A PHILOSOPHICAL STUDY OF THE VALUE OF MATHEMATICAL KNOWLEDGE AND THE PLACE OF MATHEMATICS ON THE CURRICULUM

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ABSTRACT

In basing the school curriculum on the view that some value can be attributed to knowledge most arguments have centred on either the contingent consequences of studying particular disciplines, the claim that knowledge can be differentiated into distinct forms and that all students should be introduced to them, or the claim that some knowledge can be valued for its own sake or for its power in developing the mind. In the case of mathematics common justifications given for teaching it are that it is useful, that it promotes intellectual development or that it is intrinsically worthwhile.

But a recent view argues that some knowledge is valuable because it provides people with such an understanding that allows them to reflect on questions concerning the nature and meaning of life and to be in a position to best determine what they will do with their lives. The role that mathematics plays here is investigated by an examination of the nature and foundations of mathematical knowledge. Dominant views on mathematics have nearly all stressed its *a priori* nature but they all have serious objections to them. By a comparison with views on the nature of scientific change a recent view on the nature of mathematical knowledge has been articulated that describes it to be in a process of evolution. At any particular time there exists a mathematical practice which consists of a language component, a metamathematical view component,

and sets of accepted reasonings, questions and statements. The mathematical practice of today has evolved from a set of beliefs about simple manipulations of physical objects and consists of idealized ways of operating on the world.

It is concluded that while all students should be introduced to the minimal mathematical language that is useful to everyone they should also come to understand the cultural significances of mathematics as it has evolved through man's attempts to solve problems within his environment. This comes through a study of the influences that mathematics has had on different cultures and the way that man has looked to mathematics as providing a method of solution to problems within his culture. Unlike earlier justifications given for teaching mathematics the justification based on the cultural significances of mathematics centres on all five components of the mathematical practice of the day and provides important considerations for the structure and presentation of mathematics courses in schools.

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CHAPTER I

THE PROBLEM OF THE PLACE OF MATHEMATICS ON THE CURRICULUM

Nature of the Problem

Whenever there is debate over the purposes of education and the value of particular educational activities, it becomes important to look at the traditional subject disciplines, like mathematics, and the claims made for their placement on the curriculum. For in the case of mathematics, the value judgements we make regarding such things as reforms to traditional curricula, the selection of mathematical topics, and the best methods of instruction, are all related to the fundamental concerns of why we value mathematical knowledge and the justification we can give for teaching mathematics in schools.

In a discussion on the aims of teaching mathematics at a meeting of the British Association for the Advancement of Science, held at Glasgow in 1901, a Professor J. Perry said:

> The Study of Mathematics began because it was useful, continues because it is useful and is valuable to the world because of the usefulness of its results, while the mathematicians, who determine what the teacher will do, hold that the subject should be studied for its own sake.
> (Quoted in Griffiths and Howson, 1974, p. 17)

Seventy years later there was no apparent change in the purposes of studying mathematics, as the Assistant Masters Association in England said:

> There is a twofold purpose in the study of mathematics. Firstly, and of prime importance, is the pursuit of the subject for its own worth. ... Secondly, ... we must regard mathematics as

> a study of a service subject to science, technology, commerce, politics and economics, and even the arts.
> (Assistant Masters Association, 1973, p. 205)

More recently, the justifications given for the teaching of mathematics have increased in number though great importance is still attached to the usefulness to which the knowledge can be put. As K. Selkirk says:

> The teaching of mathematics in schools may be justified in a number of ways. It is, for example, part of the cultural background of our civilisation, and as such should rank with art, music, literature and similar aspects of our heritage. Again it is a logical and efficient system of deduction and this may well transfer to problems outside the immediate area of the subject. The justification which appeals particularly to those whose primary interests lie outside the subject is, however, that it is useful. At a time when the limitations of our national and global resources are only too painfully apparent, this usefulness must be a major justification for the teaching of the subject in schools and for its important share in the total school curriculum.
> (In M. Cornelius (ed.), 1982, p. 186)

But these comments leave important problems unresolved. While we might agree, for example, that all students should be taught the mathematics that will be useful to them later on in life, we are still left with the problem of deciding what mathematics is useful to all people. It might be that the amount of mathematics that is useful to all people is so minimal that it requires very little time at all on the curriculum. And if we are to argue that students should be taught the mathematics that will be useful to them in their future occupations then we must confront the problem of predicting the future for these students and deciding whether the mathematical knowledge they need is not better taught "on the job" or in specific vocational training institutions.

The claim that mathematics should be studied for its own sake is unclear. Does it mean that people enjoy studying mathematics and that, therefore, it is worthwhile pursuing? If so, how does one answer the student who says that he doesn't enjoy studying it and that, therefore, it is not worthwhile? If one is to claim that, irrespective of any preference of the learner, mathematics is an intrinsically worthwhile activity and should be studied by all students, then we must decide on what makes it intrinsically worthwhile. Why can we claim that mathematical knowledge is valuable to all people without any reference to the use to which they can put that knowledge?

If it is claimed that mathematics should be studied by all students because it develops the mind and promotes intellectual development, then it must be clearly established in what ways mathematical knowledge is necessary for intellectual development to proceed. Are all paths to intellectual development, for example, dependent upon a certain minimal training in mathematics?

Finally, if mathematics is ranked with activities like art, music and literature, then why is this sufficient reason for requiring all students to study it at school? Why should schools be concerned about the cultural significances of mathematics?

The answers to these questions are important because the reasons we give for teaching mathematics in schools have implications for how we teach it as well as for the selection of content of mathematics courses. But a critical examination of the claims for the justification for teaching mathematics must be based on views about the value of knowledge in general, and the nature and value of mathematical knowledge in particular. Is there a strict dichotomy of knowledge into that which is valuable because of its extrinsic usefulness, and that which is valuable because of its intrinsic worthwhileness? Are there other value categories of knowledge and, if so, what are they? What is the nature of mathematical knowledge? Why should mathematics constitute part of the compulsory curriculum? It is the purpose of this dissertation to investigate these questions from the philosophical perspective and to argue for the inclusion of mathematics in the curriculum, based on a view of the nature of man, the nature of mathematical knowledge, and the cultural significances of mathematics. Such a view, it is shown, will have radical implications for the mathematics curriculum.

Outline of the Argument

Chapter II is concerned with arguments for basing the curriculum on particular notions of the worthwhileness of knowledge. It considers the claim that some subjects are valuable because of their contingent consequences, the claim that the areas of knowledge represented by certain subjects are distinct forms of knowledge, and the claim that some subjects themselves possess intrinsic worthwhileness. Finding objections to all these views, an argument is then considered which rejects the dichotomy of knowledge into that which is instrumentally useful and that which is intrinsically worthwhile; but which gives value to some knowledge in assisting people to acquire a "world view" and make reasoned decisions about what they will do with their lives.

Given that argument it is then reasonable to ask what it is about mathematics that allows it to serve that purpose. So Chapter III is related to questions concerning the nature and foundations of mathematics. The dominant 20th century views on the nature of mathematics, stemming from the earlier works of philosophers such as Leibniz and Kant and even back earlier to Plato and Aristotle, are all found to have serious objections to them. This is because, it is argued, they all regard mathematics as something that is unchanging with time. A recent thesis is presented which considers mathematics to be in a process of evolution and constituting a particular element of the culture at a particular time. This view then provides the basis for an examination of the cultural significances of mathematics.

In Chapter IV some of the common justifications given for teaching mathematics are considered; namely, that it is useful, that it is intrinsically worthwhile, and that it promotes intellectual development. These justifications are found to be inadequate. A justification based on the cultural significances of mathematics is then presented together with some important considerations for school mathematics courses.

CHAPTER II

THE VALUE OF KNOWLEDGE

The aim of this chapter is to show the development of an argument which contends that we can choose a curriculum based on a particular notion of the worthwhileness of knowledge. In the first section the case for such an argument is established by considering firstly the views of those who maintain that such judgements of worthwhileness cannot or should not be made. Then there follows an examination of the utilitarian view that the promotion of happiness is the sole criterion under which man's actions, including curriculum choice, are to be judged worthwhile or not; and the pragmatist's view that sees knowledge as something that is acquired by man as he struggles to control his environment. Finally, in this section, consideration is given to the view that while educators should not make final judgements of worthwhileness they should, in fact, design the curriculum in such a way that enables students to do so.

All these arguments are found to have objections to them and so the following section considers various attempts made by theorists to give some value to knowledge and which should serve as a basis for curriculum design. The first approach considered is one which attributes worthwhileness to certain subjects based on the contingent consequences of those subjects; the contingent consequences being a list of specific objectives that, it is argued, students ought to attain. The second approach considers the justification for a curriculum based on a view of knowledge that distinguishes distinct forms of knowledge. It is claimed that the curriculum should be designed so as to introduce students to these distinct forms. A third approach is to claim that some subjects possess intrinsic worthwhileness and so they are to be valued for their own sake. Some subjects might be deemed intrinsically worthwhile, for example, because they involve a higher degree of intellectual functioning or because they are concerned with truth and rationality.

All the views are found to be inadequate as they stand and an argument is then presented which contends that we are wrong to regard all knowledge as being either instrumentally useful or as an end in itself. Some knowledge, it is argued, is valuable because it helps people determine their own ends by acquiring an understanding of things in a way that allows them to make reasoned decisions about what aims to set themselves and what they are to do with their lives. Furthermore, the knowledge that does that, it is claimed, is found in the traditional disciplines as they have evolved across generations and cultures and with the contributions of many scholars in the various fields.

This argument then provides the basis for a critical examination of the traditional subject disciplines, like mathematics, in order to elucidate their nature and their influence within different societies and cultures, and to consider what implications this might have for the curriculum.

Establishing the Argument

(i) There is a view of values, found in the works of philosophers like David Hume in the 18th century and A.J. Ayer in the 20th century, that considers all value judgements to be simply expressions of emotion. Hume maintained that reason alone cannot decide moral questions but that most people have a "moral sentiment" that is used to make decisions. The moral sentiment is pleasant if it is a feeling of approval and unpleasant if it is one of disapproval. Ayer's view, first argued in 1936, is that statements which cannot be verified by observation or analysis have no meaning. True statements are verifiable propositions and statements which are not verifiable are meaningless. Therefore, there is no way to decide between different value judgements. To say "stealing is wrong" is simply to express a feeling and the statement cannot be proven in any sense since it contains no verifiable proposition. Ayer maintains that this same analysis holds for all types of value judgements.

This emotivist view then, in relation to questions of curriculum, would maintain that there is no rational basis for choosing the elements of a curriculum. To say that something ought to be included in the curriculum is simply to express a feeling that others may or may not have. No reasons can be given, however, to justify such inclusion.

But while it may be difficult to find reasons for including something in the curriculum, this is not to say that there are none and the emotivist's point of view does have certain flaws within it.

Firstly, it is possible for our emotions and our value judgements not to coincide. One can say, "I feel like doing something but I know it is wrong", or "I don't want to do this but I know I ought to." While some emotivists might simply claim that there is a conflict of emotions here, it would seem that such occasions ought not to arise if our value judgements were just a reflection of our emotions. Secondly, we recognize that our emotions can change over time, whereas the value judgements we are attempting to make are based upon premises that we believe are unchangeable. When we make the value judgement that to steal is wrong, we are implying that it always will be, even if in the future in a particular situation, under the influence of drugs say, we adopt the attitude that to steal is an acceptable form of behaviour. Thirdly, the emotivist's philosophy is based upon the judgement of the truth of propositions in only two ways, by observations and analysis. But there is no reason to accept that these are the only ways of attesting to the truth of all propositions. The emotivist has not shown, for example, that there cannot be reasoning about values.

(ii) The argument that one ought not to decide what goes into a curriculum is closely allied with the general area of what is called "child-centred" education. It is based upon a particular view of the child and the right of the child to determine what he or she will study.

J.J. Rousseau's thinking, for example, is reflected in his fictionalised account of the child Emile, published in 1762, and involves an analogy with nature. Rousseau argues that if left to himself Emile will become what nature intended him to become. No coercion nor prompting is needed, but only support. If nurtured correctly Emile will grow up to be physically and intellectually what was ideally intended for him at birth. To direct his thinking in any way would be "to substitute authority for reason in his mind" and make him "the victim of other people's opinions" (Quoted in Boyd, 1956, p. 73). Later, Friedrich Froebel (see Lilley, (ed.) 1967) expanded Rousseau's views to develop a direct analogy with nature. The teachers in charge of their pupils should be like gardeners tending their plants, providing them with the best possible environment for growth and then allowing nature to follow its course.

A.S. Neill was not only a writer but also a practitioner in the field of education and his school, Summerhill, was meant to reflect his educational thought. His arguments centre around the freedom of the child to learn. It is the right of the child to choose what and when to study. This right of the child outweighs any arguments claiming the worthwhileness of studying a particular subject at a particular time. If the child wishes to study the subject then he will decide when to do so.

There are criticisms, however, which can be directed towards the views of writers such as Rousseau, Froebel and Neill. Firstly, as for the analogy with growing plants, it is easy to see a flower or a plant grow with the minimum of guidance and to explain that it is nature's way. It is very easy, but misleading, to say the same should be so of human beings, both physically and intellectually. The knowledge, attitudes and skills that can be acquired by a child in society do not occur naturally. The culture that man has created is very complex and deliberate guidance into it is required. To allow "nature's way" and not to attempt to direct his passage into society, is to leave the child open to detrimental influences. Secondly, if one does have reasoned views on what a child ought to know and knowledge of the capabilities of the child, and if one has reasoned views on how knowledge ought to be attained, then one should attempt to make certain that such knowledge is acquired. To allow the child always to make decisions on what and when to study leaves the child open to a choice based upon a misunderstanding of the available possibilities.

(iii) The premise of utilitarianism is that what matters most is a world in which everyone is happy. Therefore, man's actions ought to be about minimizing pain and maximizing pleasure. This philosophy was developed in the 1800s by Jeremy Bentham and John Stuart Mill and one recent educationalist, Robin Barrow, has based his educational thoughton the utilitarian premise:

> Education should seek to develop individuals in such a way that they are in a position to gain happiness for themselves, while contributing to the happiness of others, in a social setting that is designed to maintain and promote the happiness of all so far as possible.
>
> (Barrow, 1976, p. 84)

Barrow does not believe that an ideal state will be attained where everybody achieves maximum pleasure, as he points out in answering critics such as MacIntyre (1964). What matters is that man ought to be striving for such a state of affairs and that when decisions have to be made they ought to be based on the utilitarian premise.

Apart from the fact that the ideal state is not with us and may never be with us there are other difficulties with the utilitarian premise. Firstly, in claiming that pleasure should be distributed among all men, rather than a small number of men being supremely happy, the utilitarian is claiming that he attaches importance to the principle of distributive justice. Barrow admits this, but in so doing he clearly values this principle too and not just the sum total of human happiness.

Secondly, since utilitarians claim their premise to be true, and not just something to be arbitrarily accepted, they also commit themselves to valuing the truth as well as happiness.

Some utilitarians also get into difficulties when they claim that some activities, such as studying science, are more valuable than others, such as playing darts, given that each activity promotes the same amount of pleasure. A person may be drawn to pursue a certain activity because he feels it is important in some way. He may feel as if pursuing a particular scientific project is important, for example, because it will help him solve problems that he believes ought to be solved. Such problems may not even be understood by the majority of people and their solution may contribute nothing to the sum total of the happiness of society. So too a person may study philosophy because he believes it is helping him to answer questions that have concerned him and that he feels he needs to answer. The time spent on such activities may give him little or no pleasure but he is still drawn to them. Some criterion, other than the utilitarian one, must be used to decide on the worth of these various activities, thus contradicting the premise that maximizing pleasure is all that matters.

The utilitarian premise is rejected then as the sole criterion for determining which subjects should constitute the curriculum. That is not to say that we do not value happiness, but that we do not accept that simply valuing happiness is enough to select the content of the curriculum. We shall show the development of an argument which claims some knowledge as being valuable for reasons other than simply promoting happiness.

(iv) A writer who has had a great impact upon educational practice in the 20th century is John Dewey. His pragmatic philosophy rejects the dichotomy between knowledge and experience. The pragmatist believes man's intelligence is a tool used by man to control his environment. To learn something significant about the world we must do more than operate logically upon what appear to be self-evident truths. We must transform the environment as a response to problems that need resolution. Thought does provide hypothetical ideas in response to the problems but these ideas are tested in action.

The process of learning from experience is thus an active process. The learner locates and defines a problem to be solved, collects pertinent data through observation and reasoning and decides on possible solutions before finally testing them. And for Dewey it is the quality of experience that is important. So in directing his comments to educators he says:

> It is his [the educator's] business to arrange for the kind of experiences which, while they do not repel the student, but rather engage his activities are, nevertheless, more than immediately enjoyable since they promote having desirable future experiences.... Wholly independent of desire or intent, every experience lives on in further experiences.
>
> (Dewey, 1938, p. 16)

For Dewey there is only one kind of knowledge; a knowledge that may be termed either moral or scientific. Moral science:

> ... is ineradicably empirical, not theological nor metaphysical nor mathematical.... Hence physics, chemistry, history, statistics, engineering science, are a part of disciplined moral knowledge so far as they enable us to understand the conditions and agencies through which man lives, and on account of which he forms and executes his plans. Moral science is not something with a separate province. It is physical, biological and historic knowledge placed in a human context where it will illuminate and guide the activities of men.
>
> (Dewey, 1922, pp. 295-6)

So knowledge itself has no intrinsic worth. It is something that is acquired by man as he grapples with problems in his environment and eventually comes to solve these problems.

There is no division into different types or forms of knowledge. There are no absolute or universal truths that are of different kinds. Experience and problem situations have forced man to use and develop the power of thought to control his environment.

For pragmatists like Dewey then the curriculum is a process as much as a distinct body of subject matter. Dewey does not reject what might be called traditional disciplines such as mathematics and history, but claims that the student should draw upon his reflections in these areas to help solve the problems he has encountered. They have no usefulness in their own right, only in their ability to enrich the life of the student and enable him to solve the problems.

Dewey's position and the pragmatic philosophy in general do have serious objections, however. Firstly, the pragmatist is in difficulties because the reasoning behind his philosophical position is surely theoretical and not practical. It is difficult to see how he can arrive at a philosophical position other than through theoretical reasoning as distinct from practical reasoning. And yet this distinction is what the pragmatist disclaims.

Secondly, in claiming that the quality of experience is important, Dewey says that the experience is meant to lead on to other rewarding experiences:

> Hence the central problem of an education based upon experience is to select the kind of present experiences that live fruitfully and creatively in subsequent experiences.
>
> (Dewey, 1938, pp. 16-17)

Education can be identified with growth, not just physically but intellectually and morally. The objection is that one must specify the directions in which present experiences will lead the learner. For isn't it also true that some people, such as criminals, may find some experiences rewarding for themselves but that do not appeal to the rest of the community? Dewey's answer to that is that while a man may acquire great skill as a criminal through a series of experiences the question is whether this will affect his growth in general:

> Does this form of growth create conditions for further growth, or does it set up conditions that shut off the person who has grown in this particular direction from the occasions, stimuli and opportunities for continuing growth in new directions? What is the effect of growth in a special direction upon the attitudes and habits which alone open up avenues for development in other lines?
> (ibid., p. 29)

But even so it is difficult to see how the educator, mindful of the fact that different experiences are conducive to growth in different directions, can escape making qualitative judgements. Ultimately he must be able to decide between the worthwhileness of different experiences.

Finally, it is difficult to see how all intellectual activities can follow the same pattern of the experimental sciences. How, for example, can history be fitted into the same experimental patterns? But whether it is true or not, it is necessary for the pragmatist to furnish reasons for regarding them as the same, just as it is necessary for those who claim there are distinct types of activities to show how they are distinct.

(v) A different approach has been adopted by J.P. White in his book Towards a Compulsory Curriculum (1973), where he presents his argument that education ought to be about providing people with the knowledge that enables them to make meaningful choices between different activities and different ways of life. Educators do not make the final judgement of worthwhileness but design the curriculum to enable the students to do so. Surprisingly, White bases his argument for a compulsory curriculum on the concept of liberty. "Any infringement of liberty is *prima facie* morally unjustifiable" (ibid., p. 5). But it is *only prima facie* wrong to stop people doing what they want to do for there may be considerations which override this principle. So what is needed is an examination of the kinds of considerations which might justify an interference with liberty. White claims that considerations of a person's own good as well as the good of others may justify such an interference. In relation to education:

> ... it would be right to constrain a child to learn such and such only if (a) he is likely to be harmed if he does not do so, or (b) other people are likely to be harmed.
>
> (ibid., p. 6)

Case (a) is central to White's argument.

The problem now is to identify, from all the possible activities to be understood by learning, those activities that are likely to harm the child if he is not constrained to learn them. To this end White makes an important distinction between the questions, "What kinds of activities are worthwhile in themselves?" and "What kinds of activities are educationally worthwhile?" History and mathematics may not be intrinsically valuable for everyone but they may be educationally valuable. The educational value of an activity is not determined by any value intrinsic to the activity itself but by the nature of the activity.

White thus divides activities into two categories in which:

> (1) no understanding of what it is to want X is logically possible without engaging in X
> (2) some understanding of what it is to want X is logically possible without engaging in X.
>
> (ibid., p. 26)

The activities of the first category must be part of a compulsory curriculum because if the student is not compulsorily introduced to them he will have no understanding of what it is to study them and, therefore, will not be able to make a reasoned choice as to whether he will pursue them or not. In this category White includes subjects such as pure mathematics, communication in general, engaging in the (exact) physical sciences, appreciating works of art, and philosophizing.

The same justification cannot be given for the activities of the second category, however, which includes speaking a foreign language, playing organized games, cookery, painting pictures and writing poetry, as examples. These activities are not compulsory but are offered only as options. While it is important for all students to know what these activities are, it is not necessary for students to engage in them since this is not needed to understand what it is to want to pursue them.

In his book White does consider some objections to the theory, such as whether it is clearly evident what activities belong to which category, and whether one is not simply advocating one's subjective preference for autonomy in designing the curriculum. But M.A.B. Degenhardt raises further serious objections to the theory. For the learner:

> Does not coming to understand any serious activity involve coming to feel something of its 'call' or 'urgency'? Would we not be sceptical of one who said 'Oh, I know what there is in that poetry business (or science, or history, or music): I think I'll give it a miss'? Certainly it would be odd if someone said 'In my early twenties I <u>decided</u> to be interested in philosophy'. For people just do not relate to serious activities in this way.
>
> (Degenhardt, 1982, p. 79)

And for the teacher:

> It is generally thought that a good teacher must care passionately for his subject, evidencing this passion and getting pupils to share it. But who could sustain such passion if he saw himself as merely offering a <u>smorgasbrod</u> of pursuits to be sampled and then freely chosen or rejected? Does not good teaching presuppose a more positive conviction of the worth of what we teach?
>
> (ibid., p. 80)

Such comments call on us to rethink our views on knowledge and to establish what ultimate value we can give to it. Can knowledge be divided into two kinds; knowledge that is useful and knowledge that is intrinsically valuable in some way? Later, in this chapter, we see that this dichotomy is rejected by Degenhardt; that it overlooks a third possibility, that some knowledge is not necessarily extrinsically useful nor intrinsically worthwhile, but is what Degenhardt calls "serious" or "significant" knowledge. Not only that, what makes it serious or significant is based on a view of the nature of man as well as a view of the nature of knowledge. Some knowledge is valuable because it serves a distinctively human enterprise.

Contingent Consequences

In the previous section we sought to establish the argument that we can choose a particular curriculum based on the worthwhileness of knowledge by considering the views of those who argue that such judgements are not possible or that they ought not to be made. Such views were found wanting as they stand. In this section we consider the views of some who maintain that worthwhileness can be attributed to certain disciplines, but that the worthwhileness of such disciplines is related to the contingent consequences of pursuing them. In trying to be quite specific about the design of a curriculum, two recent writers have attempted to list a set of specific objectives that students ought to attain, and then to select those disciplines which assist in the attainment of those specific objectives. Such a method has been adopted by S. Nisbet (1957) in his book Purpose in the Curriculum and by the highly influential Taxonomy of Educational Objectives edited by B.S. Bloom (1956).

Nisbet classifies the "practical objectives of education" that a teacher might realistically achieve into two groups. The first group, labelled "Adjustment to Environment", consists of skills, culture, home membership, occupation, leisure, and active citizenship. The second group, labelled "Personal Growth", consists of the physical, aesthetic, social, spiritual, intellectual, and moral development of the individual.

Nisbet explains:

> Such a list is comprehensive enough to include most of what has been claimed as important in education, whatever the ultimate philosophy of those who make the claims, and yet detailed enough to provide guidance and illumination for the practical person, whatever specific functions he may have to perform.
> (Nisbet, 1957, p. 14)

He then examines the conventional curriculum subjects in turn and considers how many of the objectives are, in fact, contributed to by a study of those subjects.

Bloom's taxonomy is more detailed but the intention is the same as Nisbet's. Three domains are specified; the cognitive, the affective, and the psychomotor. Within each domain certain objectives are categorized and sub-categorized. For example, the cognitive domain is categorized into such things as knowledge of specifics, knowledge of criteria, application, analysis and evaluation. Some categories in the affective domain are awareness, willingness to respond, and satisfaction in response.

There are, however, two main criticisms that can be directed towards both Nisbet's and Bloom's approach. The first is to do with the list of objectives. While both writers agree that there may be some disagreement as to what the list of objectives should consist of, their final list is more of one achieved by consensus than by rational argument. Nisbet, for example, was concerned about high-level aims such as "to facilitate complete living" and "to promote the highest intellectual or moral development of the pupil". He was equally concerned about specific practical aims such as "to produce Macbeth" and "to make first year Latin interesting". He therefore set out to produce a comprehensive list of "intermediate practical objectives".

But it is not enough simply to specify a list of objectives that is hoped will gain acceptance by a majority of people. The objectives must be clearly stated and argued for.

While Nisbet's description of high-level or ultimate aims may be vague this does not mean that the aims should be dismissed. If there are ultimate aims then these aims should be clarified such that clear teaching objectives may be developed. To introduce "intermediate practical objectives" does not clarify ultimate aims and until these ultimate aims are clarified then there is much room for disagreement about the practical objectives. The same vagueness that characterises Nisbet's high-level aims contributes to disagreement as to the value of the practical objectives.

The second point of criticism is to do with what is said about the various subjects. The approach is to draw attention to the contingent consequences of each subject.

The study of science, Nisbet maintains, contributes to spiritual development, and arithmetic and mathematics contribute to moral development. When noting that in arithmetic the answer is right or wrong he says:

> Virtue, in the form of persistence and concentration, is rewarded by a correct answer : vice, in the shape of carelessness or listlessness or laziness, is punished quite simply by a wrong answer.
> (ibid., p. 83)

The point is that, whether one agrees with the contingent consequences or not, the subjects are being justified by considerations which have nothing to do with the subjects themselves. If there is good reason for studying mathematics, irrespective of any other subject being studied, then there must be something about the nature of mathematics that is distinct from any other subject and that which makes it worthwhile for the student to study. Considering contingent consequences does nothing to aid in the selection of subjects for inclusion in a curriculum. If different subjects have the same contingent consequences then there is no reason for necessarily valuing any one of them above any other.

We must, therefore, look at the subject itself, to seek out what is unique to that subject and to argue for its inclusion in the curriculum as contributing to the achievement of clear ultimate aims.

Forms of Knowledge

Instead of considering contingent consequences, a different approach has been adopted by P. Hirst in attempting to justify a curriculum based on a view of knowledge that distinguishes distinct forms of knowledge (see Hirst, 1974). Hirst originally identified eight forms but in subsequent revision has listed seven (see Hirst and Peters, 1970). These forms are logic and mathematics, the physical sciences, the knowledge of our own and other minds, moral knowledge, aesthetic knowledge, religious knowledge, and philosophical knowledge. The important claim is that these forms are distinguishable by four criteria:

(a) He first claims that each form involves concepts that are peculiar in character to the form.

(b) In each form the concepts provide a network of relationships giving the form a distinctive logical structure.

(c) Each form has expressions that are "testable against experience", the criteria on which the tests are based being unique to that form.

(d) Finally, the forms have a distinctive methodology for testing their expressions.

Thus the truth of propositions in different forms of knowledge is established in quite logically distinct ways.

Hirst's thesis ties in with a view of knowledge as reflecting the different ways we experience the world and the different ways we use language to communicate ideas, rather than a view of knowledge that is meant to reflect the true nature of the world. A liberal education is one which gives an understanding of the distinct forms of knowledge and, therefore, the curriculum should be so designed as to introduce students to the distinct forms.

Many of Hirst's critics have concentrated on the epistemological arguments in his thesis (see, for example, Gribble (1970), Phillips (1971), Hindess (1972) and Warnock (1977)). Barrow (1976), on the other hand, has rejected Hirst's view but developed his own, arguing that there are only two distinct forms; namely, the empirical and the philosophical. These forms are based on two distinct validation procedures. In the empirical form the truth or falsehood of propositions is arrived at by a combination of logic and reference to empirical evidence. In the philosophical form the truth or falsehood of propositions can only be determined by logical reasoning. Barrow also suggests that there are two basic "interpretive attitudes" to the world; the religious and the scientific, and a number of distinct "kinds of awareness". So the truth or falsity of every statement, according to Barrow, can be determined by reference to one of two validation procedures. The two interpretive attitudes represent two distinct fundamental conceptions of how the world and existence is to be explained. And the kinds of awareness refer to different kinds of feeling that can be aroused when contemplating particular phenomena. There can be situations where people have either a moral, aesthetic, religious or scientific awareness, for example. Even someone with a religious interpretive attitude may still have a scientific awareness provoked by a particular situation.

The type of epistemological criticism directed at Hirst's thesis could also be directed at Barrow's. The important point that is implied by such views of knowledge, however, is that if someone knows how to set about assessing whether a proposition in one of the forms is true, then he is familiar with the kind of procedure necessary to establish the truth of other propositions in that form. He may not be able to give an answer, not having studied the required topic, but he knows the kind of procedure required to establish an answer.

The concern of this dissertation is with the implications for curricula and schooling. If we accept that certain propositions do, in fact, reflect different ways we experience the world, and, in so doing, reflect different kinds of knowledge, then we must ask whether this, in itself, implies that all children should be initiated into the different forms. Is R.F. Dearden (1968) right, for example, in taking Hirst's thesis and developing "forms of understanding" that primary school children ought to be introduced to, simply because they can be categorized according to Hirst's selection criteria? If not, what other arguments can be put forward justifying initiation into the forms?

Finally, we must ask of the importance of content. If there is only one method of assessing the truth within each form does that mean that it doesn't really matter what content is presented in each form, only that the method of assessing truth is acquired? And if the content is important, under what criteria is it to be selected. If Hirst's thesis is correct one would have to be able to identify concepts as belonging to particular forms before deciding on the criteria to be used to test the expressions in which the concepts appear.

In drawing attention to a criticism of Hirst's work by R.K. Elliott (1975), M.A.B. Degenhardt (1982) considers these problems under three themes.

<u>Powers of the mind</u>. One conclusion from Hirst's work might be that in order to develop one's powers of the mind one needs to be first initiated into the forms of knowledge. This implies that a person would somehow be totally unknowledgeable of all things around him; that he would not be able to make any operations in the mind after experiencing the world through his senses, unless he was initiated into the forms. But this underrates the nature of the learner whose powers of mind and ability to understand are present before any introduction to the forms is initiated. It could be argued that the forms of knowledge have, in fact, developed from human beings being able to retain what they perceive with their senses, and to organize that information in some way, in seeking to understand those concepts that their minds apprehend.

<u>Critique of the disciplines</u>. Degenhardt observes that mastery of a discipline does not necessarily improve one's understanding of the subject matter of that discipline. He gives the example of the mathematicized nature of physics, where experts in the field have difficulty in relating that to physical reality, the assumed subject matter of physics. While such doubts can be raised about any discipline Elliott

concludes:

> These considerations suggest a task which properly belongs to Philosophy of Education, namely enquiry into the character of the disciplines with a view to assessing their educational value. It is less than just to give a student an education which encourages him to take enthusiastically to a discipline whose true character is not what it is proclaimed to be.
>
> (Elliott, 1975, p. 61)

For example, does one arrive at an awareness and understanding of people's minds by pursuing courses in psychology that consist of elaborate mathematical relationships between arbitrarily defined factors? Elliott argues that each form of knowledge, as identified by Hirst, is a distinct systematic study but which also is an extension of what he calls a corresponding "common area of everyday knowledge". It may be that the understanding of people's minds that one wishes to acquire is found in this common everyday knowledge.

What matters most? Under this heading Degenhardt considers the question of how the content of the forms might be selected. If one selects the content guided only by what best exemplifies the distinctive nature of the discipline then the discipline itself could suffer. It does not follow that those things that best exemplify the logical features of a form of knowledge are the important things for people to know about in that form. We still lack criteria for selecting worthwhile knowledge.

Intrinsic Worthwhileness

Instead of arguing then that certain subjects are worthwhile pursuing because they contribute to the attainment of certain specified objectives, or because they represent a distinct form of knowledge some writers have argued that some subjects are worth studying because those subjects have some intrinsic worthwhileness.

Before considering some of these ideas a distinction must first be made between the intrinsic worthwhileness of engaging in the study of a particular subject and the intrinsic worthwhileness of mastering a subject or attaining knowledge in that subject. As an example of the first case, we might consider that studying mathematics and trying to come to understand a mathematical concept is worthwhile in itself, irrespective of whether one succeeds in that endeavour or not. What we are concerned about, however, is the intrinsic worthwhileness of attaining knowledge in a particular discipline and judging whether success in one activity is more worthwhile than success in any other.

The question to be asked is, what reasons, related to the nature of a particular subject, can be given when claiming that some subjects are more intrinsically worthwhile than others? G.H. Bantock (1963) insists that some subjects are intrinsically worthwhile and are more valuable because their understanding involves a higher degree of intellectual functioning. He says:

> ... the fact that ... some subjects make more demands on human beings, require, for their mastery, a more complex human organization and finally produce more valuable consequences is inescapable.
> (Bantock, 1963, p. 94, footnote)

The point is, however, that while mastering higher mathematics, or appreciating poetry, may require more complex intellectual functioning than playing football, it has to be shown that the consequences of doing so are, in fact, valuable. It may be that the complex intellectual functioning that is required to engage in higher mathematics say, is valuable only in allowing one to engage in higher mathematics and nothing else. The fact that certain subjects may require a more complex intellectual functioning does not show that those subjects, in themselves, are necessarily worthwhile.

A different approach has been adopted by R.S. Peters in his book Ethics and Education (1966). While he has subsequently expressed doubts about the arguments expressed in that book (see, for example, Peters' chapter in Hirst (ed.) (1983) pp. 30-61) his views there have evoked much discussion.

In the book Peters argues that education involves the initiation of others into worthwhile activities and that the activities that are educationally worthwhile are valued for their own sake. The first problem he considers is to determine what makes some activities more worthwhile than others. What makes mathematics and history more worthwhile pursuing than football or billiards say? The first step in answering that question is to establish that there are in fact fundamental differences between activities like mathematics and football that do not exist between billiards and football or between mathematics and history. Both billiards and mathematics may be "disinterested, civilized and skilful pursuits", yet mathematics seems to earn a place on the school curriculum ahead of billiards.

Firstly, in arguing for a fundamental difference between certain activities, consideration could be given to the object of the activity. Some activities, like eating, have limits imposed upon them due to bodily conditions. Also, some activities are competitive. When one person acquires money there is less for others. But in theoretical activities, Peters argues, the object of pursuit, be it truth or creation of beauty, is not under anybody's possession and no one is prevented from pursuing truth or creating beauty if others are involved in it. There is something permanent about the object of these theoretical activities.

Theoretical activities can also be differentiated from other activities in respect of the opportunities they provide for skill and discrimination. Card games or football have a conventional objective which can be attained in many ways. But, says Peters, "truth is not an object that can be attained; it is an aegis under which there must always be progressive development." So there must be opportunities for "fresh discrimination and judgement and for the development of further skills". (Peters, 1966, p. 158)

A third consideration is to do with the cognitive nature of the activities. Knowledge can be involved in games and pastimes, but this is limited to the end of the activity. One can be knowledgeable of the rules of bridge but the purpose is to compete and win at the game. Theoretical activities, on the other hand, have a wide ranging cognitive content. In science and literature there is a huge amount to know and that knowledge contributes to how one views other things. So while they may be like games in being disinterested pursuits, sometimes pursued for intrinsic values, they are given a value that is not given to mere games or pastimes:

> They are "serious" and cannot be considered merely as if they were particularly delectable pastimes, because they consist largely in the explanation, assessment, and illumination of the different facets of life. They thus insensibly change a man's view of the world.
>
> (ibid., p. 160)

The problem now to be considered is why, when answering seriously the question "Why do this rather than that?", would someone choose those activities that are "serious" or "theoretical"? Merely establishing that certain activities are fundamentally different from others does not explain why some of them are more worthy of pursuit.

Peters claims firstly that this question can only be seriously asked by people who have some conception of what the different choices are and that this "... has been formed in the main by the differentiated forms of understanding that have been developed" (ibid., p. 161). Thus, the very activities that have been differentiated as having wide-ranging cognitive content are the ones that are necessary to <u>answer</u> the question "Why do this rather than that?".

Secondly, Peters' "serious" activities can be distinguished from other activities by their concern with truth. They are concerned with truth just as the person who asks the question "Why do this rather than that?" is concerned with the truth. It is argued that these "serious" activities, as well as being necessary in answering the question "Why do this rather than that?", are also involved in asking it.

For Peters, truth and rationality are among the ultimate human values, and so he is led to a justification for "serious" activities based on a view of the nature of man as well as the characteristics that determine these "serious" activities.

In assessing Peters' arguments Degenhardt (1982) indicates that some people can engage in "serious" activities for reasons other than because the activities are deemed to be about truth and rationality. They pursue them because in some way they find them interesting and important. People do not just decide to do something because it is about truth and rationality. Instrumental reasons aside, they decide to pursue certain activities because somehow those activities help them in solving particular problems that they consider troublesome but important in their lives. They give the person different ways of viewing problems that that person feels important to consider. For Degenhardt the question is why do people find some problems in life important to consider and why are some activities helpful in giving people answers to those important problems? He says:

> Yet part of the point of Peters' insistence on the seriousness of serious activities is that they are not just pleasing embellishments added to life, but are somehow part of what life is, or ought to be, all about. We need, it seems, to refer to more than knowledge and rationality to work out why this should be.
> (Degenhardt, 1982, p. 60)

<u>World Views and the Value of Knowledge</u>

In the preceding sections we have considered the ways that various thinkers have attempted to give some sort of worth to knowledge and what implications their ideas might have for the curriculum. In each case we have found that there are serious objections. In this section we present the ideas of M.A.B. Degenhardt who, in his book <u>Education and the Value of Knowledge</u> argues that we are wrong to regard knowledge as being either instrumentally useful or as an end in itself. He argues that this overlooks a third possibility, a way of valuing some knowledge that is related to a distinct view about the nature of man.

Degenhardt tackles the question of what constitutes a worthwhile curriculum by considering three ideas. Firstly, he rejects the dichotomy between knowledge as a means to an end and knowledge as an end in itself. A third possibility, he claims, is that some knowledge is valuable because it helps us to determine our ends. Secondly, the view of man as a free agent in the world enables him to decide what ends he sets himself and these ends are best determined by first acquiring a world view; that is, having some understanding of the nature of the world and the nature of man in that world. Such a view, he claims, should not be generated individually but should be socially inherited. So, thirdly, he argues for the great evolved bodies of knowledge to be central to the content of the curriculum, in that, as they have evolved across generations and cultures, they have become "more rigorous and self-critical, less parochial, and much enriched from the achievements of many thinkers" (Degenhardt, 1982, p. 89).

In support of his argument for the rejection of the dichotomy between knowledge as a means to an end and knowledge as an end in itself, he lists several ambiguities. Firstly, when one talks of knowledge being good in itself, it is not clear whether one is talking of the good in possessing the knowledge or the good in pursuing it. While pursuing knowledge may be worthwhile under some criteria, it should not be confused with the value inherent in possessing that knowledge.

Secondly, Degenhardt considers the ambiguity between the intrinsic worth attached to an individual person possessing knowledge and the intrinsic worth of the total knowledge possessed by humans existing and growing.

A third ambiguity concerns the claim that an introduction into the various forms of knowledge nurtures those qualities of mind that are valued. For example, an introduction into mathematics, it might be argued, develops sound deductive reasoning. But it is not clear where we can separate the qualities of mind from the subject. That is, to be able to engage in sound deductive reasoning, it might be claimed, is to be able to do mathematics and does not follow from it.

A fourth ambiguity concerns the different ways in which knowledge can be pursued for ends that are distinct from that knowledge. For example, a mathematician who engages in mathematics in order to solve practical problems involved with the construction of bridges may be said to be less concerned with mathematical knowledge as such, than someone who engages in mathematics in order to arrive at hitherto unknown solutions to mathematical problems. On the other hand, the first mathematician may be said to be more concerned with mathematical knowledge than someone who studies the subject simply to acquire qualifications to enhance his job promotion prospects.

Finally, Degenhardt points out that it is absurd to think that all knowledge can be thought of as intrinsically good. There is much pointless data, the lack of knowledge of which would not seriously disadvantage anyone.

Therefore, the claim is that inherent worth cannot be attributed to all knowledge or any knowledge, but only to bodies of "serious" or "significant" knowledge. This seriousness puts knowledge into a third value category:

> It is not valuable as an end in itself, for it is serious or significant in so far as it makes a difference to how one lives. But neither is it useful, for it is not knowledge that is to be used to some further end. Rather, it is the kind of knowledge that helps us to determine our ends. By this I mean that it gives us that picture or understanding of things in terms of which we can decide what to do with our lives, what aims to set ourselves, what ends to live for.
>
> (ibid., p. 85)

So while he gives value to some knowledge, he is also tying this value to a particular view of the nature of man as a free agent. That is, man is able to make decisions for himself about how he will conduct his life and what ends he will strive for. Such ends are determined after one has acquired a world view; an understanding of man, his world, and the universe. The acquisition of such a world view cannot be done individually, but is done as ideas are socially inherited through education, both planned and unplanned. And how one is to behave in that world, and what ends are to be strived for, can only be done in the light of the culture that has been passed on. Thus, says Degenhardt:

> Given this, it must surely follow that we should educate human beings into such a cultural inheritance as will best fit them for the distinctively human enterprise of working out what sorts of human beings they are to make of themselves.
>
> (ibid., p. 88)

It is, therefore, the traditional bodies of knowledge that have educational importance because they help man reflect on questions concerning the nature and meaning of life. And as they have evolved across generations and cultures with the contributions of a great many thinkers, they offer the best that can be given in allowing one to develop a world view and to determine one's ends. The argument is then for a curriculum that offers the evolved bodies of knowledge, not just as technical disciplines designed for instrumental usefulness, but as a means to reflect on the achievements of other thinkers, and in answering questions about man, his world, and the universe.

One criticism of this argument might be that it is too idealistic. To say that human beings ought to be educated into a cultural inheritance that will best enable them to work out what they are to do with their lives, is like claiming that everyone ought to be free from hunger; people will agree in principle but doubt that it is possible. Some people may not be in any position to determine their own ends, irrespective of the knowledge they have acquired, so the curriculum ought to be based on preparing people for the life they will lead rather than the life they decide to lead. A student might find himself in a position where he has determined what he would like to do with his life but is unable to follow that course. Ought he not, therefore, be in the best position to seek fulfilment in the life that has been determined for him?

In reply, there can be no doubt that many students will have aspirations, determined partly by the influence of schooling and studying particular subjects, that will not achieve fruition. The view that man ought to be educated to be in the best position to determine his own ends is based on a particular notion of human nature; namely, the ability of the human being to act freely on the world and where choice is inevitable; but only within the limits imposed by society. So the human being who has determined his own ends but is unable to follow that path ought to be able to see why he is unable to do so by understanding the constraints that are imposed upon him. Someone who is in the <u>best</u> position to determine his own ends could only be said to be in the best position if those ends are possible within the limits imposed.

A second criticism of Degenhardt's position might be that the programme is not practical. Given the different psychological make-up of students, are there methods of instruction that will enable them to understand and reflect on the different disciplines in the same way; a way that best fits all of them to determine their own ends? And, if not, how are we to cater for individual differences? Also, does the argument suggest that there are two distinct ways of looking at the disciplines? One can achieve technical mastery in a subject, like mathematics say, without any understanding of the nature of mathematics or its cultural significances. But is it clear that one could have such an understanding of the subject without first coming to master its technical side? Can one fully appreciate the effect that the calculus has had on society since the 1600s, for example, without first understanding the mathematical concepts involved in differentiation and integration?

Clearly, to begin to answer these sorts of questions, one is going to have to look closely at the specific disciplines; firstly, from the philosophical perspective, to elucidate their nature and foundational concepts; and, secondly, from a cultural perspective, to determine the influence that the discipline has had on society and the forces within society that have influenced the growth of the discipline. Only then will one be able to argue for or against the practicability of the programme for the various disciplines.

The next chapter is concerned then with a critical examination of various theories on the nature of mathematics and the way forces within society have influenced the evolution of mathematical concepts.

CHAPTER III

THE NATURE OF MATHEMATICS

The aim of this chapter is to show the development of a theory about mathematical knowledge which breaks with traditional thinking about the nature of mathematics. Dominant philosophies have nearly all stressed the *a priori* nature of mathematics. Mathematical knowledge is regarded as different in kind from scientific knowledge in that it can be obtained without the use of the senses. This apriorist view has been the basis of the traditional schools of thought regarding the nature and foundations of mathematics. These traditional philosophies have all been disputed at times but alternative philosophies have not been fully articulated. P. Kitcher (1983) has now developed a theory of mathematical knowledge which rejects mathematical apriorism.

To show the development of Kitcher's theory this chapter starts with a consideration of the older views of Plato and Aristotle and the 19th century views of Leibniz and Kant which anticipated the three dominant *a priori* philosophies of mathematics in the 20th century; namely, formalism, intuitionism and logicism. J.S. Mill's 19th century empiricist view of mathematics is also considered here.

The three dominant philosophies are then examined in some detail and it is concluded that while they do give some insight into the activities of mathematicians, they are not adequate in their description of the nature of mathematical knowledge. It is argued that this is because they view mathematics as something that is unchanging with time, whereas a consideration of historical episodes suggests that mathematics is in a process of evolution, and that the mathematical knowledge we have today has evolved in response to practical problems within different cultures and with the need to generalize and make rigorous the symbolic mathematical language that is being used.

It is shown how Kitcher's comparison of mathematical change with theories of scientific change, and his re-assessment of Mill's earlier empiricist view of mathematics, provide the basis for a theory of mathematical knowledge that accounts for its evolution from basic manipulations in the environment to the mathematics that we have today. For Kitcher, mathematics is a theory about the possibilities that exist in the physical world.

Finally, the last section of this chapter considers the example of the calculus from 1650 to 1900 and illustrates its development in line with Kitcher's theory of mathematical change.

Earlier Views

(i) Plato held that it was an intellectual task of man to distinguish appearance from reality. The appearance of the world around him, gained through sense experience, was ever changing, whereas reality, which could not be apprehended by the senses, was unchanging. This view was articulated by Plato in what is called his Theory of Forms and originated out of certain general ideas that featured in dialectical disputes. In any disputation one ultimately must make clear the concepts involved. When there is argument over whether, say, honesty is a virtue, we are dealing with concepts of honesty and virtuousness which have to be made clear. Similarly, when we talk of someone's honesty improving, we are comparing that person's standard of honesty to some ideal standard which is regarded as unchangeable through time. Our understanding of these ideal standards is not seemingly dependent upon our senses. When we observe phenomena we might readily agree that if our eyesight was sharper we would see things clearer and have a better knowledge of them, but general notions of honesty and virtuousness are not seemingly apprehended by the senses, and when we attain certainties about them, even if only negative ones, we do so by argument.

Through such considerations, and particularly with his mathematical orientation, Plato was led to develop the Theory of Forms. Geometrical truths about triangles were not thought of as just truths of particular triangles drawn on paper or in the sand, but as truths of all possible triangles. Geometry and arithmetic were regarded as studies of certain realities that do not have the imprecision of things that occur in the everyday world. Plato noted that dialectical disputes were also concerned with concepts that have only imperfect representations in the everyday world. So that to argue that honesty is a virtue is to argue about the concepts of honesty and virtuousness that do exist, but not in the everyday world. Whereas our everyday world contains examples of triangularity, honesty and virtuousness, the Forms of triangularity, honesty and virtuousness exist permanently and independently of man, and in a world that is not apprehended by the senses.

Some of the Forms became the domain of mathematicians. Oneness, twoness, point, line, circle, for example, are mathematical Forms and dots and marks drawn on paper are only approximations to these Forms. Not only that, there are also relationships between the Forms, and it is the job of the mathematician to discover them, just as others may seek to discover relationships between objects in the physical world. Instead of relying on his senses, however, the mathematician relies on his reason.

This view of mathematics appeals to some mathematicians, as Kőrner says:

> ... Platonism is a natural philosophical inclination of mathematicians, in particular those who think of themselves as the discoverers of new truths rather than of new ways of putting old ones or as making explicit logical consequences that were already implicit.
>
> (Kőrner, 1960, p. 15)

The proposition, that one plus two equals three, states a relationship between the Forms of "oneness", "twoness" and "threeness", and is true independent of anything we can sense in the physical world. By reason the mathematician can discover this truth of mathematics. Similarly, the proposition, that any two straight lines which are not parallel, intersect at one point, states a relationship between the Forms of "line" and "point" and no physical demonstration is needed to judge the truth of this proposition.

It is important to note that Plato did not idealize his Forms from the physical world and sense experiences. He did not, for example, idealize the Form "circle" from the many instances of circularity that he sensed in the physical world. The Form "circle" does exist, is permanent and is not apprehended by the senses. All empirical examples of circularity are only approximations to this Form.

(ii) Unlike Plato, Aristotle's philosophy stemmed from a biological orientation where he looked at different life forms and asked what the function of them was. For him, what distinguished man from other life forms was man's rationality, and what was good for man was exercising his reason in the pursuit of knowledge.

Aristotle rejected Plato's distinction between the world of physical objects and the world of ideal Forms. The subject matter of mathematics is not ideal Forms that exist independently from objects in the everyday world, rather the subject matter is what can be abstracted from what we perceive in the world. For Aristotle, the form or essence of an object is as much a part of it as its physical matter. The essence of "circularity" does not exist independently from circular objects but can only be abstracted by man from examples of circular objects. The distinction between mathematical and physical definitions can be distinguished by the example of "curve", which specifies no matter, and "snub", which specifies the curved matter, a nose. The mathematical definition "curve" is abstracted from the physical definition "snub". The subject matter of mathematics is then the result of such abstractions and these mathematical objects are, in some sense, in the things from which they are abstracted.

This notion of abstraction from physical objects avoids one criticism of Platonism; namely, if there is an ideal Form of threeness say, then what is the status of "three" when it occurs twice in the proposition "three plus three equals six"? For Aristotle this is no problem, as the abstracted mathematical object "three" can occur as many times as required. The work of the mathematician then is to idealize the relationships between mathematical objects, these objects being abstracted from the physical world.

An example of the importance of Aristotle's thought to later views on mathematics can be demonstrated by considering his ideas on infinity. The notion of infinity has caused considerable difficulties in much recent work on the foundations of mathematics.

Aristotle distinguished between two notions of the infinite, the actual and the potential. If we consider the sequence of natural numbers 1, 2, 3, ... and the possibility of always obtaining the next member in the sequence and of proceeding as far as we want to, then we have the notion of the potential infinite. We never obtain a complete sequence of all the natural numbers, but we are not stopped from going as far as we like. This, however, is in contrast to the notion of the actual infinite, where the natural numbers are deemed to be given in totality. Under this notion there exists a set, the elements of which are all the natural numbers. Aristotle favoured the use of the potential infinite and much of modern mathematics needs only this notion of infinity. The use of the notion of actual infinity, however, produces many antimonies (paradoxes). An example will be considered in the section on logicism, later in this chapter.

(iii) Unlike Plato and Aristotle, Leibniz does not take mathematical propositions to be about anything, neither mind-independent eternal objects nor abstractions from the physical world. He maintains that mathematical statements are true by virtue of the fact that their denial would be impossible.

He identifies two kinds of truths, those of reasoning and those of fact. Truths of reasoning are necessarily true by the impossibility of their denial. The denial of truths of fact, however, is possible. Their truth is contingent.

Consider two examples. The proposition that, if A is greater than B and B is greater than C, then A is greater than C; is a truth of reasoning. It would be impossible for A not to be greater than C under these constraints. But the proposition that all metals expand on heating, is a truth of fact and its denial is possible. It's just that no metal is known not to expand on heating. Leibniz thus regards the truths of mathematics akin to the truths of logic and, in this sense, he foreshadows the modern movement of logicism, which maintains that all mathematics is reducible to logic.

The relationship between pure and applied mathematics is tied up in what for Leibniz is "the best of all possible worlds". As a proposition in pure mathematics, "One plus one equals two" is true for its denial is impossible. The proposition "One apple plus one apple makes two apples" is true in this world, for anything else would not be true in the best of all possible worlds that could be created.

(iv) While Kant rejects Leibniz's dichotomy of propositions between those of reasoning and those of fact, he is concerned about the different ways of knowing. In his book Critique of Pure Reason he says:

> It is therefore a question which requires close investigation, and is not answered at first sight - whether there exists a knowledge altogether independent of experience, and even of all sensuous impressions. Knowledge of this kind is called a priori, in contradistinction to empirical knowledge, which has its sources a posteriori, that is, in experience.
> (Kant; Trans. by Meiklejohn; 1964, p. 25)

Kant then develops a three way classification of propositions.

(a) Some propositions he describes as being analytic in that their denial is self-contradictory. The truth of these propositions can be shown by analysing the terms and concepts involved in the propositions. An example is the proposition "All bachelors are unmarried". Nothing, other than the meanings of the terms involved in the proposition, is needed to judge its truth. These propositions correspond to Leibniz's propositions of reasoning. For Leibniz, all pure mathematical propositions are of this form.

(b) Kant then describes some propositions as being synthetic _a posteriori_. That is, they do describe a state of affairs in the physical world and their truth is judged by sense perceptions. An example of this type of proposition is "My pen is blue". The denial of this proposition is not self-contradictory and the truth of it is judged by using the senses.

(c) Finally, Kant considers some propositions as being synthetic _a priori_. These propositions describe a state of affairs in the physical world but they are not deemed true by use of the senses, but by reasoning. They are necessary conditions for the possibility of objective experience. That is, they are necessary in that if any proposition about the physical world is true they too must be true.

Kant was concerned about synthetic _a priori_ judgements because he believed that we make these types of judgements in physics and metaphysics as well as in mathematics. The proposition "In all changes of the material world the quantity of matter remains unchanged" is deemed to be synthetic, in telling us something about the physical world, and _a priori_, in that we make this judgement before experiencing every change. So too the proposition "All men are free to choose" is deemed to be synthetic because it gives us new knowledge about all men, and _a priori_, in that we make the judgement before experiencing all men.

Yet there is still the doubt of how we can make judgements about the state of affairs of the physical world without first experiencing that world. To solve this problem Kant hypothesized a new relationship between the mind and its objects. He did not regard the mind as passively receiving information from the objects. He regarded the mind as active and doing something with the objects it experiences, so that the mind imposes its way of knowing upon the objects. Thinking involves not only receiving impressions through the senses but also making judgements about what is experienced. The mind has the power to make judgements without first experiencing the world.

In describing the propositions of pure mathematics as being synthetic a priori, Kant introduces another classification. He distinguishes between discursive synthetic a priori propositions, which give an ordering of notions (for example, causality), and intuitive synthetic a priori propositions, which are concerned with the structure of perceptions. To this latter group, he claims, belong the propositions of pure mathematics.

Kant's argument can then be summarized as follows. Being in space and time is a necessary condition for the possibility of perception. The subject matter of pure mathematics is the structure of space and time free from empirical material. The propositions of pure mathematics are structures of perception, synthetic in describing space and time, but a priori in describing the unchanging nature of space and time and in not requiring any sense experiences to judge their truth. For Kant the mathematical proposition "Two plus three equals five" is synthetic and a priori. The logical possibility of alternatives is not denied (and, therefore, the proposition is not analytic), but any other alternative would not be a description of perceptual space and time.

But Kant's philosophy of mathematics went further than simply describing space and time. He was concerned about the possibilities that exist in space and time and the distinction between *a priori* constructions and postulations. The concept of a ten dimensional sphere, for example, can be postulated and a geometry of ten dimensions can be developed and shown to be be self-consistent. That is, axioms and propositions can be formulated leading to results which do not contain self-contradictions. The *a priori* construction of a ten dimensional sphere, however, is not possible, whereas the *a priori* construction, and not mere postulation, of a perfect three dimensional sphere is possible.

The subject matter of pure mathematics then becomes the structure of space and time and the possibilities of constructions within it. The subject matter of applied mathematics becomes the structure of space and time and the actual material filling it.

(v) In contrast to Kant's *a priori* nature of mathematics, John Stuart Mill argues that all our knowledge is empirical. To the question, "Are synthetic *a priori* judgements possible?" he answers in the negative. Firstly, he rejects the abstract notion attributed to numbers:

> All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse.
> (Mill, 1973, p. 254)

And just as numbers refer to things we experience the basic axioms of mathematics are not necessary truths but laws we accept, based on our experience:

> That things equal to the same thing are equal to one another, and that two straight lines which have once intersected one another continue to diverge, are inductive truths; resting, ... on the fact that they have been perpetually perceived to be true, and never once found to be false.
>
> (ibid., p. 609)

He thus maintained that the axioms of mathematics were inductive generalizations based on a large number of instances. In this sense they were the same as scientific hypotheses, the difference being one of degree and not kind. The subject matter of mathematics is more general than any other science and its propositions have been tested for many more times than the propositions in other sciences. But, according to Mill, we are unjustified in thinking that mathematical propositions are, therefore, qualitatively different from the hypotheses of other sciences.

Mill's philosophy of mathematics has been attacked from many quarters. Principally, the attacks came at a time when the three so-called traditionalist schools of thought on the foundations of mathematics flourished in the late 19th century and into the 20th century. These three _a priori_ philosophical positions; formalism, intuitionism, and logicism, will be examined later in the chapter. In a paper on the foundations of arithmetic, first published in 1884, G. Frege maintained that Mill did not distinguish between mathematical propositions and the use to which they could be put:

> Mill always confuses the applications that can be made of an arithmetical proposition, which often are physical and do presuppose observed facts, with the pure mathematical proposition itself.
>
> (Frege, 1968, p. 13)

And A.J. Ayer (1975), making use of Kant's dictum that though all our knowledge begins with experience, this does not mean that it all arises out of experience, claims that Mill fails to distinguish between knowing mathematical truths and coming to know them:

> We may come to discover them through an inductive process; but once we have apprehended them we see that they are necessarily true, that they hold good for every conceivable instance.
>
> (Ayer, 1975, p. 318)

He argues that we will never find an example to refute mathematical axioms because they are true by definition. They are analytic statements or tautologies.

Some philosophers have been more sympathetic to Mill's position, however. W.V. Quine says that perhaps in Mill's time classical mathematics did lie closer to experience than it does now, noting that the infinitistic reaches of set theory, which are so remote from our experiences, were not explored then (see Quine, in Benacerraf and Putnam (eds.), 1983, p. 355). In a cautious note Quine says that it is the relationship between mathematics and the empirical sciences that is important:

> I am concerned to urge the empirical character of logic and mathematics no more than the unempirical character of theoretical physics; it is rather their kinship that I am urging, and a doctrine of gradualism.
>
> (Quine, 1970, p. 100)

Mill's philosophy is an important consideration then because it is the rejection of his empiricist position and the acceptance of the a priori nature of mathematics that saw the flourishing of the three traditionalist schools of thought on the foundations of mathematics. But, in addition, when these three philosophical positions were found to have serious objections to them, some modern day philosophers sought to re-examine Mill's work. (See, for example, H.Lehman(1979) and P. Kitcher (1983)). They re-assessed Mill's position by highlighting the difficulties of using language to convey meanings, and recently Kitcher (1983) has rejected the a priori nature of mathematics and developed his own "defensible" empiricist philosophy of mathematics. Before considering Kitcher's arguments, however, we will first look at the three traditionalist theories.

Dominant 20th century views

During the first half of this century there were three main schools of thought regarding the nature and foundations of mathematics; namely, formalism, intuitionism, and logicism. This was not to say that all theorists ascribed to one of these theories but, rather, it represented a classification of the different ideas of those who worked in the area of the philosophy of mathematics. These ideas had their origins in the earlier work of Leibniz and Kant and, to a lesser extent, Plato and Aristotle. Formalists and intuitionists acknowledged the influence of Kant's philosophy of mathematics while rejecting that of Leibniz, but formalism and intuitionism subsequently evolved to differ in quite important ways. Logicists, on the other hand, were influenced by the tradition of Leibniz in regarding mathematical propositions as analytic, and demonstrating their truth by applying the principles of logic.

It must be noted that a strict three-way classification oversimplifies what is a very complex area of study. Each class has many sub-varieties and different writers in each class often disagree with one another. It is possible, however, to indicate and critically examine the main features of the three theories.

Formalism

Strict formalism is the view that mathematics is the formal (that is, rule-governed) manipulation of symbols and nothing else. Mathematics then consists of a list of terms, a list of operations which are modes of combination for forming a new term in the list from a set of given terms in the list, certain elementary propositions (axioms) which are stated to be true unconditionally, and rules of procedure for the derivation of further propositions from the axioms. D. Hilbert is regarded as the founder of the formalist movement, and developed the view in the course of research into the theorems and axioms of Euclidean geometry. The formalist system was first used by him in the paper "The foundations of mathematics" published in 1928, but he is not regarded as a strict formalist, holding that the finite combinatorial part of mathematics is meaningful and true (see Hilbert, in Benacerraf and Putnam (eds.), 1983, p. 183).

For example, he would maintain that the proposition "1 + 2 = 3" is within a formal system but that it does have meaning and is true outside that system. Accounts of the strict formalist position, which denies that any mathematical statement has a truth value, can be found in H. Curry (1951) and A. Robinson (1965).

Strict formalism rejects the idea that mathematics is about mind-independent eternal objects, and it rejects the view that it is about constructions in the mind. To the strict formalist there is no subject matter to mathematics at all, it is simply a series of manipulations of symbols. The theorems in mathematics are developed by applying the axioms to the list of terms and the list of operations. In plane geometry, for instance, we have the terms "point" and "straight line" and the axiom "Through any two points there exists exactly one straight line". But we might equally have defined the terms "glm" and "gam blyp" and the axiom "Through any two glms there exists exactly one gam blyp". Irrespective of the mental image engendered, what is meaningful is to apply the given axioms to the given terms in the correct way.

But while it may appear that mathematicians merely manipulate symbols according to pre-assigned rules, there are objections that do not allow us to accept this as an adequate account of the nature of mathematics.

Firstly, one requirement of any formal system must be that the system is consistent. This means that the system cannot allow a proposition "p" and its negation "not p" to be derived within the system, thus asserting both "p" and "not p" to be true in the system. But in 1931 K. Gödel proved that a specific contradiction can always be deduced from any proof of the impossibility of the occurrence of contradictions in a formal system. (Discussions of Gödel's paper can be found in R. Wilder (1965), S. Körner (1960) and M. Black (1933)).

Secondly, for someone who has done any mathematics at all, it is not simply an arbitrary manipulation of symbols. When grappling with a mathematical problem one is not simply dealing with symbols, but ideas and constructions in the mind. Such constructions may eventually be symbolized but they are talked about and discussed in ways which suggest they are more than simply symbols used by mathematicians. To the mathematician such mental constructions are very real. Two distinct proofs of a theorem may use quite different symbols while still embodying the same ideas or mental constructions. The mathematician can see beyond the symbols and can give meaning to the ideas represented by the symbols.

A further criticism of formalism is that if mathematics is just a game played with symbols, why is it so useful in predicting outcomes of events in the physical world? And why do we choose some axioms and not others? If it is sheer arbitrariness then there is a difficulty in explaining how formulae such as $v = u + at$ do approximate to such a degree the empirical result of the velocity of an object with a certain acceleration after a certain time. The usefulness of mathematics suggests that something other than an arbitrary collection of terms and axioms goes into it.

Likewise the nature of inference in mathematical systems needs to be explained. It seems clear that the signs we use, such as $\sim$ (negation) and = (equality), have meaning outside the formal system of mathematics. We accept the inference that if $a = b$ then $a + 1 = b + 1$, but we would not accept the inference that if $a = b$ then $a + 1 = b + 2$; whereas, presumably, such an axiom could occur in some formalised system. It has to be explained why some axioms appear to have meaning and are useful while others appear to have no meaning at all.

Formalism is rejected then as an adequate philosophy of mathematics in that it offers no explanation as to the usefulness of mathematics, and in that it denies the existence of mental constructions, which are not formal and not symbolic, but do have structure and, to the mathematician, are real.

Intuitionism

The intuitionist's view of mathematics is that it consists solely of mental (intuitive) constructions. Mathematics is thus a production of the human mind. The existence of mathematical objects can only be guaranteed if they can be determined by thought, and the properties of mathematical objects are only properties if they can be discerned by thought. The symbolic mathematical language that is used is simply a device for communicating thoughts and allowing oneself, or others, to follow mathematical ideas. The fundamental tenets of intuitionism were first formulated by L.E.J. Brouwer, following his inaugural address at the University of Amsterdam in 1912 (see Brouwer, 1913). More recent introductions to the intuitionist philosophy of mathematics are by A. Heyting (1956) and M. Dummett (1977).

Like the formalist, the intuitionist does not accept that mathematics is about mind-independent eternal objects. But unlike the formalist, the intuitionist recognizes the ability of the person to perform certain constructions in the mind. Initially this consists of the construction of unity and then the series of natural numbers. All mathematics is then built upon these initial constructions. And to quote Brouwer:

> ... neither the ordinary language nor any symbolic language can have any other role than that of serving as a nonmathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same set.
> (Brouwer, in Benacerraf and Putnam (eds), 1983, p. 81)

So the only mathematics that is done is done by a series of constructions in the mind. The logic engendered by the manipulation of symbols is a product of mathematics, and with ordinary language it is the means of communicating ideas so that others may effect the same mental constructions.

The mathematical logic of intuitionism is different from classical logic. In classical logic the existence of an entity can be proved by showing that the assumption of its non-existence leads to a contradiction. In intuitionistic logic the entity whose existence is to be proved must be shown to be constructible. Consider an intuitionistic proposition P as the record of a construction:

P : I have effected a construction A in my mind.

The intuitionistic negation $\neg P$ is also a construction. It is not saying that I have not constructed P; rather:

$\neg P$: I have effected a construction B in my mind which deduces a contradiction in that the construction A is brought to an end.

What is important is that they are both constructions and it need not be the case that one of them has occurred. Thus the intuitionist does not accept that all propositions can be characterized as being true or not true.

Consider an example. π is the non-recurring decimal 3.14159.... We calculate $\hat{\pi}$ in the following way. Expand π until we have a sequence of 100 successive zeros. Suppose the first run of 100 successive zeros starts in the nth digit. If n is odd, let $\hat{\pi}$ terminate in its nth digit. If n is even, let $\hat{\pi}$ have a 1 in the (n + 1)st digit and then terminate. So if n is odd then $\hat{\pi}$ is less than π. If n is even then $\hat{\pi}$ is greater than π. If no successive 100 zeros ever occurs then $\hat{\pi} = \pi$. Now let $Q = \hat{\pi} - \pi$. The question is whether Q is positive, negative or zero. Now Q is a real number, and the law of the excluded middle maintains that it must be positive negative or zero. But the intuitionist does not accept this. $\hat{\pi}$ and Q have not been constructed in the mind since it is not known if there is a sequence of 100 successive zeros in the expansion of π. To the intuitionist, the proposition:

P : Q is positive

is not true, nor false but meaningless.

The proponents of intuitionism saw its success in removing many of the paradoxes that so plagued the early infinite set theorists in mathematics. Bertrand Russell defined a set as abnormal if it contained itself as one of its elements and normal if it did not. As an example of an abnormal set consider "The set of all objects describable in exactly eleven English words". Russell then considered the set N of all normal sets. Is N abnormal or normal? If N is normal then it is one of the set of all normal sets and hence is an element of N. But this means N is abnormal. Conversely, if N is abnormal then it is one of its own elements, which are all normal sets. Hence N is normal.

The intuitionist removes the paradox by claiming that if he cannot construct in his mind a set that has itself as a member, then talk of them is meaningless and is not a part of mathematics. The beauty of the intuitionist programme then is that it does not produce ideas and concepts that the mind cannot accept.

There are criticisms, however, which suggest that intuitionism is not an adequate philosophy of mathematics. In describing mathematical activity as intuitive constructions the intuitionist denies that the inner experience refers to any external reality. But in so doing the mathematician then gives up the most powerful motivation for his work - to seek truth that can be publicly validated. A mathematician is not interested in intuitive constructions for their own sake but for the new truths they enable him to find. As Goodman says:

> Just as the constructions lie behind the symbols and give them their interest and meaning, so there is something behind the constructions - mathematical truth.
> (Goodman, 1979, p. 545)

The mathematician is not free to take any arbitrary set of rules and apply them to his mental constructions. Mathematical rigour is a restriction on that freedom, and mathematical truth does not exist in the mind of the mathematician. When we evaluate a mathematical argument we determine whether the argument works - that is, whether it convinces us of the truth of its conclusion.

One critic of the intuitionist programme is L. Wittgenstein. He seeks to explain the nature of mathematical truth through the collective behaviour of the people who use the rules, and as originating from simple manipulations of objects in the environment. Firstly, he argues that understanding mathematics is not just a mental state but an ability to do something; namely, an ability to apply what the person claims to know. This overt behaviour is necessary before claiming understanding in mathematics. Wittgenstein says:

> We are trying to get hold of the mental process of understanding which seems to be hidden behind the coarser and therefore more readily visible accompaniments [the overt behaviours]. But we do not succeed; or rather, it does not get as far as a real attempt. For even supposing I had found something that happened in all those cases of understanding, - why should <u>it</u> be the understanding?
> (Wittgenstein, 1978, p. 60)

Secondly, Wittgenstein argues that the process of inference is something that need not happen "in the head".

Inferring consists in the transition from one assertion to another, but:

> Misled by the special use of the verb 'infer' we readily imagine that inferring is a peculiar activity, a process in the medium of the understanding, as it were a brewing of the vapour out of which the deduction arises. But let's look at what happens here. - There is a transition from one proposition to another *via* other propositions, ... This may go on on paper, orally, or 'in the head'. - The conclusion may however also be drawn in such a way that the one proposition is uttered after the other, without any such process; ...
> (Wittgenstein, 1967, p. 5)

Wittgenstein then goes on to argue that the mental constructions cannot serve as a foundation for mathematical inference because they cannot give us a way of deciding whether the inferences are correct or not. When multiplying, for example, different people may have different mental constructions but "the correct multiplication is the pattern of the way we all work" (ibid., p. 95). And "... 'calculating right' does not mean calculating with a clear understanding or smoothly; it means calculating *like this*" (ibid., p. 180). So the criteria of what is right or wrong are established on the basis of the collective behaviour of those who use the rules. So correct inference is simply the way we all do in fact infer. "This is use and custom among us, or a fact of our natural history" (ibid., p. 20).

The problem is, however, that there is never going to be perfect agreement on how to infer and if people disagree and infer differently there needs to be a way of deciding which process of inference is correct. Wittgenstein's answer is that the rules of inference we accept are not just arbitrary but are influenced by empirical conditions. The origins of our mathematical practices are found in the simple processes of counting objects and comparing sets of objects in the environment. These simple processes provide the genesis of mathematical truth.

This then leads to a further objection to intuitionism; that it has nothing to say about the growth of mathematics and about cultural significances. If mathematics is simply constructs of the mind, what is the nature of the higher mathematics that has developed today and that did not exist 100 or 1000 years ago? Accounts of historical episodes in mathematics indicate that for certain periods of time mathematics develops cumulatively. The theory of the calculus today can be seen to have developed from the initial work of Newton and Leibniz in the 1600s, even though some of the symbolic language that we use today would have been unknown to them. The language involved with limits, for example, was developed following the problems Newton, Leibniz and their followers had dealing with "infinitesimals". This symbolic language is regarded as essential for a full understanding of the calculus as we know it today. Any philosophy of mathematics should give an account of why this language has developed the way it has and is not simply an arbitrary system chosen to represent mental constructions. It must also explain why this language is regarded as a better system for expressing the mental constructions involved in the calculus than the one originally used.

Logicism

Logicism is the view that mathematics consists of certain truths and the arguments that establish those truths, of the formal manipulations of symbols that express those arguments and truths, and of nothing else. The logicist denies that there is any subject matter to which mathematical truths refer. They are simply true by their own internal structure; that is, they are analytic. If one had complete knowledge of logical propositions then one could deduce through logical means all the theorems in mathematics. Mathematics becomes one giant tautology.

Logicism has made a great contribution to mathematics, as a lot of mathematics as we know it is just logic, and logicism has given the impetus to simplify and unify basic mathematical notions. But rather than simply stating a belief that all mathematics is just logic, logicists, like G. Frege (1964, 1968), B. Russell (1903, 1919) and R. Carnap (1931), have attempted to demonstrate that a system of logic could, in fact, generate the theorems of mathematics. One would start with a list of fundamental logical laws and a list of permissible methods to deduce the truth or falsity of propositions. The symbolic expressions marking the first stages of the deduction would only be logical symbols. A system of logical concepts to be employed for the logicist's thesis has been given by Carnap (1931) and is summarized in the following tables.

Concepts from propositional calculus

Concept	Symbols	Read
The negation of a sentence p	$\sim p$	not p
The disjunction of two sentences	$p \vee q$	p or q
The conjunction of two sentences	$p . q$	p and q
Implication	$p \supset q$	if p then q

Concepts from functional calculus

Concept	Symbols	Read
Property f belongs to object a	$f(a)$	f of a
Property f belongs to every object	$(x) f(x)$	For every x, f of x
f belongs to at least one object	$(\exists x) f(x)$	There is an x such that f of x

Concept of identity

Concept	Symbols	Read
a and b are the names of the same object	$a = b$	a is b

Carnap then goes on to say:

> It is the logicist thesis, then, that the logical concepts just given suffice to define all mathematical concepts, that over and above them no specifically mathematical concepts are required for the construction of mathematics.
> (Carnap, in Benacerraf and Putnam (eds.), 1983, p.42)

If we accept the logicist's thesis, then some form of transition is needed to get from the logical symbolization to a more familiar notation such as 1 + 2 = 3, and for the logicist, this involves the use of definition. Russell uses definition as a notational device where one symbol stands for another symbol or combinations of symbols. His definition of the number 2, for example, is a case in point. Initially the symbol $2_m(f)$ is defined to mean that at least two objects fall under the concept f in the following way (where "$=_{Df}$" is read "means by definition"):

$$2_m(f) =_{Df} (\exists x)(\exists y)\left[\sim(x = y) .\ f(x).f(y)\right]$$

This is read: there is an x and there is a y such that x is not identical with y, and f belongs to x and f belongs to y. Similarly, we can define $3_m(f)$, $4_m(f)$ and so on. The number two is then defined:

$$2(f) =_{Df} 2_m(f) . \sim 3_m(f)$$

which reads: at least two, but not at least three, objects fall under f. In like manner all the natural numbers can be defined, as well as negative integers, fractions, real and complex numbers, and the operations of addition, subtraction, multiplication and division, and eventually the concepts in higher mathematics such as convergence, limit, differential, integral and so on.

The next part of the logicist's programme is to show that the theorems of mathematics can be derived from logical axioms through logical deductions. Thus, every sentence in mathematics, involving mathematical symbols, should be translated into a sentence containing logical symbols and should be proved in logic. But the logicist's programme raises serious objections that suggest that logicism is not an adequate philosophy of mathematics.

Firstly, the logicist claims that he is not discovering mathematical structures by proving their existence, but constructing them by definition. In discussing real numbers, for example, Carnap says of the logicist:

> .... through explicit definitions, he produces logical constructions that have, by virtue of these definitions, the usual properties of the real numbers.
> (ibid., p.44)

But what is not clear here is the relationship between the logical constructions and the "usual properties of the real numbers". How do we know the usual properties of the real numbers? Russell's definitions are mere notational devices, where one symbol stands for a combination of symbols. But when giving a definition two things stand out. Firstly, it is implied that what is defined is worthy of consideration. It would be possible to define all sorts of new concepts using different combinations of logical symbols, but why do we choose the ones we do, and why do we consider them important? Secondly, it is hard to envisage a situation where one would wish to define something that one did not have some prior idea about. The definition of the natural numbers is a case in point. The question should be asked as to whether it is possible to have no knowledge of what we mean by natural numbers, to read and understand the logicist's definition, and then to assert that one did now know what natural numbers were. We are still left with the belief that the logicist's definition does refer to something, some concept that we can grasp without any knowledge of the logicist's programme. Logicism does not explain the nature of these natural numbers, nor how we can acquire such knowledge about them.

The second objection to logicism concerns the view that all the theorems of mathematics can be derived from logical axioms through logical deductions. This view holds that every sentence in mathematics involving mathematical symbols can be translated into a sentence containing logical symbols and can be proved in logic. For the elementary theorems of arithmetic this is easily shown, but it has also been shown that some theorems in mathematics require special axioms known as the Axiom of Infinity and the Axiom of Choice.

The Axiom of Infinity states that to any class of n elements there exists a class of n + 1 elements. That means we can always add an element to a set that is not already contained in it. This then stipulates the existence of infinitely many elements, for we never reach the stage of not being able to add one more element to the class. The Axiom of Choice states that if α is any collection of sets $\{A,B,C,\ldots\}$ and no set in α is empty then there exists a set Z consisting of precisely one element from A, one element from B, and so on. This axiom is quite plausible if α is finite but since the axiom is stated for any collection of sets one must take on faith the possibility of forming such a set Z if α is infinite. Not only may α be infinite, it may be infinite and non-denumerable; that is, it may be incapable of being put in a one-to-one correspondence with the natural numbers.

The point is that some theorems of mathematics use these axioms for their proofs. While some philosophers accept them as principles of logic others do not. C. Hempel says:

> All the theorems of mathematics can be deduced from those definitions [of concepts of mathematics] by means of the principles of logic (including the axioms of infinity and choice).
> (Hempel, in Benacerraf and Putnam (eds), 1983, p.389)

But in a cautioning note he draws attention to the paradoxes that are found when the axioms of infinity and choice are included as principles of logic. And S. Körner says, in relation to Russell's definition of number:

> He not only <u>defines</u> every natural number <u>n</u> as <u>having</u> a unique successor <u>n + 1</u>, but has to <u>assume as a non-logical hypothesis</u> the axiom of infinity... The programme was to reduce mathematics to logic and not to logic <u>plus</u> non-logical hypotheses.
> (Kőrner, 1960, p.59)

To avoid the paradoxes, Russell introduced his Axiom of Reducibility and his Theory of Types, which did not allow elements of sets to be the sets themselves. But the final outcome was to base the foundations of mathematics not on logic, in the sense of the rules of correct reasoning, but on logic plus axioms which were needed in order to justify the notion of mathematics that we already had. In this sense it was a failure and drew Russell to despair:

> .... after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I

> could do in the way of making mathematical knowledge indubitable.
> (Russell, as quoted in Davis and Hersh, 1981, p.333)

Attacking logicism from another angle, S. Körner (1960) argues that it is a mistake to attempt to show that mathematics is reducible to logic by virtue of its logical character. His position is that mathematics and logic have the same structure and yet are two separate fields of study. They are two separate a priori disciplines. Whereas logic has no subject matter, mathematics does, though not in the same sense, say, as zoology. For mathematics the subject matter is obtained by postulation. One can postulate the existence of Euclidean points and lines, for example, and then derive results from the nature of these postulated entities. The nature of the derivation of these results parallels the nature of derivation of results in logic.

The Evolution of Mathematics

In the previous sections we looked at some of the ideas of those who attempted to answer the question "What is mathematics about?" We considered Plato's mind-independent Forms, Aristotle's abstractions, Kant's synthetic a priori knowledge, and Leibniz's claim that mathematics is not about anything at all. For the formalist it is the symbolic language and its associated rules that is mathematics, for the intuitionist it is the mental constructions, and for the logicist mathematics is just one giant tautology.

Finding serious objections to all these points of view we might then wonder whether it is possible to give an answer to the question "What is mathematics?" If we assume that mathematics is something absolute, unchanging with place and time, then we may believe that eventually we will be able to give a precise answer. But a consideration of historical and cultural factors suggests that the nature of mathematics is notso absolute. When we take mathematics to be one particular element of a culture at a particular time it is possible to get a clearer picture of the nature of that activity.

Now the activities of man are not dependent upon being characterized under a particular label such as "mathematics". It is in the nature of man to engage in particular activities and some of them are grouped together and assigned the name "mathematics" to distinguish between them and other activities within the culture. The activities are passed on from one generation to another and across cultures, and are greatly influenced by other cultural elements such as agriculture, warfare, philosophy, physics, astronomy and so on.

The work of the formalists, intuitionists and logicists all give some insight into the activities of mathematicians, but instead of trying to give a precise answer to the question "What is mathematics?" we should be seeking to explain how certain activities of man have become grouped together, how these activities have been passed on from culture to culture and generation to generation, and, in so doing, have evolved to what we call the mathematics of today.

In his book Proofs and Refutations published posthumously in 1976, Imre Lakatos sets out a dialogue between a teacher and a class of students who are discussing the Euler-Descartes formula for polyhedra:

$$V - E + F = 2$$

where V is the number of vertices of the polyhedron,

E is the number of edges,

F is the number of faces.

The teacher presents the proof of the formula, whereby the polyhedron is stretched out on a plane. The students follow up with a series of counter-examples and the proof of the formula is corrected and elaborated. The development so presented by Lakatos is seen by him to be a model for the development of mathematics in general. His argument is that the development does not consist of the accumulation of undeniable truths but consists of a series of conjectures and attempts made to prove them (by reducing them to other conjectures), or by attempts made to produce counter-examples. In this book and the paper "Infinite regress and the foundations of mathematics" (1962), he draws heavily on K. Popper's philosophy of science (see Popper, 1959, 1974). He admits to a theory of mathematical fallibility holding that mathematics is a science that grows by a process of successive criticism and refinements of theories and the advancement of new and competing theories:

> The logical theory of mathematics is an exciting, sophisticated speculation, like any scientific theory. It is an empiricist theory and thus can be either shown to be false of can remain conjectural for ever.
>
> (Lakatos: 1962, p.178)

What is now needed is a philosophy of mathematics that is made explicit and that seeks to establish what mathematics is about and what forces operate to advance new theories.

In this section we consider mathematics as a cultural system and develop the argument of a recent view, by P. Kitcher, that regards it as growing and evolving through a series of rational transitions to the present day. Then in the next section we look at a particular historical episode; the development of mathematical analysis from 1650 to 1900; and see how changes occurred in response to the needs of the mathematics that was a part of the culture of the time.

(i) A consideration of the cultural influences on mathematics derives its impetus from a study of mathematical history and the relationship between mathematics and other elements of the culture. M. Kline (1962) and (1972), R. Wilder (1965), (1975) and (1981), and R. Marks (ed) (1964) for example, all stress the dependence of mathematics on the cultural life of the civilization which nourished it. The classical period of Greek culture from 600 B.C. to 300 B.C. and the rational quality of its philosophy and its sculptural and architectural ideals, is compared to the concern of the mathematicians of that age to reason abstractly and to contemplate the ideal. Practically minded Rome, and its concern with administration and conquest, produced little that was truly creative and original (See Kline, 1972, pp. 11-12). And so the general character of an age is seen to be closely related to its mathematical activity. In our age mathematics has attained an extraordinary range and applicability.

While a study of these historical episodes suggests an intimate relationship between mathematics and other cultural elements, it is necessary to delve deeper and to try and find the forces within cultures that influence the shape of mathematics and that cause it to change. Two principal factors emerge; the nature of the problems to be solved, and the nature of the symbolic language that is being used. Much of the algebra that we use today, for example, in studying the theorems of mathematics, was unknown to the early Greek mathematicians. But there was no reason for the Greeks to develop any new symbolism for problems that had already been handled satisfactorily by geometrical methods. External forces were also at play. When mechanical gadgets began to appear in Greek culture, such as siphons, fire engines and "an automatic machine for sprinkling holy water when a five-drachma coin was inserted" (Kline, 1972, p.62), the society no longer looked to mathematics for solutions to its problems. As Wilder says:

> One can justifiably conclude that it was those cultural stresses, external to mathematics, that came to dominate the course of evolution of the entire Western culture, which were chiefly the cause for the gradual dying out of Hellenic mathematics. And that, as happened later during an era of ingenious mechanical experimentation in France, ideas having great potential 'died on the vine' because of a lack of demand for them in the cultural environment. To put it another way, science had more than satisfied the demands created by the cultural stresses of the period.
> (Wilder, 1975, p.155)

To take another example, for most of its history mathematics avoided the use of the notion of infinity. Even Euclid's basic axiom said "Every line can be extended" rather than "Every line is infinitely long". Similarly he proved that "Given any finite set of prime numbers there is another prime not in the set" rather than "There are infinitely many primes". But the study of wave motion in acoustics and heat theory and the like, led to a consideration of trigonometric series, which further led to questions concerning the foundations of analysis that could only be explained by considering infinite collections. What proved to be troublesome for mathematics and was avoided for so long could not stand up against the cultural forces of the day. This ultimately led the German mathematician G. Cantor to develop his theory of the so-called transfinite numbers, and a new branch of mathematics took hold as it offered new ways of looking at troublesome fundamental problems. Such became the importance of the concept of infinity to mathematics that, in 1949, H. Weyl was led to call mathematics "the science of the infinite" (Weyl, 1949, p.66).

We might now ask whether some theory can be given that provides an explanation as to how mathematical change occurs. Wilder (1975) lists eleven forces that are discernible in the development of mathematics. These he labels environmental stress (physical and cultural), hereditary stress, symbolization, diffusion, abstraction, generalization, consolidation, diversification, cultural lag, cultural resistance, and selection. Problems in the environment, for example, suggest new problems in mathematics to be investigated. Hereditary stresses describe forces that operate within current mathematical practices, like the challenge to solve previously unsolved problems, and the subsequent need for the creation of new concepts. New symbols are sometimes needed to describe new concepts and these concepts are sometimes diffused from one mathematical field to another in order to fill a need in the receiving field. Attempts are then made to generalize the results and to consolidate diverse mathematical systems by encompassing them under one system. Cultural lag and cultural resistance then describe the forces operating within communities of mathematicians that prompts them to continue with previously tried methods and current symbol usage. Selection then comes into play when a choice has to be made among many competing ways of describing solutions to problems and the concepts and symbols involved. Wilder later takes these forces as the basis for twenty three laws governing the evolution of mathematics (1981). More recently however, P. Kitcher (1983) has given a view of the nature of mathematics that does not argue for its absoluteness, that accounts for the role of the human agent in its formulation, and that explains the growth of mathematics through a series of rational changes. It is to these views that we now turn.

(ii) Kitcher rejects the view that mathematical knowledge is _a priori_ knowledge and starts with the thesis that mathematics is descriptive of the structure of the physical world that we perceive through our senses. He begins by claiming that children learn the meaning of the terms set, number, addition, and the like, by initially engaging in the activities of collecting, segregating objects and so on. Rather than seeing this as a way of acquiring some knowledge of abstract objects, he sees the simple arithmetical truths as true in virtue of these operations:

> .... we might consider arithmetic to be true in virtue not of what <u>we can do</u> to the world but rather of what <u>the world</u> will let us do <u>to it</u>.
> (Kitcher, 1983, p.108)

But if the only arithmetical truths are those that we perform then how do we explain the apparent truth of the proposition 1000 + 3000 = 4000 without having physically performed some segregation and collection of objects? Kitcher's answer is that mathematics is an "idealizing theory". The truths of arithmetic are those ideal operations performed by an ideal agent. The important point is that one arrives at a conception of those ideal operations only through actual operations with actual objects. It is this reaction with the physical world then upon which all of mathematics is derived, and in doing the mathematics that we do we are describing a possible state of affairs of the world:

> I propose that the view that mathematics describes the structure of reality should be articulated as the claim that mathematics describes the operational activity of an ideal subject...
> (ibid., p.111)

Such a view gives rise to questions concerning the distinction between mathematics and science. If arithmetic is the idealization of manipulations of objects say, and the theory of the laws of ideal gases is the idealization of the properties of actual gases, why is arithmetic part of mathematics and the theory of ideal gases part of physical science? The key here is the role of the human agent.

Arithmetic is the idealization of actions, such as grouping and matching, that human beings make. The theory of the laws of ideal gases is an idealization of the way gases react under certain conditions. Mathematics, as distinct from science, is the idealization of the possible outcomes that can occur when a human agent engages in the operations of collecting, grouping, matching and so on, with the physical environment.

While we have concentrated the discussion on arithmetic, in an analagous manner, Kitcher derives the basis of geometrical knowledge from the observations and manipulations of shapes in the physical world. The idealization of such possible manipulations leads to the propositions of geometry.

Previously, in this chapter, it was noted that J.S. Mill attempted to lay the foundations of mathematics in the empirical sciences and this brought much criticism from philosophers like Frege the logicist. Kitcher, however, re-examines Mill's work and, in fact, develops his programme around the language of Mill's arithmetic. Primitive notions such as <u>one-operation</u>, <u>successor of an operation</u>, <u>additions of operations</u> and <u>matchability</u> are used to develop the axioms of Mill's arithmetic, and the results we use in our familiar idealized arithmetic of the natural numbers. In like manner, Kitcher develops the axioms of the real numbers and the axioms of geometry.

To summarize, Kitcher's main argument, backed by his developed programme of Mill's arithmetic, claims that, initially, our mathematical knowledge is obtained by physically manipulating the world and describing the manipulations. It is because of this that he describes mathematics as being empirically based. Mathematics is not about mind-independent objects, nor is it about constructions in the mind. Mathematics is an idealization. It is about the possibilities of ideal manipulations by ideal agents.

But more is needed to explain the development of mathematics. If there is nothing else to the story then why didn't the ancient Greeks know the mathematics we know today? Presumably we both start with the same crude manipulations. There is evidently more to the story and something which Kitcher takes up. He takes our current knowledge to be explained by the transmission of knowledge from one society to its successor and from the society to the individual, and as for the manipulation and observation of physical objects, he says:

> Since I claim that the knowledge of the mathematical tradition is grounded in the experiences of those who initiated the tradition, what I have offered can best be regarded as an attempt to explain how the arithmetical knowledge of our remote ancestors might have been obtained.
> (ibid., p.119)

Then, in order to develop a theory that explains how mathematics has evolved from these crude beginnings, he compares changes in mathematics to theories of change in science by examining recent philosophies of science. By looking at particular episodes in the history of mathematics, he then illustrates his theory of mathematical change and shows how mathematics has evolved through a series of rational transitions. In order to follow his argument then, we first look at recent theories of scientific change.

(iii) A simplified empiricist view of science is that it involves a set of observations and a set of theoretical statements inferred from these observations. As science develops, the set of observations accumulates and the theoretical statements are modified to account for the new observations. The presumption is that without new observations science would be static. However, this simplified view of science is one that is rejected in recent philosophies and is rejected by Kitcher.

The work of K. Popper and T. Kuhn has served as a springboard for a new way of looking at science. Popper's first views were published in 1934 in his book The Logic of Scientific Discovery (English edition 1959). He claims that the difficulties of inductive logic, of moving from the particular to the general, are insurmountable. No amount of observations can justify one's belief in theoretical statements. Popper's rejection of induction thus led him to reject the verification of theories. Theories of the world are not discovered in the sense that observations of singular events lead to them, nor are they verified in the sense that once put they can be shown to be true by observing singular events. For Popper hypotheses can only be "corroborated":

> Instead of discussing the 'probability' of a hypothesis we should try to assess what tests, what trials, it has withstood; that is, we should try to assess how far it has been able to prove its fitness to

> survive by standing up to tests. In brief, we should try to assess how far it has been 'corroborated'.
> (Popper, 1974, p.251)

The power of the hypotheses is assessed by their ability to account for previous observations and their ability to stand up to tests of rejection.

Kuhn's work makes a distinction between what he terms "normal" science and revolutionary elements in the development of science. He says:

> ... 'normal science' means research firmly based upon one or more past scientific achievements, achievements that some particular scientific community acknowledges for a time as supplying the foundations for its further practice.
> (Kuhn, 1962, p.10)

Revolutionary elements, such as the transition from Aristotle's view of cosmology to Copernican cosmology, and the transition from Newtonian physics to the theory of relativity, occur as anomalies arise within the practice of "normal" science. Questions need to be asked in different ways and quite often a new language and accepted procedures of reasoning are needed to answer the questions. This often involves a new outlook on the nature of science. Revolutionary methods gain their approval by the scientific community accepting their ability to explain hitherto unexplained problems. But the introduction of a new language and reasoning poses new problems that become part of "normal" science. New observations are made and new hypotheses are considered. Scientific change is thus thought of as occurring through additional observations and also through internal stresses caused by the new language and reasoning. For Kitcher these same procedures can account for changes in mathematics.

An anticipated objection to Kitcher's view is that it appears that mathematical theories have a higher rate of survival than scientific theories. The reply is that this is due to the nature of the different theories. In hypothesizing, the scientist is attempting to arrive at the correct theory of explanation. The mathematician is offering an idealization of what is possible in the physical world.

Kitcher contends that our basic mathematical knowledge is derived perceptually and then grows through our attempts to idealize the possibilities of the physical world. So mathematics consists of idealized theories of ways in which we can operate on the world. This way of idealizing becomes what Kitcher calls a "mathematical practice" and consists of five components. These components are identified as:

L - a language component,

M - a set of metamathematical views,

Q - a set of accepted questions,

R - a set of accepted reasonings,

S - a set of accepted statements.

He introduces the symbol $\langle L,M,Q,R,S\rangle$ to stand for an arbitrary mathematical practice and says:

> The problem of accounting for the growth of mathematical knowledge becomes that of understanding what makes a transition from a practice $\langle L,M,Q,R,S\rangle$ to an immediately succeeding practice $\langle L',M',Q',R',S'\rangle$ a rational transition.
> (Kitcher, 1983, p.164)

Consider now the five components in turn.

(a) The language component. There are at least three ways in which rational change occurs in the language component of a mathematical practice. Firstly, there are simple notational changes in symbols, where new symbols are introduced to stand for accepted concepts. Secondly, there are examples of disputes arising in mathematics due to different meanings being attached to the same word or symbol. The resolution of such disputes involves a change in the language so that the different concepts under discussion can be referred to by distinct words and symbols. Thirdly, in attempting to think about problems in a different way, one is forced to introduce symbols that appear to have no referents. In Cantor's work on number theory and transfinite numbers, for example, he introduced the symbol "ω" to stand for the first number immediately following the series 1,2,3,.... And $i = \sqrt{-1}$ is introduced to stand for the solution to the equation $x^2 + 1 = 0$.

(b) The metamathematical views. The metamathematics of a practice includes the standards of proof, the scope of mathematics and the relative value of particular types of inquiry. These views of a practice become most evident when other transitions suggest a revolution is under way.

(c) The accepted questions. The set of accepted questions are formulated in the language of the practice and are regarded as unanswered and worth answering. They may be instrumentally worthwhile answering because their solution will aid the solution of other problems in mathematics and science. Alternatively, they may not appear to have immediate instrumental value but do have clear cut answers. An example would be the question "Are there only a finite number of twin primes?"

(d) <u>The accepted reasonings</u>. The set of accepted reasonings is the sequence of statements that the mathematicians put forward in support of the statements they assert. These accepted reasonings are ultimately connected to the metamathematical views of the practice.

(e) <u>The accepted statements</u>. The set of accepted statements is the set of sentences, formulated in the mathematical language of the practice, to which all conversant readers would assent. The types of changes that could occur here involve a reformulation of statements in line with changes in the language. For example, before the division of numbers into real and complex, mathematicians would have assented to the statement "There is no number whose square is -1." Now mathematicians deny that statement. The accepted reformulation is "There is no <u>real</u> number whose square is -1." The thing to which the word "number" refers has changed.

To summarize Kitcher's view then, a rational change in one of the components of a mathematical practice is intimately tied to the current view of the other components and more often than not involves changes in them too. In particular, for a certain mathematical practice, a method is proposed for answering certain questions. This introduces a new language which advances new statements and threatens existing statements. It may also advance new ways of reasoning. However, the proposal for change is accepted because of its power in answering important questions, and the practice is extended to encompass it. While the search for new methods of rigorous reasoning begins, involving changes in the language, prior metamathematical views may be overthrown. The resultant product is a mathematical practice which may appear completely different from that which initiated the process. A rational transition has occurred.

Before examining a particular case in the history of mathematics, we conclude this section by matching what we have said about rational mathematical change to the question of how people have the mathematical knowledge that they do. Given the orthodox philosophical position that knowledge is warranted true belief, Kitcher's view is that most cases of mathematical belief are warranted in virtue of them having been explicitly taught by a community authority, or by virtue of having derived them by types of inference that have been explicitly taught. Kitcher then envisages a chain of communities beginning with a community whose beliefs are perceptually warranted. He thus sees the growth of mathematical knowledge as a process by which:

> .... a scattered set of beliefs about manipulations of physical objects, gives rise to a succession of multi-faceted practices through rational transitions, leading ultimately to the mathematics of today.
>
> (ibid., p.226)

Thus, to judge the individual's claim to having mathematical knowledge is to judge the community authority's claim that the mathematical beliefs it has are warranted. For Kitcher they are warranted because they have been derived initially from beliefs that are perceptually warranted; that is, from physical manipulations of the environment; and via transitions that are rational.

A Case Study: Analysis from 1650 to 1900

Present day mathematical analysis originated from the calculus of the 17th century. The development of calculus independently by Newton and Leibniz introduced a new language into mathematics, accompanied by new reasonings, new statements and new questions. But their work was accepted for it gave answers to questions that mathematicians had been asking for years before that. These questions involved such things as finding the tangents to curves, computing areas, and finding the maxima and minima of functions. The techniques of differentiation and integration, developed by Newton and Leibniz, led to algorithms for the solution of many of these problems. The power to answer questions was great enough for the new language and reasonings to be accepted. What followed can be viewed as a series of transitions, as the components of the mathematical practice of the time changed to encompass this new way of successfully dealing with previously unsolved problems.

Newton introduced the notions of fluent and fluxion. A fluent is any quantity which is in the process of changing, and the fluxion is its rate of change. The problem becomes one of determining the fluxion given the fluent (differentiation), and determining the fluent given the fluxion (integration).

As an example, imagine a particle moving along the curve $y = x^3$ where y denotes the position of the particle at time x. Assume that through a small interval of time, θ, the velocity remains constant. In this time y increases to $y + \dot{y}\theta$ and x increases to $x + \dot{x}\theta$. Then

$$y + \dot{y}\theta = (x + \dot{x}\theta)^3$$

$$x^3 + \dot{y}\theta = (x + \dot{x}\theta)^3$$

$$\dot{y} = \frac{(x + \dot{x}\theta)^3 - x^3}{\theta}$$

$$= \frac{3x^2\dot{x}\theta + 3x\dot{x}^2\theta^2 + \theta^3}{}$$

$$= 3x^2\dot{x} + 3x\dot{x}^2\theta + \theta^2$$

$$= 3x^2\dot{x}$$

as we can omit terms containing θ since they are infinitesimally small.

$$\frac{\dot{y}}{\dot{x}} = 3x^2$$

This says the velocity at time x is $3x^2$. But questions remain concerning the reasonings behind the method. Why can we assume that fluxions remain constant through a small interval of time? Why are we allowed to neglect some terms? Either θ equals zero or it does not. If it does then we cannot divide by it, and if it doesn't then the terms $3x\dot{x}^2\theta$ and θ^2 are not zero and cannot be omitted.

Leibniz's work developed in a similar way, except that he did not adopt a kinematic approach and his symbolism was different. But he effectively arrived at the same conclusion. If $y = x^3$ then $\frac{dy}{dx} = 3x^2$. From the work of Newton and Leibniz the techniques of differentiation and integration were developed and many important results were found. Leibniz and his followers found that they could compute the sums of infinite series. For example,

$$\int_0^1 \frac{1}{1 + x^2}\,dx = \frac{\pi}{4}$$

But the power series expansion of $\frac{1}{1 + x^2}$ was known to be $1 - x^2 + x^4 - x^6 + \ldots\ldots$ and integrating this term by term gives

$$\int_0^1 \frac{1}{1 + x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

Therefore, $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

The power of the new language and reasonings in answering certain problems assured their acceptance, even though there were some anomalies. For example, Leibniz claimed that

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots\ldots$$

Putting $x = 1$ gives

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \ldots\ldots$$

But surely
$$\begin{aligned} 1 - 1 + 1 - 1 + \ldots &= (1 - 1) + (1 - 1) + \ldots \\ &= 0 + 0 + \ldots \\ &= 0 \end{aligned}$$

or,
$$\begin{aligned} 1 - 1 + 1 - 1 + 1 - \ldots &= 1 - (1 - 1) - (1 - 1) - \ldots \\ &= 1 - 0 - 0 - \ldots \\ &= 1 \end{aligned}$$

In attempting to explain the anomaly, Euler writing some forty years after Leibniz, suggests that the expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ holds only when $x < 1$, if x is positive.

The work of Newton and his successors followed a different path to Leibniz's. In 1734 Berkeley wrote a scathing criticism of Newton's work demanding more rigour (see Boyer, 1959, pp.224-9). He claimed that while the methods were successful, no explanation for their success had been given. This call was answered by a large number of writings, providing a geometrical interpretation of the algebraic techniques employed. It was reasonable for Newton's successors to do so, given their metamathematical views of the time and the criticisms levelled against their work. But Berkeley's criticisms on the foundations of Newton's work had long term consequences. For Leibniz's successors on the continent the criticisms took second place to the algebraic interpretation of the calculus and its power in answering questions. And, as Kitcher puts it:

> Priding itself on its rigor and its maintenance of a proper geometrical approach to mathematics, the British mathematical community fell further and further behind.
> (Kitcher, 1983, p.240)

In the 1820s Cauchy introduced the algebraic concept of a limit to tackle the problems of the calculus.

> When the values successively attributed to the same variable approach indefinitely a fixed value, eventually differing from it by as little as one could wish, that fixed value is called the <u>limit</u> of all others.
> (Cauchy, translated by Birkhoff, 1973, p.2)

An infinitesimal was then defined to be a variable that has zero as its limit. The notions of continuous functions, derivatives and convergent and divergent series, which had been used extensively by Euler and his contemporaries, could now be given a more rigorous definition in terms of limits. The important point is that Cauchy was not concerned about rigour for rigour's sake but for the use of convergent series in answering questions that were considered important in the mathematical practice of the day. Such questions centred around the work done on vibrating strings and Fourier's work on representing some functions as the sum of trigonometric functions. The problem then became one of whether any function could be given a trigonometric series representation. So, rather than proceeding as a response to a call for securer foundations to mathematics, as Kitcher says:

> .... I think that examination of this episode will underscore my thesis that foundational work is not usually undertaken by mathematicians because of apriorist epistemological ideas, but because of mathematical needs.
> (Kitcher, 1983, p.246)

The power of Cauchy's work in answering important questions in the mathematical practice of the time led to a flurry of writing. But the original work itself provided inconsistencies both in language and results. Cauchy's use of infinitesimals as constants led to problems and his solution to the Fourier problem was not complete. Abel, for example, in 1826 showed that there existed an infinite series of continuous functions that was not everywhere continuous.

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots$$

is not continuous at each value of $x = (2m + 1)\pi$.

It was left to Weierstrass to introduce a formulation of the limit that banished the troublesome talk of infinitesimals. Cauchy's criterion for convergence was formulated in Weierstrass's terminology as:

$\sum_{i=1}^{\infty} u_i$ is convergent if and only if

for all $\varepsilon > 0$ there is an N such

that, for all $r > 0$, $\left|\sum_{i=N}^{N+r} u_i\right| < \varepsilon$.

The 1860s and 1870s saw a proliferation of theorems in analysis by Weierstrass and his students and Weierstrass's formulation is now common to all elementary textbooks on analysis.

Cauchy's work also led to Dedekind's analysis of the continuity of the real numbers, providing a transition from the geometrical interpretation of considering real numbers as spread out on a straight line, to a purely algebraic one. Dedekind introduced the notion of a "cut", which is a separation of the real numbers into two classes A and B such that for an x and y, if x belongs to A and y belongs to B then $x < y$. A cut, designated by (A,B), is uniquely determined by a real number.

This new language allowed the derivation of familiar theorems about real numbers as well as some limit existence theorems that Cauchy had failed to prove, such as the result that a monotonically increasing sequence, bounded above, is convergent. Furthermore, Dedekind's work itself raised questions concerning the existence of sets which were taken up by Cantor and others at the end of the 19th century.

For Kitcher, the transition from one mathematical practice to another does not follow from a response to epistemological aims, but to the needs of mathematical research. The calculus of Newton and Leibniz was warranted because it was a procedure that answered important questions in the mathematical practice of the time. Successive investigations were prompted by the use to which the calculus could be put. The new language and accepted reasonings satisfied the needs of research while spawning new questions and new problems of rigour. Axioms and definitions were accepted because they systematized previously accepted problem solutions, and studies in the foundations of mathematics were motivated by the pragmatic concerns of working mathematicians.

To conclude then, this case study has illustrated Kitcher's theory by noting some of the rational transitions that occurred to make the mathematical practice of 1900 different from the one of 1650. The next chapter considers the implications that this theory about the nature and evolution of mathematics might have for the school curriculum. Common justifications given for teaching mathematics need to be re-assessed in light of the views developed in Chapter II on the value of knowledge and the views developed in this chapter on the nature of mathematics. The role that mathematics plays in allowing people to develop a world view and be in the best position to determine their own ends must be articulated.

CHAPTER IV

CURRICULUM CONSIDERATIONS AND CONCLUSIONS

In Chapter II, M.A.B. Degenhardt's argument, which contends that a curriculum ought to be chosen based on a particular notion of the worthwhileness of knowledge, was developed. The idea that some knowledge is valuable in educating human beings into a cultural inheritance that puts them in the best position to work out what to do with their lives, involves developing a "world view" by studying the traditional evolved bodies of knowledge.

In Chapter III, the nature of mathematical knowledge in particular was considered, and the chapter concluded with P. Kitcher's argument that mathematics began as crude manipulations by man and his attempts to describe those manipulations. The *a priori* nature of mathematics was rejected and replaced by Kitcher's "defensible empiricism". Mathematics, it was argued, is an idealization, about the possibilities of ideal manipulations by ideal agents. A mathematical practice consists of a language component, a metamathematical view component, and sets of accepted questions, reasonings and statements. Such a practice is passed on from one community to another but is subject to rational transitions. These transitions occur in response to the practice's power in answering important questions of the day and in the subsequent work in making rigorous other components within the practice.

In this chapter some implications of the conclusions reached so far are considered. The important point is that in most schools mathematics forms part of the compulsory curriculum for a great deal of time, and we first need to look at the justifications given for teaching so much mathematics to so many students. The justifications first considered, and rejected, are the claims that so much mathematics ought to be taught to all students based on its usefulness, its intrinsic worthwhileness, and its power in developing the mind. It is then argued that mathematics, by its very nature and its cultural significances, can contribute to human beings developing a world view, and thereby put them in a better position to determine their own ends. Finally, some implications that such a justification for teaching mathematics might have on the school curriculum are considered.

Common Justifications Given for Teaching Mathematics

(i) One justification given for teaching mathematics is that mathematics is useful. In order to examine this argument it is necessary to look at the various ways in which mathematics might be claimed to be useful.

(a) It might be claimed, for instance, that studying algebra in grade 9 is useful because it will be used in studying calculus in grade 11. Of course the objection is that no justification is given for studying calculus in grade 11. Ultimately, the justification, in terms of usefulness, must lie outside the subject.

(b) A second claim might be that mathematics ought to be studied because it is useful in other subjects. Technical subjects apply the rules of measurement and ratio, for example. The graphical representation of data and the determination of statistics, such as mean, median and mode, are useful in the social sciences. And calculus is used in calculating rates of change in physics. In Chapter III we saw that a mathematical practice evolved from crude manipulations of objects and as a response to questions that were thought to be important in some way. The view that mathematics is the language describing the possibilities of time and space and the view that science is about forming and testing hypotheses about the nature of time and space clearly implies that mathematics is useful to scientists.

Two questions remain however. Firstly, can we establish that science, or any other subject where mathematics is useful, is itself worthwhile? Secondly, is mathematics necessary for acquiring knowledge in that subject? If it can be shown that the subject is worthwhile, but that mathematics is not necessary for acquiring knowledge in that subject, then there is no justification here for making mathematics compulsory. The claim that mathematics ought to be studied on the grounds that it is useful in other subjects simply forces us to look at the value of those other subjects, and the necessity of mathematics in acquiring knowledge in those subjects. It is not in itself, however, a justification for teaching so much mathematics to all students.

(c) A third claim might be that mathematics is useful for people in their employment. It is true that many jobs require a lot of mathematics and perhaps all jobs require some mathematics. Highly skilled careers in technology quite obviously use a lot of mathematics, since technology has developed from man's manipulation of the environment, and the language component of a mathematical practice describes those manipulations. But while some mathematical knowledge may be necessary for some jobs, the compulsion in studying mathematics would demand that the mathematics taught is necessary for all students in whatever job they secured. It is not difficult to think of many positions where very little mathematics is used. So, in attempting to justify mathematics for all on the grounds that it will be used by all in future occupations, we have arrived at a minimal amount of knowledge that warrants very little time at all on the curriculum, certainly not the current amount of time spent in most schools.

Another objection to be considered is that if the mathematics required for certain jobs is quite specific, and if it is valued only for its usefulness in those jobs, then while it ought to be a part of job training it need not occur at school. In fact it can be argued that such mathematical knowledge is better taught "on the job" and by practitioners in the field, where the user can see immediately the use to which the mathematical knowledge can be put. For any given class of mathematics students at school, there is a large range of occupations that those students might end up in. It is impractical to present to all students all the specific mathematics that they might use in such a large occupational range.

(d) A fourth claim of usefulness might be that mathematics is useful for everyday living. That is to say, that in the daily activities that one concerns oneself with, there is some mathematics to be used. And this is clearly so. The simple manipulation of objects and the communication of ideas requires us to use the language that has evolved to describe those operations. This includes the basic operations of counting, adding, subtracting, multiplying and dividing, and the ability to read and understand the presentation of information from tables and graphs. Such mathematical knowledge is worthwhile on the grounds that without it one would not be in the best position to participate in the community as we know it. The ability to handle simple financial transactions, for example, would be regarded as essential for all members of the community, and the ability to understand economic issues on a wider scale when deciding how to vote at elections would be regarded as highly desirable. It seems reasonable to expect that schools ought to be about the business of compulsorily introducing students to this language. But the question is whether this requires the current amount of time spent on mathematics. Such mathematical knowledge that is useful for everyday living is acquired by most people well before the end of compulsory education.

So while accepting that some mathematics is useful to all people and that some people use a lot of mathematics, we reject the claim that the justification for teaching mathematics for the current length of time to all students is based on the usefulness of mathematics.

(ii) A second justification given for teaching mathematics to all is that it is intrinsically worthwhile; that mathematical knowledge is valuable for its own sake.

If intrinsic worthwhileness is simply a matter of individual psychology, whereby the learner claims to like doing mathematics and that's all, then there can be no claim here for teaching mathematics to everyone. Someone who does not like doing mathematics can equally claim that it is not intrinsically worthwhile and that there is no justific- ation in studying it. The justification for teaching mathematics to all, based on its intrinsic worthwhileness, must centre on the nature of mathematics and what makes it an intrinsic worthwhile activity irrespective of any preference of the learner.

But then is it the knowledge of the accepted mathemat- ical statements of the practice couched in the practice's language that is valued? For instance, is simply knowing that "4 + 2 = 6" or that

$$\text{"}\int x \ln x \, dx = \frac{x^2}{2} (\ln x - \tfrac{1}{2}) + C$$

for some constant C"

intrinsically worthwhile? Quite clearly many mathematical statements would not be valuable without an understanding of the reasoning behind the statements and an ability to arrive at the mathematical statements by using the practice's reasonings. But the question remains as to why this is intrinsically worthwhile and justifies the teaching of mathematics to all.

In Chapter II, arguments for basing the school curriculum on the intrinsic worthwhileness of some subjects were considered. G.H. Bantock (1963) argued that some subjects are more valuable than others because their understanding involves a higher degree of intellectual functioning. R.S. Peters (1966 and (ed.) 1973) argued that some subjects are intrinsically worthwhile because they are concerned with truth and rationality. But these positions were found to have serious objections as they stand. If we claim that mathematics ought to be on the school curriculum for all students then we must present an argument based on the educational value of mathematics. Some activities may be intrinsically worthwhile but form no part of the school curriculum, while other activities, such as writing and using calculators, may be deemed to be highly worthwhile educationally while possessing little or no intrinsic worthwhileness. So the claim that pursuing and possessing mathematical knowledge are intrinsically worthwhile activities is rejected as a justification for teaching mathematics to all.

(iii) A third justification given for teaching mathematics to all is that it develops the mind or promotes intellectual development. The value of mathematical knowledge lies outside of the usefulness to which the knowledge can be put and the intrinsic worthwhileness of simply possessing or pursuing the knowledge. If P. Hirst's thesis is correct, and there are distinct forms of knowledge, then it may be argued that there are distinct kinds of developed minds. A mind can be highly developed in the sphere of moral knowledge, for example, but quite undeveloped in the field of the physical sciences. If there are logically distinct forms of knowledge, and if a developed mind is defined to be a mind developed in any one or more of those distinct forms, then it is not necessary for all students to study mathematics to develop their minds. To justify mathematics learning for all, on the grounds that it does develop the mind, implies that one values a developed mind and that mathematical learning is necessary for the mind to develop. It is appropriate, therefore, to consider the arguments which contend that mathematics is necessary for the mind to develop.

(a) It may be argued, for instance, that the more mathematical sentences in the mathematical practice that are accumulated in the mind, the more developed the mind is. But it is difficult to see how we can place value on a mind which simply possesses these statements and allows one to write them out on demand, as it were. We would hold, at least, that one should arrive at the sentences through accepted reasonings and see how they are used to solve the accepted questions of the mathematical practice. But even so, we would still be left with the problem of explaining why we value such a mind with this ability.

(b) If we argue that mathematics is needed to develop the mind in general, then we are saying that mathematical knowledge is somehow needed in order that development proceeds in other areas. This sort of argument would contend that while there may be distinct forms of knowledge that can be developed in the mind independently, mathematics is granted a higher status, in that all other forms are dependent upon it. Arguments granting this higher status to mathematics might centre around the logical processes involved in mathematics. But as we saw in Chapter III, a mathematical practice is not just logic, even though its reasonings may well apply logical principles. Mathematics is an idealization of the possibilities existing within time and space and it attempts to form generalizations by describing these possibilities in the language of the practice. It is this very generalization, however, that makes it appear that mathematics is required for all spheres of intellectual development. But the sorts of reasonings that are required in other areas of intellectual development can be acquired within that subject and with no prior knowledge of the mathematical practice of the day.

(c) Instead of arguing that mathematical knowledge is necessary for developing the mind, we might simply argue that it is helpful in other areas of intellectual development. This is to suggest that development in mathematics might somehow be transferred to other areas of development. The attractiveness of this argument stems from a view of mathematics as being about problem solving in the abstract, where training in mathematics is regarded as acquiring problem solving techniques which can be transferred to other disciplines. There are, however, objections to this argument.

Firstly mathematics is about something. It is not problem solving in the abstract. It is about the possibilities that exist in time and space and it is about learning a particular language of mathematics that describes those possibilities. Secondly, there is no conclusive psychological evidence for the transfer of learning. (See, for example, the earlier paper by R.M. Gagné in H.F. Clarizio et al. (eds.), 1981, pp.117-126, and papers by J. Baron, J.R. Hayes and D. Meichenbaum in S.F. Chipman et al. (eds.), 1985, pp.365-426.) It has been easy to show that some people solve problems more easily than do others, but it has not been easy to show the transference of problem solving. Deliberate attempts "to teach students to think" have not been successful. Certainly there is no clear evidence that mathematics training is necessary for development in other intellectual areas.

Thirdly, even if it is true that there is some transfer between different disciplines, the task should be to establish what sort of a developed mind one hopes to promote, and then to teach specifically for it. If mathematics is to be valued for its role in intellectual development then the sorts of reasons given could range from the view that mathematical knowledge is necessary for any kind of development to proceed, to the view that a mind developed in mathematics alone is sufficient to claim intellectual development. The psychological evidence, and P. Hirst's thesis on forms of knowledge, suggests that mathematical knowledge is not necessary for development in other areas and we have not yet established in what sense having mathematical knowledge contributes to the promotion of a developed mind.

So the claim that mathematics develops the mind or promotes intellectual development is also rejected as a justification for teaching mathematics to all. In the next section we focus on arguments based on the cultural significances of mathematics. It is shown that these arguments are warranted given the views developed so far on the value of knowledge and the nature of mathematical practices.

The Cultural Importance of Mathematics

So far in this chapter we have attempted to give some justification for teaching a lot of mathematics to all students and considered arguments in light of our views on the nature of mathematics. Mathematics is an evolving body of knowledge which, at any particular time, constitutes a mathematical practice, consisting of a set of accepted statements in a mathematical language. These statements are arrived at by certain accepted reasonings. A mathematical practice also has important questions that may be unanswered and certain metamathematical views that illustrate the scope of mathematics and the nature of particular types of mathematical inquiry. So far our claims for justification have centred on only three of the components of a mathematical practice. We have considered the use to which statements can be put, the intrinsic worth of possessing statements in the language of the practice, and the value that the statements, the language and the reasonings of the practice have in developing the mind. Such justifications were found wanting. The fourth kind of justification that is argued for here considers all five components of a mathematical practice, and shows how mathematics teaching can be justified from the point of view of the cultural importance of mathematics.

In order to do that, it is necessary, firstly, to clarify what is meant by the term "culture". The word has been used in many different contexts, sometimes synonomous with the word "society" as in "Australian culture", or sometimes meaning one refined in tastes and manners as in "a cultured gentleman". The word is used here to mean that, at any particular time, a given society possesses a "culture" which is a collection of customs rituals and beliefs, and different languages; spoken, written and symbolized in different ways; that allow communication of ideas between members of the society. It is argued that mathematics has been culturally important in that it has contributed to the way individuals have interpreted the world by influencing their beliefs, their problem solving techniques and the language they have used to communicate ideas.

(i) Firstly, we could argue that mathematics is itself one of many great cultural achievements and not to be aware of these achievements is not to have an understanding of the culture. But this in itself is not enough for it doesn't take into account our view of the nature of mathematics. The formalist, and his view that mathematics is simply the formal manipulation of symbols; the intuitionist, and his view that mathematics is the manipulation of symbols together with mental constructions; and the logicist, who regards mathematics as one giant tautology; all could look at the mathematics we have today and claim it as a great cultural achievement. We have argued, though, for the evolution of mathematics through rational transitions and the development of a language and accepted reasonings that answered important questions within the culture. The mathematical practice also had with it metamathematical views on the scope of mathematics. To regard mathematics simply as an impressive cultural achievement in its own right is not to understand how mathematics has influenced the evolution of other aspects of the culture, and how the problems considered important within a culture have influenced the evolution of mathematics. Advances made in mathematics are importantly linked to advances made in other parts of the culture.

(ii) So the second point to be made is that to fully understand how a culture evolved one must be able to understand how the evolution of mathematics has been a part of that wider cultural evolution. To justify the place of mathematics on the curriculum by this argument would have radical consequences for the content of mathematics courses. Rather than simply presenting a language of mathematics and a set of statements, one would have to present, from an historical perspective, an account of how individuals have influenced aspects of the culture by their work in mathematics. For example, the study of projective geometry by Pascal and Desargues in the early 1600s was influenced by painters' attempts to construct an optical system of perspective. Navigators then used this projective geometry to design new map projections. And the writings of Descartes, Galileo and Newton in a precise logical style, free from metaphor and symbolism, influenced the prose style of many literary scholars in the late 1600s and early 1700s. To come to understand mathematics in this way is quite different from "doing" the mathematics of the present day, which means using the language and reasonings of the current practice to answer previously solved problems and arrive at already known statements. It is one thing to be able to "do" problems in dynamics and another to be able to understand how the work of Newton and Leibniz contributed greatly to our understanding of the motion of moving objects.

(iii) The third point to be made is in relation to the meta-mathematical views associated with a mathematical practice, and the power and scope of mathematics as perceived within the culture. At particular times in history, man has looked to the current mathematical practice and seen within it a mode of thinking that he has applied to other elements of the culture. The success of mathematics in answering important questions in the physical sciences, for instance, not only affected the mathematical practice of the day, it also influenced man's thinking in other areas such as the social sciences and art. The economists of the 18th century, for example, sought to "mathematize" economic theory. Thomas Malthus and David Ricardo attempted to identify the factors that influenced economic life and to discover natural laws of economy. When this failed, economists concentrated on specific phenomena where they applied their mathematical techniques to deduce conclusions. The modern movement now has seen a massive amount of symbolism used to describe and predict economic behaviour. But in attempting to provide natural laws of economy it can only be said that, so far, mathematics has failed. The activities of man and the factors that influence his behaviour have not been neatly packaged and predicted with certainty. But this very fact has also influenced mathematical activity. The idea of nature being unpredictable and composed of chance events has seen a rise in the mathematical theories of probability and statistics.

The important point is that the mathematical practice of the day becomes culturally important partly in light of the metamathematical view associated with it. While its scope may prove to be limited, and while it may not be successful in its application to all other elements of the culture, the fact that man has looked to the mathematical mode of thinking as an answer to various problems, is in itself significant. An understanding of the culture would not be complete unless one had an understanding of the various ways man has attempted to answer problems within the culture.

The aim of this chapter was to consider the nature of mathematics and to argue for its inclusion in the curriculum because of its contribution in allowing human beings to develop a world view and determine their own ends. It has not been argued that there is a "mathematical view" of the world but, rather, a view that takes in the achievements of mathematics together with other human endeavours. If we want students to develop a world view and be in the best possible position to decide what to do with their lives, then students ought to be introduced to the combined achievements of these endeavours. If one is to have a world view then it would be deficient if it lacked knowledge of how mathematics has influenced culture. But it would also be deficient if it saw mathematics as an isolated element of the culture that develops in its own right without the influence of other cultural forces.

CONCLUSIONS

The conclusions reached can be summarized as follows:

1. While the mathematical philosophies of formalism, intuitionism and logicism all give some insight into the activities of mathematicians, the nature of mathematics is that it consists of idealized theories of ways we can operate on the world and, at any particular time, constitutes a mathematical practice with the following five components:
   (a) a language component,
   (b) a set of metamathematical views,
   (c) a set of accepted questions,
   (d) a set of accepted reasonings,
   (e) a set of accepted statements.

2. Mathematics has evolved from a set of beliefs about simple manipulations of physical objects, and through a series of rational transitions, to the mathematical practice of today.

3. There is a language and a set of accepted reasonings in today's mathematical practice, that is useful to everyone and that all students ought to be initiated into. This includes basic numeracy, operations with numbers and fractions, and the ability to read and interpret the presentation of data in tables, graphs and simple formulae.

4. All students should come to see mathematics not as a fixed body of knowledge to be discovered, but one that has evolved and continues to evolve as man attempts to understand the nature of his environment.

5. All students should be initiated into the influence that mathematics has had on our culture; firstly, by the contribution it has already made to the solution of problems posed within the culture; and secondly, by the way man has sought to use it in other endeavours.

DISCUSSION

The implications that these conclusions might have for the mathematics curriculum will be considered by looking at the case of the calculus, whose evolution to analysis from Newton's and Leibniz's initial work was shown in Chapter III. There is a large proportion of students who do not reach the stage of studying the calculus in their school years, and a significant proportion who proceed well beyond it, but in most countries students in the advanced levels are presented with an introduction to the calculus in their final years of schooling. The arguments considered below, however, could be modified and applied to other topics within mathematics syllabuses.

We saw that Newton's and Leibniz's work was motivated by practical problems and was accepted into the mathematical practice of the day because it was successful in answering important questions within the practice. The call for rigorous reasoning then led to Cauchy's work on limits, and Weierstrass's work on the terminology led to a new language within the practice. Dedekind followed by an analysis of the real numbers in algebraic form and Cantor was led to investigate problems associated with sets. But in teaching the calculus these topics are frequently presented in the reverse order. Real numbers are studied early in the course, some definitions of concepts are given, and theorems follow concerning limits. A definition of the derivative is given and some rules for finding derivatives are proved. Finally, some questions are posed with the view of demonstrating the power of the technique in solving practical problems.

In general, the presentation of the material within any topic is designed with a view of passing on the mathematical language component of the mathematical practice. Students are graded into levels on their ability to use this mathematical language and they pass on to the next set of work by showing an understanding of the language and an ability to use the associated reasonings within the practice. It is not intended that students "rediscover" mathematics as it were, by confronting them with the problems that Newton and Leibniz were confronted with, and for them to derive successfully the new mathematical practice. Once the rigour of the language has been arrived at the material is presented in the most expedient way. From no experience at all of the calculus students acquire, within weeks, a language that took decades to evolve after Newton's and Leibniz's initial work. What is missing, however, is a study of the forces behind the transition to a new mathematical practice.

At this point we can consider a possible criticism. It might be said that it is not practicable to design a curriculum in order that all students come to see the cultural significances of mathematics. Many teachers might argue that it is difficult enough to get some students to understand the language and reasonings of the current mathematical practice and to apply them to simple problems, without even beginning to think about explaining the development of the language and reasonings, the practical problems that influenced their development and the effect they have had on other aspects of the culture.

In reply it is agreed that some students will be able to grasp cultural significances easier than others. Just as students differ in their abilities in the present subjects so they will in any future courses. But the fact that students do differ in their ability to understand concepts within the current mathematical practice does not render impracticable any programme designed to explain the cultural significances of mathematics. What is required is a programme that takes into account these individual differences. What has to be decided is the question of when to present an account of the significance of mathematical knowledge and how to incorporate it into the curriculum. Should it be part of each mathematics lesson? Should it be combined with a study of the value of other disciplines such as science or history? Other factors to be considered concern the preparation of teachers and how a study of the cultural significances of mathematics is to be incorporated into their training. But while such concerns indicate that much thought is needed to develop a system whereby the cultural significances of mathematics form an integral part of the mathematics curriculum, they do not, in themselves, show that the programme is impracticable.

Now if we are to argue for a curriculum that should be presented to all students then we must ask to what extent the current mathematics courses suffice. The higher level syllabuses are a preparation ground for a community of scholars to engage in pursuing significant questions in the practice, but lower levels become "watered down" versions of this type, by selecting a language component that is presumably easier to understand (though not for all students), and with a view to showing the applicability of the language to "real life" situations, which more often than not never occur to students after they leave school. From a recent publication:

> Most teachers are aware that when these subjects [mathematics and the sciences] are presented as theoretical and abstract studies many students are put off, and only those students with a special inclination towards the subjects bother to pursue them. This cannot be allowed to continue, given the role that the subjects play in the world. All students should continue studies in mathematics and science as long as possible. The range of mathematical and scientific studies should be extended to cover their applications in daily life and the work-place, and to cover also a wider range of student abilities.
> (Education Department of Tasmania: 1986, p.12)

But as we have argued, the amount of mathematics that can be seen to be useful to <u>all</u> students in daily life involves no more than basic numeracy and the ability to read and interpret presentations of data. Instead of "watered down" versions of higher level syllabuses, all students should be presented with a study of the cultural forces behind the development of mathematics and its relationship to other areas of human endeavour. For those with the interest and ability to pursue a study of the language and reasonings of the mathematical practice, then the current syllabuses exist with the specialist teachers in the field. Advances in mathematics occur as a response to cultural forces and much new mathematics is supplied by scholars in many other fields, such as the sciences, engineering and computer technology. It ought to be the goal of mathematics educators firstly, to ensure as far as possible, that all students come to an understanding of the nature of mathematics, its evolution and cultural significances; and secondly, to prepare, through the expertise of specialist mathematics teachers, that community of scholars which contains the practitioners of the future in the new mathematical practice.

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