ldots \cup R_k \cup y ,$$

where $R_1, \ldots, R_k$ are circuits of $M(S).(S \setminus z)$ which do not contain y. Since $Y \setminus z$ is a symmetric difference of $C_1 \setminus z$ and C_2 and $C_1 \setminus z$ is a circuit of $M(S).(S \setminus z)$ and C_2 is the disjoint union of at most two circuits of $M(S).(S \setminus z)$, $Y \setminus z$ is a symmetric difference of at most three circuits of $M(S).(S \setminus z)$. Then

$$y = R_1 \,\Delta \ldots \Delta\, R_k \,\Delta\, (Y \setminus z)$$

is a symmetric difference of circuits of $M(S).(S \setminus z)$. Since $M(S).(S \setminus z)$ is binary, y is a circuit of $M(S).(S \setminus z)$. By Lemma 4.2.8 either y or $y \cup z$ is a circuit of $M(S)$ which is a contradiction.

Hence r is odd and thus by the induction hypothesis for each $i = 2, 4, \ldots, r-1$, $S_{i-1} \,\Delta\, S_i$ is a disjoint union of circuits of $M(S)$. Then we can write $S_1 \Delta \ldots \Delta S_{r-1}$ as $\bigcup_{i=1}^{p} C_i'$ where all C_i' are disjoint circuits of $M(S)$ and therefore $Y = (\bigcup_{i=1}^{p} C_i') \cup S_r \cup \ldots \cup S_t$ is a disjoint union of circuits of $M(S)$ as required. //

5. GAMMOIDS AND BASE ORDERABLE MATROIDS

Strict gammoids, that is, matroids arising from directed graphs were introduced by Mason [72]. We show their relationship to transversal matroids.

The class of gammoids is closed under the taking of minors and under duality and it also contains transversal matroids. Thus the class of gammoids is the closures of the class of transversal matroids under contraction, restriction and dual.

Finally a class of base orderable matroids is discussed.

5.1 STRICT GAMMOIDS AND GAMMOIDS

A *path* in a directed graph (more briefly : digraph) $G = (V,E)$, where V is the set of vertices and E the set of edges, is a sequence $P = (v_0, v_1, \ldots, v_k)$ of a pairwise distinct vertices of G such that $k \geq 0$ and $(v_{i-1}, v_i) \in E$, $1 \leq i \leq k$. The vertices v_0 and v_k are respectively the *initial* and *terminal vertices* of P. We say that $(v_{i-1}, v_i) \in P$ and v_{i-1}, v_i are on P, $1 \leq i \leq k$. Two paths are *disjoint* if their vertex sets are disjoint.

Let $A, B \subseteq V$. A *linking* of A onto B is a bijection $\alpha : A \to B$ such that there are pairwise disjoint paths $(P_x / x \in A)$, where P_x has initial vertex x and terminal vertex $\alpha(x) \in B$. Before we present the Linkage Lemma due to Ingleton and Piff [73] we define for any $Z \subseteq V$ the set

$$\tilde{Z} = Z \cup \{ v \in V / (z, v) \in E \text{ for some } z \in Z \},$$

and for each $v \in V$ we denote by A_v the set $\tilde{v}$.

If $\mathcal{A}$ is the family of sets $(A_v \,/\, v \in V)$ we denote by $\mathcal{A}_X$ the subfamily $(A_v \,/\, v \in X)$, where $X \subseteq V$.

Throughout this chapter any digraph considered is finite.

5.1.1 THE LINKAGE LEMMA. Let $G = (V, E)$ be a digraph. If X, Y are subsets of V then X can be linked onto Y in G if and only if $V \setminus X$ is a transversal of the family $\mathcal{A}_{V \setminus Y}$.

PROOF. First suppose that X is linked onto Y in G by pairwise disjoint paths $(P_v \,/\, v \in X)$. Define a function $\alpha : V \setminus X \to V \setminus Y$ by

$$\alpha(u) = \begin{cases} v & \text{if } (v, u) \in P_x \text{ for some } x \in X, \\ u & \text{otherwise}. \end{cases}$$

Then α is well defined, since the paths $(P_v \,/\, v \in X)$ are pairwise disjoint, and is an injection. For each $u \in V \setminus X$ we see that $u \in A_{\alpha(u)}$ which belongs to $\mathcal{A}_{V \setminus Y}$. Since $\alpha(u_1) \neq \alpha(u_2)$ if $u_1 \neq u_2$, $V \setminus X$ is a transversal of $\mathcal{A}_{V \setminus Y}$.

Conversely let $V \setminus X$ be a transversal of $\mathcal{A}_{V \setminus Y}$. Then there is a bijection $\alpha : V \setminus X \to V \setminus Y$ such that $u \in A_{\alpha(u)}$ for all $u \in V \setminus X$. Consider any $v \in Y \setminus X$. We show that there is a path joining a point in $X \setminus Y$ to v. As $v \in Y \setminus X$, $v \in V \setminus X$ so that $v \in A_{\alpha(v)}$ and hence $(\alpha(v), v) \in E$. If $\alpha(v) \notin X$, then $\alpha(v) \in V \setminus X$ so that $\alpha(v) \in A_{\alpha(\alpha(v))}$ which implies $(\alpha^2(v), \alpha(v)) = (\alpha(\alpha(v)), \alpha(v)) \in E$. Thus either there exists k with $\alpha^k(v) \in X$ and $\alpha^r(v) \notin X$, where $r < k$ or we obtain an infinite sequence $\{\alpha^r(v)\}_{r=1}^{\infty}$. Now $\alpha^r(v) \in V \setminus Y$ for all r. Since G is finite we have $\alpha^r(v) = \alpha^s(v)$ for some $r < s$. Choose the minimal r with $\alpha^r(v) = r^s(v)$, $r < s$. Then $\alpha(\alpha^{r-1}(v)) = \alpha(\alpha^{s-1}(v))$, contradicting the minimality of r. Thus

$\alpha^k(v) \in X$ for some k and $\alpha^r(v) \notin X$ for all $r < k$. Thus we obtain a path $(\alpha^k(v), \alpha^{k-1}(v), \ldots, v)$ from $\alpha^k(v) \in X \setminus Y$ to v. Since α is injective, the paths $((\alpha^k(v), \ldots, v) \,/\, v \in Y \setminus X)$ are pairwise disjoint. We adjoin the trivial paths (v), for $v \in X \cap Y$ to the above paths to get a linking of X onto Y. //

5.1.2 THEOREM. Given a digraph $G = (V, E)$. Denote by $L(G, B)$ the collection of all subsets of V which can be *linked into a fixed subset B of V*. That is $X \in L(G, B)$ if and only if there exists $Y \subseteq B$ such that there is a linking of X onto Y. Then $L(G, B)$ is the collection of independent sets of a matroid on V. We call this a *Strict gammoid.*

We always denote a strict gammoid by $L(G, B)$ with G and B as above. Observe that $B \in L(G, B)$ and so $r(L(G, B)) = |B|$.

PROOF. By the Linkage Lemma, $X \in L(G, B)$ if and only if $V \setminus X$ is a transversal of the family $\mathcal{A}_{V \setminus B'}$, for some $B' \subseteq B$. Then

$$\begin{aligned} X \in L(G, B) &\iff V \setminus X \text{ is a transversal of the family } \mathcal{A}_{V \setminus B'}, \text{ for some } B' \subseteq B, \\ &\iff V \setminus X \text{ contains a transversal of } \mathcal{A}_{V \setminus B}, \\ &\iff V \setminus X \text{ is spanning in the transversal matroid } M[\mathcal{A}_{V \setminus B}]. \end{aligned}$$

Since the complement of a spanning set of a matroid is an independent set of its dual matroid, $L(G, B)$ is the set of independent sets of the dual of $M[\mathcal{A}_{V \setminus B}]$. //

In fact we have proved.

5.1.3 THEOREM. A matroid $M(S)$ is a strict gammoid if and only if $M^*(S)$ is transversal. //

5.1.4 EXAMPLE. Consider the following digraph $G = (V, E)$.

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$$

Then $L(G, \{6\})$ has as bases all singletons of V while $L(G, \{3, 6\})$ has as bases all sets $\{x, y\}$, where $1 \leq x \leq 3$ and $1 \leq y \leq 6$.

5.1.5 Given a digraph $G = (V, E)$ and $B \subseteq V$; by the *strict gammoid presentation* of $L(G, B)^*$ we mean the family $(\mathcal{A}_v \,/\, v \notin B)$ and write $L(G, B)^*$ for $M[\mathcal{A}_{V \setminus B}]$.

5.1.6 LEMMA. The strict gammoid presentation of any transversal matroid exists.

PROOF. Let $M[A_1, \ldots, A_r]$ be any transversal matroid of rank r on a set V. Choose a basis $V \setminus B = \{v_1, \ldots, v_r\}$ of $M(V)$, where $v_i \in A_i$, $1 \leq i \leq r$. Construct the digraph $G = (V, E)$ as follows:

$$(v_i, x) \in E \iff x \neq v_i,\ x \in A_i, \qquad 1 \leq i \leq r$$

Then it is clear that $L(G, B)^* = M[A_1, \ldots, A_r]$. //

Thus for any strict gammoid we can obtain a presentation of its dual as a transversal matroid and conversely.

5.1.7 A *gammoid* is any restriction (submatroid) of a strict gammoid.

5.1.8 LEMMA. Any transversal matroid is a gammoid.

PROOF. Let $M[A_1, \ldots, A_n]$ be any transversal matroid on S. Put $I = \{1, \ldots, n\}$. Construct the digraph $G = (V, E)$ as follows:

Let $V = S \cup I$.

For each $x \in S$ join x to $i \in I \iff x \in A_i$. Consider $L(G, I)$. We easily see that $M[A_1, \ldots, A_r]$ is the restriction of $L(G, I)$ to S. //

For convenience in notation the restriction of M(S) to any subset T of S is denoted by M(S) / T.

5.1.9 Lemma. (i) Any minor of a gammoid is a gammoid.

(ii) The dual of any gammoid is a gammoid.

PROOF. (i) It suffices to show that any restriction and any contraction of a gammoid is a gammoid. Let M(S) be a gammoid. Then there exists a digraph $G = (V, E)$ with $M(S) = L(G, B)/S$ for some subset B of V and some subset S of V. Thus for any $T \subseteq S$ we have

$$M(S) / T = (L(G, B)/ S) / T = L(G, B) / T$$

and so $M(S) / T$ is a gammoid.

To show that a contraction $M(S) . T$ is a gammoid we use the fact that for any $M(S)$ and $A \subseteq B \subseteq S$ we have

$$(M(S) / B). A = (M(S).(S \setminus (B \setminus A))) / A.$$

Then $M(S). T = (N(S') / S). T$, where N is a strict gammoid on some $S' \supseteq S$. By the above, $M(S). T = (N(S'). T')/ T$, where $T' = S' \setminus (S \setminus T)$. Now $(N(S').T') = (N(S').T')^{**} = (N^*(S')/ T')^*$ and since $N(S')$ is a strict gammoid, $N^*(S')$ is transversal and hence $N^*(S')/T'$. Therefore $N(S').T'$ is a strict gammoid and so its restriction, $M(S).T$, is a gammoid.

The following theorem which we state without proof is due to Ingleton and Piff [73].

5.1.10 THEOREM. (i) Every matroid of rank 1 or 2 is a strict gammoid.

(ii) Every gammoid of rank 3 is a strict gammoid.

(iii) Every matroid of rank $n - 1$ or $n - 2$ on a set of n elements is transversal.

(iv) Every gammoid of rank $n - 3$ on a set of n elements is transversal.

5.2 BASE ORDERABLE MATROIDS

5.2.1 A matroid $M(S)$ is *base orderable* if for any two bases B_1, B_2 of $M(S)$ there exists a bijection $\theta : B_1 \to B_2$ such that for each $x \in B_1$,

$(B_1 \setminus x) \cup \Theta(x)$ and $(B_2 \setminus \Theta(x)) \cup x$ are bases of $M(S)$.

The function Θ is an *exchange ordering* for B_1, B_2.

5.2.2 EXAMPLE. $M(\mathcal{B}_n)$ is base orderable.

PROOF. Let B_1, B_2 be distinct bases of $M(\mathcal{B}_n)$. We can assume that $B_1 \cap B_2 = \phi$. Suppose $B_1 = \{a, b\}$ and $B_2 = \{c, d\}$. If there exists a pair of elements one from B_1 and one from B_2 such that this pair is F_i, $\{a, c\} = F_i$ say. Then $\{a, d\} \varepsilon \mathcal{B}_n$ and $\{b, c\} \varepsilon \mathcal{B}_n$. Hence $\Theta : B_1 \rightarrow B_2$ defined by $\Theta(a) = c$, $\Theta(b) = d$ is an exchange ordering for B_1, B_2. In the other case any injection from B_1 onto B_2 is an exchange ordering for B_1, B_2. //

5.2.3 LEMMA. Not every matroid is base orderable.

PROOF. We show $M(\mathcal{S}_7)$ is not. Let $B_1' = \{x_1, x_4, x_6\}$ and $B_2' = \{x_2, x_5, x_6\}$ be two triples in $\mathcal{S}_7$. Then there exists x_3 such that $B_3' = \{x_2, x_3, x_4\}$ is the triple containing x_2, x_4. Put $B_1 = \{x_1, x_2, x_3\}$, $B_2 = \{x_4, x_5, x_6\}$. Then B_1 and B_2 are bases of $M(\mathcal{S}_7)$. Since $\mathcal{S}_7$ contains 7 triples and every element in S_7 is contained in exactly 3 triples of $\mathcal{S}_7$, the only triples that are not subsets of $B_1 \cup B_2$ are the three triples containing 7. Hence $B_1 \cup B_2$ contains another triple different from B_1', B_2' and B_3'. We claim that the triple B containing x_1, x_3 is a subset of $B_1 \cup B_2$. Suppose not, then $B = \{x_1, x_3, x_7\}$. Thus the triple B' containing x_1, x_2 does not contain x_7 so that it is a subset of $B_1 \cup B_2$. Therefore $B' = \{x_1, x_2, x_3\}$ or $\{x_1, x_2, x_4\}$ or $\{x_1, x_2, x_5\}$ or $\{x_1, x_2, x_6\}$ which is impossible. Thus $B \subseteq B_1 \cup B_2$ and B must be equal to $\{x_1, x_3, x_5\}$.

If $M(\mathcal{S}_7)$ is base orderable, then there exists a **bijection** $f : B_1 \to B_2$ such that $(B_1 \setminus x) \cup f(x)$ and $(B_2 \setminus f(x)) \cup x$ are bases of $M(\mathcal{S}_7)$, $\forall x \in B_1$. Now $f(x_1) \neq x_5$ and $f(x_2) \neq x_4$ (otherwise $(B_2 \setminus f(x_1)) \cup x_1 = \{x_1, x_4, x_6\} \in \mathcal{S}_7$ and $(B_2 \setminus f(x_2)) \cup x_2 = \{x_2, x_5, x_6\} \in \mathcal{S}_7$). Thus $f(x_1) = x_4$ or x_6 and $f(x_2) = x_5$ or x_6 and so we have all three possible **bijections** f_1, f_2, f_3 from B_1 onto B_2 defined as follows:

$$f_1(x_1) = x_6 \quad , \quad f_1(x_2) = x_5 \quad , \quad f_1(x_3) = x_4$$

$$f_2(x_1) = x_4 \quad , \quad f_2(x_2) = x_6 \quad , \quad f_2(x_3) = x_5$$

$$f_3(x_1) = x_4 \quad , \quad f_3(x_2) = x_5 \quad , \quad f_3(x_3) = x_6$$

Then we have $(B_1 \setminus x_2) \cup f_1(x_2) = \{x_1, x_3, x_5\} \in \mathcal{S}_7$

or $(B_1 \setminus x_1) \cup f_2(x_1) = \{x_2, x_3, x_4\} \in \mathcal{S}_7$

or $(B_1 \setminus x_1) \cup f_3(x_1) = \{x_2, x_3, x_4\} \in \mathcal{S}_7$

which is not so.

Therefore $M(\mathcal{S}_7)$ is not base orderable. //

In fact if $n \equiv 1$ or $3 \pmod 6$ and $n \notin N_0$ a non - base orderable matroid $M(\mathcal{S}_n)$ exists since $\mathcal{S}_n$ contains $\mathcal{S}_7$ and we have

5.2.4 LEMMA. Any restriction of a base orderable matroid is base orderable.

PROOF. Let $M(S)/T$ be any restriction of a base orderable matroid $M(S)$ on a subset T. If B_1, B_2 are bases of $M(S)|T$, then there exist bases A_1, A_2 of $M(S)$ with $B_1 \subseteq A_1$, $B_2 \subseteq A_2$. Let θ be an exchange ordering for A_1, A_2. Suppose that there exists $x \in B_1$ with $\theta(x) \in A_2 \setminus B_2$. Then $(A_2 \setminus \theta(x)) \cup x$ is a basis of $M(S)$ so that $x \notin B_2$. Thus $B_2 \cup x$ is independent in $M(S)/T$. A contradiction. Hence $\theta(x) \in B_2$, $\forall x \in B_1$ and so $\theta|B_1$ is an exchange ordering for B_1, B_2. //

5.2.5 LEMMA. The dual of any base orderable matroid is base orderable.

PROOF. Let $M(S)$ be any base orderable matroid. Let B_1, B_2 be bases of $M^*(S)$. Thus $S \setminus B_1$ and $S \setminus B_2$ are bases of $M(S)$ so that there exists an exchange ordering θ for $S \setminus B_1$, $S \setminus B_2$. If there exists $x \in B_2 \setminus B_1$ with $\theta(x) \notin B_1$, then $\theta(x) \in S \setminus B_1$ so that $|(S \setminus B_1 \setminus x) \cup \theta(x)| = |S \setminus B_1| - 1$. But $((S \setminus B_1) \setminus x)) \cup \theta(x)$ is a basis of $M(S)$ and so $|((S \setminus B_1) \setminus x) \cup \theta(x)| = |S \setminus B_1|$. A contradiction. Hence $\forall x \in B_2 \setminus B_1$, $\theta(x) \in B_1 \setminus B_2$. That is $B_2 \setminus B_1 = \theta^{-1}(B_1 \setminus B_2)$.

Define $\psi : B_1 \to B_2$ by

$$\psi(x) = \begin{cases} x & \text{if } x \in B_1 \cap B_2, \\ \theta^{-1}(x) & \text{if } x \in B_1 \setminus B_2 . \end{cases}$$

Then for any $x \in B_1 \setminus B_2$ we have $(B_1 \setminus x) \cup \psi(x) = (B_1 \setminus x) \cup \theta^{-1}(x)$. Now $\theta^{-1}(x) \in B_2 \setminus B_1 \Rightarrow \exists\, y \in B_1 \setminus B_2$ with $\theta(y) = x$ and so $((S \setminus B_1) \setminus y) \cup \theta(y)$ is a basis of $M(S)$. That is $((S \setminus B_1) \setminus \theta^{-1}(x)) \cup x$ is a basis of $M(S)$ so that $(B_1 \setminus x) \cup \theta^{-1}(x) = (B_1 \setminus x) \cup \psi(x)$ is a basis of $M^*(S)$.

Similarly we can show that $(B_2 \setminus \psi(x) \cup x$ is a basis of $M(S)^*$

and the result is proved. //

As a consequence of two above lemmas we obtain

5.2.6 LEMMA. Any minor of a base orderable matroid is base orderable.

6 PREGEOMETRY PRODUCTS WITH APPLICATIONS

6.1 FIRST PRODUCT

Given a matroid $M(S_1)$ and a pregeometry $G(S_2)$. For any basis B of $M(S_1)$ we consider the collection $\mathcal{D}_B$, the collection of sets of the form

$$D = (\bigcup_{e \in B} e \times B_e) \cup (\bigcup_{e \in S_1 \setminus B} e \times (B_e \setminus f)),$$

where for each $e \in S_1$, some basis B_e of $G(S_2)$ is selected and further for each $e \in S_1 \setminus B$ some element $f \in B_e$ is selected.

We vary the construction of Lim [77] (see section 2) to obtain a pregeometry from a given matroid $M(S_1)$ and a pregeometry $G(S_2)$ by proving

6.1.1 THEOREM. $\mathcal{D}_B$ is the collection of bases of a pregeometry $G_B(S_1 \times S_2)$ defined on $S_1 \times S_2$.

PROOF. We see from the definition that $\mathcal{D}_B$ is a nonempty collection of finite subsets of $S_1 \times S_2$, each of the same size. We show that $\mathcal{D}_B$ satisfies the basis axiom (B). Let $D, D' \in \mathcal{D}_B$. Then

$$D = (\bigcup_{e \in B} e \times B_e) \cup (\bigcup_{e \in S_1 \setminus B} e \times (B_e \setminus f)),$$

$$D' = (\bigcup_{e \in B} e \times B'_e) \cup (\bigcup_{e \in S_1 \setminus B} e \times (B'_e \setminus f')).$$

Consider any particular $(e, x) \in D \setminus D'$. We show that there exists $(e', x') \in D' \setminus D$ such that $(D \setminus (e, x)) \cup (e', x') \in \mathcal{D}_B$. There are two possibilities ; (i) $e \in B$, (ii) $e \notin B$.

(i) Suppose $e \in B$. Because B_e and B'_e are bases of $G(S_2)$ and $x \notin B'_e$, from the basis axiom (B), there exists $g \in B'_e \setminus B_e$ such that $(B_e \setminus x) \cup g$ is a basis of $G(S_2)$. Then by changing D only in selecting $(B_e \setminus x) \cup g$ in place of the original B_e we have another member $(D \setminus e \times B_e) \cup (e \times ((B_e \setminus x) \cup g))$ of $\mathcal{D}_B$ which differs from D only in that (e, x) is replaced by (e, g) and $(e, g) \in e \times (B'_e \setminus B_e)$ is in $D' \setminus D$ as required.

(ii) Lastly suppose $e \notin B$. Thus $(e, x) \in e \times (B_e \setminus f)$ and so $x \neq f$. Now $(B_e \setminus f) \setminus x$ and $B'_e \setminus f'$ are both independent in $G(S_2)$ and of size $r(S_2)-2$ and $r(S_2)-1$ respectively, and $x \notin (B'_e \setminus f')$. Hence there exists $x \neq g \in B'_e \setminus f'$ such that $((B_e \setminus f) \setminus x) \cup g$ is independent in $G(S_2)$. Then by changing D only in selecting $((B_e \setminus f) \setminus x) \cup g$ in place of the original $(B_e \setminus f)$ corresponding to e we have another member

$(D \setminus (e \times (B_e \setminus f))) \cup (e \times (((B_e \setminus f) \setminus x) \cup g))$ of $\mathcal{D}_B$ which differs from D only in that (e, x) is replaced by (e, g) and
$(e, g) \in e \times ((B'_e \setminus f') \setminus (B_e \setminus f \setminus x))$ is in $D' \setminus D$ as required. //

We noted in the proof that ranks r of $G_B(S_1 \times S_2)$ is given by

$$r = r(S_1)\, r(S_2) + (|S_1| - r(S_1))(r(S_2) - 1),$$

the common size of each D .

6.1.2 LEMMA . The circuits of $G_B(S_1 \times S_2)$ are exactly the sets of the following forms

(i) $e \times C_2$, where $e \in B$ and C_2 is a circuit of $G(S_2)$,

(ii) $e \times C_2$, where $e \notin B$ and C_2 is a circuit of $G(S_2)$ of rank strictly less than $r(S_2)$.

PROOF. We see from the definition of $\mathcal{D}_B$ that any subset of $S_1 \times S_2$ of the form (i) or (ii) is a circuit of $G_B(S_1 \times S_2)$. Let C be a circuit of $G_B(S_1 \times S_2)$. We show that C has the form (i) or (ii)

Suppose $C = \bigcup_{i=1}^{m} (e_i \times G_i)$, where all $G_i \neq \emptyset$, $m \geq 2$, $e_i = e_j \Leftrightarrow i = j$. Then since all $e_i \times G_i$ is independent in $G(S_1 \times S_2)$, all G_i are independent in $G(S_2)$. But if all G_i are not bases of $G(S_2)$, it implies that C is contained in a basis of $G_B(S_1 \times S_2)$ which is not so. Thus there exists G_i which is a basis of $G(S_2)$ and so $C = \{e_i \,/\, G_i$ is a basis of $G(S_2)\} \neq \emptyset$. If all e_i in C_1 belong to B, then C in contained in a basis of $G_B(S_1 \times S_2)$. Thus there exists $e_i \in C_1$, e_1 say, with $e_1 \notin B$. Put $x = (e_2, c)$ for some $c \in G_2$. Now the dependent set $e_1 \times G_1$ is contained in $C \setminus x$. This is a contradiction.

Thus $C = e \times C_2$, where $C_2 \subsetneq S_2$. We consider two possibilities : (i) $e \in B$, (ii) $e \notin B$.

(i) Suppose $e \in B$. Then C_2 is dependent in $G(S_2)$ and C_2 must be a circuit of $G(S_2)$ (otherwise C contains a proper dependent subset). Thus C has the form (i).

(ii) Lastly suppose $e \notin B$. Also C_2 is a circuit of $G(S_2)$. For any $x \in C_2$, $e \times (C_2 \setminus x)$ is independent in $G_B(S_1 \times S_2)$ so that there exists a basis B_x of $G(S_2)$ with $C_2 \setminus x \subsetneq B_x \setminus f$. Thus C_2 has rank strictly less than $r(S_2)$ as required. //

6.2 SECOND PRODUCT

Given a matroid $M(S_1)$ and a pregeometry $G(S_2)$ we define

$\mathcal{D} = \bigcup_B \mathcal{D}_B$, for all bases B of $M(S_1)$.

6.2.1 THEOREM. $\mathcal{D}$ is the collection of bases of a pregeometry, $G(S_1 \times S_2)$, defined on $S_1 \times S_2$.

PROOF. We see from the definition of $\mathcal{D}$ that $\mathcal{D}$ is a nonempty collection of finite subsets of $S_1 \times S_2$ of the same size. We show that $\mathcal{D}$ satisfies the basis axiom (B). Let $D, D' \in \mathcal{D}$. Then there exist bases B, B' of $M(S_1)$ such that

$$D = \Big(\bigcup_{e \in B} e \times B_e\Big) \cup \Big(\bigcup_{e \in S_1 \setminus B} e \times (B_e \setminus f)\Big),$$

$$D' = \Big(\bigcup_{e' \in B'} e' \times B'_{e'}\Big) \cup \Big(\bigcup_{e' \in S_1 \setminus B'} e' \times (B'_{e'} \setminus f')\Big).$$

Consider any particular $(e, x) \in D \setminus D'$. We show that there exists $(e', x') \in D' \setminus D$ such that $(D \setminus (e,x)) \cup (e', x') \in \mathcal{D}$. There are four possibilities : (i) $e \in B \cap B'$, (ii) $e \in (S_1 \setminus B) \cap (S_1 \setminus B')$, (iii) $e \in (S_1 \setminus B) \cap B'$, (iv) $e \in B \cap (S_1 \setminus B')$.

(i) Suppose $e \in B \cap B'$. Since B_e, B'_e are bases of $G(S_2)$ and $x \in B_e \setminus B'_e$, there exists $g \in B'_e \setminus B_e$ so that $(B_e \setminus x) \cup g$ is a basis of $G(S_2)$. Then by changing D only in selecting $(B_e \setminus x) \cup g$ in place of the original B_e we have another member $(D \setminus e \times B_e) \cup (e \times ((B_e \setminus x) \cup g))$ of $\mathcal{D}$ which differs from D only in that (e, x) is replaced by (e, g) and $(e, g) \in (B'_e \setminus B_e)$ is in $D' \setminus D$ as required.

(ii) Now suppose $e \in (S_1 \setminus B) \cap (S_1 \setminus B')$. Thus $(e, x) \in e \times (B_e \setminus f)$ and so $x \neq f$. Now $(B_e \setminus f) \setminus x$ and $(B'_e \setminus f')$ are both independent in $G(S_2)$ of size $r(S_2) - 2$ and $r(S_2) - 1$ respectively and $x \notin (B'_e \setminus f')$. Hence there exists $x \neq g \in ((B'_e \setminus f') \setminus ((B_e \setminus f) \setminus x))$ so that $((B_e \setminus f) \setminus x) \cup g$ is independent in $G(S_2)$ of size $r(S_2) - 1$. Then by changing D only in selecting $((B_e \setminus f) \setminus x) \cup g$ in place of the original $(B_e \setminus f)$ corresponding to e we have another member $(D \setminus (e \times (B_e \setminus f))) \cup (e \times (((B_e \setminus f) \setminus x) \cup g))$ of $\mathcal{D}$ which differs from D only in that (e, x) is replaced by (e, g), and $(e, g) \in e \times ((B'_e \setminus f') \setminus (B_e \setminus f \setminus x))$ is in $D' \setminus D$ as required.

(iii) Now suppose $e \in (S_1 \setminus B) \cap B'$.

Since B'_e , $B_e \setminus f$ are both independent in $G(S_2)$ of size $r(\mathcal{G}(S_2))$ and $r(\mathcal{G}(S_2)) - 1$ respectively, there exists $g \in (B'_e \setminus (B_e \setminus f)$ so that $(B_e \setminus f) \cup g$ is a basis of $G(S_2)$. But $x \notin B'_e$, so $x \neq g$ and since $x \in (B_e \setminus f)$, $((B_e \setminus f) \setminus x) \cup g$ is independent in $G(S)$ of size $r(\mathcal{G}(S_2)) - 1$. Then by changing D only in selecting $((B_e \setminus f) \setminus x) \cup g$ in place of the original $(B_e \setminus f)$ corresponding to e we have another member $(D \setminus (e \times (B_e \setminus f))) \cup (e \times (((B_e \setminus f) \setminus x) \cup g))$ of $\mathcal{A}$ which differs from D only in that (e, x) is replaced by (e,g) and $(e, g) \in e \times (B'_e \setminus (B_e \setminus f))$ is in $D' \setminus D$ as required.

(iv) Lastly suppose $e \in B \cap (S_1 \setminus B')$ Then $e \in B \setminus B'$ and hence there exists $e' \in B' \setminus B$ so that $(B \setminus e) \cup e'$ is a basis of $\mathcal{M}(S_1)$. Now $B_{e'} \setminus f$ and $B'_{e'}$ are independent in $G(S_2)$ of size $r(\mathcal{G}(S_2)) - 1$ and $r(\mathcal{G}(S_2))$ respectively and hence there exists $g \in (B'_{e'} \setminus (B_{e'} \setminus f)$ so that $(B_{e'} \setminus f) \cup g$ is a basis of $G(S_2)$. Then by changing D only in selecting the basis $(B \setminus e) \cup e'$ in place of the basis B, and selecting $(B_{e'} \setminus f) \cup g$ corresponding to e', and selecting $B_e \setminus x$ corresponding to e we have another member $((D \setminus (e \times B_e) \setminus (e' \times (B_{e'} \setminus f))) \cup (e' \times ((B_{e'} \setminus f) \cup g))$ $\cup (e \times (B_e \setminus x))$ of $\mathcal{A}$ which differs from D only in that (e, x) is replaced by (e', g) and $(e', g) \in e' \times (B'_{e'} \setminus (B_{e'} \setminus f)$ is in $D' \setminus D$ as required. //

Theorem 6.2.1 was obtained for matroids by Lim [77] The proof makes no use of the fact that S_2 is infinite. We have based the contruction in section 1 on it. We note that

6.2.2 LEMMA. $\mathcal{D}$ is the disjoint union of the $\mathcal{D}_B$, for all bases B of $M(S_1)$.

PROOF. We are left to show that $\mathcal{D}_{B_1} \cap \mathcal{D}_{B_2} = \emptyset$ if B_1 and B_2 are distinct bases of $M(S_1)$. Suppose that there exists $D \in \mathcal{D}_{B_1} \cap \mathcal{D}_{B_2}$ with $B_1 \neq B_2$. Assume $r((S_1)) = r$.

Let $B_1 = \{e_1, \ldots, e_r\}$, $B_2 = \{e'_1, \ldots, e'_r\}$. Then D can be written in the forms

$$D = \left(\bigcup_{i=1}^{r} e_i \times B_{e_i}\right) \cup \left(\bigcup_{e \in S_1 \setminus B_1} e \times (B_e \setminus f)\right)$$

$$D = \left(\bigcup_{i=1}^{r} e'_i \times B_{e'_i}\right) \cup \left(\bigcup_{e' \in S_1 \setminus B_2} e' \times (B'_{e'} \setminus f')\right)$$

For each i, $1 \leq i \leq r$, choose $x_i \in e_i \times B_{e_i}$. If

$\{x_1, \ldots, x_r\} \subseteq \bigcup_{i=1}^{r} (e'_j \times B'_{e'_j})$, it follows that $B_1 = B_2$.

Hence there exists x_i and $e' \in S_1 \setminus B_2$ with $x_i \in e' \times (B'_{e'} \setminus f')$ and so $e_i = e'$. Pick an element $y \in ((e_i \times B_{e_i}) \setminus (e' \times (B'_{e'} \setminus f')))$. Now either $\exists$ j with $y \in e'_j \times B'_{e'_j}$ or $\exists$ $e'' \neq e'$ and $e' \in S_1 \setminus B_2$ with $y \in (e'' \times (B'_{e''} \setminus f'))$. Thus we have $e'_j = e'$ or $e'' = e'$ which is a contradiction in either case.

Hence $\mathcal{D}_{B_1} \cap \mathcal{D}_{B_2} = \emptyset$ if B_1 and B_2 are distinct bases of $M(S_1)$. //

We note from the definition of $\mathcal{D}$ that the rank r of $G(S_1 \times S_2)$ is given by

$$r = r((S_1)) \, r((S_2)) + (|S_1| - r((S_1)))(r((S_2)) - 1),$$

as for $G_B(S_1 \times S_2)$.

6.2.3 LEMMA. The circuits of $G(S_1 \times S_2)$ are of the following forms

(i) $e \times C_2$, where $e \in S_1$ and C_2 is a circuit of $G(S_2)$.

(ii) $\bigcup_{e \in C_1} (e \times B_e)$, where C_1 is a circuit of $M(S_1)$ and each B_e is a basis of $G(S_2)$.

PROOF. We see from the definition of $\mathcal{D}$ that any subset of $S_1 \times S_2$ of the form (i) or (ii) is a circuit of $G(S_1 \times S_2)$. Let C be any circuit of $G(S_1 \times S_2)$. We show that C has the form (i) or (ii).

case 1. $C = e \times J$ for some $e \in S_1$ and $J \subsetneq S_2$. If e is not a loop of $M(S_1)$, then J is dependent in $G(S_2)$ (otherwise C is independent in $G(S_1 \times S_2)$). For any $x \in J$, if $J \setminus x$ is dependent, then $e \times (J \setminus x)$ is a proper dependent subset of C which is not so. Thus $J \setminus x$ is independent and so J is a circuit of $G(S_2)$. Hence C has the form (i).

If e is a loop of $M(S_1)$. We show that J is either a circuit or a basis of $G(S_2)$. For any $x \in J$, $e \times (J \setminus x)$ is independent in $G(S_1 \times S_2)$. But e is not contained in any basis of $M(S_1)$, thus

$e \times (J \setminus x) \subsetneq e \times (B_e \setminus f)$ for some basis B_e of $G(S_2)$ and so $J \setminus x$ is independent in $G(S_2)$. Then either J is dependent or independent. Suppose that J is dependent in $G(S_2)$ and so C has the form (i). If J is independent we see that J must be a basis of $G(S_2)$ (otherwise C is independent in $G(S_1 \times S_2)$) and so C has the form (ii).

case 2. $C = \bigcup_{i=1}^{m} (e_i \times G_i)$, where $G_i \neq \emptyset$ for all i, $m \geq 2$ and $e_i = e_j \iff i = j$. Then since all $e_i \times G_i$ are independent in $G(S_1 \times S_2)$, all G_i are independent in $G(S_2)$. By the same argument as in the proof of Lemma 6.1.2 the set $C_1 = \{e_i \,/\, G_i$ is a basis of $G(S_2)\} \neq \emptyset$ and C_1 is dependent in $M(S_1)$ (otherwise C is independent in $G(S_1 \times S_2)$). Suppose that C_1 properly contains a dependent subset C_2. Then $\bigcup_{e_i \in C_2} (e_i \times G_i)$ is a proper dependent subset of C which is a contradiction. Thus C_1 is a circuit of $M(S_1)$ and so $\bigcup_{e_i \in C_1} (e_i \times G_i)$ is circuit of $G(S_1 \times S_2)$. But $\bigcup_{e_i \in C_1} (e_i \times G_i) \subsetneq C$, hence $\bigcup_{e_i \in C_1} (e_i \times G_i) = C$. //

Indeed Lim [77] proved the following three hereditary properties of $M(S_1 \times S_2)$ with bases $\mathcal{B}$ – writing $M(S_1 \times S_2)$(when S_2 is finite) for $G(S_1 \times S_2)$.

6.2.4 THEOREM. $M(S_1 \times S_2)$ with $|S_1| \geq 2$ is connected if and only if $M(S_1)$ is connected.

6.2.5 THEOREM. $M(S_1 \times S_2)$ is base orderable if and only if $M(S_1)$ and $M(S_2)$ are each base orderable.

6.2.6 THEOREM. $M(S_1 \times S_2)$ is binary if and only if the following are satisfied.

(i) $M(S_1)$ and $M(S_2)$ are binary.

(ii) If $M(S_1)$ and $M(S_2)$ both have a circuit, then every circuit of $M(S_2)$ is of cardinality two.

6.3 APPLICATIONS TO GROUPS

We now apply this last construction to matroids $M(S_1)$ and $M(S_2)$ defined on subgroups S_1 and S_2 which are direct summands of the group $S = S_1S_2$ (Although we write the group operations we consider additively for convenience). Thus we obtain a matroid $M(S_1S_2) = M(S_1 \times S_2)$. We show that this example posseses some of the hereditary properties discussed in the previous section. We also obtain the size of the the group of its geometric automorphisms.

For any positive integer $m > 1$, denote by Z_m the cyclic group of integers 0, 1, ..., m-1 with respect to addition modulo m.

Let $\mathcal{B} < m >$ be the collection of 2 - subsets of Z_m of the form $\{ r_1, r_2 \}$, where r_1 is odd and r_2 is even.

6.3.1 LEMMA. $\mathcal{B} < m >$ is the collection of bases of a loopless matroid $M(Z_m)$ on Z_m which is binary and base orderable but not connected.

PROOF. That $\mathcal{B}\langle m \rangle$ is the collection of bases of a loopless matroid on Z_m is clear from its definition.

The circuits of $M(Z_m)$ are the collection of sets of two odd integers or two even integers. Thus a set of an odd integer and an even integer is not contained in a circuit so that $M(Z_m)$ is not connected. We easily see that the symmetric difference of distinct circuits contains a circuit. Hence $M(Z_m)$ is binary.

For any two bases B_1, B_2 of $M(Z_m)$ the bijection $\theta : B_1 \to B_2$ sending the odd integer in B_1 to the odd integer in B_2 is an exchange ordering for B_1, B_2. Thus $M(Z_m)$ is base orderable. //

Given $m \geq 2$, $n \geq 2$ and $(m, n) = 1$. Consider the subgroups $\langle m \rangle$ and $\langle n \rangle$ of Z_{mn} generated by m and n respectively. By Moore [67, p 118] Z_{mn} is the internal direct product of $\langle m \rangle$ and $\langle n \rangle$. Moreover $Z_m \cong \langle n \rangle$ by an isomorphism $r \to nr$. Also $Z_n \cong \langle m \rangle$. Then we obtain.

6.3.2 LEMMA. Let $B = \{ mr_1, mr_2\} \in \mathcal{B}\langle m \rangle$ and $mk \in \langle m \rangle$, where $k < n$. Then

(i) $mkB \in \mathcal{B}\langle m \rangle$ if n is even.

(ii) $mkB \in \mathcal{B}\langle m \rangle \iff (k + r_1 \leq n, k + r_2 \leq n)$ or $(k + r_1 > n, k + r_2 > n)$ if n is odd.

PROOF. (i) We may assume that r_1 is odd and r_2 is even. By the Euclidean algorithm $\exists\, r'_1, r'_2, 0 \leq r'_1, r'_2 < n$ with $k + r_1 = n + r'_1$, $k + r_2 = n + r'_2$. If k is odd, then r'_1 is even and r'_2 is odd. Now $m(k + r_1) = m(n + r'_1) = m\, r'_1$ and $m(k + r_2) = m\, r'_2$ so that $mkB = \{ mr'_1, m\, r'_2\} \in \mathcal{B}\langle m \rangle$. Similarly if k is even we can show that $mkB \in \mathcal{B}\langle m \rangle$.

We first show that if either $(k + r_1 \leq n, k + r_2 > n)$ or $(k + r_1 > n_1, k + r_2 \leq n)$, then $m\, k\, B \notin \mathcal{B}\langle m \rangle$. Assume that $k + r_1 \leq n$ and $k + r_2 > n$. We can assume that r_1 is odd and r_2 is even. If k is odd, then $k + r_1$ is even. There exists r'_2, $0 \leq r'_2 < n$ with $k + r_2 = n + r'_2$. Now $k + r_2$ is odd so that r'_2 is even (as n is odd) and hence $m\, k\, B = \{ m\,(k + r_1), m\, r'_2\} \notin \mathcal{B}\langle m \rangle$. Similarly if k is even we can show that $m\, k\, B \in \mathcal{B}\langle m \rangle$.

If $k + r_1 > n$ and $k + r_2 \leq n$ we show by the same argument as above that $m\, k\, B \notin \mathcal{B}\langle m \rangle$. Thus $m\, k\, B \in \mathcal{B}\langle m \rangle$ implies that either $(k + r_1 \leq n, k + r_2 \leq n)$ or $(k + r_1 > n, k + r_2 > n)$.

We next show that either $(k + r_1 \leq n, k + r_2 \leq n)$ or $(k + r_1 > n, k + r_2 > n)$ implies $m\, k\, B \in \mathcal{B}\langle m \rangle$. Assume $k + r_1 \leq n$, $k + r_2 \leq n$. Then it is obvious that only one of $k + r_1$ and $k + r_2$ is odd is the case so that $m\, k\, B = \{ m\,(k + r_1), m(k + r_2)\} \in \mathcal{B}\langle m \rangle$.

If $k + r_1 > n$ and $k + r_2 > n$. Then $k + r_1 = n + r'_1$, $k + r_2 = n + r'_2$, for some $r'_1, r'_2, 0 \leq r'_1, r'_2 < n$. We may assume that r_1 is odd and r_2 is even. If k is odd, then r'_1 is odd and r'_2 is even.

If k is even, then r'_1 is even and r'_2 is odd. In either case we have $m\,k\,B = \{m\,r'_1, m\,r'_2\} \in \mathcal{B} < m >$. //

By Herstein [75], a group S which is an internal direct product of subgroups S_1 and S_2 is isomorphic to the external direct product $S_1 \times S_2$ by an isomorphism $ab \to (a,b)$, $\forall a \in S_1$, $\forall b \in S_2$. Thus for given $M(S_1)$ and $G(S_2)$ if we replace the cartesian product $e \times B_e$ in D by eB_e, then the collection $\mathcal{D}$ is the collection of bases of a pregeometry on $S = S_1S_2$ which is isomorphic to $G(S_1 \times S_2)$ in the obvious natural way, and we do not distinguish between them.

We are now ready for the example.

6.3.3 LEMMA. Given $m \geq 2$, $n \geq 2$ and $(m,n) = 1$. Let $M(S_1)$ be the matroid on $< m >$ with bases $\mathcal{B} < m >$ and let $M(S_2)$ be the matroid on $< n >$ with bases $\mathcal{B} < n >$. Then $M(S_1 \times S_2)$ is binary and base orderable but not connected. Moreover for any $A = m\,s_1\{n\,r_1, n\,r_2\} \cup m\,s_2\{n\,r'_1, n\,r'_2\} \cup m\,s_3\,\{n\,r_3\} \in \mathcal{D}$ and for any $m\,e \in < m >$, $n\,k \in < n >$ we have

(i) $n\,k\,A \in \mathcal{D}$ if m is even and $m\,e\,A \in \mathcal{D}$ if n is even

(ii) $n\,k\,A \in \mathcal{D} \iff ((k + r_1 \leq m, k + r_2 \leq m)$ or $(k + r_1 > m, k + r_2 > m))$ and $((k + r'_1 \leq m, k + r'_2 \leq m)$ or $(k + r'_1 > m, k + r'_2 > m))$ if m is odd.

(iii) $m\,e\,A \in \mathcal{D} \iff (e + S_1 \leq n, e + s_2 \leq n)$ or $(e + s_1 > n, e + s_2 > n)$ if n is odd.

(iv) $(m\ e\ n\ k)\ A \in \mathcal{D} \iff (e + s_1 \leq n,\ e + s_2 \leq n)$ or $(e + s_1 > n,\ e + s_2 > n)$ if m is even and n is odd.

(v) $(m\ e\ n\ k)\ A \in \mathcal{D} \iff ((k + r_1 \leq m,\ k + r_2 \leq m)$ or $(k + r_1 > m,\ k + r_2 > m))$ and $((k + r'_1 \leq m,\ k + r'_2 \leq m)$ or $(k + r'_1 > m,\ k + r'_2 > m))$ if m is odd and n is even.

(vi) $(m\ e\ n\ k)\ A \in \mathcal{D} \iff$ R.H.S(iv) and R.H.S(v) if m and n are odd.

(vii) $(m\ e\ n\ k)\ A \in \mathcal{D}$ if m and n are even.

PROOF. That $M(S_1 \times S_2)$ is binary and base orderable but not connected follows from Lemma 6.3.1. That $M(S_1 \times S_2)$ satisfies (i) - (vii) follows from Lemma 6.3.2. //

6.3.4 An *automorphism* σ of a pregeometry G(S) is a permutation on S such that B is a basis if and only if σ (B) is a basis.

We note that the set of all automorphisms of G(S) is a group under composition.

6.3.5 LEMMA. The automorphism group A(M) of $M(Z_n)$ has size given by

$$A(M) = \begin{cases} 2(\frac{n}{2}!)\ (\frac{n}{2}!) & \text{if n is even,} \\ (\frac{n+1}{2}!)\ (\frac{n-1}{2}!) & \text{if n is odd.} \end{cases}$$

PROOF. First assume that n is even. Then the number of even integers in Z_n and the number of odd integers in Z_n are equal and is equal to $\frac{n}{2}$. Put I = set of all even integers in Z_n. Thus a permutation α on I and a permutation β on $Z_n \setminus I$ define an automorphism Θ of $M(Z_n)$ by $\Theta/_I = \alpha$ and $\Theta/_{Z_n \setminus I} = \beta$. Also a bijection $\beta : I \to Z_n \setminus I$ and a bijection $\gamma : Z_n \setminus I \to I$ define an automorphism σ of $M(Z_n)$ by $\sigma/_I = \beta$ and $\sigma/_{Z_n \setminus I} = \gamma$. Thus we have $2(\frac{n}{2}!)(\frac{n}{2}!)$ different automorphisms defined this way.

Suppose that θ is an automorphism of $M(Z_n)$. We show that either $\theta(I) = I$ or $\theta(I) = Z_n \setminus I$. If $\theta(I) \neq I$, then there exists $x \in I$ such that $\theta(x) = y \in Z_n \setminus I$. For each $a \in Z_n \setminus I$ we have $\{\theta(a), y\} \in \mathcal{B}\langle n \rangle$ so that $\theta(a) \in I$. By the same argument we can show that for each $b \in I$, $\theta(b) \in Z_n \setminus I$. Thus $\theta(I) = Z_n \setminus I$ and $\theta(Z_n \setminus I) = I$. Therefore either $\theta(I) = I$ or $\theta(I) = Z_n \setminus I$. In either case we see that Θ is one of the automorphisms defined as above.

We next assume that n is odd. Also a permutation α on the set I of even integers in Z_n and a permutation β on $Z_n \setminus I$ define an automorphism Θ of $M(Z_n)$ by $\Theta/I = \alpha$, $\Theta/Z_n \setminus I = \beta$. Thus we have $\frac{(n+1!)}{2} \frac{(n-1!)}{2}$ different automorphisms of $M(Z_n)$ defined this way.

Let Θ be any automorphism of $M(Z_n)$. Suppose that there exists $x \in I$ such that $\Theta(x) \in Z_n \setminus I$. Then for each $a \in Z_n \setminus I$ we have $\Theta(a) \in I$ and so $|Z_n \setminus I| = |I|$ which is not so. Thus $\Theta(I) = I$ and $\Theta(Z_n \setminus I) = Z_n \setminus I$. Hence $|A(M)| = \frac{(n+1!)}{2} \frac{(n-1!)}{2}$. //

6.3.6 The *wreath product* of a permutation group G on A by a permutation group H on B is the group of all permutations Θ on $A \times B$ of the following kind

$\Theta(a,b) = (\gamma_b(a), \eta(b))$, $a \in A$, $b \in A$, where for each $b \in B$, γ_b is a permutation of G on A, but for different b's the choices of the permutations γ_b are independent. The permutation η is a permutation of H on B. (cf. Hall [76], p 81).

The relation of the automorphism group of $M(Z_{mn})$, where $m \geq 3$, to the wreath product of the automorphism group of $M(Z_m)$ by the automorphism group of $M(Z_n)$, was obtained, as the following result, by Lim [77].

6.3.7 THEOREM. The automorphism group of $M(S_1 \times S_2)$ is the wreath product of the automorphism group of $M(S_1)$ by the automorphism group of $M(S_2)$ if and only if the following conditions hold.

(i) $M(S_2)$ is 1 - uniform implies that every 2 - element subset of E is independent.

(ii) $M(S_2)$ is not connected implies that $M(S_1)$ has the property that for every two distinct elements e_1, e_2 of S_1 there exists a circuit C with $|C \cap \{e_1, e_2\}| = 1$ //

6.4 AUTOMORPHISMS

We now give an example of a pregeometry defined on a group S so that multiplication is a geometric automorphism i.e. so that the collection $\mathcal{B}$ of bases is *preserved under the group operation*. That is, $B \in \mathcal{B} \iff gB \in \mathcal{B}$. We also show that the products of the previous sections have such geometric automorphism.

6.4.1 EXAMPLE. Let H be a non-trivial proper subgroup of a group S, of finite index r. Denote all distinct left cosets of H in S by $g_1 H, \ldots, g_r H$. Define $\mathcal{B}$ to be the collection of all subsets of S of the form $\{b_1, \ldots, b_r\}$, where $b_i \in g_i H$, $i = 1, \ldots, r$. Then $\mathcal{B}$ is the collection of bases of a transversal pregeometry G(S) on S such that

$$B \in \mathcal{B} \iff gB \in \mathcal{B}, \quad \forall g \in S.$$

Moreover G(S) is a pregeometry which is (i) loopless, (ii) binary and (iii) base orderable.

PROOF. It follows from the definition of $\mathcal{B}$ that it satisfies (B) so that $\mathcal{B}$ is the collection of bases of a transversal pregeometry G(S) with a presentation $[g_1 H, \ldots, g_r H]$.

We show that $B \in \mathcal{B} \iff gB \in \mathcal{B}$, $\forall g \in S$. Let $B \in \mathcal{B}$ and $g \in S$. To show that $gB \in \mathcal{B}$ it suffices to show that $(gg_1 H) \mathring{\cup} \ldots \mathring{\cup} (gg_r H) = S$. For $i = 1, \ldots, r$, put $g_i' = gg_i$. We first show that $(g_i' H) \cap (g_j' H) = \phi$ if $i \neq j$. Suppose that $\exists x \in (g_i' H) \cap (g_j' H)$. Then there exist h_1, h_2 in H with

$x = gg_i h_1 = gg_j h_2$ and hence $y = g^{-1}x \in (g_i H) \cap (g_j H)$ which is a contradiction. Thus $(g_i' H) \cap (g_j' H) = \phi$ if $i \neq j$. Now $g_1' H \cup \ldots \cup g_r' H \subseteq S$. For any $x \in S$ we have $g^{-1}x \in S$ so that $g^{-1}x \in g_i H$ for some i and so $x \in gg_i H$. Thus $S = (gg_1 H) \mathring{\cup} \ldots \mathring{\cup} (gg_r H)$.

Let $g B \in \mathcal{B}$. Suppose $B = \{ b_1, \ldots, b_r \}$. Then $gb_i \in g_j H$ for some j. Since $gb_i \neq gb_j$ and $(g_i H \cap g_j H) = \phi$ if $i \neq j$, we can assume that $gb_i \in g_i H$, $i = 1, \ldots, r$. Thus $b_i \in g^{-1} g_i H$. By the above $(g^{-1} g_1 H) \mathring{\cup} \ldots \mathring{\cup} (g^{-1} g_r H) = S$ so that $\{g^{-1} g_1 H, \ldots, g^{-1} g_r H\} = \{ g_1 H, \ldots, g_r H \}$ and hence $B \in \mathcal{B}$.

(i) Since $S = \bigcup_{i=1}^{r} g_i H$, every element of S is contained in a basis and so $G(S)$ is loopless.

(ii) Notice that a circuit of $G(S)$ is just any set of two elements from the same coset. Let C_1, C_2 be distinct circuits of $G(S)$. Then since $|C_1| = |C_2| = 2$, $|C_1 \Delta C_2| \geq 2$. If C_1, C_2 are in the same coset, then clearly $C_1 \Delta C_2$ contains a circuit. But if C_1, C_2 are in different cosets, then $C_1 \Delta C_2 = C_1 \quad C_2$ and so $C_1 \Delta C_2$ contains a circuit. By Theorem 4.2.4 $G(S)$ is binary.

(iii) Let $B_1 = \{ b_1, \ldots, b_r \}$ and $B_2 = \{ b_1', \ldots, b_r' \}$ be any two bases, where $b_i \in g_i H$, $b_i' \in g_i H$, $i = 1, \ldots, r$. Then the function $\Theta : B_1 \to B_2$ defined by $\Theta(b_i) = b_i'$, $\forall i = 1, \ldots, r$, is an exchange ordering for B_1, B_2. Thus $G(S)$ is base orderable. //

6.4.2 LEMMA. Any automorphism σ of the pregeometry in Example 6.4.1 is of the form

(*) $\quad \sigma / g_i H = g_{\Theta(i)} H \qquad i = 1, \ldots, r,$

where Θ is any permutation on $\{1, \ldots, r\}$.

PROOF. First we show that any function σ on S which satisfies (*) is an automorphism of G(S). It is clear that σ is a permutation on G. Let $B = \{ g_1 h_1, \ldots, g_r h_r \} \in \mathcal{B}$. Then for $i \neq j$ we have $\sigma(g_i h_i)$ and $\sigma(g_j h_j)$ in different cosets since Θ is a permutation on $\{1, \ldots, r\}$. Thus $\sigma(B)$ intersects every coset of H in S in exactly one element. Thus $\sigma(B) \in \mathcal{B}$. Similarly if $\sigma(B) \in \mathcal{B}$ we can show that $B \in \mathcal{B}$. Hence σ is an automorphism of G(S).

Suppose that α is an automorphism of G(S) which does not satisfy (*). Then $\exists\, a \neq b$, $a \in g_i H$, $b \in g_i H$ with $\alpha(a) \in g_j H$ and $\alpha(b) \in g_k H$, where $j \neq k$. Thus $\alpha(a)$ and $\alpha(b)$ are in different cosets. Choose a basis B containing $\alpha(a)$ and $\alpha(b)$. Since α^{-1} is also an automorphism of G(S), $\alpha^{-1}(B) \in \mathcal{B}$. Now $ab \subsetneq \alpha^{-1}(B)$. But a, b are in the same coset which is a contradiction. Thus any automorphism of G(S) satisfies (*). //

Thus we have

6.4.3 LEMMA. The automorphism group of the Example 6.4.1 has size $(r!) \left(\frac{|S|}{r} ! \right)$ if S is finite, where r is the index of H in S.

We finally prove

6.4.4 LEMMA. If S_1 is a subgroup of S and multiplication by $h \in S_1$ is a geometric automorhism of some G(S) then it is also a geometric automorphism of $G_S(S_1)$.

PROOF. Let $\mathcal{B}$ and $\mathcal{B}'$ be the collection of bases of

$G(S)$ and $G_S(H)$ respectively. Then

$B' \in \mathcal{B}' \Rightarrow B' = B \cap H$, where $B \in \mathcal{B}$,

$\Rightarrow hB' = (hB) \cap H$ and $hB \in \mathcal{B}$, $|hB'| = |B'|$, for any $h \in H$

$\Rightarrow hB' \in \mathcal{B}'$. //

Conversely when we deal with a matroid $\mathcal{M}(S_1 \times S_2)$ obtained as in 6.1 we have

6.4.5 LEMMA. If multiplication $h \in S_i$ is a geometric automorphism of $M(S_i)$, $i = 1, 2$, then it is also a geometric automorphism of $\mathcal{M}(S_1 \times S_2)$.

PROOF. Let D be any basis of $M(S_1 \times S_2)$. Let $g = (g_1, g_2)$ be any element of $S_1 \times S_2$. Then by the definition of $M(S_1 \times S_2)$ there exists a basis B of $M(S_1)$ such that

$$D = \Big(\bigcup_{e \in B} e \times B_e\Big) \cup \Big(\bigcup_{e \in S_1 \setminus B} e \times (B_e \setminus f)\Big)$$

Thus $gD = \Big(\bigcup_{e \in B} eg_1 \times g_2B_e\Big) \cup \Big(\bigcup_{e \in S_1 \setminus B} eg_1 \times (g_2\,(B_e \setminus f))\Big)$

Since B is a basis of $M(S_1)$, g_1B is a basis of $M(S_1)$. Also $g_2(B_e \setminus f)$ is independent in $M(S_2)$ of rank $r(M(S_2)) - 1$.

Hence gD is a basis of $M(S_1 \times S_2)$ as required. //

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