The Fock-Schwinger Gauge

by

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DECLARATION

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any university. To the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by any person, except where due reference is made in the text of the thesis.

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To Santi and Kinda

Abstract

Unlike some other gauge choices the Fock-Schwinger gauge condition $x.A(x) = 0$ uniquely fixes the gauge potentials in terms of the Maxwell fields through the so-called inversion formula. Thus the Fock-Schwinger gauge potentials in some simple configurations can be derived by making use of this formula and contrasted with the familiar Coulomb gauge potentials. Two important consequences are that Fock-Schwinger potentials of electrostatic systems are no longer static and (unlike the Lorentz gauge potentials) that Fock-Schwinger potentials corresponding to plane electromagnetic waves are not plane waves.

To apply the Fock-Schwinger gauge to perturbation theory the gauge propagator is first derived by the use of two different gauge fixing to the Lagrangian mechanism. The first one corresponds to adding a gauge fixing term while the second makes use of auxiliary or Lagrange multiplier fields. The auxiliary method leads to two components of the propagator: the physical and the unphysical. The physical component in the second method coincides with the propagator in the first one. Symmetry properties of the above propagators are also derived and provide considerable improvement of Kummer and Weiser's analysis.

The fact that the Fock-Schwinger gauge theory is a ghost-free theory enables one to derive the Slavnov-Taylor identities without using the language of BRST transformations. Nevertheless BRST identities are also obtained.

The main focus and content of the thesis are perturbation calculations in the Fock-Schwinger gauge. The most important one-loop corrections in electrodynamics and chromodynamics have been computed and compared with the more standard translation-invariant gauge choices. The on-mass-shell equivalence of these calculations with more conventional gauge choices has been established in detail.

CONTENTS

FIGURES

Chapter 1

Introduction

This chapter is intended as a literature review on the Fock-Schwinger gauge as well as to give a global perspective on how the chapters of the thesis relate to one other.

1.1 Review on the Fock-Schwinger Gauge

The less familiar gauge condition

$$(x - x_0) \cdot A(x) = 0 \tag{1.1}$$

where x_0 is a certain space time point, that without loss of generality may be set to zero, has various names: the Fock-Schwinger gauge [Nov 84, Ohr 85, Kum 86, Zuk 86, Sch 87, Kar 87, Mod 90], the Fock gauge [Ska 85], the Schwinger gauge [Nik 82, Niko 82, Sch 89], the complete Lorentz covariant gauge [Cro 80, Men 84, Oka 84], the coordinate gauge [Shi 80, Dur 82, Men 84, Mod 90], the fixed-point gauge [Dub 81], the Constrõm-Dubovikov-Smilga (CDS) gauge [Ita 81, Men 84, Hau 84], the Poincaré gauge [Bri 82, Ska 85, Gal 89, Gal 90], the homogenous gauge [Aza 81] and the multipolar gauge [Kob 82, Kob 83, Ell 90]*. This gauge condition is the subject of the thesis.

*The noncovariant version of the gauge, namely $\vec{r} \cdot \vec{A}(r) = 0$, is sometimes called the radial gauge [Mod 90] since the radial component of the potential vanishes.

It is worth noting that the above mentioned gauge is only a special choice of a set of gauges [Jac 78, Men 84]

$$f^\mu(x)A_\mu(x) = 0 \tag{1.2}$$

where

$$f^\mu(x) = a^\mu + bx^\mu + \omega^{\mu\nu}x_\nu + 2x^\mu c^\nu x_\nu - c^\mu x^2; \qquad \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{1.3}$$

are conformal Killing vectors satisfying

$$\partial_\mu f_\nu + \partial_\nu f_\mu = \frac{1}{2} g_{\mu\nu}\partial_\alpha f^\alpha. \tag{1.4}$$

Since the gauge condition (1.1) was originated a long time ago by V. A. Fock [Foc 37] and then rediscovered by J. Schwinger [Sch 70], the Fock-Schwinger gauge condition seems to be the best name for it and we will adopt it hereafter for $x \cdot A(x) = 0$.

In spite of its relative unfamiliarity the Fock-Schwinger gauge choice has some interesting properties that sometimes make it attractive to field theorists. It is a ghost-free theory since, in this gauge, the Faddeev-Popov ghost action does not depend on the gauge fields and thus its effects can be absorbed into the normalization factor of the entire generating functional. As a result all Feynman diagrams involving ghost loops vanish. Some complications found in the axial-type gauges [Sch 89, Lei 84, Lei 87, Bas 89, Bass 89, Bas 90], which are also ghost-free, cause field theorists to look for other ghost-free gauges like the Fock-Schwinger gauge [Kum 86].

One of the most interesting points about the Fock-Schwinger gauge is that there is a unique relationship between the gauge potentials $A_\mu(x)$ and the field strength $F_{\mu\nu}(x)$ [Hal 79, Men 84, Kum 86, Sch 87]*. Such a relation, the inversion formula [Ita 81, Gal 89], seems to be the most enticing feature in applications to problems in quantum field theory. For example, the formula allows formulations of gauge theory directly in terms of field strengths $F_{\mu\nu}(x)$ instead of gauge potentials $A_\mu(x)$ [Hal 79,

*Notice that another gauge, the so-called fixed axial gauge [Hal 79], with its gauge conditions $A_0(t, x_0, y_0, z_0) = A_1(t, x, y, z_0) = A_2(t, x_0, y, z_0) = A_3(t, x, y, z) = 0$ with x_0, y_0, z_0 fixed, also has an inversion formula. However the inversion formula in the Fock-Schwinger gauge is simpler.

Ita 81, Dur 82, Men 84, Sch 87]. It can also be employed to obtain gluonic mean fields in Hartree approximation where the gluonic mean field is generally generated by vacuum expectation values of gluonic operators [Sch 87]. Coefficients of gluon operators in the operator product expansions (OPE)

$$\Pi_{12}(x,0) = T : \overline{q}(x)\Gamma_1 q(x) :: \overline{q}(0)\Gamma_2 q(0) : \tag{1.5}$$

with Γ_1 and Γ_2 are any two Dirac matrices ($\Gamma_1, \Gamma_2 = 1, \gamma_\mu$ and $\sigma_{\mu\nu}$ for scalar, vector and tensor amplitudes respectively) and $q(x)$ is a quark field may be calculated by utilizing the inversion formula as well [Hub 82]. In addition, for $\Gamma_1 = \gamma_\mu$ and $\Gamma_2 = \gamma_\nu$, the OPE itself can be computed by making use of the complete quark propagator up to first order of the inversion formula showing that the resulting function is transverse [Nov 84].

Shifman utilizes the inversion formula to consider the behaviour of the Wilson loop average [Shi 80]. By rewriting the Wilson loop [Wil 69] as a power series of gauge field strengths the series may be grouped into two sets of terms. The first set consists of terms with no derivatives while terms in the second set contain derivatives of gauge field strengths. Analysis then shows that the first set depends only on the area of a contour of integration and (besides its dependence on the area) the second set also depends on the shape of the contour.

Another example, duality transformations, that change a given (original) theory with the coupling constant g into a (dual) theory with $(\frac{1}{g})$ as its coupling constant and play an important role in discussing strong coupling theories [Sav 80, Ita 81], benefit from the inversion relation. After applying the inversion formula to pure Yang-Mills theories it is found that [Ita 81, Miz 82] the dual theory goes over the original theory in the weak coupling limit and vice versa in the strong coupling limit. In scalar quantum chromodynamics, Mizrachi has concluded that, apart from the usual Lagrangian in the generating functional for dual fields, there occur self interactions of gauge fields and extra antisymmetric tensor fields coupled to the dual gauge fields and scalar fields. Finally generalization of the inversion formula to supersymmetric theories has been presented by Ohrndorf [Ohr 85]. Here he has shown that the gauge connection as well as the prepotential can be expressed in terms of the supersymmetric field strength.

Although the Fock-Schwinger gauge has positive features there are disadvantages as well. The non-translational invariance [Nov 84, Kum 86, Sch 87, Sch 89] and the nonlocality of the inversion formula [Kum 86, Ell 90, Wit 62] are, perhaps, the main drawbacks, producing complexities in the Fock-Schwinger gauge propagator and, as a result, difficulties in perturbation theory calculations. Such difficulties were exhibited by Kummer and Weiser [Kum 86] on their work on one-loop graphs in spinor quantum electrodynamics. Despite the intricacies of the computations they found the interesting result that up to first order the scattering matrix in the Fock-Schwinger gauge is equivalent, on mass-shell, to that in the Feynman gauge. The main aim of the thesis is to widen the application of the perturbation methods to scalar electrodynamics and more significantly quantum chromodynamics, and specifically to calculate the significant one-loop graphs in those gauge theories.

1.2 The Thesis

The thesis consists of six chapters. The first chapter is given for introduction while the final chapter is devoted to conclusions. The remaining ones constitute the main body of the thesis.

Inversion formulas are the main ingredient of Chapter 2. After a formal derivation of the formula and its necessary and sufficient conditions, we go to consider some simple classical systems. The Fock-Schwinger gauge potentials for these systems are then computed by making use of the formula. In addition, the scattering of quantized charged particles by the Fock-Schwinger gauge potential is calculated.

In Chapter 3, the Fock-Schwinger gauge as well as the Lorentz and axial-type gauge propagators are derived by two methods or two different choices of gauge fixing Lagrangians. The first method is the familiar one of adding gauge fixing term while auxiliary or Lagrange multiplier fields are introduced in the second method. Their symmetry properties are also obtained.

Those propagators are then employed to derive the Ward-Takahashi and the BRST identities. Since the Fock-Schwinger gauge theory is a ghost-free theory the BRST identities are derived in two ways: with and without introducing ghost fields. These ideas are carried through in Chapter 4.

The fifth chapter is the most substantial part of the thesis. Perturbation calculations to one-loop order in quantum electrodynamics and quantum chromodynamics are presented and compared with more familiar translational-invariant gauge choices. It is then possible to prove the on-mass-shell equivalence of the two treatments.

The general notations used throughout the thesis and various details of calculations are contained in the Appendices.

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Chapter 2

Fock-Schwinger Gauge Potentials

The main goals of this chapter are to derive the inversion formula and, by its use, to obtain the Fock-Schwinger potentials for some classical systems. As an illustration the scattering of quantized charged particles in a Fock-Schwinger (FS) potential will be derived and proved to be identical to the Coulomb scattering.

2.1 Gauge Transformations

Any theory of fundamental nature of matter must be consistent with quantum theory as well as relativity [Ryd 85]. Therefore we must frame the theory in its Lorentz covariant form.

In electrodynamics, for example, it is necessary to reformulate the noncovariant form of the Maxwell equations into the covariant one by introducing field strength tensors $F_{\mu\nu}(x)$ and four-vector potentials A_μ

$$\begin{aligned} F^{\mu\nu}(x) = &\ \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \\ = &\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \end{aligned} \tag{2.1}$$

The four three-vector Maxwell equations combine to

$$\partial_\mu F^{\mu\nu} = j^\nu; \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.2}$$

and are now covariant. Here $j^\mu = (\rho, \vec{j})$ and $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The Lagrangian of electromagnetic fields then reads

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{2.3}$$

which is automatically covariant. Thus the Lorentz covariance of the theory is complete and explicit.

One important consequence of introducing potentials $A_\mu(x)$ is that if one transforms $A_\mu(x)$ into $A'_\mu(x)$ according to

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x) \tag{2.4}$$

for some arbitrary function $\Lambda(x)$ the Lagrangian (2.3) remains unchanged. One then says that the theory of electromagnetism is invariant under the transformation (2.4) which is then called a gauge transformation. Hence the Maxwell theory is a gauge invariant theory. The name gauge field is ascribed to the potential $A_\mu(x)$ for historical reasons.

In 1954 Yang and Mills [Yan 54, Mil 89] proposed a new theory for strong nuclear interactions, very similar to the electromagnetic theory. The difference between both theories is in respect of their gauge groups. This leads to somewhat different properties of the gauge fields. The gauge fields in electromagnetism are Abelian while they are non-Abelian in Yang-Mills theory.

The nonuniqueness of potentials due to the gauge invariance of the theory allows us to choose certain conditions which make the potential unique. These are often called the gauge fixing conditions or simply the gauge choices. Clever choices of gauge lead to interesting simplifications but can also destroy manifest covariance [Itz 80].

One of the most familiar gauges is the Lorentz gauge $\partial^\mu A_\mu = 0$. Although it has some advantages such as relativistic invariance and uniform Feynman's $i\epsilon$-prescription for the momentum space singularities of propagators [Lei 87] there are also disadvantages: ghost particles should arise in non-Abelian theories which considerably complicate perturbation calculations. It is also difficult to handle certain topical models such as supersymmetric Yang-Mills and superstring theories in the Lorentz gauge [Lei 87]. Such complexities have led many theoretical physicists to examine other gauges like the axial-type gauges and the less familiar FS gauge.

Some properties of the FS gauge

$$x^{\mu}A_{\mu}(x) = 0 \tag{2.5}$$

have been mentioned in the Introduction. The relativistic covariance of the condition is obvious since the dot (scalar) product between two four-vectors, in this case, x^{μ} and $A^{\mu}(x)$, is Lorentz invariant (see for example [Jac 75]). Further, the FS gauge theory is ghost-free and the proof of this will be given in chapter 4. The inversion formula will be derived after the next section. Our immediate task is to consider the attainability and completeness of the FS gauge.

2.2 Attainability and Completeness of the Fock-Schwinger Gauge

Consider the gauge transformations in non-Abelian theories, where $U(x)$ refers to some internal group unitary change,

$$A_{\mu} \longrightarrow A'_{\mu} = UA_{\mu}U^{-1} - \frac{i}{g}(\partial_{\mu}U)U^{-1}. \tag{2.6}$$

Suppose that potential $A_{\mu}(x)$ does not obey the gauge condition (2.5) but $A'_{\mu}(x)$ does satisfy it. Then one has

$$\frac{i}{g}(x^{\mu}\partial_{\mu}U(x)) = U(x)x^{\mu}A_{\mu}. \tag{2.7}$$

Replacing $x_{\mu} \rightarrow \alpha x_{\mu}$ where $\alpha \in [0,1]$ is a parameter, equation (2.7) reads

$$x^{\mu}\partial_{\mu}U(\alpha x) = \alpha\frac{d}{d\alpha}U(\alpha x) = -igU(\alpha x)\alpha x^{\mu}A_{\mu}(\alpha x) \tag{2.8}$$

or

$$\frac{d}{d\alpha}U(\alpha x) = -igU(\alpha x)x^{\mu}A_{\mu}(\alpha x). \tag{2.9}$$

Hence

$$U(x) = P\left[exp\left(-ig\int_0^1 d\alpha x^{\mu}A_{\mu}(\alpha x)\right)\right]U(0). \tag{2.10}$$

P denotes path ordering in the variable α and $U(0)$ is an arbitrary initial value for $U(x)$. Thus the gauge condition (2.5) is attainable [Cro 80, Zuk 86, Gal 89, Gal 90] which means that one can always find gauge potentials satisfying condition (2.5) by means of some appropriate $U(x)$ as given by (2.10).

Let one now suppose that both $A'_\mu(x)$ and $A_\mu(x)$ satisfy the FS gauge condition. Accordingly the right-hand side of equation (2.7) vanishes

$$x^\mu \partial_\mu U(x) = 0. \tag{2.11}$$

Transforming $x_\mu \to \alpha x_\mu$, equation (2.11) becomes

$$x^\mu \partial_\mu U(\alpha x) = 0. \tag{2.12}$$

Equations (2.11) and (2.12) allow us to conclude that

$$U(x) = U(\alpha x) = constant \tag{2.13}$$

i.e. U is a homogenous function of zeroth degree in x. Thus, apart from constant gauge transformations, the gauge condition (2.5) is a complete gauge condition [Cro 80, Zuk 86, Gal 89, Gal 90]. The homogeneity of U (of zeroth degree) is the reason, explaining why the gauge condition (2.5) is called the homogenous gauge choice [Aza 81]. Another name, the complete Lorentz-covariant gauge [Cro 80], refers to its properties: complete and Lorentz covariant.

2.3 Inversion Formulae

The inversion formula to which we have alluded is nothing but the expression of potentials $A_\mu(x)$ in terms of their field strengths $F_{\mu\nu}(x)$. It is called inversion since the familiar relationship between both is in the expression of field strengths $F_{\mu\nu}(x)$ in terms of (the derivative of) potentials $A_\mu(x)$ as is seen in (2.1); inversion is the converse. The derivation of the inversion formula in the FS gauge goes as follows. Consider the electromagnetic field strength (2.1). One then has

$$x^\mu F_{\mu\nu}(x) = (1 + x^\mu \partial_\mu) A_\nu(x) - \partial_\nu x^\mu A_\mu(x). \tag{2.14}$$

Replacing x^μ by αx^μ where $\alpha \in [0,1]$ is a parameter and then integrating over α from 0 to 1, equation (2.14) becomes

$$\begin{aligned}\int_0^1 d\alpha\, \alpha x^\mu F_{\mu\nu}(\alpha x) &= \int_0^1 d\alpha \left[(1+x\partial)A_\nu(\alpha x) - \frac{1}{\alpha}\partial_\nu \alpha x^\mu A_\mu(\alpha x)\right] \\ &= \int_0^1 d\alpha \left[\frac{d}{d\alpha}\alpha A_\nu(\alpha x) - \frac{1}{\alpha}\partial_\nu K(\alpha x)\right]. \qquad (2.15)\end{aligned}$$

Hence

$$A_\mu(x) = -\int_0^1 d\alpha\, \alpha x^\nu F_{\mu\nu}(\alpha x) + \int_0^1 \frac{d\alpha}{\alpha}\partial_\mu K(\alpha x) \qquad (2.16)$$

where $K(x)$ is a function of x defined by

$$x^\mu A_\mu(x) = K(x). \qquad (2.17)$$

Equation (2.16) is the inversion formula in the inhomogenous FS gauge (2.17). Setting $K(x) = 0$ leads one to the inversion formula

$$A_\mu(x) = -\int_0^1 d\alpha\, \alpha x^\nu F_{\mu\nu}(\alpha x). \qquad (2.18)$$

It turns out that the gauge potentials $A_\mu(x)$ at a point x not only recieve contributions from the field strengths $F_{\mu\nu}$ at point x but also by all those points αx along a straight line between point x and the origin. In this sense the gauge potentials $A_\mu(x)$ are nonlocal [Ber 56, Wit 62, Ell 90]. It is worth mentioning that the inversion formula (2.18) may be derived by a simple geometrical argument [Dur 82] as well as by applying Poincaré lemma* in a star shaped region in a modern differential geometry [Bri 82] (see Appendix D). From now on we only consider the homogenous FS gauge condition (2.5) and therefore the inversion formula (2.18).

To ensure that $A_\mu(x)$ in (2.18) are really the electromagnetic potentials one should be able to derive the field strengths $F_{\mu\nu}$ from (2.18). Operating the curl on (2.18), one has

$$\begin{aligned}\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) &= -\int_0^1 d\alpha\, \alpha[\partial_\mu x^\beta F_{\nu\beta}(\alpha x) - \partial_\nu x^\beta F_{\mu\beta}(\alpha x)] \qquad (2.19) \\ &= -\int_0^1 d\alpha\, \alpha[2F_{\nu\mu}(\alpha x) + x^\beta \partial_\mu F_{\nu\beta}(\alpha x) - x^\beta \partial_\nu F_{\mu\beta}(\alpha x)].\end{aligned}$$

*This is the reason why the gauge condition (2.5) is sometimes called the Poincaré gauge [Bri 82, Ska 85, Gal 89, Gal 90].

Now the first term on the right-hand side of (2.19) may be integrated by parts

$$\begin{aligned} -2\int_0^1 d\alpha\ \alpha F_{\nu\mu}(\alpha x) &= \alpha^2 F_{\mu\nu}(\alpha x)|_{\alpha=0}^1 - \int_0^1 d\alpha\ \alpha^2 \frac{d}{d\alpha} F_{\mu\nu}(\alpha x) \\ &= F_{\mu\nu}(x) - \int_0^1 d\alpha\ \alpha x^\beta \partial_\beta F_{\mu\nu}(\alpha x). \end{aligned} \tag{2.20}$$

Putting (2.20) into (2.19), equation (2.19) yields

$$\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{\mu\nu}(x) - \int_0^1 d\alpha\ \alpha^2 x^\beta [\partial'_\beta F_{\mu\nu}(\alpha x) + \partial'_\mu F_{\nu\beta}(\alpha x) + \partial'_\nu F_{\beta\mu}(\alpha x)] \tag{2.21}$$

where $\partial'_\mu = \frac{1}{\alpha}\partial_\mu$. One now notices that $A_\mu(x)$ in (2.18) will represent the electromagnetic potentials with $F_{\mu\nu}(x)$ as their field strength tensors provided that the last term of equation (2.21) vanishes,

$$\partial_\beta F_{\mu\nu}(x) + \partial_\mu F_{\nu\beta}(x) + \partial_\nu F_{\beta\mu}(x) = 0. \tag{2.22}$$

But these are nothing but the homogenous Maxwell equations (2.2b) or the Bianchi identities in Abelian theories. Hence the Bianchi identities (2.22) are the necessary and sufficient conditions for the inversion formula (2.18) to be relations between field potentials $A_\mu(x)$ and their field strengths $F_{\mu\nu}(x)$ [Dur 82].

The inversion formula (2.18) also holds for non-Abelian theories

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)] \tag{2.23}$$

because if one multiplies x^μ and $F_{\mu\nu}(x)$ in (2.23) the commutator terms vanish and equation (2.14) remains unchanged due to gauge condition (2.5). However the identities (2.22) are not correct in non-Abelian theories. To obtain the right constraints let us go back to equation (2.21) but with $F_{\mu\nu}(x)$ defined in (2.23). It turns out that the second term in the right-hand side of equation (2.21) does not vanish because the left-hand side of equation (2.21) is not equal to $F_{\mu\nu}(x)$ anymore. If one adds to both sides of (2.21) the identity (remember $x \cdot A(x) = 0$)

$$
\begin{aligned}
& ig \int_0^1 d\alpha \; \alpha^2 x^\beta \{[A_\mu(\alpha x), F_{\nu\beta}(\alpha x)] + [A_\nu(\alpha x), F_{\beta\mu}(\alpha x)]\} \\
= \; & ig \int_0^1 d\alpha \; \alpha^2 x^\beta \frac{1}{\alpha} \{[A_\mu(\alpha x), \partial_\nu A_\beta(\alpha x)] - [A_\mu(\alpha x), \partial_\beta A_\nu(\alpha x)] \\
& + [A_\nu(\alpha x), \partial_\beta A_\mu(\alpha x)] - [A_\nu(\alpha x), \partial_\mu A_\beta(\alpha x)]\} \\
= \; & ig \int_0^1 d\alpha \; \alpha \{-2[A_\mu(\alpha x), A_\nu(\alpha x)] - \alpha \frac{d}{d\alpha}[A_\mu(\alpha x), A_\nu(\alpha x)]\} \\
= \; & -ig \int_0^1 d\alpha \frac{d}{d\alpha} \alpha^2 [A_\mu(\alpha x), A_\nu(\alpha x)] \\
= \; & -ig[A_\mu(x), A_\nu(x)]
\end{aligned}
\tag{2.24}
$$

one arrives at

$$
\begin{aligned}
& \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)] \\
= \; & F_{\mu\nu}(x) - \textstyle\int_0^1 d\alpha \; \alpha^2 x^\beta [D_\beta F_{\mu\nu}(\alpha x) + D_\mu F_{\nu\beta}(\alpha x) + D_\nu F_{\beta\mu}(\alpha x)]
\end{aligned}
\tag{2.25}
$$

where

$$D_\beta F_{\mu\nu} = \partial_\beta F_{\mu\nu} - ig[A_\beta, F_{\mu\nu}]. \tag{2.26}$$

Equation (2.25) allows us to conclude that the Bianchi identities

$$D_\beta F_{\mu\nu}(x) + D_\mu F_{\nu\beta}(x) + D_\nu F_{\beta\mu}(x) = 0 \tag{2.27}$$

are the necessary and sufficient conditions for the inversion formula (2.18) to succeed in non-Abelian theories [Cro 80, Dur 82]. Notice that equation (2.18) is actually the general solutions of equations (2.1) and (2.23) without fixing the gauge (thus they hold for all gauges in the vicinity of the origin) [Itz 80]. This is obvious since around the origin, $x_\mu \approx 0$, the last term on the right-hand side of equation (2.14) and the commutator term which appears in non-Abelian theories may be neglected.

One important point that should be mentioned is that for Abelian theories the inversion formula in (2.18) is only a special case of the more general relations discovered by Cornish [Cor 84], namely that the potential can be expressed in terms of the field strength via

$$A_\alpha(x) = \int_0^1 d\lambda F_{\mu\nu}(S) \frac{\partial S^\mu}{\partial \lambda} \frac{\partial S^\nu}{\partial x^\alpha}, \tag{2.28}$$

where $S = S(\lambda, x)$ is a two dimensional surface with parameter $\lambda \in [0,1]$. The proof is as follows. Consider a one parameter closed path in space time

$$\begin{aligned} x^\alpha = & \; w^\alpha(\nu) \qquad\qquad \nu_1 \leq \nu \leq \nu_2 \\ w^\alpha(\nu_1) = & \; w^\alpha(\nu_2) \end{aligned} \tag{2.29}$$

with $\frac{dw^\alpha}{d\nu}$ is continuous. Let also define a two dimensional surface S

$$x^\alpha = S^\alpha(\lambda, w(\nu)) \qquad\qquad \nu_1 \leq \nu \leq \nu_2; \quad 0 \leq \lambda \leq 1 \tag{2.30}$$

which is differentiable and satisfies boundary conditions

$$S^\alpha(0, w^\alpha(\nu)) = 0; \qquad\qquad S^\alpha(1, w(\nu)) = w^\alpha(\nu). \tag{2.31}$$

Now one obtains the following integral over S

$$\begin{aligned} \int_S F_{\rho\sigma} d\Omega^{\rho\sigma} &= \int_{\nu_1}^{\nu_2} \int_0^1 F_{\rho\sigma}\{S[\lambda, w(\nu)]\} \left(\frac{\partial S^\rho}{\partial \lambda} \frac{\partial S^\sigma}{\partial \nu} - \frac{\partial S^\rho}{\partial \nu} \frac{\partial S^\sigma}{\partial \lambda} \right) d\lambda d\nu \\ &= 2 \int_{\nu_1}^{\nu_2} \int_0^1 \left[F_{\rho\sigma}\{S[\lambda, w(\nu)]\} \frac{\partial S^\rho}{\partial \lambda} \frac{\partial S^\sigma}{\partial w^\alpha} d\lambda \right] \frac{\partial w^\alpha}{\partial \nu} d\nu \,; \end{aligned} \tag{2.32}$$

on the other hand

$$\begin{aligned} \int_S F_{\rho\sigma} d\Omega^{\rho\sigma} &= \int_S (\partial_\rho A_\sigma - \partial_\sigma A_\rho) d\Omega^{\rho\sigma} \\ &= 2 \int_S \partial_\rho A_\sigma d\Omega^{\rho\sigma} = 2 \int_w A_\nu dw^\nu = 2 \int_{\nu_1}^{\nu_2} A_\alpha \frac{\partial w^\alpha}{\partial \nu} d\nu \end{aligned} \tag{2.33}$$

according to Stokes' theorem. By equating (2.32) and (2.33) one finally arrives at equation (2.28) above. Notice that formula (2.28) was derived by choosing a surface which belongs to a class of surfaces satisfying boundary conditions (2.31). Since the number of such surfaces is infinite there are still many degrees of freedom. One therefore can say that choosing a certain surface S is equivalent to choosing a certain gauge. By setting $S^\alpha = \lambda x^\alpha$, for example, equations (2.28) lead to the FS gauge potentials (2.18)[†]. However not all familiar gauges, such as the Lorentz gauge, belongs to class of gauges (2.28) [Cor 84]. Note too that the Bianchi identities (2.22) act as conditions for relations (2.28) (see Appendix D). A formula similar to (2.28),

[†]The noncovariant version of the FS gauge $\vec{r} \cdot \vec{A}(\vec{r}, t) = 0$ is equivalent to the choice of $S^i = \lambda x^i$ and $S^0 = x^0$.

differing only in boundary conditions, was proposed for the first time by De Witt [Wit 62, Aha 62, Man 62] when he formulated quantum theories without potentials. This formula may be derived from choosing the gauge parameter [Ell 90, Wit 62, Aha 62, Hea 79, Bel 62, Roh 65]

$$\Lambda(x) = -\int_{P_0}^{P(x)} A_\mu(z)dz^\mu = \int_0^1 A_\mu(z(\lambda,x))\frac{\partial z^\mu}{\partial\lambda}d\lambda. \tag{2.34}$$

The inversion formula (2.18) can be written in the form of infinite series by Taylor-expanding the field strength $F_{\mu\nu}(\alpha x)$ around the origin. We have

$$\begin{aligned} A_\mu(x) &= -\int_0^1 d\alpha\, \alpha x^\nu F_{\mu\nu}(\alpha x) \qquad (2.35) \\ &= -\sum_{n=0}^{\infty}\frac{1}{n!(n+2)}x^\nu x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}F_{\mu\nu}(0) \end{aligned}$$

‡ Because of the condition (2.5) the identity (see Appendix D)

$$x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}F_{\mu\nu}(0) = x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_n}F_{\mu\nu}(0) \tag{2.36}$$

holds, and we then come to [Shi 80, Hub 82, Nov 84, Zuk 86]

$$\begin{aligned} A_\mu(x) &= -\sum_{n=0}^{\infty}\frac{1}{n!(n+2)}x^\nu x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_n}F_{\mu\nu}(0) \\ &= \frac{1}{2}x^\nu F_{\nu\mu}(0) + \frac{1}{3}x^\nu x^\alpha D_\alpha F_{\nu\mu}(0) + \frac{1}{8}x^\nu x^\alpha x^\beta D_\alpha D_\beta F_{\nu\mu}(0) + \cdots \qquad (2.37) \end{aligned}$$

i.e., the gauge condition (2.5) enables one to replace ordinary derivatives in (2.35) by their covariant ones. The elegant appearance of this series has attracted many theoretical physicists to take advantage of them, even though only the first few terms are usually taken.

There is no doubt that the Coulomb and Lorentz gauges are the most well known gauges as one can see that almost all texts on electrodynamics are written in terms of those gauge choices. The reason is that at the classical level, for example, both gauges play an important role in simplifying some problems. It is therefore of some interest to calculate the FS gauge potentials fixed by the FS gauge condition for some simple classical systems and then compare them with the familiar ones, the Coulomb/Lorentz potentials. The following section is devoted to this.

‡The name multipolar gauge [Kob 82, Kob 83, Ell 90] is also given to the gauge (2.5) because the expression (2.35) looks like a multipole expansion.

2.4 Fock-Schwinger Potentials in Simple Classical Systems

2.4.1 General Formulae

In this section we will exploit the inversion formula (2.18) to obtain the FS gauge potentials for some classical systems. According to formula (2.18) the FS scalar potential is

$$A_0(x) = -\int_0^1 d\alpha\ \alpha x^\nu F_{0\nu}(\alpha x) = -\int_0^1 d\alpha\ \alpha\vec{r}\cdot\vec{E}(\alpha x) \tag{2.38}$$

and the FS vector potential is

$$A^i(x) = -\int_0^1 d\alpha\ \alpha x_\nu F^{i\nu}(\alpha x) = -\int_0^1 d\alpha\ \alpha[x_0 E^i(\alpha x) + \epsilon^{ijk}x_j B_k(\alpha x)] \tag{2.39}$$

or

$$\vec{A}(x) = -\int_0^1 d\alpha\ \alpha[x_0\vec{E}(\alpha x) + \vec{r}\times\vec{B}(\alpha x)]. \tag{2.40}$$

It turns out that the vector potential $\vec{A}(x)$ does not depend purely on the magnetic field $\vec{B}(x)$ but also is dependent on the electric field $\vec{E}(x)$. This additional term is the major difference between the vector potential in the FS gauge and that in the Coulomb gauge. Both formulas (2.38) and (2.40) are quite general and will be applied to some classical charge/current configurations.

2.4.2 Electrostatics

Here the electric field is $\vec{E}(\vec{r})$, independent of time, and the magnetic field $\vec{B}(\vec{r})$ vanishes. The FS potentials of electrostatic systems reduce to

$$A_0(\vec{r}) = -\int_0^1 d\alpha\ \alpha\vec{r}\cdot\vec{E}(\alpha\vec{r}) \tag{2.41}$$

$$\vec{A}(x_0,\vec{r}) = -\int_0^1 d\alpha\ \alpha x_0\vec{E}(\alpha\vec{r}). \tag{2.42}$$

It is clear that, because of its dependence on the electric field, the vector potential $\vec{A}(x)$ is no longer zero. Another important fact is that it is proportional to time x_0. Thus, in FS terms electrostatic systems are not static, they depend on time!

Let us now relate the FS and the Coulomb potentials. As is well known, the Coulomb potential $V(\vec{r})$ is defined by

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}). \tag{2.43}$$

The corresponding Fock-Schwinger potentials become

$$A_0(\vec{r}) = \int_0^1 d\alpha\, \alpha \vec{r} \cdot \frac{\nabla}{\alpha} V(\alpha \vec{r}) = V(\vec{r}) + V_1(\vec{r}) \tag{2.44}$$

with $V_1(\vec{r}) = -\int_0^1 d\alpha\, V(\alpha\vec{r})$, and

$$\vec{A}(x_0, \vec{r}) = x_0 \int_0^1 d\alpha\, \alpha \frac{1}{\alpha} \nabla V(\alpha\vec{r}) = -x_0 \nabla V_1(\vec{r}). \tag{2.45}$$

Thus the FS scalar potential differs from the Coulomb potential by $V_1(\vec{r})$ which explains how the nonvanishing FS vector potential (2.45) comes about.

a) n point charges

The electric field $\vec{E}(\vec{r})$ measured at the point $\vec{r}$, due to n point charges q_s at $\vec{r}_s$ where s runs from 1 to n is defined as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{s=0}^{n} \frac{q_s}{|\vec{r} - \vec{r}_s|^3} \; (\vec{r} - \vec{r}_s) \tag{2.46}$$

and the corresponding Coulomb potential reads

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{s=1}^{n} \frac{q_s}{|\vec{r} - \vec{r}_s|} + constant. \tag{2.47}$$

Hence, the FS gauge potentials are

$$\begin{aligned} A_0(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{s=1}^{n} \frac{q_s}{|\vec{r} - \vec{r}_s|} - \frac{1}{4\pi\epsilon_0} \sum_{s=1}^{n} \int_0^1 d\alpha \frac{q_s}{|\alpha\vec{r} - \vec{r}_s|} && (2.48) \\ \vec{A}(x_0, \vec{r}) &= \frac{x_0}{4\pi\epsilon_0} \sum_{s=1}^{n} \int_0^1 d\alpha \nabla \frac{q_s}{|\alpha\vec{r} - \vec{r}_s|}. && (2.49) \end{aligned}$$

b) Electric dipole

This is nothing but two point charges q and $-q$ infinitesimally separated:

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - \vec{r'} - \vec{l}|} - \frac{1}{|\vec{r} - \vec{r'}|} \right] \simeq \frac{q}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r'}) \cdot \vec{l}}{|\vec{r} - \vec{r'}|^3}. \tag{2.50}$$

Here the charges q and $-q$ are at the points $\vec{r'}+\vec{l}$ and $\vec{r'}$ respectively and $|\vec{l}| \ll |\vec{r}-\vec{r'}|$. The FS gauge potentials (2.44) and (2.45) read

$$A_0(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r'})\cdot\vec{l}}{|\vec{r}-\vec{r'}|^3} - \frac{q}{4\pi\epsilon_0}\int_0^1 d\alpha \frac{(\alpha\vec{r}-\vec{r'})\cdot\vec{l}}{|\alpha\vec{r}-\vec{r'}|^3} \tag{2.51}$$

$$\vec{A}(x_0,\vec{r}) = \frac{qx_0}{4\pi\epsilon_0}\int_0^1 d\alpha\nabla\frac{(\alpha\vec{r}-\vec{r'})\cdot\vec{l}}{|\alpha\vec{r}-\vec{r'}|^3}. \tag{2.52}$$

c) Infinite line charge along z-axis

The electric field in this case is defined by

$$\vec{E}(\vec{r}) = \lim_{a,b\to\infty}\frac{\lambda}{4\pi\epsilon_0}\int_{-a}^{b}\frac{dz'}{|\vec{r}-\vec{r'}|^3}(\vec{r}-\vec{r'}) \tag{2.53}$$

where λ is a charge density chosen to be constant. After integration over z' one has

$$\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\epsilon_0}\frac{x\vec{i}+y\vec{j}}{x^2+y^2} \tag{2.54}$$

and accordingly

$$V(\vec{r}) = -\frac{\lambda}{4\pi\epsilon_0}\ln(x^2+y^2) + constant. \tag{2.55}$$

The FS gauge potentials are

$$A_0(\vec{r}) = -\frac{\lambda}{4\pi\epsilon_0}\ln(x^2+y^2) + \frac{\lambda}{4\pi\epsilon_0}\int_0^1 d\alpha\ \ln(\alpha^2x^2+\alpha^2y^2) = -\frac{\lambda}{2\pi\epsilon_0}, \tag{2.56}$$

a constant, and

$$\vec{A}(x_0,\vec{r}) = -\frac{\lambda x_0}{4\pi\epsilon_0}\int_0^1 d\alpha\nabla\ln(\alpha^2x_1^2+\alpha^2x_2^2) = -\frac{\lambda x_0}{4\pi\epsilon_0}\frac{x^1\vec{i}+x^2\vec{j}}{x_1^2+x_2^2} = -x_0\vec{E}(\vec{r}). \tag{2.57}$$

d) Charged ring (with z-axis as its symmetry axis)

In this case the Coulomb gauge scalar potential is given by

$$V(\vec{r}) = \frac{\lambda}{4\pi\epsilon_0}\oint\frac{dl_1}{|\vec{r}-\vec{r_1}|} + constant. \tag{2.58}$$

Since

$$\begin{aligned} dl_1 &= ad\phi_1 \\ (\vec{r}-\vec{r_1}) &= (x-a\cos\phi_1)\vec{i}+(y-a\sin\phi_1)\vec{j}+z\vec{k}, \end{aligned}$$

a being the radius of the ring, one has

$$V(\vec{r}) = \frac{a\lambda}{2\epsilon_0}\frac{1}{\sqrt{r^2+a^2}} + constant \qquad \text{for} \quad r \gg a \tag{2.59}$$

and therefore

$$\begin{aligned} A_0(\vec{r}) &= \frac{a\lambda}{2\epsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2}} - \int_0^1 d\alpha \frac{1}{\sqrt{\alpha^2 r^2 + a^2}}\right\} \simeq \frac{a\lambda}{2r\epsilon_0}(1 - \ln\frac{2r}{a}) & (2.60) \\ \vec{A}(x_0,\vec{r}) &= \frac{a\lambda x_0}{2\epsilon_0}\int_0^1 d\alpha \nabla \frac{1}{\sqrt{\alpha^2 r^2 + a^2}} = -\frac{a\lambda x_0}{2\epsilon_0}\int_0^1 d\alpha\alpha^2 \frac{\vec{r}}{(\alpha^2 r^2 + a^2)^{\frac{3}{2}}} \\ &\simeq \frac{a\lambda x_0}{2r^3\epsilon_0}\vec{r}\,(1 - \ln\frac{2r}{a}) \qquad \text{for} \quad r \gg a & (2.61) \end{aligned}$$

2.4.3 Magnetostatics

Since in magnetostatic systems the magnetic field $\vec{B}(\vec{r})$ is independent of time and the electrostatic field vanishes, the FS potentials (2.38) and (2.40) reduce to

$$\begin{aligned} A_0(x_0,\vec{r}) &= 0 & (2.62) \\ \vec{A}(\vec{r}) &= -\int_0^1 d\alpha\; \alpha\vec{r} \times \vec{B}(\alpha\vec{r}). & (2.63) \end{aligned}$$

Thus whereas in electrostatic systems both the FS scalar potential and the FS vector potential have different value from those in the Coulomb gauge, in systems of magnetostatics only the vector potentials in both gauges are different. It is worth noting that unlike electrostatics, the FS potentials in the magnetostatic systems are indeed static.

In order to obtain the difference between the Coulomb gauge and the FS gauge potentials let us define a vector potential $\vec{A}^f(\vec{r})$ satisfying

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}^f(\vec{r}). \tag{2.64}$$

The superscript f is to remind one that there is an infinite number of vectors satisfying equation (2.64). Those vectors may be written as

$$\vec{A}^f(\vec{r}) = \vec{A}^c(\vec{r}) + \nabla f(\vec{r}) \tag{2.65}$$

where the arbitrariness resides in the function $f(\vec{r})$. The superscript c in the first term on the right-hand side of (2.65) will be associated with the Coulomb gauge condition later, but now, it is just to distinguish the potential $\vec{A}^f(\vec{r})$ on the left-hand side of (2.65). Substituting (2.64) and (2.65) into (2.63) one obtains

$$\vec{A}(\vec{r}) = -\int_0^1 d\alpha\alpha\vec{r} \times [\frac{1}{\alpha}\nabla \times \vec{A}^c(\alpha\vec{r})] = \vec{A}^c(\vec{r}) - \int_0^1 d\alpha\nabla[\vec{r}\cdot\vec{A}^c(\alpha\vec{r})]. \tag{2.66}$$

In (2.66) $\vec{A}(\vec{r})$ is the FS gauge vector potential and $\vec{A}^c(\vec{r})$ is a vector potential which has not been gauge-fixed yet. Therefore if one is restricted to a certain gauge condition on $\vec{A}^c(\vec{r})$, equation (2.66) describes the relationship between the FS gauge vector potential and other-gauge vector potential. If one takes the "other-gauge" as the FS gauge, the second term on the right-hand side of (2.66) vanishes and $\vec{A}(\vec{r}) = \vec{A}^c(\vec{r})$ as expected.

Let one now choose $\vec{A}^c(\vec{r})$ as a vector potential in the Coulomb gauge and then calculate the FS gauge vector potential for some simple systems.

a) Infinite steady current

The magnetic field at point $\vec{r}$ due to a flow of steady current I along the x-axis is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} dx' \frac{\vec{\imath}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0 I}{2\pi}\,\frac{-z\vec{\jmath}+y\vec{k}}{y^2+z^2}. \tag{2.67}$$

The corresponding Coulomb gauge vector potential is

$$\vec{A}^c(\vec{r}) = -\frac{\mu_0 I}{4\pi}\vec{\imath}\,\ln(y^2+z^2) + \nabla f(x,y,z); \qquad \nabla^2 f(\vec{r}) = 0 \tag{2.68}$$

and the FS gauge vector potential is

$$\begin{aligned}\vec{A}(\vec{r}) &= -\frac{\mu_0 I}{4\pi}\vec{\imath}\,\ln(y^2+z^2) + \nabla f(\vec{r}) + \frac{\mu_0 I}{4\pi}\nabla\int_0^1 d\alpha\, x\ln[\alpha^2(y^2+z^2)] \\ &\quad -\nabla\int_0^1 d\alpha\ \vec{r}\cdot\frac{1}{\alpha}\nabla f(\alpha\vec{r}) \\ &= -\frac{\mu_0 I}{2\pi}\vec{\imath} + [xB_z\vec{\jmath} - xB_y\vec{k}]. \end{aligned}\tag{2.69}$$

It should be noted here that the FS gauge vector potential (2.69) is free of the gauge parameter $f(\vec{r})$, whereas the Coulomb potential (2.68) is not. Another observation is that the FS vector potential is perpendicular to the magnetic field. Unless one chooses $f = f(r)$ rather than $f = f(\vec{r})$, the Coulomb gauge vector potential and its associated magnetic field are not perpendicular.

b) Steady current ring

The vector potential of a system of a closed steady current I in the Coulomb gauge is of the form

$$\vec{A}^c(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}'}{|\vec{r} - \vec{r}'|}. \tag{2.70}$$

It is a solution of

$$\nabla(\nabla \cdot \vec{A}^c) - \nabla^2 \vec{A}^c = -\nabla^2 \vec{A}^c = \mu_0 \vec{\jmath} \tag{2.71}$$

with $\vec{\jmath} dV \to I d\vec{r}$. For a system of a current ring of radius r',

$$d\vec{l}' = r' d\phi' (-\vec{\imath} \sin\phi' + \vec{\jmath} \cos\phi')$$

the Coulomb vector potential (2.70) reads

$$\vec{A}^c(\vec{r}) = \frac{\mu_0 I}{4\pi} \frac{r'}{\sqrt{r^2 + r'^2}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(2n-1)!!}{n!} \binom{n}{k} \left(\frac{y}{x}\right)^k \left(\frac{x r'}{r^2 + r'^2}\right)^n$$
$$\int_0^{2\pi} d\phi' (-\vec{\imath} \sin\phi' + \vec{\jmath} \cos\phi') \sin^k \phi' \cos^{n-k} \phi'. \tag{2.72}$$

For $r' \ll r$, it becomes

$$\vec{A}^c(\vec{r}) = \frac{\mu_0 I}{4} \frac{r'^2}{(r^2 + r'^2)^{3/2}} (-y\vec{\imath} + x\vec{\jmath}) \tag{2.73}$$

and its corresponding FS gauge vector potential is

$$\vec{A}(\vec{r}) = \vec{A}^c(\vec{r}) - \nabla \vec{r} \cdot \int_0^1 d\alpha\, \vec{A}^c(\alpha\vec{r}) = \vec{A}^c(\vec{r}); \qquad r \gg r'. \tag{2.74}$$

Thus the FS gauge potential and the Coulomb gauge potential are equal in the far region. Of course the Coulomb and the FS gauge conditions hold asymptotically,

$$\nabla \cdot \vec{A}(\vec{r}) = \vec{r} \cdot \vec{A}(\vec{r}) = 0. \tag{2.75}$$

2.4.4 Constant Electromagnetic Fields

In this system the electric and magnetic fields are constant

$$\vec{E}(x) = \vec{E} = constant; \qquad \vec{B}(x) = \vec{B} = constant. \tag{2.76}$$

So the FS potentials, according to (2.38) and (2.40), are

$$A_0(\vec{r}) = -\int_0^1 d\alpha\alpha \; \vec{r}\cdot\vec{E} = -\frac{1}{2}\vec{r}\cdot\vec{E} \tag{2.77}$$

$$\vec{A}(x_0,\vec{r}) = -\int_0^1 d\alpha\alpha \; [x_0\vec{E} + \vec{r}\times\vec{B}] = -\frac{1}{2}(x_0\vec{E} + \vec{r}\times\vec{B}). \tag{2.78}$$

Thus the scalar potentials in the FS gauge and the Coulomb gauge only differ by a scale; the Coulomb scalar potential is twice the FS scalar potential. The vector potentials, on the other hand, are different because of the extra term $-\frac{1}{2}x_0\vec{E}$.

2.4.5 Plane Electromagnetic Waves

The plane electromagnetic fields have the forms

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_0 e^{-ikx}; \quad \vec{B}(\vec{r},t) = \vec{B}_0 e^{-ikx} \tag{2.79}$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant, and

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{k_0}; \quad \vec{E}_0\cdot\vec{k} = \vec{B}_0\cdot\vec{k} = \vec{E}_0\cdot\vec{B}_0 = 0; \quad k^2 = 0. \tag{2.80}$$

According to general formulae (2.38) and (2.40), the FS potentials are

$$\begin{aligned} A_0(x) &= -\vec{r}\cdot\vec{E}_0\int_0^1 d\alpha\alpha\, e^{-i\alpha kx} = \frac{\vec{r}\cdot\vec{E}_0}{(kx)^2}[1-(1+ikx)e^{-ikx}] \\ &= \vec{r}\cdot\vec{E}_0 e^{-ikx}\sum_{n=0}^{\infty}\frac{(ikx)^n}{(n+2)!} \end{aligned} \tag{2.81}$$

$$\begin{aligned} \vec{A}(x) &= (x_0\vec{E}_0 + \vec{r}\times\vec{B}_0)\frac{1}{(kx)^2}\left[1-(1+ikx)e^{-ikx}\right] \\ &= (x_0\vec{E}_0 + \vec{r}\times\vec{B}_0)e^{-ikx}\sum_{n=0}^{\infty}\frac{(ikx)^n}{(n+2)!} \end{aligned} \tag{2.82}$$

Equations (2.81) and (2.82), describing the FS potentials in a system of plane waves (2.79), are no longer plane waves! Note that the zero order of equations (2.81) and (2.82) have a similar form to equations (2.77) and (2.78), the FS potentials in a system of constant electromagnetic fields.

2.5 Potential Scattering of Charged Particles

We have calculated the FS gauge potentials for some classical electromagnetic systems. It was found that, in general, the FS gauge potentials and the Coulomb gauge potentials differ. Since both kind of potentials were derived from the same physical quantities, the electric and magnetic fields, the difference in their values cannot produce any physical effects. This also holds for quantum systems. The scattering of charged particles due to either the FS gauge potentials or the Coulomb gauge potentials must produce the same result. This last section will be used to derive the differential cross section of charged particles scattered by the FS gauge potentials.

Consider the transition amplitude [Itz 80] between the initial state

$$\psi_i(x) \propto u^{(\alpha)}(p_i)e^{-ip_i x} \tag{2.83}$$

and the final state

$$\psi_f(x) \propto u^{(\alpha)}(p_f)e^{-ip_f x} \tag{2.84}$$

of an electron scattered by the FS gauge potentials (2.48) and (2.49) with $K = \frac{q}{4\pi\epsilon_0}$, $s = 1$ and $q_s = q$

$$\begin{aligned} S_{fi} &= -ie \int d^4x \overline{u}^{(\alpha)}(p_f)\gamma_\mu A^\mu(x) e^{i(p_f-p_i)\cdot x} u^\beta(p_i) \\ &= S_{fi}(Coul) + ieK \int d^4x \overline{u}^{(\alpha)}(p_f)\gamma^0 \int_0^1 d\lambda \frac{1}{|\lambda\vec{r}-\vec{r'}|} e^{i(p_f-p_i)\cdot x} u^{(\beta)}(p_i) \qquad (2.85) \\ &\quad -ieK \int d^4x \overline{u}^{(\alpha)}(p_f) x_0 \gamma^i \partial_i \int_0^1 d\lambda \frac{1}{|\lambda\vec{r}-\vec{r'}|} e^{i(p_f-p_i)\cdot x} u^\beta(p_i) \end{aligned}$$

where $S_{fi}(Coul)$ is the transition amplitude of the electron due to the Coulomb potential $V(r) = \frac{K}{r}$ with $\vec{A}(\vec{r}) = 0$. Integration over x_0 in the last two terms on the right-hand side of equation (2.85) can be easily done. One obtains

$$
\begin{aligned}
& 2\pi ieK\left\{\overline{u}^{(\alpha)}(p_f)\gamma^0 u^{(\beta)}(p_i)\delta(E_f-E_i)\int_0^1 d\lambda\int d^3r\frac{e^{-i(\vec{p}_f-\vec{p}_i)\cdot\vec{r}}}{|\lambda\vec{r}-\vec{r}'|}\right. \\
& \left.-\,\overline{u}^{(\alpha)}(p_f)\gamma^i u^{(\beta)}(p_i)\left[\frac{\partial}{i\partial(E_f-E_i)}\delta(E_f-E_i)\right]i(p_f-p_i)_i\int_0^1 d\lambda\int d^3r\frac{e^{-i(\vec{p}_f-\vec{p}_i)\cdot\vec{r}}}{|\lambda\vec{r}-\vec{r}'|}\right\} \\
= \; & 2\pi ieK\left\{\overline{u}^{(\alpha)}(p_f)\gamma^0 u^{(\beta)}(p_i)\delta(E_f-E_i)\right. \qquad (2.86) \\
& \left.+\,\overline{u}^{(\alpha)}(p_f)\gamma^0(E_f-E_i)u^{(\beta)}(p_i)\frac{\partial}{\partial(E_f-E_i)}\delta(E_f-E_i)\right\}\int_0^1 d\lambda\int d^3r\frac{e^{-i(\vec{p}_f-\vec{p}_i)\cdot\vec{r}}}{|\lambda\vec{r}-\vec{r}'|}
\end{aligned}
$$

where the mass-shell condition

$$
\overline{u}^{(\alpha)}(p_f)\,(\not{p}_f-\not{p}_i)\,u^{(\beta)}(p_i)=0 \qquad (2.87)
$$

has been used. It turns out that equation (2.86) vanishes since it is proportional to

$$
\delta(E)+E\frac{\partial}{\partial E}\delta(E)=\frac{\partial}{\partial E}E\delta(E-0)=0 \qquad (2.88)
$$

with $E=E_f-E_i$. Hence, the only non-zero term of equation (2.85) is $S_{fi}(Coul)$:

$$
S_{fi}=S_{fi}(Coul) \qquad (2.89)
$$

i.e., the transition amplitude of an electron scattered by the FS gauge potentials is equal to that by the Coulomb gauge potentials and likewise for differential cross-sections.

To summarize we have calculated potentials for some classical systems which obey the FS gauge condition. Even though they differ from the Coulomb/Lorentz gauge potentials the physical content is the same and this has been verified in scattering of quantized charged particles. The remaining chapters will be devoted to higher order corrections in the relativistic quantum theory. Here quantities such as propagators play a crucial role and hence the next chapter concentrates on the FS propagators.

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Chapter 3

Fock-Schwinger Gauge Propagators

In this chapter the Lorentz, axial and FS gauge propagators will be derived in coordinate space by the use of two different gauge-fixing of Lagrangians $\mathcal{L}_{\mathcal{GF}_1} = -\frac{1}{2\lambda}(G\cdot A)^2$ and $\mathcal{L}_{\mathcal{GF}_2} = CG\cdot A + \frac{\lambda}{2}C^2$ where $C(x)$ is an auxiliary or Lagrange multiplier field. The undoubtedly popular gauges, Lorentz and axial-type, are incorporated here in order to check the calculations. The FS gauge propagator will also be presented in momentum space and various symmetry properties will be derived. The first section is devoted to a brief review of generating functionals in order to introduce the basic theoretical idea which underlies the derivations. This review is based on Bailin and Love [Bai 86], Ryder [Ryd 85], Burden [Bur 90] and Nash [Nas 78].

3.1 Review on Generating Functionals

The transition amplitude of a (non-relativistic) quantum system in which its state are $|q', t'>$ and $|q'', t''>$ at time t' and $t'' > t'$ respectively is defined by

$$< q'', t''|q', t' >=< q''| \exp[-i\hat{H}(t'' - t')]|q' > . \tag{3.1}$$

Here $|q>$ is an eigenstate of position operator Q with its eigenvalue q in the Schrö dinger picture

$$Q|q>=q|q> . \tag{3.2}$$

The Hamiltonian operator $\hat{H}$ is time independent. By dividing the time interval $t''-t'$ into $(N+1)$ interval of equal length ϵ and putting $\epsilon \to 0$ (or $N \to \infty$) one may write

$$\begin{aligned} <q'',t''|q',t'>= \lim_{N\to\infty} \prod_{j=1}^{N} \int dq_j <q'',t''|q_N,t_N><q_N,t_N|q_{N-1},t_{N-1}> \\ \cdots <q_1,t_1|q',t'> . \end{aligned} \tag{3.3}$$

According to equation (3.1), and when the Hamiltonian operator $\hat{H}$ has the form $H(Q,P) = \frac{P^2}{2m} + V(Q)$, one has, after some algebra,

$$\begin{aligned} <q_{j+1},t_{j+1}|q_j,t_j>= & \int \frac{dp_j}{2\pi} \exp\left\{i \int_{t'}^{t''} dt\,[p_j\dot{q}_j - H(q_j,p_j)]\right\} \\ \propto & \exp\left\{i \int_{t'}^{t''} dt\; L(q_j,\dot{q}_j)\right\} \end{aligned} \tag{3.4}$$

for every $j = 1,2,\cdots,N$. Hence, by putting $q_0 = q'$ and $q_{N+1} = q''$, inserting (3.4) into (3.3) leads to

$$<q'',t''|q',t'> \propto \int \mathcal{D}q \; \exp\left\{i \int_{t'}^{t''} dt\; L(q,\dot{q})\right\} \tag{3.5}$$

where the integration is over all functions $q(t)$ with boundary conditions $q(t) = q'$ and $q(t'') = q''$.

In the presence of an external source $j(t)$, the transition amplitude (3.5) becomes

$$<q'',t''|q',t'>^j \propto \int \mathcal{D}q \; \exp\left\{i \int_{t'}^{t''} dt\,[L(q,\dot{q}) + j(t)q]\right\}. \tag{3.6}$$

Now if the source $j(t)$ is non-zero only in the interval $t'' > t_a > t > t_b > t'$ the left-hand side of (3.6) can be written as

$$\begin{aligned} <q'',t''|q',t'>^j= & \sum_{n,m} \int dq_a dq_b <q'',t''|n><n|q_a,t_a><q_a,t_a|q_b,t_b>^j \\ & <q_b,t_b|m><m|q',t'> \\ = & \int dq_a dq_b \psi_0(q'',t'')\psi_0^*(q_a,t_a) <q_a,t_a|q_b,t_b>^j \psi_0(q_b,t_b)\psi_0^*(q',t') \end{aligned} \tag{3.7}$$

where $|n>$ are the energy eigenstates

$$H|n>=E_n|n>, \qquad E_n > E_0 \tag{3.8}$$

and

$$\psi_n(q,t) = < q,t|n> = e^{-iE_nt} < q|n> . \tag{3.9}$$

It turns out that in the limit $it'' \to \infty$ and $it' \to -\infty$, $< q'',t''|q',t'>^j$ is dominated by contribution from the vacuum $|0>$. Let one now define a functional $Z[j]$ as follows

$$\begin{aligned} Z[j] &= \int dq_a dq_b \psi_0^*(q_a,t_a) < q_a,t_a|q_b,t_b>^j \psi_0(q_b,t_b) \\ &\propto \; < q'',t''|q',t'>^j . \end{aligned} \tag{3.10}$$

Thus $Z[j]$ is nothing but the vacuum expectation value of the transition amplitude which can be taken in the limit where t_a and $-t_b$ (hence t' and $-t''$) $\to \infty$. By recalling (3.6), functional differentiation of (3.10) with respect to $j(t)$ n times leads to

$$\begin{aligned} \left.\frac{\delta^n Z[j]}{\delta j(t_1)\delta j(t_2)\cdots\delta j(t_n)}\right|_{j=0} &= i^n \int dq_a dq_b \psi_0^*(q_a,t_a) \\ & \quad < q_a,t_a|T[Q(t_1)Q(t_2)\cdots Q(t_n)]|q_b,t_b> \psi_0(q_b,t_b) \\ &= i^n < 0|T[Q(t_1)Q(t_2)\cdots Q(t_n)]|0> . \end{aligned} \tag{3.11}$$

This means that the vacuum expectation value of the time ordered product of any number of operators $Q(t)$ can be obtained by functional differentiating the vacuum to vacuum amplitude $Z[j]$. Thus if $Z[j]$ is known the vacuum expectation value of the time ordered product of operators $Q(t)$ may be obtained. This is why $Z[j]$ is called the generating functional.

The transition from the non-relativistic to the relativistic quantum theory is done just by replacing $q(t) \to \phi(t,\vec{x}) = \phi(x)$ and $Q(t) \to \hat{Q}(x)$. Therefore, e.g. in a quantum field theory of scalar fields,

$$Z[j] = \int \mathcal{D}\phi \; \exp\left\{ i \int dx \; [\mathcal{L}(\phi,\partial_\mu\phi) + j\phi] \right\} \tag{3.12}$$

$$\frac{\delta^n Z[0]}{\delta j(t_1)\delta j(t_2)\cdots\delta j(t_n)} = i^n < 0|T[\hat{Q}(t_1)\hat{Q}(t_2)\cdots\hat{Q}(t_n)]|0> . \tag{3.13}$$

To normalise the generating functional (3.12) the condition $Z[0] = 1$ is added. The Lagrangian density $\mathcal{L}$ is defined from expression $L = \int d^3x\mathcal{L}$. Now the Taylor series of the generating functional $Z[j]$ is given by

$$Z[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 dx_2 \cdots dx_n j(x_1) j(x_2) \cdots j(x_n) \tau(x_1, x_2, \cdots, x_n) \tag{3.14}$$

where

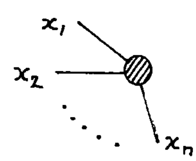
$$\begin{aligned} \tau(x_1, x_2, \cdots, x_n) &= \frac{\delta^n Z[0]}{i^n \delta j(t_1) \delta j(t_2) \cdots \delta j(t_n)} \\ &= < 0|T[\hat{Q}(t_1)\hat{Q}(t_2) \cdots \hat{Q}(t_n)]|0 > \end{aligned} \tag{3.15}$$

is called the n-point (Green's) function. Another Green's function which is called the connected Green's function is defined by

$$\tau_C(x_1, x_2, \cdots, x_n) = \frac{\delta^n W[0]}{i^n \delta j(t_1) \delta j(t_2) \cdots \delta j(t_n)}. \tag{3.16}$$

Graphically,

$$\tau_C(x_1, x_2, \cdots, x_n) = \tag{3.17}$$



with

$$W[j] = -\ln Z[j]. \tag{3.18}$$

The relationship between the two above Green's functions is given by

$$\tau(x_1, x_2, \cdots, x_n) = \tau_C(x_1, x_2, \cdots, x_n) + disc. \tag{3.19}$$

Here *disc.* stands for disconnected diagrams, i.e., the sum over all possible partition $\tau_C(x_1, x_2, \cdots, x_n) : \tau_C(x_1, x_2)\tau_C(x_3, \cdots, x_n), \tau_C(x_1, x_2, x_3)\tau_C(x_4, \cdots, x_n)$,etc.

Since only the Hamiltonian of the type $H = \frac{p^2}{2m} + V(q)$ has been used in the derivation the (free field) Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ (with $V = 0$) corresponds to the bilinear

$$\mathcal{L} = \frac{1}{2} \int dy \phi(x) \Delta^{-1}(x, y) \phi(y) \tag{3.20}$$

where $\Delta^{-1}(x, y)$ is a differential operator. Thus, after some algebra, equation (3.20) brings equation (3.12) into the form

$$Z[j] = \exp\left\{ -\frac{i}{2} \int dx dy \; j(x) \Delta(x, y) j(y) \right\} \tag{3.21}$$

upon using the relation

$$\phi(x) = -\int \Delta(x,y) j(y) dy. \tag{3.22}$$

The two point function is simply

$$\tau(x_1, x_2) =< 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)]|0 >= i\Delta(x_1, x_2). \tag{3.23}$$

Similarly, for vector fields $A_\mu^a(x)$, in which (3.12) becomes

$$Z[j] = \int \mathcal{D}A \ \exp\left\{ i \int dx \ [\mathcal{L}_0(A_\mu^a, \partial_\mu A_\mu^a) + j^{a\mu} A_\mu^a] \right\} \tag{3.24}$$

one has

$$iG_{\mu\nu}^{ab}(x,y) =< 0|T[A_\mu^a(x) A_\nu^b(y)]|0 > \tag{3.25}$$

and

$$A_\mu^a(x) = -\int dy G_{\mu\nu}^{ab}(x,y) j^{b\nu}(y). \tag{3.26}$$

3.2 Gauge-fixing Lagrangian Terms

Consider now the generating functional (3.24) with $j = 0$

$$Z[0] = \int \mathcal{D}A \ e^{i\int dx \mathcal{L}_0}; \qquad \mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^a(x) F^{a\mu\nu}(x). \tag{3.27}$$

As can be seen in the Appendix F the Lagrangian $\mathcal{L}_0$ is invariant under the gauge transformation

$$A_\mu'^a = A_\mu^{(\theta)a} = A_\mu^a + f^{abc}\theta^b A_\mu^c - \frac{1}{g}\partial_\mu \theta^a. \tag{3.28}$$

Because of this invariance the generating functional (3.27) can be written as

$$\begin{aligned} Z[0] &= \int \mathcal{D}A^{(\theta)} \ \exp \ i \int dx \mathcal{L}_0(A^{(\theta)}, \partial A^{(\theta)}) \\ &= \int \mathcal{D}A^{(\theta)} \ \exp \ i \int dx \mathcal{L}_0(A, \partial A). \end{aligned} \tag{3.29}$$

Accordingly, integration over the gauge-transformed field $A^{(\theta)}$ in (3.29) diverges because it includes an infinite gauge freedom volume factor $\int \prod_{x,a} d\theta^a(x)$. This factor should be factorized out before using the perturbation theory [Fad 67,Mut 87,Lei 87];

otherwise it will lead to ill-defined Green's functions [Lei 87]. To eliminate this factor Faddeev and Popov [Fad 67] introduced a functional $\Delta[A]$ via

$$\Delta[A] \int \mathcal{D}\theta \delta(G^{\mu} A_{\mu}^{(\theta)}) = 1 \tag{3.30}$$

where in the argument of the delta function we write G^{μ} instead of ∂^{μ} because we want to generalise this identity from the Lorentz gauge to other gauge choices. Here $\Delta[A]$ is gauge invariant since when we transform A into $A^{(\theta)}$ $\Delta[A]$ becomes

$$\Delta[A^{(\theta)}] = \left\{ \int \mathcal{D}\theta' \delta(G \cdot A^{(\theta\theta')}) \right\}^{-1} = \left\{ \int \mathcal{D}(\theta\theta') \delta(G \cdot A^{(\theta\theta')}) \right\}^{-1} = \Delta[A] \tag{3.31}$$

where the second equality comes from the fact that exists the gauge group identity [Fra 70, Gil 74, Ryd 85]

$$\int \mathcal{D}\theta f(\theta) = \int \mathcal{D}\theta f(\theta\theta'). \tag{3.32}$$

Now the identity (3.30) may be inserted, after replacing $\Delta[A]$ by $\Delta[A^{(\theta)}]$, into the generating functional (3.29)

$$\begin{aligned} Z[0] &= \int \mathcal{D}\theta \mathcal{D}A^{(\theta)} \Delta[A^{(\theta)}] \delta(G \cdot A^{(\theta)}) \text{ exp } i \int dx \mathcal{L}_0(A^{(\theta)}, \partial A^{(\theta)}) \\ &= \int \mathcal{D}\theta \mathcal{D}A \, \Delta[A] \delta(G \cdot A) \text{ exp } i \int dx \mathcal{L}_0(A, \partial A). \end{aligned} \tag{3.33}$$

We see that this generating functional is explicitly proportional to the volume $\int \mathcal{D}\theta$. Therefore we can now factor out the volume and the generating functional becomes

$$Z[0] = \int \mathcal{D}A \, \Delta[A] \delta(G \cdot A) \text{ exp } i \int dx \mathcal{L}_0(A, \partial A). \tag{3.34}$$

Consider now the factor $\delta(G \cdot A)$. This is nothing but the homogenous gauge condition $G \cdot A = 0$ with $G^{\mu} = \partial^{\mu}, n^{\mu}$ or x^{μ} in the Lorentz gauge, the axial-type gauges or the FS gauge. It is advantageous to replace the homogenous gauge condition by the inhomogenous one

$$G \cdot A = B. \tag{3.35}$$

Accordingly the generating functional (3.34) becomes

$$Z[0] = \int \mathcal{D}A \, \Delta[A] \delta(G \cdot A - B) \text{ exp } i \int dx \mathcal{L}_0(A, \partial A). \tag{3.36}$$

Field translation ensures that $Z[0]$ is independent of $B(x)$ so one may integrate the right-hand side of (3.36) over $B(x)$ with the help of a chosen weight function [Hoo 71, Lei 87]. The result is

$$Z[0] = \int \mathcal{D}A \, \Delta[A] \exp i \int dx(\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}}) \tag{3.37}$$

where $\mathcal{L}_{\mathcal{GF}}$ is called the gauge-fixing term of the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}}$. Adding a source term in (3.37) one has

$$Z[0] = \int \mathcal{D}A \, \Delta[A] \exp i \int dx(\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + jA). \tag{3.38}$$

This is the general form of the generating functional for Yang-Mills theories. The functional $\Delta[A]$ will be derived in detail in the next chapter. The explicit form of the gauge-fixing Lagrangian $\mathcal{L}_{\mathcal{GF}}$ depends on the chosen weight function. If one chooses a Gaussian weight function

$$\exp\left(-\frac{i}{2\lambda}\int dx \, B^2(x)\right) \tag{3.39}$$

and integrates over B one ends up with

$$\mathcal{L}_{\mathcal{GF}} = -\frac{1}{2\lambda}(G \cdot A)^2. \tag{3.40}$$

This Lagrangian may also be written in different form, namely in the form of auxiliary fields $C^a(x)$ [Nak 66, Mut 87]:

$$\mathcal{L}_{\mathcal{GF}} = CG \cdot A + \frac{\lambda}{2}C^2 \tag{3.41}$$

where in this formulation functional integration over C^a must be added to the generating functional (3.38)

$$Z[0] = \int \mathcal{D}A\mathcal{D}C \, \Delta[A] \exp i \int dx(\mathcal{L}_0(A) + \mathcal{L}_{\mathcal{GF}}(A, C) + jA). \tag{3.42}$$

The equivalence between the Lagrangian (3.40) and (3.41) is trivially proved by making use of the identity

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a} \tag{3.43}$$

in the functional (3.42). Note that $C(x)$ is an auxiliary field and therefore all physical observables should not depend on it [Fra 70]. This field is also called the Lagrange multiplier field [Fra 70], especially in the context of gauge-fixing Lagrangian

$$\mathcal{L}_{\mathcal{GF}} = CG \cdot A \tag{3.44}$$

[Del 74, Kum 75, Kon 77, Itz 80, Cap 86, Kum 76]. In fact the Lagrangian (3.44) is only a special case of (3.41), i.e. the case when $\lambda \to 0$.

The next section will be devoted to derivations of Green's functions or propagators. The derivations will be presented by recalling the bilinearity of the Lagrangian, equation (3.20) and (3.21), and by applying Euler-Lagrange equations and equation (3.26).

3.3 Gauge Field Propagators

The gauge field propagators depend significantly on the gauge fixing term of Lagrangian. By taking the gauge-fixing Lagrangian (3.41) into account the Lagrangian of the gauge fields reads

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + C^a G^\mu A^a_\mu + \frac{\lambda}{2} (C^a)^2. \tag{3.45}$$

The propagator will be obtained first by deriving the fields $A^{a\mu}$ and C^a in the form of their external sources via the Euler-Lagrange equations. Then, according to equation (3.26), the propagator emerges automatically. Using this method we ought to add external source terms to the Lagrangian (3.45)

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + C^a G^\mu A^a_\mu + \frac{\lambda}{2} (C^a)^2 + A^{a\mu} j^a_\mu + K^a C^a \tag{3.46}$$

where $j^{a\mu}$ and K^a are the external sources of the fields $A^{a\mu}$ and C^a respectively. The Euler-Lagrange equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu C^a)} &= 0 \\ \frac{\partial \mathcal{L}}{\partial A^{a\nu}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^{a\nu})} &= 0 \end{aligned} \tag{3.47}$$

lead to the field equations

$$G \cdot A^a = -(K^a + \lambda C^a) \tag{3.48}$$

$$(\Box g^{\mu\nu} - \partial^\mu \partial^\nu) A^a_\nu \mp G^\mu C^a = -j^{a\mu}. \tag{3.49}$$

The ($\mp$) factor in the second term on the left-hand side of (3.49) comes from the second term of (3.46): in the axial and FS gauges $CG^\mu A_\mu = +(G^\mu C)A_\mu$ while, after omitting surface term, $CG^\mu A_\mu = -(G^\mu C)A_\mu$ in the Lorentz gauge. Thus the upper sign, in this case (-), in the last term on the left-hand side of equation (3.49) is for the Lorentz gauge whereas the lower sign is given for the axial and FS gauges. Note that equation (3.48) is nothing but the inhomogenous gauge condition. Now operating with $\Box$ on (3.48) and ∂_μ on (3.49) we obtain

$$-\Box(K + \lambda C^a) = G^\mu \Box A^a_\mu + 2I_{FS} \partial \cdot A^a \tag{3.50}$$

$$C^a = \pm(\partial \cdot G)^{-1} \partial \cdot j^a \tag{3.51}$$

where

$$\begin{aligned} I_{FS} = \ & 1 \quad \text{for the FS gauge} \\ = \ & 0 \quad \text{for other (Lorentz and axial) gauges.} \end{aligned} \tag{3.52}$$

By operating with G_μ on (3.49) and using results (3.50) and (3.51) one obtains, after some rearrangement,

$$\partial \cdot A^a = (\partial G - 2I_{FS})^{-1}[-\Box K^a - (G^2 \pm \lambda\Box)(\partial G)^{-1}\partial \cdot j^a + G \cdot j^a]. \tag{3.53}$$

Combination of (3.49), (3.51) and (3.53) leads to

$$\begin{aligned} A^{a\mu}(x) = & -(\partial G - 3I_{FS})^{-1}\partial^\mu K^a - \Box^{-1}\{g^{\mu\nu} - (\partial G - I_{FS})^{-1}(G^\mu \partial^\nu + \partial^\mu G^\nu) + \\ & (\partial G - I_{FS})^{-2}\partial^\mu G^2 \partial^\nu \pm \lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial^\mu\partial^\nu\} j^a_\nu. \end{aligned} \tag{3.54}$$

In obtaining expression (3.54) the identities

$$\partial G = G\partial + 4I_{FS} \tag{3.55}$$

$$\partial_\mu(\partial G + a)^{\pm 1} = (\partial G + a + I_{FS})^{\pm 1}\partial_\mu \tag{3.56}$$

$$x_\mu(\partial G + a)^{\pm 1} = (\partial G + a - I_{FS})^{\pm 1} x_\mu \tag{3.57}$$

$$\partial_\mu G_\nu = G_\nu \partial_\mu + I_{FS} g_{\mu\nu} \tag{3.58}$$

with a an arbitrary number have been employed. Finally using the definition (3.26) one extracts the propagators

$$\begin{aligned}
G^{ab}_{\mu\nu}(x,y) &= -\frac{\delta A^a_\mu(x)}{\delta j^{b\nu}(y)} \\
&= \Box^{-1}\{g_{\mu\nu} - (\partial G - I_{FS})^{-1}(G_\mu \partial_\nu + \partial_\mu G_\nu) + (\partial G - I_{FS})^{-2}\partial_\mu G^2 \partial_\nu + \\
&\qquad \pm \lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial_\mu\partial_\nu\}\delta^{ab}(x-y), \qquad (3.59)\\
G^{ab}_{\mu 4}(x,y) &= -\frac{\delta A^a_\mu(x)}{\delta j^{b4}(y)} = -\frac{\delta A^a_\mu(x)}{\delta K^b(y)} = (\partial G - 3I_{FS})^{-1}\partial_\mu \delta^{ab}(x-y), \qquad (3.60)\\
G^{ab}_{4\mu}(x,y) &= -\frac{\delta A^a_4(x)}{\delta j^{b\mu}(y)} = -\frac{\delta C^a(x)}{\delta j^{b\mu}(y)} = \mp(\partial G)^{-1}\partial_\mu \delta^{ab}(x-y), \qquad (3.61)\\
G^{ab}_{44}(x,y) &= -\frac{\delta A^a_4(x)}{\delta j^{b4}(y)} = -\frac{\delta C^a(x)}{\delta K^b(y)} = 0. \qquad (3.62)
\end{aligned}$$

Thus we have the Lorentz, axial and FS gauge propagators

$$\begin{aligned}
G^{ab\mu\nu}_L(x,y) &= \Box^{-1}[g^{\mu\nu} + (\lambda - 1)\Box^{-1}\partial^\mu\partial^\nu]\delta^{ab}(x-y) \\
G^{ab\mu 4}_L(x,y) &= -G^{ab4\mu}_L(x,y) = \Box^{-1}\partial^\mu \delta^{ab}(x-y) \\
G^{44}_L(x,y) &= 0
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
G^{ab\mu\nu}_A(x,y) &= \Box^{-1}\left[g^{\mu\nu} - \frac{\partial^\mu n^\nu + \partial^\nu n^\mu}{\partial \cdot n} + \frac{n^2 - \lambda\Box}{(\partial \cdot n)^2}\partial^\mu\partial^\nu\right]\delta^{ab}(x-y) \\
G^{ab\mu 4}_A(x,y) &= G^{ab4\mu}_A(x,y) = (\partial n)^{-1}\partial^\mu \delta^{ab}(x-y) \\
G^{ab44}_A(x,y) &= 0
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
G^{ab\mu\nu}_{FS}(x,y) &= \Box^{-1}\left\{g^{\mu\nu} - (\partial x - 1)^{-1}(x^\mu\partial^\nu + \partial^\mu x^\nu) + (\partial x - 1)^{-2}\partial^\mu x^2 \partial^\nu \right.\\
&\qquad \left. - \lambda(\partial x - 1)^{-1}(\partial x + 3)^{-1}\Box\partial^\mu\partial^\nu\right\}\delta^{ab}(x,y) \\
G^{ab\mu 4}_{FS}(x,y) &= (\partial x - 3)^{-1}\partial^\mu \delta^{ab}(x-y) \\
G^{ab4\mu}_{FS}(x,y) &= (\partial x)^{-1}\partial^\mu \delta^{ab}(x-y) \\
G^{ab44}_{FS}(x,y) &= 0.
\end{aligned} \tag{3.65}$$

The inverse propagator $G^{-1^{abKL}}(x,y)$ where $K, L = 0, \cdots 4$ may be obtained in a straightforward way by recalling the bilinearity (equation (3.20)) of the Lagrangian (3.45):

$$\mathcal{L} = -\frac{1}{2}[A^{a\mu}(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)A^{a\nu} \mp A^{a\mu}G_\mu C^a + C^a G_\mu A^{a\mu} + C^a\lambda C^a] \tag{3.66}$$

after discarding the surface terms. Again, the upper sign in the second term is associated with the Lorentz gauge while the lower sign is for the axial and FS gauges. Hence the inverse of propagators (3.59-3.62) can be read off:

$$G^{-1^{ab}}_{\mu\nu}(x,y) = (\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\delta^{ab}(x-y) \tag{3.67}$$

$$G^{-1^{ab}}_{\mu 4}(x,y) = \mp G_\mu \delta^{ab}(x-y) \tag{3.68}$$

$$G^{-1^{ab}}_{4\mu}(x,y) = G_\mu \delta^{ab}(x-y) \tag{3.69}$$

$$G^{-1^{ab}}_{44}(x,y) = \lambda\delta^{ab}(x-y). \tag{3.70}$$

The above inverse propagator may also be derived by the use of identity

$$\int dy\; G^{ab}_{KL}(x,y)G^{-1^{bcLM}}(y,z) = \delta^M_K\delta^{ac}(x-z). \tag{3.71}$$

However such derivation is not straightforward and a few pages are needed to perform all the calculations (see Appendix E).

In the limit $\lambda \to 0$ we have

$$\mathcal{L}_{\mathcal{GF}} = C\, G\cdot A \tag{3.72}$$

and

$$\begin{aligned} G^{ab}_{\mu\nu}(x,y) = \Box^{-1}\{g_{\mu\nu} - (\partial G - I_{FS})^{-1}(G_\mu\partial_\nu + \partial_\mu G_\nu) +& \\ (\partial G - I_{FS})^{-2}\partial_\mu G^2\partial_\nu\}\delta^{ab}(x-y),& \end{aligned} \tag{3.73}$$

$$G^{ab}_{\mu 4}(x,y) = (\partial G - 3I_{FS})^{-1}\partial_\mu\delta^{ab}(x-y), \tag{3.74}$$

$$G^{ab}_{4\mu}(x,y) = \mp(\partial G)^{-1}\partial_\mu\delta^{ab}(x-y), \tag{3.75}$$

$$G^{ab}_{44}(x,y) = 0, \tag{3.76}$$

and their inverse

$$G^{-1^{ab}}_{\mu\nu}(x,y) = (\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\delta^{ab}(x-y), \tag{3.77}$$

$$G^{-1^{ab}}_{\mu 4}(x,y) = \mp G_\mu \delta^{ab}(x-y), \tag{3.78}$$

$$G^{-1^{ab}}_{4\mu}(x,y) = G_\mu \delta^{ab}(x-y), \tag{3.79}$$

$$G^{-1^{ab}}_{44}(x,y) = 0. \tag{3.80}$$

In this limit the FS gauge propagator and its inverse are

$$\begin{aligned}
G^{ab\mu\nu}_{FS}(x,y) = {} & \Box^{-1}\{g^{\mu\nu} - (\partial x - 1)^{-1}(x^\mu\partial^\nu + \partial^\mu x^\nu) \\
& +(\partial x - 1)^{-2}\partial^\mu x^2 \partial^\nu\}\,\delta^{ab}(x-y) \\
G^{ab\mu 4}_{FS}(x,y) = {} & (\partial x - 3)^{-1}\partial^\mu \delta^{ab}(x-y) \\
G^{ab4\mu}_{FS}(x,y) = {} & (\partial x)^{-1}\partial^\mu \delta^{ab}(x-y) \\
G^{ab44}_{FS}(x,y) = {} & 0
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
G^{-1^{ab}}_{FS\mu\nu}(x,y) = {} & (\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\delta^{ab}(x-y) \\
G^{-1^{ab}}_{FS\mu 4}(x,y) = {} & G^{-1^{ab}}_{FS4\mu}(x,y) = x_\mu \delta^{ab}(x-y) \\
G^{-1^{ab}}_{FS44}(x,y) = {} & 0.
\end{aligned}$$

To end this section let us compare the above propagators to those associated with the gauge-fixing Lagrangian $\mathcal{L}_{\mathcal{GF}_1} = -\frac{1}{2\lambda}(G\cdot A)^2$:

$$\begin{aligned}
G^{ab}_{\mu\nu}(x,y) = \Box^{-1}\Big\{ & g_{\mu\nu} - (\partial G - I_{FS})^{-1}(\partial_\mu G_\nu + G_\mu \partial_\nu) \; + \\
& (\partial G - I_{FS})^{-2}\partial_\mu G^2 \partial_\nu \\
& \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial_\mu\partial_\nu\Big\}\,\delta^{ab}(x-y)
\end{aligned} \tag{3.82}$$

and

$$G^{-1^{ab}}_{\mu\nu}(x,y) = (\Box g_{\mu\nu} - \partial_\mu\partial_\nu \pm \frac{1}{\lambda}G_\mu G_\nu)\delta^{ab}(x-y). \tag{3.83}$$

The derivation of these propagators can be carried out in a similar way as above (see Appendix E) and therefore we only show the final result here.

It turns out that the propagator associated with $\mathcal{L}_{\mathcal{GF}_1}$ is equal to (μ, ν) components of the propagator associated with $\mathcal{L}_{\mathcal{GF}_2}$. This equality is understood since both gauge-fixing Lagrangians are equivalent in the context of generating functional. On the other hand, the physical components of the inverse propagator (3.67) is simpler: it is free from the gauge parameter λ and is equal to the gauge parameter-free term of the inverse propagator (3.83). The appearance of extra components, $G^{ab4\mu}$ and $G^{ab\mu 4}$, in the propagator associated with $\mathcal{L}_{\mathcal{GF}_2}$ is due to the introduction of the auxiliary field $C(x)$. However these extra components will not contribute to the scattering matrix and thus we may call them the unphysical components of the corresponding propagator but the remainder, (μ, ν), are called the physical components.

In perturbation calculations we will not use the FS gauge propagators (3.65) or (3.82) but because of its simplicity we will employ the FS gauge propagator and its inverse (3.81).

Symmetry properties of propagators play an important role in simplifying perturbation calculations and therefore we should derive them for the above propagators. The next section is devoted to the derivation of those symmetries.

3.4 Properties of Gauge Field Propagators

The symmetry properties of the propagator $G^{ab}_{KL}(x,y)$ can be deduced immediately from identities below

$$\Box'^{-1}\delta(x-x') = \Box^{-1}\delta(x-x') \tag{3.84}$$

$$G'_\mu\delta(x-x') = \mp G_\mu\delta(x-x') \tag{3.85}$$

$$(\partial' G' + a)^{-1}\delta(x-x') = (\pm\partial G + a + 4I_{FS})^{-1}\delta(x-x') \tag{3.86}$$

$$(\partial G + a)^{\pm}\Box^{-1} = \Box^{-1}(\partial G + a + 2I_{FS})^{\pm} \tag{3.87}$$

$$G_\mu\Box^{-1} = \Box^{-1}G_\mu + 2I_{FS}\Box^{-2}\partial_\mu \tag{3.88}$$

where a is constant and G'_μ, ∂'_μ and $\Box'$ refer to the variable x'. The proof of the above identities can be found in the Appendix B. By applying (3.85), (3.86) and then (3.56) the unphysical components of the propagator $G^{ab}_{KL}(x,x')$ have the property

$$G^{ab}_{\mu 4}(x',x) = G^{ab}_{4\mu}(x,x'). \tag{3.89}$$

Similarly, but with a few more lines of calculations, all the identities (3.84-3.88) and (3.55-3.58) can be applied to show that the physical components of the propagator obey

$$G^{ab}_{\mu\nu}(x',x) = G^{ab}_{\nu\mu}(x,x'). \tag{3.90}$$

Thus we conclude that the propagators are symmetrical under interchanging both $x \leftrightarrow x'$ and $K \leftrightarrow L$ simultaneously

$$G^{ab}_{KL}(x,x') = G^{ab}_{LK}(x',x), \tag{3.91}$$

as is consistent with Bose symmetry. Another symmetry

$$G^{ab}_{\mu\nu}(x,x') = G^{ab}_{\mu\nu}(-x,-x') \tag{3.92}$$

also holds for the physical components of the propagators (but it does not hold for the unphysical components) as is easily seen from (3.60-3.61) or (3.74-3.75).

The physical components have another important property. If G_μ is operated on them the result is

$$G^\mu G^{ab}_{\mu\nu}(x,x') = \lambda(\partial G)^{-1}\partial_\nu \delta^{ab}(x-x'). \tag{3.93}$$

Hence in the limit $\lambda \to 0$ the physical components of the propagator $G^{ab}_{\mu\nu}(x,x')$ are orthogonal to G^μ,

$$G^\mu G^{ab}_{\mu\nu}(x,x') = 0; \qquad \lambda \to 0. \tag{3.94}$$

3.5 Fock-Schwinger Gauge Propagators in Coordinate and Momentum Spaces

The explicit form of the physical components ot the FS gauge propagator (3.81) is complicated and will cause difficulties in scattering calculations. However these difficulties may be diminished by taking advantage of the symmetry properties of the propagator without using its detailed form. In this section we will rewrite the propagator (3.81) in a more useful way.

Let us consider the physical components of the FS gauge propagator (3.81)

$$G_{\mu\nu}(x,x') = \Box^{-1}\left\{g_{\mu\nu} - (\partial x - 1)^{-1}(x_\mu\partial_\nu + \partial_\mu x_\nu) + (\partial x - 1)^{-2}\partial_\mu x^2\partial_\nu\right\}\delta(x - x') \tag{3.95}$$

where for simplicity we have dropped index FS and color indices a and b. The first term on the right-hand side is the Feynman gauge propagator $G_{F_{\mu\nu}}(x,x')$ and the remaining terms, $G'_{\mu\nu}(x,x')$, are the contributions associated with the FS gauge condition. In this section we only pay attention to $G'_{\mu\nu}(x,x')$.

$$\begin{aligned} G_{\mu\nu}(x,x') &= G_{F_{\mu\nu}}(x,x') + G'_{\mu\nu}(x,x') &(3.96)\\ G'_{\mu\nu}(x,x') &= \Box^{-1}\left\{-(\partial x - 1)^{-1}(x_\mu\partial_\nu + \partial_\mu x_\nu) + (\partial x - 1)^{-2}\partial_\mu x^2\partial_\nu\right\}\delta(x - x'). &(3.97)\end{aligned}$$

When evaluating the scattering matrix, the fermion (or boson) propagator as well as the gauge field propagator play a crucial role. The intricacies of perturbation calculations depend significantly on these propagators. Since the basic form of the fermion propagator is an inverse of differential operator ∂_μ (inverse of $\Box$ in the case of the boson propagator) the gauge field propagator will facilitate perturbation calculations if it contains factors of differential operators. Because of this reason we should cast the propagator $G'_{\mu\nu}(x,x')$ into derivatives of some functions. Since $G_{\mu\nu}(x,x')$ is a two point function one may relate indices μ and ν with derivatives with respect to x and x' respectively. In this way one can arrive at the more useful form,

$$G'_{\mu\nu}(x,x') = \partial_\mu f_{1\nu}(x,x') + \partial'_\nu f_{2\mu}(x,x') \tag{3.98}$$

where $\partial'_\nu = \frac{\partial}{\partial x'^\nu}$, and

$$\begin{aligned} f_{1\mu}(x,x') &= -\Box^{-1}x_\mu(\partial x - 1)^{-1}\delta(x - x') + \frac{1}{2}\Box^{-1}x^2\partial_\mu(\partial x - 1)^{-2}\delta(x - x') &(3.99)\\ f_{2\mu}(x,x') &= +\Box^{-1}(\partial x - 1)^{-1}x_\mu\delta(x - x') - \frac{1}{2}\Box^{-1}(\partial x - 1)^{-2}\partial_\mu x^2\delta(x - x') &(3.100)\end{aligned}$$

Identities (3.56) and (3.57) have been used to obtain (3.100). According to the symmetry (3.90), or else by applying (3.55), (3.56) and (3.57) on (3.100) directly, one has

$$f_{1\mu}(x,x') = f_{2\mu}(x',x). \tag{3.101}$$

The momentum space form of (3.97) or (3.98) can also be obtained. Recalling the symmetry (3.101) and the definition (3.100) one has, after some algebra,

$$\begin{aligned}\partial'_\nu f_{1\mu}(x',x) = & \ \partial'_\nu f_{2\mu}(x,x') \\ = & \ \Box^{-2}\partial_\mu\partial_\nu(\partial x - 1)^{-1}\delta(x-x') + \tfrac{1}{2}x_\mu\Box^{-1}\partial_\nu G_3(x',x) \\ & + \tfrac{1}{2}\Box^{-1}(x'_\mu\partial'_\nu + g_{\mu\nu})G_1(x,x') + \tfrac{1}{2}\Box^{-1}(x'^2\partial'_\mu\partial'_\nu + 2x'_\nu\partial'_\mu)H_1(x,x')\end{aligned} \tag{3.102}$$

where we have defined

$$(\partial x - n)G_n(x,x') = \delta(x-x') \tag{3.103}$$

$$(\partial x - n)^{-1}G_n(x,x') = H_n(x,x'). \tag{3.104}$$

$G_n(x,x')$ and $H_n(x,x')$ may be obtained as follows. After introducing a parameter β via a replacement $x \rightarrow \beta x$ in equation (3.103) the operator $x\partial$ in (3.103) can be replaced by an operator $\beta\frac{d}{d\beta}$ because $x\partial$ acts on a function of βx, viz. $G_n(\beta x, x')$. Thus, equation (3.103) becomes

$$\left[\beta\frac{d}{d\beta} - (n-4)\right] G_n(\beta x, x') = \delta(\beta x - x'). \tag{3.105}$$

To simplify this equation we may replace the parameter β by another parameter $\alpha = \frac{1}{\beta}$. After some algebra we arrive at

$$\frac{d}{d\alpha}\alpha^{n-4}G_n(\frac{x}{\alpha}, x') = -\alpha^{n-1}\delta(x - \alpha x'). \tag{3.106}$$

(Note that in deriving equation (3.106) we used a trick, namely, we multiply both-sides of equation (3.105) by a factor α^{n-5}). Equation (3.106) yields a solution

$$G_n(x,x') = \lim_{\delta\rightarrow+0}\int_1^\infty d\alpha\, e^{-\alpha\delta}\alpha^{n-1}\delta(x-\alpha x'). \tag{3.107}$$

Accordingly, the function $H_n(x,x')$ defined in (3.104) follows immediately

$$\begin{aligned}H_n(x,x') &= \lim_{\delta\rightarrow+0}\int_1^\infty d\alpha\, e^{-\alpha\delta}\alpha^{n-1}(\partial x - n)^{-1}\delta(x-\alpha x') \\ &= \lim_{\delta\rightarrow+0}\int_1^\infty d\alpha\, e^{-\alpha\delta}\alpha^{n-1}G_n(x,\alpha x').\end{aligned} \tag{3.108}$$

A few simple cases are,

$$G_1(x,x') = \int \bar{d}k \int_1^\infty d\alpha\; e^{-\alpha\delta} e^{-ik(x-\alpha x')} \tag{3.109}$$

$$H_1(x,x') = \int \bar{d}k \int_1^\infty d\alpha\; e^{-\alpha\delta} \ln\alpha\, e^{-ik(x-\alpha x')} \tag{3.110}$$

$$G_3(x,x') = \int \bar{d}k \int_1^\infty d\alpha\; e^{-\alpha\delta} \alpha^2 e^{-ik(x-\alpha x')} \tag{3.111}$$

where $\bar{d}k = (2\pi)^{-4}d^4k$ and from now on we neglect the specification $\lim_{\delta\to+0}$, except where needed. Putting (3.109), 3.110) and (3.111) into equation (3.102) we get

$$\begin{aligned} &\partial'_\nu f_{1\mu}(x',x) \\ &= \Box^{-2}\partial_\mu\partial_\nu(\partial x-1)^{-1}\delta(x-x') - \tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\; e^{-\alpha\delta}\, i\alpha x_\mu k_\nu\, e^{-ik(\alpha x-x')} \\ &\quad +\tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\; e^{-\alpha\delta}[g_{\mu\nu} + i\alpha x'_\mu k_\nu + 2i\alpha x'_\nu k_\mu \ln\alpha \\ &\qquad\qquad -y^2 k_\mu k_\nu \alpha^2 \ln\alpha]\; e^{-ik(x-\alpha x')}. \end{aligned} \tag{3.112}$$

This is the second term of (3.98). The first term of (3.98) can be obtained from (3.112) by interchanging $x \leftrightarrow x'$ and $\mu \leftrightarrow \nu$ and the symmetry (3.101). Hence the propagator (3.98) reads

$$\begin{aligned} &G'_{\mu\nu}(x,x') \\ &= \tfrac{1}{2}\int_1^\infty d\alpha\; e^{-\alpha\delta}\int \tfrac{\bar{d}k}{-k^2} e^{-ik(x-\alpha x')}\times \\ &\qquad [g_{\mu\nu} + i\alpha x'_\mu k_\nu + i\alpha(1+2\ln\alpha)x'_\nu k_\mu - x'^2 k_\mu k_\nu \alpha^2\ln\alpha] \\ &\quad +\tfrac{1}{2}\int_1^\infty d\alpha\; e^{-\alpha\delta}\int \tfrac{\bar{d}k}{-k^2} e^{-ik(\alpha x-x')}\times \\ &\qquad [g_{\mu\nu} - i\alpha x_\nu k_\mu - i\alpha(1+2\ln\alpha)x_\mu k_\nu - x^2 k_\mu k_\nu \alpha^2 \ln\alpha] \end{aligned} \tag{3.113}$$

where we have used the identity

$$\Box'^{-2}\partial'_\mu\partial'_\nu(\partial' x'-1)^{-1}\delta(x-x') = -\Box^{-2}\partial_\mu\partial_\nu(\partial x-1)^{-1}\delta(x-x'). \tag{3.114}$$

The propagator (3.113) agrees with Kummer and Weiser [Kum 86]. In their derivation they did not introduce formula (3.98) but directly worked out (3.71) by rewriting the propagator $G_{AB}(x,x')$ in the form

$$G_{AB}(x,x') = \int \bar{d}k\, e^{-ikx}\tilde{G}_{AB}(k,x') \tag{3.115}$$

with

$$\begin{aligned}\tilde{G}_{\mu\nu}(k,x') &= Ag_{\mu\nu} + Bk_\mu k_\nu + Cx'_\mu x'_\nu + Dx'_\mu k_\nu + Ek_\mu x'_\nu \\ \tilde{G}_{4\mu}(k,x') &= ak_\mu + bx'_\mu \\ \tilde{G}_{\mu 4}(k,x') &= ck_\mu + dx'_\mu\end{aligned} \tag{3.116}$$

where the coefficients $A, B, \cdots, a, b, \cdots$ are functions of k^2, kx' and x'^2. They found that the resulting propagator does not obey the symmetry (3.91). The contradiction is understandable because in order to keep $G_{\mu\nu}(x,x')$ symmetric $\tilde{G}_{\mu\nu}(k,x')$ in (3.115) must be necessarily free from x'_μ; it is only a function of forms like $\exp(ikx')\tilde{\tilde{G}}_{\mu\nu}(k,k')$. However since the inverse propagator contains x'_μ, x'_μ appears in $\tilde{G}_{\mu\nu}(k,x')$ too. To escape the contradiction Kummer and Weiser then proposed a symmetric propagator $\hat{G}_{\mu\nu}(x,x')$:

$$\hat{G}_{\mu\nu}(x,x') = \frac{1}{2}[G_{\mu\nu}(x,x') + G_{\nu\mu}(x',x)] \tag{3.117}$$

after claiming that $G_{BA}(x',x)$ is also a solution of (3.71). They were then able to obtain (3.113).

The propagator in momentum space (3.113) can be obtained from coordinate space as a Fourier transform,

$$G'_{\mu\nu}(x,x') = \int \frac{\bar{d}k}{-k^2}\int_1^\infty d\beta\ \left[f_\nu(\beta,k,\partial_k,x')k_\mu e^{-i\beta kx} + g_\mu(\beta,k,\partial_k,x)k_\nu e^{i\beta kx'}\right] \tag{3.118}$$

although all reference to the coordinates does *not* go into exponential. Above,

$$\begin{aligned}f_\mu(\beta,k,\partial_k,x) = {} & \tfrac{1}{2}e^{-\beta\delta}\{e^{ikx}\tfrac{\partial}{\partial k^\mu} + \delta(\beta-1)\textstyle\int_1^\infty d\alpha\, e^{i\alpha kx}\times \\ & [i\alpha(1+2\ln\alpha)x_\mu - \alpha^2\ln\alpha k_\mu x^2]\} \\ g_\mu(\beta,k,\partial_k,x) = {} & \tfrac{1}{2}e^{-\beta\delta}\{e^{-ikx}\tfrac{\partial}{\partial k^\mu} + \delta(\beta-1)\textstyle\int_1^\infty d\alpha\, e^{-i\alpha kx}\times \\ & [-i\alpha(1+2\ln\alpha)x_\mu - \alpha^2\ln\alpha k_\mu x^2]\}\end{aligned} \tag{3.119}$$

with their properties

$$\begin{aligned}f_\mu(\beta,k,\partial_k,x) = {} & -g_\mu(\beta,-k,-\partial_k,x), \\ f_\mu(\beta,k,\partial_k,-x) = {} & +g_\mu(\beta,k,\partial_k,x).\end{aligned} \tag{3.120}$$

Hence,

$$G'_{\mu\nu}(x,x') = \int \frac{\bar{d}k}{-k^2}\int_1^\infty d\beta\ \left[g_\nu(\beta,k,\partial_k,x')k_\mu e^{i\beta kx} + (\mu\leftrightarrow\nu, x\leftrightarrow x')\right]. \tag{3.121}$$

To summarize, the physical components of the Green's function (3.81) have the form:

Coordinate space

$$
\begin{aligned}
G^{ab\mu\nu}(x,x') =& \; G_F^{ab\mu\nu}(x,x') + \partial^\mu f_1^{ab\nu}(x,x') + \partial'^\nu f_2^{ab\mu}(x,x') \\
f_1^{ab\mu}(x,x') =& \; f_2^{ab\mu}(x',x) \\
=& \; \Box^{-1}[-x_\mu(\partial x - 1)^{-1} + \tfrac{1}{2}x^2\partial_\mu(\partial x - 1)^{-2}]\delta^{ab}(x,x') \\
f_1^{ab\mu}(x,x') =& \; \delta^{ab} f_1^\mu(x,x').
\end{aligned}
\tag{3.122}
$$

Momentum space

$$
\begin{aligned}
G_{\mu\nu}^{ab}(x,x') =& \; G_{F_{\mu\nu}}^{ab}(x,x') + \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \, \Big[f_\nu^{ab}(\beta,k,\partial_k,x')k_\mu \, e^{-\imath\beta kx} \\
& + g_\mu^{ab}(\beta,k,\partial_k,x)k_\nu \, e^{i\beta kx'} \Big] \\
f_\mu^{ab}(\beta,k,\partial_k,x) =& \; -g_\mu^{ab}(\beta,-k,-\partial_k,x) \\
f_\mu^{ab}(\beta,k,\partial_k,-x) =& \; g_\mu^{ab}(\beta,k,\partial_k,x) \\
f_\mu^{ab}(\beta,k,\partial_k,x) =& \; \delta^{ab} f_\mu(\beta,k,\partial_k,x)
\end{aligned}
\tag{3.123}
$$

where $G_F^{ab\mu\nu}(x,x') = \delta^{ab} g^{\mu\nu} \Box^{-1} \delta(x-x')$ is the Feynman gauge propagator. Note that the Green functions (3.123) are not fully in momentum space because the space-time coordinates x and x' in (3.123) still exist.

The propagators we have derived here will be used in the next chapter to obtain the Ward-Takahashi and the BRST identities.

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Chapter 4

Ward-Takahashi and BRST Identities

The Ward-Takahashi and the BRST identities will be derived here. Since the FS gauge theory is a ghost-free theory its BRST identity will also be derived without introducing the ghost fields. Both kind of BRST identities will be compared. The ghost-free version of the FS gauge theory will be obtained in the first part of the chapter.

4.1 Ghost-free Fock-Schwinger-Gauge Formulations

The generating functional in the Yang-Mills theory

$$Z[J] = \int \mathcal{D}A \, \triangle[A] \, e^{i \int dx \, (\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + JA)} \tag{4.1}$$

has been derived in the previous chapter. Here we will focus mainly on the functional $\triangle[A]$. Its general form will be derived and its responsibility for the appearance of the nonphysical, ghost, fields will be discussed. The derivation of this functional will follow the work of Muta [Mut 87].

Let us begin with the definition of $\triangle[A]$ as given in the previous chapter

$$(\triangle[A])^{-1} = \int \prod_b \mathcal{D}s^b \delta(G \cdot A^{(s)^a}) \tag{4.2}$$

where s^a is the gauge parameter of the gauge transformation

$$\delta A^a_\mu(x) = f^{abc} s^b(x) A^c_\mu(x) - \frac{1}{g}\partial_\mu s^a(x). \tag{4.3}$$

f^{abc} being the structure constant of the gauge group and g a coupling constant. The above integral may be written as

$$\begin{aligned}(\triangle[A])^{-1} &= \int \prod_b \mathcal{D}(G^\mu A^{(s)^c}_\mu) \left[\frac{\delta(G^\mu A^{(s)^c}_\mu)}{\delta s^b}\right]^{-1} \delta(G \cdot A^{(s)^a}) \\ &= (\det M_G)^{-1}.\end{aligned} \tag{4.4}$$

Hence

$$\triangle[A] = \det M_G \tag{4.5}$$

and

$$(M_G(x,y))^{ab} = \frac{\delta(G^\mu A^{(s)^a}_\mu(x))}{\delta s^b(y)}. \tag{4.6}$$

Since other integrands in (4.1) are in an exponential form it is advantagous to write $\triangle[A]$ in such a form. Fortunately the determinant of a matrix may be so expressed. For the matrix M_G one may write

$$\det iM_G = \int \mathcal{D}\chi \mathcal{D}\chi^* \, \exp\left(-i\int dx \int dy \, \chi^{*^a}(x) M^{ab}_G(x,y)\chi^b(y)\right). \tag{4.7}$$

Here χ and χ^* are two independent fictitious fields called the Faddeev-Popov ghost fields. They are anticommuting like fermions. The explicit derivation of the above expression can be seen in many textbooks (see for example [Ryd 85]) and, thus, no derivation is needed here. Now by inserting (4.7) into (4.5) and then (4.5) into (4.1) the generating functional (4.1) becomes, up to irrelevant factor

$$\begin{aligned}Z[J,\xi,\xi^*] = \int \mathcal{D}[A\chi\chi^*] \, \exp\Big\{ i \int dx \, [\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} \\ - \int dy \chi^{*^a}(x) M^{ab}_G(x,y)\chi^b(y) + A^a J^a + \chi^{*^a}\xi^a + \xi^{*^a}\chi^a \Big]\Big\}\end{aligned} \tag{4.8}$$

where $\mathcal{D}[A\chi\chi^*] = \mathcal{D}A\mathcal{D}\chi\mathcal{D}\chi^*$ for short and ξ^a and ξ^{*^a} are the external sources for the ghost fields χ^{*^a} and χ^a respectively. It turns out that $\triangle[A]$ is responsible for the appearance of the ghost fields χ^{*^a} and χ^a.

Let us now find the explicit form of the matrix element $M_G^{ab}(x,y)$ for some gauge conditions. Inserting (4.3) into (4.6) one has

$$M_G^{ab}(x,y) = \left(f^{abc}G^\mu A_\mu^c - \frac{1}{g}\delta^{ab}G^\mu\partial_\mu\right)\delta(x-y)$$
$$= -\frac{1}{g}G^\mu D_\mu^{ab}\delta(x-y) \tag{4.9}$$

where

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - gf^{abc}A_\mu^c. \tag{4.10}$$

The expression (4.9) can be written as

$$M_G^{ab}(x,y) = \left(f^{abc}A_\mu^c G^\mu - \frac{1}{g}\delta^{ab}G^\mu\partial_\mu\right)\delta(x-y) \tag{4.11}$$

subject to the homogenous gauge condition $G \cdot A = 0$. It turns out that the dependence of M_G^{ab} on A_μ^a arises through the first term of the right-hand side of equation (4.11). When this term vanishes the element matrix M_G^{ab} is independent of the gauge field A_μ^a. As a consequence integrations over χ and χ^* in the generating functional (4.8) reduce to just a number that can be absorbed into the normalization factor. In that case, theories with the generating functional (4.8) are ghost-free. It is obvious that Abelian theories are ghost-free theories since $f^{abc} = 0$. In non-Abelian theories, on the other hand, f^{abc} is not zero in general and thus the independence of M_G^{ab} on the gauge field A_μ^a hinges on the value of $A^{c\mu}G_\mu$. Accordingly, non-Abelian theories in the Lorentz gauge are not ghost-free since $A_\mu^c G^\mu = A_\mu^c\partial^\mu \neq 0$. On the other hand, $A_\mu^c x^\mu = x^\mu A_\mu^c = 0$ means that the FS gauge theory is not haunted by ghost.

4.2 Local Gauge and BRST Invariances of Lagrangians

Consider the Lagrangian density of a system of quarks and massless gluons

$$\mathcal{L}_0 = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{4.12}$$

where

$$\begin{aligned} D_\mu = &\ \partial_\mu - igT^a A^a_\mu \\ F_{\mu\nu} = &\ D_\mu A_\nu - D_\nu A_\mu \\ [T^a, T^b] = &\ if^{abc}T^c. \end{aligned} \tag{4.13}$$

Under the infinitesimal local gauge transformations

$$\begin{aligned} \psi(x) \to \psi'(x) = &\ U(x)\psi(x) = (1 - igT^a\Lambda^a(x))\psi(x) \\ \overline{\psi}(x) \to \overline{\psi'}(x) = &\ \overline{\psi}(x)U^*(x) = \overline{\psi}(x)(1 + igT^a\Lambda^a(x)) \\ A^a_\mu(x) \to A'^a_\mu(x) = &\ A^a_\mu - D^{ab}_\mu \Lambda^b \end{aligned} \tag{4.14}$$

with

$$\begin{aligned} U(x) = &\ e^{-igT^a\Lambda^a(x)} \\ D^{ab}_\mu = &\ \delta^{ab}\partial_\mu - gf^{abc}A^c_\mu \\ \Lambda^a = &\ \tfrac{1}{g}s^a \end{aligned} \tag{4.15}$$

the Lagrangian density $\mathcal{L}_0$ is invariant because $F^a_{\mu\nu}F^{a\mu\nu}$, $\overline{\psi}\psi$ and $\overline{\psi}D_\mu\psi$ are gauge invariant quantities.

If we add $\mathcal{L}_{\mathcal{GF}} = CG \cdot A + \frac{\lambda}{2}C^2$ to the Lagrangian $\mathcal{L}_0$, the new Lagrangian $\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}}$ is no longer gauge invariant due to the non-gauge invariance of $\mathcal{L}_{\mathcal{GF}}$. Nevertheless when the Faddeev-Popov ghost Lagrangian $\mathcal{L}_{\mathcal{FP}} = -\chi^{*^a}M^{ab}\chi^b$ is also added one can find larger transformations which make the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + \mathcal{L}_{\mathcal{FP}} \tag{4.16}$$

invariant. These transformations consist of the local gauge transformations (4.14) and transformations related to χ, χ^* and C

$$\begin{aligned} \chi'^a &= \chi^a + \delta\chi^a &&= \chi^a - \tfrac{1}{2} g\theta f^{abc}\chi^b\chi^c \\ \chi^{*'^a} &= \chi^{*^a} + \delta\chi^{*^a} &&= \chi^{*a} + \theta C^a \\ C'^a &= C^a + \delta C^a &&= C^a \end{aligned} \tag{4.17}$$

where θ is defined from

$$\Lambda^a(x) = -\theta\chi^a(x). \tag{4.18}$$

θ must be a Grassmann number since χ^a is an anticommuting ghost field. In order to keep the reality of Λ^a, θ is restricted by relations

$$(\theta\chi^a)^\dagger = \theta\chi^a. \tag{4.19}$$

The local gauge trasformations (4.14) together with the transformations (4.17) are well-known and are called the BRST transformations following the work of Becchi, Rouet, Stora [Bec 74, Bec 76] and Tyutin [Tyu 75]. The detailed proof of the BRST invariance of the Lagrangian (4.16) is given in Appendix F.

4.3 Ward-Takahashi Identities

The invariance of a Lagrangian under certain transformations produces some consequences. In quantum electrodynamics the invariance of the Lagrangian $\mathcal{L}_0$ under the gauge transformation will result in the so-called the Ward-Takahashi identities [War 50, Tak 57]. In this section those identities will be derived. We will find that the Ward-Takahashi identities will be slightly different with the gauge-fixing Lagrangian $\mathcal{L}_{\mathcal{GF}}$. However they all imply orthogonality of photon self-energy.

To obtain the Ward-Takahashi identities let us consider the generating functional of quantum electrodynamics

$$Z[J,\eta,\overline{\eta},K] = \int \mathcal{D}[A\psi\overline{\psi}C]\ \exp i\int dx(\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + AJ + \overline{\psi}\eta + \overline{\eta}\psi + CK) \tag{4.20}$$

where $\mathcal{L}_{\mathcal{GF}} = CG\cdot A + \frac{\lambda}{2}C^2$ is the chosen gauge fixing Lagrangian and K represents the external source of the auxiliary or multiplier field C. Here $\triangle[A]$ described in the previous chapter is just a constant and included in the normalization constant. The above generating functional remains unchanged under arbitrary field variables transformations

$$Z[J,\eta,\overline{\eta},K] = \int \mathcal{D}[A'\psi'\overline{\psi}'C'] \, \exp i\int dx(\mathcal{L}'_0+\mathcal{L}'_{\mathcal{GF}}+A'J+\overline{\psi}'\eta+\overline{\eta}\psi'+C'K). \quad (4.21)$$

If the local gauge transformations (4.14), with g the electron charge e and one color index, are chosen as the field transformations the generating functional (4.21) becomes

$$\begin{aligned} Z[J,\eta,\overline{\eta},K] &= \int \mathcal{D}[A\psi\overline{\psi}C] \, \exp i\int dx(\mathcal{L}_0 + \mathcal{L}'_{\mathcal{GF}} + A'J + \overline{\psi}'\eta + \overline{\eta}\psi' + C'K) \\ &= \int \mathcal{D}[A\psi\overline{\psi}C] \, \exp\left[i\int dx(\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + AJ + \overline{\psi}\eta + \overline{\eta}\psi + CK)\right] \times \\ &\quad \left\{1 + i\int dx \, \left[\delta\mathcal{L}_{\mathcal{GF}} + (\delta A)\cdot J + (\delta\overline{\psi})\eta + \overline{\eta}\delta\psi\right] + \cdots\right\}. \end{aligned} \quad (4.22)$$

In obtaining the above expression we use the fact that the integral measure $\mathcal{D}[A\psi\overline{\psi}C]$ and the Lagrangian $\mathcal{L}_0$ are gauge transformation invariant (see Appendix F). Equating (4.20) and (4.22) one gets

$$\begin{aligned} 0 = \delta Z[J,\eta,\overline{\eta},K] = \int \mathcal{D}[A\psi\overline{\psi}C] \int dx \, [\delta\mathcal{L}_{\mathcal{GF}} + (\delta A)\cdot J + (\delta\overline{\psi})\eta \, + \\ \overline{\eta}\delta\psi + (\delta C)K] \, e^{iS} \end{aligned} \quad (4.23)$$

with

$$S = \int dx(\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + AJ + \overline{\psi}\eta + \overline{\eta}\psi + CK). \quad (4.24)$$

To rewrite the identity (4.23) in a more useful form let us combine the variation of $\mathcal{L}_{\mathcal{GF}}$ and CK

$$\delta\mathcal{L}_{\mathcal{GF}} + (\delta C)K = (\delta C)(K + G\cdot A + \lambda C) + CG^{\mu}\delta A_{\mu} = CG^{\mu}\delta A_{\mu} = \mp(G^{\mu}C)\delta A_{\mu}. \quad (4.25)$$

The δC term in the above expression vanishes since, according to the Euler-Lagrange equation,

$$K + G\cdot A + \lambda C = 0. \quad (4.26)$$

This also means that the action S is invariant under any transformation of C. Now the identity (4.23) becomes

$$0 = \int dx\Lambda(x)[\mp\partial_\mu G^\mu C + \partial \cdot J + ie\overline{\psi}\eta - ie\overline{\eta}\psi]Z[J,\eta,\overline{\eta},K]$$
$$= \pm i\partial_\mu G^\mu \frac{\delta Z}{\delta K} + \partial \cdot JZ + e\eta\frac{\delta Z}{\delta\eta} - e\overline{\eta}\frac{\delta Z}{\delta\overline{\eta}}. \tag{4.27}$$

This is the Ward-Takahashi identity in the functional form. The upper (lower) sign in the first term on the right-hand side of equation (4.27) is associated with the Lorentz gauge (the axial and FS gauges). It turns out that the Ward-Takahashi identities differ only slightly with gauge choices.

As has been mentioned in the previous chapter, the generating functional $Z[J,\eta,\overline{\eta},K]$ consist of connected and disconnected Feynman diagrams. Since only the connected diagrams contribute to $S-1$, the nontrivial part of the scattering matrix S, it is advantageous to rewrite the Ward-Takahashi identity (4.27) in the terms of the connected generating functional $W[J,\eta,\overline{\eta},K]$. The relation between both kind of functionals is given by

$$Z[J,\eta,\overline{\eta},K] = \exp iW[J,\eta,\overline{\eta},K]. \tag{4.28}$$

Since

$$\frac{\delta Z}{\delta J^\mu} = iZ\frac{\delta W}{\delta J^\mu}; \quad \frac{\delta Z}{\delta\eta} = iZ\frac{\delta W}{\delta\eta}; \quad \frac{\delta Z}{\delta\overline{\eta}} = iZ\frac{\delta W}{\delta\overline{\eta}}; \quad \frac{\delta Z}{\delta K} = iZ\frac{\delta W}{\delta K} \tag{4.29}$$

the Ward-Takahashi identity (4.27) becomes

$$\pm i\partial_\mu G^\mu \frac{\delta W}{\delta K} - i\partial \cdot J + e\eta\frac{\delta W}{\delta\eta} - e\overline{\eta}\frac{\delta W}{\delta\overline{\eta}} = 0. \tag{4.30}$$

It is instructive and useful to express the identity in terms of another functional called the effective action $\Gamma[A,\psi,\overline{\psi},C]$ which generates one-particle irreducible diagrams. This is defined by

$$\Gamma[A,\psi,\overline{\psi},C] = W[J,\eta,\overline{\eta},K] - \int dx[J\cdot A + \overline{\psi}\eta + \overline{\eta}\psi + CK] \tag{4.31}$$

with

$$\begin{array}{llll} \frac{\delta W}{\delta J^\mu} = A_\mu; & \frac{\delta W}{\delta\eta} = -\overline{\psi}; & \frac{\delta W}{\delta\overline{\eta}} = \psi; & \frac{\delta W}{\delta K} = C \\ \frac{\delta\Gamma}{\delta A^\mu} = -J_\mu; & \frac{\delta\Gamma}{\delta\overline{\psi}} = -\eta; & \frac{\delta\Gamma}{\delta\psi} = \overline{\eta}; & \frac{\delta\Gamma}{\delta C} = -K. \end{array} \tag{4.32}$$

Accordingly the Ward-Takahashi identity (4.30) changes into

$$\pm i\partial_\mu G^\mu C + i\partial^\mu \frac{\delta\Gamma}{\delta A^\mu} - e\overline{\psi}\frac{\delta\Gamma}{\delta\overline{\psi}} + e\psi\frac{\delta\Gamma}{\psi} = 0. \tag{4.33}$$

Equations (4.30) and (4.33) form a complete formulation of the Ward-Takahashi identities of quantum electrodynamics. The identity (4.30) is the Ward-Takahashi identity in the form of external sources whereas the corresponding identity given in the form of field variables is shown in equations (4.33).

Finally consider some consequences relating to the identity (4.33). Functionally differentiating the identity with respect to $\overline{\psi}(x_1)$ and $\psi(x_2)$ and setting $A = \overline{\psi} = \psi = C = 0$ one gets

$$i\partial^\mu \frac{\delta^3\Gamma[0]}{\delta\overline{\psi}(x_1)\delta\psi(x_2)\delta A^\mu(x)} = e[\delta(x-x_1) - \delta(x-x_2)]\frac{\delta^2\Gamma[0]}{\delta\overline{\psi}(x_1)\delta\psi(x_2)} \tag{4.34}$$

Differentiating (4.33) with respect to $A_\mu(y)$ and then putting $A = \overline{\psi} = \psi = 0$, one obtains

$$0 = \partial_\mu \frac{\delta^2\Gamma[0]}{\delta A_\mu(x)\delta A_\nu(y)} = \partial_\mu[G^{-1^{\mu\nu}}(x,y) - \Pi^{\mu\nu}(x,y)]. \tag{4.35}$$

This gives

$$\partial_\mu \Pi^{\mu\nu}(x,y) = 0 \tag{4.36}$$

because

$$G^{-1^{\mu\nu}}(x,y) = [\Box g^{\mu\nu} - \partial^\mu\partial^\nu]\delta(x-y). \tag{4.37}$$

Lastly, functionally differentiate (4.33) with respect to $C(y)$ and take $C = A = \overline{\psi} = \psi = 0$; we have a trivial result

$$0 = \pm\partial_\mu G^\mu \delta(x-y) + \partial^\mu \frac{\delta^2\Gamma[0]}{\delta A^\mu(x)\delta C(y)} = \pm\partial_\mu G^\mu\delta(x-y) \mp \partial_\mu G^\mu \delta(x-y). \tag{4.38}$$

4.4 Slavnov-Taylor Identities

The Ward-Takahashi identities in non-Abelian theories were first derived by Taylor [Tay 71] and Slavnov [Sla 72]. Since then the identities often bear their names. The derivation of the Slavnov-Taylor identity is analogous to the derivation of the Ward-Takahashi identity previously discussed. This section is devoted to the derivation of it.

Instead of the generating functional (4.20) we begin with the generating functional

$$Z[J,K] = \int \mathcal{D}[AC]\,\triangle[A] \exp i \int dx (\mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + AJ + CK). \tag{4.39}$$

We leave out the fermion fields because they are irrelevant to our discussion. $\triangle[A]$ is crucial in non-Abelian theories as it is dependent on the gauge fields A^μ and thus cannot be included in the normalization factor.

Taking advantage of the gauge invariance of $\mathcal{D}[AC]$ and $\triangle_F[A]$ the generating functional (4.39) is equivalent to

$$Z[J,K] = \int \mathcal{D}[AC]\,\triangle[A] \exp i \int dx (\mathcal{L}_0 + \mathcal{L}'_{\mathcal{GF}} + A'J + C'K), \tag{4.40}$$

and therefore

$$0 = \delta Z = \int dx\, (\delta\mathcal{L}_{\mathcal{GF}} + J\cdot\delta A + (\delta C)K)Z. \tag{4.41}$$

Even though its general form is similar to the identity (4.23) there is a difference. The difference between both is associated with the form of δA. In non-Abelian theories, δA is dependent on the gauge field A while in Abelian theories it is not. This dependence of δA on A in (4.41) leads to difficulties in reformulating this identity into the way we have treated with the Ward-Takahashi identity. We show this difficulty below.

Now let us define

$$\Omega^a = \delta(G\cdot A^a) = -G^\mu D^{ab}_\mu \Lambda^b. \tag{4.42}$$

Accordingly we have

$$\delta A^a_\mu = -D^{ab}_\mu \Lambda^b = D^{ab}_\mu (G\cdot D)^{-1^{bc}} \Omega^c \tag{4.43}$$

and

$$\delta\mathcal{L}_{\mathcal{GF}} + K\delta C = (\delta C^a)(G\cdot A^a + \lambda C^a + K^a) + C^a\delta(G\cdot A^a) = C^a\Omega^a. \tag{4.44}$$

Hence we obtain the Slavnov-Taylor identity

$$0 = \left[\frac{\delta}{i\delta K^a} + \int dy J^{b\mu}(y) D_\mu^{bc}(\frac{\delta}{i\delta J}) M^{-1^{ca}}(y, x; \frac{\delta}{i\delta J})\right] Z. \tag{4.45}$$

It turns out that the appearance of $M(y, x; \frac{\delta}{i\delta J})$ leads to difficulties in expressing the Slavnov-Taylor identity (4.45) compactly for one-particle irreducible functions [Lee 76, Itz 80, Tho 82]. This problem was resolved by Becchi, Rouet, Stora [Bec 74, Bec 76] and Tyutin [Tyu 75] who replaced the gauge transformations with their extended BRST transformations.

One important point that should be noted is that in ghost-free gauges like the FS gauge the quantity M is independent of $\frac{\delta}{\delta J}$. In that case the Slavnov-Taylor identity (4.45) may be translated easily into an identity for the one-particle irreducible functions. Thus reformulation of the above identity into the Ward-Takahashi-like form may be carried out and is given in the last part of the next section. It happens to be identical to the BRST identity in the ghost-free gauges.

4.5 BRST Identities

The derivation of the BRST identity is similar to the derivation of the Ward-Takahashi identity. Instead of the gauge transformation (4.14) the BRST transformations (4.14) and (4.17) are used as the symmetry of the Lagrangian. The identity will be more complicated compared to the Ward-Takahashi identity. However, in ghost-free theories such as the FS gauge theory, the ghost fields may be discarded. As a result the BRST transformations simplify to the local gauge transformation version. In this section we will derive the BRST identities in both: by keeping and excluding the ghost terms in the generating functional

$$Z[J, \eta, \overline{\eta}, \xi, \xi^*, K] = \int \mathcal{D}[A\psi\overline{\psi}\chi\chi^* C] \exp\left\{i \int dx \left[\mathcal{L} + A \cdot J + \overline{\eta}\psi + \overline{\psi}\eta \right.\right.$$
$$\left.\left. + \chi^*\xi + \xi^*\chi + CK\right]\right\} \tag{4.46}$$

with

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + \mathcal{L}_{\mathcal{FP}}. \tag{4.47}$$

Since the Lagrangian $\mathcal{L}$ and the integral measure $\mathcal{D}[A\psi\overline{\psi}\chi\chi^* C]$ are BRST invariant (see Appendix F), one has

$$
\begin{aligned}
0 &= \delta Z[J,\eta,\overline{\eta},\xi,\xi^*,K] \\
&= \int dx \left[\delta A\cdot J + \overline{\eta}\delta\psi + (\delta\overline{\psi})\eta + (\delta\chi^*)\xi + \xi^*\delta\chi + K\delta C\right] Z[J,\eta,\overline{\eta},\xi,\xi^*,K] \\
&= \int \mathcal{D}[A\psi\overline{\psi}\chi\chi^* C] \int dx \left[\theta J^{a\mu} D_\mu^{ab}\chi^b + ig\overline{\eta}\theta T^a\chi^a\psi - ig\theta\overline{\psi}T^a\chi^a\eta + \theta C^a\xi^a \right. \\
&\qquad \left. - \frac{g}{2} f^{abc}\xi^{*a}\theta\chi^b\chi^c\right] \exp iS
\end{aligned}
\tag{4.48}
$$

with

$$
S = \int dx \left[\mathcal{L} + A\cdot J + \overline{\eta}\psi + \overline{\psi}\eta + \chi^*\xi + \xi^*\chi + CK\right]. \tag{4.49}
$$

To rewrite (4.48) in the form of external source variations one should introduce new anticommuting source $u^{a\mu}$ and commuting sources v^a, ω and $\overline{\omega}$ in the action S,

$$
\begin{aligned}
S = \int dx \Big[&\mathcal{L} + A\cdot J + \overline{\eta}\psi + \overline{\psi}\eta + \chi^*\xi + \xi^*\chi + CK + u^{a\mu}D_\mu^{ab}\chi^b \\
&- \frac{g}{2} v^a f^{abc}\chi^b\chi^c + ig\chi^a\overline{\omega}T^a\psi - ig\overline{\psi}\chi^a T^a\omega\Big].
\end{aligned}
\tag{4.50}
$$

This new action does not lead to different identities, i.e. the identity (4.48) remains unchanged since $D_\mu^{ab}\chi^b$, $f^{abc}\chi^b\chi^c$, $\chi\psi$ and $\overline{\psi}\chi$ are BRST invariant as is checked in Appendix F:

$$
\delta(D_\mu^{ab}\chi^b) = \delta(f^{abc}\chi^b\chi^c) = \delta(\chi\psi) = \delta(\overline{\psi}\chi) = 0. \tag{4.51}
$$

Note also that under the BRST transformations the following equations hold

$$
\delta^2 A^{a\mu} = \delta^2\psi = \delta^2\overline{\psi} = \delta^2\chi^a = \delta^2\chi^{*a} = \delta^2 C^a = 0. \tag{4.52}
$$

This means that the BRST transformations are nilpotent. Now under the new action (4.50) the identity (4.48) reads

$$
\int dx \left[J^{a\mu}\frac{\delta Z}{\delta u^{a\mu}} + \frac{\delta Z}{\delta K^a}\xi^a - \xi^{*a}\frac{\delta Z}{\delta v^a} - \overline{\eta}\frac{\delta Z}{\delta\overline{\omega}} + \frac{\delta Z}{\delta\omega}\eta\right] = 0 \tag{4.53}
$$

or in the form of connected generating functional $W = -i\ln Z$

$$
\int dx \left[J^{a\mu}\frac{\delta W}{\delta u^{a\mu}} + \xi^a\frac{\delta W}{\delta K^a} - \xi^{*a}\frac{\delta W}{\delta v^a} - \overline{\eta}\frac{\delta W}{\delta\overline{\omega}} + \frac{\delta W}{\delta\omega}\eta\right] = 0 \tag{4.54}
$$

Equations (4.53) and (4.54) are the BRST identities in the generating and connected generating functional forms respectively. If we define the effective action Γ

$$\begin{aligned}\Gamma[A,\psi,\overline{\psi},\chi,\chi^*,C,u,v,\omega,\varpi] &= W[J,\eta,\overline{\eta},\xi,\xi^*,K,u,v,\omega,\varpi]\\ &\quad -\int dx[J\cdot A+\chi^*\xi+\xi^*\chi+\overline{\psi}\eta+\overline{\eta}\psi+CK]\end{aligned} \tag{4.55}$$

where

$$\begin{aligned}
\frac{\delta W}{\delta J^{a\mu}} &= A^a_\mu; & \frac{\delta W}{\delta \xi^a} &= -\chi^{*^a}; & \frac{\delta W}{\delta \xi^{*^a}} &= \chi^a;\\
\frac{\delta W}{\delta \eta} &= -\overline{\psi}; & \frac{\delta W}{\delta \overline{\eta}} &= \psi; & \frac{\delta W}{\delta K} &= C;\\
\frac{\delta \Gamma}{\delta A^{a\mu}} &= -J^a_\mu; & \frac{\delta \Gamma}{\delta \chi^{*^a}} &= -\xi^a; & \frac{\delta \Gamma}{\delta \chi^a} &= \xi^{*^a};\\
\frac{\delta \Gamma}{\delta \overline{\psi}} &= -\eta; & \frac{\delta \Gamma}{\delta \psi} &= \overline{\eta}; & \frac{\delta \Gamma}{\delta C} &= -K;\\
\frac{\delta W}{\delta t} &= \frac{\delta \Gamma}{\delta t} = \frac{1}{iZ}\frac{\delta Z}{\delta t}, & & t = u^{a\mu}, v^a, \omega, \varpi
\end{aligned} \tag{4.56}$$

equation (4.54) becomes

$$\int dx\left[\frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta u^{a\mu}}+C^a\frac{\delta\Gamma}{\delta\chi^{*^a}}+\frac{\delta\Gamma}{\delta\chi^a}\frac{\delta\Gamma}{\delta v^a}+\frac{\delta\Gamma}{\delta\psi}\frac{\delta\Gamma}{\delta\varpi}+\frac{\delta\Gamma}{\delta\omega}\frac{\delta\Gamma}{\delta\overline{\psi}}\right]=0. \tag{4.57}$$

The functional derivative with respect to χ^{*^a} may be replaced by functional derivative with respect to $u^{a\mu}$ by the use of the ghost field equation

$$0=<-G^\mu D^{ab}_\mu\chi^b+\xi^a>=-G^\mu\frac{\delta\Gamma}{\delta u^{a\mu}}-\frac{\delta\Gamma}{\delta\chi^{*^a}}. \tag{4.58}$$

Hence

$$\int dx\left\{\frac{\delta\Gamma}{\delta u^{a\mu}}\left[\frac{\delta\Gamma}{\delta A^a_\mu}\pm G^\mu C^a\right]+\frac{\delta\Gamma}{\delta\chi^a}\frac{\delta\Gamma}{\delta v^a}+\frac{\delta\Gamma}{\delta\psi}\frac{\delta\Gamma}{\delta\varpi}+\frac{\delta\Gamma}{\delta\omega}\frac{\delta\Gamma}{\delta\overline{\psi}}\right\}=0, \tag{4.59}$$

after performing integration by parts in the second term for the Lorentz gauge.

The identity like (4.36) can be obtained by functionally differentiating equation (4.59) with respect to $A^{b\nu}(y)$ and $\chi^c(z)$ and setting all fields to zero. We obtain,

$$0=\int dx\frac{\delta^2\Gamma[0]}{\delta\chi^c(z)\delta u^{a\mu}(x)}\frac{\delta^2\Gamma[0]}{\delta A^a_\mu(x)\delta A^b_\nu(y)}=\partial_\mu\Pi^{ab\mu\nu}(x,y)=0 \tag{4.60}$$

after recalling (4.37), i.e. the self-energy remains transversal even in non-Abelian theories. Other identities similar to (4.34) may also be derived, namely by functionally differentiating (4.59) with respect to $\chi^b(y)$, $\psi(z)$ and $\overline{\psi}(t)$.

We have derived the BRST identities in the Lorentz, axial-type and FS gauges. In deriving the BRST identity ghost fields must be included in the Lorentz gauge. In ghost-free gauges such as the axial and FS gauges, on the other hand, the BRST identities may be derived by neglecting the ghost fields. Since this derivation has not been carried out the BRST identity in the FS gauge without using the ghost fields will now be obtained.

By excluding the ghost terms the generating functional (4.46) reduces to (4.39). Since we are still working with the non-Abelian theory color indices a should be retained in the generating functional (4.20), but $\triangle[A]$ in (4.39) may be omitted in ghost-free cases. The exclusion of ghost fields effectively reduces the BRST transformations to the local gauge transformations (4.14). Therefore the identity that we are looking for is

$$0 = \delta Z[J^a, \eta, \overline{\eta}, K^a] = \int \mathcal{D}[A^a \psi \overline{\psi} C^a] \int dx \left[\delta \mathcal{L}_{\mathcal{GF}} + (\delta A^a) \cdot J^a + (\delta \overline{\psi})\eta \right.$$
$$\left. + \overline{\eta} \delta \psi\right] e^{iS}. \tag{4.61}$$

This is nothing but the identity (4.41) (after including fermion terms) or the non-Abelian version of the identity (4.23). Note that the term $K^a \delta C^a$ in (4.61) is excluded since $\delta C^a = 0$.

To derive the above BRST identity (and thus the Slavnov-Taylor identity) in the FS ghost-free gauge more explicitly, let us consider the first two terms of identity (4.61). According to the gauge transformation (4.14) these terms become

$$\begin{aligned} \delta \mathcal{L}_{\mathcal{GF}} + \delta A^{a\mu} J^a_\mu = & \ (C^a x_\mu + J^a_\mu) \delta A^{a\mu} \\ = & \ \Lambda^b [\delta^{ab} \partial^\mu + g f^{abc} A^{c\mu}] (C^a x_\mu + J^a_\mu) \end{aligned} \tag{4.62}$$

after discarding the surface terms. Inserting (4.62) into (4.61) and recalling the variation of the fermion fields according to the gauge transformation (4.14) we obtain the BRST identity

$$\begin{aligned} 0 = & \ i\delta^{ab} \partial^\mu x_\mu \frac{\delta Z}{\delta K^a} + g f^{abc} x_\mu \frac{\delta^2 Z}{\delta J^c_\mu \delta K^a} - \delta^{ab} \partial_\mu J^{a\mu} Z \\ & + ig f^{abc} J^{a\mu} \frac{\delta Z}{\delta J^{c\mu}} - g T^b \eta \frac{\delta Z}{\delta \eta} + g T^b \overline{\eta} \frac{\delta Z}{\delta \overline{\eta}}. \end{aligned} \tag{4.63}$$

In terms of the connected generating functional W and the effective action Γ the BRST identity (4.63) is given by

$$\begin{aligned} 0 = \; & i\delta^{ab}\partial^\mu x_\mu \frac{\delta W}{\delta K^a} + g f^{abc} x_\mu \left(i \frac{\delta W}{\delta J^c_\mu} \frac{\delta W}{\delta K^a} + \frac{\delta^2 W}{\delta J^c_\mu \delta K^a} \right) \\ & + i\delta^{ab}\partial_\mu J^{a\mu} + ig f^{abc} J^{a\mu} \frac{\delta W}{\delta J^{c\mu}} - gT^b \eta \frac{\delta W}{\delta \eta} + gT^b \overline{\eta} \frac{\delta W}{\delta \overline{\eta}} \end{aligned} \tag{4.64}$$

or

$$\begin{aligned} 0 = \; & i\delta^{ab}\partial^\mu x_\mu C^a + ig f^{abc} x_\mu A^{c\mu} C^a - i\delta^{ab}\partial^\mu \frac{\delta \Gamma}{\delta A^{a\mu}} \\ & - ig f^{abc} A^{c\mu} \frac{\delta \Gamma}{\delta A^{a\mu}} + gT^b \overline{\psi} \frac{\delta \Gamma}{\delta \overline{\psi}} - gT^b \psi \frac{\delta \Gamma}{\delta \psi}. \end{aligned} \tag{4.65}$$

In contrast to the BRST identity (4.59), the BRST identity (4.65) is simpler in the sense that the latter identity does not contain composite sources u, v, ω and ϖ. Their difference from the Ward-Takahashi identity (4.33) is due to the f^{abc}-terms. Thus when we change non-Abelian theories into Abelian theories the BRST identity (4.65) will reduce to the Ward-Takahashi identity (4.33).

Now suppose the identity (4.65) is functionally differentiated with respect to $A^{d\nu}(y)$ and then setting all fields to zero. In that case the only term in the identity that survives is the third term, and it becomes

$$\partial^\mu \frac{\delta^2 \Gamma[0]}{\delta A^{b\mu}(x) \delta A^{d\nu}(y)} = 0. \tag{4.66}$$

This equation is the same as the equation (4.60), thus it gives the same result

$$\partial_\mu \Pi^{bd\mu\nu}(x,y) = 0. \tag{4.67}$$

It is obvious that the contributions of fermion terms, such as the identity like (4.34), are the same as found previously.

To end this chapter let us briefly summarize our results. We have rederived the Ward-Takahashi identity in quantum electrodynamics. In non-Abelian gauges the difficulties in expressing the Slavnov-Taylor identities (the equivalence of the Ward-Takahashi identities) on one-particle irreducible functions leads one to introducing a new symmetry, the BRST symmetry. The BRST invariance of the Lagrangian results in the BRST identity, replacing the Slavnov-Taylor identity. The BRST identity in the FS gauge has been derived in two cases. The first version includes the ghost fields and composite sources such as u, v, ω and ϖ. The resulting identity therefore consists of such sources as well as the original ones (gauge, fermion and ghost sources). In the second case the derivation has been carried out by excluding the ghost fields. As a result the identity reduces to the Slavnov-Taylor identity and no composite sources need be introduced; it just consists of the original sources and is much simpler.

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Chapter 5

One-Loop Graphs in the Fock-Schwinger Gauge

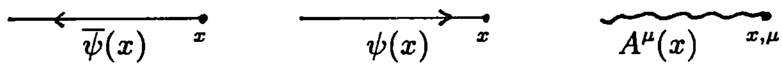
The FS gauge propagator with the gauge parameter $\lambda \rightarrow 0$ will be used to work out all the following perturbative calculations. In scalar and spinor quantum electrodynamics calculations will be performed both in "momentum space" and coordinate space. Problems with translational invariance in quantum chromodynamics lead to difficulties in carrying out the scattering computations in momentum space; In this particular case the evaluations will be done only in coordinate space. Throughout the chapter we only write the basic form of each diagram and its final form before and after putting the diagram on the mass-shell (ms) condition. Details of calculations which are sometimes tedious can be seen in the Appendix G.

5.1 Feynman Rules

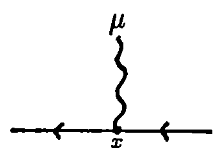
The Feynman rules of spinor and scalar quantum electrodynamics and quantum chromodynamics stemming from the books of Itzykson and Zuber [Itz 80] and Muta [Mut 87] are summarized below.

5.1.1 Spinor Electrodynamics

Fields

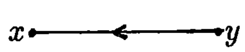


Vertex



$-ie\gamma^{\mu}$

Propagators

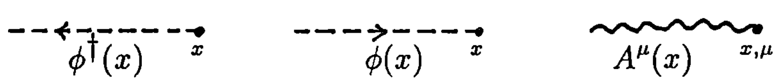


$\psi(x)\overline{\psi}(y) = iS_F(x-y) =< 0|T\psi(x)\overline{\psi}(y)|0 >$

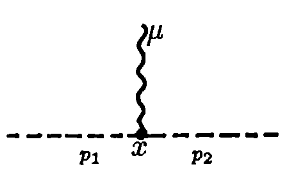
$A_{\mu}(x)A_{\nu}(y) = iG_{\mu\nu}(x,y) =< 0|TA_{\mu}(x)A_{\nu}(y)|0 >$

5.1.2 Scalar Electrodynamics

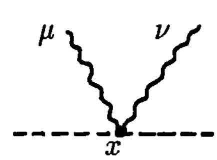
Fields



Vertices

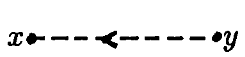


$e\,\overleftrightarrow{\partial}^{\mu}$



$2ie^2g^{\mu\nu}$

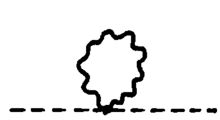
Propagators



$\phi(x)\phi^{\dagger}(y) = iS_B(x-y) =< 0|T\phi(x)\phi^{\dagger}(y)|0 >$

$A_{\mu}(x)A_{\nu}(y) = iG_{\mu\nu}(x,y) =< 0|TA_{\mu}(x)A_{\nu}(y)|0 >$

Symmetry factor

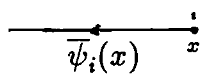


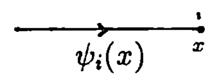
$\frac{1}{2!}$

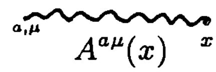
note: $\overleftrightarrow{\partial}^{\mu}$ acts purely on scalar fields.

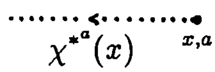
5.1.3 Quantum Chromodynamics

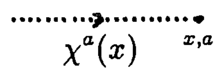
Fields



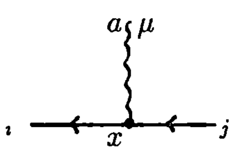




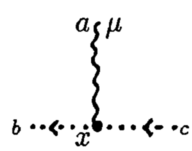




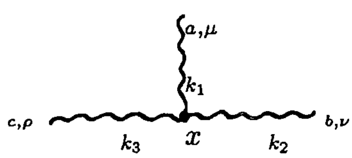
Vertices



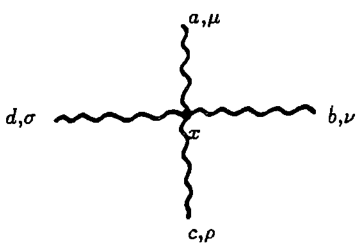
$ig\gamma^\mu(T^a)_{ij}$



$\mp igf^{abc}G^{b\mu}$

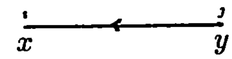


$$
\begin{aligned}
&igf^{abc}V^{(a,b,c)}_{\mu\nu\rho}(\partial_x)\\
&= igf^{abc}[g_{\mu\nu}(\partial^{(a)}_{x\rho} - \partial^{(b)}_{x\rho}) + g_{\nu\rho}(\partial^{(b)}_{x\mu} - \partial^{(c)}_{x\mu})\\
&\qquad +g_{\rho\mu}(\partial^{(c)}_{x\nu} - \partial^{(a)}_{x\nu})]
\end{aligned}
$$



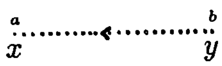
$$
\begin{aligned}
&-ig^2W^{abcd}_{\mu\nu\rho\sigma}\\
&= -ig^2[f^{eab}f^{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\\
&\qquad +f^{eac}f^{edb}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma})\\
&\qquad +f^{ead}f^{ebc}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})]
\end{aligned}
$$

Propagators



$iS_Q^{ij}(x-y) = i\delta^{ij} S_F(x-y)$

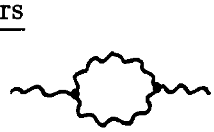
$iG_{\mu\nu}^{ab}(x,y) = i\delta^{ab} G_{\mu\nu}(x,y)$



$iS_G^{ab}(x-y) = i\delta^{ab} S_B(x-y), \quad m=0$

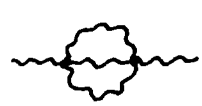
(Lorentz gauge)

Symmetry factors



$\frac{1}{2!}$

$\frac{1}{2!}$



$\frac{1}{3!}$

5.2 One-loop Corrections in Spinor Electrodynamics

The FS gauge propagators which have been derived in Chapter 3 can be written in a general form

$$G^{\mu\nu}(x,y) = G_F^{\mu\nu}(x,y) + G'^{\mu\nu}(x,y) \tag{5.1}$$

where $G_F^{\mu\nu}(x,y)$ are the Feynman gauge propagators

$$G_F^{\mu\nu}(x,y) = g^{\mu\nu}\Box^{-1}\delta(x-y) = g^{\mu\nu}\int \frac{\bar{d}k}{-k^2} e^{-ik(x-y)}. \tag{5.2}$$

Since calculations of scattering diagrams in the Feynman gauge can be found in almost all textbooks on quantum field theory, it is sufficient to only consider the corrections due to $G'^{\mu\nu}(x,y)$ when one works diagrams which are linear in gauge propagators.

5.2.1 Calculations in Momentum Space

Here the modification to the propagators that we use are

$$G'^{\mu\nu}(x,y) = \int_1^\infty d\beta \int \frac{\bar{d}k}{-k^2}[f_\nu(\beta,k,\partial_k,y)k_\mu e^{-i\beta kx} + g_\mu(\beta,k,\partial_k,x)k_\nu e^{i\beta ky}]. \quad (5.3)$$

1. Electron-electron scattering (Born term)

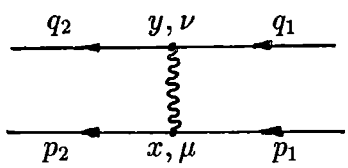


Figure 5.1

$$\begin{aligned} S' =& -ie^2\overline{u}(p_2)\gamma^\mu u(p_1)\overline{u}(q_2)\gamma^\nu u(q_1)\int dx\int dy G'_{\mu\nu}(x,y)e^{ix(p_2-p_1)+iy(q_2-q_1)} \\ =& \; ie^2\int_1^\infty \frac{d\beta}{\beta}\int\frac{dk}{-k^2}\int dy g_\mu(\beta,k,\partial_k,y) \\ & \{\overline{u}(p_2)(\not p_2 - \not p_1)u(p_1)\overline{u}(q_2)\gamma^\mu u(q_1)\delta(p_2-p_1+\beta k)e^{iy(q_2-q_1)} \\ & +\overline{u}(p_2)\gamma^\mu u(p_1)\overline{u}(q_2)(\not q_2 - \not q_1)u(q_1)\delta(q_2-q_1+\beta k)e^{iy(p_2-p_1)}\} \\ \overset{ms}{=}& \; 0. \end{aligned} \quad (5.4)$$

2. Electron self-energy

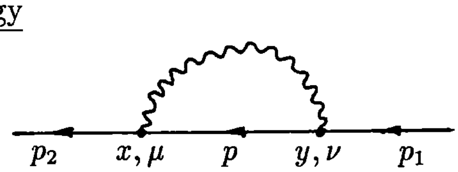


Figure 5.2

$$\begin{aligned} S' =& \; e^2\int \bar{d}p\overline{u}(p_2)\gamma^\mu(\not p - m + i\epsilon)^{-1}\gamma^\nu u(p_1)\int dx\int dy e^{ix(p_2-p)+iy(p-p_1)}G'_{\mu\nu}(x,y) \\ =& \; e^2\int_1^\infty\frac{d\beta}{\beta}\int\frac{dk}{-k^2}\int\bar{d}y\int dp g_\mu(\beta,k,\partial_k,y) \\ & [\overline{u}(p_2)(\not p - \not p_2)(\not p - m + i\epsilon)^{-1}\gamma^\mu u(p_1)\delta(p_2-p+\beta k)e^{iy(p-p_1)} + \\ & \overline{u}(p_2)\gamma^\mu(\not p - m + i\epsilon)^{-1}(\not p_1 - \not p)u(p_1)\delta(p-p_1+\beta k)e^{iy(p_2-p)}] \\ \overset{ms}{=}& \; 0. \end{aligned} \quad (5.5)$$

Off the mass-shell, the expressions $(\not p - \not p_2)(\not p - m + i\epsilon)^{-1}\gamma^\mu$ and $\gamma^\mu(\not p - m + i\epsilon)^{-1}$ $(\not p - \not p_2)$ cannot be further simplified. Also, the parameter β introduced in the FS gauge propagator does not play any role in simplifying the whole expression.

Accordingly, S' does *not* vanish off the mass-shell. On the mass-shell, however, $(\not{p}-\not{p}_2)(\not{p}-m+i\epsilon)^{-1}$ reduces to unity and leaves a factor $\overline{u}(p_2)\gamma^\mu u(p_1)$. Such factor can also be obtained in the second term. In this case cancellation between both terms depends on the rest (exponential and delta functions) and indeed occurs:

$$S' \stackrel{ms}{=} 0. \tag{5.6}$$

In most diagrams that we will come across, simplifications also happen in a similar way on the mass-shell. In the first order of vertex corrections, for example, it takes place between all three possible diagrams. This will be shown below.

3. Vertex corrections

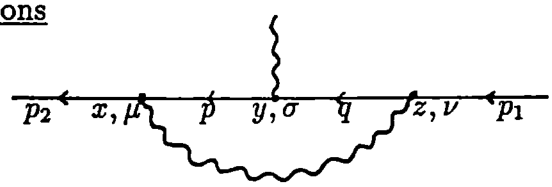


Figure 5.3

$$\begin{aligned}
S_1' = \; & e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz \overline{u}(p_2)\gamma^\mu(\not{p}-m+i\epsilon)^{-1} \not{A}(y)(\not{q}-m+i\epsilon)^{-1} \times \\
& \gamma^\nu u(p_1) e^{ix(p_2-p)+iy(p-q)+iz(q-p_1)} G'_{\mu\nu}(x,z) \\
= \; & e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int \bar{d}p \int \bar{d}q \int dy \int dz g_\nu(\beta,k,\partial_k,z)\overline{u}(p_2) \\
& [(\not{p}-\not{p}_2)(\not{p}-m+i\epsilon)^{-1} \not{A}(y)(\not{q}-m+i\epsilon)^{-1}\gamma^\nu \\
& \qquad \delta(p_2-p+\beta k) e^{iy(p-q)+iz(q-p_1)} \\
& +\gamma^\nu(\not{p}-m+i\epsilon)^{-1} \not{A}(y)(\not{q}-m+i\epsilon)^{-1}(\not{p}_1-\not{q}) \\
& \qquad \delta(q-p_1+\beta k) e^{iy(p-q)+iz(p_2-p)}]u(p_1) \\
\stackrel{ms}{=} \; & e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int dp \int \bar{d}y \int \bar{d}z g_\mu(\beta,k,\partial_k,z)\overline{u}(p_2) \times \\
& [\not{A}(y)(\not{p}-m+i\epsilon)^{-1}\gamma^\mu e^{iy(p_2-p+\beta k)+iz(p-p_1)} \\
& -\gamma_\mu(\not{p}-m+i\epsilon)^{-1} \not{A}(y) e^{iy(p-p_1+\beta k)+iz(p_2-p)}]u(p_1).
\end{aligned} \tag{5.7}$$

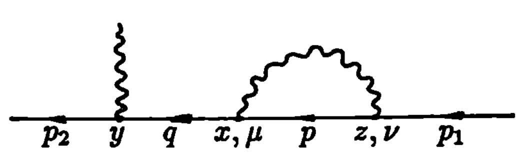


Figure 5.4

$$\begin{aligned}
S_2' = \; & e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz \overline{u}(p_2) \not{A}(y)(\not{q}-m+i\epsilon)^{-1}\gamma^\mu(\not{p}-m+i\epsilon)^{-1} \times \\
& \gamma^\nu u(p_1) e^{ix(q-p)+iy(p_2-q)+iz(p-p_1)} G'_{\mu\nu}(x,z)
\end{aligned}$$

$$
\begin{aligned}
= \quad & -e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dq \int dy \int dz g_\nu(\beta, k, \partial_k, z) \overline{u}(p_2) \; \not{A}(y) \times \\
& \qquad e^{iy(p_2-q)+iz(q-p_1+\beta k)} \times \\
& \quad \{(\not{q} - m + i\epsilon)^{-1}[(\not{q} - m) - (\not{p} - m)](\not{p} - m + i\epsilon)^{-1}\gamma^\nu \delta(q - p + \beta k) \\
& \quad +(\not{q} - m + i\epsilon)^{-1}\gamma^\nu(\not{p} - m + i\epsilon)^{-1}(\not{p} - \not{p}_1)\delta(p - p_1 + \beta k)\}u(p_1) \\
\overset{ms}{=} \quad & -e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dy \int dz g_\mu(\beta, k, \partial_k, z) \overline{u}(p_2) \; \not{A}(y)(\not{p} - m + i\epsilon)^{-1} \times \\
& \qquad \gamma^\mu u(p_1) e^{iy(p_2-p+\beta k)+iz(p-p_1)}.
\end{aligned}
\tag{5.8}
$$

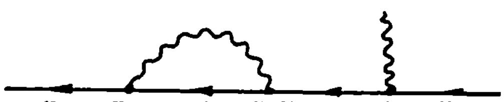


Figure 5.5

$$
\begin{aligned}
S_3' = \quad & e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz \overline{u}(p_2)\gamma^\mu(\not{p} - m + i\epsilon)^{-1}\gamma^\nu(\not{q} - m + i\epsilon)^{-1} \times \\
& \quad \not{A}(y)u(p_1)e^{ix(p_2-p)+iy(q-p_1)+iz(p-q)} G'_{\mu\nu}(x, z) \\
= \quad & e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int \bar{d}p \int \bar{d}q \int dy \int dz g_\nu(\beta, k, \partial_k, z) \overline{u}(p_2) e^{iy(q-p_1)+iz(p_2-q+\beta k)} \\
& \{(\not{p} - \not{p}_2)(\not{p} - m + i\epsilon)^{-1}\gamma^\nu(\not{q} - m + i\epsilon)^{-1}\delta(p_2 - p + \beta k) \\
& \quad -\gamma^\nu(\not{p} - m + i\epsilon)^{-1}[(\not{p} - m) - (\not{q} - m)](\not{q} - m + i\epsilon)^{-1}\delta(p - q + \beta k)\} \\
& \qquad \not{A}(y)u(p_1)
\end{aligned}
$$

$$
\begin{aligned}
\overset{ms}{=} \quad & e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dy \int dz g_\mu(\beta, k, \partial_k, z) \overline{u}(p_2)\gamma^\mu(\not{p} - m + i\epsilon)^{-1} \times \\
& \qquad \not{A}(y)u(p_1)e^{iy(p-p_1+\beta k)+iz(p_2-p)}.
\end{aligned}
\tag{5.9}
$$

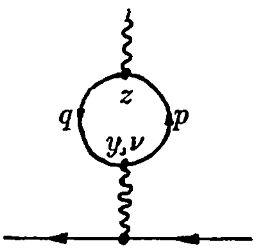


Figure 5.6

$$
\begin{aligned}
S_4' = \quad & e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz \overline{u}(p_2)\gamma^\mu u(p_1) \mathrm{Tr}(\not{q} - m + i\epsilon)^{-1}\gamma^\nu \times \\
& \quad (\not{p} - m + i\epsilon)^{-1} \; \not{A}(z) e^{ix(p_2-p_1)+iy(p-q)+iz(q-p)} G'_{\mu\nu}(x, y)
\end{aligned}
$$

$$
\begin{aligned}
= & \ e^3 \int_1^\infty d\beta \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dq \int dy \int dz g_\nu(\beta, k, \partial_k, y) e^{iz(q-p)} \times \\
& \{\overline{u}(p_2)(\not{p}_1 - \not{p}_2) u(p_1) \mathrm{Tr}(\not{q} - m + i\epsilon)^{-1} \gamma^\nu (\not{p} - m + i\epsilon)^{-1} \times \\
& \quad \delta(p_2 - p_1 + \beta k) e^{iy(p-q)} \\
& + \overline{u}(p_2) \gamma^\nu u(p_1) \mathrm{Tr}(\not{q} - m + i\epsilon)^{-1} [(\not{q} - m) - (\not{p} - m)](\not{p} - m + i\epsilon)^{-1} \\
& \quad \delta(p - q + \beta k) e^{iy(p_2 - p_1)}\} \not{A}(z) \\
\overset{ms}{=} & \ 0.
\end{aligned}
\tag{5.10}
$$

We obtain

$$
S' = S_1' + \cdots + S_4' \overset{ms}{=} 0. \tag{5.11}
$$

5.2.2 Calculations in Coordinate Space

Instead of (5.3) computations in coordinate space will be based on

$$
G'_{\mu\nu}(x, y) = \partial_\mu f_{1\nu}(x, y) + \partial_\nu f_{2\mu}(x, y). \tag{5.12}
$$

This has advantages. Firstly, when the derivative, say $\partial_{x\mu}$, meets the Dirac matrix γ^μ it is possible, for some expression, that a form like $(\not{\partial}_x + im) S_F(x - y)$ is generated on the mass-shell whereupon it will simply reduce to the Dirac delta function $\delta(x - y)$. Another benefit comes from the functions $f_{1\mu}(x, y)$ and $f_{2\mu}(x, y)$, which sometimes produce cancellations due to the symmetry property $f_{1\mu}(x, y) = f_{2\mu}(y, x)$.

In this section we will start with truncated diagrams, i.e. diagrams without external lines, and then cast them into a form such that when external lines are added to the diagrams one can find immediately their simple form on mass-shell.

Truncated Diagrams

All truncated diagrams will be written into forms like $[\cdot(\overleftarrow{\not{\partial}}_x + \overrightarrow{\not{\partial}}_x)\cdot]$ or $[\cdot(\overleftarrow{\not{\partial}}_x + \overrightarrow{\not{\partial}}_x) S_F(x - y)\cdot]$ where the dots are given just to recall that when we turn to the corresponding completed diagrams we just put external (fermion) fields on those dots. The usefulness of the above expressions is that on the mass-shell those forms will reduce to vanish or to Dirac delta functions. Here we will use notation:

directed derivatives such as $\overleftarrow{\not\partial}_x$ and $\overrightarrow{\not\partial}_x$ do *not* act on photon propagators, i.e. on $f_{1\mu}(x,y)$ and $f_{2\mu}(x,y)$.

1. Electron self-energy

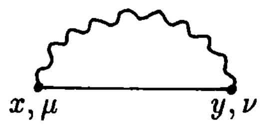


Figure 5.7

$$\begin{aligned} e^{-2}\Sigma'(x,y) = & \ [\cdot\gamma^{\mu}S_F(x-y)\gamma^{\nu}\cdot]G'_{\mu\nu}(x,y) \\ = & \ -[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-y)\gamma^{\mu}\cdot]f_{1\mu}(x,y) \\ & -[\cdot\gamma^{\mu}S_F(x-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\cdot]f_{2\mu}(x,y). \end{aligned} \tag{5.13}$$

2. Vertex corrections

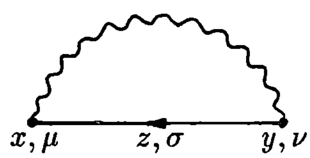


Figure 5.8

$$\begin{aligned} -ie^{-2}\Gamma_1'^{\sigma}(x,y,z) = & \ [\cdot\gamma^{\mu}S_F(x-z)\gamma^{\sigma}S_F(z-y)\gamma^{\nu}\cdot]G'_{\mu\nu}(x,y) \\ = & \ -[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-z)\gamma^{\sigma}S_F(z-y)\gamma^{\nu}\cdot]f_{1\nu}(x,y) \\ & -[\cdot\gamma^{\mu}S_F(x-z)\gamma^{\sigma}S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\cdot]f_{2\mu}(x,y). \end{aligned} \tag{5.14}$$

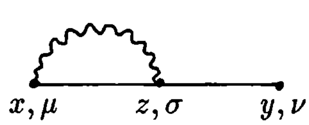


Figure 5.9

$$\begin{aligned} -ie^{-2}\Gamma_2'^{\nu}(x,y,z) = & \ [\cdot\gamma^{\mu}S_F(x-z)\gamma^{\sigma}S_F(z-y)\gamma^{\nu}\cdot]G'_{\mu\sigma}(x,z) \\ = & \ -[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-z)\gamma^{\mu}S_F(z-y)\gamma^{\nu}\cdot]f_{1\mu}(x,z) \\ & -i\{\cdot\gamma^{\mu}[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^{\nu}\cdot\}f_{2\mu}(x,z). \end{aligned} \tag{5.15}$$

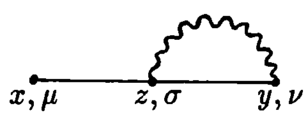


Figure 5.10

$$\begin{aligned}
-ie^{-2}\Gamma_3'^{\mu}(x,y,z) = & \ [\cdot\gamma^{\mu} S_F(x-z)\gamma^{\sigma} S_F(z-y)\gamma^{\nu}\cdot] G'_{\sigma\nu}(z,y) \\
= & \ -i\{\cdot\gamma^{\mu}[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^{\nu}\cdot\} f_{1\nu}(z,y) \qquad (5.16) \\
& -[\cdot\gamma^{\mu} S_F(x-z)\gamma^{\sigma} S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\cdot] f_{2\sigma}(z,y).
\end{aligned}$$

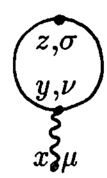


Figure 5.11

$$\begin{aligned}
-ie^{-2}\Gamma_4'^{\sigma}(x,y,z) = & \ [\cdot\gamma^{\mu} \mathrm{Tr} S_F(z-y)\gamma^{\nu} S_F(y-z)\gamma^{\sigma}\cdot] G'_{\mu\nu}(x,y) \\
= & \ -[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)\cdot]\mathrm{Tr} S_F(z-y)\gamma^{\nu} S_F(y-z)\gamma^{\sigma} f_{1\nu}(x,y) \qquad (5.17) \\
& -[\cdot\ f_2(x,y)\cdot]\mathrm{Tr} S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y) S_F(y-z)\gamma^{\sigma}.
\end{aligned}$$

On-shell Diagrams

Results in the previous section are used to evaluate the corresponding (on mass-shell) diagrams.

1. Electron-electron scattering (Born term)

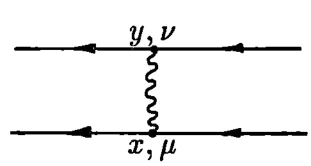


Figure 5.12

$$\begin{aligned}
S' = & \ -ie^2 \int dx \int dy \overline{\psi}(x)\gamma^{\mu}\psi(x)\overline{\psi}(y)\gamma^{\nu}\psi(y) G'_{\mu\nu}(x,y) \\
= & \ ie^2 \int dx \int dy \Big\{ \Big(\overline{\psi}(x)[(\overleftarrow{\not\partial}_x - im) + (\overrightarrow{\not\partial}_x + im)]\psi(x)\Big) \overline{\psi}(y)\ f_1(x,y)\psi(y) \\
& + \overline{\psi}(x)\ f_2(x,y)\psi(x) \Big(\overline{\psi}(y)[(\overleftarrow{\not\partial}_y - im) + (\overrightarrow{\not\partial}_y + im)]\psi(y)\Big)\Big\} \\
\overset{ms}{=} & \ 0.
\end{aligned}$$

(5.18)

2. Electron self-energy

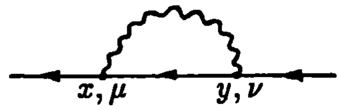


Figure 5.13

$$
\begin{aligned}
S' =&\ \int dx \int dy \overline{\psi}(x)\Sigma'(x,y)\psi(y) \\
=&\ -e^2 \int dx \int dy \left\{ \left[\overline{\psi}(x)(\overleftarrow{\partial\!\!\!/}_x + \overrightarrow{\partial\!\!\!/}_x)S_F(x-y)\gamma^\mu\psi(y)\right] f_{1\mu}(x,y) \right. \\
&\left. + \left[\overline{\psi}(x)\gamma^\mu S_F(x-y)(\overleftarrow{\partial\!\!\!/}_y + \overrightarrow{\partial\!\!\!/}_y)\psi(y)\right] f_{2\mu}(x,y) \right\} \\
\overset{ms}{=}&\ 0.
\end{aligned}
\tag{5.19}
$$

3. Vertex corrections

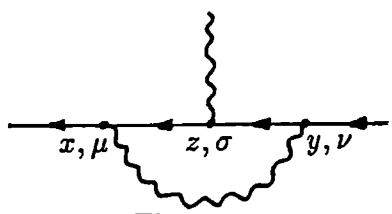


Figure 5.14

$$
\begin{aligned}
S_1' =&\ -ie \int dx \int dy \int dz \overline{\psi}(x)\Gamma_1'^{\sigma}(x,y,z)A_\sigma(z)\psi(y) \\
=&\ -e^3 \int dx \int dy \int dz A_\sigma(z) \\
&\{[\overline{\psi}(x)(\overleftarrow{\partial\!\!\!/}_x + \overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\psi(y)]f_{1\nu}(x,y) \\
&+[\overline{\psi}(x)\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y + \overrightarrow{\partial\!\!\!/}_y)\psi(y)]f_{2\mu}(x,y)\} \\
\overset{ms}{=}&\ ie^3 \int dx \int dy \int dz \{\overline{\psi}(x)\delta(x-z)\ A\!\!\!/(z)S_F(z-y)\ f\!\!\!/_1(x,y)\psi(y) \\
&-\overline{\psi}(x)\ f\!\!\!/_2(x,y)S_F(x-z)\ A\!\!\!/(z)\delta(z-y)\psi(y)\}.
\end{aligned}
\tag{5.20}
$$

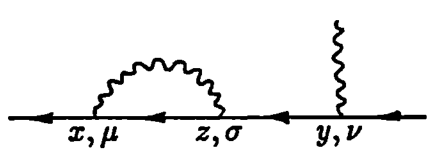


Figure 5.15

$$
\begin{aligned}
S_2' =&\ -ie \int dx \int dy \int dz \overline{\psi}(x)\Gamma_2''^{\nu}(x,y,z)A_\nu(y)\psi(y) \\
=&\ -e^3 \int dx \int dy \int dz A_\nu(y) \\
&[\overline{\psi}(x)(\overleftarrow{\partial\!\!\!/}_x + \overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\mu S_F(z-y)\gamma^\nu\psi(y)]f_{1\mu}(x,z) \\
&+i\{\overline{\psi}(x)\gamma^\mu[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^\nu\psi(y)\}f_{2\mu}(x,z)\} \\
\overset{ms}{=}&\ ie^3 \int dx \int dy \int dz \overline{\psi}(x)\ f\!\!\!/_2(x,z)\delta(z-y)S_F(x-y)\ A\!\!\!/(y)\psi(y).
\end{aligned}
\tag{5.21}
$$

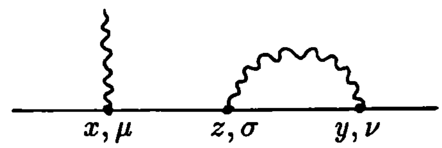


Figure 5.16

$$
\begin{aligned}
S_3' = &\ -ie \int dx \int dy \int dz \overline{\psi}(x) \Gamma_3'^{\mu}(x,y,z) A_\mu(x) \psi(y) \\
= &\ -e^3 \int dx \int dy \int dz A_\mu(x) \\
&\quad i\{[\overline{\psi}(x)\gamma^\mu[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^\nu\psi(y)]f_{1\nu}(z,y) \\
&\quad +[\overline{\psi}(x)\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y+\overrightarrow{\partial\!\!\!/}_y)\psi(y)]f_{2\sigma}(z,y)\} \\
\overset{ms}{=} &\ -ie^3 \int dx \int dy \int dz \overline{\psi}(x)\ A\!\!\!/(x)\delta(x-z)S_F(z-y)\ f\!\!\!/_1(z,y)\psi(y).
\end{aligned}
\tag{5.22}
$$

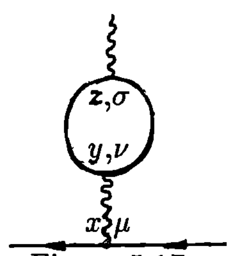


Figure 5.17

$$
\begin{aligned}
S_4' = &\ -ie \int dx \int dy \int dz \overline{\psi}(x) \Gamma_4'^{\sigma}(x,y,z) A_\sigma(z) \psi(x) \\
= &\ -e^3 \int dx \int dy \int dz \overline{\psi}(x)\{(\overleftarrow{\partial\!\!\!/}_x+\overrightarrow{\partial\!\!\!/}_x)\mathrm{Tr}S_F(z-y)\ f\!\!\!/_1(x,y)\times \\
&\qquad S_F(y-z)\gamma^{\sigma} \\
&\ + f\!\!\!/_2(x,y)\mathrm{Tr}S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y+\overrightarrow{\partial\!\!\!/}_y)S_F(y-z)\gamma^\sigma\}A(z)\psi(x) \\
\overset{ms}{=} &\ 0.
\end{aligned}
\tag{5.23}
$$

Hence we have the same result as in momentum space (obviously):

$$
S' = S_1' + S_2' + S_3' + S_4' \overset{ms}{=} 0. \tag{5.24}
$$

5.3 One-loop Corrections in Scalar Electrodynamics

Computations in scalar electrodynamics are basically similar to those in spinor electrodynamics and require little elaboration.

5.3.1 Calculations in Momentum Space

1. Meson-meson scattering (Born term)

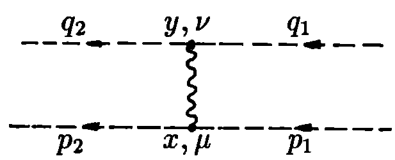


Figure 5.18

$$
\begin{aligned}
S' = & -ie^2 \int dxdy(p_1+p_2)^\mu(q_1+q_2)^\nu G'_{\mu\nu}(x,y)e^{ix(p_2-p_1)+iy(q_2-q_1)} \\
= & -ie^2 \int \frac{\bar{d}k}{-k^2}\int_1^\infty \frac{d\beta}{\beta}\int dy[(q_1+q_2)^\nu g_\nu(\beta,k,\partial_k,y)(p_1^2-p_2^2)\cdot \\
& \qquad \bar{\delta}(p_2-p_1+\beta k)e^{iy(q_2-q_1)}+(p\leftrightarrow q)] \\
\overset{ms}{=} & \ 0.
\end{aligned} \tag{5.25}
$$

2. Meson self-energy

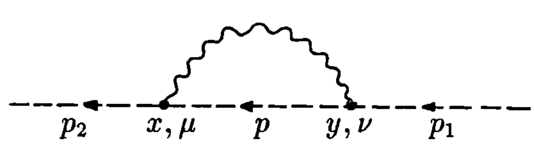


Figure 5.19

$$
\begin{aligned}
S_1' = & \ e^2\int dx\int dy\int \bar{d}p(p_2+p)^\mu(p+p_1)^\nu(p^2-m^2+i\epsilon)^{-1}G'_{\mu\nu}(x,y) \\
& \qquad e^{ix(p_2-p)+iy(p-p_1)} \\
= & \ e^2\int \frac{\bar{d}k}{-k^2}\int_1^\infty \frac{d\beta}{\beta}\int dy\int dp(p^2-m^2+i\epsilon)^{-1}[g_\nu(\beta,k,\partial_k,y)(p+p_1)^\nu(p^2-p_2^2)\cdot \\
& \ \delta(p_2-p+\beta k)e^{iy(p-p_1)}+(p\leftrightarrow -p,\ p_1\leftrightarrow -p_2)] \\
\overset{ms}{=} & \ 2e^2\int \frac{\bar{d}k}{-k^2}\int_1^\infty d\beta\int dy g_\nu(\beta,k,\partial_k,y)k^\nu e^{iy(p_2-p_1+\beta k)}.
\end{aligned} \tag{5.26}
$$

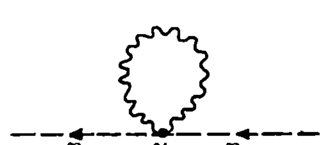


Figure 5.20

$$
\begin{aligned}
S_2' = & -e^2\int dy e^{iy(p_2-p_1)}g^{\mu\nu}G'_{\mu\nu}(y,y) \\
= & -2e^2\int \frac{\bar{d}k}{-k^2}\int_1^\infty d\beta\int dy g_\nu(\beta,k,\partial_k,y)k^\nu e^{iy(p_2-p_1+\beta k)}.
\end{aligned} \tag{5.27}
$$

We have

$$S' = S_1' + S_2' \overset{ms}{=} 0. \tag{5.28}$$

3. Vertex corrections

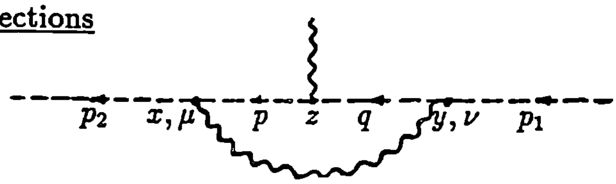


Figure 5.21

$$
\begin{aligned}
S_1' = \;& e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} (p_2+p)^\mu (p+q)^\sigma (q+p_1)^\nu A_\sigma(z) \\
& \qquad G'_{\mu\nu}(x,y) e^{ix(p_2-p)+iy(q-p_1)+iz(p-q)} \\
= \;& e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}y \int \bar{d}z \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} A_\sigma(z) g_\nu(\beta,k,\partial_k,y) \\
& [(p^2-p_2^2)(p+q)^\sigma (q+p_1)^\nu \delta(p_2-p+\beta k) e^{iy(q-p_1)+iz(p-q)} \\
& \qquad -(p \leftrightarrow -q,\;\; p_1 \leftrightarrow -p_2)] \\
\overset{ms}{=} \;& e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}x \int \bar{d}y \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} A_\sigma(z) g_\nu(\beta,k,\partial_k,x) \\
& \quad [(p+p_2+\beta k)^\sigma (p+p_1)^\nu e^{ix(p-p_1)+iy(p_2-p+\beta k)} \\
& \qquad -(p+p_1-\beta k)^\sigma (p+p_2)^\nu e^{ix(p_2-p)+iy(p-p_1+\beta k)}].
\end{aligned}
\tag{5.29}
$$

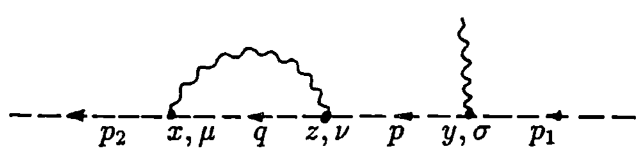


Figure 5.22

$$
\begin{aligned}
S_2' = \;& e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} (p_2+q)^\mu (q+p)^\nu (p+p_1)^\sigma A_\sigma(y) \\
& \qquad G'_{\mu\nu}(x,z) e^{ix(p_2-q)+iy(p-p_1)+iz(q-p)} \\
= \;& e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}y \int \bar{d}x \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} A_\sigma(y) (p+p_1)^\sigma e^{iy(p-p_1)} \\
& [g_\nu(\beta,k,\partial_k,x)(q^2-p_2^2)(p_2+p+\beta k)^\nu \delta(p_2-q+\beta k) e^{ix(p_2-p+\beta k)} + \\
& \qquad (q \leftrightarrow -q,\; p_2 \leftrightarrow -p)] \\
\overset{ms}{=} \;& e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}x \int \bar{d}y \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} A_\sigma(y) g_\mu(\beta,k,\partial_k,x) \\
& [2\beta k^\mu (p+p_1)^\sigma e^{ix(p_2-p+\beta k)+iy(p-p_1)} + \\
& \quad (p+p_2)^\mu (p+p_1+\beta k)^\sigma e^{ix(p_2-p)+iy(p-p_1+\beta k)}].
\end{aligned}
\tag{5.30}
$$

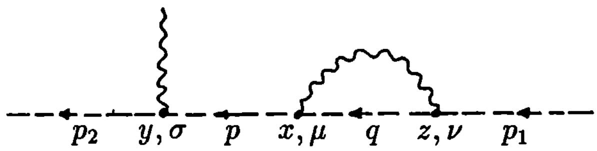


Figure 5.23

$$\begin{aligned}
S_3' = \;& e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} (p_2+p)^\sigma (p+q)^\mu (q+p_1)^\nu A_\sigma(y) \\
& \qquad G'_{\mu\nu}(x,z) e^{ix(p-q)+iy(p_2-p)+iz(q-p_1)} \\
\stackrel{ms}{=} \;& e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}x \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} A_\sigma(y) g_\mu(\beta,k,\partial_k,x) \\
& [2\beta k^\mu (p+p_2)^\sigma e^{ix(p-p_1-p+\beta k)+iy(p_2-p)} + \\
& \quad -(p+p_1)^\mu (p+p_2-\beta k)^\sigma e^{ix(p-p_1)+iy(p_2-p+\beta k)}].
\end{aligned} \tag{5.31}$$

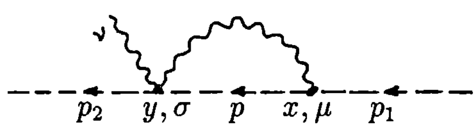


Figure 5.24

$$\begin{aligned}
S_4' = \;& -2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} (p+p_1)^\mu A^\nu(y) G'_{\mu\nu}(x,y) e^{ix(p-p_1)+iy(p_2-p)} \\
= \;& -2e^3 \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A^\nu(y) \\
& [g_\nu(\beta,k,\partial_k,y)(p_1^2-p^2)\delta(p-p_1+\beta k) e^{iy(p_2-p_1+\beta k)} + \\
& \quad \beta \int \bar{d}x g_\mu(\beta,k,\partial_k,x) k_\nu (p+p_1)^\mu e^{ix(p-p_1)+iy(p_2-p+\beta k)}] \\
\stackrel{ms}{=} \;& 2e^3 \int \bar{d}y \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A^\sigma(y) \Big\{ [g_\sigma(\beta,k,\partial_k,y) e^{iy(p_2-p_1+\beta k)} \\
& - \int \bar{d}x \int \frac{dp}{p^2-m^2+i\epsilon} g_\mu(\beta,k,\partial_k,x) \beta k_\sigma (p+p_1)^\mu e^{ix(p-p_1)+iy(p_2-p+\beta k)} \Big\}.
\end{aligned} \tag{5.32}$$

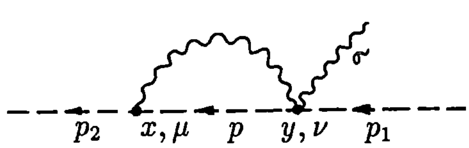


Figure 5.25

$$\begin{aligned}
S_5' = \;& -2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} (p+p_2)^\mu A^\nu(y) G'_{\mu\nu}(x,y) e^{ix(p_2-p)+iy(p-p_1)} \\
\stackrel{ms}{=} \;& 2e^3 \int \bar{d}y \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A^\sigma(y) \Big\{ [-g_\sigma(\beta,k,\partial_k,y) e^{iy(p_2-p_1+\beta k)} \\
& - \int \bar{d}x \int \frac{dp}{p^2-m^2+i\epsilon} g_\mu(\beta,k,\partial_k,x) \beta k_\sigma (p+p_2)^\mu e^{ix(p_2-p)+iy(p-p_1+\beta k)} \Big\}.
\end{aligned} \tag{5.33}$$

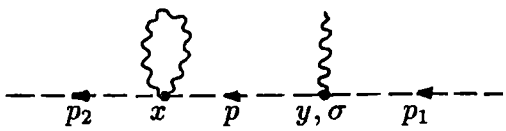


Figure 5.26

$$
\begin{aligned}
S_6' = & \; -e^3 \int dx \int dy \int \tfrac{\bar{d}p}{p^2-m^2+i\epsilon}(p+p_1)^\sigma A_\sigma(y) g^{\mu\nu} G'_{\mu\nu}(x,x) e^{ix(p_2-p)+iy(p-p_1)} \\
= & \; -2e^3 \int \bar{d}x \int \bar{d}y \int \tfrac{dp}{p^2-m^2+i\epsilon} \int \tfrac{dk}{-k^2} \int_1^\infty \tfrac{d\beta}{\beta} A_\sigma(y)(p+p_1)^\sigma g_\mu(\beta,k,\partial_k,x)\beta k^\mu \\
& \; e^{ix(p_2-p+\beta k)+iy(p-p_1)}.
\end{aligned} \tag{5.34}
$$

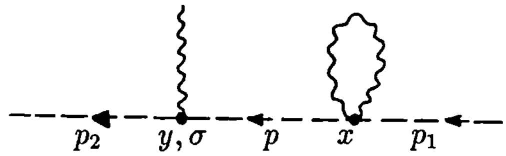


Figure 5.27

$$
\begin{aligned}
S_7' = & \; -e^3 \int dx \int dy \int \tfrac{\bar{d}p}{p^2-m^2+i\epsilon}(p+p_2)^\sigma A_\sigma(y) g^{\mu\nu} G'_{\mu\nu}(x,x) e^{ix(p-p_1)+iy(p_2-p)} \\
= & \; -2e^3 \int \bar{d}x \int \bar{d}y \int \tfrac{dp}{p^2-m^2+i\epsilon} \int \tfrac{dk}{-k^2} \int_1^\infty \tfrac{d\beta}{\beta} A_\sigma(y)(p+p_2)^\sigma g_\mu(\beta,k,\partial_k,x)\beta k^\mu \\
& \; e^{ix(p-p_1+\beta k)+iy(p_2-p)}.
\end{aligned} \tag{5.35}
$$

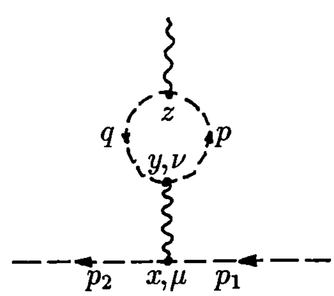


Figure 5.28

$$
\begin{aligned}
S_8' = & \; e^3 \int dx \int dy \int dz \int \tfrac{\bar{d}p}{p^2-m^2+i\epsilon} \int \tfrac{\bar{d}q}{q^2-m^2+i\epsilon}(p_1+p_2)^\mu (q+p)^\nu (p+q)^\sigma A_\sigma(z) \\
& \; G'_{\mu\nu}(x,y) e^{ix(p_2-p_1)+iy(p-q)+iz(q-p)} \\
= & \; e^3 \int \bar{d}y \int \bar{d}z \int \tfrac{dp}{p^2-m^2+i\epsilon} \int \tfrac{dq}{q^2-m^2+i\epsilon} \int \tfrac{dk}{-k^2} \int_1^\infty \tfrac{d\beta}{\beta} A_\sigma(z)(p+q)^\sigma e^{iz(q-p)} \\
& \; g_\nu(\beta,k,\partial_k,y)\{(p_1^2-p_2^2)(q+p)^\nu \delta(p_2-p_1+\beta k) e^{iy(p-q)} + \\
& \; [(q^2-m^2)-(p^2-m^2)](p_1+p_2)^\nu \delta(p-q+\beta k) e^{iy(p_2-p_1)}\} \\
\overset{ms}{=} & \; e^3 \int \bar{d}y \int \bar{d}z \int \tfrac{dp}{p^2-m^2+i\epsilon} \int \tfrac{dk}{-k^2} \int_1^\infty \tfrac{d\beta}{\beta} A_\sigma(z) g_\nu(\beta,k,\partial_k,y)(p_1+p_2)^\nu e^{iy(p_2-p_1)} \\
& \; [(2p+\beta k)^\sigma e^{i\beta kz} - (2p-\beta k)^\sigma e^{-i\beta kz}] \\
= & \; 0.
\end{aligned} \tag{5.36}
$$

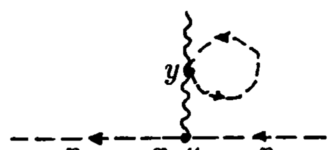


Figure 5.29

$$
\begin{aligned}
S_9' = {} & 2e^3 \int dx \int dy \int \tfrac{\bar{d}p}{p^2-m^2+i\epsilon}(p_1+p_2)^\mu A^\nu(y) G'_{\mu\nu}(x,y) e^{ix(p_2-p_1)} \\
= {} & 2e^3 \int dx \int dy \int \tfrac{\bar{d}p}{p^2-m^2+i\epsilon}(p_1^2-p_2^2) A^\nu(y) f_{1\nu}(x,y) \\
\overset{ms}{=} {} & 0.
\end{aligned}
\tag{5.37}
$$

Adding S_1, S_2,$\cdots$ S_9 we obtain

$$
S' = S_1' + \cdots + S_9' \overset{ms}{=} 0. \tag{5.38}
$$

5.3.2 Calculations in Coordinate Space

Truncated Diagrams

As in spinor electrodynamics, directed derivatives do not act on $f_{1\mu}(x,y)$ and $f_{2\mu}(x,y)$. Surface terms will be omitted and indices μ, ν and σ in derivatives are attached to variables x, y and z respectively. Thus $\partial_\mu = \partial_{x\mu}$, $\partial_\nu = \partial_{y\nu}$ and $\partial_\sigma = \partial_{z\sigma}$.

1. <u>Meson self-energy</u>

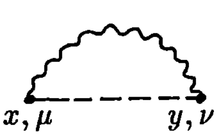


Figure 5.30

$$
\begin{aligned}
-e^{-2}\Sigma_1'(x,y) = {} & [\cdot \overleftrightarrow{\partial}^\mu S_B(x-y) \overleftrightarrow{\partial}^\nu \cdot] G'_{\mu\nu}(x,y) \\
= {} & -\{\cdot[\overrightarrow{\Box}_x - \overleftarrow{\Box}_x] S_B(x-y) \overleftrightarrow{\partial}^\nu \cdot\} f_{1\nu}(x,y) \\
& -\{\cdot \overleftrightarrow{\partial}^\mu S_B(x-y)[\overrightarrow{\Box}_y - \overleftarrow{\Box}_y]\cdot\} f_{2\mu}(x,y).
\end{aligned}
\tag{5.39}
$$

Figure 5.31

$$
\begin{aligned}
-e^{-2}\Sigma_2'(x,y) = {} & g^{\mu\nu}[\cdot\delta(x-y)\cdot] G'_{\mu\nu}(x,y) \\
= {} & [\cdot \overleftrightarrow{\partial}^\mu \delta(x-y)\cdot] f_{2\mu}(x,y) - [\cdot\delta(x-y) \overleftrightarrow{\partial}^\nu \cdot] f_{1\nu}(x,y).
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\Sigma'(x,y) = {} & \Sigma'_1(x,y) + \Sigma'_2(x,y) \\
= {} & e^2[\cdot(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x)S_B(x-y)\overleftrightarrow{\partial}^\nu \cdot]f_{1\nu}(x,y) + \\
& e^2[\cdot \overleftrightarrow{\partial}^\mu S_B(x-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\cdot]f_{2\mu}(x,y) + \\
& [\cdot\delta(x-y)\overleftrightarrow{\partial}^\nu \cdot]f_{1\nu}(x,y) - [\cdot \overleftrightarrow{\partial}^\mu \delta(x-y)\cdot]f_{2\mu}(x,y).
\end{aligned}
\tag{5.41}
$$

2. Vertex corrections

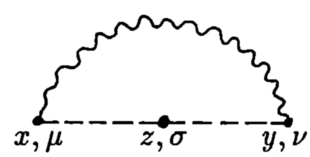


Figure 5.32

$$
\begin{aligned}
ie^{-3}\Gamma_1'^{\sigma}(x,y,z) = {} & [\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]G'_{\mu\nu}(x,y) \\
= {} & -[\cdot(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x)S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]f_{1\nu}(x,y) \\
& -[\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\cdot]f_{2\mu}(x,y).
\end{aligned}
\tag{5.42}
$$

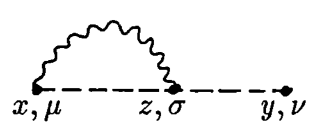


Figure 5.33

$$
\begin{aligned}
ie^{-3}\Gamma_2'^{\nu}(x,y,z) = {} & [\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]G'_{\mu\sigma}(x,z) \\
= {} & -[\cdot(\overrightarrow{\Box}_y - \overleftarrow{\Box}_x)S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]f_{1\sigma}(x,z) \\
& +[\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)(\overrightarrow{\Box}_z - \overleftarrow{\Box}_z)S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]f_{2\mu}(x,z).
\end{aligned}
\tag{5.43}
$$

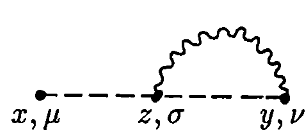


Figure 5.34

$$
\begin{aligned}
ie^{-3}\Gamma_3'^{\mu}(x,y,z) = {} & [\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]G'_{\sigma\nu}(z,y) \\
= {} & -[\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)\overleftrightarrow{\partial}^\sigma S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\cdot]f_{2\sigma}(z,y) \\
& -[\cdot \overleftrightarrow{\partial}^\mu S_B(x-z)(\overrightarrow{\Box}_z - \overleftarrow{\Box}_z)S_B(z-y)\overleftrightarrow{\partial}^\nu \cdot]f_{1\nu}(z,y).
\end{aligned}
\tag{5.44}
$$

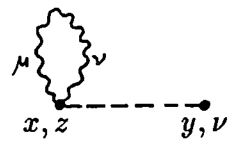


Figure 5.35

$$
\begin{aligned}
ie^{-3}\Gamma_4'^{\nu}(x,y,z) = &\ g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]G'_{\mu\sigma}(x,z) \\
= &\ [\cdot(\overleftrightarrow{\partial^\mu}\delta(x-z))S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{2\mu}(x,z) \\
&-[\cdot\delta(x-z)\overleftrightarrow{\partial^\sigma}S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\sigma}(x,z).
\end{aligned} \tag{5.45}
$$

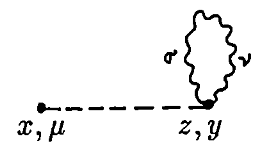


Figure 5.36

$$
\begin{aligned}
ie^{-3}\Gamma_5'^{\mu}(x,y,z) = &\ g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}S_B(x-z)\delta(z-y)\cdot]G'_{\sigma\nu}(z,y) \\
= &\ -[\cdot(\overleftrightarrow{\partial^\mu}S_B(x-z))\delta(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\nu}(z,y) \\
&+[\cdot\overleftrightarrow{\partial^\mu}S_B(x-z)\overleftrightarrow{\partial^\sigma}\delta(z-y)\cdot]f_{2\sigma}(z,y).
\end{aligned} \tag{5.46}
$$

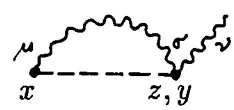


Figure 5.37

$$
\begin{aligned}
ie^{-3}\Gamma_6'^{\nu}(x,y,z) = &\ 2g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}S_B(x-z)\delta(z-y)\cdot]G'_{\mu\sigma}(x,z) \\
= &\ -2g^{\nu\sigma}\{\cdot[\overrightarrow{\Box}_x-\overleftarrow{\Box}_x]S_B(x-z)\delta(z-y)\cdot\}f_{1\sigma}(x,z) \\
&-2g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}S_B(x-z)(\overleftarrow{\partial^\sigma}+\overrightarrow{\partial^\sigma})\delta(z-y)\cdot]f_{2\mu}(x,z).
\end{aligned} \tag{5.47}
$$

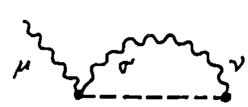


Figure 5.38

$$
\begin{aligned}
ie^{-3}\Gamma_7'^{\mu}(x,y,z) = &\ 2g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]G'_{\sigma\nu}(z,y) \\
= &\ -2g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)(\overrightarrow{\Box}_y-\overleftarrow{\Box}_y)\cdot]f_{2\sigma}(z,y) \\
&-2g^{\mu\sigma}[\cdot\delta(x-z)(\overleftarrow{\partial^\sigma}+\overrightarrow{\partial^\sigma})S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\nu}(z,y).
\end{aligned} \tag{5.48}
$$

Figure 5.39

$$
\begin{aligned}
ie^{-3}\Gamma_8'^{\sigma}(x,y,z) = & \ [\cdot \overleftrightarrow{\partial^{\mu}} \cdot]G'_{\mu\nu}(x,y)[(\partial^{\nu}S_B(y-z)) \overleftrightarrow{\partial^{\sigma}} S_B(z-y) - \\
& \qquad S_B(y-z)) \overleftrightarrow{\partial^{\sigma}} \partial^{\nu}S_B(z-y)] \\
= & \ [\cdot(\overleftarrow{\Box}_x - \overrightarrow{\Box}_x)\cdot]f_{1\nu}(x,y)\times \\
& [(\partial^{\nu}S_B(y-z)) \overleftrightarrow{\partial^{\sigma}} S_B(z-y) - S_B(y-z)) \overleftrightarrow{\partial^{\sigma}} \partial^{\nu}S_B(z-y)].
\end{aligned}
\tag{5.49}
$$

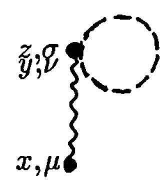


Figure 5.40

$$
\begin{aligned}
ie^{-3}\Gamma_9'^{\sigma}(x,y,z) = & \ 2[\cdot \overleftrightarrow{\partial^{\mu}} \cdot]G'_{\mu\nu}(x,y)g^{\nu\sigma}S_B(y-z)\delta(y-z) \\
= & \ 2[\cdot \overleftrightarrow{\partial^{\mu}} \cdot]S_B(y-z)\delta(y-z)g^{\nu\sigma}[\partial_{\mu}f_{1\nu}(x,y) + \partial_{\nu}f_{2\mu}(x,y)] \\
= & \ 2[\cdot(\overleftarrow{\Box}_x - \overrightarrow{\Box}_x)\cdot]S_B(y-z)\delta(y-z)g^{\nu\sigma}f_{1\nu}(x,y) + \\
& \qquad [\cdot \overleftrightarrow{\partial^{\mu}} \cdot]S_B(y-z)\delta(y-z)g^{\nu\sigma}\partial_{\nu}f_{2\mu}(x,y).
\end{aligned}
\tag{5.50}
$$

On-shell Diagrams

Examining the truncated diagrams it is obvious that each of the corresponding completed diagrams consists of terms containing factors like

$$\phi^{\dagger}(x)[\overleftarrow{\Box}_x - \overrightarrow{\Box}_x]S_B(x-y) \quad \text{or} \quad S_B(x-z)[\overleftarrow{\Box}_z - \overrightarrow{\Box}_z]S_B(z-y).$$

Difficulties in simplifications off the mass-shell are caused by those factors since they cannot be brought into simpler forms. On the mass-shell, on the other hand, those factors may be simplified into Dirac delta functions. As a result cancellations may and do take place in this case. A summary of the computations will be given below.

1. Meson-meson scattering (Born term)

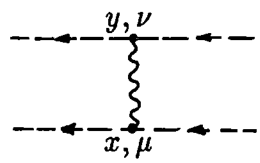


Figure 5.41

$$
\begin{aligned}
S' = &\ ie^2 \int dx \int dy \phi^\dagger(x) \overleftrightarrow{\partial^\mu} \phi(x)\phi^\dagger(y) \overleftrightarrow{\partial^\nu} \phi(y) G'_{\mu\nu}(x,y) \\
= &\ ie^2 \int dx \int dy \phi^\dagger(x) \overleftrightarrow{\partial^\mu} \phi(x)\phi^\dagger(y) \overleftrightarrow{\partial^\nu} \phi(y)[\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] \\
= &\ -2ie^2 \int dx \int dy \{\phi^\dagger(x)\partial^\mu\phi(x)\phi^\dagger(y)[(\overrightarrow{\Box}_y + m^2) - (\overleftarrow{\Box}_y + m^2)]\phi(y)\} f_{2\mu}(x,y) \\
\overset{ms}{=} &\ 0.
\end{aligned}
\tag{5.51}
$$

2. Meson self-energy

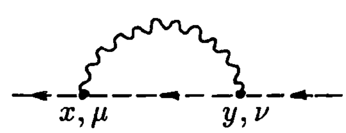


Figure 5.42

$$
\begin{aligned}
S'_1 = &\ \int dx \int dy \phi^\dagger(x)\Sigma'_1(x,y)\phi(y) \\
\overset{ms}{=} &\ -e^2 \int dx \int dy \{-\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-y)\phi(y) f_{2\mu}(x,y) \\
&\ +\phi^\dagger(x)\delta(x-y) \overleftrightarrow{\partial^\nu} \phi(y) f_{1\nu}(x,y)\}.
\end{aligned}
\tag{5.52}
$$

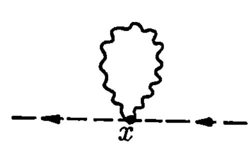


Figure 5.43

$$
\begin{aligned}
S'_2 = &\ \int dx \int dy \phi^\dagger(x)\Sigma'_2(x,y)\phi(y) \\
= &\ -e^2 \int dx \int dy \{\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-y)\phi(y) f_{2\mu}(x,y) \\
&\ -\phi^\dagger(x)\delta(x-y) \overleftrightarrow{\partial^\nu} \phi(y) f_{1\nu}(x,y)\}.
\end{aligned}
\tag{5.53}
$$

$$S' = S_1' + S_2' \overset{ms}{=} 0. \tag{5.54}$$

3. Vertex corrections

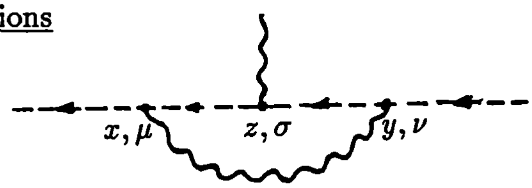


Figure 5.44

$$\begin{aligned} S_1' = & \int dx \int dy \int dz \phi^\dagger(x) \Gamma_1'^{\sigma}(x,y,z)\phi(y)A_\sigma(z) \\ \overset{ms}{=} & -ie^3 \int dx \int dy \int dz A_\sigma(z)\{[\phi^\dagger(x)\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(x,y) \\ & -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)]f_{2\mu}(x,y)\}. \end{aligned} \tag{5.55}$$

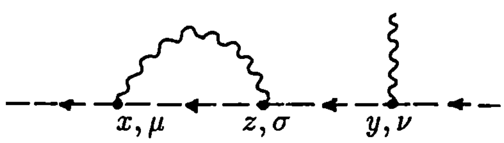


Figure 5.45

$$\begin{aligned} S_2' = & \int dx \int dy \int dz \phi^\dagger(x) \Gamma_2'^{\nu}(x,y,z)\phi(y)A_\nu(y) \\ \overset{ms}{=} & -ie^3 \int dx \int dy \int dz A_\nu(y)\{[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{2\mu}(x,z) \\ & -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z)S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{2\mu}(x,z) \\ & +[\phi^\dagger\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\sigma}(x,z)\}. \end{aligned} \tag{5.56}$$

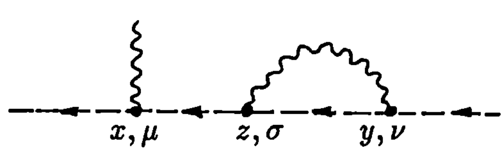


Figure 5.46

$$\begin{aligned} S_3' = & \int dx \int dy \int dz \phi^\dagger(x) \Gamma_3'^{\mu}(x,y,z)\phi(y)A_\mu(x) \\ \overset{ms}{=} & -ie^3 \int dx \int dy \int dz A_\mu(x)\{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z)S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y) \\ & -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)]f_{2\sigma}(z,y) \\ & +[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y)\}. \end{aligned} \tag{5.57}$$

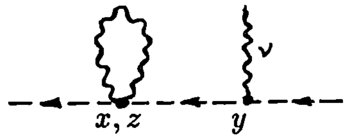


Figure 5.47

$$
\begin{aligned}
S_4' = & \int dx \int dy \int dz \phi^\dagger(x)\Gamma_4'^{\nu}(x,y,z)\phi(y)A_\nu(y) \\
= & -ie^3 \int dx \int dy \int dz A_\nu(y)\{[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z))S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{2\mu}(x,z) \\
& -[\phi^\dagger(x)\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\sigma}(x,z)\}.
\end{aligned} \tag{5.58}
$$

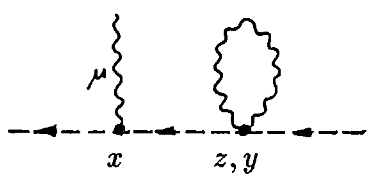


Figure 5.48

$$
\begin{aligned}
S_5' = & \int dx \int dy \int dz \phi^\dagger(x)\Gamma_5'^{\mu}(x,y,z)\phi(y)A_\mu(x) \\
= & -ie^3 \int dx \int dy \int dz A_\mu(x)\{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z))\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y) \\
& +[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)]f_{2\sigma}(z,y)\}.
\end{aligned} \tag{5.59}
$$

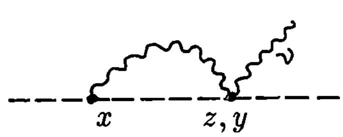


Figure 5.49

$$
\begin{aligned}
S_6' = & \int dx \int dy \int dz \phi^\dagger(x)\Gamma_6'^{\nu}(x,y,z)\phi(y)A_\nu(y) \\
\overset{ms}{=} & -ie^3 \int dx \int dy \int dz A_\nu(y)\{2g^{\nu\sigma}\phi^\dagger(x)\delta(x-z)\delta(z-y)\phi(y)f_{1\sigma}(x,z)+ \\
& -2g^{\nu\sigma}[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})\delta(z-y)\phi(y)]f_{2\mu}(x,z)\}.
\end{aligned} \tag{5.60}
$$

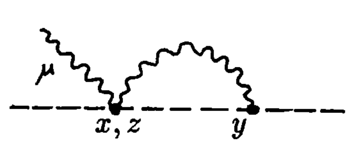


Figure 5.50

$$
\begin{aligned}
S_7' = & \int dx \int dy \int dz \phi^\dagger(x)\Gamma_7'^{\mu}(x,y,z)\phi(y)A_\mu(x) \\
\overset{ms}{=} & -ie^3 \int dx \int dy \int dz A_\mu(x)\{-2g^{\mu\sigma}\phi^\dagger(x)\delta(x-z)\delta(z-y)\phi(y)f_{2\sigma}(z,y) \\
& -2g^{\mu\sigma}[\phi^\dagger(x)\delta(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)f_{1\nu}(z,y)\}.
\end{aligned} \tag{5.61}
$$

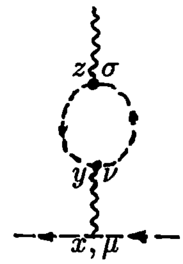


Figure 5.51

$$
\begin{aligned}
S'_8 = & \int dx \int dy \int dz \phi^\dagger(x) \Gamma'^{\sigma}_8(x,y,z) \phi(y) A_\sigma(z) \\
\stackrel{ms}{=} & \ 0.
\end{aligned}
\tag{5.62}
$$

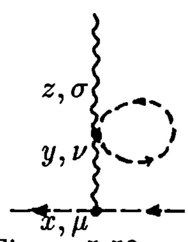


Figure 5.52

$$
\begin{aligned}
S'_9 = & \int dx \int dy \int dz \phi^\dagger(x) \Gamma'^{\sigma}_9(x,y,z) \phi(y) A_\sigma(z) \\
\stackrel{ms}{=} & \ 0.
\end{aligned}
\tag{5.63}
$$

The first order of vertex corrections is then given by

$$
\begin{aligned}
S' = & -ie^3 \int dx \int dy \int dz \phi^\dagger(x) \phi(y) \\
& \quad [\Gamma'^{\sigma}_1(x,y,z) A_\sigma(z) + \Gamma'^{\nu}_2(x,y,z) A_\nu(y) + \Gamma'^{\mu}_3(x,y,z) A_\mu(x) \\
& \quad + \Gamma'^{\nu}_4(x,y,z) A_\nu(y) + \Gamma'^{\mu}_5(x,y,z) A_\mu(x) + \Gamma'^{\nu}_6(x,y,z) A_\nu(y) \\
& \quad + \Gamma'^{\mu}_7(x,y,z) A_\mu(x) + \Gamma'^{\sigma}_8(x,y,z) A_\sigma(z) + \Gamma'^{\sigma}_9(x,y,z) A_\sigma(z)] \\
= & \ S'_1 + S'_2 + S'_3 + S'_4 + S'_5 + S'_6 + S'_7 + S'_8 + S'_9 \\
\stackrel{ms}{=} & -ie^3 \int dx \int dy \int dz \{ -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y) \phi(y)] f_{2\mu}(x,y) A_\sigma(z) \\
& \quad + [\phi^\dagger(x) \delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(x,y) A_\sigma(z) \\
& \quad + [\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z) A_\nu(y) \\
& \quad - [\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z) S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(z,y) A_\mu(x) \\
& \quad - 2[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) (\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma}) \delta(z-y) \phi(y)] f_{2\mu}(x,z) A_\sigma(y) \\
& \quad - 2[\phi^\dagger(x) \delta(x-z) (\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma}) S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(z,y) A_\sigma(x) \} \stackrel{ms}{=} 0.
\end{aligned}
\tag{5.64}
$$

5.4 One-loop Corrections in Quantum Chromodynamics

The equivalence between the electron-photon vertex in quantum electrodynamics and the quark-gluon vertex in quantum chromodynamics can be seen easily from relationship

$$-e \leftrightarrow g(T^a)_{ij}. \tag{5.65}$$

This equivalence leads one to infer that, neglecting the group factors, the quark-quark scattering and the quark self-energy diagrams are equivalent to the electron-electron scattering and the electron self-energy diagrams. Hence we can conclude that, like in quantum electrodynamics, the correction $G'^{ab\mu\nu}(x,y)$ gives no contribution, on the mass-shell, to the quark-quark scattering and the quark self-energy.

However the same conclusion does *not* apply to the sum of the four diagrams in the first order of the vertex corrections

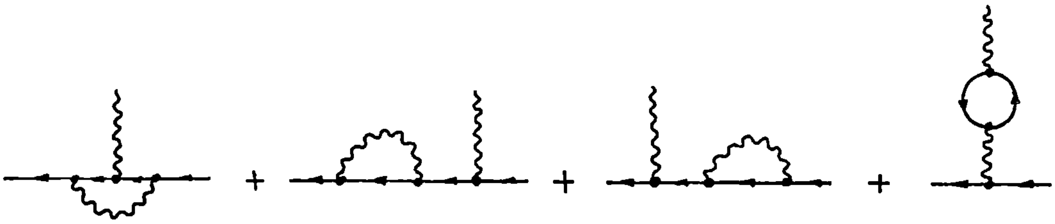


Figure 5.53

This is because the group factors for each diagram are different due to the noncommutativity of the group generators. However, these are not all of the diagrams in quantum chromodynamics. To first order the vertex correction require three other diagrams contributed by the three-gluon and four-gluon vertices.

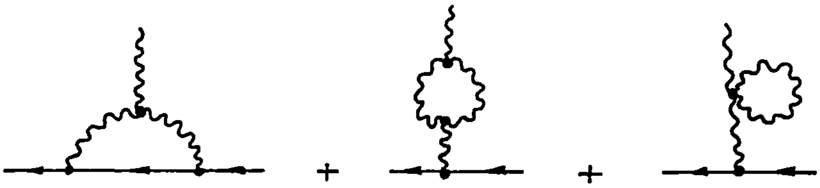


Figure 5.54

Therefore we have to work out these seven diagrams combined and see whether the same conclusion of the previous calculations will also apply. These corrections to the vertex and the gluon self-energy are the last calculations of the thesis.

First order corrections to the three-gluon and four-gluon vertices (see figures 5.55 and 5.56) have not been carried out in this thesis because the calculation is even more formidable and needs considerably more effort. The complication arises from diagrams that contain three and four gluon propagators. But we are hopeful that the same conclusion of on-mass-shell equivalence also holds here.

Three-gluon vertices

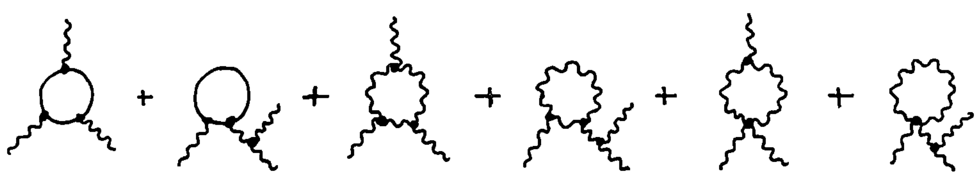


Figure 5.55

Four-gluon vertices

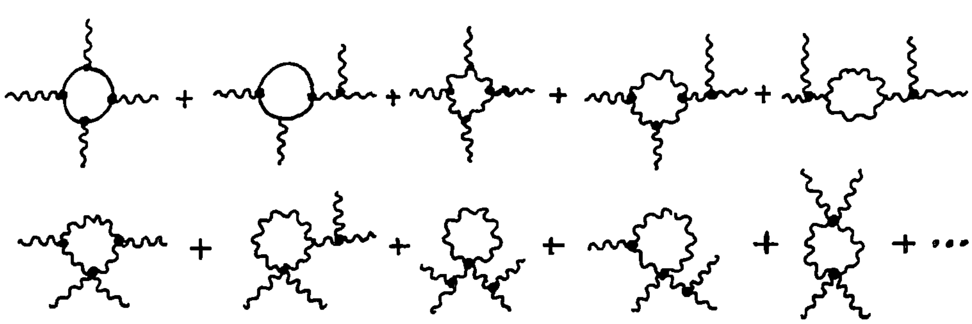


Figure 5.56

Before we go into detailed computations for each diagram considered let us take notice of the Feynman rules in quantum chromodynamics. Firstly consider the ghost vertex. $G_\mu^{(b)}$ in the ghost vertex equals $\partial_\mu^{(b)}$ and x_μ in the Lorentz gauge and the FS gauge respectively. This vertex vanishes in the FS gauge because $A^{a\mu}G_\mu = 0$ and therefore all diagrams containing ghost fields/propagators can be disregarded in scattering calculations involving external vectors. The second consideration is about the three-gluon vertex. In the case when all three wavy lines in the three-gluon vertex represent external gauge fields, the corresponding $igf^{abc}V^{(a,b,c)}_{\mu\nu\rho}(\partial_x)$ in momentum space may be readily obtained and is just

$$igf^{abc}V^{(a,b,c)}_{\mu\nu\rho}(k) = -gf^{abc}[g_{\mu\nu}(k_1 - k_2)_\rho + g_{\nu\rho}(k_2 - k_3)_\mu + g_{\rho\mu}(k_3 - k_1)_\nu]. \quad (5.66)$$

If one (or more) of the wavy lines is replaced by the FS gauge propagator, $V^{(a,b,c)}_{\mu\nu\rho}(k)$ will necessarily be more complicated. This is because unlike propagators in the Lorentz gauge that can be formulated as

$$G^{ab\mu\nu}(x,y) = \int dk f^{ab\mu\nu}(k)\exp[-ik(x-y)] \quad (5.67)$$

the general form of the FS gauge propagator is more complex,

$$G^{ab\mu\nu}(x,y) = \int\int\int d\alpha d\beta dk f^{ab\mu\nu}(\alpha,\beta,x,y,k)\exp[-i\alpha kx + i\beta ky]. \quad (5.68)$$

The amplitude $f^{ab\mu\nu}$ is still a function of space-time coordinates and thus $V^{(a,b,c)}_{\mu\nu\rho}(k)$ is a function of k, ∂_k as well as integrations over α and β. Accordingly, complications of $V^{(a,b,c)}_{\mu\nu\rho}(k)$ in the FS gauge lead to difficulties in evaluating perturbative calculations in momentum space. We avoid such difficulties by only taking perturbative calculations in quantum chromodynamics in coordinate space into account (where the gauge condition looks more natural). We start with truncated diagrams before extending them to mass-shell diagrams.

5.4.1 Truncated Diagrams

The first four diagrams can be obtained by the use of the previous results in quantum electrodynamics. We have

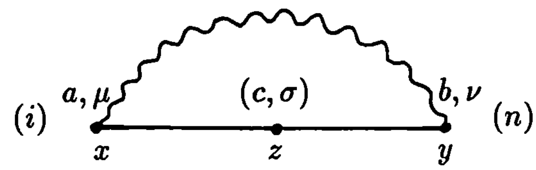


Figure 5.57

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{1in}(x,y,z) = {} & [\cdot\gamma^\mu (T^a)_{ij} S_{Qjk}(x-z)\gamma^\sigma (T^c)_{kl} S_{Qlm}(z-y)\gamma^\nu (T^b)_{mn}\cdot]\times \\
& \qquad G^{ab}_{\mu\nu}(x,y) \\
= {} & (F) + (T^aT^bT^c + if^{cbd}T^aT^d)_{in} \\
& \{-[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x) S_F(x-z)\gamma^\sigma S_F(z-y)\ \not f^{ab}_1(x,y)\cdot] \\
& -[\cdot\ \not f^{ab}_2(x,y) S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\cdot]\}.
\end{aligned}
\tag{5.69}
$$

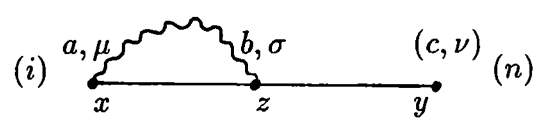


Figure 5.58

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\nu}_{2in}(x,y,z) = {} & [\cdot\gamma^\mu (T^a)_{ij} S_{Qjk}(x-z)\gamma^\sigma (T^b)_{kl} S_{Qlm}(z-y)\gamma^\nu (T^c)_{mn}\cdot]\times \\
& \qquad G^{ab}_{\mu\sigma}(x,z) \\
= {} & (F) + (T^aT^bT^c)_{in} \\
& \{-[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x) S_F(x-z)\ \not f^{ab}_1(x,z) S_F(z-y)\gamma^\nu\cdot] \\
& -i[\cdot\ \not f^{ab}_2(x,z)[\delta(x-z) - \delta(y-z)] S_F(x-y)\gamma^\nu\cdot]\}.
\end{aligned}
\tag{5.70}
$$

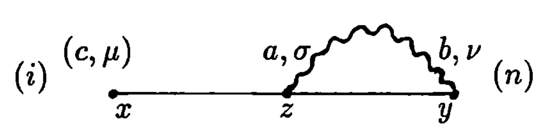


Figure 5.59

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\mu}_{3in}(x,y,z) = {} & [\cdot\gamma^\mu (T^c)_{ij} S_{Qjk}(x-z)\gamma^\sigma (T^a)_{kl} S_{Qlm}(z-y)\gamma^\nu (T^b)_{mn}\cdot]\times \\
& \qquad G^{ab}_{\sigma\nu}(z,y) \\
= {} & (F) + (T^aT^bT^c)_{in} \\
& \{-i[\cdot\gamma^\mu[\delta(x-z) - \delta(z-y)] S_F(x-y)\ \not f^{ab}_1(z,y)\cdot] \\
& -[\cdot\gamma^\mu S_F(x-z)\ \not f^{ab}_2(z,y) S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\cdot]\}.
\end{aligned}
\tag{5.71}
$$

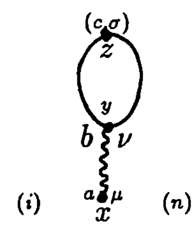


Figure 5.60

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{4in}(x,y,z) = \;& [\cdot\gamma^{\mu}\cdot](T^a)_{in}G^{ab}_{\mu\nu}(x,y) \\
& \quad \mathrm{Tr}[S_{Qjk}(z-y)\gamma^{\nu}(T^b)_{kl}S_{Qlm}(y-z)\gamma^{\sigma}(T^c)_{mj}] \\
= \;& (F) - \tfrac{1}{2}\delta^{bc}(T^a)_{in}[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)\cdot]\times \\
& \quad \mathrm{Tr}S_F(z-y)\ \not f^{ab}_1(x,y)S_F(y-z)\gamma^{\sigma}] \\
& -\tfrac{1}{2}(T^a)_{in}[\cdot\ \not f^{ac}_2(x,y)\cdot]\,TrS_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)S_F(y-z)\gamma^{\sigma}
\end{aligned} \tag{5.72}
$$

The terms (F) in each above diagrams stand for the Feynman gauge version of the corresponding diagrams. Now we turn to diagrams contributed by the three-gluon vertex. We just transcribe the final result given in the Appendix G here. The first diagram is

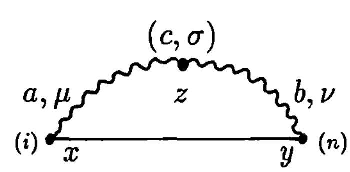


Figure 5.61

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{5in}(x,y,z) = \;& f^{ced}V^{(c,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^{\mu}(T^a)_{ij}S_{Qjl}(x-y)\gamma^{\nu}\cdot](T^b)_{ln} \\
& \quad G^{ae}_{\mu\alpha}(x,z)G^{db}_{\beta\nu}(z,y) \\
= \;& (F) + f^{ced}(T^aT^b)_{in}O^{(d+e)\sigma\beta}_{z}\times \\
& \quad \{G^{db}_{F\beta\nu}(z,y)[\cdot\ \not f^{ae}_2(x,z)S_F(x-y)\gamma^{\nu}\cdot] \\
& \quad -[\cdot\gamma^{\mu}S_F(x-y)\ \not f^{db}_1(z,y)\cdot]G^{ae}_{\mu\beta}(x,z)\}
\end{aligned} \tag{5.73}
$$

$$
\begin{aligned}
&-f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\times \\
&\qquad \{f^{ae}_{1\alpha}(x,z)G^{db}_{F\beta\nu}(z,y)[\cdot(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)S_F(x-y)\gamma^\nu\cdot] \\
&\qquad +f^{db}_{2\beta}(z,y)G^{ae}_{\mu\alpha}(x,z)[\cdot\gamma^\mu S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\cdot]\} \\
&-f^{ced}(T^aT^b)_{in}O^{(d)\sigma\beta}_z f^{db}_{2\beta}(z,y)\times \\
&\qquad [\cdot\ \not f^{ae}_2(x,z)S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\cdot] \\
&+f^{ced}(T^aT^b)_{in}[\delta^\sigma_\mu - z^\sigma(\partial z)^{-1}\partial^{(e)}_{z\mu}]\times \\
&\qquad \{[\cdot\ \not f^{ad}_2(x,z)S_F(x-y)\gamma^\mu\cdot]\delta^{eb}(z-y) \\
&\qquad +[\cdot\gamma^\mu S_F(x-y)\ \not f^{db}_1(z,y)\cdot]\delta^{ae}(z-x)\}.
\end{aligned}
\tag{5.74}
$$

The last diagram contributed by the three-gluon vertex is

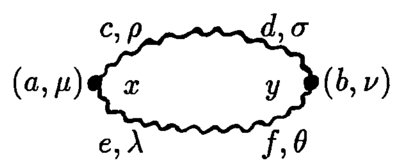


Figure 5.62

$$
\begin{aligned}
\Pi^{ab\mu\nu}_1(x,y) =\ & \tfrac{1}{2}g^2 f^{ace}V^{(a,c,e)\mu\rho\lambda}(\partial_x)f^{bfd}V^{(b,f,d)\nu\theta\sigma}(\partial_y)G^{cd}_{\rho\sigma}(x,y)G^{ef}_{\lambda\theta}(x,y) \\
=\ & (F)+(ghost) \\
& +\tfrac{1}{2}g^2 f^{ace}f^{bfd}\Big\{O^{\nu\sigma}_y\Big[V^{(-,c,e)\mu\rho\lambda}(\partial_x)\Big(G^{cd}_{\rho\lambda}(x,y)f^{ef}_{2\lambda}(x,y) \\
&\qquad -G^{ef}_{F\lambda\sigma}(x,y)f^{cd}_{2\rho}(x,y)\Big)+O^{(e)\mu\lambda}_x f^{cd}_{1\sigma}(x,y)f^{ef}_{2\lambda}(x,y)\Big] \\
&\quad +O^{\mu\rho}_x\Big[-V^{(-,f,d)\nu\theta\sigma}(\partial_y)\Big(G^{cd}_{\rho\sigma}(x,y)f^{ef}_{1\theta}(x,y) \\
&\qquad -G^{ef}_{F\rho\theta}(x,y)f^{cd}_{1\sigma}(x,y)\Big)+O^{(f)\nu\theta}_y f^{cd}_{2\rho}(x,y)f^{ef}_{1\theta}(x,y)\Big]+ \\
&\quad +y^\nu\Big[V^{(-,c,e)\mu\rho\lambda}(\partial_x)\Big(f^{ef}_{2\lambda}(x,y)(\partial y)^{-1}\partial^{(d)}_{y\rho}\delta^{cd}(x-y) \\
&\qquad\qquad -f^{cd}_{2\rho}(x,y)(\partial y)^{-1}\partial^{(f)}_{y\lambda}\delta^{ef}(x-y)\Big) \\
&\qquad +2((\partial y)^{-1}\delta^{ef}(x-y))\Box^{-1}_x\partial^\mu_x\delta^{cd}(x-y) \\
&\qquad -x^\mu((\partial x)^{-1}\delta^{cd}(x-y))(\partial y)^{-1}\delta^{ef}(x-y)\Big] \\
&\quad +x^\mu\Big[V^{(-,f,d)\nu\theta\sigma}(\partial_y)\Big(f^{cd}_{1\sigma}(x,y)(\partial x)^{-1}\partial^{(e)}_{x\theta}\delta^{ef}(x-y) \\
&\qquad\qquad -f^{ef}_{1\theta}(x,y)(\partial x)^{-1}\partial^{(c)}_{x\sigma}\delta^{cd}(x-y)\Big) \\
&\qquad +2((\partial x)^{-1}\delta^{cd}(x-y))\Box^{-1}_y\partial^\nu_y\delta^{ef}(x-y) \\
&\qquad -y^\nu((\partial x)^{-1}\delta^{cd}(x-y))(\partial y)^{-1}\delta^{ef}(x-y)\Big] \\
&\quad +2\ [2g^{\mu\nu}\partial^{(f)\lambda}_y - g^{\nu\lambda}\partial^{(f)\mu}_y - g^{\lambda\mu}\partial^{(f)\nu}_y]f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y)\Big\}.
\end{aligned}
\tag{5.75}
$$

Here $(ghost)$ in $\Pi_1^{ab\mu\nu}(x,y)$ equals

$$
\begin{aligned}
(ghost) = \;& g^2 f^{ace} f^{bdf} [\Box_x^{-1} \partial_x^\mu \delta^{cd}(x-y)][\Box_y^{-1} \partial_y^\nu \delta^{ef}(x-y) \\
= \;&
\end{aligned}
\tag{5.76}
$$

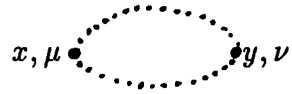


Figure 5.63

It is quite interesting that $\Pi_1^{ab\mu\nu}(x,y)$ in the FS ghost-free gauge contains implicitly the ghost term associated with the Feynman gauge.

The contribution of the four-gluon vertex is

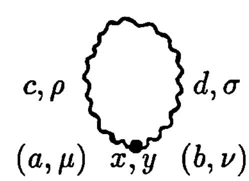


Figure 5.64

$$
\begin{aligned}
\Pi_{2\mu\nu}^{ab}(x,y) = \;& \tfrac{1}{2} g^2 W_{\mu\nu\rho\sigma}^{abcd} \delta(x-y) G^{cd\rho\sigma}(x,y) \\
= \;& (F) \\
& + g^2 f^{ace} f^{bfd} [-2 g_{\mu\nu} \partial_{y\lambda}^{(f)} + g_{\nu\lambda} \partial_{y\mu}^{(f)} + g_{\lambda\mu} \partial_{y\nu}^{(f)}] f_2^{ef\lambda}(x,y) \delta^{cd}(x-y).
\end{aligned}
\tag{5.77}
$$

Thus we have a part (the other part is contributed by fermion loop diagram) of the gluon self-energy

$$
\begin{aligned}
\Pi^{ab\mu\nu}(x,y) = \;& \Pi_1^{ab\mu\nu}(x,y) + \Pi_2^{ab\mu\nu}(x,y) \\
= \;& (F) + (ghost) + \\
& \tfrac{1}{2} f^{ace} f^{bfd} [O_y^{\nu\sigma} H_{1\sigma}^{cdef\mu}(x,y) + O_x^{\mu\sigma} H_{1\sigma}^{cdef\nu}(y,x) \\
& \qquad + y^\nu H_2^{cdef\mu}(x,y) + x^\mu H_2^{cdef\nu}(y,x)]
\end{aligned}
\tag{5.78}
$$

where

$$
\begin{aligned}
H_{1\sigma}^{cdef\mu}(x,y) = \;& O_x^{(e)\mu\lambda} f_{1\sigma}^{cd}(x,y) f_{2\lambda}^{ef}(x,y) \\
& + V^{(-,c,e)\mu\rho\lambda}(\partial_x)[G_{\rho\sigma}^{cd}(x,y) f_{2\lambda}^{ef}(x,y) - G_{F\lambda\sigma}^{ef} f_{2\rho}^{cd}(x,y)] \\
H_2^{cdef\mu}(x,y) = \;& V^{(-,c,e)\mu\rho\lambda}(\partial_x)[f_{2\lambda}^{ef}(x,y)(\partial y)^{-1} \partial_{y\rho} \delta^{cd}(x-y) \\
& \qquad - f_{2\rho}^{cd}(x,y)(\partial y)^{-1} \partial_{y\lambda} \delta^{ef}(x-y)] \\
& + 2[(\partial y)^{-1} \delta^{ef}(x-y)] \Box_x^{-1} \partial_x^\mu \delta^{cd}(x-y) \\
& - x^\mu [(\partial x)^{-1} \delta^{cd}(x-y)][(\partial y)^{-1} \delta^{ef}(x-y)].
\end{aligned}
\tag{5.79}
$$

Note that $(F) + (ghost)$ above is nothing but the gluon selfenergy $\Pi_F^{ab\mu\nu}(x,y)$ in the Feynman gauge. Finally the last two diagrams of the first order correction of the quark-gluon vertex are combined into

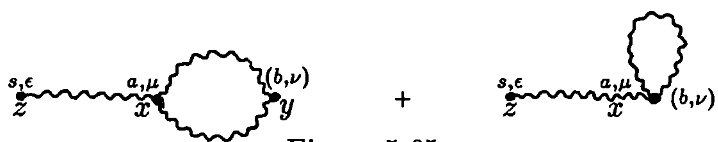


Figure 5.65

$$\begin{aligned}
\Gamma_{6in}^{b\nu}(x,y,z) = &\ \Gamma_{6_1in}^{b\nu}(x,y,z) + \Gamma_{6_2in}^{b\nu}(x,y,z) \\
= &\ (F) + (ghost) + i[\cdot\gamma^{\epsilon}\cdot](T^s)_{in}\partial_{z\epsilon} f_{1\mu}^{sa}(z,x)\Pi^{ab\mu\nu}(x,y) \\
&+ \tfrac{i}{2}g^2 f^{ace} f^{bfd}[\cdot\gamma^{\epsilon}\cdot](T^s)_{in} G_{\epsilon\mu}^{sa}(z,x)\times \\
&\qquad [O_y^{\nu\sigma} H_{1\sigma}^{cdef\mu}(x,y) + O_x^{\mu\sigma} H_{1\sigma}^{cdef\nu}(y,x) + y^{\nu} H_2^{cdef\mu}(x,y)].
\end{aligned} \tag{5.80}$$

5.4.2 On-shell Diagrams

<u>Gluon self-energy</u>

The gluon self-energy in the FS gauge consists of three diagrams below

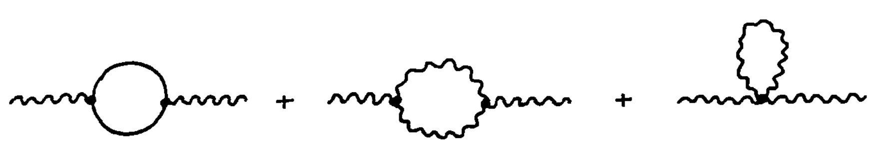


Figure 5.66

The first diagram is gauge propagator independent. It is also gauge independent because of transversality of its truncated diagram. This means that this diagram in the FS gauge is exactly equal to that in the Feynman gauge. According to the previous result, equation (5.78), it turns out that the last two diagrams equal the same diagrams in the Feynman gauge plus the ghost terms

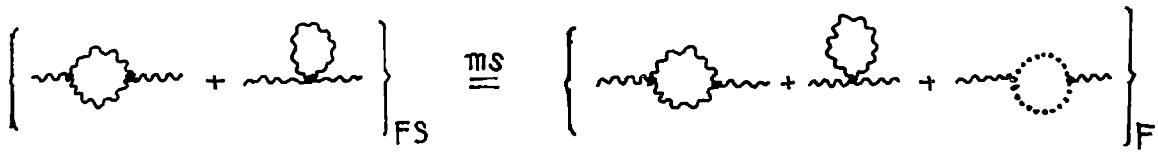


Figure 5.67

because

$$A(y) \cdot y = 0 \tag{5.81}$$

and

$$O_y^{\sigma\nu} A_\nu(y) \stackrel{J\equiv 0}{=} 0. \tag{5.82}$$

Since the gluon self-energy in the Feynman gauge is transverse, see for example [Mut 87], we conclude that the equality leads to the transversality of the gluon self-energy in the FS gauge, as already anticipated.

Quark-gluon vertex corrections

Now consider the first three diagrams depicted below

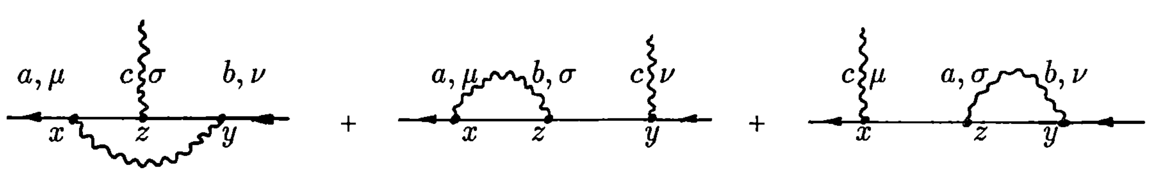


Figure 5.68

It turns out that the terms in equations (5.69), (5.70) and (5.71) which each has factors $T^aT^bT^c$ are proportional to equations (5.14), (5.15) and (5.16) in spinor quantum electrodynamics respectively. As a consequence, because the sum of those terms in quantum electrodynamics vanishes on the mass-shell, the only term in the three above diagrams that give a contribution on the mass-shell is the term in (5.69) which has group factors $f^{cad}T^aT^d = f^{ced}T^aT^b\delta^{ae}\delta^{db}$. Accordingly

$$
\begin{aligned}
S_1 + S_2 + S_3 = \; & ig \int dx \int dy \int dz \overline{\psi}_i(x) \\
& [\Gamma^{c\sigma}_{1in}(x,y,z) A^c_\sigma(z) + \Gamma^{b\nu}_{2in}(x,y,z) A^b_\nu(y) + \Gamma^{a\mu}_{3in}(x,y,z) A^a_\mu(x)] \psi_n(y) \\
\stackrel{ms}{=} \; & (F) + g^3 (f^{ced} T^a T^b)_{in} \int dx \int dy \int dz \\
& \{[\overline{\psi}_i(x) \delta^{ae}(x-z) \; \not{A}^c(z) S_F(z-y) \; \not{f}_1^{db}(x,y) \psi_n(y)] \\
& -[\overline{\psi}_i(x) \; \not{f}_2^{ae}(x,y) S_F(x-z) \; \not{A}^c(z) \delta^{db}(z-y) \psi_n(y)]\}.
\end{aligned}
\tag{5.83}
$$

Similarly, since the second term of $\Gamma^{c\sigma}_{4in}(x,y,z)$ in (5.72) is proportional to (5.17) and thus does not give any contribution to S_4 on mass-shell, we have

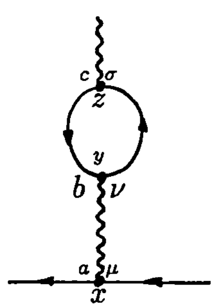


Figure 5.69

$$
S_4 \stackrel{ms}{=} (F). \tag{5.84}
$$

Now we come to the last three diagrams. Their details of calculation can be seen in the Appendix G. The final results are given as follows.

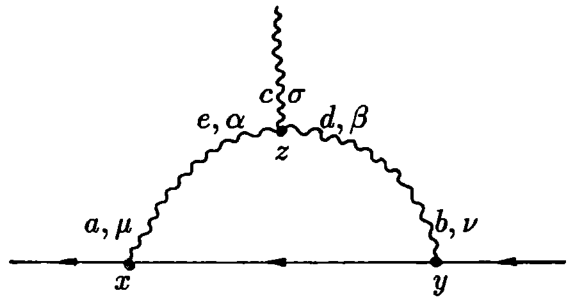


Figure 5.70

$$
\begin{aligned}
S_5 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(x)\Gamma^{c\sigma}_{5in}(x,y,z)A^c_\sigma(z)\psi_n(y) \\
\overset{ms}{=}\;& (F) - g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z) V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\times \\
& \quad \{2i f^{ae}_{1\alpha}(x,z) G^{db}_{\beta\nu}(z,y)\overline{\psi}_i(x)\delta(x-y)\gamma^\nu\psi_n(y) \\
& \quad -i f^{ae}_{1\alpha}(x,z) f^{db}_{2\beta}(z,y)\overline{\psi}_i(x)(\overrightarrow{\not\partial}_x + im)\delta(x-y)\psi_n(y)\} \\
& + g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z) O^{(d)\sigma\beta}_z \times \\
& \quad 2i f^{db}_{2\beta}(z,y)\overline{\psi}_i(x)\; f^{ae}_2(x,z)\delta(x-y)\psi_n(y) \\
& - g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z) \\
& \quad \{-\overline{\psi}_i(x)\; f^{ae}_2(x,z) S_F(x-y)\gamma^\sigma\psi_n(y)\delta^{db}(z-y) \\
& \quad + \overline{\psi}_i(x)\gamma^\sigma S_F(x-y)\; f^{db}_1(z,y)\psi_n(y)\delta^{ae}(z-x)\}.
\end{aligned} \tag{5.85}
$$

The last

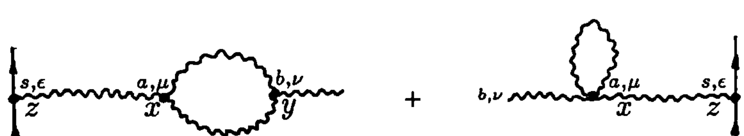


Figure 5.71

$$
\begin{aligned}
S_6 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(z)\Gamma^{b\nu}_{6in}(x,y,z)A^b_\nu(y)\psi_n(z) \\
= \;& (F) + (ghost) - ig^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z)\overline{\psi}_i(x)\times \\
& \{-2V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\delta(x-y)\gamma^\nu G^{db}_{\beta\nu}(z,y) f^{ae}_{1\alpha}(x,z) \\
& + 2\delta(x-y)O^{(d)\sigma\beta}_z\; f^{ae}_2(x,z) f^{db}_{2\beta}(z,y) \\
& - V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[(\overrightarrow{\not\partial}_y - im)\delta(x-y)] f^{ae}_{1\alpha}(x,z) f^{db}_{2\beta}(z,y)\}\psi_n(y)\,.
\end{aligned} \tag{5.86}
$$

Remember that $(ghost)$ is the ghost contribution in the Feynman gauge:

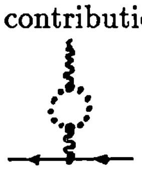


Figure 5.72

Summing up equations (5.83)-(5.86), we get

$$\begin{aligned} S = &\ S_1 + S_2 + S_3 + S_4 + S_5 + S_6 \\ \stackrel{ms}{=} &\ (F) + (ghost). \end{aligned} \tag{5.87}$$

To conclude: off the mass-shell, all QED diagrams considered in the FS gauge are different from those in the Feynman gauge. The correction $G'_{\mu\nu}(x,y)$ is responsible for it. These correction terms, though they are quite complicated, contain helpful factors like

$$\overline{\psi}(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-y) \quad \text{and} \quad S_F(x-z)(\overleftarrow{\not\partial}_z + \overrightarrow{\not\partial}_z)S_F(z-y)$$

in spinor QED and

$$\phi^\dagger(x)(\overleftarrow{\Box}_x - \overrightarrow{\Box}_x)S_B(x-y) \quad \text{and} \quad S_B(x-z)(\overleftarrow{\Box}_z - \overrightarrow{\Box}_z)S_B(z-y)$$

in scalar QED. Such factors will reduce to Dirac delta functions on the mass-shell. This reduction leads to cancellations in the correction terms. As a result both the FS gauge and the Feynman gauge are identical on mass-shell

$$\begin{aligned} (\text{selfenergy})_{FS} &\stackrel{ms}{=} (\text{selfenergy})_F \\ (\text{vertex})_{FS} &\stackrel{ms}{=} (\text{vertex})_F. \end{aligned}$$

The same conclusion holds for quantum chromodynamics. One interesting point here is that since the FS gauge is ghost-free while the Feynman gauge on the other hand contains ghost loops the above equality means that when one shifts the FS gauge into the Feynman gauge ghost terms in the Feynman gauge emerge automatically. Diagramatically gluon selfenergy

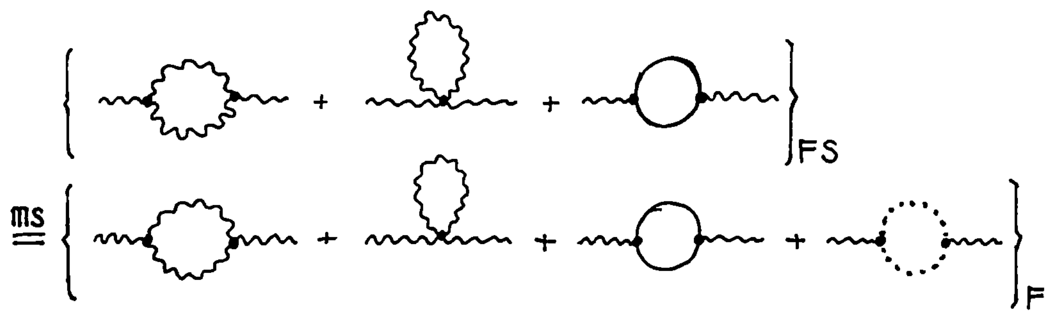


Figure 5.73

quark-gluon vertex corrections

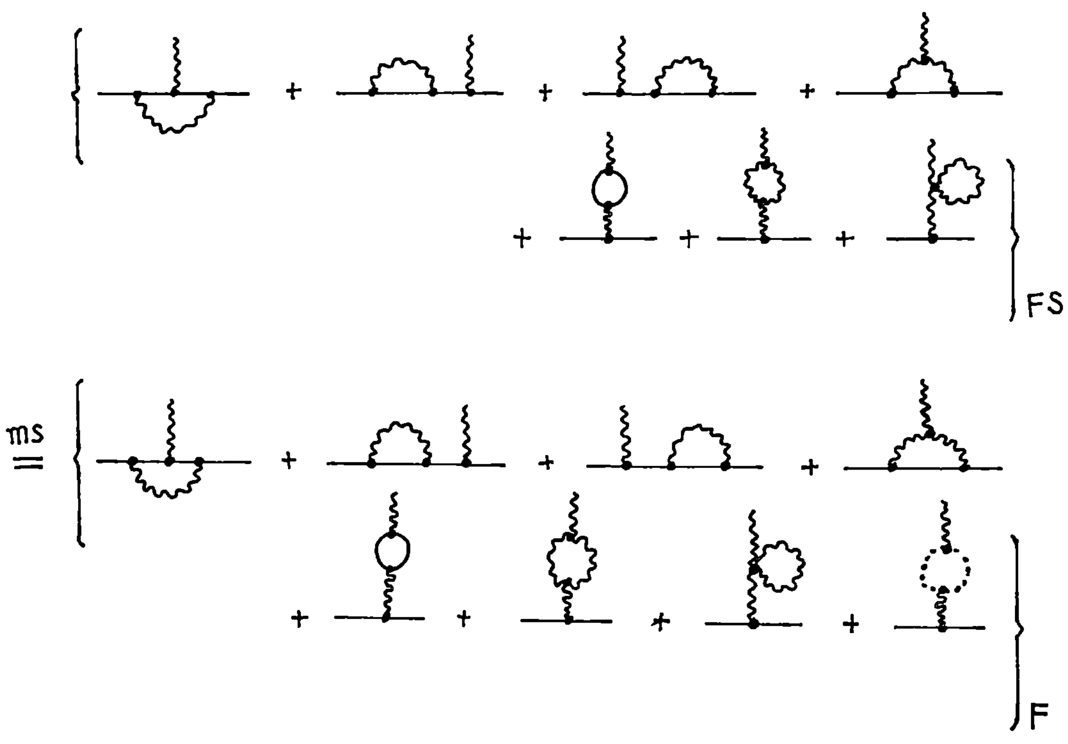


Figure 5.74

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Chapter 6

Conclusion

6.1 Summary

In Chapter 2 we reviewed the derivation of inversion formulas together with their sufficient and necessary conditions both in Abelian and non-Abelian gauge theories. These formulas then were employed to obtain the so-called FS gauge potentials for some classical configurations. We found that in electrostatics the FS vector potentials are nonzero whereas (like the familiar Coulomb gauge) there are no scalar potentials in magnetostatic systems. One important result is that since according to the inversion formula the FS gauge vector potential depends on time in the language of FS gauge potentials electrostatic systems are no longer static! In addition, the FS gauge potentials in systems of plane electromagnetic waves are not plane waves. The fact that scattering of charged particles due to the FS gauge potential is identical to the Coulomb scattering lets us conclude that the FS gauge potentials (as expected) do not produce any (new) physical consequences.

Chapter 3 is the starting point of the discussion on quantum field theory. Green functions which play an important role in quantum field theory were derived in coordinate space in two different gauge fixing terms of Lagrangian $\mathcal{L}_{\mathcal{GF}_1} = -\frac{1}{2\lambda}(G \cdot A)^2$ and $\mathcal{L}_{\mathcal{GF}_2} = C(G \cdot A) + \frac{\lambda}{2}C^2$ and where $G_\mu = \partial_\mu, x_\mu$ or n_μ in the Lorentz, FS and axial gauges respectively and C is an auxiliary or Lagrange multiplier field. The Green's function derived by the use of $\mathcal{L}_{\mathcal{GF}_1}$ is the most familiar one. It is a 4×4 matrix and thus it only contains (μ, ν)-elements. The last Green's function is a 5×5 matrix. Besides (μ, ν)-elements there are $(\mu, 4)$ and $(4, \mu)$-elements. These last two elements are called the unphysical part of the Green's function since they are not found in scattering matrices. Accordingly the (μ, ν)-elements are contained in the physical part of the Green's function. We showed that the first Green's function is equal to the physical part of the second Green's function. This equality is understood since in the generating functional $\mathcal{L}_{\mathcal{GF}_1}$ is effectively equal to $\mathcal{L}_{\mathcal{GF}_2}$.

The symmetry properties of the Green functions or propagators were found to be $G_{AB}(x, y) = G_{BA}(y, x)$. In the case when $\lambda = 0$ the physical component of the Green's function has another property: it is orthogonal to G_μ. The derivation of the physical part of the propagator, for $\lambda \to 0$, in "momentum space" from that in coordinate space was based on the above symmetry and therefore the resulting propagator did not lose its symmetry. Our derivation is a definite improvement on what Kummer and Weiser [Kum 86] did. They did not make use the symmetry property from the beginning and as a result they found that the symmetry does not obviously appear in their resulting propagator; this forced them to propose new propagators $\hat{G}_{\mu\nu}(x, y)$ which obey the symmetry.

In Chapter 4 the local gauge and the BRST transformations were reviewed. The BRST transformations were derived by the use of $\mathcal{L}_{\mathcal{GF}_2}$. Based on the local gauge and BRST symmetries of Lagrangians the Ward-Takahashi, Slavnov-Taylor and BRST identities were then derived.

The fact that the FS gauge is a ghost-free gauge was demonstrated in the first section of the chapter. Since the ghost fields in ghost-free gauges like the FS gauge may be disregarded the BRST identities in such gauges can be simplified into the Slavnov-Taylor identities or the non-Abelian version of the Ward-Takahashi identities. However, the content of all these identities, such as the transversality of the gauge field self-energy, naturally remains the same.

Investigations on up to one-loop diagrams in spinor and scalar quantum electrodynamics and quantum chromodynamics were done in Chapter 5. We found that in scalar and spinor quantum electrodynamics the extra propagator $G'_{\mu\nu}(x, y)$ contributions disappear on mass-shell and leaving only the terms containing the Feynman gauge propagator $G_{F\mu\nu}(x, y)$. Therefore as far as the mass-shell perturbation calculations are concerned, the FS gauge theory is equivalent to the Feynman gauge theory. This is hardly surprising. The same conclusion is also true for quantum chromodynamics. Here, extra diagrams contributed by ghost fields in the Feynman gauge are contained in the FS gauge propagator and the transversality of the full gluon self-energy is in agreement with the BRST identity.

We anticipate that the conclusion holds (on mass-shell) to all order of scattering matrix but highly complicated nature of FS calculations indicates that it is better to consider the Feynman gauge rather than the FS gauge in perturbation calculations. However some properties of the FS gauge such as the inversion formula may make it useful for certain nonperturbative computations.

6.2 Outlook

The conclusion obtained in Chapter 5 is based on the lowest order diagrams we have considered. Thus it is essential to justify whether this conclusion also well applies for (at least up to a few) higher order diagrams. Of course this is not an easy task. The easiest task is, perhaps, to examine the first order three-gluon vertex correction which remains to be done.

The investigations are based on the propagator with $\lambda = 0$. It would be quite interesting if computations are carried out by making use of propagators with a general value of the gauge parameter λ and discover what role the gauge parameter term plays in perturbation theories. As in the axial gauge, [Cap 82, Lei 87], the final results will of course be more complicated. We are confident that in the limit $\lambda \to 0$ the final result will shift to the result found in Chapter 5.

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APPENDICES

A General Notations

Summation over repeated (Greek or Latin) indices is understood. Units with $c = \hbar = h/2\pi = 1$ are used throughout the thesis.

Metric tensor:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Four-vectors:

$$x^\mu = (t, x, y, z) = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$$

Gradient:

$$\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \nabla)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_0, -\nabla)$$

Divergence and curl:

$$\nabla \cdot A = \partial_i A^i$$

$$B^i = (\nabla \times \vec{A})^i = \epsilon^{ijk} \partial_j A^k$$

Levi-Civita tensors:

$$\epsilon^{ijk} = -\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma} \begin{cases} 1 & \text{if } (\mu\nu\rho\sigma) \text{ is an even permutation of } (0123) \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

Electromagnetic field tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$$F^{0i} = -E^i; \qquad F^{ij} = \epsilon^{ijk} B_k.$$

B Identities Relating to Operators ∂_μ and G_μ

First we list important formulae and later we indicate their proofs.

$$I_{FS} = \begin{cases} 1 & \text{if } G^\mu = x^\mu \\ 0 & \text{if } G^\mu = \partial^\mu \text{ or } n^\mu \end{cases}$$

$$\begin{array}{lrl}
(1) & \partial_\mu G_\nu = & G_\mu \partial_\nu + I_{FS} g_{\mu\nu} \\
(2) & \partial G = & G\partial + 4I_{FS} \\
(3) & G^\mu(\partial G + a)^{\pm 1} = & (\partial G + a - I_{FS})^{\pm 1} G^\mu \\
(4) & \partial^\mu(\partial G + a)^{\pm 1} = & (\partial G + a + I_{FS})^{\pm 1} \partial^\mu \\
(5) & \Box G^\mu = & G^\mu \Box + 2I_{FS}\partial^\mu \\
(6) & \Box G_\mu G_\nu = & G_\mu G_\nu \Box + 2I_{FS}(\partial_\mu G_\nu + G_\mu \partial_\nu) \\
(7) & \Box(\partial G + a)^{\pm 1} = & (\partial G + a + 2I_{FS})^{\pm 1}\Box \\
(8) & G^\mu \Box^{-1} = & \Box^{-1} G^\mu + 2I_{FS}\Box^{-2}\partial^\mu \\
(9) & G_\mu G_\nu \Box^{-1} = & \Box^{-1} G_\mu G_\nu + 2I_{FS}\Box^{-1}(\partial_\mu G_\nu + G_\mu \partial_\nu)\Box^{-1} \\
(10) & G_\mu G_\nu \Box^{-1} = & \Box^{-1} G_\mu G_\nu + 2I_{FS}\Box^{-2}(\partial_\mu G_\nu + G_\mu \partial_\nu) \\
 & & +8I_{FS}\Box^{-3}\partial_\mu \partial_\nu \\
(11) & G^2\Box^{-1} = & \Box^{-1} G^2 + 4I_{FS} G\partial \Box^{-2} \\
(12) & G^\mu \Box^{-2} = & \Box^{-2} G^\mu + 4I_{FS}\Box^{-3}\partial^\mu \\
(13) & (\partial G + a)^{\pm 1}\Box^{-1} = & \Box^{-1}(\partial G + a + 2I_{FS})^{\pm 1} \\
(14) & G'^2 \delta(x - x') = & G^2 \delta(x - x') \\
(15) & G'^\mu \delta(x - x') = & \mp G^\mu \delta(x - x') \\
(16) & (\partial' G' + a)^{\pm 1}\delta(x - x') = & \pm(\partial G \pm a - 4I_{FS})^{\pm 1}\delta(x - x')
\end{array}$$

where the upper sign of $(\mp)$ or $(\pm)$ in the coefficient of the righthand side of the last two equations is given for the Lorentz gauge and the lower sign is for the axial and Fock-Schwinger gauges. Note that a is a number.

Proofs

(1) and (2) obvious.

$$\text{(3)} \qquad \begin{aligned}(\partial G + a - I_{FS})G_\mu &\overset{(2)}{=} (G\partial + a + 3I_{FS})G_\mu \\ &\overset{(1)}{=} G^\nu(G_\mu\partial_\nu + I_{FS}g_{\mu\nu}) + G_\mu(a + 3I_{FS}) \\ &= G_\mu(G\partial + a + 4I_{FS}) \overset{(2)}{=} G_\mu(\partial G + a).\end{aligned}$$

Multiplying both sides by $(\partial G + a - I_{FS})^{-1}$ from the left and $(\partial G + a)^{-1}$ from the right the above identity is converted into

$$G_\mu(\partial G + a)^{-1} = (\partial G + a - I_{FS})^{-1}G_\mu.$$

$$\text{(4)} \qquad \partial^\mu(\partial G + a) \overset{(1)}{=} (\partial G + I_{FS})\partial^\mu + a\partial^\mu = (\partial G + a + I_{FS})\partial^\mu.$$

The same multiplication as in (3) leads to

$$(\partial G + a + I_{FS})^{-1}\partial^\mu = \partial^\mu(\partial G + a)^{-1}.$$

$$\begin{aligned}\text{(5)} \qquad \Box G_\mu &\overset{(1)}{=} \partial^\nu(G_\mu\partial_\nu + I_{FS}g_{\mu\nu}) \\ &\overset{(1)}{=} (G_\mu\partial_\nu + I_{FS}g_{\mu\nu})\partial^\nu + I_{FS}\partial_\mu = G_\mu\Box + 2I_{FS}\partial_\mu. \\ \text{(6)} \qquad \Box G_\mu G_\nu &\overset{(5)}{=} (G_\mu\Box + 2I_{FS}\partial_\mu)G_\nu \overset{(5)}{=} G_\mu(G_\nu\Box + 2I_{FS}\partial\nu) + 2I_{FS}\partial\mu G_\nu \\ &= G_\mu G_\nu\Box + 2I_{FS}(\partial_\mu G_\nu + G_\mu\partial_\nu). \\ \text{(7)} \quad \Box(\partial G + a)^{\pm1} &\overset{(4)}{=} \partial^\mu(\partial G + a + I_{FS})^{\pm1}\partial_\mu \overset{(4)}{=} (\partial G + a + 2I_{FS})^{\pm1}\Box.\end{aligned}$$

(8) Multiplying $\Box^{-1}$ from both the left and right on both sides of (5) we have

$$G_\mu\Box^{-1} = \Box^{-1}G_\mu + 2I_{FS}\Box^{-2}\partial\mu.$$

(9) Same treatment as (6).

(10) = (9) according to (8).

$$(11) \qquad G^2\Box^{-1} \overset{(10)}{=} \Box^{-1}G^2 + 2I_{FS}\Box^{-2}(\partial G + G\partial) + 8I_{FS}\Box^{-2}$$
$$= \Box^{-1}G^2 + 4I_{FS}\Box^{-2}\partial G \overset{(13)}{=} \Box^{-1}G^2 + 4I_{FS}G\partial\Box^{-2}.$$
$$(12) \qquad G_\mu\Box^{-2} \overset{(8)}{=} (\Box^{-1}G_\mu + 2I_{FS}\Box^{-2}\partial\mu)\Box^{-1} \overset{(8)}{=} \Box^{-1}(\Box^{-1}G_\mu + 2I_{FS}\Box^{-2}\partial\mu)$$
$$+2I_{FS}\Box^{-3}\partial\mu = \Box^{-2}G_\mu + 4I_{FS}\Box^{-3}\partial\mu.$$

(13) Same treatment as in (8) for (7).

(14) and (15) obvious.

$$(16) \qquad (\partial' G' + a)\delta(x - x') \overset{(15)}{=} (\mp\partial' G + a)\delta(x - x') = (\mp G\partial' + a)\delta(x - x')$$
$$\overset{(15)}{=} (\pm G\partial + a)\delta(x - x') = (\pm(\partial G - 4I_{FS}) + a)$$
$$\delta(x - x')$$
$$= \pm(\partial G \pm a - 4I_{FS})\delta(x - x').$$

Multiplying $\pm(\partial G \pm a - 4I_{FS})^{-1}(\partial' G' + a)^{-1}$ on both sides the identity becomes

$$\pm(\partial G \pm a - 4I_{FS})^{-1}\delta(x - x') = (\partial' G' + a)^{-1}\delta(x - x').$$

C Special Unitary Group SU(N)

The generators T^a are hermitian and traceless, $a = 1, 2, 3, \cdots, N^2 - 1$. They obey the Lie algebra

$$[T^a, T^b] = i f^{abc} T^c$$

where f^{abc} are the antisymmetric structure constants. Other useful relations are

$$\begin{aligned} \{T^a, T^b\} = & \ \tfrac{1}{N}\delta^{ab} + d^{abc}T^c \\ T^a T^b = & \ \tfrac{1}{2N}\delta^{ab} + \tfrac{1}{2}d^{abc}T^c + \tfrac{1}{2}i f^{abc}T^c \\ d^{abc} = & \ 2Tr[\{T^a, T^b\}T^c]. \end{aligned}$$

Traces of product of generators:

$$\begin{aligned} Tr[T^a] = & \ 0 \\ Tr[T^a T^b] = & \ \tfrac{1}{2}\delta^{ab} \\ Tr[T^a T^b T^c] = & \ \tfrac{1}{4}(d^{abc} + i f^{abc}) \\ Tr[T^a T^b T^c T^d] = & \ \tfrac{1}{4N}\delta^{ab}\delta^{cd} + \tfrac{1}{8}(d^{abe} + i f^{abe})(d^{cde} + i f^{cde}). \end{aligned}$$

Jacobi identities:

$$\begin{aligned} [T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = & \ 0 \\ [T^a, \{T^b, T^c\}] + [T^b, \{T^c, T^a\}] + [T^c, \{T^a, T^b\}] = & \ 0 \end{aligned}$$

or

$$\begin{aligned} f^{abe}f^{cde} + f^{cbe}f^{dae} + f^{dbe}f^{ace} = & \ 0 \\ f^{abe}d^{cde} + f^{cbe}d^{dae} + f^{dbe}d^{ace} = & \ 0. \end{aligned}$$

D Inversion Formulae and Their Conditions

D.1 Derivations of identity $\alpha\frac{d}{d\alpha}f(\alpha x) = x \cdot \partial_x f(\alpha x)$

For any function $f(\alpha x)$ in n dimensional space $x = (x^1, x^2, \cdots, x^n)$ with an arbitrary parameter α one has

$$\alpha\frac{d}{d\alpha}f(\alpha x) = \alpha\frac{ds^i}{d\alpha}\frac{\partial}{\partial s^i}f(s) = \alpha x^i\frac{\partial}{\alpha\partial x^i}f(\alpha x) = x^i\frac{\partial}{\partial x^i}f(\alpha x) \tag{1}$$

where $s = \alpha x$ and $i = 1, 2, \cdots, n$.

D.2 Derivations of conditions for inversion formulae in non-Abelian theories

We start from equation

$$\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{\mu\nu}(x) - \int_0^1 d\alpha\alpha^2 x^\beta[\partial'_\beta F_{\mu\nu}(\alpha x) + \partial'_\mu F_{\nu\beta}(\alpha x) + \partial'_\nu F_{\beta\mu}(\alpha x)]$$

with $\partial'_\beta = \frac{1}{\alpha}\partial_\beta$. In non-Abelian gauge theories the left-hand side of the above equation is not equal to $F_{\mu\nu}(x)$. Adding $-ig[A_\mu(x), A_\nu(x)]$ to both sides of the above equation one has

$$\begin{aligned} & \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)] \\ = \; & -ig[A_\mu(x), A_\nu(x)] + F_{\mu\nu}(x) \\ & - \textstyle\int_0^1 d\alpha\alpha^2 x^\beta[\partial'_\beta F_{\mu\nu}(\alpha x) + \partial'_\mu F_{\nu\beta}(\alpha x) + \partial'_\nu F_{\beta\mu}(\alpha x)]. \end{aligned} \tag{2}$$

The commutator can be written as follows

$$\begin{aligned} [A_\mu(x), A_\nu(x)] = & \textstyle\int_0^1 d\alpha\frac{d}{d\alpha}\alpha^2[A_\mu(\alpha x), A_\nu(\alpha x)] \\ = & \textstyle\int_0^1 d\alpha\alpha\{2[A_\mu(\alpha x), A_\nu(\alpha x)] + \alpha\frac{d}{d\alpha}[A_\mu(\alpha x), A_\nu(\alpha x)]\}. \end{aligned} \tag{3}$$

By recalling the Fock-Schwinger gauge condition $x \cdot A = 0$, $A_\mu(\alpha x)$ and $A_\nu(\alpha x)$ in the first term on the right-hand side can be written as

$$-A_\mu(\alpha x) = \alpha x^\beta \partial'_\mu A_\beta(\alpha x)$$

while, according to the identity (1), $\alpha\frac{d}{d\alpha}$ in the second term can be replaced by $x_\beta\partial^\beta = x'_\beta\partial'^\beta$. Equation (3) now reads

$$\begin{aligned}[A_\mu(x), A_\nu(x)] = & \int_0^1 d\alpha\alpha^2 x^\beta\{-[\partial'_\mu A_\beta(\alpha x), A_\nu(\alpha x)] - [A_\mu(\alpha x), \partial'_\nu A_\beta(\alpha x)] \\ & +\partial'_\beta[A_\mu(\alpha x), A_\nu(\alpha x)]\} \\ = & \int_0^1 d\alpha\alpha^2 x^\beta\{-[\partial'_\mu A_\beta(\alpha x) - \partial'_\beta A_\mu(\alpha x), A_\nu(\alpha x)] \\ & -[A_\mu(\alpha x), \partial'_\nu A_\beta(\alpha x)] - \partial'_\beta A_\nu(\alpha x)]\} \\ = & \int_0^1 d\alpha\alpha^2 x^\beta\{[F_{\beta\mu}(\alpha x), A_\nu(\alpha x)] - [A_\mu(\alpha x), F_{\nu\beta}(\alpha x)]\}.\end{aligned} \tag{4}$$

Inserting (4) into (2) one has

$$\begin{aligned} & \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)] \\ = & F_{\mu\nu}(x) - \int_0^1 d\alpha\alpha^2 x^\beta\{\partial'_\beta F_{\mu\nu}(\alpha x) - ig[A_\beta(\alpha x), F_{\mu\nu}(\alpha x)] + \partial'_\mu F_{\nu\beta}(\alpha x) \\ & -ig[A_\mu(\alpha x), F_{\nu\beta}(\alpha x)] + \partial'_\nu F_{\beta\mu}(\alpha x)] - [A_\nu(\alpha x), F_{\beta\mu}(\alpha x)]\} \\ = & F_{\mu\nu}(x) - \int_0^1 d\alpha\alpha^2 x^\beta[D_\beta F_{\mu\nu} + D_\mu F_{\nu\beta} + D_\nu F_{\beta\mu}](\alpha x)\end{aligned} \tag{5}$$

where

$$D_\beta F_{\mu\nu} = \partial_\beta F_{\mu\nu} - ig[A_\beta, F_{\mu\nu}].$$

Equation (5) agrees with the Bianchi identities

$$[D_\beta F_{\mu\nu} + D_\mu F_{\nu\beta} + D_\nu F_{\beta\mu}] = 0.$$

D.3 Validity of $A_\alpha(x) = \int_0^1 d\lambda F_{\mu\nu}(S)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\nu}{\partial x^\alpha}$

The inversion formula

$$A_\alpha(x) = \int_0^1 d\lambda F_{\mu\nu}(S)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\nu}{\partial x^\alpha} \tag{6}$$

is only for abelian theories [Cor 84]. The derivation of necessary and sufficient conditions of the inversion formula (6) is as follows. Applying derivatives on $A_\alpha(x)$, one has

$$\begin{aligned}\partial_\beta A_\alpha(x) = \int_0^1 d\lambda \Bigg\{ & \frac{\partial F_{\mu\nu}(S)}{\partial S_\rho}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\nu}{\partial x^\alpha} + F_{\mu\nu}(S)\frac{\partial^2 S^\mu}{\partial x^\beta\partial\lambda}\frac{\partial S^\nu}{\partial x^\alpha} + \\ & F_{\mu\nu}(S)\,\frac{\partial S^\mu}{\partial\lambda}\frac{\partial^2 S^\nu}{\partial x^\beta\partial x^\alpha}\Bigg\}.\end{aligned}$$

Therefore

$$\partial_\beta A_\alpha(x) - \partial_\alpha A_\beta(x) = \int_0^1 d\lambda \left\{ \frac{\partial F_{\mu\nu}(S)}{\partial S_\rho}\frac{\partial S^\mu}{\partial\lambda}\left(\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha} - \frac{\partial S^\rho}{\partial x^\alpha}\frac{\partial S^\nu}{\partial x^\beta}\right) + \right.$$

$$\left. F_{\mu\nu}(S)\left(\frac{\partial^2 S^\mu}{\partial x^\beta \partial\lambda}\frac{\partial S^\nu}{\partial x^\alpha} - \frac{\partial^2 S^\mu}{\partial x^\alpha \partial\lambda}\frac{\partial S^\nu}{x^\beta}\right)\right\}.$$

The first term on the right-hand side may be written as

$$\frac{\partial F_{\mu\nu}(S)}{\partial S_\rho}\frac{\partial S^\mu}{\partial\lambda}\left(\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha} - \frac{\partial S^\rho}{\partial x^\alpha}\frac{\partial S^\nu}{\partial x^\beta}\right) = \left(\frac{\partial F_{\mu\nu}(S)}{\partial S^\rho} + \frac{\partial F_{\rho\mu}(S)}{\partial S^\nu}\right)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha}$$

$$= \left(\partial'_\rho F_{\mu\nu}(S) + \partial'_\mu F_{\nu\rho}(S) + \partial'_\nu F_{\rho\mu}(S)\right)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha} - \frac{\partial F_{\nu\rho}(S)}{\partial\lambda}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha}$$

where $\partial'_\mu = \frac{\partial}{\partial S^\mu}$. In consequence,

$$\begin{aligned}
&\partial_\beta A_\alpha(x) - \partial_\alpha A_\beta(x) \\
&= \int_0^1 d\lambda \left\{ \left(\partial'_\rho F_{\mu\nu}(S) + \partial'_\mu F_{\nu\rho}(S) + \partial'_\nu F_{\rho\mu}(S)\right)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha} \quad + \right. \\
&\qquad \left. F_{\mu\nu}(S)\left(\frac{\partial^2 S^\mu}{\partial x^\beta \partial\lambda}\frac{\partial S^\nu}{\partial x^\alpha} + \frac{\partial^2 S^\nu}{\partial x^\alpha \partial\lambda}\frac{\partial S^\mu}{x^\beta}\right) + \frac{\partial F_{\mu\nu}(S)}{\partial\lambda}\frac{\partial S^\mu}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha}\right\} \\
&= \int_0^1 d\lambda \left\{ \left(\partial'_\rho F_{\mu\nu}(S) + \partial'_\mu F_{\nu\rho}(S) + \partial'_\nu F_{\rho\mu}(S)\right)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha} \quad + \right. \\
&\qquad \left. \frac{d}{d\lambda}\left[F_{\mu\nu}(S)\frac{\partial S^\mu}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha}\right]\right\} \\
&= F_{\beta\alpha}(x) + \int_0^1 d\lambda \left(\partial'_\rho F_{\mu\nu}(S) + \partial'_\mu F_{\nu\rho}(S) + \partial'_\nu F_{\rho\mu}(S)\right)\frac{\partial S^\mu}{\partial\lambda}\frac{\partial S^\rho}{\partial x^\beta}\frac{\partial S^\nu}{\partial x^\alpha}.
\end{aligned}$$

This final result states once again that the Bianchi identities

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$$

are the necessary and sufficient conditions for the inversion relation (6) to hold.

D.4 Expansions of $A_\mu(x)$

The expansion formula for $A_\mu(x)$ follows the work of Shifman [Shi 80]. Consider the inversion formula

$$A_\mu(x) = -\int_0^1 d\alpha\alpha x^\nu F_{\mu\nu}(\alpha x) \tag{7}$$

$F_{\mu\nu}(\alpha x)$ in the integrand can be Taylor-expanded around $x = 0$

$$F_{\mu\nu}(\alpha x) = F_{\mu\nu}(0) + \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha x)^{\alpha_1} \cdots (\alpha x)^{\alpha_n} \left(\partial'_{\alpha_1} \cdots \partial'_{\alpha_n} F_{\mu\nu}(\alpha x)|_{\alpha x=0} \right)$$

$$= F_{\mu\nu}(0) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} x^{\alpha_1} \cdots x^{\alpha_n} \left(\partial_{\alpha_1} \cdots \partial_{\alpha_n} F_{\mu\nu}(x)|_{x=0} \right) \tag{8}$$

where $\partial'_{\alpha_1} = \frac{1}{\alpha}\partial_{\alpha_1}$. After replacing $F_{\mu\nu}(\alpha x)$ in (7) by series (8) integration over α can be done easily. The formula (7) becomes

$$A_\mu(x) = \frac{1}{2} x^\nu F_{\nu\mu}(0) + \sum_{n=1}^{\infty} \frac{1}{n!(n+2)} x^{\alpha_1} \cdots x^{\alpha_n} \partial_{\alpha_1} \cdots \partial_{\alpha_n} F_{\nu\mu}(0). \tag{9}$$

The ordinary derivatives in (9) can be replaced by the covariant ones

$$A_\mu(x) = \frac{1}{2} x^\nu F_{\nu\mu}(0) + \sum_{n=1}^{\infty} \frac{1}{n!(n+2)} x^{\alpha_1} \cdots x^{\alpha_n} D_{\alpha_1} \cdots D_{\alpha_n} F_{\nu\mu}(0) \tag{10}$$

because in the FS gauge the identity

$$x^{\alpha_1} \cdots x^{\alpha_n} \partial_{\alpha_1} \cdots \partial_{\alpha_n} F_{\nu\mu}(0) = x^{\alpha_1} \cdots x^{\alpha_n} D_{\alpha_1} \cdots D_{\alpha_n} F_{\nu\mu}(0) \tag{11}$$

holds.

The proof of identity (10) is as follows. Consider the following expression

$$x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_n} [\partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_n} F_{\nu\mu}(x) - D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_n} F_{\nu\mu}(x)]. \tag{12}$$

The equation (11) holds automatically if expression (12) vanishes for every n. For $n = 1$

$$x^{\alpha_1}[\partial_{\alpha_1} - \partial_{\alpha_1} + igA_{\alpha_1}]F_{\nu\mu}(x) = igx^{\alpha_1} A_{\alpha_1} F_{\nu\mu}(x) = 0$$

because $x \cdot A = 0$. For $n = 2$ expression (12) reads

$$x^{\alpha_1} x^{\alpha_2} [\partial_{\alpha_1} \partial_{\alpha_2} - (\partial_{\alpha_1} \partial_{\alpha_2} - ig\partial_{\alpha_1} A_{\alpha_2} - igA_{\alpha_1} \partial_{\alpha_2} - g^2 A_{\alpha_1} A_{\alpha_2})] F_{\nu\mu}(x).$$

The last two terms vanish because of the gauge condition. The remaining term also vanishes

$$x^{\alpha_1} x^{\alpha_2} \partial_{\alpha_1} A_{\alpha_2} = x^{\alpha_1} (\partial_{\alpha_1} x \cdot A - \delta^{\alpha_2}_{\alpha_1} A_{\alpha_2}) = 0. \tag{13}$$

Thus (12) is also zero for $n = 2$. By the use of (13) it can be easily shown that for $n = 3$ the remaining term has the form $x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}\partial_{\alpha_1}\partial_{\alpha_2}A_{\alpha_3}$. However, this is also zero

$$\begin{aligned} x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}\partial_{\alpha_1}\partial_{\alpha_2}A_{\alpha_3} &= x^{\alpha_1}x^{\alpha_2}[\partial_{\alpha_1}x^{\alpha_3}\partial_{\alpha_2} - \delta^{\alpha_3}_{\alpha_1}\partial_{\alpha_2}]A_{\alpha_3} \\ &= x^{\alpha_1}x^{\alpha_2}\partial_{\alpha_1}\partial_{\alpha_2}x\cdot A - x^{\alpha_1}x^{\alpha_3}\partial_{\alpha_1}A_{\alpha_3} - x^{\alpha_2}x^{\alpha_3}\partial_{\alpha_2}A_{\alpha_3} = 0. \end{aligned} \tag{14}$$

The similarity between identities (13) and (14) enables us to prove (11) for higher n by mathematical induction. Suppose that this expression holds for n

$$x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}x^{\beta}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}A_{\beta}(x) = 0.$$

Accordingly, for $n+1$

$$\begin{aligned} & x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}x^{\alpha_{n+1}}x^{\beta}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}\partial_{\alpha_{n+1}}A_{\beta}(x) \\ = \; & x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_{n+1}}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_{n+1}}(x\cdot A) \\ & -x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_{n+1}}[\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}A_{\alpha_{n+1}}(x) + \partial_{\alpha_1}\cdots\partial_{\alpha_{n-1}}\partial_{\alpha_{n+1}}A_{\alpha_n}(x) \\ & \qquad + \cdots + \partial_{\alpha_2}\partial_{\alpha_3}\cdots\partial_{\alpha_{n+1}}A_{\alpha_1}(x)] \\ = \; & 0. \end{aligned}$$

Thus expression (11) is true for all n and therefore expansion (10) is valid.

D.5 Two other derivations of the inversion formulae

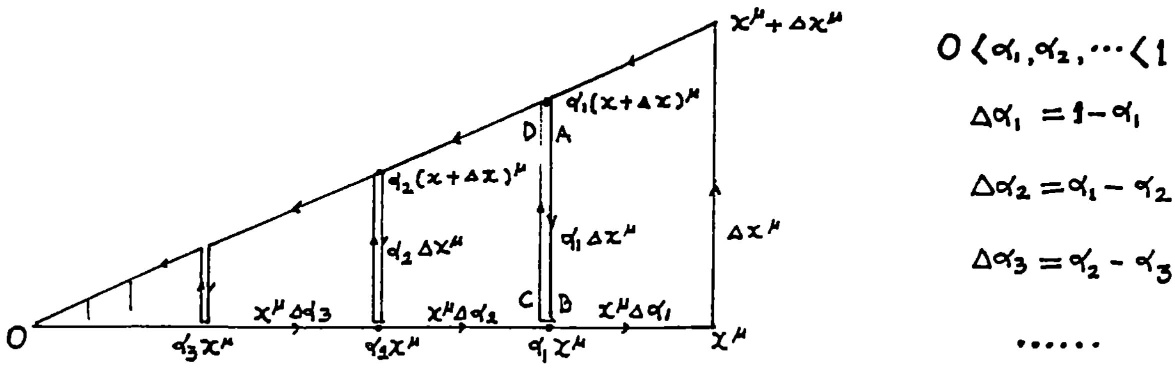
We may derive the inversion formulae by two other ways: by employing Stokes' theorem [Dur 82] and by using the language of differential geometry [Bri 81]. Both derivations will be given below for completeness.

D.5.1 Stokes' theorem methods

Relationships between potentials $A_\mu(x)$ and their field strength tensors $F_{\mu\nu}(x)$ based on the Stokes' theorem are given by expression

$$\oint_C A_\mu dx^\mu = \int_S F_{\mu\nu}d\tau^{\mu\nu}$$

where the line integral is along a closed path C and the surface integral is over a surface S around C. For our purposes it is sufficient to choose a closed path C as depicted below



Now the line integral reads

$$
\begin{aligned}
\oint_C A_\mu(x)dx^\mu &= A_\mu(x)\Delta x^\mu + A_\mu(\alpha_1 x)x^\mu\Delta\alpha_1 + A_\mu(\alpha_2 x)x^\mu\Delta\alpha_2 + \cdots \\
&\quad - A_\mu(x+\Delta x)\,(x+\Delta x)^\mu\Delta\alpha_1 - A_\mu[\alpha_1(x+\Delta x)]\,(x+\Delta x)^\mu\Delta\alpha_2 \\
&\quad - \cdots \\
&= A_\mu(x)\Delta x^\mu
\end{aligned}
$$

subject to the gauge condition $x \cdot A(x) = 0$. Note that every pair of paths such as AB and CD does not give any contribution to the integral because of their opposite directions. The final result tells us that the only path which contributes to the integral is the base Δx^μ. On the other hand the surface around the above closed path leads the surface integral into

$$
\begin{aligned}
\int_S F_{\mu\nu}(x)d\tau^{\mu\nu} &= F_{\mu\nu}(\alpha_1 x)(x^\mu\Delta\alpha_1)(\alpha_1\Delta x^\nu) + F_{\mu\nu}(\alpha_2 x)(x^\mu\Delta\alpha_2)(\alpha_2\Delta x^\nu) + \cdots \\
&= \sum_{i=1}^{N} \alpha_i\Delta\alpha_i x^\mu F_{\mu\nu}(\alpha_i x)\Delta x^\nu .
\end{aligned}
$$

If we divide the surface into infinite number of trapezoids the integral becomes

$$\int_S F_{\mu\nu}d\tau^{\mu\nu} = \int_0^1 d\alpha\alpha x^\mu F_{\mu\nu}(\alpha x)\Delta x^\nu = -\int_0^1 d\alpha\alpha x^\nu F_{\mu\nu}(\alpha x)\Delta x^\mu .$$

Equating both expressions one arrives at the result

$$A_\mu(x) = -\int_0^1 d\alpha\alpha x^\nu F_{\mu\nu}(\alpha x).$$

D.5.2 Differential geometry

The derivation is based on a choice of a region called a star-shaped region [Spi 65]. A star-shaped region $S \subset R^n$ is a set of $x \in S$ with conditions $\lambda x \in S$ for $0 \leq \lambda \leq 1$. Now if ω is a closed m-form on the star-shaped S, that is $d\omega = 0$, then, according to Poincaré lemma [Spi 65], ω is exact, i.e. ω can be written as $\omega = d\Omega$ with Ω is an $(m-1)$ form on S. In a star-shaped region Ω can be defined as

$$\Omega = I\omega = \sum_{1\leq i_1<i_2<\cdots<i_m\leq n} \sum_{r=1}^{m} (-1)^{r-1} \int_0^1 d\alpha \alpha^{m-1} x^{i_r}$$
$$\omega_{i_1\cdots i_m}(\alpha x) dx^{i_1} \wedge \cdots \wedge \hat{dx}^{i_r} \wedge \cdots \wedge dx^{i_m}$$

where the hat symbol over dx^{i_r} indicates that it is omitted. The above expression can be applied to gauge field theories since in the language of differential geometry the gauge field strength tensors are closed (see for example [Ryd 85]):

$$dF = 0$$

where

$$F = \sum_{0\leq\mu<\nu\leq 3} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

and according to the Poincaré lemma

$$F = dA$$

with

$$A = A_\mu dx^\mu.$$

In integral form

$$\begin{aligned} A = IF &= \sum_{0\leq\mu<\nu\leq 3} \int_0^1 d\alpha \alpha F_{\mu\nu}(\alpha x) \left(x^\mu dx^\nu - x^\nu dx^\mu\right) \\ &= \frac{1}{2} \int_0^1 d\alpha \alpha F_{\mu\nu}(\alpha x) \left(x^\mu dx^\nu - x^\nu dx^\mu\right) \\ &= -\int_0^1 d\alpha \alpha F_{\mu\nu}(\alpha x) x^\nu dx^\mu \end{aligned}$$

yielding

$$A_\mu(x) = -\int_0^1 d\alpha \alpha x^\nu F_{\mu\nu}(\alpha x).$$

E Fock-Schwinger Gauge Propagators

E.1 Propagators with $\mathcal{L}_{\mathcal{GF}} = -\frac{1}{\lambda}(G\cdot A)^2$

The inverse propagators are given by

$$G^{-1^{ab\mu\nu}}(x,y) = [\Box g^{\mu\nu} - \partial^\mu\partial^\nu \pm \frac{1}{\lambda}G^\mu G^\nu]\delta^{ab}(x-y).$$

The general form of the corresponding propagators must be

$$G^{ab\mu\nu}(x,y) = [Ag^{\mu\nu} + \partial^\mu\partial^\nu B + G^\mu G^\nu C + \partial^\mu G^\nu D + G^\mu\partial^\nu E]\delta^{ab}(x-y)$$

with $A, B, \cdots, E$ are in general functions of ∂^μ and G^μ. These quantities can be solved by the use of identities

$$\int dy G^{-1^{ab\mu\nu}}(x,y)G^{bc}_{\nu\alpha}(y,z) = \delta^\mu_\alpha\delta^{ac}(x-z).$$

We get, after integration over y,

$$\begin{aligned}
\delta^\mu_\alpha\delta(x-z) = \;& [\Box g^{\mu\nu} - \partial^\mu\partial^\nu \pm \tfrac{1}{\lambda}G^\mu G^\nu][Ag_{\nu\alpha} + \partial_\nu\partial_\alpha B + G_\nu G_\alpha C \\
& \qquad\qquad\qquad +\partial_\nu G_\alpha D + G_\nu\partial_\alpha E]\delta(x-y) \\
= \;& \{\Box A\delta^\mu_\alpha + \Box G^\mu G_\alpha C + \Box G^\mu\partial_\alpha E - \partial^\mu\partial_\alpha A - \partial^\mu\partial\cdot GG_\alpha C \\
& -\partial^\mu\partial\cdot G\partial_\alpha E \pm \tfrac{1}{\lambda}[G^\mu G_\alpha A + G^\mu G\cdot\partial\partial_\alpha B + G^\mu G^2 G_\alpha C \\
& \qquad\qquad +G^\mu G\cdot\partial G_\alpha D + G^\mu G^2\partial_\alpha E]\}\delta(x-z) \\
= \;& \{\Box A\delta^\mu_\alpha + G^\mu G_\alpha[\Box C \pm \tfrac{1}{\lambda}(A + G^2 C(\partial G - 3I_{FS})D - 2I_{FS}E)] \\
& +\partial^\mu G_\alpha[2I_{FS}C - (\partial G + I_{FS})C] \\
& +G^\mu\partial_\alpha[2I_{FS}C + \Box E \pm \tfrac{1}{\lambda}((\partial G - 5I_{FS})B + G^2 E)] \\
& +\partial^\mu\partial_\alpha[2I_{FS}E - A - (\partial G - I_{FS})E]\}\delta(x-z)
\end{aligned}$$

where we have used identities derived in the appendix B. We conclude that

$$\begin{aligned}
A = \Box^{-1}; \qquad\qquad C =& \;\; 0, \\
A + (\partial G - 3I_{FS})D - 2I_{FS}E =& \;\; 0, \\
(\Box \pm \tfrac{1}{\lambda}G^2)E \pm \tfrac{1}{\lambda}(\partial G - 5I_{FS})B =& \;\; 0, \\
-A - (\partial G - 3I_{FS})E =& \;\; 0.
\end{aligned}$$

Recalling identities in the appendix B we have

$$\begin{aligned}
E = & -(\partial G - 3I_{FS})^{-1}\Box^{-1} = -\Box^{-1}(\partial G - I_{FS})^{-1}, \\
D = & +(\partial G - 3I_{FS})^{-1}[-\Box^{-1} - 2I_{FS}\Box^{-1}(\partial G - I_{FS})^{-1}] \\
= & -\Box^{-1}(\partial G - I_{FS})^{-2}(\partial G + I_{FS}), \\
B = & \mp\lambda(\partial G - 5I_{FS})^{-1}(\Box \pm \tfrac{1}{\lambda}G^2)(-\Box^{-1})(\partial G - I_{FS})^{-1} \\
= & \pm\lambda\Box^{-1}(\partial G - 3I_{FS})^{-1}[\Box \pm \tfrac{1}{\lambda}(G^2 + 4I_{FS}\Box^{-1}\partial G)](\partial G - I_{FS})^{-1}.
\end{aligned}$$

Accordingly

$$\begin{aligned}
\partial_\mu\partial_\nu B = & \pm\lambda\Box^{-1}\partial_\mu\partial_\nu(\partial G - 3I_{FS})^{-1}[\Box \pm \tfrac{1}{\lambda}(G^2 + 4I_{FS}\Box^{-1}\partial G)](\partial G - I_{FS})^{-1} \\
= & \Box^{-1}(\partial G - I_{FS})^{-1}[\pm\lambda(\partial G + 3I_{FS})^{-1}\Box\partial\mu\partial_\nu + (\partial G - I_{FS})^{-1}\partial_\mu G^2\partial_\nu] + \\
& 2I_{FS}\Box^{-1}(\partial G - I_{FS})^{-2}\partial_\mu G_\nu + 4I_{FS}\Box^{-2}(\partial G + I_{FS})^{-2}(\partial G + 2I_{FS})\partial_\mu\partial_\nu \\
\partial_\mu G_\nu D = & -\partial_\mu G_\nu\Box^{-1}(\partial G - I_{FS})^{-2}(\partial G + I_{FS}) \\
= & -\Box^{-1}(\partial G - I_{FS})^{-1}\partial_\mu G_\nu - 2I_{FS}\Box^{-1}(\partial G - I_{FS})^{-2}\partial_\mu G_\nu \\
& -2I_{FS}\Box^{-2}(\partial G + I_{FS})^{-2}(\partial G + 3I_{FS})\partial_\mu\partial_\nu \\
G_\mu\partial_\nu E = & -G_\mu\partial_\nu\Box^{-1}(\partial G - I_{FS})^{-1} \\
= & -\Box^{-1}(\partial G - I_{FS})^{-1}G_\mu\partial_\nu - 2I_{FS}\Box^{-2}(\partial G + I_{FS})^{-1}\partial_\mu\partial_\nu.
\end{aligned}$$

Finally

$$\begin{aligned}
G^{ab\mu\nu}(x,y) = & [Ag^{\mu\nu} + \partial^\mu\partial^\nu B + G^\mu G^\nu C + \partial^\mu G^\nu D + G^\mu\partial^\nu E]\delta^{ab}(x-y) \\
= & \Box^{-1}\{g^{\mu\nu} - (\partial G - I_{FS})^{-1}(\partial^\mu G^\nu + G^\mu\partial^\nu) + (\partial G - I_{FS})^{-2}\partial^\mu G^2\partial^\nu \\
& \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial^\mu\partial^\nu\}\delta^{ab}(x-y).
\end{aligned}$$

E.2 Inverse propagators when $\mathcal{L}_{\mathcal{GF}} = CG\cdot A + \frac{\lambda}{2}C^2$

In this case the propagators are of the form

$$\begin{aligned}
G^{ab\mu\nu}(x,y) = & \Box^{-1}\{g^{\mu\nu} - (\partial G - I_{FS})^{-1}(\partial^\mu G^\nu + G^\mu\partial^\nu) + (\partial G - I_{FS})^{-2}\partial^\mu G^2\partial^\nu \\
& \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial^\mu\partial^\nu\}\delta^{ab}(x-y), \\
G^{ab\mu 4}(x,y) = & (\partial G - 3I_{FS})^{-1}\partial^\mu\delta^{ab}(x-y), \\
G^{ab4\mu}(x,y) = & \mp(\partial G)^{-1}\partial^\mu\delta^{ab}(x-y), \\
G^{ab44}(x,y) = & 0.
\end{aligned}$$

In general the inverse of the corresponding propagators are

$$
\begin{aligned}
G^{-1^{ab\mu\nu}}(x,y) = &\ [Ag^{\mu\nu} + (\partial^\mu G^\nu + \partial^\nu G^\mu)B + \partial^\mu\partial^\nu C + G^\mu G^\nu D]\delta^{ab}(x-y), \\
G^{-1^{ab\mu 4}}(x,y) = &\ G^\mu E\delta^{ab}(x-y), \\
G^{-1^{ab4\mu}}(x,y) = &\ G^\mu F\delta^{ab}(x-y), \\
G^{-1^{ab44}}(x,y) = &\ H\delta^{ab}(x-y),
\end{aligned}
$$

where $A, B, \cdots, H$ are in general functions of G^μ and ∂^μ. These functions can be obtained by making use of identities

$$
\int dy G^{ab}_{KL}(x,y)G^{-1^{bcLM}}(y,z) = \delta_K^M \delta^{ac}(x-z)
$$

with $K, L, M = 0, 1, \cdots, 4$. More explicitly

$$
\begin{aligned}
\delta(x-z) =& \int dy G_{4\mu}(x,y)G^{-1^{\mu 4}}(y,z), \\
0 =& \int dy G_{4\mu}(x,y)G^{-1^{\mu\nu}}(y,z), \\
0 =& \int dy [G_{\mu\nu}(x,y)G^{-1^{\nu 4}}(y,z) + G_{\mu 4}(x,y)G^{-1^{44}}(y,z)], \\
\delta^\alpha_\mu \delta(x-z) =& \int dy [G_{\mu\nu}(x,y)G^{-1^{\nu\alpha}}(y,z) + G_{\mu 4}(x,y)G^{-1^{4\alpha}}(y,z)],
\end{aligned}
$$

where $G^{abKL}(x,y) = \delta^{ab}G^{KL}(x,y)$. Now the first identity reads

$$
\delta(x-z) = \mp(\partial G)^{-1}\partial_\mu G^\mu E\delta(x-z) = \mp E\delta(x-z);
$$

thus

$$
E = \mp 1.
$$

The second identity

$$
\begin{aligned}
0 = &\ \mp(\partial G)^{-1}\partial_\mu[Ag^{\mu\nu} + (\partial^\mu G^\nu + \partial^\nu G^\mu)B + \partial^\mu\partial^\nu C + G^\mu G^\nu D]\delta^{ab}(x-z) \\
= &\ (\partial G)^{-1}[\partial^\nu(A + \partial G B + \Box C) + (\Box G^\nu B + \partial G G^\nu D)]\delta(x-z).
\end{aligned}
$$

The third identity

$$
\begin{aligned}
0 = &\ \Box^{-1}\{g^{\mu\nu} - (\partial G - I_{FS})^{-1}(\partial^\mu G^\nu + G^\mu\partial^\nu) + (\partial G - I_{FS})^{-2}\partial^\mu G^2\partial^\nu \\
&\ \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial^\mu\partial^\nu\}G^\nu E\delta(x-y) + \\
&\ (\partial G - 3I_{FS})^{-1}\partial_\mu H\delta(x-z) \\
= &\ (\partial G - 3I_{FS})^{-1}\partial_\mu(H \pm \lambda E)\delta(x-z).
\end{aligned}
$$

Hence

$$H = \mp\lambda E = \lambda.$$

The last identity

$$\begin{aligned}
\delta^\alpha_\mu = \; & \Box^{-1}\{g_{\mu\nu} - (\partial G - I_{FS})^{-1}(\partial_\mu G_\nu + G_\mu \partial_\nu) + (\partial G - I_{FS})^{-2}\partial_\mu G^2 \partial_\nu \\
& \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial_\mu\partial_\nu\}\{Ag^{\nu\alpha} + (\partial^\nu G^\alpha + \partial^\alpha G^\nu)B \\
& +\partial^\nu\partial^\alpha C + G^\nu G^\alpha D\} + (\partial G - 3I_{FS})^{-1}\partial_\mu G^\alpha F \\
= \; & (A+B)\delta^\alpha_\mu + G_\mu G^\alpha \Box(\partial G - I_{FS})^{-1}B \\
& +G_\mu\partial^\alpha[B - (\partial G - I_{FS})^{-1}(A + 2I_{FS} + \Box C)] \\
& +\partial_\mu G^\alpha\{B - (\partial G - I_{FS})^{-1}[A - B + (\partial G - 3I_{FS})B \\
& \qquad \mp\lambda(\partial G + 3I_{FS})^{-1}\Box^2 B \mp \lambda\Box D + \Box F] \\
& \qquad +(\partial G - I_{FS})^{-2}[-2I_{FS}A + G^2\Box B - 4I_{FS}B \\
& \qquad -2I_{FS}\partial G B - 2I_{FS}\Box C)\} \\
& +\partial_\mu\partial^\alpha\{C + (\partial G - 3I_{FS})^{-1}[G^2 B - (\partial G - 5I_{FS})C + (\partial G - 3I_{FS})^{-1}G^2 \\
& \qquad (A + 2I_{FS}B + \Box C) \pm \lambda(\partial G + I_{FS})^{-1}\Box A \\
& \qquad \pm 4I_{FS}\lambda(\partial G + I_{FS})^{-1}\Box B \pm \lambda(\partial G + I_{FS})^{-1}\Box^2 C \\
& \qquad \pm 2I_{FS}\lambda D + 2I_{FS}F] \\
& \qquad \pm\lambda\partial G(\partial G - 5I_{FS})^{-1}(\partial G - I_{FS})^{-1}B\}.
\end{aligned}$$

Thus we have

$$A = \Box; \qquad B = D = 0; \qquad C = -1; \qquad F = 1.$$

The second identity agrees with this result.

Now we have obtained all the quantities $A, B, \cdots, H$. These lead to the inverse propagators (which can be read off the Lagrangian in fact)

$$\begin{aligned}
G^{-1^{ab\mu\nu}}(x,y) = \; & (\Box g^{\mu\nu} - \partial^\mu\partial^\nu)\delta^{ab}(x-y), \\
G^{-1^{ab\mu 4}}(x,y) = \; & \mp G^\mu \delta^{ab}(x-y), \\
G^{-1^{ab4\mu}}(x,y) = \; & G^\mu \delta^{ab}(x-y), \\
G^{-1^{ab44}}(x,y) = \; & \lambda\delta^{ab}(x-y).
\end{aligned}$$

E.3 Symmetry properties of propagators

Consider $G^{ab\mu\nu}(x,x')$

$$G^{ab\mu\nu}(x,y) = \Box^{-1}\{g^{\mu\nu} - (\partial G - I_{FS})^{-1}(\partial^\mu G^\nu + G^\mu \partial^\nu) + (\partial G - I_{FS})^{-2}\partial^\mu G^2 \partial^\nu \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial^\mu\partial^\nu\}\delta^{ab}(x-y).$$

We have

$$G^{ab\mu\nu}(-x,-x') = G^{ab\mu\nu}(x,x')$$

because $\Box^{-1}$, $(\partial G - I_{FS})^{-1}\partial_\mu G_\nu$ and $\partial_\mu G^2\partial_\nu$ are invariant under transformations $x \to -x$ and $x' \to -x'$. Also

$$G^{ab}_{\nu\mu}(x',x) = \Box'^{-1}\{g_{\mu\nu} - (\partial' G' - I_{FS})^{-1}(\partial'_\nu G'_\mu + G'_\nu \partial'_\mu) + (\partial' G' - I_{FS})^{-2}\partial'_\nu G'^2 \partial'_\mu \pm\lambda(\partial' G' - I_{FS})^{-1}(\partial' G' + 3I_{FS})^{-1}\Box'\partial'_\nu\partial'_\mu\}\delta^{ab}(x-x').$$

Rearranging and making use of identities given in the appendix B,

$$\begin{aligned}
G^{ab}_{\nu\mu}(x',x) = \ & \{\Box'^{-1} g_{\mu\nu} - (\partial' G' - 3I_{FS})^{-1}[\partial'_\nu(G'_\mu\Box'^{-1} - 2I_{FS}\Box'^{-2}\partial'_\mu) \\
& \qquad + (G'_\nu\Box'^{-1} - 2I_{FS}\Box'^{-2}\partial'_\nu)\partial'_\mu] \\
& + (\partial' G' - 3I_{FS})^{-2}\partial'_\nu[G'^2\Box'^{-1} - 4I_{FS}G'\partial'\Box'^{-2}]\partial'_\mu \\
& \pm\lambda(\partial' G' - 3I_{FS})^{-1}(\partial' G' + I_{FS})^{-1}\partial'_\nu\partial'_\mu\}\delta^{ab}(x-x') \\
= \ & \Box^{-1}\{g_{\mu\nu} \mp [\pm G_\mu\partial_\nu \pm \partial_\mu G_\nu - 4I_{FS}\Box^{-1}\partial_\mu\partial_\nu]\cdot \\
& \qquad (\partial G \mp 3I_{FS} - 4I_{FS})^{-1} \\
& + [\partial_\mu G^2\partial_\nu \mp 4I_{FS}\Box^{-1}\partial_\mu\partial G\partial_\nu](\pm)^2(\partial G \mp 3I_{FS} - 4I_{FS})^{-2} \\
& \pm\lambda(\partial G - 3I_{FS})^{-1}(\partial G + I_{FS})^{-1}\partial_\mu\partial_\nu\}\delta^{ab}(x-x') \\
= \ & \Box^{-1}\{g_{\mu\nu} - [G_\mu\partial_\nu + \partial_\mu G_\nu \mp 4I_{FS}\Box^{-1}\partial_\mu\partial_\nu](\partial G - I_{FS})^{-1} \\
& + [\partial_\mu G^2\partial_\nu \mp 4I_{FS}\Box^{-1}\partial_\mu\partial G\partial_\nu](\partial G - I_{FS})^{-2} \\
& \pm\lambda(\partial G - I_{FS})^{-1}(\partial G + 3I_{FS})^{-1}\Box\partial_\mu\partial_\nu\}\delta^{ab}(x-x') \\
= \ & G^{ab}_{\mu\nu}(x,x').
\end{aligned}$$

Such symmetry also holds for the nonphysical propagators

$$\begin{aligned}
G^{ab\mu 4}(x',x) = \ & (\partial' G' - 3I_{FS})^{-1}\partial'^\mu\delta^{ab}(x-x') \\
= \ & \mp(\partial G)^{-1}\partial^\mu\delta^{ab}(x-x') = G^{ab4\mu}(x,x').
\end{aligned}$$

However

$$G^{ab\mu 4}(-x,-x') = -G^{ab\mu 4}(x,x').$$

Hence we conclude that the symmetry properties of the propagators are

$$\begin{aligned}
G^{abKL}(x,y) &= \ G^{abLK}(y,x), \\
G^{ab\mu\nu}(x,y) &= +G^{ab\mu\nu}(-x,-y), \\
G^{ab\mu 4}(x,y) &= -G^{ab\mu 4}(-x,-y), \\
G^{ab4\mu}(x,y) &= -G^{ab4\mu}(-x,-y).
\end{aligned}$$

E.4 Fock-Schwinger propagators in momentum space

The non-Feynman gauge part of the Fock-Schwinger gauge propagators in the case $\lambda \to 0$

$$G'_{\mu\nu}(x,x') = \Box^{-1}[-(\partial x-1)^{-1}(\partial_\mu x_\nu + x_\mu \partial_\nu) + (\partial x - 1)^{-2}\partial_\mu x^2 \partial_\nu]\delta(x-x')$$

can be written in the form of derivatives ∂_μ and ∂'_μ of some functions as follows:

$$\begin{aligned}
G'_{\mu\nu}(x,x') = &\ \Box^{-1}[-\partial_\mu x_\nu(\partial x-1)^{-1} + (\partial x-1)^{-1}x_\mu\partial'_\nu + \tfrac{1}{2}\partial_\mu x^2\partial_\nu(\partial x-1)^{-2} \\
&-\tfrac{1}{2}(\partial x-1)^{-2}\partial_\mu x^2\partial'_\nu\,]\delta(x-x') \\
= &\ +\partial_\mu[-\Box^{-1}x_\nu(\partial x-1)^{-1}\delta(x-x') + \tfrac{1}{2}\Box^{-1}x^2\partial_\nu(\partial x-1)^{-2}\delta(x-x')] \\
&+\partial'_\nu[\Box^{-1}(\partial x-1)^{-1}x_\mu\delta(x-x') - \tfrac{1}{2}\Box^{-1}(\partial x-1)^{-2}\partial_\mu x^2\delta(x-x')] \\
= &\ \partial_\mu f_{1\nu}(x,x') + \partial'_\nu f_{2\mu}(x,x')
\end{aligned}$$

where

$$\begin{aligned}
f_{1\mu}(x,x') = &\ -\Box^{-1}x_\mu(\partial x-1)^{-1}\delta(x-x') + \tfrac{1}{2}\Box^{-1}x^2\partial_\mu(\partial x-1)^{-2}\delta(x-x') \\
f_{2\mu}(x,x') = &\ +\Box^{-1}(\partial x-1)^{-1}x_\mu\delta(x-x') - \tfrac{1}{2}\Box^{-1}(\partial x-1)^{-2}\partial_\mu x^2\delta(x-x') \\
= &\ -\Box'^{-1}x'_\mu(\partial' x'-1)^{-1}\delta(x-x') + \tfrac{1}{2}\Box'^{-1}x'^2\partial'_\mu(\partial' x'-1)^{-2}\delta(x-x') \\
= &\ f_{1\mu}(x',x).
\end{aligned}$$

Now

$$
\begin{aligned}
\partial'_\nu f_{1\mu}(x',x) = & \ \partial'_\nu f_{2\mu}(x,x') \\
= & \ \tfrac{1}{2}[2\Box^{-1}\partial'_\nu x'_\mu(\partial x-1)^{-1}+\Box^{-1}\partial'_\nu x'^2\partial'_\mu(\partial x-1)^{-2}]\delta(x-x') \\
= & \ \tfrac{1}{2}[\Box^{-1}(\partial'_\nu x'_\mu + x'_\mu\partial'_\nu + g_{\mu\nu})(\partial x-1)^{-1} \\
& +\Box^{-1}(x'^2\partial'_\nu\partial'_\mu + 2x'_\nu\partial'_\mu)(\partial x-1)^{-2}]\delta(x-x') \\
= & \ \tfrac{1}{2}[\Box^{-1}(-x_\mu\partial_\nu + x'_\mu\partial'_\nu + g_{\mu\nu})(\partial x-1)^{-1} \\
& +\Box^{-1}(x'^2\partial'_\nu\partial'_\mu + 2x'_\nu\partial'_\mu)(\partial x-1)^{-2}]\delta(x-x') \\
= & \ \tfrac{1}{2}[(-x_\mu\Box^{-1}+2\Box^{-2}\partial_\mu)\partial_\nu(\partial x-1)^{-1}+\Box^{-1}(x'_\mu\partial'_\nu+g_{\mu\nu})(\partial x-1)^{-1} \\
& +\Box^{-1}(x'^2\partial'_\nu\partial'_\mu + 2x'_\nu\partial'_\mu)(\partial x-1)^{-2}]\delta(x-x') \\
= & \ [\tfrac{1}{2}x_\mu\Box^{-1}\partial_\nu(\partial' x'-3)^{-1}+\Box^{-2}\partial_\mu\partial_\nu(\partial x-1)^{-1}]\delta(x-x') \\
& +\tfrac{1}{2}\Box^{-1}[(x'_\mu\partial'_\nu+g_{\mu\nu})(\partial x-1)^{-1}+(x'^2\partial'_\nu\partial'_\mu+2x'_\nu\partial'_\mu)(\partial x-1)^{-2}] \\
& \qquad \delta(x-x') \\
= & \ \Box^{-2}\partial_\mu\partial_\nu(\partial x-1)^{-1}\delta(x-x')+\tfrac{1}{2}x_\mu\Box^{-1}\partial_\nu G_3(x',x) \\
& +\tfrac{1}{2}\Box^{-1}(x'_\mu\partial'_\nu+g_{\mu\nu})G_1(x,x')+\tfrac{1}{2}\Box^{-1}(x'^2\partial'_\nu\partial'_\mu+2x'_\nu\partial'_\mu)H_1(x,x')
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
(\partial x-n)G_n(x,x') = & \ \delta(x-x') \\
(\partial x-n)^{-1}G_n(x,x') = & \ H_n(x,x').
\end{aligned}
$$

$G_n(x,x')$ and $H_n(x,x')$ may be obtained as follows. First of all we rewrite operator $(\partial x-n)$ which acts on $G_n(x,x')$ as $[x\partial-(n-4)]$. Then we introduce a parameter β by replacing $x\to\beta x$. This gives

$$[x\partial-(n-4)]G_n(\beta x,x') = \delta(\beta x-x').$$

Now $x\partial$ may be replaced by $\beta\frac{d}{d\beta}$

$$[\beta\frac{d}{d\beta}-(n-4)]G_n(\beta x,x') = \delta(\beta x-x').$$

Furthermore, we can change to parameter $\alpha=\frac{1}{\beta}$

$$-[\alpha\frac{d}{d\alpha}+(n-4)]G_n(\frac{x}{\alpha},x') = \alpha^4\delta(x-\alpha x').$$

The left-hand side may be reduced to single term by multiplying α^{n-5} into the equation

$$\frac{d}{d\alpha}\alpha^{n-4}G_n(\frac{x}{\alpha},x') = -\alpha^{n-1}\delta(x-\alpha x').$$

The last steps to obtain $G_n(x,x')$ come by multiplying both sides by $e^{-\alpha\delta}$ with $\delta \to +0$ and then integrating over α from 1 to $+\infty$. We have

$$G_n(x,x') = \int_1^{\infty} d\alpha\, e^{-\alpha\delta}\alpha^{n-1}\delta(x-\alpha x').$$

$H_n(x,x')$ follows immediately

$$\begin{aligned} H_n(x,x') = &\ (\partial x - n)^{-1}G_n(x,x') \\ = &\ \textstyle\int_1^{\infty} d\alpha\, e^{-\alpha\delta}\alpha^{n-1}(\partial x - n)^{-1}\delta(x-\alpha x') \\ = &\ \textstyle\int_1^{\infty} d\alpha\, e^{-\alpha\delta}\alpha^{n-1}G_n(x,\alpha x'). \end{aligned}$$

Particular n-values are

$$\begin{aligned} G_1(x,x') = &\ \textstyle\int_1^{\infty} d\alpha\, e^{-\alpha\delta}\delta(x-\alpha x') = \int \bar{d}k \int_1^{\infty} d\alpha\, e^{-\alpha\delta}e^{-ik(x-\alpha x')} \\ = &\ -\textstyle\int \bar{d}k \frac{e^{-ik(x-x')}}{ikx'} \\ H_1(x,x') = &\ \textstyle\int_1^{\infty} d\alpha\, e^{-\alpha\delta}G_1(x,\alpha x') = -\int_1^{\infty} d\alpha\, e^{-\alpha\delta}\int \bar{d}k \frac{e^{-ik(x-\alpha x')}}{i\alpha kx'} \\ = &\ \textstyle\int \bar{d}k \int_1^{\infty} d\alpha e^{-\alpha\delta}\ln\alpha\, e^{-ik(x-\alpha x')} \\ G_3(x,x') = &\ \textstyle\int_1^{\infty} d\alpha\, e^{-\alpha\delta}\alpha^2\delta(x-\alpha x') = \int \bar{d}k \int_1^{\infty} d\alpha e^{-\alpha\delta}\alpha^2\, e^{-ik(x-\alpha x')}. \end{aligned}$$

Now we have

$$\begin{aligned} \partial'_\nu f_{2\mu}(x,x') = &\ \Box^{-2}\partial_\mu\partial_\nu(\partial x - 1)^{-1}\delta(x-x') \\ &+\tfrac{1}{2}\textstyle\int \frac{\bar{d}k}{-k^2}\int_1^{\infty} d\alpha\, e^{-\alpha\delta}i\alpha x_\mu k_\nu e^{-ik(x'-\alpha x)} \\ &+\tfrac{1}{2}\textstyle\int \frac{\bar{d}k}{-k^2}\int_1^{\infty} d\alpha\, e^{-\alpha\delta}[i\alpha x'_\mu k_\nu + g_{\mu\nu} - x'^2 k_\nu k_\mu \alpha^2 \ln\alpha \\ &\qquad\qquad +2i\alpha x'_\nu k_\mu \ln\alpha]\, e^{-ik(x-\alpha x')} \\ = &\ \Box^{-2}\partial_\mu\partial_\nu(\partial x - 1)^{-1}\delta(x-x') \\ &-\tfrac{1}{2}\textstyle\int \frac{\bar{d}k}{-k^2}\int_1^{\infty} d\alpha\, e^{-\alpha\delta}i\alpha x_\mu k_\nu e^{-ik(\alpha x-x')} \\ &+\tfrac{1}{2}\textstyle\int \frac{\bar{d}k}{-k^2}\int_1^{\infty} d\alpha\, e^{-\alpha\delta}[g_{\mu\nu} + i\alpha x'_\mu k_\nu + 2i\alpha x'_\nu k_\mu \ln\alpha \\ &\qquad\qquad -x'^2 k_\mu k_\nu \alpha^2 \ln\alpha]\, e^{-ik(x-\alpha x')}. \end{aligned}$$

Since $\partial_\mu f_{1\nu}(x,x')$ can be obtained from $\partial'_\nu f_{2\mu}(x,x')$ by replacements $x \leftrightarrow x'$ and

$\mu \leftrightarrow \nu$, the propagator $G'_{\mu\nu}(x,x')$ becomes

$$\begin{aligned} G'_{\mu\nu}(x,x') = & \ [\Box^{-2}\partial_\mu\partial_\nu(\partial x-1)^{-1} + \Box'^{-2}\partial'_\mu\partial'_\nu(\partial' x'-1)^{-1}]\delta(x-x') \\ & +\tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\, e^{-\alpha\delta} g_{\mu\nu}(e^{-ik(x-\alpha x')} + e^{-ik(\alpha x - x')}) \\ & +\tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\, e^{-\alpha\delta}[i\alpha x'_\mu k_\nu + i\alpha(1+2\ln\alpha)x'_\nu k_\mu \\ & \qquad -x'^2 k_\mu k_\nu \alpha^2 \ln\alpha]\, e^{-ik(x-\alpha x')} \\ & +\tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\, e^{-\alpha\delta}[-i\alpha x_\nu k_\mu - i\alpha(1+2\ln\alpha)x_\mu k_\nu \\ & \qquad -x'^2 k_\mu k_\nu \alpha^2 \ln\alpha]\, e^{-ik(\alpha x - x')}. \end{aligned}$$

The first term vanishes because of the equality

$$\begin{aligned} \Box'^{-2}\partial'_\mu\partial'_\nu(\partial' x'-1)^{-1}\delta(x-x') = & \ -\Box'^{-2}\partial'_\mu\partial'_\nu(\partial x-3)^{-1}\delta(x-x') \\ = & \ -(\partial x-3)^{-1}\Box'^{-2}\partial_\mu\partial_\nu\delta(x-x') \\ = & \ -\Box^{-2}\partial_\mu\partial_\nu(\partial x-1)^{-1}\delta(x-x'). \end{aligned}$$

Hence

$$\begin{aligned} G'_{\mu\nu}(x,x') = & \ \tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\, e^{-\alpha\delta} g_{\mu\nu}(e^{-ik(x-\alpha x')} + e^{-ik(\alpha x - x')}) + \\ & \tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\, e^{-\alpha\delta}[i\alpha x'_\mu k_\nu + i\alpha(1+2\ln\alpha)x'_\nu k_\mu \\ & \qquad -x'^2 k_\mu k_\nu \alpha^2 \ln\alpha]\, e^{-ik(x-\alpha x')} \\ & \tfrac{1}{2}\int \tfrac{\bar{d}k}{-k^2}\int_1^\infty d\alpha\, e^{-\alpha\delta}[-i\alpha x_\nu k_\mu - i\alpha(1+2\ln\alpha)x_\mu k_\nu \\ & \qquad -x'^2 k_\mu k_\nu \alpha^2 \ln\alpha]\, e^{-ik(\alpha x - x')}. \end{aligned}$$

More neatly rewrite

$$G'_{\mu\nu}(x,x') = \int \frac{\bar{d}k}{-k^2}\int_1^\infty d\beta\,[f_\nu(\beta,k,\partial_k,x')k_\mu e^{-i\beta kx} + g_\mu(\beta,k,\partial_k,x)k_\nu e^{i\beta kx'}]$$

where

$$\begin{aligned} f_\mu(\beta,k,\partial_k,x) = & \ \tfrac{1}{2}e^{-\beta\delta}\Big\{e^{ikx}\tfrac{\partial}{\partial k^\mu} + \delta(\beta-1)\int_1^\infty d\alpha\, e^{-\alpha\delta}\cdot \\ & \qquad e^{i\alpha kx}[i\alpha(1+2\ln\alpha)x_\mu - x^2 k_\mu\alpha^2\ln\alpha]\Big\} \\ g_\mu(\beta,k,\partial_k,x) = & \ \tfrac{1}{2}e^{-\beta\delta}\Big\{e^{-ikx}\tfrac{\partial}{\partial k^\mu} + \delta(\beta-1)\int_1^\infty d\alpha\, e^{-\alpha\delta}\cdot \\ & \qquad e^{-i\alpha kx}[-i\alpha(1+2\ln\alpha)x_\mu - x^2 k_\mu\alpha^2\ln\alpha]\Big\}. \end{aligned}$$

It is readily verified that

$$\begin{aligned} g_\mu(\beta,-k,-\partial_k,x) = & \ -f_\mu(\beta,k,\partial_k,x) \\ g_\mu(\beta,k,\partial_k,-x) = & \ f_\mu(\beta,k,\partial_k,x). \end{aligned}$$

F Local Gauge and BRST Invariances

F.1 The local gauge invariance of Lagrangians

The infinitesimal local gauge transformatians

$$\begin{aligned}
\psi'(x) =& \ \psi(x) - igT^a\Lambda^a(x)\psi(x) \\
\overline{\psi'}(x) =& \ \overline{\psi}(x) + ig\overline{\psi}(x)T^a\Lambda^a(x) \\
A'_\mu(x) =& \ A_\mu(x) - T^a D^{ab}_\mu \Lambda^b(x) \\
=& \ UA_\mu U^* - \tfrac{i}{g}(\partial_\mu U)U^*
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
U(x) =& \ 1 - igT^a\Lambda^a(x) \\
D_\mu =& \ \partial_\mu - igT^a A^a_\mu \\
D^{ab}_\mu =& \ \delta^{ab}\partial_\mu - gf^{abc}A^c_\mu
\end{aligned}$$

lead to the field strength $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ and the covariant derivative of the fermion field $D_\mu\psi$ which transform into

$$\begin{aligned}
F'_{\mu\nu} =& \ UF_{\mu\nu}U^* \\
D'_\mu\psi' =& \ UD_\mu\psi.
\end{aligned}$$

It follows that quantities such as $F^{a\mu\nu}F^a_{\mu\nu}$, $\overline{\psi}\psi$ and $\overline{\psi}D_\mu\psi$ are invariant. Accordingly the quark-gluon Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi$$

is invariant under the local gauge transformations (1).

F.2 The BRST invariance of Lagrangians

Consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathcal{GF}} + \mathcal{L}_{\mathcal{FP}} \tag{2}$$

with

$$\begin{aligned}
\mathcal{L}_0 =& \ -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \\
\mathcal{L}_{\mathcal{GF}} =& \ CG\cdot A + \tfrac{\lambda}{2}C^2 \\
\mathcal{L}_{\mathcal{FP}} =& \ -\chi^{*^a}G^\mu D^{ab}_\mu \chi^b
\end{aligned}$$

We wish to prove the invariance of the above Lagrangian under infinitesimal BRST transformations

$$\begin{aligned}
\delta\psi(x) &= ig\theta T^a\chi^a(x)\psi(x) \\
\delta\overline{\psi}(x) &= -ig\overline{\psi}(x)\theta T^a\chi^a(x) \\
\delta A^a_\mu(x) &= \theta D^{ab}_\mu\chi^b(x) \\
\delta\chi^a(x) &= -\tfrac{1}{2}g\theta f^{abc}\chi^b(x)\chi^c(x) \\
\delta\chi^{*^a}(x) &= \theta C^a(x) \\
\delta C^a(x) &= -\tfrac{n}{\lambda}\theta G^\mu D^{ab}_\mu\chi^b
\end{aligned} \tag{3}$$

where $n = 0$ when C is an auxiliary field, and $n = 1$ when

$$C = -\frac{1}{\lambda}G\cdot A^a, \tag{4}$$

i.e. when $\mathcal{L}_{\mathcal{GF}} = -\frac{1}{2\lambda}(G\cdot A)^2$.

As a matter of fact the first three transformations are essentially the local gauge transformations with $\Lambda = -\theta\chi$. Since $\mathcal{L}_0$ is local gauge invariant it is also BRST invariant. Thus we need only to prove that $\mathcal{L}_{\mathcal{GF}} + \mathcal{L}_{\mathcal{FP}}$ is BRST invariant. The variation of $\mathcal{L}_{\mathcal{GF}} + \mathcal{L}_{\mathcal{FP}}$ under the BRST transformations is given by

$$\begin{aligned}
\delta\mathcal{L}_{\mathcal{GF}} + \delta\mathcal{L}_{\mathcal{FP}} &= \delta C^a(G\cdot A^a + \lambda C^a) + C^aG^\mu\delta A^a_\mu - (\delta\chi^{*^a})G^\mu D^{ab}_\mu\chi^b \\
&\quad -\chi^{*^a}G^\mu\delta(D^{ab}_\mu\chi^b) \\
&= \delta C^a(G\cdot A^a + \lambda C^a) - \chi^{*^a}G^\mu\delta(D^{ab}_\mu\chi^b) \\
&= -\left(\tfrac{n\theta}{\lambda}G\cdot A^a + n\theta C^a\right)G^\mu D^{ab}_\mu\chi^b - \chi^{*^a}G^\mu\delta(D^{ab}_\mu\chi^b).
\end{aligned} \tag{5}$$

Since $D^{ab}_\mu\chi^b$ is BRST invariant, namely

$$\begin{aligned}
\delta(D^{ab}_\mu\chi^b) &= D^{ab}_\mu\delta\chi^b - gf^{abc}(\delta A^c_\mu)\chi^b \\
&= -\tfrac{1}{2}g\theta f^{acd}(\partial_\mu\chi^c)\chi^d - \tfrac{1}{2}g\theta f^{acd}\chi^c\partial_\mu\chi^d \\
&\quad -g\theta f^{acd}(\partial_\mu\chi^d)\chi^c + \tfrac{1}{2}g^2\theta\left(f^{cdb}f^{bea} + 2f^{acb}f^{bde}\right)A^e_\mu\chi^c\chi^d \\
&= +g\theta f^{acd}(\partial_\mu\chi^d)\chi^c - g\theta f^{acd}(\partial_\mu\chi^d)\chi^c \\
&\quad +\tfrac{1}{2}g^2\theta\left(f^{cdb}f^{bea} + f^{acb}f^{bde} + f^{adb}f^{bec}\right)A^e_\mu\chi^c\chi^d \\
&= 0
\end{aligned} \tag{6}$$

upon remembering the Jacobi identity

$$f^{cdb}f^{bea} + f^{acb}f^{bde} + f^{adb}f^{bec} = 0$$

the equation (5) becomes

$$\delta\mathcal{L}_{\mathcal{GF}} + \delta\mathcal{L}_{\mathcal{FP}} = -\left(\frac{n\theta}{\lambda}G \cdot A^a + n\theta C^a\right) G^\mu D_\mu^{ab}\chi^b. \tag{7}$$

It turns out that when C^a is a (auxiliary) field and is independent of A_μ^a, $\delta\mathcal{L}_{\mathcal{GF}} + \delta\mathcal{L}_{\mathcal{FP}}$ vanishes since in this case $n = 0$. The same conclusion also holds when we choose (4). In the latter case n must be equal to 1 in order to match δC^a in (3) and δC^a derived from (4). Thus we have proved that the Lagrangian (2) is invariant under the BRST transformations (3).

F.3 BRST-nilpotencies

Other quantities which are BRST invariant are $f^{abc}\chi^b\chi^c$, $\chi\psi$ and $\overline{\psi}\chi$. The proofs are as follows

$$\begin{aligned} \delta(f^{abc}\chi^b\chi^c) = & \; f^{abc}(\delta\chi^b)\chi^c + f^{abc}\chi^b\delta\chi^c \\ = & \; -\tfrac{1}{2}gf^{abc}f^{bde}\theta\chi^d\chi^e\chi^c - \tfrac{1}{2}gf^{abc}f^{cde}\chi^b\theta\chi^d\chi^e \\ = & \; g\theta f^{abc}f^{cde}\chi^b\chi^d\chi^e = 0 \end{aligned} \tag{8}$$

because

$$\begin{aligned} -f^{abc}f^{cde}\chi^b\chi^d\chi^e = & \; \left(f^{adc}f^{ceb} + f^{aec}f^{cbd}\right)\chi^b\chi^d\chi^e \\ = & \; f^{abc}f^{cde}\left(\chi^b\chi^d\chi^e + \chi^b\chi^d\chi^e\right) \\ = & \; 2f^{abc}f^{cde}\chi^b\chi^d\chi^e \end{aligned}$$

$$\begin{aligned} \delta(\chi\psi) = & \; T^a(\delta\chi^a)\psi + T^a\chi^a\delta\psi \\ = & \; \tfrac{1}{2}g\theta[T^b, T^c]\chi^b\chi^c\psi - ig\theta T^aT^b\chi^a\chi^b\psi \\ = & \; ig\theta T^bT^c\chi^b\chi^c\psi - ig\theta T^bT^c\chi^b\chi^c\psi = 0. \end{aligned} \tag{9}$$

The variation of $\overline{\psi}\chi$ is similar to that of $\chi\psi$, thus

$$\delta(\overline{\psi}\chi) = 0. \tag{10}$$

Results (6-10) establish the nilpotency of the BRST transformations

$$\begin{aligned}
\delta^2 A^a_\mu =& \ \theta\delta(D^{ab}_\mu \chi^b) = 0 \\
\delta^2 C^a =& \ -\tfrac{n\theta}{\lambda} G^\mu \delta(D^{ab}_\mu \chi^b) = 0 \\
\delta^2 \psi =& \ ig\theta\delta(\chi\psi) = 0 \\
\delta^2 \overline{\psi} =& \ -ig\theta\delta(\overline{\psi}\chi) = 0 \\
\delta^2 \chi^a =& \ -\tfrac{1}{2} g\theta\delta(f^{abc}\chi^b\chi^c) = 0 \\
\delta^2 \chi^{*^a} =& \ \theta\delta C^a = 0.
\end{aligned}$$

F.4 Invariance of integral measures

Let us consider two integral measures

$$\begin{aligned}
&(1) \quad \mathcal{D}[A\overline{\psi}\psi C] \\
&(2) \quad \mathcal{D}[A\overline{\psi}\psi\chi\chi^* C].
\end{aligned}$$

The former will be related to the local gauge symmetry while the latter will refer to the BRST transformation.To check the invariance of the above integral measures under their corresponding transformations we need only to prove the unity of their corresponding Jacobians. Recalling the infinitesimal local gauge transformations (1) the Jacobian related to the first integral measure is

$$\begin{aligned}
&\frac{\partial(A'^a_\mu(x), \overline{\psi}'(x), \psi'(x), C'^a(x))}{\partial(A^{b\nu}(y), \overline{\psi}(y), \psi(y), C^b(y))} \\
=& \ \det\Big\{[g_{\mu\nu}(\delta^{ab} - gf^{abc}\Lambda^c)\delta(x-y)][(1+ig\Lambda)\delta(x-y)]\times \\
& \quad [(1-ig\Lambda)\delta(x-y)][\delta^{ab}(x-y)]\Big\}
\end{aligned}$$

which is independent of Λ and is equal to $\delta(x-y)$. Hence $\mathcal{D}[A\overline{\psi}\psi C]$ is gauge invariant. Similarly by recalling the BRST transformations (3) the Jacobians for the last integral measure is given by

$$\begin{aligned}
&\frac{\partial(A'^a_\mu(x), \overline{\psi}'(x), \psi'(x), \chi'^a(x), \chi^{*'^a}(x), C'^a(x))}{\partial(A^{b\nu}(y), \overline{\psi}(y), \psi(y), \chi(y), \chi^*(y), C^b(y))} \\
=& \ \det\Big\{[g_{\mu\nu}(\delta^{ab} + gf^{abc}\chi^c)\delta(x-y)][(1-igT^c\theta\chi^c)\delta(x-y)]\times \\
& \quad [(1+igT^c\theta\chi^c)\delta(x-y)][(\delta^{ab} + g\theta f^{abc}\chi^c)\delta^{ab}(x-y)]^2[\delta^{ab}(x-y)]\Big\}.
\end{aligned}$$

This determinant is also unity and therefore $\mathcal{D}[A\overline{\psi}\psi\chi\chi^* C]$ is BRST invariant.

G Some Details of Perturbation Calculations

All diagrams considered in scalar and spinor quantum electrodynamics are linear in $G_{\mu\nu}(x,y)$. This linearity enables us to write each of the diagrams into two terms: the $G_{F\mu\nu}(x,y)$ and the $G'_{\mu\nu}(x,y)$ terms. Since the familiar Feynman diagram are assumed to have been calculated the whole propagator $G_{\mu\nu}(x,y)$ will not be taken into account but, rather, the corrections $G'_{\mu\nu}(x,y)$.

In quark-gluon vertex corrections, however, the full $G_{\mu\nu}(x,y)$ must be used rather than $G'_{\mu\nu}(x,y)$ because diagrams contributed by the three and four-gluon vertices are not linear but quadratic in $G_{\mu\nu}(x,y)$, producing terms which consist of multiplication of $G_{F\mu\nu}(x,y)$ and $G'_{\mu\nu}(x,y)$ that cannot be neglected.

Perturbation calculations in scalar and spinor quantum electrodynamics will be carried out in "momentum space" as well as in coordinate space. In quantum chromodynamics, on the other hand, we will only consider calculations in coordinate space due to difficulties in combining the three-gluon momentum vertex with $G'_{\mu\nu}(x,y)$.

G.1 Scalar Quantum Electrodynamics

G.1.1 Momentum Space

Meson-meson scattering (Born term) (Figure 5.18)

$$\begin{aligned}
S' = & -ie^2 \int dxdy(p_1+p_2)^\mu (q_1+q_2)^\nu G'_{\mu\nu}(x,y) e^{ix(p_2-p_1)+iy(q_2-q_1)} \\
= & -ie^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int dx \int dy (p_1+p_2)^\mu (q_1+q_2)^\nu [g_\nu(\beta,k,\partial_k,y) k_\mu e^{i\beta kx} + \\
& \qquad (\mu \leftrightarrow \nu,\ x \leftrightarrow y)] e^{ix(p_2-p_1)+iy(q_2-q_1)} \\
= & -ie^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int dy (p_1+p_2)^\mu (q_1+q_2)^\nu [g_\nu(\beta,k,\partial_k,y) k_\mu \cdot \\
& \qquad \bar{\delta}(p_2-p_1+\beta k) e^{iy(q_2-q_1)} + (\mu \leftrightarrow \nu,\ p \leftrightarrow q)] \\
= & -ie^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int dy [(q_1+q_2)^\nu g_\nu(\beta,k,\partial_k,y)(p_1^2-p_2^2) \cdot \\
& \qquad \bar{\delta}(p_2-p_1+\beta k) e^{iy(q_2-q_1)} + (p \leftrightarrow q)] \\
\overset{ms}{=} & \ 0.
\end{aligned}$$

Meson self-energy (Figures 5.19-5.20)

$$
\begin{aligned}
S_1' = &\; e^2 \int dx \int dy \int \bar{d}p (p_2+p)^\mu (p+p_1)^\nu (p^2-m^2+i\epsilon)^{-1} G'_{\mu\nu}(x,y) \\
&\qquad e^{ix(p_2-p)+iy(p-p_1)} \\
= &\; e^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int dx \int dy \int \bar{d}p (p_2+p)^\mu (p+p_1)^\nu (p^2-m^2+i\epsilon)^{-1} \cdot \\
&\; [g_\nu(\beta,k,\partial_k,y) k_\mu e^{i\beta kx} + (x,\mu \leftrightarrow y,\nu)] e^{ix(p_2-p)+iy(p-p_1)} \\
= &\; e^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int dy \int dp (p_2+p)^\mu (p+p_1)^\nu (p^2-m^2+i\epsilon)^{-1} [g_\nu(\beta,k,\partial_k,y) \cdot \\
&\; k_\mu \cdot \delta(p_2-p+\beta k) e^{iy(p-p_1)} + (p \leftrightarrow -p,\ p_1 \leftrightarrow -p_2,\ \mu \leftrightarrow \nu)] \\
= &\; e^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int dy \int dp (p^2-m^2+i\epsilon)^{-1} [g_\nu(\beta,k,\partial_k,y)(p+p_1)^\nu (p^2-p_2^2) \cdot \\
&\; \delta(p_2-p+\beta k) e^{iy(p-p_1)} + (p \leftrightarrow -p,\ p_1 \leftrightarrow -p_2)] \\
\overset{ms}{=} &\; e^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int dy [g_\nu(\beta,k,\partial_k,y)(p_2+p_1+\beta k)^\nu e^{iy(p_2-p_1+\beta k)} \\
&\qquad +(p_1 \leftrightarrow -p_2)] \\
= &\; 2e^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int dy g_\nu(\beta,k,\partial_k,y) k^\nu e^{iy(p_2-p_1+\beta k)}.
\end{aligned}
$$

$$
\begin{aligned}
S_2' = &\; -e^2 \int dy e^{iy(p_2-p_1)} g^{\mu\nu} G'_{\mu\nu}(y,y) \\
= &\; -2e^2 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int dy g_\nu(\beta,k,\partial_k,y) k^\nu e^{iy(p_2-p_1+\beta k)}
\end{aligned}
$$

$$
S' = S_1' + S_2' \overset{ms}{=} 0.
$$

Vertex corrections (Figures 5.21-5.29)

$$
\begin{aligned}
S_1' = &\; e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} (p_2+p)^\mu (p+q)^\sigma (q+p_1)^\nu A_\sigma(z) \\
&\qquad G'_{\mu\nu}(x,y) e^{ix(p_2-p)+iy(q-p_1)+iz(p-q)} \\
= &\; e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta A_\sigma(z)(p+q)^\sigma \\
&\; (p_2+p)^\mu (q+p_1)^\nu [g_\nu(\beta,k,\partial_k,y) k_\mu e^{i\beta kx} + (x \leftrightarrow y,\ \ \mu \leftrightarrow \nu)] \\
&\qquad e^{ix(p_2-p)+iy(q-p_1)+iz(p-q)} \\
= &\; e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}y \int \bar{d}z \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dq}{q^2-m^2+i\epsilon} A_\sigma(z) g_\nu(\beta,k,\partial_k,y) \\
&\; [(p^2-p_2^2)(p+q)^\sigma (q+p_1)^\nu \delta(p_2-p+\beta k) e^{iy(q-p_1)+iz(p-q)} \\
&\qquad -(p \leftrightarrow -q,\ \ p_1 \leftrightarrow -p_2)]
\end{aligned}
$$

$$\begin{aligned}
&\overset{ms}{=} e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}y \int \bar{d}z \int \frac{dq}{q^2-m^2+i\epsilon} A_\sigma(z) g_\nu(\beta, k, \partial_k, y) \\
&\quad [(p_2+q+\beta k)^\sigma (q+p_1)^\nu e^{iy(q-p_1)+iz(p_2-q+\beta k)} - (q \leftrightarrow -q,\ p_1 \leftrightarrow -p_2)] \\
&= e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}x \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} A_\sigma(z) g_\nu(\beta, k, \partial_k, x) \\
&\quad\quad [(p+p_2+\beta k)^\sigma (p+p_1)^\nu e^{ix(p-p_1)+iy(p_2-p+\beta k)} \\
&\quad\quad\quad -(p+p_1-\beta k)^\sigma (p+p_2)^\nu e^{ix(p_2-p)+iy(p-p_1+\beta k)}].
\end{aligned}$$

$$\begin{aligned}
S_2' =&\ e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} (p_2+q)^\mu (q+p)^\nu (p+p_1)^\sigma A_\sigma(y) \\
&\quad\quad G'_{\mu\nu}(x,z) e^{ix(p_2-q)+iy(p-p_1)+iz(q-p)} \\
=&\ e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta A_\sigma(y)(p+p_1)^\sigma \\
&\ (p_2+q)^\mu (q+p)^\nu [g_\nu(\beta, k, \partial_k, z) k_\mu e^{i\beta kx} + (x \leftrightarrow z,\ \mu \leftrightarrow \nu)] \\
&\quad\quad e^{ix(p_2-q)+iy(p-p_1)+iz(q-p)} \\
=&\ e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta \int \bar{d}y \int \bar{d}z \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dq}{q^2-m^2+i\epsilon} A_\sigma(y)(p+p_1)^\sigma e^{iy(p-p_1)} \\
&\ (p_2+q)^\mu (q+p)^\nu [g_\nu(\beta, k, \partial_k, z) k_\mu \delta(p_2-q+\beta k) e^{iz(q-p)} + \\
&\quad\quad (q \leftrightarrow -q,\ p_2 \leftrightarrow -p,\ \mu \leftrightarrow \nu)] \\
=&\ e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}y \int \bar{d}x \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dq}{q^2-m^2+i\epsilon} A_\sigma(y)(p+p_1)^\sigma e^{iy(p-p_1)} \\
&\ [g_\nu(\beta, k, \partial_k, x)(q^2-p_2^2)(p_2+p+\beta k)^\nu \delta(p_2-q+\beta k) e^{ix(p_2-p+\beta k)} + \\
&\quad\quad (q \leftrightarrow -q,\ p_2 \leftrightarrow -p)] \\
\overset{ms}{=}&\ e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}y \int \bar{d}x \int \frac{dp}{p^2-m^2+i\epsilon} \int dq A_\sigma(y)(p+p_1)^\sigma e^{iy(p-p_1)} \\
&\ [g_\nu(\beta, k, \partial_k, x)(p_2+p+\beta k)^\nu \delta(p_2-q+\beta k) e^{ix(p_2-p+\beta k)} + \\
&\quad g_\mu(\beta, k, \partial_k, x) \frac{(q^2-m^2)-(p^2-m^2)}{q^2-m^2+i\epsilon} (-p-p_2+\beta k)^\mu \delta(q-p+\beta k) e^{ix(p_2-p+\beta k)}] \\
=&\ e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}x \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} A_\sigma(y) g_\mu(\beta, k, \partial_k, x) \\
&\ [2\beta k^\mu (p+p_1)^\sigma e^{ix(p_2-p+\beta k)+iy(p-p_1)} + \\
&\quad (p+p_2)^\mu (p+p_1+\beta k)^\sigma e^{ix(p_2-p)+iy(p-p_1+\beta k)}].
\end{aligned}$$

By diagram-inspection we can conclude that the diagram S_3' can be obtained from the diagram S_2' by transformations

$$[p_2 \leftrightarrow -p_1,\ p \leftrightarrow -p,\ q \leftrightarrow -q,\ x \leftrightarrow z,\ \mu \leftrightarrow \nu,\ e \leftrightarrow -e].$$

Thus calculations in S_3' are very similar to those in S_2'. The result is

$$\begin{aligned}
S_3' = & \; e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} (p_2+p)^\sigma (p+q)^\mu (q+p_1)^\nu A_\sigma(y) \\
& \qquad G'_{\mu\nu}(x,z) e^{ix(p-q)+iy(p_2-p)+iz(q-p_1)} \\
= & \; e^3 \int \frac{\bar{d}k}{-k^2} \int_1^\infty \frac{d\beta}{\beta} \int \bar{d}x \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} A_\sigma(y) g_\mu(\beta,k,\partial_k,x) \\
& [2\beta k^\mu (p+p_2)^\sigma e^{ix(p-p_1-p+\beta k)+iy(p_2-p)} + \\
& \quad -(p+p_1)^\mu (p+p_2-\beta k)^\sigma e^{ix(p-p_1)+iy(p_2-p+\beta k)}].
\end{aligned}$$

$$\begin{aligned}
S_4' = & \; -2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} (p+p_1)^\mu A^\nu(y) G'_{\mu\nu}(x,y) e^{ix(p-p_1)+iy(p_2-p)} \\
= & \; -2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta A^\nu(y)(p+p_1)^\mu e^{ix(p-p_1)+iy(p_2-p)} \\
& \qquad [g_\nu(\beta,k,\partial_k,y) k_\mu e^{i\beta kx} + (x \leftrightarrow y,\ \mu \leftrightarrow \nu)] \\
= & \; -2e^3 \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A^\nu(y) \\
& [g_\nu(\beta,k,\partial_k,y)(p_1^2-p^2)\delta(p-p_1+\beta k) e^{iy(p_2-p_1+\beta k)} + \\
& \qquad \beta \int \bar{d}x g_\mu(\beta,k,\partial_k,x) k_\nu (p+p_1)^\mu e^{ix(p-p_1)+iy(p_2-p+\beta k)}] \\
\overset{ms}{=} & \; 2e^3 \int \bar{d}y \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A^\sigma(y) \Big\{ [g_\sigma(\beta,k,\partial_k,y) e^{iy(p_2-p_1+\beta k)} \\
& - \int \bar{d}x \int \frac{dp}{p^2-m^2+i\epsilon} g_\mu(\beta,k,\partial_k,x) \beta k_\sigma (p+p_1)^\mu e^{ix(p-p_1)+iy(p_2-p+\beta k)} \Big\}.
\end{aligned}$$

S_5' can be obtained from S_4' by replacements

$$[p_2 \leftrightarrow -p_1,\ p \leftrightarrow -p,\ e \leftrightarrow -e].$$

We have

$$\begin{aligned}
S_5' = & \; -2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} (p+p_2)^\mu A^\nu(y) G'_{\mu\nu}(x,y) e^{ix(p_2-p)+iy(p-p_1)} \\
\overset{ms}{=} & \; 2e^3 \int \bar{d}y \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A^\sigma(y) \Big\{ [-g_\sigma(\beta,k,\partial_k,y) e^{iy(p_2-p_1+\beta k)} \\
& - \int \bar{d}x \int \frac{dp}{p^2-m^2+i\epsilon} g_\mu(\beta,k,\partial_k,x) \beta k_\sigma (p+p_2)^\mu e^{ix(p_2-p)+iy(p-p_1+\beta k)} \Big\}.
\end{aligned}$$

$$\begin{aligned}
S_6' = & \; -e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} (p+p_1)^\sigma A_\sigma(y) g^{\mu\nu} G'_{\mu\nu}(x,x) e^{ix(p_2-p)+iy(p-p_1)} \\
= & \; -e^3 g^{\mu\nu} \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta A_\sigma(y)(p+p_1)^\sigma e^{ix(p_2-p)+iy(p-p_1)} \\
& \qquad [g_\nu(\beta,k,\partial_k,x) k_\mu e^{i\beta kx} + (\mu \leftrightarrow \nu)] \\
= & \; -2e^3 \int \bar{d}x \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A_\sigma(y)(p+p_1)^\sigma g_\mu(\beta,k,\partial_k,x) \beta k^\mu \\
& \qquad e^{ix(p_2-p+\beta k)+iy(p-p_1)}.
\end{aligned}$$

Similarly S_7' can be obtained from S_6' as we obtained S_5' from S_4'.

$$\begin{aligned}
S_7' = & -e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon}(p+p_2)^\sigma A_\sigma(y) g^{\mu\nu} G'_{\mu\nu}(x,x) e^{ix(p-p_1)+iy(p_2-p)} \\
= & -2e^3 \int \bar{d}x \int \bar{d}y \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A_\sigma(y)(p+p_2)^\sigma g_\mu(\beta,k,\partial_k,x)\beta k^\mu \\
& e^{ix(p-p_1+\beta k)+iy(p_2-p)}.
\end{aligned}$$

$$\begin{aligned}
S_8' = & \; e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon}(p_1+p_2)^\mu (q+p)^\nu (p+q)^\sigma A_\sigma(z) \\
& G'_{\mu\nu}(x,y) e^{ix(p_2-p_1)+iy(p-q)+iz(q-p)} \\
= & \; e^3 \int dx \int dy \int dz \int \frac{\bar{d}p}{p^2-m^2+i\epsilon} \int \frac{\bar{d}q}{q^2-m^2+i\epsilon} \int \frac{\bar{d}k}{-k^2} \int_1^\infty d\beta A_\sigma(z)(p+q)^\sigma (p_1+p_2)^\mu \\
& (q+p)^\nu [g_\nu(\beta,k,\partial_k,y) k_\mu e^{i\beta kx} + (x \leftrightarrow y,\ \mu \leftrightarrow \nu)] e^{ix(p_2-p_1)+iy(p-q)+iz(q-p)} \\
= & \; e^3 \int \bar{d}y \int \bar{d}z \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dq}{q^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A_\sigma(z)(p+q)^\sigma (p_1+p_2)^\mu \\
& (q+p)^\nu e^{iz(q-p)} [g_\nu(\beta,k,\partial_k,y)(p_1-p_2)_\mu \delta(p_2-p_1+\beta k) e^{iy(p-q)} \\
& + (q \leftrightarrow p_1,\ p \leftrightarrow p_2,\ \mu \leftrightarrow \nu)] \\
= & \; e^3 \int \bar{d}y \int \bar{d}z \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dq}{q^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A_\sigma(z)(p+q)^\sigma e^{iz(q-p)} \\
& g_\nu(\beta,k,\partial_k,y)\{(p_1^2-p_2^2)(q+p)^\nu \delta(p_2-p_1+\beta k) e^{iy(p-q)} + \\
& [(q^2-m^2)-(p^2-m^2)](p_1+p_2)^\nu \delta(p-q+\beta k) e^{iy(p_2-p_1)}\} \\
\overset{ms}{=} & \; e^3 \int \bar{d}y \int \bar{d}z \int \frac{dp}{p^2-m^2+i\epsilon} \int \frac{dk}{-k^2} \int_1^\infty \frac{d\beta}{\beta} A_\sigma(z) g_\nu(\beta,k,\partial_k,y)(p_1+p_2)^\nu e^{iy(p_2-p_1)} \\
& [(2p+\beta k)^\sigma e^{i\beta kz} - (2p-\beta k)^\sigma e^{-i\beta kz}] \\
= & \; 0
\end{aligned}$$

$$\begin{aligned}
S_9' = & \; 2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon}(p_1+p_2)^\mu A^\nu(y) G'_{\mu\nu}(x,y) e^{ix(p_2-p_1)} \\
= & \; 2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon}(p_1+p_2)^\mu A^\nu(y) \\
& [\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] e^{ix(p_2-p_1)} \\
= & \; 2e^3 \int dx \int dy \int \frac{\bar{d}p}{p^2-m^2+i\epsilon}(p_1^2-p_2^2) A^\nu(y) f_{1\nu}(x,y) \\
\overset{ms}{=} & \; 0
\end{aligned}$$

Observe that

$$S' = S_1' + \cdots + S_9' \overset{ms}{=} 0.$$

G.1.2 Coordinate Space

Notations: Directed derivatives $\overleftrightarrow{\partial}_\mu$, $\overrightarrow{\partial}_\mu$, $\overleftarrow{\partial}_\mu$, $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ do not act on photon propagators ($f_{1\mu}(x,y)$ and $f_{2\mu}(x,y)$). Surface terms will be discarded. Indices μ, ν and σ in derivatives are attached to variables x, y and z respectively: $\partial_\mu = \partial_{x^\mu}$, $\partial_\nu = \partial_{y^\nu}$ and $\partial_\sigma = \partial_{z^\sigma}$. $Q = Q_F + Q'$ for any quantities Q and Q_F in the Fock-Schwinger gauge and Feynman gauge respectively.

1. Truncated Diagrams

Meson self-energy (Figures 5.30-5.31)

$$\begin{aligned}
-e^{-2}\Sigma_1'(x,y) = {} & [\cdot \overleftrightarrow{\partial^\mu} S_B(x-y) \overleftrightarrow{\partial^\nu} \cdot] G'_{\mu\nu}(x,y) \\
= {} & [\cdot \overleftrightarrow{\partial^\mu} S_B(x-y) \overleftrightarrow{\partial^\nu} \cdot][\partial_\mu f_{1\nu} + \partial_\nu f_{2\mu}(x,y)] \\
= {} & -\{\cdot[\overrightarrow{\Box}_x - \overleftarrow{\Box}_x] S_B(x-y) \overleftrightarrow{\partial^\nu} \cdot\} f_{1\nu}(x,y) \\
& -\{\cdot \overleftrightarrow{\partial^\mu} S_B(x-y)[\overrightarrow{\Box}_y - \overleftarrow{\Box}_y]\cdot\} f_{2\mu}(x,y).
\end{aligned}$$

$$\begin{aligned}
-e^{-2}\Sigma_2'(x,y) = {} & g^{\mu\nu}[\cdot\delta(x-y)\cdot] G'_{\mu\nu}(x,y) \\
= {} & g^{\mu\nu}[\cdot\delta(x-y)\cdot][\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] \\
= {} & -g^{\mu\nu}[\cdot \overleftarrow{\partial^\mu} \delta(x-y) \cdot + \cdot (\partial_\mu \delta(x-y))\cdot] f_{1\nu}(x,y) \\
& -g^{\mu\nu}[\cdot(\overrightarrow{\partial^\nu} \delta(x-y)) \cdot + \cdot \delta(x-y) \overrightarrow{\partial^\nu} \cdot] f_{2\mu}(x,y) \\
= {} & -g^{\mu\nu}\{[\cdot \overleftarrow{\partial^\mu} \delta(x-y)\cdot] f_{2\nu}(x,y) - [\cdot(\delta(x-y) \overleftarrow{\partial^\nu})\cdot] f_{1\mu}(x,y)\} \\
& -g^{\mu\nu}\{-[\cdot(\overrightarrow{\partial^\mu} \delta(x-y))\cdot] f_{2\nu}(x,y) + [\cdot\delta(x-y) \overrightarrow{\partial^\nu} \cdot] f_{1\mu}(x,y)\} \\
= {} & [\cdot \overleftrightarrow{\partial^\mu} \delta(x-y)\cdot] f_{2\mu}(x,y) - [\cdot\delta(x-y) \overleftrightarrow{\partial^\nu} \cdot] f_{1\nu}(x,y).
\end{aligned}$$

$$\begin{aligned}
\Sigma'(x,y) = {} & \Sigma_1'(x,y) + \Sigma_2'(x,y) \\
= {} & e^2[\cdot(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x) S_B(x-y) \overleftrightarrow{\partial^\nu} \cdot] f_{1\nu}(x,y) + \\
& e^2[\cdot \overleftrightarrow{\partial^\mu} S_B(x-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\cdot] f_{2\mu}(x,y) + \\
& [\cdot\delta(x-y) \overleftrightarrow{\partial^\nu} \cdot] f_{1\nu}(x,y) - [\cdot \overleftrightarrow{\partial^\mu} \delta(x-y)\cdot] f_{2\mu}(x,y).
\end{aligned}$$

Vertex corrections (Figures 5.32-5.40)

$$\begin{aligned}
ie^{-3}\Gamma_1'^{\sigma}(x,y,z) &= [\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] G'_{\mu\nu}(x,y) \\
&= [\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot][\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] \\
&= -[\cdot(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x) S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] f_{1\nu}(x,y) \\
&\quad -[\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\cdot] f_{2\mu}(x,y).
\end{aligned}$$

$$\begin{aligned}
ie^{-3}\Gamma_2'^{\nu}(x,y,z) &= [\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] G'_{\mu\sigma}(x,z) \\
&= [\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} [S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot \partial_\mu f_{1\sigma}(x,z) + \\
&\qquad S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot \partial_\sigma f_{2\mu}(x,z)].
\end{aligned}$$

This form actually has the same form as $\Sigma_1'(x,y)$ because the above expression can be obtained from $\Sigma_1'(x,y)$ by changing

$$\begin{aligned}
\overleftrightarrow{\partial^\nu} &\rightarrow \overleftrightarrow{\partial^\sigma} \\
S_B(x-y) &\rightarrow S_B(x-z) \\
f_{1\nu}(x,y) &\rightarrow S_B(z-y) \overleftrightarrow{\partial^\nu} f_{1\sigma}(x,z) \\
f_{2\mu}(x,y) &\rightarrow S_B(z-y) \overleftrightarrow{\partial^\nu} f_{2\mu}(x,z).
\end{aligned}$$

Thus by making these replacements into $\Sigma_1'(x,y)$ one has the final form of Γ_2'

$$\begin{aligned}
ie^{-3}\Gamma_2'^{\nu}(x,y,z) &= -[\cdot(\overrightarrow{\Box}_y - \overleftarrow{\Box}_x) S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] f_{1\sigma}(x,z) + \\
&\quad [\cdot \overleftrightarrow{\partial^\mu} S_B(x-z)(\overrightarrow{\Box}_z - \overleftarrow{\Box}_z) S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] f_{2\mu}(x,z).
\end{aligned}$$

From the diagrams we can see that Γ_3' can be deduced from Γ_2' by hermitean conjugation (to change the direction of boson lines) and then by transforming $(x,\mu) \leftrightarrow (y,\nu)$. We have

$$\begin{aligned}
ie^{-3}\Gamma_3'^{\mu}(x,y,z) &= [\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] G'_{\sigma\nu}(z,y) \\
&= -[\cdot \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\cdot] f_{2\sigma}(z,y) + \\
&\quad -[\cdot \overleftrightarrow{\partial^\mu} S_B(x-z)(\overrightarrow{\Box}_z - \overleftarrow{\Box}_z) S_B(z-y) \overleftrightarrow{\partial^\nu} \cdot] f_{1\nu}(z,y).
\end{aligned}$$

$$
\begin{aligned}
ie^{-3}\Gamma_4^{\prime\nu}(x,y,z) = {} & g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]G'_{\mu\sigma}(x,z) \\
= {} & g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot][\partial_\mu f_{1\sigma}(x,z)+\partial_\sigma f_{2\mu}(x,z)] \\
= {} & -g^{\mu\sigma}[\cdot(\overrightarrow{\partial^\mu}\,\delta(x-z))S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot+ \\
& \qquad\cdot\overleftarrow{\partial^\mu}\,\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\sigma}(x,z) \\
& -g^{\mu\sigma}[\cdot(\delta(x-z)\overleftarrow{\partial^\sigma})S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot+ \\
& \qquad\cdot\delta(x-z)\overrightarrow{\partial^\sigma}\,S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{2\mu}(x,z) \\
= {} & g^{\mu\sigma}[\cdot(\overrightarrow{\partial^\mu}\,\delta(x-z))S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{2\sigma}(x,z) \\
& -g^{\mu\sigma}[\cdot\delta(x-z)\overrightarrow{\partial^\sigma}\,S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\mu}(x,z) \\
& +g^{\mu\sigma}[\cdot\delta(x-z)\overleftarrow{\partial^\sigma}\,S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\mu}(x,z) \\
& -g^{\mu\sigma}[\cdot\overleftarrow{\partial^\mu}\,\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{2\sigma}(x,z) \\
= {} & [\cdot(\overleftrightarrow{\partial^\mu}\,\delta(x-z))S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{2\mu}(x,z) \\
& -[\cdot\delta(x-z)\overleftrightarrow{\partial^\sigma}\,S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\sigma}(x,z).
\end{aligned}
$$

The process used in deriving Γ'_3 from Γ'_2 may be applied to obtain Γ'_5 from Γ'_4. Therefore, from Γ'_4 we get

$$
\begin{aligned}
ie^{-3}\Gamma_5^{\prime\mu}(x,y,z) = {} & g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}\,S_B(x-z)\delta(z-y)\cdot]G'_{\sigma\nu}(z,y) \\
= {} & -[\cdot(\overleftrightarrow{\partial^\mu}\,S_B(x-z))\delta(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\nu}(z,y) \\
& +[\cdot\overleftrightarrow{\partial^\mu}\,S_B(x-z)\overleftrightarrow{\partial^\sigma}\,\delta(z-y)\cdot]f_{2\sigma}(z,y).
\end{aligned}
$$

$$
\begin{aligned}
ie^{-3}\Gamma_6^{\prime\nu}(x,y,z) = {} & 2g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}\,S_B(x-z)\delta(z-y)\cdot]G'_{\mu\sigma}(x,z) \\
= {} & 2g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}\,S_B(x-z)\delta(z-y)\cdot][\partial_\mu f_{1\sigma}(x,z)+\partial_\sigma f_{2\mu}(x,z)] \\
= {} & -2g^{\nu\sigma}\{\cdot[\overrightarrow{\Box}_x-\overleftarrow{\Box}_x]S_B(x-z)\delta(z-y)\cdot\}f_{1\sigma}(x,z) \\
& -2g^{\nu\sigma}[\cdot\overleftrightarrow{\partial^\mu}\,S_B(x-z)(\overleftarrow{\partial^\sigma}+\overrightarrow{\partial^\sigma})\delta(z-y)\cdot]f_{2\mu}(x,z).
\end{aligned}
$$

Γ'_7 follows immediately from Γ'_6

$$
\begin{aligned}
ie^{-3}\Gamma_7^{\prime\mu}(x,y,z) = {} & 2g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]G'_{\sigma\nu}(z,y) \\
= {} & -2g^{\mu\sigma}[\cdot\delta(x-z)S_B(z-y)(\overrightarrow{\Box}_y-\overleftarrow{\Box}_y)\cdot]f_{2\sigma}(z,y) \\
& -2g^{\mu\sigma}[\cdot\delta(x-z)(\overleftarrow{\partial^\sigma}+\overrightarrow{\partial^\sigma})S_B(z-y)\overleftrightarrow{\partial^\nu}\cdot]f_{1\nu}(z,y).
\end{aligned}
$$

$$
\begin{aligned}
ie^{-3}\Gamma_8'^{\sigma}(x,y,z) = & \ [\cdot \overleftrightarrow{\partial^\mu} \cdot]G'_{\mu\nu}(x,y)[(\partial^\nu S_B(y-z)) \overleftrightarrow{\partial^\sigma} S_B(z-y) - \\
& \qquad\qquad\qquad S_B(y-z)) \overleftrightarrow{\partial^\sigma} \partial^\nu S_B(z-y)] \\
= & \ [\cdot \overleftrightarrow{\partial^\mu} \cdot]\partial_\mu f_{1\nu}(x,y) + \partial^\nu f_{2\mu}(x,y)] \times \\
& [(\partial^\nu S_B(y-z)) \overleftrightarrow{\partial^\sigma} S_B(z-y) - S_B(y-z)) \overleftrightarrow{\partial^\sigma} \partial^\nu S_B(z-y)] \\
= & \ [\cdot(\overleftarrow{\Box}_x + m^2) - (\overrightarrow{\Box}_x + m^2)\cdot] f_{1\nu}(x,y) \times \\
& [(\partial^\nu S_B(y-z)) \overleftrightarrow{\partial^\sigma} S_B(z-y) - S_B(y-z)) \overleftrightarrow{\partial^\sigma} \partial^\nu S_B(z-y)] \\
& -[\cdot \overleftrightarrow{\partial^\mu} \cdot]\{[(\overrightarrow{\Box}_y + m^2) S_B(y-z)] \overleftrightarrow{\partial^\sigma} S_B(z-y) - \\
& \qquad\qquad S_B(y-z)) \overleftrightarrow{\partial^\sigma} [(\overrightarrow{\Box}_y + m^2) S_B(z-y)]\} f_{2\mu}(x,y) \\
= & \ [\cdot(\overleftarrow{\Box}_x - \overrightarrow{\Box}_x)\cdot] f_{1\nu}(x,y) \times \\
& [(\partial^\nu S_B(y-z)) \overleftrightarrow{\partial^\sigma} S_B(z-y) - S_B(y-z)) \overleftrightarrow{\partial^\sigma} \partial^\nu S_B(z-y)].
\end{aligned}
$$

$$
\begin{aligned}
ie^{-3}\Gamma_9'^{\sigma}(x,y,z) = & \ 2[\cdot \overleftrightarrow{\partial^\mu} \cdot]G'_{\mu\nu}(x,y) g^{\nu\sigma} S_B(y-z)\delta(y-z) \\
= & \ 2[\cdot \overleftrightarrow{\partial^\mu} \cdot] S_B(y-z)\delta(y-z) g^{\nu\sigma}[\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] \\
= & \ 2[\cdot(\overleftarrow{\Box}_x - \overrightarrow{\Box}_x)\cdot] S_B(y-z)\delta(y-z) g^{\nu\sigma} f_{1\nu}(x,y) + \\
& \qquad\qquad [\cdot \overleftrightarrow{\partial^\mu} \cdot] S_B(y-z)\delta(y-z) g^{\nu\sigma} \partial_\nu f_{2\mu}(x,y).
\end{aligned}
$$

2. On-shell Diagrams

Meson-meson scattering (Born term) (Figure 5.41)

$$
\begin{aligned}
S' = & \ ie^2 \int dx \int dy \phi^\dagger(x) \overleftrightarrow{\partial^\mu} \phi(x) \phi^\dagger(y) \overleftrightarrow{\partial^\nu} \phi(y) G'_{\mu\nu}(x,y) \\
= & \ ie^2 \int dx \int dy \phi^\dagger(x) \overleftrightarrow{\partial^\mu} \phi(x) \phi^\dagger(y) \overleftrightarrow{\partial^\nu} \phi(y)[\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] \\
= & \ -2ie^2 \int dx \int dy \{\phi^\dagger(x)\partial^\mu \phi(x) \phi^\dagger(y)[(\overrightarrow{\Box}_y + m^2) - (\overleftarrow{\Box}_y + m^2)]\phi(y)\} f_{2\mu}(x,y) \\
\overset{ms}{=} & \ 0.
\end{aligned}
$$

Meson self-energy (Figures 5.42-5.43)

$$
\begin{aligned}
S_1' = &\ \int dx \int dy \phi^\dagger(x)\Sigma_1'(x,y)\phi(y) \\
= &\ -e^2 \int dx \int dy \{-\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\phi(y) f_{2\mu}(x,y) + \\
&\qquad -\phi^\dagger(x)(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x) S_B(x-y) \overleftrightarrow{\partial^\nu} \phi(y) f_{1\nu}(x,y)\} \\
\overset{ms}{=} &\ -e^2 \int dx \int dy \{-\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-y)\phi(y) f_{2\mu}(x,y) + \\
&\qquad \phi^\dagger(x)\delta(x-y) \overleftrightarrow{\partial^\nu} \phi(y) f_{1\nu}(x,y)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_2' = &\ \int dx \int dy \phi^\dagger(x)\Sigma_2'(x,y)\phi(y) \\
= &\ -e^2 \int dx \int dy \{\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-y)\phi(y) f_{2\mu}(x,y) \\
&\qquad -\phi^\dagger(x)\delta(x-y) \overleftrightarrow{\partial^\nu} \phi(y) f_{1\nu}(x,y)\}.
\end{aligned}
$$

$$
S' = S_1' + S_2'.
$$

Vertex corrections (Figures 5.44-5.52)

$$
\begin{aligned}
S_1' = &\ \int dx \int dy \int dz \phi^\dagger(x)\Gamma_1'^{\sigma}(x,y,z)\phi(y)A_\sigma(z) \\
= &\ -ie^3 \int dx \int dy \int dz A_\sigma(z) \\
&\ \{+[-\phi^\dagger(x)(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x) S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(x,y) \\
&\ -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\phi(y)] f_{2\mu}(x,y)\} \\
\overset{ms}{=} &\ -ie^3 \int dx \int dy \int dz A_\sigma(z)\{[\phi^\dagger(x)\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(x,y) \\
&\qquad -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)] f_{2\mu}(x,y)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_2' = &\ \int dx \int dy \int dz \phi^\dagger(x)\Gamma_2'^{\nu}(x,y,z)\phi(y)A_\nu(y) \\
= &\ -ie^3 \int dx \int dy \int dz A_\nu(y) \\
&\ \{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)[(\overrightarrow{\Box}_z + m^2) - (\overleftarrow{\Box}_z + m^2)] S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z) \\
&\ -[\phi^\dagger(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x) S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\sigma}(x,z)\} \\
\overset{ms}{=} &\ -ie^3 \int dx \int dy \int dz A_\nu(y)\{[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z) \\
&\ -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z) S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z) \\
&\ +[\phi^\dagger \delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\sigma}(x,z)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_3' = &\int dx \int dy \int dz \phi^\dagger(x)\Gamma_3'^{\mu}(x,y,z)\phi(y)A_\mu(x) \\
= &-ie^3 \int dx \int dy \int dz A_\mu(x) \\
&\{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)[(\overrightarrow{\Box}_z + m^2) - (\overleftarrow{\Box}_z + m^2)]S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y) \\
&-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\phi(y)]f_{2\sigma}(z,y) \\
\overset{ms}{=} &-ie^3 \int dx \int dy \int dz A_\mu(x)\{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z)S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y) \\
&-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)]f_{2\sigma}(z,y) \\
&+[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_4' = &\int dx \int dy \int dz \phi^\dagger(x)\Gamma_4'^{\nu}(x,y,z)\phi(y)A_\nu(y) \\
= &-ie^3 \int dx \int dy \int dz A_\nu(y)\{[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z))S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{2\mu}(x,z) \\
&-[\phi^\dagger(x)\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\sigma}(x,z)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_5' = &\int dx \int dy \int dz \phi^\dagger(x)\Gamma_5'^{\mu}(x,y,z)\phi(y)A_\mu(x) \\
= &-ie^3 \int dx \int dy \int dz A_\mu(x)\{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z))\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)]f_{1\nu}(z,y) \\
&+[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)]f_{2\sigma}(z,y)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_6' = &\int dx \int dy \int dz \phi^\dagger(x)\Gamma_6'^{\nu}(x,y,z)\phi(y)A_\nu(y) \\
= &-ie^3 \int dx \int dy \int dz A_\nu(y) \\
&\{-2g^{\nu\sigma}\phi^\dagger(x)(\overrightarrow{\Box}_x - \overleftarrow{\Box}_x)S_B(x-z)\delta(z-y)\phi(y)f_{1\sigma}(x,z)+ \\
&-2g^{\nu\sigma}[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})\delta(z-y)\phi(y)]f_{2\mu}(x,z)\} \\
\overset{ms}{=} &-ie^3 \int dx \int dy \int dz A_\nu(y)\{2g^{\nu\sigma}\phi^\dagger(x)\delta(x-z)\delta(z-y)\phi(y)f_{1\sigma}(x,z)+ \\
&-2g^{\nu\sigma}[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})\delta(z-y)\phi(y)]f_{2\mu}(x,z)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_7' = &\int dx \int dy \int dz \phi^\dagger(x)\Gamma_7'^{\mu}(x,y,z)\phi(y)A_\mu(x) \\
= &-ie^3 \int dx \int dy \int dz A_\mu(x) \\
&\{-2g^{\mu\sigma}\phi^\dagger(x)\delta(x-z)S_B(z-y)(\overrightarrow{\Box}_y - \overleftarrow{\Box}_y)\phi(y)f_{2\sigma}(z,y) \\
&-2g^{\mu\sigma}[\phi^\dagger(x)\delta(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)f_{1\nu}(z,y)\} \\
\overset{ms}{=} &-ie^3 \int dx \int dy \int dz A_\mu(x)\{-2g^{\mu\sigma}\phi^\dagger(x)\delta(x-z)\delta(z-y)\phi(y)f_{2\sigma}(z,y) \\
&-2g^{\mu\sigma}[\phi^\dagger(x)\delta(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)f_{1\nu}(z,y)\}.
\end{aligned}
$$

$$
\begin{aligned}
S'_8 = & \int dx \int dy \int dz \phi^\dagger(x)\Gamma'^{\sigma}_8(x,y,z)\phi(y)A_\sigma(z) \\
\stackrel{ms}{=} & \; 0.
\end{aligned}
$$

$$
\begin{aligned}
S'_9 = & \int dx \int dy \int dz \phi^\dagger(x)\Gamma'^{\sigma}_9(x,y,z)\phi(y)A_\sigma(z) \\
\stackrel{ms}{=} & \; 0.
\end{aligned}
$$

Adding $S'_1, S'_2 \cdots S'_9$,

$$
\begin{aligned}
S' = & \; S'_1 + S'_2 + S'_3 + S'_4 + S'_5 + S'_6 + S'_7 + S'_8 + S'_9 \\
\stackrel{ms}{=} & \; -ie^3 \int dx \int dy \int dz \\
& \{-[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)] f_{2\mu}(x,y)A_\sigma(z) \\
& \quad +[\phi^\dagger(x)\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(x,y)A_\sigma(z) \\
& \quad +[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z)A_\nu(y) \\
& \quad -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z)S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(z,y)A_\mu(x) \\
& \quad -2[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})\delta(z-y)\phi(y)] f_{2\mu}(x,z)A_\sigma(y) \\
& \quad -2[\phi^\dagger(x)\delta(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(z,y)A_\sigma(x)\}.
\end{aligned}
$$

The last two terms can be written as (neglecting surface terms)

$$
\begin{aligned}
& 2[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})\delta(z-y)\phi(y)] f_{2\mu}(x,z)A_\sigma(y) \\
= & \; 2[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftarrow{\partial^\sigma} \delta(z-y)\phi(y)] f_{2\mu}(x,y)A_\sigma(z) \\
& \; -2[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z)A_\nu(y) \\
= & \; -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z) \overleftrightarrow{\partial^\sigma} \delta(z-y)\phi(y)] f_{2\mu}(x,y)A_\sigma(z) \\
& \; +[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} S_B(x-z)\delta(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{2\mu}(x,z)A_\nu(y)
\end{aligned}
$$

$$
\begin{aligned}
& 2[\phi^\dagger(x)\delta(x-z)(\overleftarrow{\partial^\sigma} + \overrightarrow{\partial^\sigma})S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(z,y)A_\sigma(x) \\
= & \; -[\phi^\dagger(x) \overleftrightarrow{\partial^\mu} \delta(x-z)S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(z,y)A_\mu(x) \\
& \; +[\phi^\dagger(x)\delta(x-z) \overleftrightarrow{\partial^\sigma} S_B(z-y) \overleftrightarrow{\partial^\nu} \phi(y)] f_{1\nu}(x,y)A_\sigma(z).
\end{aligned}
$$

Thus we get

$$
S' = S'_1 + \cdots + S'_9 \stackrel{ms}{=} 0.
$$

G.2 Spinor Quantum Electrodynamics

G.2.1 Momentum Space

The calculations here are very similar to the previous ones. Therefore it is sufficient to show their initial and final expressions.

Electron-electron scattering (Born term) (Figure 5.1)

$$\begin{aligned}
S' = & -ie^2\overline{u}(p_2)\gamma^\mu u(p_1)\overline{u}(q_2)\gamma^\nu u(q_1)\int dx\int dy G'_{\mu\nu}(x,y)e^{ix(p_2-p_1)+iy(q_2-q_1)} \\
= & \; ie^2\int_1^\infty \frac{d\beta}{\beta}\int\frac{dk}{-k^2}\int dy g_\mu(\beta,k,\partial_k,y) \\
& \{\overline{u}(p_2)(\not{p}_2-\not{p}_1)u(p_1)\overline{u}(q_2)\gamma^\mu u(q_1)\delta(p_2-p_1+\beta k)e^{iy(q_2-q_1)} \\
& +\overline{u}(p_2)\gamma^\mu u(p_1)\overline{u}(q_2)(\not{q}_2-\not{q}_1)u(q_1)\delta(q_2-q_1+\beta k)e^{iy(p_2-p_1)}\} \\
\overset{ms}{=} & \; 0.
\end{aligned}$$

Electron self-energy (Figure 5.2)

$$\begin{aligned}
S' = & \; e^2\int \bar{d}p\overline{u}(p_2)\gamma^\mu(\not{p}-m+i\epsilon)^{-1}\gamma^\nu u(p_1)\int dx\int dy e^{ix(p_2-p)+iy(p-p_1)}G'_{\mu\nu}(x,y) \\
= & \; e^2\int_1^\infty d\beta\int\frac{\bar{d}k}{-k^2}\int \bar{d}p\overline{u}(p_2)\gamma^\mu(\not{p}-m+i\epsilon)^{-1}\gamma^\nu u(p_1)\int dx\int dy \\
& e^{ix(p_2-p)+iy(p-p_1)}[g_\nu(\beta,k,\partial_k,y)k_\mu e^{i\beta kx}+(\mu\leftrightarrow\nu,\, x\leftrightarrow y)] \\
= & \; e^2\int_1^\infty\frac{d\beta}{\beta}\int\frac{dk}{-k^2}\int \bar{d}y\int dp g_\mu(\beta,k,\partial_k,y) \\
& [\overline{u}(p_2)(\not{p}-\not{p}_2)(\not{p}-m+i\epsilon)^{-1}\gamma^\mu u(p_1)\delta(p_2-p+\beta k)e^{iy(p-p_1)} \\
& +\overline{u}(p_2)\gamma^\mu(\not{p}-m+i\epsilon)^{-1}(\not{p}_1-\not{p})u(p_1)\delta(p-p_1+\beta k)e^{iy(p_2-p)}] \\
\overset{ms}{=} & \; 0.
\end{aligned}$$

Vertex corrections (Figures 5.3-5.6)

$$\begin{aligned}
S_1' = & \; e^3\int\bar{d}p\int\bar{d}q\int dx\int dy\int dz\overline{u}(p_2)\gamma^\mu(\not{p}-m+i\epsilon)^{-1}\not{A}(y)(\not{q}-m+i\epsilon)^{-1}\times \\
& \gamma^\nu u(p_1)e^{ix(p_2-p)+iy(p-q)+iz(q-p_1)}G'_{\mu\nu}(x,z) \\
= & \; e^3\int_1^\infty d\beta\int\frac{\bar{d}k}{-k^2}\int\bar{d}p\int\bar{d}q\int dx\int dy\int dz\overline{u}(p_2)\gamma^\mu(\not{p}-m+i\epsilon)^{-1}\not{A}(y)\times \\
& (\not{q}-m+i\epsilon)^{-1}\gamma^\nu u(p_1)e^{ix(p_2-p)+iy(p-q)+iz(q-p_1)}\times \\
& [g_\nu(\beta,k,\partial_k,z)k_\mu e^{i\beta kx}+(\mu\leftrightarrow\nu,\, x\leftrightarrow z)]
\end{aligned}$$

$$
\begin{aligned}
&= e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int \bar{d}p \int \bar{d}q \int dy \int dz g_\nu(\beta, k, \partial_k, z)\overline{u}(p_2) \\
&\quad [(\not p - \not p_2)(\not p - m + i\epsilon)^{-1} \not A(y)(\not q - m + i\epsilon)^{-1}\gamma^\nu \\
&\qquad\qquad \delta(p_2 - p + \beta k)e^{iy(p-q)+iz(q-p_1)} \\
&\quad +\gamma^\nu(\not p - m + i\epsilon)^{-1} \not A(y)(\not q - m + i\epsilon)^{-1}(\not p_1 - \not q) \\
&\qquad\qquad \delta(q - p_1 + \beta k)e^{iy(p-q)+iz(p_2-p)}]u(p_1) \\
&\overset{ms}{=} e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int dp \int \bar{d}y \int \bar{d}z g_\mu(\beta, k, \partial_k, z)\overline{u}(p_2)\times \\
&\quad [\not A(y)(\not p - m + i\epsilon)^{-1}\gamma^\mu e^{iy(p_2-p+\beta k)+iz(p-p_1)} \\
&\quad -\gamma_\mu(\not p - m + i\epsilon)^{-1} \not A(y)e^{iy(p-p_1+\beta k)+iz(p_2-p)}]u(p_1).
\end{aligned}
$$

$$
\begin{aligned}
S_2' &= e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz\overline{u}(p_2) \not A(y)(\not q - m + i\epsilon)^{-1}\gamma^\mu(\not p - m + i\epsilon)^{-1}\times \\
&\quad \gamma^\nu u(p_1)e^{ix(q-p)+iy(p_2-q)+iz(p-p_1)}G'_{\mu\nu}(x,z) \\
&= e^3 \int_1^\infty d\beta \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz\overline{u}(p_2) \not A(y)(\not q - m + i\epsilon)^{-1}\gamma^\mu\times \\
&\quad (\not p - m + i\epsilon)^{-1}\gamma^\nu u(p_1)e^{ix(q-p)+iy(p_2-q)+iz(p-p_1)}\times \\
&\quad [g_\nu(\beta, k, \partial_k, z)k_\mu e^{i\beta kx} + (\mu \leftrightarrow \nu,\, x \leftrightarrow z)] \\
&= -e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dq \int dy \int dz g_\nu(\beta, k, \partial_k, z)\overline{u}(p_2) \not A(y)\times \\
&\qquad\qquad e^{iy(p_2-q)+iz(q-p_1+\beta k)}\times \\
&\quad \{(\not q - m + i\epsilon)^{-1}[(\not q - m) - (\not p - m)](\not p - m + i\epsilon)^{-1}\gamma^\nu\delta(q - p + \beta k) \\
&\quad +(\not q - m + i\epsilon)^{-1}\gamma^\nu(\not p - m + i\epsilon)^{-1}(\not p - \not p_1)\delta(p - p_1 + \beta k)\}u(p_1) \\
&\overset{ms}{=} -e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dy \int dz g_\mu(\beta, k, \partial_k, z)\overline{u}(p_2) \not A(y)(\not p - m + i\epsilon)^{-1}\times \\
&\qquad\qquad \gamma^\mu u(p_1)e^{iy(p_2-p+\beta k)+iz(p-p_1)}.
\end{aligned}
$$

$$
\begin{aligned}
S_3' &= e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz\overline{u}(p_2)\gamma^\mu(\not p - m + i\epsilon)^{-1}\gamma^\nu(\not q - m + i\epsilon)^{-1}\times \\
&\quad \not A(y)u(p_1)e^{ix(p_2-p)+iy(q-p_1)+iz(p-q)}G'_{\mu\nu}(x,z) \\
&= e^3 \int_1^\infty d\beta \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz\overline{u}(p_2)\gamma^\mu(\not p - m + i\epsilon)^{-1}\gamma^\nu\times \\
&\quad (\not q - m + i\epsilon)^{-1} \not A(y)u(p_1)e^{ix(p_2-p)+iy(q-p_1)+iz(p-q)}\times \\
&\quad [g_\nu(\beta, k, \partial_k, z)k_\mu e^{i\beta kx} + (\mu \leftrightarrow \nu,\, x \leftrightarrow z)] \\
&= e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int \bar{d}p \int \bar{d}q \int dy \int dz\overline{u}(p_2)\gamma^\mu(\not p - m + i\epsilon)^{-1}\gamma^\nu\times \\
&\quad (\not q - m + i\epsilon)^{-1} \not A(y)u(p_1)e^{iy(q-p_1)}\times \\
&\quad [g_\nu(\beta, k, \partial_k, z)(p - p_2)_\mu\delta(p_2 - p + \beta k)e^{iz(p_2-q+\beta k} \\
&\qquad\qquad +(\mu \leftrightarrow \nu,\, p \leftrightarrow -p,\, q \leftrightarrow -p_2)]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int \bar{d}p \int \bar{d}q \int dy \int dz g_\nu(\beta, k, \partial_k, z) \overline{u}(p_2) e^{iy(q-p_1)+iz(p_2-q+\beta k)} \\
& \{(\not{p} - \not{p}_2)(\not{p} - m + i\epsilon)^{-1}\gamma^\nu(\not{q} - m + i\epsilon)^{-1}\delta(p_2 - p + \beta k) \\
& \quad -\gamma^\nu(\not{p} - m + i\epsilon)^{-1}[(\not{p} - m) - (\not{q} - m)](\not{q} - m + i\epsilon)^{-1}\delta(p - q + \beta k)\} \\
& \qquad \qquad \not{A}(y) u(p_1) \\
\overset{ms}{=} \quad & e^3 \int_1^\infty \frac{d\beta}{\beta} \int \frac{dk}{-k^2} \int \bar{d}p \int dy \int dz g_\mu(\beta, k, \partial_k, z) \overline{u}(p_2)\gamma^\mu(\not{p} - m + i\epsilon)^{-1} \times \\
& \qquad \qquad \not{A}(y) u(p_1) e^{iy(p-p_1+\beta k)+iz(p_2-p)}.
\end{aligned}
$$

$$
\begin{aligned}
S_4' = \quad & e^3 \int \bar{d}p \int \bar{d}q \int dx \int dy \int dz \overline{u}(p_2)\gamma^\mu u(p_1) \mathrm{Tr}(\not{q} - m + i\epsilon)^{-1}\gamma^\nu \times \\
& \quad (\not{p} - m + i\epsilon)^{-1} \not{A}(z) e^{ix(p_2-p_1)+iy(p-q)+iz(q-p)} G'_{\mu\nu}(x, y) \\
= \quad & e^3 \int_1^\infty d\beta \int \frac{\bar{d}k}{-k^2} \int \bar{d}p \int dq \int dy \int dz g_\nu(\beta, k, \partial_k, y) e^{iz(q-p)} \times \\
& \{\overline{u}(p_2)(\not{p}_1 - \not{p}_2) u(p_1) \mathrm{Tr}(\not{q} - m + i\epsilon)^{-1}\gamma^\nu(\not{p} - m + i\epsilon)^{-1} \times \\
& \quad \delta(p_2 - p_1 + \beta k) e^{iy(p-q)} \\
& +\overline{u}(p_2)\gamma^\nu u(p_1) \mathrm{Tr}(\not{q} - m + i\epsilon)^{-1}[(\not{q} - m) - (\not{p} - m)](\not{p} - m + i\epsilon)^{-1} \\
& \quad \delta(p - q + \beta k) e^{iy(p_2-p_1)}\} \not{A}(z) \\
\overset{ms}{=} \quad & 0.
\end{aligned}
$$

Hence

$$
S' = S_1' + \cdots + S_4' \overset{ms}{=} 0.
$$

G.2.2 Coordinate Space

The same notations as in scalar electrodynamics apply.

1. Truncated Diagrams

Electron self-energy (Figure 5.7)

$$
\begin{aligned}
e^{-2}\Sigma'(x,y) = & \ [\cdot\gamma^\mu S_F(x-y)\gamma^\nu\cdot]G'_{\mu\nu}(x,y) \\
= & \ [\cdot\gamma^\mu S_F(x-y)\gamma^\nu\cdot][\partial_\mu f_{1\nu}(x,y)+\partial_\nu f_{2\mu}(x,y)] \\
= & \ -[\cdot(\overleftarrow{\partial\!\!\!/}_x+\overrightarrow{\partial\!\!\!/}_x)S_F(x-y)\gamma^\mu\cdot]f_{1\mu}(x,y) \\
& \ -[\cdot\gamma^\mu S_F(x-y)(\overleftarrow{\partial\!\!\!/}_y+\overrightarrow{\partial\!\!\!/}_y)\cdot]f_{2\mu}(x,y).
\end{aligned}
$$

Vertex corrections (Figures 5.8-5.11)

$$
\begin{aligned}
-ie^{-2}\Gamma_1'^\sigma(x,y,z) = & \ [\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\cdot]G'_{\mu\nu}(x,y) \\
= & \ [\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\cdot][\partial_\mu f_{1\nu}(x,y)+\partial_\nu f_{2\mu}(x,y)] \\
= & \ -[\cdot(\overleftarrow{\partial\!\!\!/}_x+\overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\cdot]f_{1\nu}(x,y) \\
& \ -[\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y+\overrightarrow{\partial\!\!\!/}_y)\cdot]f_{2\mu}(x,y). \\
-ie^{-2}\Gamma_2'^\nu(x,y,z) = & \ [\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\cdot]G'_{\mu\sigma}(x,z) \\
= & \ [\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\cdot][\partial_\mu f_{1\sigma}(x,z)+\partial_\sigma f_{2\mu}(x,z)] \\
= & \ -[\cdot(\overleftarrow{\partial\!\!\!/}_x+\overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\mu S_F(z-y)\gamma^\nu\cdot]f_{1\mu}(x,z) \\
& \ -\{\cdot\gamma^\mu S_F(x-z)[(\overleftarrow{\partial\!\!\!/}_z-im)+(\overrightarrow{\partial\!\!\!/}_z+im)]S_F(z-y)\gamma^\nu\cdot\} \\
& \qquad f_{2\mu}(x,z) \\
= & \ -[\cdot(\overleftarrow{\partial\!\!\!/}_x+\overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\mu S_F(z-y)\gamma^\nu\cdot]f_{1\mu}(x,z) \\
& \ -i\{\cdot\gamma^\mu[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^\nu\cdot\}f_{2\mu}(x,z). \\
-ie^{-2}\Gamma_3'^\mu(x,y,z) = & \ [\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\cdot]G'_{\sigma\nu}(z,y) \\
= & \ -\{\cdot\gamma^\mu S_F(x-z)[(\overleftarrow{\partial\!\!\!/}_z-im)+(\overrightarrow{\partial\!\!\!/}_z+im)]S_F(z-y)\gamma^\nu\cdot\} \\
& \qquad f_{1\nu}(z,y) \\
& \ -[\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y+\overrightarrow{\partial\!\!\!/}_y)\cdot]f_{2\sigma}(z,y) \\
= & \ -i\{\cdot\gamma^\mu[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^\nu\cdot\}f_{1\nu}(z,y) \\
& \ -[\cdot\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y+\overrightarrow{\partial\!\!\!/}_y)\cdot]f_{2\sigma}(z,y).
\end{aligned}
$$

$$
\begin{aligned}
-ie^{-2}\Gamma_4^{\prime\sigma}(x,y,z) = \;& [\cdot\gamma^\mu \mathrm{Tr} S_F(z-y)\gamma^\nu S_F(y-z)\gamma^\sigma\cdot] G'_{\mu\nu}(x,y) \\
= \;& -[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)\mathrm{Tr} S_F(z-y)\gamma^\nu S_F(y-z)\gamma^\sigma\cdot] f_{1\nu}(x,y) \\
& -\{\cdot\gamma^\mu \mathrm{Tr} S_F(z-y)[(\overleftarrow{\not\partial}_y - im) + (\overrightarrow{\not\partial}_y + im)] S_F(y-z)\gamma^\sigma\cdot\} \\
& \qquad\qquad f_{2\mu}(x,y) \\
= \;& -[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)\mathrm{Tr} S_F(z-y)\gamma^\nu S_F(y-z)\gamma^\sigma\cdot] f_{1\nu}(x,y).
\end{aligned}
$$

2. On-shell Diagrams

Electron-electron scattering (Born term) (Figure 5.12)

$$
\begin{aligned}
S' = \;& -ie^2 \int dx \int dy \overline{\psi}(x)\gamma^\mu\psi(x)\overline{\psi}(y)\gamma^\nu\psi(y) G'_{\mu\nu}(x,y) \\
= \;& -ie^2 \int dx \int dy \overline{\psi}(x)\gamma^\mu\psi(x)\overline{\psi}(y)\gamma^\nu\psi(y)[\partial_\mu f_{1\nu}(x,y) + \partial_\nu f_{2\mu}(x,y)] \\
= \;& ie^2 \int dx \int dy \left\{ \left(\overline{\psi}(x)[(\overleftarrow{\not\partial}_x - im) + (\overrightarrow{\not\partial}_x + im)]\psi(x)\right)\overline{\psi}(y)\; \not f_1(x,y)\psi(y) \right. \\
& \left. + \overline{\psi}(x)\; \not f_2(x,y)\psi(x) \left(\overline{\psi}(y)[(\overleftarrow{\not\partial}_y - im) + (\overrightarrow{\not\partial}_y + im)]\psi(y)\right)\right\} \\
\overset{ms}{=} \;& 0.
\end{aligned}
$$

Electron self-energy (Figure 5.13)

$$
\begin{aligned}
S' = \;& \int dx \int dy \overline{\psi}(x)\Sigma'(x,y)\psi(y) \\
= \;& -e^2 \int dx \int dy \{[\overline{\psi}(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x) S_F(x-y)\gamma^\mu\psi(y)] f_{1\mu}(x,y) \\
& + [\overline{\psi}(x)\gamma^\mu S_F(x-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\psi(y)] f_{2\mu}(x,y)\} \\
\overset{ms}{=} \;& -e^2 \int dx \int dy \{[\overline{\psi}(x)(\overrightarrow{\not\partial}_x + im) S_F(x-y)\gamma^\mu\psi(y)] f_{1\mu}(x,y) \\
& + [\overline{\psi}(x)\gamma^\mu S_F(x-y)(\overleftarrow{\not\partial}_y - im)\psi(y)] f_{2\mu}(x,y)\} \\
= \;& ie^2 \int dx \int dy \{\overline{\psi}(x)\delta(x-y)\gamma^\mu\psi(y) f_{1\mu}(x,y) \\
& - \overline{\psi}(x)\gamma^\mu\delta(x-y)\psi(y) f_{2\mu}(x,y)\} \\
= \;& 0.
\end{aligned}
$$

Vertex corrections (Figures 5.14-5.17)

$$\begin{aligned}
S_1' = & -ie \int dx \int dy \int dz \overline{\psi}(x)\Gamma_1'^{\sigma}(x,y,z)A_\sigma(z)\psi(y) \\
= & -e^3 \int dx \int dy \int dz A_\sigma(z) \\
& \{[\overline{\psi}(x)(\overleftarrow{\partial\!\!\!/}_x + \overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\sigma S_F(z-y)\gamma^\nu\psi(y)]f_{1\nu}(x,y) \\
& +[\overline{\psi}(x)\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y + \overrightarrow{\partial\!\!\!/}_y)\psi(y)]f_{2\mu}(x,y)\} \\
\overset{ms}{=} & \; ie^3 \int dx \int dy \int dz \{\overline{\psi}(x)\delta(x-z)\ A\!\!\!/(z)S_F(z-y)\ f\!\!\!/_1(x,y)\psi(y) \\
& \qquad -\overline{\psi}(x)\ f\!\!\!/_2(x,y)S_F(x-z)\ A\!\!\!/(z)\delta(z-y)\psi(y)\}
\end{aligned}$$

$$\begin{aligned}
S_2' = & -ie \int dx \int dy \int dz \overline{\psi}(x)\Gamma_2'^{\nu}(x,y,z)A_\nu(y)\psi(y) \\
= & -e^3 \int dx \int dy \int dz A_\nu(y) \\
& [\overline{\psi}(x)(\overleftarrow{\partial\!\!\!/}_x + \overrightarrow{\partial\!\!\!/}_x)S_F(x-z)\gamma^\mu S_F(z-y)\gamma^\nu\psi(y)]f_{1\mu}(x,z) \\
& +i\{\overline{\psi}(x)\gamma^\mu[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^\nu\psi(y)\}f_{2\mu}(x,z)\} \\
\overset{ms}{=} & \; ie^3 \int dx \int dy \int dz \overline{\psi}(x)\ f\!\!\!/_2(x,z)\delta(z-y)S_F(x-y)\ A\!\!\!/(y)\psi(y)
\end{aligned}$$

$$\begin{aligned}
S_3' = & -ie \int dx \int dy \int dz \overline{\psi}(x)\Gamma_3'^{\mu}(x,y,z)A_\mu(x)\psi(y) \\
= & -e^3 \int dx \int dy \int dz A_\mu(x) \\
& i\{[\overline{\psi}(x)\gamma^\mu[\delta(x-z)-\delta(z-y)]S_F(x-y)\gamma^\nu\psi(y)]f_{1\nu}(z,y) \\
& +[\overline{\psi}(x)\gamma^\mu S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\partial\!\!\!/}_y + \overrightarrow{\partial\!\!\!/}_y)\psi(y)]f_{2\sigma}(z,y)\} \\
\overset{ms}{=} & -ie^3 \int dx \int dy \int dz \overline{\psi}(x)\ A\!\!\!/(x)\delta(x-z)S_F(z-y)\ f\!\!\!/_1(z,y)\psi(y).
\end{aligned}$$

$$\begin{aligned}
S_4' = & -ie \int dx \int dy \int dz \overline{\psi}(x)\Gamma_4'^{\sigma}(x,y,z)A_\sigma(z)\psi(x) \\
= & -e^3 \int dx \int dy \int dz \overline{\psi}(x)(\overleftarrow{\partial\!\!\!/}_x + \overrightarrow{\partial\!\!\!/}_x)\mathrm{Tr}S_F(z-y)\ f\!\!\!/_1(x,y)\times \\
& \qquad S_F(y-z)\ A\!\!\!/(z)\psi(x) \\
\overset{ms}{=} & \; 0.
\end{aligned}$$

Adding $S_1' \cdots S_4'$,

$$S' = S_1' + S_2' + S_3' + S_4' \overset{ms}{=} 0.$$

G.3 Quantum Chromodynamics

Calculations on quark-quark scattering and quark self-energy are similar to those on electron-electron scattering and electron self-energy previously done. Here we will only carry out gluon self-energy diagrams and the first order correction of the quark-gluon vertex. The first order corrections of the three-gluon and four-gluon vertices contain up to three and four gluon propagators and thus are very complicated. We do not consider such vertices in this thesis. As a starting point general notations presented below need to be introduced.

G.3.1 General Notations

In order to shorten some mathematical expressions in quark-gluon (and gluon-gluon) vertices some notations are needed. (Directed derivatives are still assumed not to act on $f_{1\mu}^{ab}$ and $f_{2\mu}^{ab}$.) We define $\partial_x^{(a)\mu} = \partial_{x(a)}^{\mu}$ as an operator ∂_x^{μ} that only acts on functions that contain index a. As an example we may write, for any functions $F^{ab}(x)$, $G^{bc}(x)$ and $H^d(x)$,

$$\begin{aligned}
\partial_x^{(a)\mu} F^{ab}(x) G^{bc}(x) H^d(x) = &\ F^{ab}(x) G^{bc}(x) H^d(x) \partial_x^{(a)\mu} = F^{ab}(x) \partial_x^{(a)\mu} G^{bc}(x) H^d(x) \\
= &\ F^{ab}(x) G^{bc}(x) \partial_x^{(a)\mu} H^d(x) = [\partial_x^{\mu} F^{ab}(x)] G^{bc}(x) H^d(x).
\end{aligned}$$

Thus $\partial_x^{(a)\mu}$ can be placed around functions F, G and H just like a number. We also define $\partial^{(a+b)\mu} = \partial^{(a)\mu} + \partial^{(b)\mu}$. By this notation we can write $\partial^{(a+d)\mu} F^{ab} G^{bc} H^d$ instead of $\partial^{(a)\mu} F^{ab} G^{bc} H^d + \partial^{(d)\mu} F^{ab} G^{bc} H^d$ or $[\partial^{\mu} F^{ab}] G^{bc} H^d + F^{ab} G^{bc} [\partial^{\mu} H^d]$. Notice that superscripts (a) in $\partial^{(a)\mu}$ and a in F^{ab} signify different things. Unlike indices a there is no summation over repeated indices (a). Thus it is understood, for example, that $\Box^{(a)} = \partial^{(a)\mu} \partial_{\mu}^{(a)}$. Other notations:

$$\begin{aligned}
V_{(c,e,d)}^{\sigma\alpha\beta}(\partial_x) = &\ V^{(c,e,d)\sigma\alpha\beta}(\partial_x) \\
= &\ g^{\sigma\alpha}[\partial_x^{(c)} - \partial_x^{(e)}]^{\beta} + g^{\alpha\beta}[\partial_x^{(e)} - \partial_x^{(d)}]^{\sigma} + g^{\beta\sigma}[\partial_x^{(d)} - \partial_x^{(c)}]^{\alpha} \\
V_{(-,e,d)}^{\sigma\alpha\beta}(\partial_x) = &\ V_{(c=-e-d,e,d)}^{\sigma\alpha\beta}(\partial_x) \\
= &\ -[2g^{\sigma\alpha}\partial_x^{(e)\beta} - g^{\alpha\beta}\partial_x^{(e)\sigma} - g^{\beta\sigma}\partial_x^{(e)\alpha}] \\
&\ +[2g^{\beta\sigma}\partial_x^{(d)\alpha} - g^{\sigma\alpha}\partial_x^{(d)\beta} - g^{\alpha\beta}\partial_x^{(d)\sigma}]
\end{aligned}$$

$$
\begin{aligned}
O_{x\alpha\beta} = & \ \Box_x g_{\alpha\beta} - \partial_{x\alpha}\partial_{x\beta} \\
O^{(e)}_{x\alpha\beta} = & \ \Box^{(e)}_x g_{\alpha\beta} - \partial^{(e)}_{x\alpha}\partial^{(e)}_{x\beta}.
\end{aligned}
$$

These identities follow immediately,

$$
\begin{aligned}
V^{\sigma\beta\alpha}_{(-,d,e)}(\partial_x) = & \ -V^{\sigma\alpha\beta}_{(-,e,d)}(\partial_x) \\
V^{\sigma\alpha\beta}_{(c,e,d)}(\partial_x) = & \ V^{\alpha\beta\sigma}_{(e,d,c)}(\partial_x) = V^{\beta\sigma\alpha}_{(d,c,e)}(\partial_x) \\
V^{\sigma\alpha\beta}_{(-,e,d)}(\partial_x)\partial^{(e)}_{x\alpha} = & \ O^{(e+d)\sigma\beta}_x - O^{(d)\sigma\beta}_x \\
V^{\sigma\alpha\beta}_{(-,e,d)}(\partial_x)\partial^{(d)}_{x\beta} = & \ -[O^{(e+d)\sigma\alpha}_x - O^{(e)\sigma\alpha}_x] \\
V^{\sigma\alpha\beta}_{(c,e,d)}(\partial_x)f^{(c)}(x)g^{(e)}(x)h^{(d)}(x) = & \ V^{\sigma\alpha\beta}_{(-,e,d)}(\partial_x)f^{(c)}(x)g^{(e)}(x)h^{(d)}(x) \\
& +0(\text{surface} \quad \text{terms}) \\
O_{x\alpha\sigma}G^{ab\sigma\beta}(x,y) = & \ \delta^\beta_\alpha\delta^{ab}(x-y) - x_\alpha(\partial x)^{-1}\partial^\beta_x\delta^{ab}(x-y).
\end{aligned}
$$

The last identity is derived from identities

$$
\int dy G^{-1^{ab}}_{\alpha K}(x,y)G^{bcK\beta}(t,z) = \delta^\beta_\alpha\delta^{ac}(x-z)
$$

in the FS gauge. Notice that here $O_{x\alpha\sigma}\delta^{ab}(x-y) = G^{-1^{ab}}_{\alpha\sigma}(x,y)$.

G.3.2 Truncated Diagrams

The first four diagrams below are similar to those in spinor quantum electrodynamics. Therefore we can carry over the previous results to this case. The Feynman gauge propagator $G^{ab}_{F\mu\nu}(x,y)$-terms in those diagrams will be included. (F) will stand for the corresponding diagrams that we are discussing but in the Feynman gauge. $S_{Qjk}(x-y) = \delta_{jk}S_F(x-y)$ is a quark propagator.Indices i and n will refer to external quark fields ($\Gamma_1, \cdots \Gamma_5$ refer to Figures 5.57,$\cdots$ 5.61).

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{1in}(x,y,z) = & \ [\cdot\gamma^\mu(T^a)_{ij}S_{Qjk}(x-z)\gamma^\sigma(T^c)_{kl}S_{Qlm}(z-y)\gamma^\nu(T^b)_{mn}\cdot]\times \\
& \qquad G^{ab}_{\mu\nu}(x,y) \\
= & \ (F) + (T^aT^bT^c + if^{cbd}T^aT^d)_{in} \\
& \{-[\cdot(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-z)\gamma^\sigma S_F(z-y) \ f^{ab}_1(x,y)\cdot] \\
& -[\cdot \ f^{ab}_2(x,y)S_F(x-z)\gamma^\sigma S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\cdot]\}.
\end{aligned}
$$

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\nu}_{2in}(x,y,z) = & \ [\cdot\gamma^{\mu}(T^a)_{ij}S_{Qjk}(x-z)\gamma^{\sigma}(T^b)_{kl}S_{Qlm}(z-y)\gamma^{\nu}(T^c)_{mn}\cdot]\times \\
& \qquad\qquad\qquad\qquad G^{ab}_{\mu\sigma}(x,z) \\
= & \ (F)+(T^aT^bT^c)_{in} \\
& \quad \{-[\cdot(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)S_F(x-z)\ f_1^{ab}(x,z)S_F(z-y)\gamma^{\nu}\cdot] \\
& \quad -i[\cdot\ f_2^{ab}(x,z)[\delta(x-z)-\delta(y-z)]S_F(x-y)\gamma^{\nu}\cdot]\}.
\end{aligned}
$$

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\mu}_{3in}(x,y,z) = & \ [\cdot\gamma^{\mu}(T^c)_{ij}S_{Qjk}(x-z)\gamma^{\sigma}(T^a)_{kl}S_{Qlm}(z-y)\gamma^{\nu}(T^b)_{mn}\cdot]\times \\
& \qquad\qquad\qquad\qquad G^{ab}_{\sigma\nu}(z,y) \\
= & \ (F)+(T^aT^bT^c)_{in} \\
& \quad \{-i[\cdot\gamma^{\mu}[\delta(x-z)-\delta(z-y)]S_F(x-y)\ f_1^{ab}(z,y)\cdot] \\
& \quad -[\cdot\gamma^{\mu}S_F(x-z)\ f_2^{ab}(z,y)S_F(z-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\cdot]\}.
\end{aligned}
$$

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{4in}(x,y,z) = & \ [\cdot\gamma^{\mu}\cdot](T^a)_{in}G^{ab}_{\mu\nu}(x,y) \\
& \quad \mathrm{Tr}[S_{Qjk}(z-y)\gamma^{\nu}(T^b)_{kl}S_{Qlm}(y-z)\gamma^{\sigma}(T^c)_{mj}] \\
= & \ (F)-\tfrac{1}{2}\delta^{bc}(T^a)_{in}[\cdot(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)\cdot] \\
& \quad \mathrm{Tr}S_F(z-y)\ f_1^{ab}(x,y)S_F(y-z)\gamma^{\sigma}].
\end{aligned}
$$

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{5in}(x,y,z) = & \ f^{ced}V^{(c,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^{\mu}(T^a)_{ij}S_{Qjl}(x-y)\gamma^{\nu}\cdot](T^b)_{ln} \\
& \qquad\qquad G^{ae}_{\mu\alpha}(x,z)G^{db}_{\beta\nu}(z,y) \\
= & \ f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^{\mu}S_F(x-y)\gamma^{\nu}\cdot]\times \\
& \quad [G^{ae}_{F\mu\alpha}(x,z)+\partial^{(a)}_{x\mu}f^{ae}_{1\alpha}(x,z)+\partial^{(e)}_{z\alpha}f^{ae}_{2\mu}(x,z)]\times \\
& \quad [G^{db}_{F\beta\nu}(z,y)+\partial^{(d)}_{z\beta}f^{db}_{1\nu}(z,y)+\partial^{(b)}_{y\nu}f^{db}_{2\beta}(z,y)] \\
= & \ f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^{\mu}S_F(x-y)\gamma^{\nu}\cdot]\times \\
& \quad [G^{ae}_{F\mu\alpha}(x,z)G^{db}_{F\beta\nu}(z,y)+ \\
& \quad +(\partial^{(a)}_{x\mu}f^{ae}_{1\alpha}(x,z)+\partial^{(e)}_{z\alpha}f^{ae}_{2\mu}(x,z))G^{db}_{F\beta\nu}(z,y) \\
& \quad +G^{ae}_{\mu\alpha}(x,z)(\partial^{(d)}_{z\beta}f^{db}_{1\nu}(z,y)+\partial^{(b)}_{y\nu}f^{db}_{2\beta}(z,y))] \\
= & \ (F)+A^{c\sigma}_{in}(x,y,z)+B^{c\sigma}_{in}(x,y,z)
\end{aligned}
$$

with

$$
\begin{aligned}
A_{in}^{c\sigma}(x,y,z) = \quad & f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^\mu S_F(x-y)\gamma^\nu\cdot]\times \\
& [\partial_{x\mu}^{(a)}f_{1\alpha}^{ae}(x,z)+\partial_{z\alpha}^{(e)}f_{2\mu}^{ae}(x,z)]G_{F\beta\nu}^{db}(z,y) \\
B_{in}^{c\sigma}(x,y,z) = \quad & f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^\mu S_F(x-y)\gamma^\nu\cdot]\times \\
& [\partial_{z\beta}^{(d)}f_{1\nu}^{db}(z,y)+\partial_{y\nu}^{(b)}f_{2\beta}^{db}(z,y))]G_{\mu\alpha}^{ae}(x,z).
\end{aligned}
$$

Now

$$
\begin{aligned}
A_{in}^{c\sigma}(x,y,z) = \quad & f^{ced}(T^aT^b)_{in}G_{F\beta\nu}^{db}(z,y) \\
& \{[V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\partial_{z\alpha}^{(e)}][\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot] \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot(\overleftarrow{\not{\partial}}_x+\overrightarrow{\not{\partial}}_x)S_F(x-y)\gamma_\nu\cdot]f_{1\alpha}^{ae}(x,z)\} \\
= \quad & f^{ced}(T^aT^b)_{in}G_{F\beta\nu}^{db}(z,y) \\
& \{[O_z^{(d+e)\sigma\beta}-O_z^{(d)\sigma\beta}][\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot] \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot(\overleftarrow{\not{\partial}}_x+\overrightarrow{\not{\partial}}_x)S_F(x-y)\gamma^\nu\cdot]f_{1\alpha}^{ae}(x,z)\} \\
= \quad & f^{ced}(T^aT^b)_{in}G_{F\beta\nu}^{db}(z,y) \\
& \{O_z^{(d+e)\sigma\beta}[\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot] \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot(\overleftarrow{\not{\partial}}_x+\overrightarrow{\not{\partial}}_x)S_F(x-y)\gamma^\nu\cdot]f_{1\alpha}^{ae}(x,z)\} \\
& -f^{ced}(T^aT^b)_{in}[G_{\beta\nu}^{db}(z,y)-\partial_\beta^{(d)}f_{1\nu}^{db}(z,y)-\partial_\nu^{(b)}f_{2\beta}^{db}(z,y)]\times \\
& O_z^{(d)\sigma\beta}[\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot] \\
= \quad & f^{ced}(T^aT^b)_{in}\{O_z^{(d+e)\sigma\beta}G_{F\beta\nu}^{db}(z,y)[\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot] \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)f_{1\alpha}^{ae}(x,z)G_{F\beta\nu}^{db}(z,y)[\cdot(\overleftarrow{\not{\partial}}_x+\overrightarrow{\not{\partial}}_x)S_F(x-y)\gamma^\nu\cdot] \\
& -O_z^{(d)\sigma\beta}f_{2\beta}^{db}(z,y)[\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)(\overleftarrow{\not{\partial}}_y+\overrightarrow{\not{\partial}}_y)\cdot] \\
& -[\cdot\ \not{f}_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot][\delta_\nu^\sigma-z^\sigma(\partial z)^{-1}\partial_{z\nu}^{(d)}]\delta^{db}(z-y)\}
\end{aligned}
$$

$$
\begin{aligned}
B_{in}^{c\sigma}(x,y,z) = \quad & f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^\mu S_F(x-y)\gamma^\nu\cdot]\times \\
& [\partial_{z\beta}^{(d)}f_{1\nu}^{db}(z,y)+\partial_{y\nu}^{(b)}f_{2\beta}^{db}(z,y)]G_{\mu\alpha}^{ae}(x,z) \\
= \quad & f^{ced}(T^aT^b)_{in}\{[V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\partial_{z\beta}^{(d)}[\cdot\gamma^\mu S_F(x-y)\ \not{f}_1^{db}(z,y)\cdot]\times \\
& G_{\mu\alpha}^{ae}(x,z) \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^\mu S_F(x-y)(\overleftarrow{\not{\partial}}_y+\overrightarrow{\not{\partial}}_y)\cdot]\times \\
& f_{2\beta}^{db}(z,y)G_{\mu\alpha}^{ae}(x,z)\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & f^{ced}(T^aT^b)_{in}\{-O_z^{(e+d)\sigma\alpha}[\cdot\gamma^\mu S_F(x-y)\ f_1^{db}(z,y)\cdot]G^{ae}_{\mu\alpha}(x,z) \\
& +[\cdot\gamma^\mu S_F(x-y)\ f_1^{db}(z,y)\cdot]\times \\
& [\delta^\sigma_\mu - z^\sigma(\partial z)^{-1}\partial^{(e)}_{z\mu}]\delta^{ae}(x-z) \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[\cdot\gamma^\mu S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\cdot]\times \\
& f^{db}_{2\beta}(z,y)G^{ae}_{\mu\alpha}(x,z)\}.
\end{aligned}
$$

Collecting terms we get

$$
\begin{aligned}
-ig^{-2}\Gamma^{c\sigma}_{5in}(x,y,z) = \quad & (F)+f^{ced}(T^aT^b)_{in}O_z^{(d+e)\sigma\beta}\times \\
& \{G^{db}_{F\beta\nu}(z,y)[\cdot\ f_2^{ae}(x,z)S_F(x-y)\gamma^\nu\cdot] \\
& -[\cdot\gamma^\mu S_F(x-y)\ f_1^{db}(z,y)\cdot]G^{ae}_{\mu\beta}(x,z)\} \\
& -f^{ced}(T^aT^b)_{in}V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\times \\
& \{f^{ae}_{1\alpha}(x,z)G^{db}_{F\beta\nu}(z,y)[\cdot(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)S_F(x-y)\gamma^\nu\cdot] \\
& +f^{db}_{2\beta}(z,y)G^{ae}_{\mu\alpha}(x,z)[\cdot\gamma^\mu S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\cdot]\} \\
& -f^{ced}(T^aT^b)_{in}O_z^{(d)\sigma\beta}f^{db}_{2\beta}(z,y)\times \\
& [\cdot\ f_2^{ae}(x,z)S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\cdot] \\
& +f^{ced}(T^aT^b)_{in}[\delta^\sigma_\mu - z^\sigma(\partial z)^{-1}\partial^{(e)}_{z\mu}]\times \\
& \{[\cdot\ f_2^{ad}(x,z)S_F(x-y)\gamma^\mu\cdot]\delta^{eb}(z-y) \\
& +[\cdot\gamma^\mu S_F(x-y)\ f_1^{db}(z,y)\cdot]\delta^{ae}(z-x)\}.
\end{aligned}
$$

Also (see Figure 5.62)

$$
\begin{aligned}
g^{-2}\Pi_1^{ab\mu\nu}(x,y) = \quad & \tfrac{1}{2}f^{ace}V^{(a,c,e)\mu\rho\lambda}(\partial_x)f^{bfd}V^{(b,f,d)\nu\theta\sigma}(\partial_y)G^{cd}_{\rho\sigma}(x,y)G^{ef}_{\lambda\theta}(x,y) \\
= \quad & \tfrac{1}{2}f^{ace}f^{bfd}V^{(-,c,e)\mu\rho\lambda}(\partial_x)V^{(-,f,d)\nu\theta\sigma}(\partial_y) \\
& \{G^{cd}_{\rho\sigma}(x,y)[\partial^{(e)}_{x\lambda}f^{ef}_{1\theta}(x,y)+\partial^{(f)}_{y\theta}f^{ef}_{2\lambda}(x,y)] \\
& +[\partial^{(c)}_{x\rho}f^{cd}_{1\sigma}(x,y)+\partial^{(d)}_{y\sigma}f^{cd}_{2\rho}(x,y)]G^{ef}_{F\lambda\theta}(x,y) \\
& +G^{cd}_{F\rho\sigma}(x,y)G^{ef}_{F\lambda\theta}(x,y)\} \\
= \quad & (F)+\tfrac{1}{2}f^{ace}f^{bfd}\Big\{O_y^{\nu\sigma}\Big[V^{(-,c,e)\mu\rho\lambda}(\partial_x)\Big(G^{cd}_{\rho\lambda}(x,y)f^{ef}_{2\lambda}(x,y) \\
& -\ G^{ef}_{F\lambda\sigma}(x,y)f^{cd}_{2\rho}(x,y)\Big)+O_x^{(e)\mu\lambda}f^{cd}_{1\sigma}(x,y)f^{ef}_{2\lambda}(x,y)\Big] \\
& +O_x^{\mu\rho}\Big[-V^{(-,f,d)\nu\theta\sigma}(\partial_y)\Big(G^{cd}_{\rho\sigma}(x,y)f^{ef}_{1\theta}(x,y) \\
& -\ G^{ef}_{F\rho\theta}(x,y)f^{cd}_{1\sigma}(x,y)\Big)+O_y^{(f)\nu\theta}f^{cd}_{2\rho}(x,y)f^{ef}_{1\theta}(x,y)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+y^{\nu}V^{(-,c,e)\mu\rho\lambda}(\partial_x)\Big[f^{ef}_{2\lambda}(x,y)\Big((\partial y)^{-1}\partial^{(d)}_{y\rho}\delta^{cd}(x-y)\Big)\\
&\qquad\qquad - f^{cd}_{2\rho}(x,y)\Big((\partial y)^{-1}\partial^{(f)}_{y\lambda}\delta^{ef}(x-y)\Big)\Big]\\
&+x^{\mu}V^{(-,f,d)\nu\theta\sigma}(\partial_y)\Big[f^{cd}_{1\sigma}(x,y)\Big((\partial x)^{-1}\partial^{(e)}_{x\theta}\delta^{ef}(x-y)\Big)\\
&\qquad\qquad - f^{ef}_{1\theta}(x,y)\Big((\partial x)^{-1}\partial^{(c)}_{x\sigma}\delta^{cd}(x-y)\Big)\Big]\\
&-2V^{(-,c,e)\mu\nu\lambda}(\partial_x)f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y)\\
&+2V^{(-,f,d)\nu\lambda\mu}(\partial_y)f^{ef}_{1\lambda}(x,y)\delta^{cd}(x-y)\\
&-2\,O^{(f)\nu\theta}_{y}O^{(c)\mu\rho}_{x}f^{ef}_{1\theta}(x,y)f^{cd}_{2\rho}(x,y)\Big\}.
\end{aligned}
$$

The last term can be written as

$$
\begin{aligned}
& O^{(f)\nu\theta}_{y}O^{(c)\mu\rho}_{x}f^{ef}_{1\theta}(x,y)f^{cd}_{2\rho}(x,y)\\
= \;& \delta^{cd}\delta^{ef}\left[\left(O^{\mu\rho}_{x}f_{2\rho}(x,y)\right)(x\leftrightarrow y,\;\;\mu\leftrightarrow\nu)\right]\\
= \;& \delta^{cd}\delta^{ef}\Big\{\left(\Box_x g^{\mu\rho}-\partial^{\mu}_{x}\partial^{\rho}_{x}\right)\Big[\Box^{-1}_{x}x_{\rho}(\partial x)^{-1}\delta(x-y)\\
&\qquad - \tfrac{1}{2}\Box^{-1}_{x}\partial_{x\rho}x^{2}(\partial x)^{-2}\delta(x-y)\Big]\,(x\leftrightarrow y,\;\;\mu\leftrightarrow\nu)\Big\}\\
= \;& x^{\mu}y^{\nu}[(\partial x)^{-1}\delta^{cd}(x-y)][(\partial y)^{-1}\delta^{ef}(x-y)]\\
& -x^{\mu}(\partial x)^{-1}\delta^{cd}(x-y)][\Box^{-1}_{y}\partial^{\nu}_{y}\delta^{ef}(x-y)]\\
& -y^{\nu}[(\partial y)^{-1}\delta^{ef}(x-y)][\Box^{-1}_{x}\partial^{\mu}_{x}\delta^{cd}(x-y)]\\
& +[\Box^{-1}_{x}\partial^{\mu}_{x}\delta^{cd}(x-y)][\Box^{-1}_{y}\partial^{\nu}_{y}\delta^{ef}(x-y)].
\end{aligned}
$$

The $\partial^{\mu}_{x}\partial^{\nu}_{y}$-term is equivalent to the ghost diagram $[x \;\; y]$ in the Lorentz gauge. Further, the third term from the last of $\Pi^{ab\mu\nu}_{1}(x,y)$ can be written as

$$
\begin{aligned}
& -V^{(-,c,e)\mu\nu\lambda}(\partial_x)f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y)\\
= \;& [(2g^{\mu\nu}\partial^{(c)\lambda}_{x}-g^{\nu\lambda}\partial^{(c)\mu}_{x}-g^{\lambda\mu}\partial^{(c)\nu}_{x})-(2g^{\lambda\mu}\partial^{(e)\nu}_{x}-g^{\mu\nu}\partial^{(e)\lambda}_{x}-g^{\nu\lambda}\partial^{(e)\mu}_{x})]\\
&\qquad\qquad f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y)\\
= \;& [-2g^{\mu\nu}\partial^{(d)\lambda}_{y}+g^{\nu\lambda}\partial^{(d)\mu}_{y}+g^{\lambda\mu}\partial^{(d)\nu}_{y}]f^{ef}_{1\lambda}(x,y)\delta^{cd}(x-y)\\
& [-2g^{\mu\nu}\partial^{(f)\lambda}_{y}+g^{\nu\lambda}\partial^{(f)\mu}_{y}+g^{\lambda\mu}\partial^{(f)\nu}_{y}]f^{ef}_{1\lambda}(x,y)\delta^{cd}(x-y)\\
& [2g^{\mu\nu}\partial^{(f)\lambda}_{y}-g^{\nu\lambda}\partial^{(f)\mu}_{y}-g^{\lambda\mu}\partial^{(f)\nu}_{y}]f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y)\\
& [-2g^{\lambda\mu}\partial^{(f)\nu}_{y}+g^{\mu\nu}\partial^{(f)\lambda}_{y}+g^{\nu\lambda}\partial^{(f)\mu}_{y}]f^{ef}_{1\lambda}(x,y)\delta^{cd}(x-y).
\end{aligned}
$$

Therefore

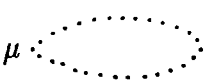
$$
\begin{aligned}
& -V^{(-,c,e)\mu\nu\lambda}(\partial_x)f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y)+V^{(-,f,d)\nu\lambda\mu}(\partial_y)f^{ef}_{1\lambda}(x,y)\delta^{cd}(x-y)\\
= \;& [2g^{\mu\nu}\partial^{(f)\lambda}_{y}-g^{\nu\lambda}\partial^{(f)\mu}_{y}-g^{\lambda\mu}\partial^{(f)\nu}_{y}]f^{ef}_{2\lambda}(x,y)\delta^{cd}(x-y).
\end{aligned}
$$

The final form of $\Pi_1^{ab\mu\nu}(x,y)$ is

$$
\begin{aligned}
\Pi_1^{ab\mu\nu}(x,y) = \ & (F) + (ghost) \\
& +\tfrac{1}{2} g^2 f^{ace} f^{bfd} \Bigl\{ O_y^{\nu\sigma} \Bigl[V^{(-,c,e)\mu\rho\lambda}(\partial_x) \Bigl(G_{\rho\lambda}^{cd}(x,y) f_{2\lambda}^{ef}(x,y) \\
& \qquad - G_{F\lambda\sigma}^{ef}(x,y) f_{2\rho}^{cd}(x,y) \Bigr) + O_x^{(e)\mu\lambda} f_{1\sigma}^{cd}(x,y) f_{2\lambda}^{ef}(x,y) \Bigr] \\
& \qquad + O_x^{\mu\rho} \Bigl[-V^{(-,f,d)\nu\theta\sigma}(\partial_y) \Bigl(G_{\rho\sigma}^{cd}(x,y) f_{1\theta}^{ef}(x,y) \\
& \qquad - G_{F\rho\theta}^{ef}(x,y) f_{1\sigma}^{cd}(x,y) \Bigr) + O_y^{(f)\nu\theta} f_{2\rho}^{cd}(x,y) f_{1\theta}^{ef}(x,y) \Bigr] \\
& \qquad + y^\nu \Bigl[V^{(-,c,e)\mu\rho\lambda}(\partial_x) \Bigl(f_{2\lambda}^{ef}(x,y)(\partial y)^{-1} \partial_{y\rho}^{(d)} \delta^{cd}(x-y) \\
& \qquad\qquad\qquad\qquad - f_{2\rho}^{cd}(x,y)(\partial y)^{-1} \partial_{y\lambda}^{(f)} \delta^{ef}(x-y) \Bigr) \\
& \qquad\qquad + 2((\partial y)^{-1} \delta^{ef}(x-y)) \Box_x^{-1} \partial_x^\mu \delta^{cd}(x-y) \\
& \qquad\qquad - x^\mu ((\partial x)^{-1} \delta^{cd}(x-y))(\partial y)^{-1} \delta^{ef}(x-y) \Bigr] \\
& \qquad + x^\mu \Bigl[V^{(-,f,d)\nu\theta\sigma}(\partial_y) \Bigl(f_{1\sigma}^{cd}(x,y)(\partial x)^{-1} \partial_{x\theta}^{(e)} \delta^{ef}(x-y) \\
& \qquad\qquad\qquad\qquad - f_{1\theta}^{ef}(x,y)(\partial x)^{-1} \partial_{x\sigma}^{(c)} \delta^{cd}(x-y) \Bigr) \\
& \qquad\qquad + 2((\partial x)^{-1} \delta^{cd}(x-y)) \Box_y^{-1} \partial_y^\nu \delta^{ef}(x-y) \\
& \qquad\qquad - y^\nu ((\partial x)^{-1} \delta^{cd}(x-y))(\partial y)^{-1} \delta^{ef}(x-y) \Bigr] \\
& \qquad + 2\ [2g^{\mu\nu} \partial_y^{(f)\lambda} - g^{\nu\lambda} \partial_y^{(f)\mu} - g^{\lambda\mu} \partial_y^{(f)\nu}] f_{2\lambda}^{ef}(x,y) \delta^{cd}(x-y) \Bigr\}
\end{aligned}
$$

where $(ghost)$ in $\Pi_1^{ab\mu\nu}(x,y)$ equals

$$
(ghost) = \ g^2 f^{ace} f^{bdf} [\Box_x^{-1} \partial_x^\mu \delta^{cd}(x-y)][\Box_y^{-1} \partial_y^\nu \delta^{ef}(x-y)
$$

$= \quad x, \mu$  y, ν

The contribution of the four-gluon vertex is (see Figure 5.64)

$$
\begin{aligned}
\Pi_{2\mu\nu}^{ab}(x,y) = \ & \tfrac{1}{2} g^2 W_{\mu\nu\rho\sigma}^{abcd} \delta(x-y) G^{cd\rho\sigma}(x,y) \\
= \ & \tfrac{1}{2} g^2 [f^{eab} f^{ecd} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f^{eac} f^{edb} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma} \\
& \quad + f^{ead} f^{ebc} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}] \times \\
& \qquad [G_F^{\rho\sigma}(x,y) + \partial_x^\rho f_1^\sigma(x,y) + \partial_x^\sigma f_2^\rho(x,y)] \delta^{cd}(x-y) \\
= \ & (F) + \tfrac{1}{2} g^2 f^{ace} f^{bfd} [g_{\nu\lambda} \partial_{y\mu}^{(f)} + g_{\mu\lambda} \partial_{y\nu}^{(f)} - 2 g_{\mu\nu} \partial_{y\lambda}^{(f)} + g_{\mu\lambda} \partial_{y\nu}^{(f)} \\
& \qquad + g_{\nu\lambda} \partial_{y\mu}^{(f)} - 2 g_{\mu\nu} \partial_{y\lambda}^{(f)}] f_2^{ef\lambda}(x,y) \delta^{cd}(x-y) \\
= \ & (F) \\
& + g^2 f^{ace} f^{bfd} [-2 g_{\mu\nu} \partial_{y\lambda}^{(f)} + g_{\nu\lambda} \partial_{y\mu}^{(f)} + g_{\lambda\mu} \partial_{y\nu}^{(f)}] f_2^{ef\lambda}(x,y) \delta^{cd}(x-y).
\end{aligned}
$$

Summing,

$$\begin{aligned}
\Pi^{ab\mu\nu}(x,y) = \;& \Pi_1^{ab\mu\nu}(x,y) + \Pi_2^{ab\mu\nu}(x,y) \\
= \;& (F) + (ghost) + \\
& \tfrac{1}{2} f^{ace} f^{bfd} [O_y^{\nu\sigma} H_{1\sigma}^{cdef\mu}(x,y) + O_x^{\mu\sigma} H_{1\sigma}^{cdef\nu}(y,x) \\
& \qquad + y^\nu H_2^{cdef\mu}(x,y) + x^\mu H_2^{cdef\nu}(y,x)]
\end{aligned}$$

where

$$\begin{aligned}
H_{1\sigma}^{cdef\mu}(x,y) = \;& O_x^{(e)\mu\lambda} f_{1\sigma}^{cd}(x,y) f_{2\lambda}^{ef}(x,y) \\
& + V^{(-,c,e)\mu\rho\lambda}(\partial_x)[G_{\rho\sigma}^{cd}(x,y) f_{2\lambda}^{ef}(x,y) - G_{F\lambda\sigma}^{ef} f_{2\rho}^{cd}(x,y)] \\
H_2^{cdef\mu}(x,y) = \;& V^{(-,c,e)\mu\rho\lambda}(\partial_x)[f_{2\lambda}^{ef}(x,y)(\partial y)^{-1}\partial_{y\rho}\delta^{cd}(x-y) \\
& \qquad - f_{2\rho}^{cd}(x,y)(\partial y)^{-1}\partial_{y\lambda}\delta^{ef}(x-y)] \\
& + 2[(\partial y)^{-1}\delta^{ef}(x-y)]\Box_x^{-1}\partial_x^\mu \delta^{cd}(x-y) \\
& - x^\mu[(\partial x)^{-1}\delta^{cd}(x-y)][(\partial y)^{-1}\delta^{ef}(x-y)].
\end{aligned}$$

The term $(F)+(ghost)$ above is nothing but the gluon self-energy $\Pi_F^{ab\mu\nu}(x,y)$ in the Feynman gauge. From $\Pi^{ab\mu\nu}(x,y)$ we obtain (see Figure 5.65)

$$\begin{aligned}
\Gamma_{6in}^{b\nu}(x,y,z) = \;& \Gamma_{6_1in}^{b\nu}(x,y,z) + \Gamma_{6_2in}^{b\nu}(x,y,z) \\
= \;& i[\cdot\gamma^\epsilon\cdot] G_{\epsilon\mu}^{sa}(z,x) \Pi^{ab\mu\nu}(x,y)(T^s)_{in} \\
= \;& (F) + (ghost) \\
& + i[\cdot\gamma^\epsilon\cdot](T^s)_{in}[\partial_{z\epsilon}^{(s)} f_{1\mu}^{sa}(z,x) + \partial_{x\mu}^{(a)} f_{2\epsilon}^{sa}(z,x)]\Pi^{ab\mu\nu}(x,y) \\
& + \tfrac{i}{2} g^2 f^{ace} f^{bfd} [\cdot\gamma^\epsilon\cdot](T^s)_{in} G_{F\epsilon\mu}^{sa}(z,x) \times \\
& \qquad [O_y^{\nu\sigma} H_{1\sigma}^{cdef\mu}(x,y) + O_x^{\mu\sigma} H_{1\sigma}^{cdef\nu}(y,x) \\
& \qquad + y^\nu H_2^{cdef\mu}(x,y) + x^\mu H_2^{cdef\nu}(y,x)] \\
= \;& (F) + (ghost) + \\
& + i[\cdot\gamma^\epsilon\cdot](T^s)_{in}[\partial_{z\epsilon}^{(s)} f_{1\mu}^{sa}(z,x) + \partial_{x\mu}^{(a)} f_{2\epsilon}^{sa}(z,x)]\Pi_F^{ab\mu\nu}(x,y) \\
& + \tfrac{i}{2} g^2 f^{ace} f^{bfd} [\cdot\gamma^\epsilon\cdot](T^s)_{in} G_{\epsilon\mu}^{sa}(z,x) \times \\
& \qquad [O_y^{\nu\sigma} H_{1\sigma}^{cdef\mu}(x,y) + O_x^{\mu\sigma} H_{1\sigma}^{cdef\nu}(y,x) \\
& \qquad + y^\nu H_2^{cdef\mu}(x,y) + x^\mu H_2^{cdef\nu}(y,x)]
\end{aligned}$$

$$
\begin{aligned}
= \;& (F) + (ghost) + \\
& +i[\cdot\gamma^{\epsilon}\cdot](T^s)_{in}\Big\{\partial_{z\epsilon}[f^{sa}_{1\mu}(z,x)\Pi^{ab\mu\nu}(x,y)] \\
& \qquad +\partial_{x\mu}[f^{sa}_{2\epsilon}(z,x)\Pi^{ab\mu\nu}_{F}(x,y)] - f^{sa}_{2\epsilon}(z,x)\partial_{x\mu}\Pi^{ab\mu\nu}_{F}(x,y)\Big\} \\
& +\tfrac{i}{2}g^2 f^{ace} f^{bfd}[\cdot\gamma^{\epsilon}\cdot](T^s)_{in}G^{sa}_{\epsilon\mu}(z,x)\times \\
& \qquad [O^{\nu\sigma}_{y} H^{cdef\mu}_{1\sigma}(x,y) + O^{\mu\sigma}_{x} H^{cdef\nu}_{1\sigma}(y,x) + y^{\nu} H^{cdef\mu}_{2}(x,y)].
\end{aligned}
$$

Since the inner vertex is represented by the x variable, the term

$$\partial_{x\mu}[f^{sa}_{2\epsilon}(z,x)\Pi^{ab\mu\nu}_{F}(x,y)]$$

is just a surface term that may be discarded. The term containing a factor

$$\partial_{x\mu}\Pi^{ab\mu\nu}_{F}(x,y)$$

vanishes because the gluon self-energy in the Feynman gauge is transverse. Thus we get

$$
\begin{aligned}
\Gamma^{b\nu}_{6in}(x,y,z) = \;& (F) + (ghost) + i[\cdot\gamma^{\epsilon}\cdot](T^s)_{in}\partial_{z\epsilon}f^{sa}_{1\mu}(z,x)\Pi^{ab\mu\nu}(x,y) \\
& +\tfrac{i}{2}g^2 f^{ace} f^{bfd}[\cdot\gamma^{\epsilon}\cdot](T^s)_{in}G^{sa}_{\epsilon\mu}(z,x)\times \\
& \qquad [O^{\nu\sigma}_{y} H^{cdef\mu}_{1\sigma}(x,y) + O^{\mu\sigma}_{x} H^{cdef\nu}_{1\sigma}(y,x) + y^{\nu} H^{cdef\mu}_{2}(x,y)].
\end{aligned}
$$

G.3.3 On-shell Diagrams

Vertex corrections (Figures 5.68-5.71)

$$
\begin{aligned}
S_1 = \;& ig\int dx\int dy\int dz\overline{\psi}_i(x)\Gamma^{c\sigma}_{1in}(x,y,z)A^c_\sigma(z)\psi_n(y) \\
= \;& (F) - g^3(T^aT^bT^c + if^{cbd}T^aT^d)_{in}\int dx\int dy\int dz \\
& \quad \{-[\overline{\psi}_i(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-z)\ \not A^c(z)S_F(z-y)\ \not f^{ab}_1(x,y)\psi_n(y)] \\
& \quad -[\overline{\psi}_i(x)\ \not f^{ab}_2(x,y)S_F(x-z)\ \not A^c(z)S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\psi_n(y)]\} \\
\overset{ms}{=} \;& (F) - ig^3(T^aT^bT^c + if^{cbd}T^aT^d)_{in}\int dx\int dy\int dz \\
& \quad \{[\overline{\psi}_i(x)\delta(x-z)\ \not A^c(z)S_F(z-y)\ \not f^{ab}_1(x,y)\psi_n(y)] \\
& \quad -[\overline{\psi}_i(x)\ \not f^{ab}_2(x,y)S_F(x-z)\ \not A^c(z)\delta(z-y)\psi_n(y)]\}.
\end{aligned}
$$

$$
\begin{aligned}
S_2 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(x)\Gamma^{c\nu}_{2in}(x,y,z)A^c_\nu(y)\psi_n(y) \\
= \;& (F) - g^3(T^aT^bT^c)_{in} \int dx \int dy \int dz \\
& \{-[\overline{\psi}_i(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-z)\ \not f_1^{ab}(x,z)S_F(z-y)\ \not A^c(y)\psi_n(y)] \\
& -i[\overline{\psi}_i(x)\ \not f_2^{ab}(x,z)[\delta(x-z)-\delta(y-z)]S_F(x-y)\ \not A^c(y)\psi_n(y)]\} \\
= \;& (F) - ig^3(T^aT^bT^c)_{in} \int dx \int dy \int dz \\
& \overline{\psi}_i(x)\ \not f_2^{ab}(x,z)\delta(y-z)S_F(x-y)\ \not A^c(y)\psi_n(y).
\end{aligned}
$$

$$
\begin{aligned}
S_3 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(x)\Gamma^{c\mu}_{3in}(x,y,z)A^c_\mu(x)\psi_n(y) \\
= \;& (F) - g^3(T^aT^bT^c)_{in} \int dx \int dy \int dz \\
& \{-i[\overline{\psi}_i(x)\ \not A^c(x)[\delta(x-z)-\delta(z-y)]S_F(x-y)\ \not f_1^{ab}(z,y)\psi_n(y)] \\
& -[\overline{\psi}_i(x)\ \not A^c(x)S_F(x-z)\ \not f_2^{ab}(z,y)S_F(z-y)(\overleftarrow{\not\partial}_y + \overrightarrow{\not\partial}_y)\psi_n(y)]\} \\
= \;& (F) + ig^3(T^aT^bT^c)_{in} \int dx \int dy \int dz \\
& \overline{\psi}_i(x)\ \not A^c(x)\delta(x-z)S_F(x-y)\ \not f_1^{ab}(z,y)\psi_n(y).
\end{aligned}
$$

$$
\begin{aligned}
S_4 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(x)\Gamma^{c\sigma}_{4in}(x,y,z)A^c_\sigma(z)\psi_n(y) \\
= \;& (F) + \tfrac{1}{2}g^3\delta^{bc}(T^a)_{in} \int dx \int dy \int dz \\
& \overline{\psi}_i(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)\psi_n(y)\mathrm{Tr}S_F(z-y)\ \not f_1(x,y)S_F(y-z)\ \not A^c(z) \\
= \;& (F) + \tfrac{1}{2}g^3\delta^{bc}(T^a)_{in} \int dx \int dy \int dz \\
& \overline{\psi}_i(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)\psi_n(y)\mathrm{Tr}S_F(z-y)\ \not f_1^{ab}(x,y)S_F(y-z)\ \not A^c(z) \\
\overset{ms}{=} \;& (F).
\end{aligned}
$$

$$
\begin{aligned}
S_5 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(x)\Gamma^{c\sigma}_{5in}(x,y,z)A^c_\sigma(z)\psi_n(y) \\
= \;& (F) - g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z)O_z^{(d+e)\sigma\beta} \\
& \{G^{db}_{F\beta\nu}(z,y)\overline{\psi}_i(x)\ \not f_2^{ae}(x,z)S_F(x-y)\gamma^\nu\psi_n(y) \\
& -G^{ae}_{\mu\beta}(x,z)\overline{\psi}_i(x)\gamma^\mu S_F(x-y)\ \not f_1^{db}(z,y)\psi_n(y) \\
& -f^{ae}_{1\alpha}(x,z)\overline{\psi}_i(x)(\overleftarrow{\not\partial}_x + \overrightarrow{\not\partial}_x)S_F(x-y)\ \not f_1^{db}(z,y)\psi_n(y)\}
\end{aligned}
$$

$$
\begin{aligned}
&+g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z) V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\times\\
&\qquad\{f^{ae}_{1\alpha}(x,z)G^{db}_{\beta\nu}(z,y)\overline{\psi}_i(x)(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)S_F(x-y)\gamma^\nu\psi_n(y)\\
&\qquad-f^{ae}_{1\alpha}(y,z)G^{db}_{\beta\nu}(z,x)\overline{\psi}_i(x)\gamma^\nu S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\psi_n(y)\\
&\qquad+f^{ae}_{1\alpha}(x,z)f^{db}_{2\beta}(z,y)\overline{\psi}_i(x)(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\psi_n(y)\}\\
&+g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z) O^{(d)\sigma\beta}_z\times\\
&\qquad\{f^{db}_{2\beta}(z,y)\overline{\psi}_i(x)\ \not f^{ae}_2(x,z)S_F(x-y)(\overleftarrow{\not\partial}_y+\overrightarrow{\not\partial}_y)\psi_n(y)\\
&\qquad-f^{db}_{2\beta}(z,x)\overline{\psi}_i(x)(\overleftarrow{\not\partial}_x+\overrightarrow{\not\partial}_x)S_F(x-y)\ \not f^{ae}_1(z,y)\psi_n(y)\}\\
&-g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z)\\
&\qquad\{-\overline{\psi}_i(x)\ \not f^{ae}_2(x,z)S_F(x-y)\gamma^\sigma\psi_n(y)\delta^{db}(z-y)\\
&\qquad+\overline{\psi}_i(x)\gamma^\sigma S_F(x-y)\ \not f^{db}_1(z,y)\psi_n(y)\delta^{ae}(z-y)\}\\
&+g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z) z^\sigma\partial^{(e)}_{z\mu}\\
&\qquad\{\overline{\psi}_i(x)\ \not f^{ad}_2(x,z)S_F(x-y)\gamma^\mu\psi_n(y)(\partial z)^{-1}\delta^{eb}(z-y)\\
&\qquad+\overline{\psi}_i(x)\gamma^\mu S_F(x-y)\ \not f^{db}_1(z,y)\psi_n(y)(\partial z)^{-1}\delta^{ae}(z-x)\}
\end{aligned}
$$

$$
\begin{aligned}
\overset{ms}{=}\ &(F)-g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz\{[O^{\beta\sigma}_z A^c_\sigma(z)]A^{abde}_\beta(x,y,z)\\
&\qquad\qquad\qquad\qquad+z^\sigma A^c_\sigma(z)B^{abde}(x,y,z)\}\\
&-g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z) V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\times\\
&\qquad\{2if^{ae}_{1\alpha}(x,z)G^{db}_{\beta\nu}(z,y)\overline{\psi}_i(x)\delta(x-y)\gamma^\nu\psi_n(y)\\
&\qquad-if^{ae}_{1\alpha}(x,z)f^{db}_{2\beta}(z,y)\overline{\psi}_i(x)(\overrightarrow{\not\partial}_x+im)\delta(x-y)\psi_n(y)\}\\
&+g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z) O^{(d)\sigma\beta}_z\times\\
&\qquad 2if^{db}_{2\beta}(z,y)\overline{\psi}_i(x)\ \not f^{ae}_2(x,z)\delta(x-y)\psi_n(y)\\
&-g^3 f^{ced}(T^aT^b)_{in}\int dx\int dy\int dz A^c_\sigma(z)\\
&\qquad\{-\overline{\psi}_i(x)\ \not f^{ae}_2(x,z)S_F(x-y)\gamma^\sigma\psi_n(y)\delta^{db}(z-y)\\
&\qquad+\overline{\psi}_i(x)\gamma^\sigma S_F(x-y)\ \not f^{db}_1(z,y)\psi_n(y)\delta^{ae}(z-x)\}
\end{aligned}
$$

$$
\begin{aligned}
\overset{ms}{=} \;& (F) - g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z) V^{(-,e,d)\sigma\alpha\beta}(\partial_z) \times \\
& \{2i f^{ae}_{1\alpha}(x,z) G^{db}_{\beta\nu}(z,y) \overline{\psi}_i(x) \delta(x-y) \gamma^\nu \psi_n(y) \\
& - i f^{ae}_{1\alpha}(x,z) f^{db}_{2\beta}(z,y) \overline{\psi}_i(x) (\overrightarrow{\not\partial}_x + im) \delta(x-y) \psi_n(y)\} \\
& + g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z) O^{(d)\sigma\beta}_z \times \\
& 2i f^{db}_{2\beta}(z,y) \overline{\psi}_i(x) \; f^{ae}_2(x,z) \delta(x-y) \psi_n(y) \\
& - g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z) \\
& \{-\overline{\psi}_i(x) \; f^{ae}_2(x,z) S_F(x-y) \gamma^\sigma \psi_n(y) \delta^{db}(z-y) \\
& + \overline{\psi}_i(x) \gamma^\sigma S_F(x-y) \; f^{db}_1(z,y) \psi_n(y) \delta^{ae}(z-x)\}.
\end{aligned}
$$

In reaching the above result, we use the free field equation

$$
\begin{aligned}
O^{\sigma\beta}_z A^c_\sigma(z) = \;& (\Box_z g^{\sigma\beta} - \partial^\sigma_z \partial^\beta_z) A^c_\sigma(z) = -J^{c\beta}(z) - z^\beta C \\
= \;& -J^{c\beta}(z) - z^\beta (\partial z)^{-1} \partial_z \cdot J^c(z) \overset{J=0}{=} 0
\end{aligned}
$$

and the gauge condition $z^\sigma A^c_\sigma(z) = 0$. Above,

$$
\begin{aligned}
A^{abde}_\beta(x,y,z) = \;& \{G^{db}_{F\beta\nu}(z,y) \overline{\psi}_i(x) \; f^{ae}_2(x,z) S_F(x-y) \gamma^\nu \psi_n(y) \\
& - G^{ae}_{\mu\beta}(x,z) \overline{\psi}_i(x) \gamma^\mu S_F(x-y) \; f^{db}_1(z,y) \psi_n(y) \\
& + i f^{ae}_{1\alpha}(x,z) \overline{\psi}_i(x) \delta(x-y) \; f^{db}_1(z,y) \psi_n(y)\} \\
B^{abde}(x,y,z) = \;& -\partial^{(e)}_{z\mu} \{\overline{\psi}_i(x) \; f^{ad}_2(x,z) S_F(x-y) \gamma^\mu \psi_n(y) (\partial z)^{-1} \delta^{eb}(z-y) \\
& + \overline{\psi}_i(x) \gamma^\mu S_F(x-y) \; f^{db}_1(z,y) \psi_n(y) (\partial z)^{-1} \delta^{ae}(z-y)\}.
\end{aligned}
$$

$$
\begin{aligned}
S_6 = \;& ig \int dx \int dy \int dz \overline{\psi}_i(z) \Gamma^{b\nu}_{6in}(x,y,z) A^b_\nu(y) \psi_n(z) \\
= \;& (F) + (ghost) + \\
& -g(T^s)_{in} \int dx \int dy \int dz \overline{\psi}_i(z) \gamma^\epsilon \psi_n(z) A^b_\nu(y) \times \\
& \{\partial_{z\epsilon} f^{sa}_{1\mu}(z,x) \Pi^{ab\mu\nu}(x,y) + \tfrac{1}{2} g^2 f^{ace} f^{bfd} G^{sa}_{\epsilon\mu}(z,x) \times \\
& [O^{\nu\sigma}_y H^{cdef\mu}_{1\sigma}(x,y) + O^{\mu\sigma}_x H^{cdef\nu}_{1\sigma}(y,x) + y^\nu H^{cdef\mu}_2(x,y)]\} \\
= \;& (F) + (ghost) - g(T^s)_{in} \int dx \int dy \int dz \\
& \{[\overline{\psi}_i(z)(\overleftarrow{\not\partial}_z + \overrightarrow{\not\partial}_z) \psi_n(z)] A^b_\nu(y) f^{sa}_{1\mu}(z,x) \Pi^{ab\mu\nu}(x,y) \\
& + \tfrac{1}{2} g^2 f^{ace} f^{bfd} \overline{\psi}_i(z) \gamma^\epsilon \psi_n(z) A^b_\nu(y) [O^{\mu\sigma}_x G^{sa}_{\epsilon\mu}(z,x)] H^{cdef\nu}_{1\sigma}(y,x)\}
\end{aligned}
$$

after recalling $A(y)\cdot y = 0$ and $O_y^{\nu\sigma} A_\nu^b \overset{J=0}{=} 0$.

Now

$$
\begin{aligned}
S_6 \overset{ms}{=} \;& (F) + (ghost) - \tfrac{1}{2} g^3 f^{ace} f^{bfd} (T^s)_{in} \int dx \int dy \int dz \overline{\psi}_i(z) \gamma^\epsilon \psi_n(z) A_\nu^b(y) \times \\
& \quad [\delta_\epsilon^\sigma + x^\sigma (\partial x)^{-1} \partial_{z\epsilon}] \delta^{sa}(z-x) H_{1\sigma}^{cdef\nu}(y,x) \\
= \;& (F) + (ghost) - \tfrac{1}{2} g^3 f^{ace} f^{bfd} (T^s)_{in} \int dx \int dy \int dz H_{1\sigma}^{cdef\nu}(y,x) A_\nu^b(y) \times \\
& \quad \overline{\psi}_i(z) [\gamma^\sigma \delta^{sa}(z-x) - \overline{\psi}_i(z) (\overleftarrow{\not\partial}_z + \overrightarrow{\not\partial}_z) x^\sigma (\partial x)^{-1} \delta^{sa}(z-x)] \psi_n(z) \\
= \;& (F) + (ghost) + \\
& -\tfrac{1}{2} g^3 f^{ace} f^{bfd} (T^s)_{in} \int dx \int dy \int dz \overline{\psi}_i(z) \gamma^\sigma \psi_n(z) A_\nu^b(y) \delta^{sa}(z-x) \times \\
& \quad \{ V^{(-,d,f)\nu\rho\sigma}(\partial_y) [G_{\sigma\rho}^{cd}(x,y) f_{1\lambda}^{ef}(x,y) - G_{F\sigma\lambda}^{ef}(x,y) f_{1\rho}^{cd}(x,y)] \\
& \qquad + O_y^{(f)\nu\lambda} f_{2\lambda}^{ef}(y,x) f_{2\sigma}^{cd}(x,y) \} \\
= \;& (F) + (ghost) + \\
& -\tfrac{1}{2} g^3 f^{ace} f^{bfd} (T^s)_{in} \int dx \int dy \int dz \overline{\psi}_i(z) \gamma^\sigma \psi_n(z) A_\nu^b(y) \delta^{sa}(z-x) \times \\
& \quad \{ V^{(-,d,f)\nu\rho\sigma}(\partial_y) [G_{\sigma\rho}^{cd}(x,y) f_{1\lambda}^{ef}(x,y) + \\
& \qquad - (G_{\sigma\lambda}^{ef}(x,y) - \partial_{x\sigma}^{(e)} f_{1\lambda}^{ef}(x,y) - \partial_{y\lambda}^{(f)} f_{2\sigma}^{ef}(x,y)) f_{1\rho}^{cd}(x,y)] \\
& \qquad + O_y^{(f)\nu\lambda} f_{2\lambda}^{ef}(y,x) f_{2\sigma}^{cd}(x,y) \} \\
= \;& (F) + (ghost) + \\
& -\tfrac{1}{2} g^3 f^{ace} f^{bfd} (T^s)_{in} \int dx \int dy \int dz \{ \overline{\psi}_i(z) \gamma^\sigma \psi_n(z) A_\nu^b(y) \delta^{sa}(z-x) \times \\
& \quad [V^{(-,d,f)\nu\rho\lambda}(\partial_y) (G_{\sigma\rho}^{cd}(x,y) f_{1\lambda}^{ef}(x,y) - G_{\sigma\lambda}^{ef}(x,y) f_{1\rho}^{cd}(x,y)) + \\
& \quad -(O_y^{(d+f)\nu\rho} - O_y^{(d)\nu\rho}) f_{2\sigma}^{ef}(x,y)) f_{1\rho}^{cd}(x,y) + O_y^{(f)\nu\lambda} f_{2\lambda}^{ef}(y,x) f_{2\sigma}^{cd}(x,y)] \\
& \quad - V^{(-,d,f)\nu\rho\lambda}(\partial_y) \overline{\psi}_i(z) [(\overrightarrow{\not\partial}_x + im) \delta^{sa}(x-z)] f_{1\lambda}^{ef}(x,y) f_{1\rho}^{cd}(z,y) A_\nu^b(y) \}
\end{aligned}
$$

$$
\begin{aligned}
= \;& (F) + (ghost) - \tfrac{1}{2} g^3 f^{ace} f^{bfd} (T^s)_{in} \int dx \int dy \int dz \overline{\psi}_i(z) \times \\
& \quad \{ 2\gamma^\sigma A_\nu^b(y) \delta^{sa}(x-z) V^{(-,d,f)\nu\rho\lambda}(\partial_y) G_{\sigma\rho}^{cd}(x,y) f_{1\lambda}^{ef}(x,y) \\
& \quad + 2\gamma^\sigma A_\nu^b(y) \delta^{sa}(x-z) O_y^{(d)\nu\rho} f_{2\sigma}^{ef}(x,y) f_{1\rho}^{cd}(x,y) \\
& \quad - V^{(-,d,f)\nu\rho\lambda}(\partial_y) [(\overrightarrow{\not\partial}_x + im) \delta^{sa}(x-z)] f_{1\lambda}^{ef}(x,y) f_{1\rho}^{cd}(z,y) A_\nu^b(y) \} \psi_n(z).
\end{aligned}
$$

The group factors can be combined as follows,

$$
f^{ace} f^{bfd} T^s \delta^{sa} \delta^{cd} \delta^{ef} = 2i T^e T^c f^{bfd} \delta^{cd} \delta^{ef} = 2i T^a T^c f^{bfd} \delta^{cd} \delta^{af}.
$$

Hence

$$\begin{aligned}
S_6 = \;& (F) + (ghost) - ig^3 f^{bfd}(T^aT^c)_{in} \int dx \int dy \int dz \overline{\psi}_i(z) \times \\
& \{2\gamma^\sigma A^b_\nu(y)\delta(x-z)V^{(-,d,f)\nu\rho\lambda}(\partial_y)G^{cd}_{\sigma\rho}(x,y)f^{af}_{1\lambda}(x,y) \\
& +2\gamma^\sigma A^b_\nu(y)\delta(x-z)O^{(d)\nu\rho}_y f^{af}_{2\sigma}(x,y)f^{cd}_{1\rho}(x,y) \\
& -V^{(-,d,f)\nu\rho\lambda}(\partial_y)[(\overrightarrow{\not\partial}_x + im)\delta(x-z)]f^{af}_{1\lambda}(x,y)f^{cd}_{1\rho}(z,y)A^b_\nu(y)\}\psi_n(z) \\
= \;& (F) + (ghost) - ig^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz A^c_\sigma(z)\overline{\psi}_i(x) \times \\
& \{-2V^{(-,e,d)\sigma\alpha\beta}(\partial_z)\delta(x-y)\gamma^\nu G^{db}_{\beta\nu}(z,y)f^{ae}_{1\alpha}(x,z) \\
& +2\delta(x-y)O^{(d)\sigma\beta}_z \; f^{ae}_2(x,z)f^{db}_{2\beta}(z,y) \\
& -V^{(-,e,d)\sigma\alpha\beta}(\partial_z)[(\overrightarrow{\not\partial}_y - im)\delta(x-y)]f^{ae}_{1\alpha}(x,z)f^{db}_{2\beta}(z,y)\}\psi_n(y).
\end{aligned}$$

Thus we obtain in all

$$\begin{aligned}
S = \;& S_1 + S_2 + S_3 + S_4 + S_5 + S_6 \\
\overset{ms}{=} \;& (F) + (ghost) + \\
& -ig^3(T^aT^bT^c)_{in} \int dx \int dy \int dz \overline{\psi}_i(x) \\
& \quad \{\delta(x-z) \; \not A^c(z)S_F(z-y) \; \not f^{ab}_1(x,y) + \\
& \quad - \; \not f^{ab}_2(x,y)S_F(x-z) \; \not A^c(z)\delta(z-y) \\
& \quad + \; \not f^{ab}_2(x,z)\delta(y-z)S_F(x-y) \; \not A^c(y) + \\
& \quad - \; \not A^c(x)\delta(x-z)S_F(x-y) \; \not f^{ab}_1(z,y)\}\psi_n(y) \\
& -g^3 f^{ced}(T^aT^b)_{in} \int dx \int dy \int dz \overline{\psi}_i(x)A^c_\sigma(z) \\
& \quad \{-\gamma^\sigma S_F(z-y) \; \not f^{db}_1(x,y)\delta^{ae}(x-z) + \\
& \quad + \; \not f^{ae}_2(x,y)S_F(x-z)\gamma^\sigma\delta^{db}(z-y) + \\
& \quad +\delta^{ae}(x-z)\gamma^\sigma S_F(z-y) \; \not f^{db}_1(x,y) + \\
& \quad -2i\delta(x-y)O^{(d)\sigma\beta}_z \; \not f^{ae}_2(x,z)f^{db}_{2\beta}(z,y) + \\
& \quad +2iV^{(-,e,d)\sigma\alpha\beta}(\partial_z)f^{ae}_{1\alpha}(x,z)\gamma^\nu G^{db}_{\beta\nu}(z,y)\delta(x-y) + \\
& \quad -\delta^{db}(z-y) \; \not f^{ae}_2(x,z)S_F(x-y)\gamma^\sigma + \\
& \quad +iV^{(-,e,d)\sigma\alpha\beta}(\partial_z)f^{ae}_{1\alpha}(x,z)f^{db}_{2\beta}(z,y)[(\not\partial_y - im)\delta(x-y)] \\
& \quad -2iV^{(-,e,d)\sigma\alpha\beta}(\partial_z)\delta(x-y)\gamma^\nu G^{db}_{\beta\nu}(z,y)f^{ae}_{1\alpha}(x,z) + \\
& \quad +2i\delta(x-y)O^{(d)\sigma\beta}_z \; \not f^{ae}_2(x,z)f^{db}_{2\beta}(z,y) \\
& \quad -iV^{(-,e,d)\sigma\alpha\beta}(\partial_z)[(\overrightarrow{\not\partial}_y - im)\delta(x-y)]f^{ae}_{1\alpha}(x,z)f^{db}_{2\beta}(z,y)\}\psi_n(y) \\
= \;& (F) + (ghost).
\end{aligned}$$

Gluon self-energy (Figure 5.67)

$$
\begin{aligned}
S = & \int dx \int dy A_\mu^a(x) A_\nu^b(y) \Pi^{ab\mu\nu}(x,y) \\
= & (F) + (ghost) + \\
& \tfrac{1}{2} g^2 f^{ace} f^{bfd} \int dx \int dy A_\mu^a(x) A_\nu^b(y) \\
& \qquad [O_y^{\nu\sigma} H_{1\sigma}^{cdef\mu}(x,y) + y^\nu H_2^{cdef\mu}(x,y) + (x \leftrightarrow y, \ \mu \leftrightarrow \nu)] \\
= & (F) + (ghost).
\end{aligned}
$$

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