A Topological Model For A 3-Dimensional Spatial Information System

by

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Statement

Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any university, and to the best of my knowledge and belief, it contains no copy or paraphrase of material previously published or written by another person, except where due reference is made.

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Abstract

This thesis proposes the topological theory necessary to extend the conventional topological models used in geographic information systems (GIS), computer-aided design (CAD) and computational geometry, to a 3-dimensional spatial information system (SIS) which supports query and analysis of spatial relationships.

To encompass a wide range of applications and minimize fragmentation, we define a spatial object as a cell complex, where each k-cell is homeomorphic to a Euclidean k-manifold with one or more subdivided $(k-1)$-manifold boundary cycles. The simplicial and regular cell complexes currently used in topology and many spatial information systems, are restricted forms of these generalized regular cell complexes.

Spatial relationships between the cells of the generalized regular cell complex are expressed in terms of their boundary and coboundary cells. To support query and traversal of the neighborhood of any cell via orderings of its cobounding cells, we embed the generalized regular k-cell complex in a Euclidean n-manifold which we represent as a 'world' n-cell.

Spatial relationships between spatial objects can be expressed in terms of the boundary and coboundary relations between the cells of another complex formed from the union of the generalized regular cell complexes. If this complex is embedded in a Euclidean n-manifold, then cobounding cells may also be ordered. The cells of this complex have 'singular manifold' or 'pseudomanifold' boundary cycles, which we classify into three primitive types using identification spaces. The cell complex is known as the generalized singular cell complex - generalized regular, regular and simplicial complexes are restricted forms of this complex.

To represent these cell complexes, we extend the implicit cell-tuple of Brisson (1990) since it encapsulates the boundary-coboundary relations and the ordering information.

Topological operators are defined to construct spatial objects. Since the set of spatial objects has few restrictions, we define topological operators which consistently construct both subdivided manifolds and manifolds with boundary, from the strong deformation retract of a manifold with boundary. The theory underlying these operators is based on combinatorial homotopy. Generic versions of these topological construction operators can then be used to join these subdivided manifolds or manifolds with boundary, to form the generalized regular cell complex.

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Publications

1. Pigot, S., 1991, Topological Models for 3D Spatial Information Systems, *Proceedings of AutoCarto-10,* Baltimore, Maryland, USA pp. 368-392

2. Pigot, S., 1992, A Topological Model for a 3D Spatial Information System, *Proceedings of the 5th International Symposium on Spatial Data Handling (ed. D. Cowen),* Charleston, South Carolina, USA, vol. 1, pp. 344-360

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4. Pigot, S., 1994, Generalized Singular 3-Cell Complexes, *Proceedings of the 6th International Symposium on Spatial Data Handling (ed. T.C. Waugh & R.G. Healey)*, Edinburgh, UK, vol. 1, pp. 89-111

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Chapter 1
Introduction

1.1 Introduction

This thesis proposes the theory necessary to extend the topological spatial models used in geographic information systems (GIS) to three-dimensional applications which require integrated representation and analysis of spatial objects of different dimensions. Although many of the problems discussed in this research are motivated by geoscientific applications, the use of the term spatial information system (SIS) indicates that many of the concepts and the expected applications are common to other fields such as computer-aided design (CAD) and visualization of scientific models.

Many current and proposed 3-dimensional SIS are based around the grid (or raster) model which implicitly represents the boundaries of spatial objects within a regular subdivision of the modelling space. This is in contrast to the topological (or vector) approach, where the boundaries of spatial objects are explicitly represented by irregular building blocks known in topology as cells. Both approaches have proven to be useful abstractions of reality and the choice between them should be based on the demands of the application. For example, any application which requires representation of spatial objects of different dimensions (multidimensional) and access to spatial relationships between them would be better represented by the topological model to be described in this research. Such an application could be the representation of faulting models and the structural geology of rock layers as discussed in Youngmann (1988). Alternatively, if the application requires representation and fast comparison of spatial objects of the same dimension (without detailed shape analysis) then a grid method with an appropriate compression scheme may be a better choice. Such applications for example, could be reservoir analysis or overlay of 3-dimensional representations of ore grade distribution in an underground ore body eg. Kavouras & Masry (1987). Hybrid representations which mix vector and raster concepts, such as vector octrees, also known as extended octrees (Navazo 1986) or polytrees (Carlbom et. al. 1985), have been proposed as an alternative which combines the advantages of both the vector and the raster approaches - see Jones (1989) for an application of vector octrees to geology.

Compared with the raster/grid approach, little attention has been given to the representation and manipulation of 3-dimensional geographic or 'natural' data using the vector approach. This inattention also inhibits the development of hybrid methods since they are dependent upon knowledge of both the vector and the raster approaches. This research attempts to redress this imbalance by focusing on vector models or as they are better known: topological models.

The application based motivation for this research comes from five specific needs of the geosciences.

1. Representation of spatial objects of different dimensions within the same topological model in order to support query and analysis of the spatial relationships between them. For example, geologists are interested in the relationships between drill holes and the ore bodies they intersect. These intersections may be quite complex. For example, a drill hole may pierce an ore body outline or intersect its boundary. The evidence from GIS applications is that topology and topological relationships are well suited to answering such queries. Also it should be noted that not all objects being modelled by a spatial information system are physically realizable, yet there is often a need to ask questions about their spatial relationships. For example, representing the dip and strike of an ore body using symbology about which it is possible to ask questions, or fault directions, or air flow indicators in the drives and stopes of an underground mine etc. These spatial objects can only be represented, integrated with other spatial objects and queried within such a spatial model.

2. Integration of 'triangulated' (eg. triangulated surfaces or tetrahedral subdivisions of solids) and more general 'polygonized' (eg. polygonized surfaces and bounded solids) data within the same topological model. Most surfaces generated from sampled data are triangulated, since a triangulation gives a good approximation of the topography or shape and guarantees that functions defined at each of the three vertices of a triangle can be extended over the whole triangle and thus over the whole surface; eg. interpolation of elevation (a scalar function) at any point on the surface - see Saalfeld (1987) for an account of this property and vector-valued functions. Other authors consider construction of triangulations with different shapes from sets of sample points eg. the alpha shapes of Edelsbrunner and Mücke (1994). Triangulations of any dimension are the 'lowest common denominator' - the more general polygonizations can always be reduced to triangulations, but:

i) triangulations are not naturally generated or preserved by many construction techniques commonly used in the geosciences eg. extrusion techniques do not preserve triangulations - see Jones & Wright (1990), Paoluzzi & Cattani (1990).

ii) while surface data is often triangulated, volumetric data is not.

iii) engineering data generated from computer-aided design systems is rarely in the form of a triangulation.

In this research a topological model which can support both 'triangulations' and 'polygonizations' is specified.

3. Support for interpreted and incomplete spatial objects that may be used, for example, to highlight spatial relationships or provide important information about physical processes. Much of the data that geologists/geoscientists deal with is sparse and incomplete. For example, partial constructions of solid objects (eg. pieces of their bounding surfaces, sections and fence diagrams) need to be integrated and analyzed with known spatial data in order to aid geoscientists in their interpretation of the processes that will affect their model. Such situations may also result from analysis operations like the boolean set operators (eg. intersect, union, negation etc). Similar conditions have also partially driven the development of the more advanced topological models currently used in computer-aided design - see Weiler (1986) and the introduction to Gursoz et. al. (1991).

4. Operators to construct and manipulate spatial objects without using quantitative methods. For example, many spatial objects in geological applications such as ore body shapes are generated from quantitative techniques such as kriging. These quantitative techniques are often statistical approximations of very complex processes (eg. mineralization). The results are often inappropriate because important factors such as geological structure, cannot be easily incorporated within these processes. Many geologists would like to modify or update the resulting extents of such ore bodies using their empirical knowledge of local geological structure and mineralization. Topology and topological operators are well suited to such problems because they can be designed to construct and/or modify the topography of a spatial object without requiring metric concepts such as distance and direction.

5. In any physical process which collapses/expands the boundaries of a spatial object, a 'singularity' may be formed ie. the boundary self-intersects. Continuous spatio-temporal applications must have the ability to model such 'singularities'. See for example, the introduction to the Surface Evolver software manual of Brakke (1993).

The topological model put forward in this research is expected to provide the foundation for an implementation which will meet these needs - a spatial information system for 3-dimensional applications.

1.2 A Spatial Information System for 3-Dimensional Applications

A spatial information system (SIS) is a mathematical model or abstraction of some aspect of 'reality'. The mathematical model (or spatial model as it is sometimes known) should define the basic spatial objects, their spatial relationships (as perceived by humans) and allow formal definition and explanation of any required operations. One of the advantages of defining such a model is that it can be used to abstract simple, efficient (in terms of storage and computation), extendible (for new requirements) application independent data structures.

The spatial model, data structures and operations can be combined to form the usual mechanistic definition of an SIS as a database system in which most of the data is indexed and there exists a set of procedures to answer queries about spatial entities and their attributes stored within the database.

The task of deriving a spatial model and abstracting the data structures with the properties listed above is non-trivial. Many existing 2-dimensional SIS have efficiency, versatility and integration problems which can be traced back to problems in the underlying spatial model (Gold 1992). For 3-dimensional applications, the increase in data volumes, complexity of analysis and spatial relationships will only exacerbate these difficulties. To attempt to overcome these difficulties and meet the demands we have outlined in section 1.1, we focus on developing and improving one important component of the underlying spatial model: topology.

1.3 Topology

Topology, together with set theory and geometry, forms the mathematical basis of the spatial model underlying a spatial information system. Topology is an important part of the spatial model due mainly to its generality. Mathematically speaking, topology is the 'most general' geometry because it is the study of those properties of a spatial object which are invariant under very general mappings. Other geometries study the invariants of more restrictive mappings. Using topology we can capture the most general properties of spatial objects such as their connectivity or genus. In addition, topology provides a combinatorial and algebraic toolkit consisting of a set of primitive spatial objects known as 'cells'. Cells are used as 'building blocks' to construct complex spatial objects thereby simplifying both their representation in a computer system *and* the calculation of their topological properties. In this thesis we focus on:

1. defining and representing appropriate cells and cell complexes;

2. defining basic topological operators for traversing and constructing cells and cell complexes.

1.4 Spatial Objects and Cell Complexes

We consider four basic types of 'cell' each of which may be distinguished from the others by dimension:

0-dimensional - the object has a position in space but no length eg. a point or vertex.

1-dimensional - an object having length and width but no area and bounded by two basic 0-dimensional objects. eg. a line segment, arc, string or edge.

2-dimensional - an object having length and width, bounded by one or more cycles of 1-dimensional cells. eg. a triangle, polygon or face.

3-dimensional - an object having length, width, height/depth and bounded by one or more cycles of 2-dimensional cells. eg. a tetrahedron, a cube.

In this research, a k-dimensional cell (or k-cell) is topologically equivalent to a subset of Euclidean k-space with one or more (k-1)-manifold boundary cycles.

In general, a k-dimensional *manifold* (or k-manifold) is a space in which each point has a neighbourhood which is topologically equivalent to a k-dimensional disk ie. it can be stretched, bent and otherwise deformed (without tearing or cutting) so that it matches a k-dimensional disk. For example, each point of a 2-manifold or surface has a neighbourhood which looks like a mildly bent 2-dimensional disk (figure 1.1(a)) as does each point in the Euclidean plane. To remove topological oddities such as one-sided surfaces, we assume that each k-manifold is two sided.

Each point of a *k-manifold with boundary* has the same property as a point in a manifold, except for those points on the boundary itself. These points have neighbourhoods which are topologically equivalent to a k-dimensional half disk or hemi-disk (figure 1.1(b)).

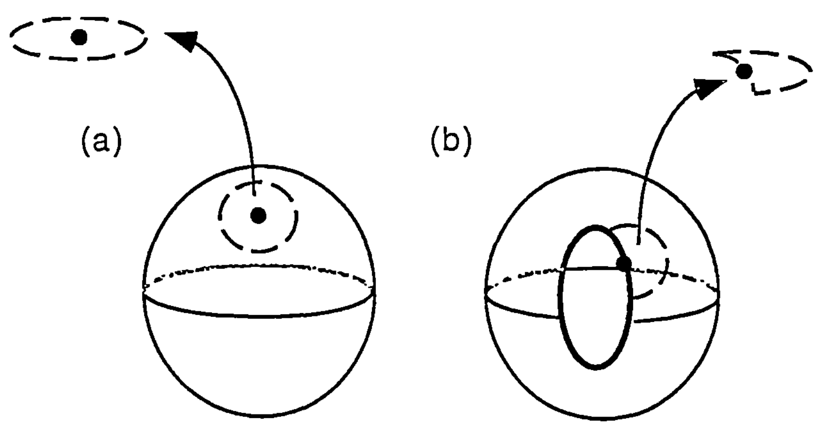


Figure 1.1 - 2-manifold and 2-manifold with boundary (a) a 2-manifold with 2-dimensional disk neighbourhood of a point (b) a 2-manifold with boundary with 2-dimensional hemidisk neighborhood of a point on the boundary

For dimensions greater than two, the definition of a 'cell' we use is more general than that usually used in topology, where a k-cell is topologically equivalent to a Euclidean k-manifold with an (k-1)-sphere boundary cycle. The reason is that a k-sphere is just one instance of a k-manifold. As an example, figure 1.2 shows two 3-cells, both of which are Euclidean 3-manifolds with 2-manifold boundary cycles. However only the 3-cell in figure1.2(b) has a boundary cycle topologically equivalent to a 2-sphere.

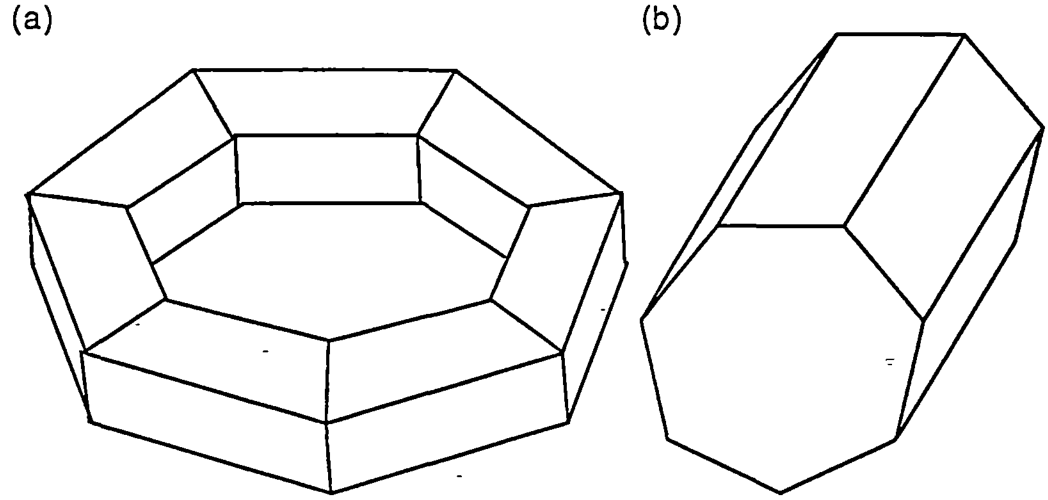


Figure 1.2 - 3-cell as a Euclidean 3-manifold with 2-manifold boundary (a) the 2-manifold boundary cycle is a torus (b) the 2-manifold boundary is a 2-sphere.

Spatial objects are constructed by joining one or more of these cells along their boundaries. For example: 1-cells could be connected along their boundaries such that they form a ring or subdivided 1-manifold, or a tree, or in general, a graph; 2-cells could be connected along their boundaries to form a surface. Examples of spatial objects are illustrated in figure 1.3. Spatial objects can thus be described as cell complexes. The advantage of using a more general 'cell' than that traditionally used in topology, is that fewer cells are required to represent the spatial object.

Spatial relationships can be grouped into two types: between the cells in the cell complex (*intra* cell complex) or between distinct cell complexes (*inter* cell complex).

1.4.1 Relationships between the cells in the cell complex

Corbett (1979) notes that intra cell complex relationships can be completely expressed using two fundamental relations: boundary and coboundary. The *boundary* of an n-cell consists of the set of $(n-1)$-cells incident to it. For example, the set of 1-cells incident to a 2-cell forms its boundary. The *coboundary* of an n-cell consists of the $(n+1)$-cells incident to it. For example, the two 3-cells that are incident to either side of a 2-cell form its coboundary set. The boundary and coboundary relationships are in fact what most authors call the 'topology' of the object. The adjacency relationships of Baer et. al. (1979) are an alternative but much less concise form of the boundary-coboundary relationships for a 2-cell complex subdividing a 2-manifold.

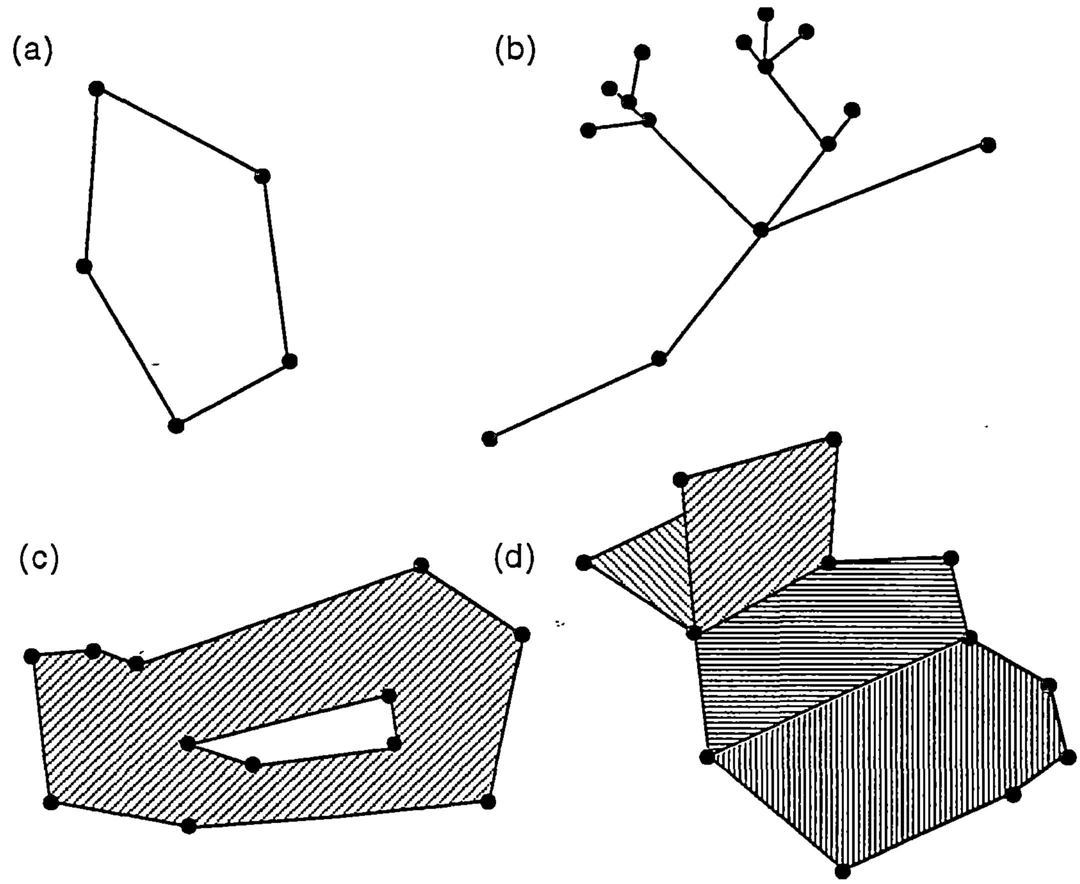


Figure 1.3 - some complex spatial objects for 3-dimensional applications - (a) a ring or 1-manifold (b) a complex 1-dimensional spatial object (c) a complex 2-dimensional cell (d) a complex 2-dimensional spatial object in R^3

If the k-cell complex subdivides a k-manifold (a subset of the class of complex spatial objects given above), then intuitively the cells that intersect a 'small' k-dimensional neighborhood of any j-cell ($k-2 \leq j \leq k-1$) may be ordered 'about' the j-cell. This ordering is consistent because every point of the underlying manifold is guaranteed to have a k-dimensional 'disk' neighbourhood (see the definition of a manifold given above). In practice, the cells that intersect the 'small' k-dimensional neighborhood of the j-cell are the $(j+2)$-cells and $(j+1)$-cells returned by iterative evaluation of the coboundary relation of j-cell ie. 'the coboundary of the coboundary' of the j-cell (White 1978). Other researchers implicitly use this fact; eg. Hanrahan (1985) defines a set of 'ordered' adjacency relationships based on an extension of the original unordered adjacency relationships (see Baer et. al. 1979) to 3-cell complexes carried by 3-manifolds. However, Brisson (1990) proved the existence of these circular orderings in subdivided n-manifolds and described many previously obscure cases, eg. when $j = -1$ in a subdivided 1-manifold, the ordering consists of alternating 0-cells and 1-cells about an abstract cell of dimension -1. Other (well-known) examples include: $j = 0$, $k = 2$ - the circular ordering of alternating 1-cells and 2-cells about a 0-cell in a subdivided 2-manifold; $j = 1$, $k = 2$ - since there are no 3-cells in a subdivided 2-manifold the circular ordering degrades to a 'two-sided' ordering consisting of two 2-cells; and $j = 1$, $k = 3$, the circular ordering of alternating 2-cells and 3-cells about a 1-cell in a subdivided 3-manifold.

Different directions in these orderings (left, right, clockwise, counter-clockwise etc.) are derived by propagating the ordering of the 0-cell boundaries of a 1-cell, to 2-cells and then to 3-cells etc.

Examples of the 'two-sided' coboundary and the circular orderings formed by 'the coboundary of the coboundary' are shown in figures 1.4-1.6. It is particularly interesting to note that for subdivided 3-manifolds (figure 1.5), there is no known ordering of 1-cells, 2-cells and 3-cells in the 3-disk neighborhood of a 0-cell. However there may be more than one circular ordering of 1-cells and 2-cells about such a 0-cell. We refer to such circular orderings as 'subspace' orderings.

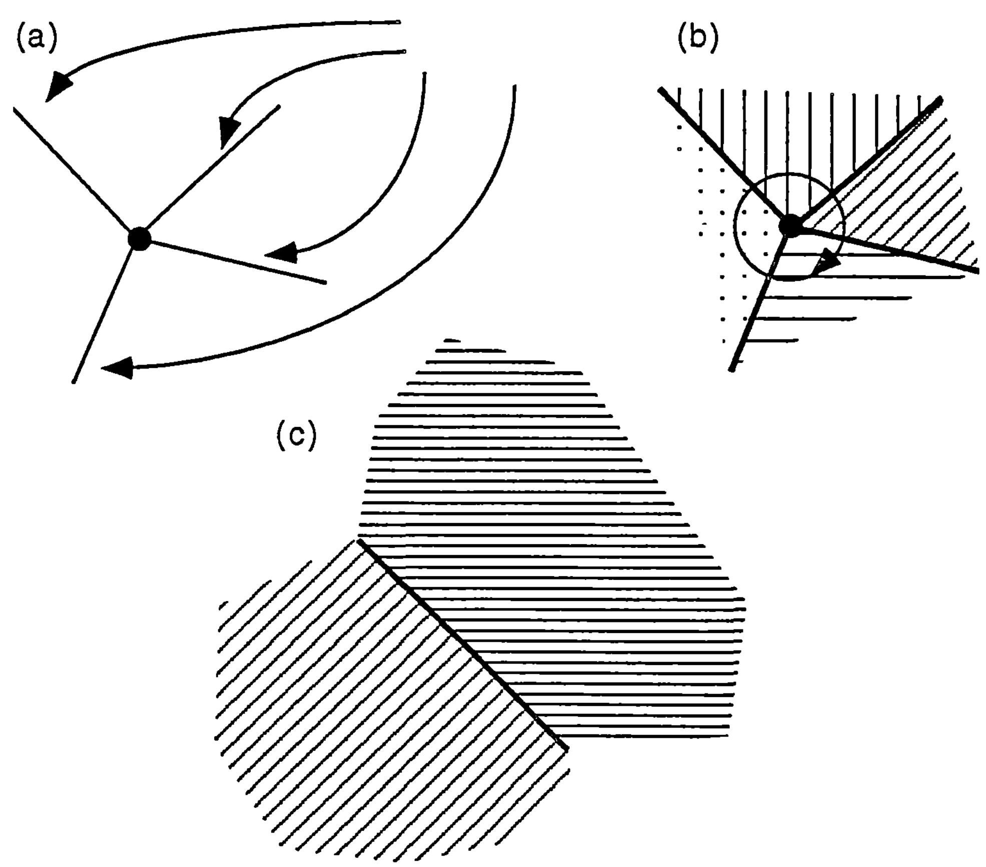


Figure 1.4 - coboundary orderings in a subdivided 2-manifold **(a)** the coboundary of a 0-cell **(b)** the circular ordering of 1-cells and 2-cells about the 0-cell and **(c)** the coboundary of a 1-cell ie. a 'two-sided' ordering

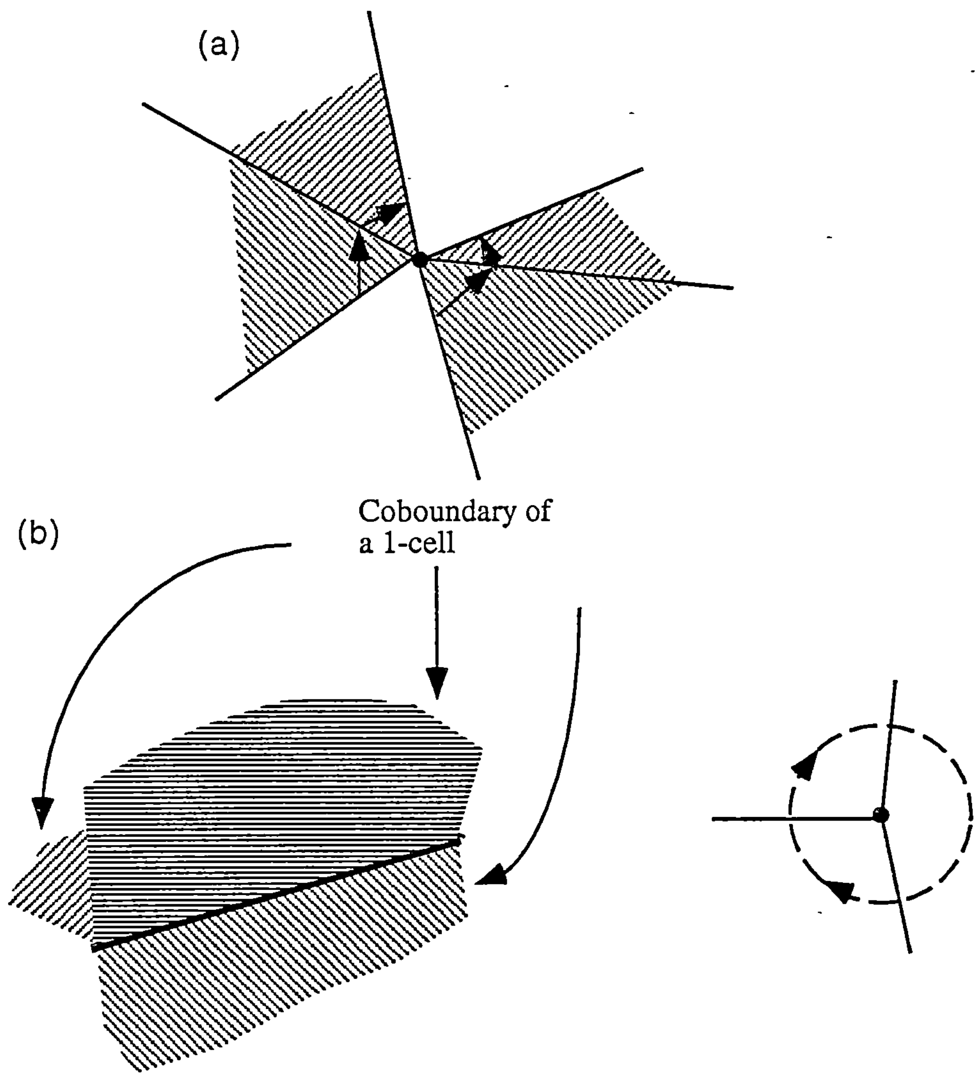


Figure 1.5 - coboundary orderings in a subdivided 3-manifold **(a)** two subspace circular orderings for a 0-cell (shaded disks when flattened). There is no general ordering of 1-cells, 2-cells and 3-cells about the 0-cell. **(b)** the coboundary of a 1-cell and an 'end on view' of the circular ordering of 2-cells and 3-cells about a 1-cell. NOTE the coboundary of a 2-cell (ie. two 3-cells in a 'two-sided' ordering) in a subdivided 3-manifold is not shown.

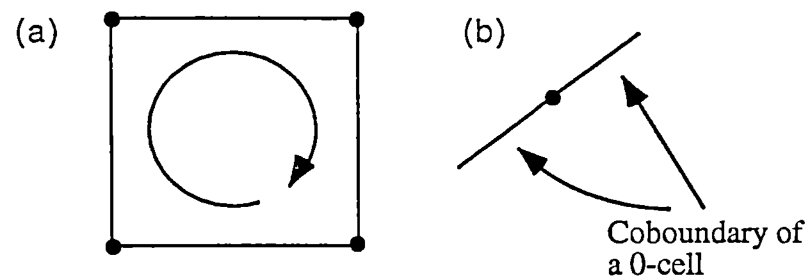


Figure 1.6 - coboundary orderings in a subdivided 1-manifold (a) the circular ordering of 0-cells and 1-cells about a (-1)-cell (the '(-1)-cell' is not shown for obvious reasons!) (b) the coboundary of a 0-cell

These orderings form a simple and powerful method for traversing all subcomplexes of a cell complex which subdivides a manifold.

Unfortunately, the spatial objects we described earlier (ie. collections of 'cells') do not necessarily form subdivided manifolds. For example, the tree shown in figure 1.3(b) is not a subdivided 1-manifold and the 2-dimensional spatial object in figure 1.3(c) is not a subdivided 2-manifold. Consequently we cannot make any assertions about the shape of the neighborhoods of all points in the space subdivided by the spatial object and thus, the coboundary ordering results cannot be applied to all cells in their cell complexes.

In 2-dimensional GIS applications, the problem is circumvented by 'embedding' the spatial object in a Euclidean 2-manifold which is represented by a special 2-cell of effectively 'infinite extent' known as the 'world' 2-cell. However, if the dimension of the spatial object does not match the Euclidean 2-manifold (eg. in the case of a 1-dimensional spatial object such as a tree) then special cases of the coboundary orderings result. The basis for these special ordering results is the embedding of the spatial object in the Euclidean 2-manifold, guaranteeing that each point has a 2-dimensional disk neighborhood.

We will adopt the same approach in this research: spatial objects will be embedded in a Euclidean 3-manifold (represented by a 'world' 3-cell) in order to obtain the ordering results based on the 3-dimensional 'disk-like' neighborhood. Once again, if the dimension of the spatial object does not match the Euclidean 3-manifold, then special cases of the coboundary orderings result. We shall study the special cases in detail.

1.4.2 Relationships between distinct cell complexes

As mentioned above we are also interested in answering questions about spatial relationships between spatial objects. These spatial relationships should be expressed in terms of the cells in the different cell complexes representing the spatial objects (inter cell complex). One solution used in 2-dimensional GIS applications, is to calculate the union of the cell complexes representing the spatial objects. The result is another cell complex, which when embedded in a Euclidean manifold, gives the appropriate ordering results mentioned above (including any special cases if the spatial objects are not of the same dimension as the Euclidean manifold). If the cell complex results from the union of spatial objects with different dimensions, the cells of this complex differ from those described above in the following ways:

1. The boundary cycles of the cells may no longer be manifolds

2. Cell complexes may be contained within the interior of other cells (ie. besides the 'world' cell).

Cells whose boundary cycles are not simple manifolds will be referred to as *singular cells* and what we previously referred to as a cell (ie. manifold boundary cycles) is a *regular cell.* Corbett (1975) & (1979) defined the boundary cycle of a singular cell as the image of a continuous map applied to the boundary cycle of a regular cell which results in the identification of sets of points. This identification is the singularity. Since such a general definition permits a myriad of different possibilities, Corbett's classification of singular cells is very general because it is directed at determining the coboundary neighbourhoods of the cells in the singularity. The three types of singularity given by Corbett for 2-cells in a 2-dimensional GIS are: cyclic, acyclic and interior - see figures 1.7(a),(b) and (c), respectively.

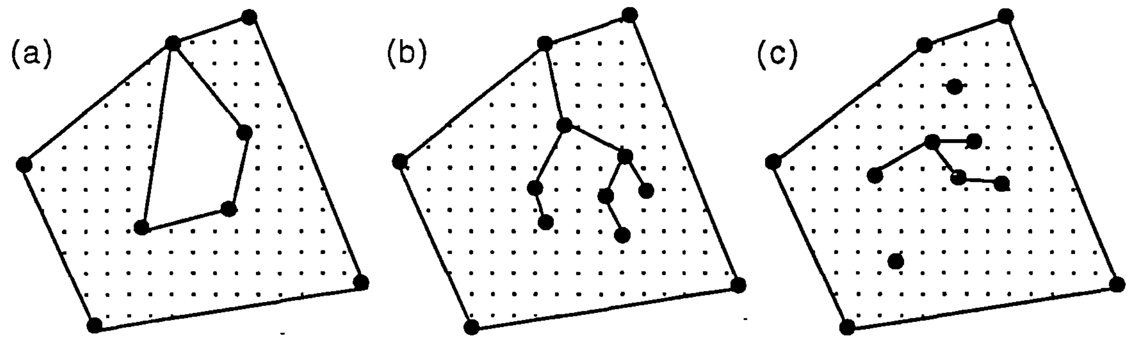


Figure 1.7 - singular 2-cells (a) 2-cell with a cyclic singular boundary (b) 2-cell with an acyclic singular boundary (c) a 2-cell with interior singularities

The main advantages of using a singular 2-cell complex to represent two or more spatial objects are:

1. The boundary-coboundary relationships between the cells implicitly define the relationships between spatial objects. For example, if a river system (a 1-cell complex) and land tenure boundaries (2-cell complex) are combined within one singular 2-cell complex, then questions such as, whether a river forms the border of, or flows through a property, can be answered by finding at least one 1-cell of the river that has the property as one or both of its cobounding 2-cells, respectively.

2. A spatial object may itself be multi-dimensional. For example a river system which contains lakes formed by dams may consist of both areal and linear features.

3. 2-manifolds are 'well-known' spaces in topology. Their properties have been classified and the presence of errors can be detected by checking the ordering of the cell neighbourhood information given above using either the primal or dual constructions. For examples see White (1978) & (1984) and Corbett (1979) and the global concept of planar enforcement.

For 3-dimensional applications, the problem of answering questions about spatial relationships between spatial objects will be handled in the same way. That is, a singular 3-cell complex will be formed from the union of the regular cell complexes representing the spatial objects. Each boundary cycle of a singular cell will be defined as a pseudomanifold and can be thought of as the image of a continuous map applied to the manifold boundary of a regular cell. Using an approach similar to that of Corbett (1979), we classify these pseudomanifolds into three primitive types by applying the theory of Whitney (1944) and identification spaces from topology. The classification reduces the immense range of possibilities to a representative set which provides the basis for the extension of the coboundary ordering results given above, to singular cell complexes.

1.4.3 Representing the Cell Complexes

To represent these cell complexes we choose to extend the cell-tuple structure of Brisson (1990). The cell-tuple has the following advantages:

1. *Implicit* - a cell complex is represented by a set of very simple 'elements'. In this case the 'elements' are tuples of integers representing sets of incident cells. Other topological models which are also implicit (but not to the same extent) include the winged-edge of Baumgart (1974), the face-edge of Hanrahan (1985) and the selective geometric complex of Rossignac and O'Connor (1991).

2. *Dimension Independence* - The cell-tuple was originally designed to represent subdivisions of n-manifolds. Consequently, the underlying principles are not dependent upon the dimension of the cell complex. Other models such as the winged-edge of Baumgart (1974) or the facet-edge of Dobkin and Laszlo (1987) are dependent upon the dimension of the cell-complex.

3. *Coboundary Ordering Information* - the cell-tuple structure captures both the boundary-coboundary relations and the ordering information for subdivided manifolds, in a simple graph structure.

1.5 Topological Operators

Operators for traversing and constructing the cell complex are just as important as the description of the representation of spatial objects using a cell complex. We describe these operators as topological operators, because they construct and query the topological relationships between the cells of a cell complex and they are not dependent upon metric concepts.

The traversal operators are dependent upon the boundary coboundary relationships described earlier. As is noted in Corbett (1975) & (1979) and White (1978), the boundary and coboundary relationships may be considered as operators which if applied to a cell (or a cell complex representing a spatial object) return the cells forming its boundary or its coboundary. It will be shown in chapter 3 and 4 that an extension of the cell-tuple of Brisson (1990) encapsulates these relationships for both regular and singular cell complexes.

The other set of topological operators that we consider are those that allow us to construct the cells and the cell complex. As mentioned earlier in this chapter, a singular cell complex results from the union of one or more regular cell complexes - each of which corresponds to a spatial object. Consequently we restrict our attention to the construction of these regular cell complexes (the union process used to form the singular cell complex is a subject for future research). A generic set of topological operators for constructing a regular k-cell complex should be able to:

1. Construct each k-cell (ie. a Euclidean k-manifold with (k-1)-manifold boundary cycles).

2. Join cells together along subsets of their boundaries in a Euclidean n-manifold $n \geq k$.

The operators described in Corbett (1985) (and by Brisson 1990, but for a more restricted domain) typify this approach and will form the basis of the extensions proposed in this research.

The advantages of this approach are application independence and simplicity, since only a small set of operators is required to construct all spatial objects.

A major problem in the construction of spatial objects with such topological operators is consistency; ie. ensuring that any spatial object falls within the domain *and/or* has the required topological properties eg. the correct number of 'holes'. In this research, the problem is compounded by the fact that the domain of spatial objects that we have chosen (see section 1.4) has very few restrictions that could be used for consistency purposes. A 'local' approach to the problem of consistency could be based on ensuring that regular cells are correctly defined and may only be joined along their boundaries in order to form the regular cell complex. Fortunately, we can do better by maintaining consistency across a slightly larger subset of the domain: the subdivided k-manifolds and k-manifolds with boundary ($k \leq 2$). This subset includes regular k-cells, the boundary cycles of regular 3-cells and the boundaries of many 3-dimensional spatial objects. As an example, a set of mine development tunnels with pillars is often represented by a surface which is topologically equivalent to a subdivided 2-sphere with handles (eg. a torus), where the number of handles equals the number of pillars. It would be useful to be able to ensure that this surface has the correct number of pillars both before and throughout its construction (not just after).

To achieve consistency across the subdivided manifolds and manifolds with boundary, we modify the generic cell complex construction operators such that they preserve a very general topological invariant known as the homotopy type (see section 3.2 for a definition). The advantage of this process is that we can use a very simple space (known as the strong deformation retract) which encapsulates important topological properties (eg. connectivity) and can be specified as a simple 1-dimensional spatial object, prior to the construction of the subdivided manifold or manifold with boundary. The modified generic construction operators are used to construct the remainder of the subdivided manifold or manifold with boundary from the strong deformation retract whilst preserving its homotopy type, thus maintaining consistency.

The local Euler operators as described in Mäntylä (1988) are a special case of this procedure specific to 2-manifolds because they rely on the initial construction of a 2-sphere, and then repeated splitting of 0-cells (ie. adding an edge) and 2-cells (ie. adding a face) within the 2-sphere, to form a subdivided 2-manifold. It will be shown that the local Euler operators actually perform a special restricted form of combinatorial homotopy.

The main disadvantage of the combinatorial homotopy operators is that they only apply to spatial objects in a subset of the domain of the spatial information system. However, a complex spatial object can be constructed using a two stage process. Firstly, the boundary cycles of cells and/or subspaces of the spatial object corresponding to subdivided manifolds or manifolds with boundary are constructed. These subspaces are then joined together in order to form the regular cell complex. It is important to note that the main reason for the restriction on the domain is that the strong deformation retract of more complex spaces is much more difficult to determine. Furthermore, the strong deformation retract and combinatorial homotopy methods are not confined to regular cell complexes. Singular cell complexes could also be constructed because the dimension of the cell being attached is unimportant.

Lastly, the importance of the fundamental group, the strong deformation retract and the combinatorial homotopy methods are emphasised by the establishment of an unrealized link to algorithms for constructing the topology of a subdivided 2-sphere from a 1-skeleton. In particular, the fundamental group is shown to be essential in extending these algorithms from simple subdivided 2-spheres to subdivided 2-manifolds and 2-manifolds with boundary.

1.6 Overview of this Thesis

The three major goals of this research are:

1. To define new cell complexes for representing individual spatial objects and combinations of spatial objects of different dimensions which optimize the representation of their geometric structure and their topological properties but maintain compatibility with the singular and regular cell complexes used in existing spatial information systems.

2. To define a general set of topological operators for consistent construction of a tractable subset of the domain of spatial objects - the subdivided k-manifolds and k-manifolds with boundary ($k \leq 2$).

3. To extend the simple implicit cell-tuple representation proposed by Brisson (1990) and the circular ordering results upon which it is based, to the new cell complexes described in this research.

In chapter 2 a simple taxonomy of existing topological models is put forward using the terms introduced in the preceding sections of this chapter. Then the important topological models are individually reviewed.

Chapter 3 presents the mathematical theory underlying this research. After describing homeomorphism and homotopy, the important topological properties of manifolds and manifolds with boundaries are given.

Chapter 4 reviews the traditional cell complexes used in topology (ie. simplicial, regular CW and normal CW) and then describes the generalized regular cell complex. The generalized regular cell complex is intended to optimize the representation of both the geometric and topological properties of individual uni-dimensional spatial objects, without losing the ability to calculate these properties. The cell-tuple of Brisson (1990) and the underlying circular orderings of cobounding cells are then extended to generalized regular cells and cell complexes embedded in Euclidean manifolds.

To model spatial objects of different dimensions within the same cell complex, chapter 5 introduces the generalized singular cell complex. The pseudomanifold boundary cycles of these singular cells are classified into three primitive types using identification spaces and the theory of Whitney (1944). The coboundary orderings implied by this classification of pseudomanifolds are then determined and the implicit cell-tuple model is extended to generalized singular cells and cell complexes.

The last section of chapter 5 introduces another cell complex which is based on the normal CW complex described in chapter 4. The cells of this complex are intended to represent the topological structure of the generalized regular and singular cells only. The main advantage is that they achieve a compression of the redundant cell neighborhood information that would normally be held in the generalized regular and singular cell complexes. The cells of this new complex are called '2-arcs' because they are a 2-dimension extension of the 1-arc concept that has been successfully applied in 2-dimensional GIS.

Chapter 6 presents the topological operators for construction of generalized regular and singular cell complexes using the notions of strong deformation retract, homotopy type, combinatorial homotopy and the generic cell complex construction operators of Corbett (1985) and Brisson (1990). Relationships between the homotopy type, these operators and various solutions to the automated reconstruction of a subset of the spaces in the domain (in particular Ganter 1981) are examined.

Chapter 7 provides a synopsis and some suggestions for future research.

Chapter 2
Review of Existing Topological Models

2.1 Introduction and Taxonomy of Topological Models

This chapter gives a detailed review of existing topological models using a number of criteria, some of which were briefly mentioned in chapter 1. These criteria give rise to a taxonomy of topological models, but like most taxonomies there are exceptions.

Uni-dimensional Domain vs. Multi-dimensional Domain - In this research, the domain (ie. set of representable spatial objects) of topological models which are able to represent uni-dimensional spatial objects (regular cell complexes) will be referred to as uni-dimensional. The domain of topological models which are able to represent spatial objects of more than one dimension will be referred to as multi-dimensional. Multi-dimensional domains must be based on some form of singular cell complex.

Cell Coboundary Orderings - If the domain consists of the subdivided n-manifolds then the coboundary ordering relationships discussed above may be applied to cells of dimension (n-2) or higher. If the domain is larger than the subdivided n-manifolds then coboundary orderings can only be obtained by 'embedding' the spatial object(s) in a Euclidean manifold of the same of higher dimension. Special cases of the ordering results must be defined.

Implicit vs. Explicit (Brisson 1990) - An *implicit* model represents the boundary-coboundary neighbourhood relationships of cells in a cell complex using a set of 'elements' eg. winged-edges (Baumgart 1974) or cell-tuples (Brisson 1990). *Explicit* models usually represent a set of basic 'elements' but also include redundant 'elements', usually to improve retrieval by removing the need to reconstruct certain critical elements (eg. the boundary cycles of polygons and solids). However there are disadvantages to the inclusion of redundant topological elements:

1. Redundancy complicates the design and increases the storage requirements of the data model (see Milne et. al. 1993 for example). Once again the problem is exacerbated in higher-dimensional applications.

2. Redundant spatial objects must be maintained throughout all operations on the model.

3. Redundant spatial objects generally have larger extents than the individual cells they are composed of. Thus they are more likely to be fragmented or cause overlap in the partitions of any spatial access scheme. Fragmentation and overlap in partitioning schemes degrade the retrieval performance that can be gained from any spatial access scheme - see Chapter 8 of Langran (1992) for a review and taxonomy of spatial access schemes.

These disadvantages indicate why very explicit models such as the radial-edge of Weiler (1986) and the tri-cyclic cusp of Gursoz et. al. (1991) are not as attractive for representing 'real-world' or 'geographic' spatial objects as they are for representing less complicated man-made objects in computer-aided design (Sword 1991).

Cells - As was noted in section 1.4, the definition of a 'cell' varies widely from the most restrictive definition in topology, to Euclidean manifolds with boundaries, as we use them in this research. We will distinguish between the different definitions used when necessary.

The rest of this chapter is laid out as follows. Section 2.2 reviews the 2-dimensional map models used in GIS. Section 2.3 reviews models that have been applied to subdivided 2-manifolds (boundaries of solids) in CAD and in computational geometry. Section 2.4 reviews models that have been applied to subdivided 3-manifolds in CAD and computational geometry. Section 2.5 reviews models that have been applied to subdivided n-manifolds and n-manifolds with boundary as proposed in computational geometry and CAD. Lastly, section 2.6 reviews models whose subdivisions need not fit into any of the previous categories.

2.2 Map Models Used in 2-Dimensional GIS

Overview

Topological models used in 2-dimensional geographic information systems (GIS) have multi-dimensional domains consisting of 0,1 and 2-dimensional spatial objects embedded in a Euclidean 2-manifold or a space homeomorphic to it, such as a map projection. The early approaches such as the DIME model of Cooke & Maxfield (1967) (see also Corbett 1975 & 1979) were implicit, storing only the two vertices bounding and the two coboundary polygons of an edge. Complex spatial objects were reconstructed using the circular ordering relationships for a subdivided 2-manifold (see section 1.4) and the dual complex (ie. that obtained by substituting points for polygons, edges for edges and polygons for points). To speed up access to commonly retrieved spatial objects (eg. boundary cycles of polygons), more recent models explicitly represent polygon boundary cycles and the cobounding 1-cells of each 0-cell. Examples of such explicit approaches can be found in Chrisman (1975) and in the arc-node model described in Aronoff (1989). A less explicit approach which is somewhat similar to the winged-edge of Baumgart (1974) is adopted in the successor to the DIME system, TIGER. In essence, TIGER stores the same basic boundary-coboundary information as the DIME segment except that instead of holding the list of boundary 1-cells for a polygon as in the arc-node model, a pointer to the first 1-cell in the boundary is stored and each 1-cell has pointers to the two forward 'wings' forming the next boundary 1-cell of the left polygon and the next boundary 1-cell of the right polygon. Polygon boundaries are now implicit (as opposed to explicit in the arc-node model) and are reconstructed by 'threading' the 1-cells together (see Moore 1985 and the Arithmicon system of White 1978 for some of the principles behind this approach).

Since all these approaches are variations of the same cells and cell complex we describe the important underlying principles only using the excellent basis provided in Corbett (1975) & (1979) and White (1978).

Subdivision

The spatial objects are finitized into discrete, piecewise linear n-cells ($0 \leq n \leq 2$) each of which (apart from 0-cells) is topologically equivalent to an n-manifold with one or more $(n-1)$-manifold boundary cycles. When combined in a Euclidean 2-manifold or "world" 2-cell, these spatial objects may intersect. The principle underlying the cell complex is that no cell intersects another except along their boundaries. Thus when the cells of spatial objects in the "world" 2-cell intersect, they are subdivided. The effect on 2-cells is that their boundary 1-cycles may also have dangling 1-cell complexes and they may have isolated interior cell complexes (as mentioned in section 1.4).

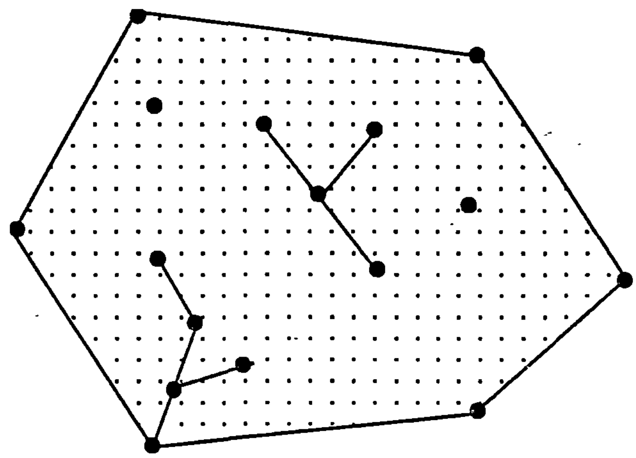


Figure 2.1 - a singular 2-cell ie. a 2-cell with isolated 1-cells and 0-cells in its interior and a boundary 1-cycle with a dangling tree structure

Corbett (1979) describes 2-cells with such boundary 1-cycles as singular 2-cells. Recall from section 1.4, that any regular n-cell is topologically equivalent to an n-disk. ie. the image of a one-to-one, onto and continuous transformation (known as a homeomorphism) applied to the n-disk. Notice how the homeomorphism is really applied to the boundary 1-cycle of the cell. Using this fact, Corbett defines a singular cell as one in which the one-to-one nature of the homeomorphism is relaxed and some points of the boundary are identified (figure 2.2).

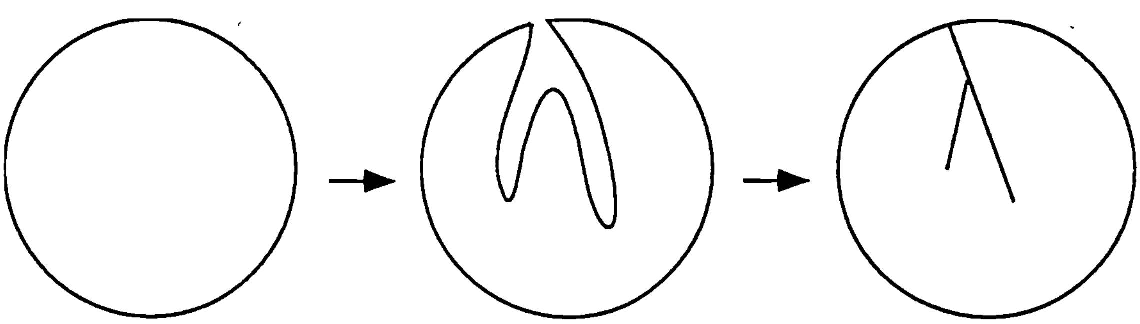


Figure 2.2 - formation of a singular cell by deforming its boundary and identifying points (see also White 1984)

Corbett then classifies these singularities into two types: acyclic and cyclic.

For *Acyclic* singularities the identification process does not produce a new 1-cycle but an internal tree structure composed of 1-cells (figure 2.2). Corbett indicates that such 1-cells may be distinguished (along with any isolated 1-cells in the interior of the 2-cell) by noting that they have the same left and right cobounding 2-cell.

Cyclic singularities may be classified into two types shown in figure 2.3. They are formed by identifying two points of the boundary 1-cycle.

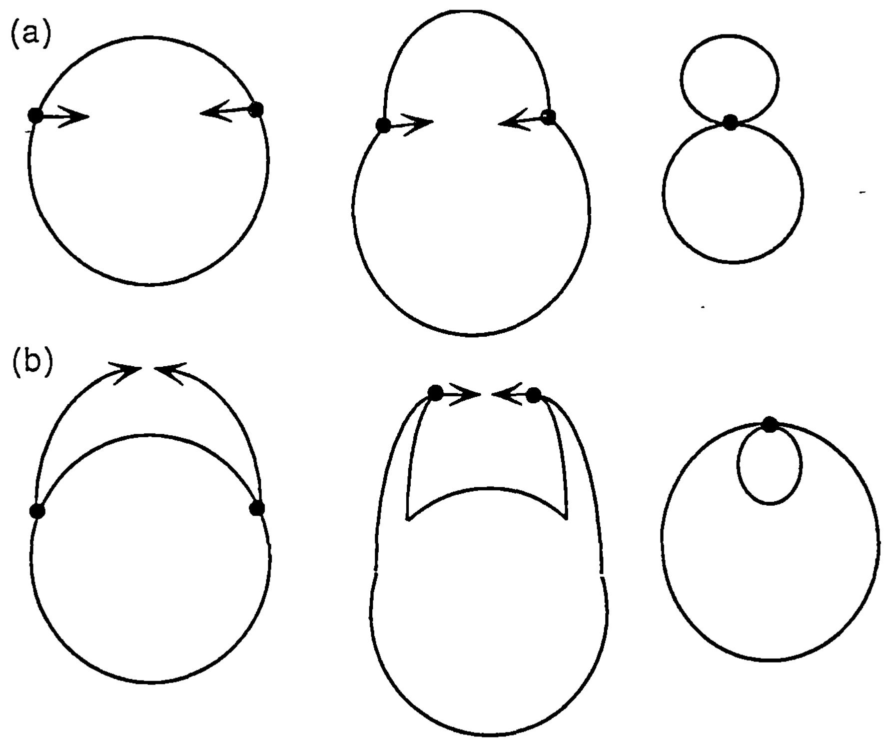


Figure 2.3 - cyclic singularities generated by identifying points - **(a)** a 'figure 8' **(b)** an interior 'hole'

Corbett recognized the importance of the dual complex as an alternative method for solving problems that are more difficult to deal with in the usual (or primal) cell complex. The dual construction is formed by 'representing' primal cells by their dual cells. For example in a subdivided 2-manifold, the primal 2-cells are represented by dual 0-cells connected by dual 1-cells (primal 1-cells are self-dual). It is interesting to note that the two types of singularity classified by Corbett above turn out to be 'duals' of one another in the singular dual cell complex (figure 2.4).

As far as the author knows the full implications of this classification of singular cells have not been realized or studied in higher dimensions - the principles described in Corbett (1979) and White (1984) form the primary basis for the approach taken to 3-dimensional applications in this research.

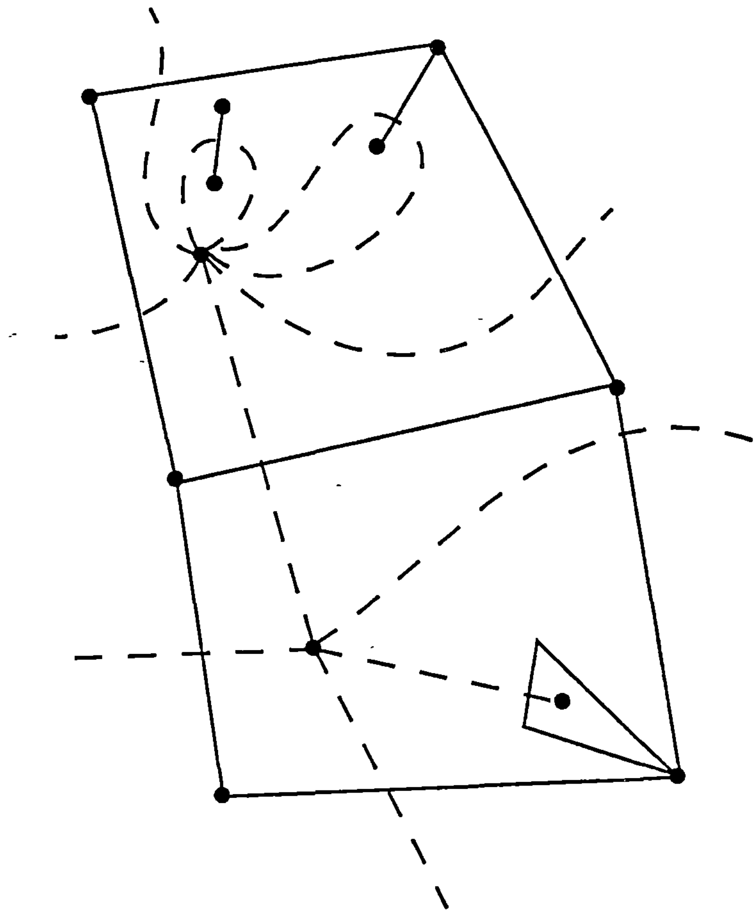


Figure 2.4 - a singular cell complex and its dual construction - notice how the two types of singularity defined by Corbett (1979) are 'duals' - see also Guibas and Stolfi (1985)

Ordering

Corbett (1975) & (1979) and White (1978) were the first to express the neighbourhood or adjacency information of cells in terms of boundary and coboundary relationships and specify how the coboundary information may be ordered. As mentioned above, the DIME structure provides explicit access to the left and right cobounding polygons of every edge and the boundary vertices or nodes of the edge and its dual. The DIME structure, like the winged-edge of Baumgart (1974), is a minimum 'template' of the neighbourhood information. To ease the cost of processing, other models explicitly store the cobounding edges of a vertex or node and the boundary edges of polygons.

One of the most important points about the boundary and coboundary operators is that they can be applied to sets of cells (ie. those that form a complex spatial object) as well as individual cells. This fact is used by many commercial GIS to define special classes of complex spatial objects for specific applications within the cell complex; eg. route systems for transportation planning and analysis.

2.3 Subdivisions of 2-manifolds

2.3.1 Winged-Edge

Overview

Early topological models for computer-aided design (CAD) systems (also known as "boundary" models) were based on that fact that subdivided 2-manifolds 'match' the surfaces of many solid objects. The first 2-manifold topological CAD model appears to have been devised by Baumgart (1974) and is known as the winged-edge model. The winged-edge is a representation of the 'minimum template' of cells required to capture the boundary-coboundary relationships and coboundary orderings in a subdivided 2-manifold (figure 2.5).

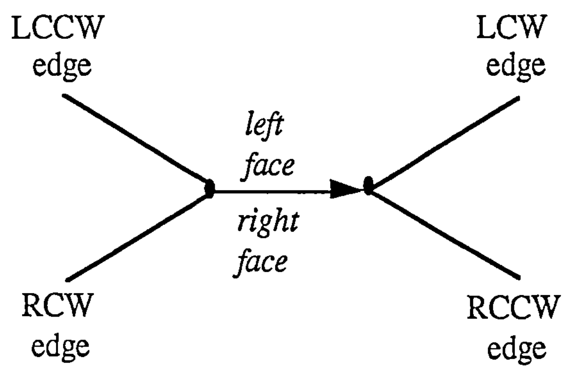


Figure 2.5 - the winged-edge template

For example, to trace the edges of the left face of an edge (a circular ordering), choose either LCW (left-clockwise) or LCCW (left-counter-clockwise) and then move the template to that edge. A well-known disadvantage of the winged-edge representation is that the arbitrary assignment of direction to the edge makes it necessary to test which of LCCW, LCW, RCCW or RCW is the next required edge.

The winged-edge representation is applied to regular 2-cell complexes. However (as we shall see) modifications have been implemented by other authors which permit the winged-edge principle to be applied to singular 2-cell complexes.

Improvements and variations (primarily in space and efficiency) on the winged-edge have been made by various authors (eg. Hanrahan 1985, Weiler 1985, Woo 1985) using modifications of a set of adjacency relationships for subdivided 2-manifolds devised by Baer et. al. (1979). However the underlying principles of the winged-edge remain the same.

2.3.2 Quad-Edge

Overview

Guibas and Stolfi (1985) introduce an implicit topological model for subdivided 2-manifolds called the quad-edge. However their approach is subtly different to the usual winged-edge approach of Baumgart (1974). The basic element of their approach is still the 1-cell, but instead of explicitly representing the left-clockwise (LCW), left-counter-clockwise (LCCW), right-clockwise (RCW) and right-counter-clockwise (RCCW) edges (see figure 2.5), they define operators based on four possible directed oriented edges that may be associated with any edge in a subdivided 2-manifold by different orderings of the edge itself and the adjacent 2-cell boundaries (figure 2.6).

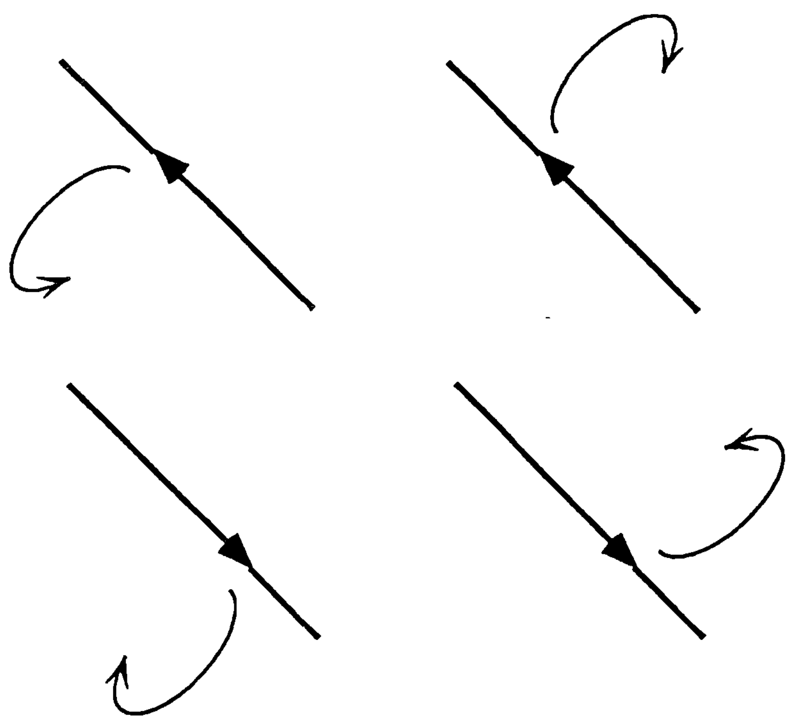


Figure 2.6 - the four directed oriented edges associated with an edge (from Brisson 1990)

Two operators: *Onext* (next edge in either clockwise or counter-clockwise direction as indicated by the orientation of the edge) and *Flip* (change from a directed oriented edge on one side of the edge to the directed oriented edge pointing in the same direction on the opposite side) can be combined or applied individually to return the RCW, RCCW, LCW and LCCW edges in the winged edge. It is this fact which underlies the claim by Brisson (1990) that the quad-edge is 'more implicit' than the winged-edge.

In addition to the primal cells, Guibas and Stolfi also define and represent the cells of the dual subdivision. An additional, operator *Rot* (short for rotate ninety degrees) moves between edges in the primal subdivision and edges in the dual subdivision, since edges are self-dual in any subdivision of a 2-manifold.

The term 'quad-edge' arises from the four possible ways of giving orientation to a primal or dual edge and its two cobounding faces.

The cells of the quad-edge may have acyclic singularities as defined by the classification of Corbett (1979) and given in section 2.2. However (as the dual subdivision shown in figure 2.4 shows) there is no reason why the quad-edge could not be extended to singular cells.

Since Brisson (1990) gives both an excellent review and a simple generalization of the quad-edge, it is not necessary to give further details, except to note that Guibas and Stolfi's research forms the foundation of the more recent implicit topological models developed in the field of computational geometry.

2.3.3 Half-Edge

Overview

The half-edge topological model of Mäntylä (1988) is a variation of the winged-edge model which represents one or more disjoint subdivided 2-manifolds embedded in a Euclidean 3-manifold. However, the half-edge is a much more explicit model than the winged-edge, since it explicitly represents the boundary cycles of the 1-cells and 2-cells and includes provisions for these 1-cells and 2-cells to be singular. The underlying principles of the model are rigorously based on the fact that any 2-manifold may be represented by a polygon with oriented edges identified in pairs (known as a plane model). The term 'half-edge' originates from the fact that in any subdivided 2-manifold, the two cobounding polygons of an edge represent 'half' the neighborhood of an edge.

Subdivision

To represent the vertices, edges and faces in a subdivided 2-manifold, an explicit five-level hierarchical model consisting of the following elements is used.

Solids - the interior point-set of a 2-manifold. Although multiple solids may exist, they must be disjoint ie. there is no provision to represent relationships between solids.

Faces (2-cells) - planar face of a polyhedron. Each face may have more than one boundary 1-cycle. The link between a face and multiple boundary 1-cycles are held explicit in the data structure by representing these 1-cycles boundaries as loops.

Loops - connected boundary 1-cycles of a 2-cell. The link between a loop and a face is held explicitly, as is the link between the loop and the first edge within it.

Edge - Associates two oppositely oriented half-edges with the physical edge to represent the identification of the faces of a plane model.

HalfEdge - The use made of an edge by both a loop and a polygon/face. Links between the half-edge and the loop it belongs to are held explicitly in the data base. The half-edge is simply a construct for maintaining the coboundary polygons of an edge. Many half-edges may share the same vertex.

Vertex - 0-cell or point to represent a position.

The basic spatial objects held in this data structure are solids, faces, loops, edges and vertices. Singular 2-cells are permitted by allowing isolated vertices within a face to be held as degenerate loops and edges to have the same face on either side. The advantage of holding isolated vertices as degenerate loops is that the face neighbourhood of the isolated vertex is held by the usual link between the loop and the face. Thus the half-edge model is really an explicit model of a subdivided 2-manifold, where the subdivision is a singular 2-cell complex. Note that the reason that singular 2-cells are permitted, is to support the application of the Euler operators that preserve the topological properties of the 2-manifold carrying the cell complex (see Mäntylä 1988 pg. 162). Thus the half-edge is an exception to the simple taxonomy given in section 2.1.

Ordering

The boundary and coboundary ordering information for the cells of the 2-cell complex is held by the links between the cells and cell boundaries of the hierarchy given above. Specifically, the two cobounding faces of any edge are held by its half-edges whilst the circular alternating ordering of edges and faces about any vertex can be derived by assembling the set of half-edges that share the vertex and analyzing their cobounding faces through the links to their parent edges.

The problems caused by the arbitrary assignment of orientation to the central edge of the winged-edge are not present in the half-edge because the half-edges carry the orientation and there are two for each edge. The arc-node topology model mentioned above also avoids these problems in a similar manner by storing the edge and its orientation in the list of boundary edges of a polygon (ie. each edge is effectively stored twice with opposing orientations).

2.4 Subdivisions of 3-Manifolds

2.4.1 Face-Edge

Hanrahan (1985) appears to have been the first to formally extend the half-edge variation of the winged-edge of Baumgart (1974) to subdivided 3-manifolds.

Initially Hanrahan derived a set of ordered adjacency relationships between the regular cells of a 3-cell complex using the work of Baer et. al. (1979) as a basis. He represented these adjacency relationships using tuples in a manner similar to the individual cell tuples of Brisson (1990). However he did not represent the coboundary ordering information as edges in an abstract graph like Brisson (see below). Instead, ordering in the form of positive or negative direction was assigned to edges only and had to be propagated to higher-dimensional cells. Hanrahan abandoned this approach because he was unable to obtain reasonable performance from a relational implementation of this concept.

In place of the cell tuples, Hanrahan developed an extension of the half-edge (see the half-edge element in Mäntylä 1988 earlier in this review) known as the face-edge. Essentially the face-edge is a minimum template of boundary-coboundary information for an edge in a subdivided 3-manifold necessary to support traversal of both the circular ordering of faces and 3-cells about the edge and the circular ordering of 0-cells and 1-cells that form the boundary cycle of a 2-cell or face (figure 2.7). Thus the face-edge is an implicit cell complex in the same vein as the winged-edge of Baumgart (1974) and its variants.

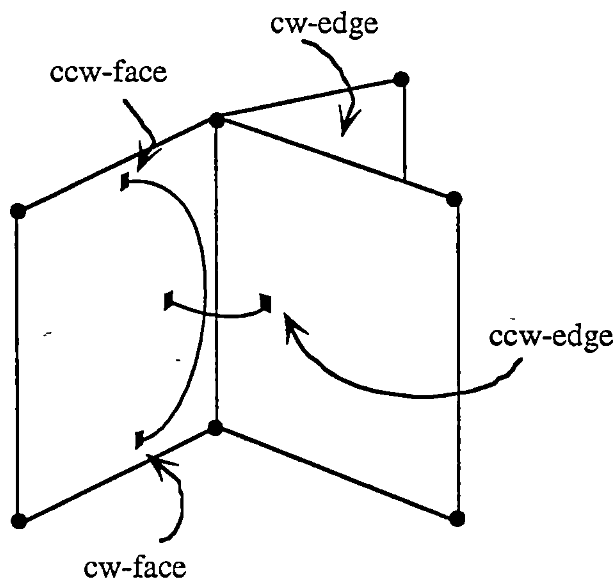


Figure 2.7 - the face-edge template

As an example, to assemble the faces of a 3-cell, the template is moved around each edge of the face via the CW-face or CCW-face pointers, then to an adjacent face not previously traversed via the CW-edge or CCW-edge pointers, according to the presence of the 3-cell between the faces etc.

The cell complex forming the subdivided 3-manifold is assumed to be a regular cell complex. Other structures, such as the star-edge of Karasick (1988), can be thought of as variations of the face-edge.

2.4.2 Facet-Edge

Overview

Dobkin and Laszlo (1987) introduce an implicit topological model for representing subdivided 3-manifolds, which is a 3-dimensional extension of the quad-edge of Guibas and Stolfi (1985). The basic element of their approach is the face-edge pair, as it is for the face-edge of Hanrahan (1985) and the star-edge of Karasick (1988) mentioned above. However the difference is that instead of directly representing the CW-face and CCW-face edges and the CW-edge and CCW-edge faces (like the face-edge of Hanrahan 1985), they define operators on the facet-edge pair, which when combined or applied individually, return these two edges and two faces. It is this fact which underlies the claim that the facet-edge is more implicit than the face-edge (just as the quad-edge is more implicit than the winged-edge).

The four operators: *Enext* (move to the next edge (ie. CW-face or CCW-face) in the face component of the facet-edge), *Fnext* (move to the next face (ie. CW-edge or CCW-edge) in the set of cobounding faces to the edge component of the facet-edge), *Rev* (reverse orientation such that the next face is given by CW-edge instead of CCW-edge or vice versa) and *Clock* (*Rev* plus a reversal of the orientation of the edge such that the next edge in the face is given by CW-face instead of CCW-face or vice versa), can be applied individually or in combination to return the two edges CW-face and CCW-face as well as the two faces CW-edge and CCW-edge and thus traverse the entire 3-cell complex.

As for the quad-edge of Guibas and Stolfi (1985), the cells of the dual subdivision are represented and manipulated as a separate complex. An additional operator *SDual* moves from a facet-edge in the primal subdivision to a facet-edge in the dual subdivision.

2.5 Subdivisions of *n*-manifolds and *n*-manifolds with boundary

2.5.1 Cell-Tuple

Overview

The cell-tuple model of Brisson (1990) is a dimension independent approach for modelling regular cell complex subdivisions of an *n*-manifold and/or an *n*-manifold with boundary. The cell-tuple is an extension of the quad-edge of Guibas and Stolfi (1985) and the facet-edge of Dobkin and Laszlo (1987) which continues and simplifies the implicit approach taken in that research. The simplification of these approaches is obtained by the use of a simple dimension independent element (the cell-tuple) and the encapsulation of both the boundary-coboundary relationships and the orderings (defined in section 1.4) within an abstract graph.

Subdivision

As specified, Brisson limits the domain of the cell-tuple to regular cell complexes forming subdivided *n*-manifolds. The cells of the subdivision are regular ie. homeomorphs of an *n*-disk and the subdivision itself is known in topology as a finite, regular CW-complex (see section 4.2).

Following Brisson (1990), given an *n*-manifold M^n and a regular cell subdivision of M^n, a *cell-tuple* t is a set of cells $\{c_{\alpha 0},\ldots\ldots,c_{\alpha n}\}$ where any k-cell $c_{\alpha k}$ is incident to $(k+1)$-cell $c_{\alpha k+1}$ (ie. is a part of a boundary of $c_{\alpha k+1}$), $0 \leq k \leq n$ and there are $n+1$ cells in each tuple. The cells within the tuple are ordered according to dimension.

The k-dimensional component of any tuple t is referred to as t_k and $t_k = c_{\alpha k}$. The set of cell-tuples representing a subdivided n-manifold is referred to as T_M.

As an example, the cell-tuples for a fragment of a subdivided 2-manifold are shown in figure 2.8. Each tuple is represented as an unlabelled point within the 2-cell it belongs to and near the other cells that it contains. Some tuples in this example are marked.

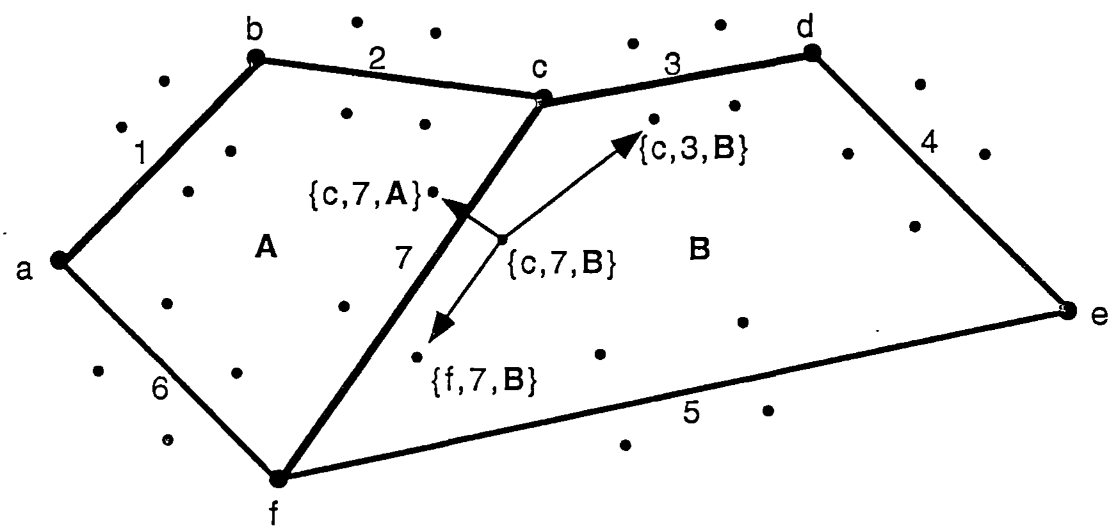


Figure 2.8 - cell-tuples in a fragment of a subdivided 2-manifold - the cell-tuples are shown as unlabelled points

Ordering

The basic operator on tuples, which also defines the circular ordering results is the *switch* operator. For any tuple $t \in T_M$, the *switch* operator essentially swaps a single i-cell in t to obtain another 'unique' cell-tuple t', where 'unique' implies both:

1. $t_i' \neq t_i$ and $t_j' = t_j$ for all $i \neq j$ and $0 \leq j \leq n$. ie. $switch_i(t) \neq t$, and,

2. for each dimension i, $switch_i(t)$ returns only one tuple t'.

For example, in figure 2.8 above, a 0-dimensional *switch* or *switch*$_0$ operator is applied to move in the direction of the arrow from {c,7,**B**} to {f,7,**B**}. The effect of the switch operator is to create edges between the cell-tuples. Thus the cell-tuples and the switch operations can be thought of as an undirected graph (G_T), where each node is a cell-tuple and each edge is a switch operation of some dimension.

The graph formed by the cell-tuples and the switch operations for figure 2.8 above, is shown below in figure 2.9.

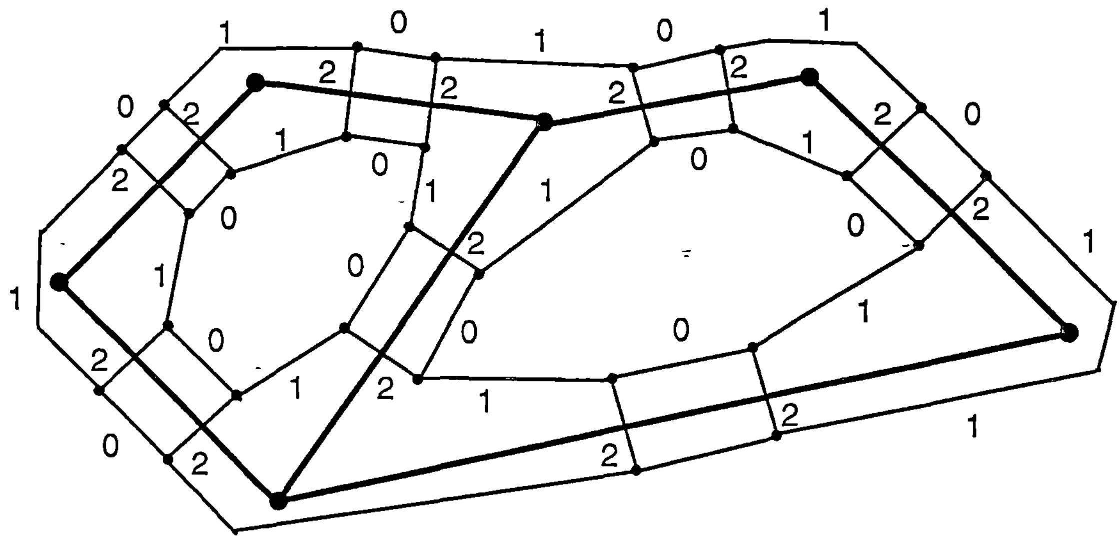


Figure 2.9 - the graph formed by cell-tuples and switch operations - cell-tuples are vertices and the $switch_0$, $switch_1$, $switch_2$ operations are edges

Apart from representing the boundary-coboundary relationships between cells, this graph encapsulates the coboundary orderings given in section 1.4. For example, each 0-cell has a circular ordering of alternating $switch_1$ and $switch_2$ operations between its cell-tuples, which represents the circular ordering of edges and faces.

We will see shortly that the cell-tuple is a similar but more flexible variation on the notion of the cusp given in the tri-cyclic cusp model of Gursoz et. al. (1991). The tri-cyclic cusp and this similarity will be discussed later in this section.

Brisson's approach is simplified by the use of regular cells from topology. For example, a regular 2-cell cannot have more than one boundary 1-cycle or internal isolated cell complexes within it. Thus the boundary of the 2-cell can be reconstructed by following the sequence of $switch_0$ and $switch_1$ operators in the set of cell tuples associated with $c_{\alpha 2}$; ie. all t for which $t_2 = c_{\alpha 2}$. The other simplifying assumption used in the cell-tuple is applied when representing the subdivision of a manifold with boundary. Brisson 'completes' the cell neighbourhoods of the manifold with boundary by representing the space 'exterior' to the manifold with boundary, as a cell. This is a simple, restricted application of the 'world' cell approach that we will use for representing spatial objects in this research. It is also a relaxation of the regular cell condition since it is not always the case that the space 'exterior' to a manifold with boundary corresponds with the definition of a regular cell.

The cell-tuple maintains both the primal and dual cell complexes simultaneously. Brisson states that the $switch_k$ operator in the primal subdivision of an n-manifold is equivalent to the $switch_{n-k}$ operator in the dual subdivision. Thus to access the dual subdivision, $switch_{n-k}$ is returned whenever $switch_k$ is asked for.

2.5.2 Winged-Representation

Overview

The winged representation of Paoluzzi et. al. (1993) represents simplicial complex subdivisions of n-manifolds and n-manifolds with boundaries. One of the advantages in using the n-simplex instead of the regular n-cell is that the number of $(n$-1)-simplexes contained in the n-simplex is equivalent to $n+1$, the number of vertices of the n-simplex. Thus the $(n$-$1)$-simplexes and their orientations may be expressed in terms of the vertices. For example in a 2-simplex or triangle with three vertices $<v_0,v_1,v_2>$, the 1-simplexes or edges are $<v_0,v_1>$, $<v_1,v_2>$ and $<v_2,v_3>$. Following Paoluzzi et. al. (1993), a formula for finding any one of the oriented $(n$-$1)$-simplexes of an n-simplex can be given as follows:

$$\Delta_{n-1,k} = (-1)^k <v_0,......,v_{k-1},v_{k+1},.....v_n>$$

where $\Delta_{n-1,k}$ and v_k are the kth $(n$-$1)$-simplex and kth vertex of Δ_n respectively. Recursive application of this formula effectively returns the $(n$-1)-skeleton of the n-simplex. For representing and efficiently traversing a regular simplicial subdivision of an n-manifold or n-manifold with boundary, this formula and the coboundary relation for each n-simplex are combined to form the basis of the winged-representation.

Simplexes and Simplicial Complex

An n-winged representation $W = (V,\{<S_n,A_n>\})$ of a simplicial subdivision of an n-manifold is constituted by its skeleton V (or the set of its indexed vertices) and an ordered set of pairs, one for each n-simplex Δ_n of M^n. The pair $\{<S_n,A_n>\}$ corresponding to each n-simplex Δ_n contains the following;

1. an ordered set $S_n = (k_0, k_1, ..., k_{n+1})$ of integers indexing the skeleton of the simplex; ie. the set of its vertices. The ordering of this set defines the orientation of the *n*-simplex

2. an ordered set A_n of integers indexing the *n*-simplexes which are (*n*-1)-adjacent (ie. form the coboundary or set of *n*-simplexes that are adjacent along (*n*-1)-simplexes) of Δ_n. If there is no adjacent *n*-simplex on a particular (*n*-1)-simplex (eg. when the (*n*-1)-simplex cobounds the modelling space), then the 'null adjacency' is indicated with the symbol $\perp$ in place of an integer.

There are two winged representations: the decompositive and boundary winged representations. The decompositive winged representation is an *n*-simplicial complex subdivision of the spatial objects. For efficiency reasons, the decompositive winged representation does not represent the Euclidean *n*-manifold that contains the spatial objects. Thus the ordering information is not complete which is the reason for the 'null adjacency' condition mentioned above.

The boundary winged representation has the same definition as the decompositive representation except that only the *n*-manifold boundary of a polyhedron ($n \geq 2$) is subdivided and the coboundary set A_n for each simplex in the boundary winged representation is specified as complete.

The stated advantages of the winged-representation are the same as those that can be given for all simplicial complexes: guaranteed combinatorial complexity resulting in fixed length data structures and consequent easy access to boundary/coboundary information. The disadvantages are that data volumes increase rapidly since many simplexes are required to construct an irregular spatial object, fragmentation makes topological properties difficult to calculate, all input data must be triangulated before it can be stored in the winged-representation and some topological operations do not preserve the simplicial structure of the complex. Paoluzzi et. al. (1993) note the last of these disadvantages by stating that the boundary winged-representation is better than the decompositive representation for extrusion (ie. the topological product).

It is interesting to note that in the excellent review of the winged representation in Paoluzzi et. al. (1993), the storage efficiency of the winged-representation is favourably compared with application of the winged-edge of Baumgart (1974) for representation of simplexes. However the storage efficiency of the decompositive winged-representation is better than the winged-edge only because it was specifically designed to take advantage of the special cases present in simplicial complexes eg. the number of vertices = number of $(n-1)$-simplexes. The winged-edge (and the other models reviewed in this research) were designed for general cell complexes and cannot make these optimizations. Given that a simplicial complex always requires more fragmentation than a cell complex, direct comparisons of storage efficiency between simplicial and cellular models are not really useful.

2.6 Other Subdivisions

2.6.1 Corbett's General Topological Model For Spatial Reference

Overview

Corbett (1985) defines a general topological model for the representation of subdivisions of any spatial object - including subdivided n-manifolds and n-manifolds with boundary. An n-cell in Corbett's model is the image of a 'topological map' of the prototypical regular n-cell from topology ie. the n-ball. The only restriction placed on the 'topological map' is that it be continuous at its boundaries. In chapter 4 it will be shown that such a definition loosely corresponds with that the of the normal CW-complex. The advantages of such a definition are that spatial objects are less fragmented than they would otherwise be if simplexes or a regular CW complex were used. It is interesting to note that this definition of a topological map permits both cyclic and acyclic singularities (Corbett 1979) but Corbett does not indicate what these cases are.

Topological maps are also used to define the geometry of a cell. We will see shortly, that the addition of geometry to Corbett's cell makes it similar to the notion of an 'extent' given in the description of the selective geometric complex by Rossignac and O'Connor (1991).

Boundary and coboundary relationships for each cell are held in a data structure which appears to be an extension of the DIME concept (see section 2.2 and Corbett 1979) ie. each cell has a complete listing of its boundary and coboundary information. Spatial objects are constructed from one or more cells by associating attributes which identify the cell with the spatial object.

Subdivision

Each k-cell has four pieces of information held with it, some of which will be variable length lists:

cell: boundary, coboundary, map, attributes

cell	The unique identifier of the cell ie. to distinguish it from other cells of the same dimension.
boundary	The list of (k-1)-cells incident to the k-cell; ie. forming its boundary.
coboundary	The list of (k+1)-cells incident to the k-cell; ie. forming its coboundary.
map	The 'topological map'; necessary to define the geometry of the cell.
attributes	User defined attributes and additional geometric information.

As noted above, the cells that result from the topological map are somewhat similar to the normal CW complex used in topology. We will see shortly that Corbett's topological maps are somewhat similar to the sewing operations used to tie together the darts of Lienhardt's n-G-map (Lienhardt 1991) discussed below.

Ordering

The relationships between cells are held implicitly in two lists of boundary and coboundary cells associated with each cell.

Corbett does not specify any explicit ordering information within these lists, because he does not embed the cell complex within a Euclidean manifold. As a consequence, there are no consistent circular and/or two-sided orderings of cobounding cells. However, if the cell complex was embedded within a Euclidean manifold, then the relative orientations of cells described by Corbett could be used to reconstruct circular orderings from the lists of cobounding cells.

To speed up access to spatial relationships (such as adjacency or containment) between individual spatial objects in a cell complex (eg. the parts of an engine), Corbett suggests that this information be represented explicitly using a construction which is loosely based on the dual complex. For example, the relationship between two 3-cells sharing a 2-cell or face can be represented by a 1-cell connecting their duals (0-cells).

2.6.2 Non-Manifold Subdivisions - The Radial Edge

Overview

The "non-manifold" model of Weiler (1986) was one of the first (see also Wesley and Markowsky 1980) to try to represent spatial objects that are not 2-manifolds; ie. those that cannot be represented by 2-manifold models such as the winged-edge of Baumgart (1974) or its variants. Weiler describes the domain (see Weiler 1988) as consisting of non-manifold surfaces (eg. a surface with one or more boundaries and/or a surface with one or more dangling edges), wire-frames (ie. graphs), regions (or volumes or solids), faces, edges and vertices. It is assumed that all these objects are contained within a Euclidean 3-manifold and intersect only along their boundaries. In terms of this research:

1. Many of the "non-manifold" conditions such as more than two faces in the coboundary of an edge (eg. when two solids are incident to an edge or when two solids are incident to a vertex) are present in 3-cell complexes.

2. The remaining "non-manifold" conditions (eg. wire-edges or dangling edges) can be represented by allowing the cells of the 3-cell complex to be singular.

As we shall see in chapter 4 and 5, the non-manifold model is really a topological model that may be represented by a singular 3-cell complex. This view of the non-manifold model simplifies the ordering relationships, allows consistency checks to be applied to the cells, facilitates derivation of an implicit model and clears up much of the confusion about what is and isn't a "non-manifold" - see Takala (1991).

Example applications of the non-manifold model in computer-aided design, include integrated modelling of solid objects and wire-frames (graphs), and the representation of the results of boolean-set or overlay operations (since the set of 2-manifolds is not closed under the boolean set operations).

Subdivision

Weiler 's model is an explicit one consisting of eight basic topological elements represented in a top-down hierarchical data structure (see Weiler 1988 pg. 19). The adjacency relationships between these elements were derived by extending the relationships given by Baer et. al. (1979). The eight elements are:

Model - a single three dimensional topological modelling space, consisting of one or more distinct regions of space. It acts as a repository for all topological elements contained in a geometric model.

Region - a volume of space. Weiler indicates that there is always at least one region in a model. If there is only one region, then it is infinite in extent and in the terminology of this research, the region is the Euclidean 3-manifold. Degenerate regions are permitted ie. a region may be a point.

Shell - if closed then it forms an oriented boundary surface of a region. A region may have more than one shell. Shells can also be open, ie. a collection of faces.

Face (2-cell) - oriented, bounded portion of a shell. The face is a piece of a surface which does not include its bounding edges. Weiler states that a face is always bounded by exactly two regions. In addition a face must be both a 2-manifold and the homeomorph of a plane (to avoid handles in faces).

Loop - oriented set of edges forming a connected boundary 1-cycle of a face. Degenerate loops are permitted; ie. a loop may be a point (see also Mäntylä 1988).

Edge (1-cell) - oriented portion of a loop between two vertices. The orientation comes from assigning a to-from order to the bounding vertices

Vertex (0-cell) - unique point position. A vertex may serve as the boundary of a face (ie. degenerate loop) or as a region (ie. degenerate region).

Ordering

Weiler (1986) was one of the first to realize the existence of circular and two-sided orderings for 3-cell complexes. He derived the underlying ordering and adjacency relationships represented by the radial-edge data structure, by extending those originally put forward for subdivided 2-manifolds by Baer et. al. (1979). However, we will express these results in terms of the boundary-coboundary relations and the coboundary ordering results, both of which are given in section 1.4. The following ordering relationships and adjacencies are included in the radial-edge:

1. Cobounding edges of every vertex. A similar concept has been used in 2-dimensional GIS (where the set of arcs at each node is held) in order to support network applications.

2. The two cobounding 3-cells of a face ie. the two-sided ordering of a face.

3. The circular ordering of faces and 3-cells about a 1-cell. This relationship is actually obtained in two stages. Firstly the circular ordering of loops about an edge is represented. The loops have pointers to their parent faces which in turn, have pointers to their cobounding 3-cells.

The explicit representation of shells, loops etc. together with their degenerate forms complicates both the topological model and the data structures, but has the advantage of quick access to adjacency information. The model is acceptable for most CAD applications where there are relatively few spatial objects (in comparison with a 'natural' scene) and each spatial object usually has a smooth surface (ie. few cells and corresponding linkages are required). Due to its complexity, the application of the radial-edge to geoscientific data has not been as successful (Sword 1991).

To support curved surface and edge environments, the radial edge model specifically permits a point (as a degenerate loop) to form the boundary of a face. The only cell-complexes that permit such situations are the normal CW-complexes used in topology (also noted in Lienhardt 1991 - see section 4.2 for more details).

2.6.3 *N*-Dimensional Generalized Maps

Overview

A combinatorial map as defined in Vince (1983) is another method for representing cell boundary-coboundary relationships in cell complexes. A barycentric subdivision of each cell in the complex is formed, then each simplex in the barycentric subdivision is assigned a number for the dimension of the cell that the simplex represents. Using the 2-dimensional example given in figure 2.10(a), the dual of the barycentric subdivision is created and the edges of the dual are assigned the dimension of the cell that does not occur at either of the two 0-cells of the primal 1-simplex (see figure 2.10b).

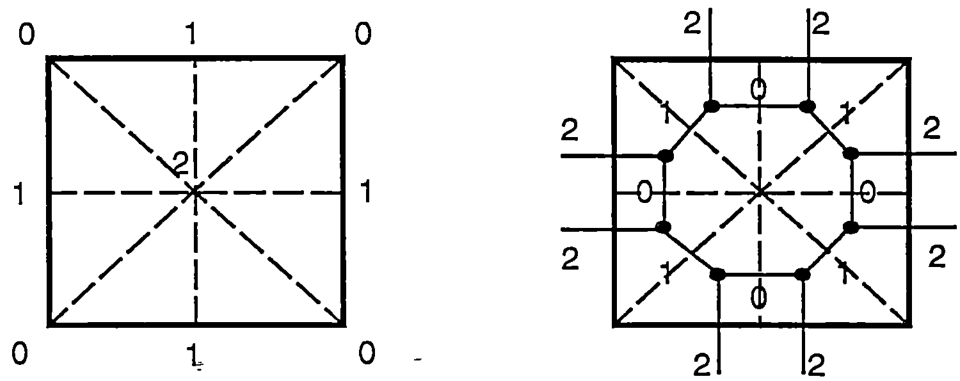


Figure 2.10 - combinatorial map of Vince (1983) (a) the assignment of cell dimensions to the simplexes of the barycentric subdivision of a 2-cell. (b) the labelled graph between dual vertices.

The n-dimensional generalized map of Lienhardt (1991) is both an extension and a simplification of this notion of a combinatorial map. Lienhardt uses an extension of the barycentric subdivision but does not explicitly represent the cells. Instead, a fundamental element known as a *dart* which corresponds to 'half an edge' (ie. an edge subdivided at its midpoint) is defined and a set of 'gathering' operations is used to combine darts to form cells and cell boundary-coboundary neighbourhoods. The cells formed by these gathering operations may be quite general. For example, they may be Euclidean manifolds with boundary as stated in section 1.4 or even more general such as manifolds with boundary. Some acyclic and cyclic singularities are permitted (see section 2.2) but interior cell complexes are not. Lienhardt states that a data structure is under development for "simultaneously handling generalized maps of different dimensions" which would (in the terminology of this research) require such cells.

Subdivision

From Lienhardt (1991):

Let $n \geq 0$, an n-G-map is defined by an $(n+2)$-tuple, $G=(B, \alpha_0, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ such that:

- B is a finite, non-empty set of darts;
- $\alpha_0, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are involutions on B (ie. $\forall\, i, 0 \leq i \leq n, \forall\, b \in B, \alpha_i^2(b) = b$)

such that:

1. $\forall\, i \in \{0,\ldots\ldots\ldots,n\text{-}1\}$ α_i is an involution without fixed points.

2. $\forall\, i \in \{0,\ldots\ldots\ldots,n\text{-}2\}, \forall\, j \in \{i+2,\ldots\ldots\ldots,n\}$, $\alpha_i\alpha_j$ is an involution.

Fixed points correspond to situations where for $b \in B$, $\alpha_i(b) = b$. For example, in a 1-G-map when $\alpha_1(b) = b$ then the dart b is at the last vertex of a 'dangling edge'. Thus, in the terminology of Corbett (1979) (see section 2.2) and this research, the first condition ensures that the singular cells are either cyclic or acyclic; ie. interior cell complexes of lower dimension which are not connected with the boundary cycle of an n-cell are not permitted. The second condition ensures that the gathering of (n-1)-cells ($n \geq 2$) is applied to the darts of the gathered (n-1)-cells.

The final statement of the definition of the n-G-map is that if α_n is an involution without fixed points then the n-G-map is an n-cycle; otherwise the n-G-map has boundaries. For example, if the 1-G-map is an involution without fixed points, then it forms a 1-cycle (or 1-manifold). Alternatively, the 1-G-map is a simple chain or path (figure 2.11(b)).

We now give some further examples to illustrate these concepts. In figure 2.11(c) the 2-G-map consists of three sets: $\alpha_0 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$, $\alpha_1 = \{\{2,3\},\{4,5\},\{6,7\},\{8,1\}\}$ and $\alpha_2 = \{\{1\},\{2\},\{3,8\},\{4,7\},\{5\},\{6\}\}$. In figure 2.11(d), $\alpha_0\alpha_2$ has a fixed point which results in the creation of a singular 2-cell.

For the application to multi-dimensional domains proposed in this research, there are two disadvantages of the n-G-map:

1. The 1-skeleton must be connected (see Lienhardt 1991 pg. 61). Thus bridges must exist between cells that have disconnected boundaries eg. a face with two boundary 1-cycles must have a bridge between them (as in figure 2.11(c)).

2. As mentioned earlier (and related with 1) the definition of the n-G-map precludes the effective use of cells with disconnected internal cell complexes. Thus many relationships between the n-G-maps ($0 \leq n \leq 3$) of spatial objects of different dimensions cannot be captured without using costly explicit global links. Research on more general combinatorial maps (as referred to in Lienhardt 1991) may be required.

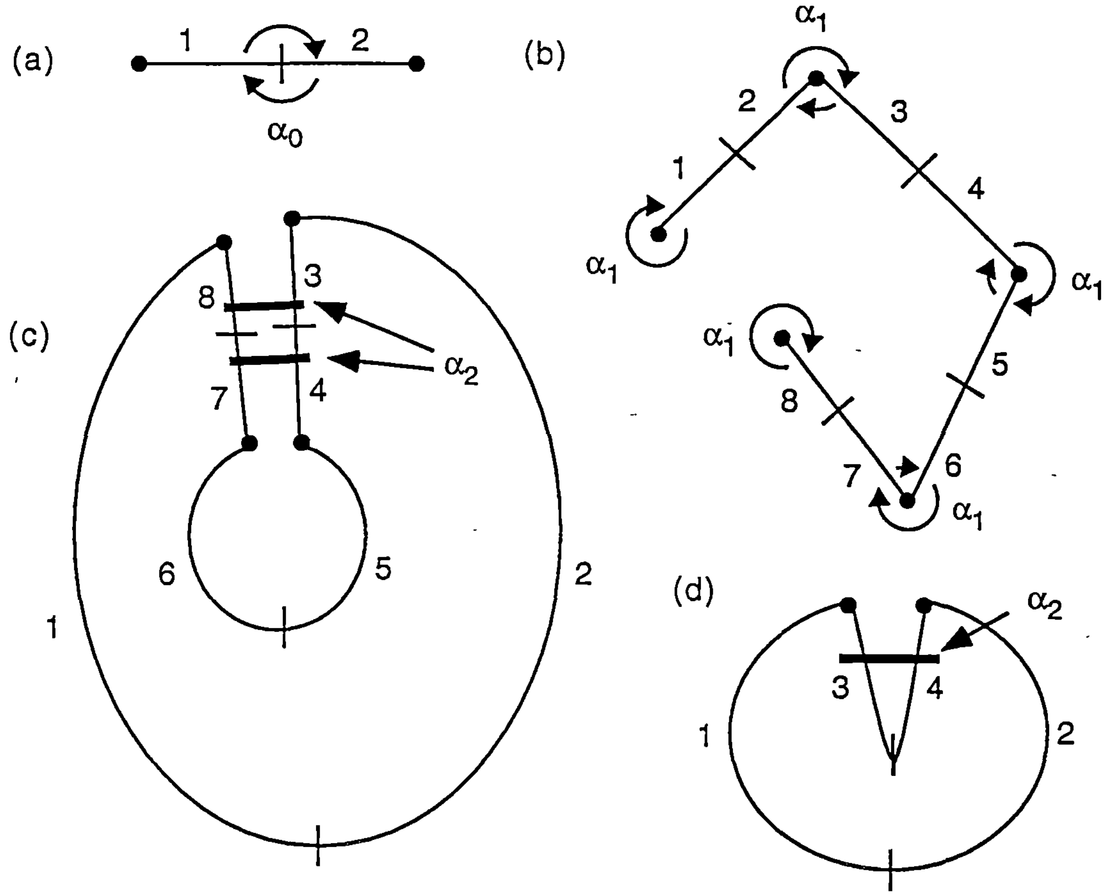


Figure 2.11 - examples of n-G-maps (a) a 0-G-map with two darts (b) a 1-G-map (with fixed points) (c) a 2-G-map showing creation of the equivalent of a 2-cell with more than one boundary cycle (d) a 2-G-map showing creation of a singular 2-cell (compare this with figure 2.2 in section 2.2. Note that every singular cell created in Lienhardt's model must be expressed in this way)

As is noted by Brisson (1990), the cells formed by the gathering operations on darts can be very general (eg. non-orientable 2-manifolds) and thus the ordering results he derived for subdivisions of manifolds and manifolds with boundary do not necessarily apply. However, if necessary the gathering operations could be restricted so that the cells are homeomorphic to n-dimensional disks and form subdivisions of manifolds and manifolds with boundaries. The ordering results for boundary-coboundary relationships could then be applied. Lienhardt describes applications of 2-G-maps to subdivisions of 2-manifolds.

Loosely speaking, n-G-maps are a record of the gathering operations used to ties darts together. The n-G-map along with the cell-tuple of Brisson (1990) (and thus the quad-edge of Guibas and Stolfi 1985 and facet-edge of Dobkin and Laszlo 1987) are the most implicit of all topological models described in this research.

Lienhardt (1991) gives an excellent review of the similarities between *n*-G-maps and these implicit models.

The similarities between the underlying research on combinatorial maps of Lienhardt (1991), the related work of Gunn (1993) and the cell-tuple of Brisson (1990) could be investigated in future research.

2.6.4 Selective Geometric Complexes

Overview

Selective geometric complexes (SGC) were first proposed in Rossignac and O'Connor (1991) as a general purpose modelling tool for complex applications involving polyhedra of more than one dimension. The SGC is essentially a collection of very general cells, each of which corresponds to a subset of a manifold. The boundary and coboundary neighborhood relationships of each cell are held in the form of an incidence graph. Spatial objects are constructed from collections of cells by associating an attribute which identifies the cell with the spatial object. Each cell of a spatial object can be marked as active or inactive, which makes possible for example, the definition of spatial objects with incomplete boundaries like a face with missing boundary 1-cells.

Subdivision

Formally an SGC is a geometric complex **K** (a finite collection of cells c_j with $j \in J$) such that:

1. $i,j \in J$ and $i \neq j$ implies $c_i \cap c_j = \varnothing$.
2. for all $c \in K$, there exists $I \subset J$ such that $\partial c = \cup\, c_i$ for each $i \in J$.
3. for all $b \in$ ***c*.boundary**, $b \subset$ ***c*.extent** or $b \cap$ ***c*.extent** $= \varnothing$.

Each cell has six pieces of information held with it, some of which will be variable length lists:

***c*.extent**	A reference to the extent in which the cell *c* lives - the extent is the subspace which the cell subdivides (for example an edge may be part of a loop thus the extent of the edge is the loop).
***c*.dimension**	Dimension of *c* and ***c*.extent**
***c*.boundary**	Collection of all cells which are in the (n-1)-skeleton of *c*.

***c*.star**	The set of all cells which have c in their boundary. This is the coboundary neighbourhood of the cell.
***c*.active**	Indicates whether a particular cell should be active as regards the total selective geometric complex.
***c*.attributes**	User defined attributes.

Notice how the extent of the cell always has the same dimension as the cell itself. The extent provides global information about a complex spatial object that the cell belongs to. Rossignac and O'Connor use the notion of extent primarily as a device for attaching geometric attributes eg. equations for describing a curve composed of a number of edges. In geographical or geoscientific contexts, extents could be used to hold (for example) parameters for a fractal description, in order to provide a more realistic rendering. The extent also provides an alternative to the attributes for speeding up access to cells that belong to a spatial object. Lastly, extents are used to provide orientation of cells. But Rossignac and O'Connor do not explain how to handle situations where, for example, an edge could be oriented in two or more different ways by the different 1-dimensional extents it belongs to. One possible solution would be to permit the **c.extent** information to be a list of extents containing the cell. Another solution could be to equate each edge in the SGC to an edge-*use*, as defined in Weiler (1986) and in many of the models mentioned earlier in this chapter.

To determine the closure of a cell, Rossignac and O'Connor use a limited neighbourhood ordering. A (k-1)-cell b belongs to the boundary cycle of a k-cell c if:

1. c is in ***b*.star** (ie. c cobounds b)
2. b is contained in ***c*.extent.**

The k-cell c can now be given a value according to whether it is on the *left*, *right* or both sides (full) of b. Rossignac and O'Connor describe a function **c.neighborhood** which returns the above values (figure 2.12).

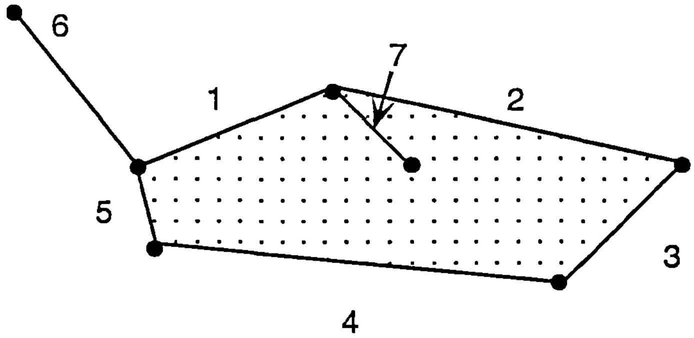


Figure 2.12 - a simple SGC consisting of a face with planar extent and an edge with linear extent. Edge 6 is not part of the singular boundary cycle because it doesn't obey conditions 1 and 2. Edge 7 may be distinguished from edges 1-5 because although it meets both conditions, its neighbourhood information is *full* (using the three value classification mentioned above).

Using this information we can give an example of a 2-dimensional SGC. In this example the information held in the incidence graph consists of:

Faces

A: extent=plane, Boundary=<1,2,3,8>,<g,a,c>
B: extent=plane, Boundary=<8,4,5,7>,<g,c,e,d>

Edges

1: extent=circle1, Boundary=<a,g>, star=<A,right>
2: extent=circle2, Boundary=<a>,star=<A,right>
3: extent=circle1, Boundary=<a,c>,star=<A,right>
4: extent=circle1, Boundary=<c,e>,star=<B,right>
5: extent=line1, Boundary=<d,e>, star=<B,full>
6: extent=line1, Boundary=<e,f>
7: extent=circle1, Boundary=<e,g>, star=<B,right>
8: extent=line2, Boundary=<g,c>, star=<A,B>
9: extent=line3, Boundary=<h,i>, star=<B,full>

Vertices

a: extent= point1, star=<1,left>,<3,right>,<2,full>,<A>
b: extent= point2, star=<A>
c: extent= point3, star=<3,left>,<4,right>,<8,left>,<A,B>
d: extent= point4, star=<5,right>,<B>
e: extent= point5, star=<5,left>,<6,right>,<B>
f: extent= point6, star=<6,left>
g: extent= point7, star=<7,left>,<1,right>,<8,right>,<A,B>
h: extent= point8, star=<9,right>,<B>
i: extent= point9, star=<9,left>,<B>

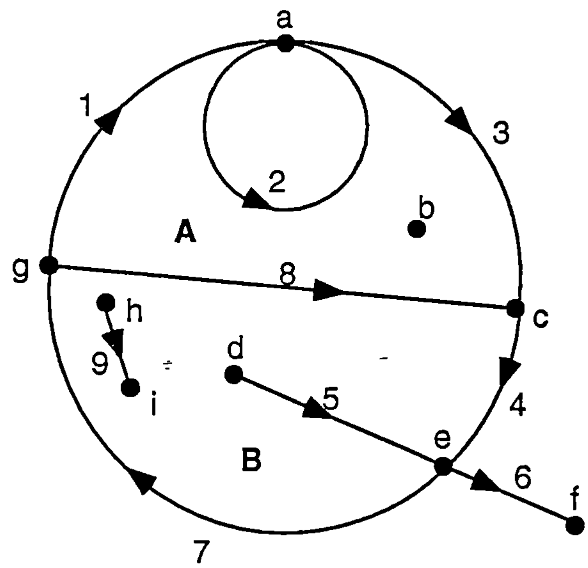


Figure 2.13 - an example of a simple 2-dimensional SGC

The example shows that the extent is a representation of the geometry of a spatial object whilst the boundary-coboundary relationships between the cells represent the topology of the spatial object.

It is interesting to note that *without* the notion of extent, the face and edge lists of the 2-dimensional example given above, are really similar to the information held in the arc-node topology model (see section 2.2). The main differences are:

1. the world 2-cell forming the Euclidean 2-manifold modelling space is not represented as a cell.

2. the polygon boundary information held by the arc-node model is reduced (ie. singularities are removed) whereas in the case of the SGC the entire (n-1)-skeleton of the cell is held.

Ordering

The ordering information explicitly held in the SGC consists of that supplied by the neighbourhood function (***c*.neighborhood**) given above ie. two-sided orderings. Circular orderings of the cobounding cells are not explicitly covered in Rossignac and O'Connor's formulation but could be added if the space containing the spatial object(s) is represented as a 'world' cell. Currently this space is implicitly present as an extent. For the 2-dimensional example given above the modelling space is the plane extent of the faces. This is one reason why orderings of the cobounding cells are not specified.

Lastly it is worth noting that if the modelling space was represented as a cell, not only would the results described in section 1.4 apply, but the specific neighbourhood information returned by **c.neighborhood** could be derived from an examination of the complete ordering of the boundary and coboundary information. For example, if the spatial objects were contained within a Euclidean 3-manifold, an edge in the boundary of a face must have one occurrence of the face in the circular ordering of volumes and faces about it. Interior edges can be detected by noting that the face will occur twice in the circular ordering of faces and volumes about the edge. Also, edges which share vertex neighbourhoods with any vertex of a face, will not have any occurrence of the face in their circular ordering of faces and volumes. Similar arguments can be given for faces and for vertices.

2.6.5 Tri-Cyclic Cusp

Overview

The tri-cyclic cusp of Gursoz et. al. (1991) has a multi-dimensional domain based on representing the spatial objects in a Euclidean 3-manifold as collections of cells, each of which corresponds to a subset of a manifold of the same dimension. The adjacency relationships of the non-manifold model and the missing vertex neighbourhood information (of what Weiler terms separation surfaces) are explicitly represented in three cycles of vertex cusps known as the loop, edge and disks cycles, some of which are degenerate (degenerate forms are known as variations). The tri-cyclic cusp captures the coboundary orderings of cells in a subdivided manifold or contained within a Euclidean manifold.

Subdivision

Gursoz et. al. describe the process of geometric modelling as one in which the subdivision (or categorization) of the geometric modelling space is based on a collection of disjoint point-sets of different dimensions. The modelling system involves the representation and manipulation of these point sets.

The procedure for defining this categorization of the modelling space R^n is defined as follows. For each k-dimensional subspace (where $0 \leq k \leq n$) there is a finite collection of k-dimensional point sets where each point set is:

1. Connected.

2. A proper subset of its closure; ie. it doesn't contain its boundary.

3. A subset of a k-manifold. Any point in the k-dimensional point set has a k-dimensional neighbourhood homeomorphic to an open set in R^k.

4. The union of all k-dimensional point sets is equal to the subspace itself.

The definition of these cells matches those given in Weiler (1986) except that Weiler adds a special case to condition 3 when applied to faces, since condition 3 does not exclude the possibility of a face containing a handle. The additional condition necessary to exclude such peculiarities, is that the interior of the face be homeomorphic to the Euclidean 2-manifold.

Ordering

As is noted above, the tri-cyclic cusp data structure essentially models all of the non-manifold adjacency relationships defined in Weiler (1986) including the ordered coboundary neighborhoods of cells in a subdivided 3-manifold (see section 1.4).

The main theme of the tri-cyclic cusp is that no topological primitive can exist without a vertex association. By definition all points in the point-sets or cells defined above have the same ordering information. Effectively the cells can be represented by such a point. A similar concept is used by Brisson (1990) in defining a single (abstract) vertex to represent a tuple of cells as is described earlier in this chapter.

In the simple example of the three-dimensional neighbourhood of a vertex in a subdivision of a 2-manifold shown in figure 2.14 (ignoring the dangling edge), the *cusps* are the *uses* made of the vertex by each face whilst a *zone* is a subset of the 3-dimensional neighbourhood of the vertex. The cycle of cusps at the vertex is known as a *disk*. In the case of the 2-manifold, the cycle of cusps has two sides and there is one disk either side. If faces or edges dangle from the vertex, then a corresponding disk climbs around them but no new zones are associated with these disks.

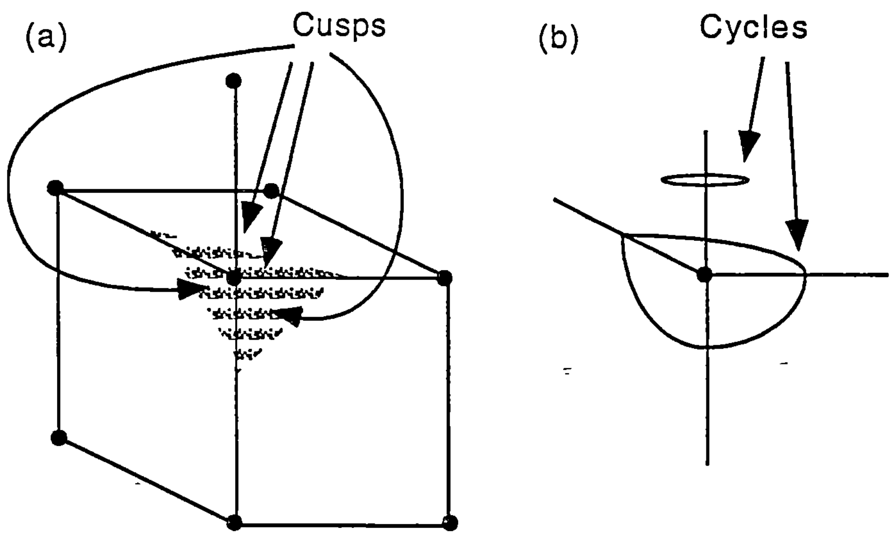


Figure 2.14 - cusps and cycles of cusps (a) the cusps at a vertex in a 2-manifold with a dangling edge. There is one cusp for each use (ie. incidence) between a cell and the vertex. (b) The cycles of cusps at the vertex. Note that although there are three cycles of cusps at this vertex there are only two zones - one inside the cube and one outside the cube.

Since edges can be oriented in two or more different ways by faces that share them, Gursoz et. al. represent the different topological *uses* of the edge separately as primitives called edge-orientations. Since faces are orientable (ie. two sided) the same method is applied. Different orientations of a face are referred to as 'walls'.

The coboundary orderings of the cells in the complex may be held in three cycles of cusps: the loop cycle, the disk cycle and the edge-orientation cycle (hence the name tri-cyclic cusp). A number of variations in the cusp entity are required to model singular cells.

1. The Loop Cycle (for walls)

The loop cycle consists of a cyclically ordered concatenation of cusps that trace each vertex and edge in the 1-skeleton of the wall; ie. all boundary 1-cycles and any isolated vertices and edges in the interior corresponding to the definition of a singular cell in this research. Each cusp is best visualized as the intersection of a 2-dimensional neighborhood with the wall centred at a vertex. As an example, for vertices in the boundary cycle of the wall, the intersection returns a hemidisk neighborhood (figure 2.15).

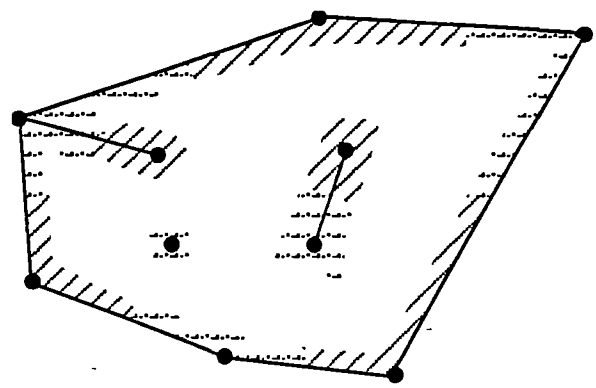


Figure 2.15 - the loop cycles of cusps of a wall

2. The Disk Cycle (for vertices)

A disk cycle is a set of cusps defined by the alternating circular ordering of edges and faces about a vertex. This cycle can belong to a single volume, in which case there are two zones, each of which has a single disk.

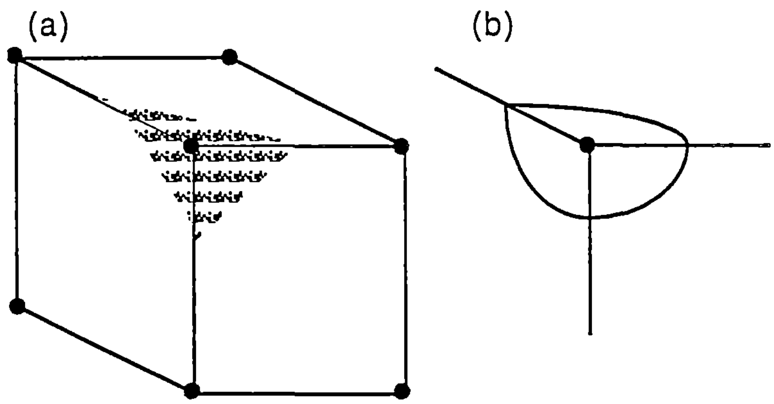


Figure 2.16 - the disk cycle of cusps (a) shaded cusps (b) the disk cycle of cusps

If there is more than one disk centred at the 0-cell (such as when two 3-cells share a common 0-cell) then the disks and zones are arranged in a hierarchy.

3. The Edge-Orientation Cycle (for edge coboundary relationships)

An edge orientation cycle is a set of cusps defined by the faces of an alternating set of faces and volumes about an edge-orientation. In the example shown in figure 2.17, there are two edge orientations each of which has an edge-orientation cycle consisting of three cusps.

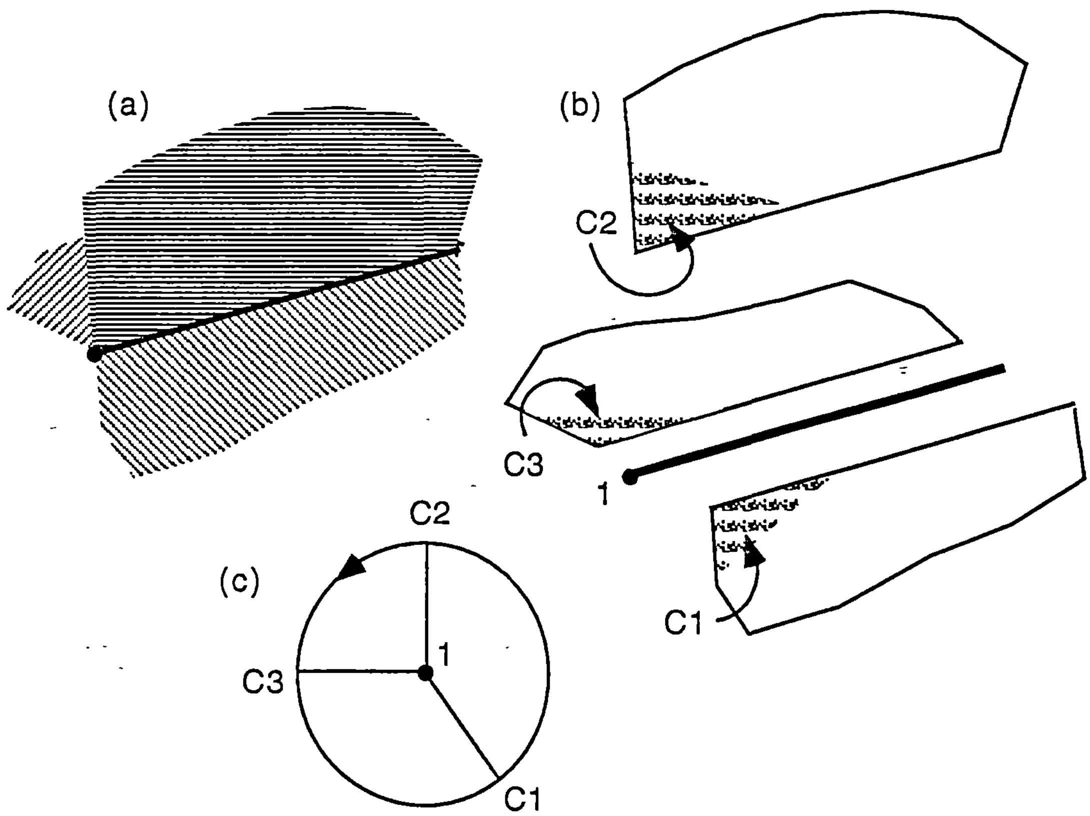


Figure 2.17 - edge orientation cycle **(a)** three faces incident to an edge **(b)** three cusps at vertex 1 **(c)** the edge-orientation cycle at vertex 1 consisting of three cusps

4. Variations in the Cusp entities (for singular cells)

The loop, disk and edge cusp cycles are sufficient to represent a regular subdivision of the 3-manifold modelling space. Degenerate forms of loops and walls are necessary to model the non-manifold adjacency relationships. The first degenerate form is the isolated vertex in a wall which forms a cusp with no edge associations. Such cusps form complete loops by themselves (see figure 2.15). The second degenerate form is an isolated edge in the Euclidean manifold or the interior of a 3-cell. The cusps of the isolated edge have edge associations but no face associations. Such cusps form a loop and wall by themselves. The third degenerate form is the isolated vertex in the modelling space. Such cusps form complete loops by themselves, which in turn form a wall and a shell.

There are nine topological elements of the tri-cyclic cusp model which support the representation of boundary-coboundary neighbourhood relationships and orderings between cells in a singular 3-cell complex, contained within the Euclidean 3-manifold:

Model - a single three dimensional topological modelling space, consisting of one or more distinct regions of space. It acts as a repository for all topological elements contained in a geometric model.

Region - a volume of space or 3-cell. There is always at least one (corresponds to the world 3-cell)

Shell - a collection of walls. Shells may be degenerate, ie. an isolated vertex is a shell in that it has a wall consisting of one loop which has a single cusp.

Zone - a local region around a vertex ie. a subset of the 3-dimensional neighbourhood of the vertex.

Wall - one of two possible orientations of a face. A wall has one or more loops. It has knowledge of the parent face and may be singular; ie. a wall with one loop and one cusp is an isolated vertex and a wall with one loop and two cusps is an isolated edge.

Edge-Orientation - one of two possible orientations of an edge (one for each vertex). An edge orientation has knowledge of the parent edge.

Disk - a cyclic list of cusps that defines a continuous surface in the vicinity of a vertex. For example, a disk forming a 2-dimensional neighbourhood can be found at any vertex of a subdivision of a 2-manifold. A disk may also be found around dangling edges and faces.

Loop - a cyclic list of cusps defining the boundaries of a wall. Each loop has knowledge of the parent wall it belongs to.

Cusp - one of many possible uses of an edge. A cusp exists around isolated vertices or may be formed by an incident edge or face, or some combination of these. The cusp is the fundamental element.

The tri-cyclic cusp is an explicit model. To avoid some of the problems that result from holding redundant data (eg. shells, loops) and the individual cycles of cusps, Franklin and Kankanhalli (1993) derive an implicit model which is based solely on the notion of cusp. Their alternative cusp consists of a quadruple of vectors (***P***,***T***,***N***,***B***) for each incident face, edge and vertex. The vectors have the following meanings:

P Cartesian coordinates of the vertex

T Unit tangent vector along the adjacent edge

N Unit vector perpendicular to **T**, in the plane of the face.

B Unit vector perpendicular to both T and N, pointing to the interior of the polyhedron

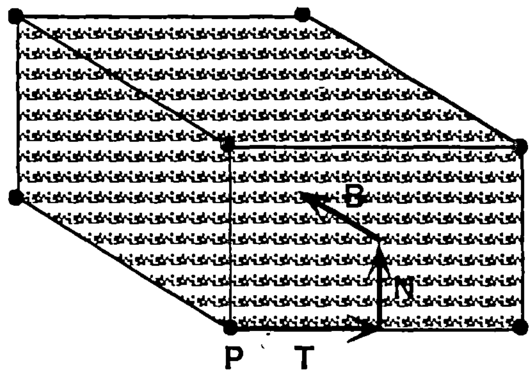


Figure 2.18 - the alternative cusp of Franklin and Kankanhalli (1993)

The alternative cusp of Franklin and Kankanhalli (1993) is very similar to the cell tuple of Brisson (1990). The difference is that the cell-tuple is more efficient for queries about cell neighbourhood relationships because the nodes representing cell tuples in Brisson's model are connected with edges which represent the ordered cell neighbourhood relationships that may be traversed by the switch operator. Notice also that when the tri-cyclic cusp is applied to regular cell complex subdivisions of manifolds and compared with the cell tuple it can be seen that the cell tuple contains the same ordering information but is less explicit and consequently less complicated. For example, individual disk cycles can be constructed by repeated $switch_1$ and $switch_2$ operations around the vertex for each 3-cell and loops in the wall of a face from the sequence of $switch_0$ and $switch_1$ operations.

Unfortunately Brisson's cell-tuple is not specified for singular cell complexes whilst both the tri-cyclic cusp and the distilled tri-cyclic cusp approach of Franklin and Kankanhalli (1993), are.

2.6.6 Single-Valued 3-Dimensional Vector Maps

Overview

Molenaar (1990) introduced an extension of the single-valued vector map (Molenaar 1989) for 3-dimensional vector maps. The domain of the model is expected to include spatial objects with different dimensions (features) specified as solids, surfaces, lines and points embedded in a Euclidean 3-manifold 'world' cell.

Each surface, line, point and solid boundary is subdivided into nodes, arcs and faces which are essentially equivalent to the 'cells' defined in section 1.4. Then a number of conventions are defined that govern the relationships between these cells. The most important of these conventions (ie. numbers 7,9,10 and 12 in Molenaar 1990) are essentially those that govern a cell complex (ie. no intersection of cells except along their boundaries) whilst conventions 5 and 6 forbid multigraph and pseudograph conditions. The other conventions describe cell geometry and some restrictions on the number of features a cell may belong to.

Ordering information for cell coboundary relationships which corresponds with the definitions given for a regular subdivision of a 3-manifold (see section 1.4) is also described. These orderings are then extended to describe the situation for a face that occurs in a surface when the domain is multidimensional ie. left and right solid are the same. However at this stage explicit modelling of some topological relationships between cells is introduced to cover some situations that Molenaar believes are difficult or not possible to recover from his model. The first relationships carried by these explicit links relate to the neighbourhoods of isolated nodes. Two links are introduced to represent a node which 'is on' a face and a node which 'is in' a solid. These are necessary since the only other possibility in Molenaar's model would be to calculate these relationships using metric (coordinate) information. The singular cell complex we describe will provide information about the neighborhoods of vertices that are 'on' faces and 'inside' solids through the coboundary ordering relationships. Molenaar also introduces two other explicit 'is on' and 'is in' relationships, this time between an arc and a face and an arc and a solid respectively. We will see that such information can also be derived from the coboundary orderings of the arcs or edges in a singular cell complex; ie. an arc which 'is in' a solid will have only one cobounding solid and an arc which 'is on' a face will have that face in its cobounding list of faces and solids exactly twice.

Apart from these two unnecessary explicit arc links the 'low-level topology' (Molenaar 1989) of the formal data structure along with the features known as bodies (ie. solids) are similar to the singular 3-cell complex we will define in this research. However, it should be noted that although some unordered boundary information is stored (begin and end node for edges, border for polygons) in Molenaar's model no coboundary information or orderings beyond the very simple two-sided orderings of a face are modelled. Apart from the explicit relationships mentioned above, the low-level topology given in Molenaar's model is somewhat similar to the selective geometric complex of Rossignac and O'Connor (1991) described above.

Chapter 3
Mathematical Background

3.1 Introduction

Algebraic topology is finding, calculating and applying functors from geometric categories (such as topological spaces and continuous maps) into algebraic categories (groups and homomorphisms of groups). The hope is that algebraists will be able to solve topological problems (ie. finding and calculating topological invariants) using algebraic methods. Point-set topology is a less pragmatic (due to the difficulty in enumerating all possible point-sets) but intuitive attempt to consistently and concisely state the invariants of a space and continuous maps between spaces (the two are interdependent) in terms of its point neighborhoods and the tools of set theory. Point-set topology essentially contributes very little to the formal solution of such problems.

For this research, most interest centres on the following concepts in topology:

- Two types of continuous maps: homeomorphism and homotopy.
- Manifold spaces - because of their generality, well-known properties and the fact that they are useful abstractions of reality
- The properties of general topological spaces which remain invariant under homeomorphisms and homotopy equivalences
- The 'algebraic toolbox' defined on the finitization of these topological spaces (the cell complex) which is used to calculate these properties (chapter 4).

This chapter gives a brief summary of homeomorphism type, homotopy type, manifolds and manifolds with boundary. These concepts will be applied throughout chapters 4, 5 and 6.

3.2 The Continuous Maps - Homeomorphism and Homotopy

An *open unit n-dimensional disk* D^n or open n-disk is defined as:

$$D^n = \left\{(x_1,.........,x_n) \in R^n \middle| (x_1^2+.........+x_n^2)^{1/2} < 1\right\}$$

An *open unit n-dimensional hemi-disk* $D^n_{1/2}$ is defined as:

$$D^n_{1/2} = \left\{(x_1,.........,x_n) \in R^n \middle| (x_1^2+.........+x_n^2)^{1/2} < 1 \text{ and } x_1 \geq 0\right\}$$

A *closed unit n-dimensional disk* $\overline{D}^n$ or closed n-disk is defined as:

$$\overline{D}^n = \left\{(x_1,.........,x_n) \in R^n \middle| (x_1^2+.........+x_n^2)^{1/2} \leq 1\right\}$$

The *unit k-dimensional sphere*, S^n is defined as:

$$S^n = \left\{(x_1,.........,x_n) \in R^n \middle| (x_1^2+.........+x_n^2)^{1/2} = 1\right\}$$

For topological spaces **X** and **Y**, a map f: **X** $\rightarrow$ **Y** is a *homeomorphism*, if f is continuous, one-to-one (injective) and onto (surjective). If **X** and **Y** are Euclidean topological spaces, then f implies that for any $x \in$ **X**, having an n-dimensional neighborhood D^n, then $y \in$ **X**, where y is the image of x under f, will also have an n-dimensional neighborhood D^n. **Y** and **X** are sometimes said to have the same *homeomorphism type*. Intuitively, a homeomorphism is a kind of elastic transformation which twists, stretches and otherwise deforms a space without cutting or tearing.

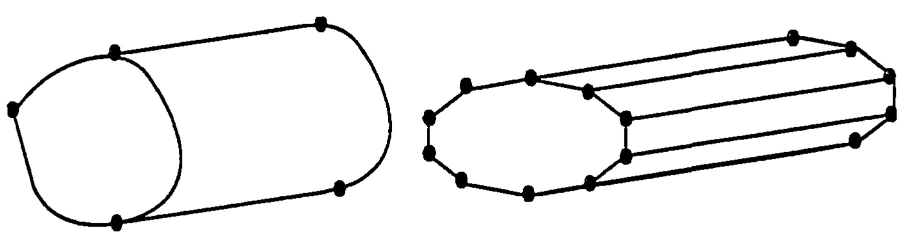


Figure 3.1 - two spatial objects having the same homeomorphism type; (ie. topologically equivalent)

Homotopy is a well-known concept in topology which captures the notion of deformation in a map. For topological spaces **X** and **Y**, f: **X** $\rightarrow$ **Y** and g: **X** $\rightarrow$ **Y** are called homotopic, if there is a continuous map h: I $\times$ **X** $\rightarrow$ **Y** such that $h(0,x)=f(x)$ and $h(1,x)=g(x)$, where I is the closed unit interval [0,1].

To illustrate the concept we use the simple example of a path between two points that can be defined as a map f: I $\rightarrow$ **Y**, where I is the interval [0,1]. The points $f(0)$ and $f(1)$ are the start and end points of f. If g: I $\rightarrow$ **Y** is a second path with the same start and end points as f, ie. $f(0) = g(0)$ and $f(1) = g(1)$, then f and g are homotopic (read "equivalent") provided that there is a function h: I $\times$ I $\rightarrow$ **Y** such that:

$$h(t,0) = f(t), \qquad h(t,1) = g(t) \qquad t \in \mathrm{I}$$
$$h(0,s) = f(0) = g(0), \qquad h(1,s) = f(1) = g(1) \qquad s \in \mathrm{I}$$

In essence the above definition states that two paths which start and end at the same points can be continuously deformed into one another, if there is a continuous function mapping the unit square (I $\times$ I) onto the subspace formed by the two paths as shown in figure 3.2.

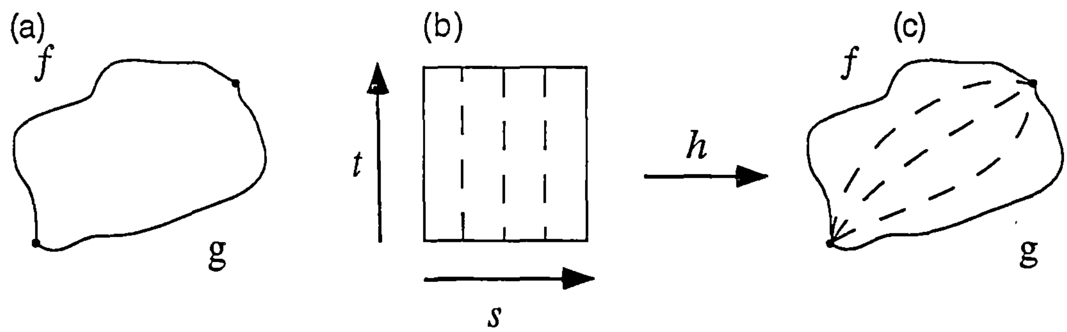


Figure 3.2 - an example of a homotopy.;
(a) The two paths in the space **Y** (b) the domain of the map h (I $\times$ I) and (c) the map itself h: I $\times$ I $\rightarrow$ **Y**. The homotopy pulls the image of f to the image of g

There are a number of important general points to note about the above example, firstly if the space **Y** is not simply connected between f and g (ie. there is a hole between f and g) then there will be no homotopy function h and the two paths are not homotopic. Secondly, the homotopy function f need not be 1:1, as is made clear by the fact that the values of s when $t = 0$ (1) are mapped to the start (end) point of f and g. In fact homotopies are more general transformations than homeomorphisms because they need not be 1:1 (ie. injective).

As mentioned earlier, two objects which are homeomorphic are said to be homeomorphically (or topologically) equivalent. A similar but less obvious equivalence can be defined using homotopy. Given topological spaces **X** and **Y**, a map f: **X** $\rightarrow$ **Y** is called a *homotopy equivalence* between **X** and **Y**, if it possesses a "homotopy inverse" g: **Y** $\rightarrow$ **X** and $g \circ f$ and $f \circ g$ are homotopic to the identity maps on X and Y respectively. If X and Y are homotopy equivalent then they are said to have the same *homotopy type*. Intuitively, we may think of the homotopy equivalence as a continuous transformation which 'collapses' or 'expands' a space X to produce Y in such a way that the homotopy type is always preserved.

In this research we are interested in spaces that result from a particular homotopy equivalence known as a *strong deformation retraction.* Intuitively, a strong deformation retraction takes each point of a space X along a continuous path into a subspace A (known as the *strong deformation retract*) with the only proviso being that the points of A do not move (see Jänich 1980 pg. 62). The fact that the points of A must remain fixed and that A is a subspace of X, makes the strong deformation retract easy to recognize in many of the spaces we are interested in. In fact, we do not need to find the functions f and g and the homotopies between them as indicated by the definition of the homotopy equivalence given above.

Formally (following Jänich 1980 pg. 62), if X is a topological space and $A \subset X$, A is called a *retract* of X if there is a continuous map $\rho: X \to A$ such that $\rho|A$ = identity map on A (ie. $\rho(a) = a, \forall\ a \in A$). If ρ is also homotopic to the identity map on X, then A is a *deformation retract* of X. If the homotopy between ρ and the identity map on X is chosen so that all points of A can be kept fixed in the course of it, then ρ is known as a *strong deformation retraction* and A a strong deformation retract of X.

As an example, consider figure 3.3 which shows a strong deformation retraction between the annulus and the circle forming its interior boundary, the result is that the strong deformation retract of the annulus is the circle. Both spaces have the same homotopy type since a strong deformation retraction is also a homotopy equivalence.

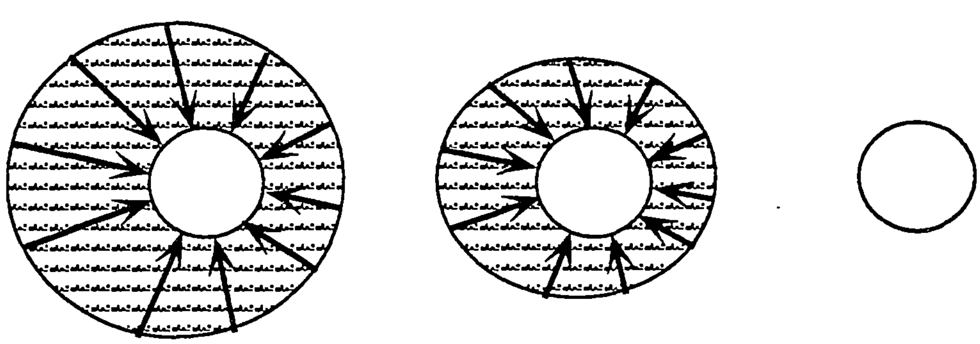


Figure 3.3 - the annulus and the circle have the same homotopy type (ie. a circle) because the annulus can be 'shrunk' to a circle via a strong deformation retraction (three stages in the strong deformation retraction are shown)

The strong deformation retract and a combinatorial equivalent to strong deformation retraction will be used in chapter 6 to define consistency constraints on operators for constructing cells and cell complexes.

Many of the important topological invariants such as the Euler-Poincaré characteristic and the fundamental group (see section 4.4) are actually invariants of the homotopy type (Armstrong 1982 pg. 103). The underlying reason is that homeomorphic spaces (ie. spaces having the same homeomorphism type) also have the same homotopy type. However, the opposite is not true; ie. spaces which have the same homotopy type do not necessarily have the same homeomorphism type. This is illustrated by the fact that a circle and an annulus in the example given above (see figure 3.3) have the same homotopy type yet are not homeomorphic. As a consequence many more spaces have the same homotopy type than homeomorphism type.

Lastly, the generality of homotopies and homeomorphisms shows clearly that topology is the 'most general' geometry - see Meserve (1955) and Takala (1991) for a good comparison of topology with other geometries. Correspondingly, the spaces that topologists most often work with are also very general. This fact which makes them attractive when specifying the cells of a cell complex and also indicates that these general spaces often directly match some of the spatial objects in the domain of a spatial information system (eg. boundaries of solids).

3.3 Manifolds and Manifolds with Boundary

An n-dimensional manifold or *n-manifold* M^n is a topological space **X** in which any point $x \in \mathbf{X}$ has a neighbourhood homeomorphic to the n-dimensional disk D^n. For example, every point on the surface of a sphere (or 2-manifold) has a neighborhood which is topologically 2-dimensional; ie. the neighborhood looks like a mildly bent disk. *Euclidean manifolds* are the simplest of all manifolds but differ from other manifolds in that they have an infinite extent. Manifolds with infinite extent (such as the Euclidean 2-manifold) are often referred to as *open* manifolds, whilst manifolds with finite extent (such as the 2-sphere) are often referred to as *closed* manifolds.

An *n-manifold* M^n *with boundary* is defined in exactly the same way as a manifold except that points on the boundary have neighbourhoods homeomorphic to an open unit n-dimensional hemi-disk $D^n{}_{1/2}$. To simplify the different abstract possibilities that such a definition implies, it is assumed in this research that the boundary of an n-manifold with boundary is an $(n$-$1)$-manifold, when considered in isolation.

However it should be noted that it is sufficient to remove a single point from a manifold to create a manifold with boundary. A manifold with boundary is *neither* closed nor open (Weeks 1990) which is why we do not use these terms in this research. Instead we use manifold, Euclidean manifold and manifold with boundary as appropriate.

For our purposes, the important topological properties of manifolds are: compactness, orientability, connectivity, two-sidedness. A property is considered to be *intrinsic* if it is dependent only on the manifold and *extrinsic* if it is dependent on the embedding of the manifold. This classification of topological properties is taken from Weeks (1990).

<u>Compactness</u> (intrinsic): a space is compact if a finite number of neighborhoods can be found which cover the space. All manifolds (except the Euclidean manifold) are assumed to be compact in this research.

<u>Connected</u> (intrinsic): a space is connected if it is not the union of two non-empty, open disjoint subspaces (Jänich 1980). All manifolds in this research are connected but the extent to which they are connected is determined by their genus and/or presence of boundaries.

<u>Orientability</u> (intrinsic): A 2-manifold is orientable if a small oriented circle (the indicatrix) placed on the surface and transported around an arbitrary closed surface always returns with the same orientation. All manifolds are assumed to be orientable in this research.

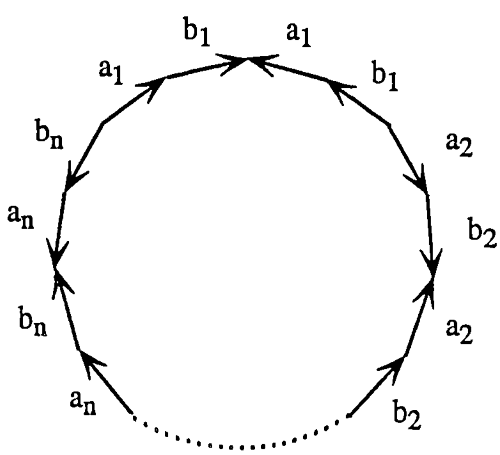
<u>Two-sided</u> (extrinsic): When orientable n-manifolds of dimension ≤ 2 are embedded in an orientable (n+1)-manifold they are always two-sided - see chapter 8, pg. 131 of Weeks (1990). Thus all manifolds are assumed to be two-sided in this research.

These properties, together with the notions of homeomorphism and homotopy equivalence defined in section 3.2, are sufficient to classify the n-manifolds and n-manifolds with boundary of interest to this research ($0 \leq n \leq 2$) into homeomorphism types (ie. spaces which cannot be distinguished under homeomorphisms) and homotopy types (ie. spaces which cannot be distinguished under homotopy equivalences). These classifications now follow.

3.3.1 Homeomorphism Types

A. Manifolds

The homeomorphism types of 0-manifolds and 1-manifolds are trivial ie. homeomorphism type of a point and a circle respectively. For 2-manifolds it turns out that the connectivity and orientability properties are sufficient to classify individual homeomorphism types. This classification was developed around the turn of the twentieth century - see Stillwell (1980) for an account. In this section we consider the homeomorphism types of 2-manifolds that can be 'embedded' in the Euclidean 3-manifold. Such 2-manifolds can be classified as having the homeomorphism type of a sphere with k-handles ($k \geq 0$). They are known as orientable (or two-sided) manifolds. Following Stillwell (1980), these 2-manifolds can be represented by a canonical polygon with oriented edges identified in pairs (figure 3.4). As an example, the torus can be constructed from the first canonical polygon by using edges a_1b_1. The resultant rectangle is glued to itself (or identified) along pairs of edges to form the surface of a torus and the normal form (set of oriented edges) is said to be $a_1b_1a_1^{-1}b_1^{-1}$.



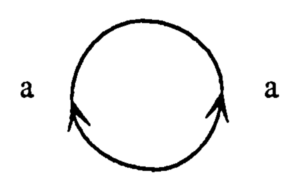


Figure 3.4 - the canonical polygons of the sphere with k-handles (k-torus) and the sphere (ie. the orientable 2-manifolds)

B. Manifolds With Boundary

The homeomorphism types of 1-manifolds with boundary are trivial (ie. an interval) and thus the focus turns to 2-manifolds with boundary. Perforating a 2-manifold (ie. adding a boundary) introduces a 'hole' in its corresponding canonical polygon - see figure 3.5.

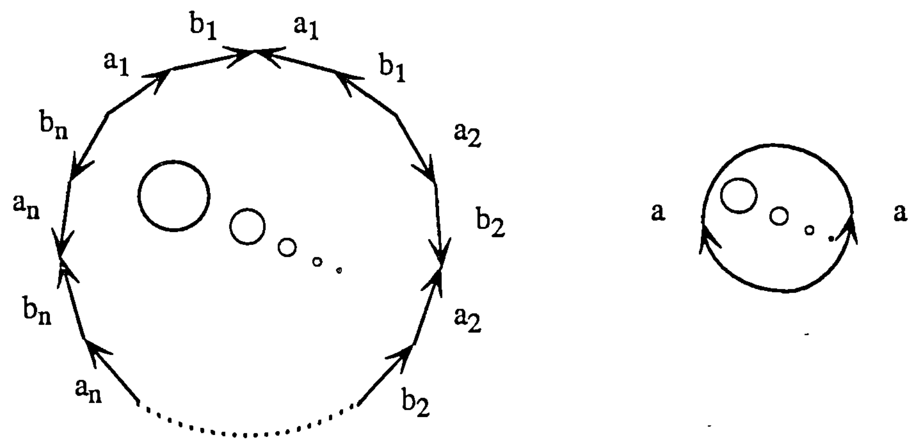


Figure 3.5 - intuitive canonical polygons for orientable 2-manifolds with boundaries

Following Massey (1967) it is possible to reduce the canonical polygons in figure 3.5 to equivalent canonical polygons without holes. Briefly, the reduction process is based firstly on making pairwise disjoint cuts $c_1, c_2, \ldots\ldots, c_m$ from the initial vertex of the canonical polygons in figure 3.5 above, to a vertex in the boundary cycle of each hole, $B_1 \ldots\ldots B_m$.

This process does not change the homeomorphism type of the 2-manifold (since it does not add any new holes or change the 2-manifold condition) but the resulting normal form for the sphere with *k*-handles ($k \geq 1$) and *m* boundaries becomes:

$$c_1 B_1 c_1^{-1} \ldots\ldots c_m B_m c_m^{-1} a_1 b_1 a_1^{-1} b_1^{-1} \ldots\ldots\ldots a_k b_k a_k^{-1} b_k^{-1}$$

and the normal form for the sphere with *m* boundaries is:

$$c_1 B_1 c_1^{-1} \ldots\ldots c_m B_m c_m^{-1} a a^{-1}$$

A canonical polygon without holes is created by 'opening up' or 'detaching' the two edges of each duplicated cut edge ie. c_1 and c_1^{-1} etc. For example, the torus with two holes (figure 3.6(a)) has normal form $c_1 B_1 c_1^{-1} c_2 B_2 c_2^{-1} a_1 b_1 a_1^{-1} b_1^{-1}$.

As described above the canonical polygon can now be 'opened up' along the cut lines c_1 and c_2 (due to their duplication) to form a simply connected canonical polygon (ie. no holes) as shown in figures 3.6(b) and (c).

Massey (1967) also notes that the homeomorphism type of any 2-manifold with boundary is dependent upon the number of boundaries *and* the homeomorphism type of the 2-manifold obtained by gluing a disk onto each boundary.

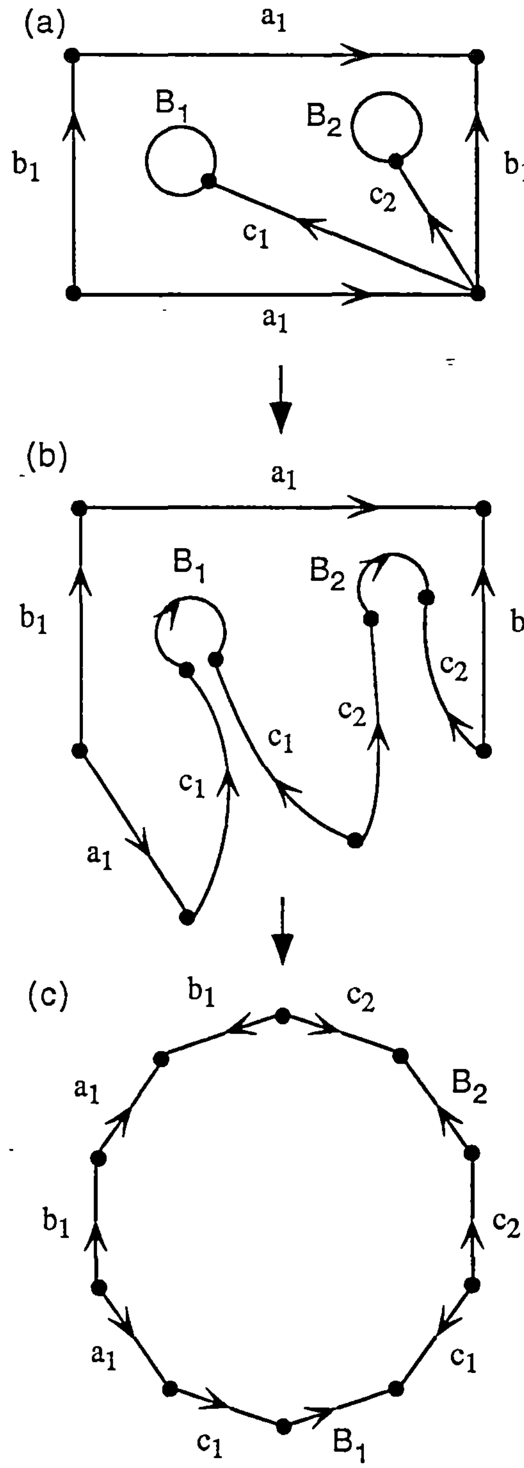


Figure 3.6 - opening up a torus with two holes (a) non-simply connected canonical polygon of a torus with two holes with cut lines c_1 and c_2 (b) opening up the non-simply connected canonical polygon along the cut lines to form (c) the modified canonical polygon for a torus with boundary

3.3.2 Homotopy Type

This section will discuss the (strong) deformation retracts of the n-manifolds and n-manifolds with boundary ($n \leq 2$). As indicated in section 3.2, a (strong) deformation retract is just one of many possible spaces having the same homotopy type as the n-manifold or n-manifold with boundary, but they are the most interesting to this research because they are the 'simplest' homotopy equivalent subspace. We will use them in chapter 6, together with a combinatorial form of strong deformation retraction, to construct any n-manifold or n-manifold with boundary.

A. Manifolds with Boundary

The strong deformation retract of any 1-manifold with a boundary is a point, since the removal of at least one point from a 1-manifold creates a space which is homeomorphic to a subset of R^1, and the point is the strong deformation retract of such a subset (see examples 1 and 2 on pg. 104 of Armstrong 1982).

The strong deformation retract of the 2-sphere with k-handles and m boundaries can be determined by modifying of the canonical polygons of both the k-torus and the 2-sphere using a process discovered by the mathematicians Dehn and Heegard in 1907. Following Stillwell (1980) pg. 78, instead of assuming that the hole or perforation (which creates the boundary of the 2-manifold) is internal to the canonical polygon of the 2-manifold (as in figures 3.5 and 3.6(a)), Dehn and Heegard situated the hole in the neighborhood of its single vertex (figure 3.7(a)) by cutting off its corners. The corners cut off form a disk (figure 3.7(b)) and the portions of the edges that remain, can be glued together to form bands on the body of another disk. This structure represents the 2-manifold with boundary (figure 3.7(c)).

Following Stillwell (1980) pg. 141, every 2-sphere with k-handles $k \geq 0$ (the 2-sphere has no handles whilst the torus has one etc) and m perforations (ie. m 1-manifold boundaries) can be expressed as a disk with $2k + (m\text{-}1)$ bands attached to its boundary using the Dehn and Heegard method described above. Since we know that the point is the strong deformation retract of the disk and a loop is the strong deformation retract of each band, the bouquet of $2k + (m - 1)$ loops is the strong deformation retract of the sphere with k-handles ($k \geq 0$) and m 1-manifold boundaries ($m \geq 1$).

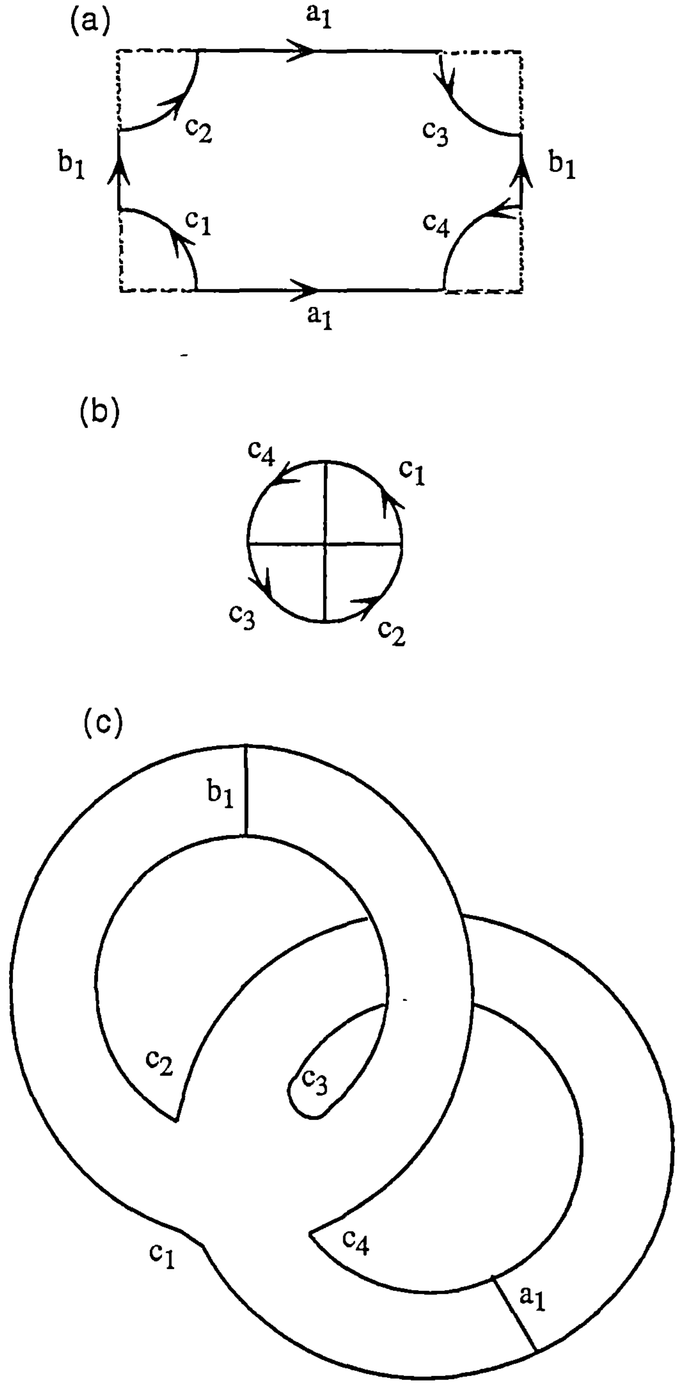


Figure 3.7 - Dehn and Heegard's construction of the torus with boundary by removing a disk (perforating) from the canonical polygon of a torus **(a)** Perforating the canonical polygon of the torus in the neighborhood of its single vertex **(b)** The disk forming the perforation **(c)** The bands that result when the remaining edges of the perforated canonical polygon are joined - this disk with bands represents the torus with boundary and we can say that **(a)** = **(b)** $\cup$ **(c)**

B. Manifolds

For a 0-manifold, the strong deformation retract is obviously the 0-manifold itself.

The 1-manifold (the 1-sphere), the 2-sphere and the non-simply connected 2-manifolds (such as the torus) have no proper subspaces (ie. a subspace not equivalent to the space itself) which could form a (strong) deformation retract. The problem with finding a proper subspace comes largely from the 'deformation' aspect of the definition of the deformation retraction specified in section 3.2, ie. the insistence that the retraction be homotopic to the identity map on the space itself. As an example, Croom 1978 pg. 54 shows that the n-sphere S^n, $(n \geq 1)$ is not contractible ie. there is no (strong) deformation retraction from S^n to a point. However, there *is* a retraction between S^n and the point. That is, a point of S^n is a retract of S^n but not a deformation retract of S^n (Hu 1959, pg. 16). Using the fact that S^n is not contractible and the fact that a deformation retraction maintains the fundamental group (the invariant that records homotopy properties), it is possible to show that unlike the 2-manifolds with boundary, a 2-manifold cannot have a graph as its deformation retract. However, for our purposes, which will become clear in chapter 6, we will list the deformation retracts of the 1-manifold and the 2-manifolds as themselves.

3.3.3 Table of Homeomorphism and Homotopy Types

Lastly, we give some examples of homeomorphism type and corresponding homotopy type of manifolds and manifolds with boundary. The spaces listed with the same homotopy type are the (strong) deformation retracts. The table is intended to illustrate the fact that many more spaces have the same homotopy type than homeomorphism type.

Homeomorphism Type	**Homotopy Type**
'Solid' Sphere (3 disk)	Point
'Solid' Torus	Loop (1 per handle)
2-manifold	
- Sphere	Sphere
- Torus	Torus
2-manifold with 1-manifold boundary cycle	
- Sphere	Point
- Torus	Bouquet of Loops (2 per handle plus 1 loop for each extra boundary cycle)
1-manifold	Loop
1-manifold with 0-manifold boundary cycle	Point

Chapter 4
Generalized Regular Cell Complexes

4.1 Introduction

As mentioned in chapter 1, a 'cell' can be defined in many different ways. Topologists define a 'cell' as a space which is homeomorphic to a disk. The topologists' aim in subdividing the space into such basic cells, is to simplify the calculation of the topological properties of the space. From the review in chapter 2, it is clear that what constitutes a cell in GIS, CAD and computational geometry, varies considerably and is often much more complex than the topological definition. For example, in 2-dimensional GIS a 1-cell is often defined as a chain of line segments and a 2-cell may have more than one boundary cycle whilst in CAD, for example, the most general form of a cell is defined as a subset of a manifold (Rossignac and O'Connor 1991). The primary reason for these different definitions is a balance between the need to simplify the calculation of topological properties (ie. the same aim as topologists) and the need to minimize the fragmentation of the spatial objects. Minimizing fragmentation has the double benefit of speeding up access and reducing storage requirements. To optimize both aims, yet retain all the previous definitions of cells (ie. simplexes etc.) for 3-dimensional applications, we define an n-cell as the homeomorph of a Euclidean n-manifold with one or more connected, orientable, two-sided, subdivided (n-1)-manifold boundary cycles ($0 \leq n \leq 3$). This cell will be called a generalized regular cell.

Section 4.2 describes the traditional simplicial, regular and CW complexes given in topology. Section 4.3 introduces the generalized regular cell complex as a compromise between the different aims given above. The boundaries of generalized regular cells are subdivided manifolds whose topological properties were given in section 3.3. Section 4.4 and 4.5 describe how the methods used to calculate important topological properties (such as connectivity) using the simplicial, regular and CW cell complexes of section 4.2, can be extended to generalized regular cell complexes. Section 4.6 extends the ordering results and the implicit cell-tuple structure of Brisson (1990) from regular cell complexes to generalized regular cell complexes.

4.2 Simplicial, Regular and CW Complexes

The study of the topological invariants of interesting topological spaces (such as the manifolds) and continuous maps of these spaces using point-set topology has been described as intuitive in chapter 3. Unfortunately it is also largely intractable due to the arbitrary nature of the point-sets and the difficulty in enumerating all possible point-sets (Stillwell 1980). The solution to this intractability is to 'approximate' the continuous topological space by a finitely describable complex of simple or 'tame' building blocks known as cells. 'Wild' spaces such as the Alexander horned sphere are explicitly excluded. Loosely speaking, the problem of enumerating all possible point-sets is reduced to the enumeration of finitely many point sets each of which constitutes a 'cell'. Based on these key results, the cell complex is used by topologists as the basis for an algebraic and combinatorial toolbox which contains methods for calculating, classifying and comparing the invariants of topological spaces and the invariants of continuous maps of these spaces. As an example, the constructions represented by the canonical polygons for a 2-manifold and a 2-manifold with boundary in figures 3.4 and 3.6 are perfect for use by topologists as cell complexes because they capture all the topological information (as we shall see later) using a very small number of cells. In the design of topological models for spatial information systems, we use the cell complex for the same purposes as topologists, but in addition the cells and the cell complex represent the geometric structure/shape of the spatial objects and provide an abstraction mechanism for data structure design.

The aim of this section is to define a cell complex where each cell is homeomorphic to a Euclidean n-manifold with one or more orientable, subdivided (n-1)-manifold boundaries, according to the properties given in chapter 3. Each cell is itself a prototypical cell complex which is why we use the term subdivided (n-1)-manifold boundary. A spatial object is a cell complex formed from one or more of these cells. The result is what we will call a generalized regular cell complex. Before developing the generalized regular cell complex we evaluate and compare the cells and cell complexes that are traditionally used in topology.

The traditional definition of a closed n-dimensional cell or n-cell c is the homeomorphic image of the closed n-dimensional disk $\overline{D}^n$. An *n-simplex* Δ is the 'smallest' n-dimensional convex point-set, where 'smallest' indicates that the interior point-set of the simplex contains at least one point. Δ is obviously homeomorphic to c. In both these definitions the homeomorphism is actually a homeomorphism of boundaries. The homeomorphism between $\overline{D}^n$ and c is known as the prototypical or characteristic map Φ: $\overline{D}^n \rightarrow c$. Because Φ is a homeomorphism, c is said to be a regular n-cell.

In point-set terms, the boundary of a regular n-cell c, ∂c, is homeomorphic to the (n-1)-sphere S^{n-1}. The combinatorial boundary of a regular n-cell c is the set of (n-1)-cells incident (ie. adjacent) to it and the coboundary of a regular n-cell c is the set of (n+1)-cells incident to it. The (n-1)-skeleton of a regular n-cell c is defined as the collection of m-cells ($0 \leq m \leq n$-1) in its boundary.

A set **K** of simplexes is called a *simplicial complex* if the following three conditions are satisfied:

- **sc1:** The set **K** is finite.
- **sc2:** If K contains a simplex then it contains all faces of this simplex.
- **sc3:** The intersection of two simplexes of **K** is either empty or a common face.

The most important condition is that simplexes cannot intersect except along their boundaries. In general, an arbitrary simplicial complex forms a space known as a polyhedron (Jänich 1980 pg. 90). However if the simplicial subdivides an n-manifold or exists within R^n, then every point in the complex has an n-dimensional neighborhood.

From the perspective of spatial information systems, simplicial complexes capture geometric shape, provide simple function extension properties (see section 1.1 and Saalfeld 1987) and through their combinatorial and geometric simplicity, are easy to control and represent. However, in common with the theory of topology, they also suffer from three main problems:

1. Many simplexes are required to represent a spatial object (even the boundary of a spatial object). This results in a significant amount of fragmentation with costly reconstruction and a large amount of data. Also many important topological invariants (eg. the fundamental group) are often obscured or 'buried' by large numbers of simplexes.

2. The topological product of two simplexes is not a simplex and thus more work must be done to re-create a simplicial complex after taking the topological product (figure 4.1). This disadvantage is critical in temporal and visualization applications since higher dimensional objects are constructed using the topological product. Paoluzzi et. al. (1993) implicitly note this fact by defining a simplicial boundary representation (instead of the full simplicial subdivision) for use with extrusion operations (a variant of the topological product).

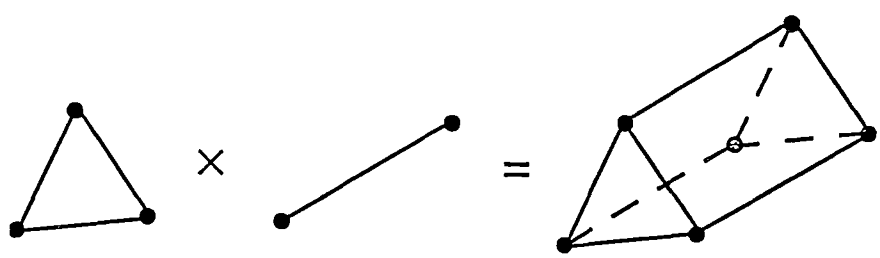


Figure 4.1 - the topological product of a 2-simplex and a 1-simplex is not a 3-simplex

3. Attaching or gluing simplexes along their boundaries does not result in a simplicial complex unless restrictions are made on the attaching map and its domain (figure 4.2 and also figure 1.13 in chapter 1).

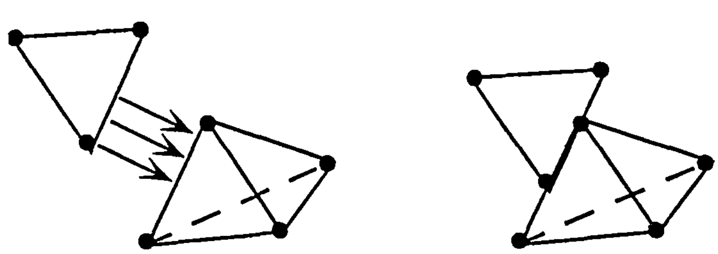


Figure 4.2 - attaching a 2-simplex by a subset of its boundary does not result in a simplicial complex - see also Massey (1967) pg. 16

These three points may be summarised by noting that simplicial complexes generate large data volumes and that the result of any operation on a simplicial complex (such as the topological product or attaching simplexes to one another) is not necessarily another simplicial complex. Additional subdivision is often required and thus tedious restrictions must be enforced.

A regular cell complex may be defined in exactly the same way as a simplicial complex by substituting the term regular cell for simplex in conditions **sc1-sc3** above. Regular cell complexes are obviously less restrictive than simplicial complexes but their definition is one example of a much more general class of cell complexes known as closure finite, weak topology or *CW complexes.* CW complexes were developed by topologists as a simplification and generalization of simplicial complexes. This simplification is best described by considering the construction of the CW complex as in Jänich (1980) and shown in figure 4.3.

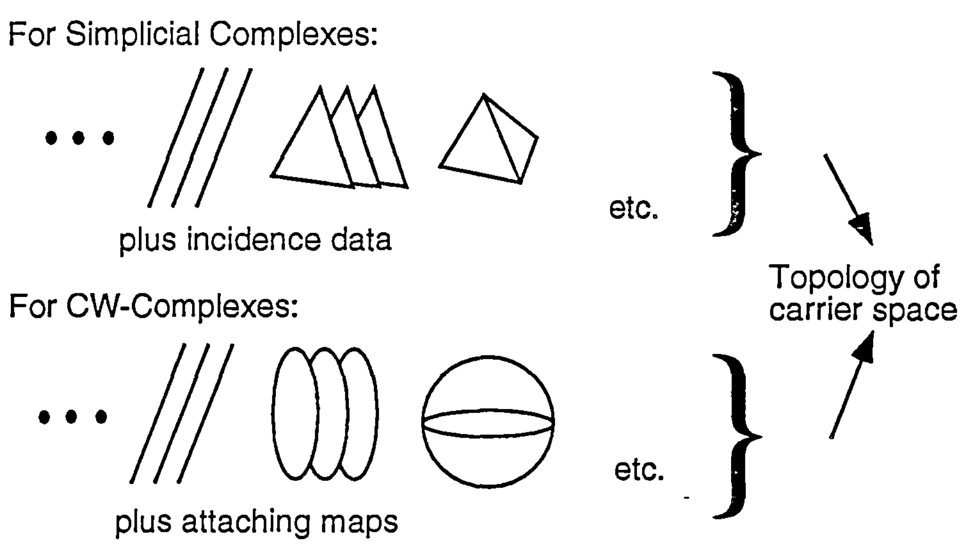


Figure 4.3 - constructing a space - the simplicial and CW approaches

Simplicial complexes are constructed from simplexes and incidences between them whilst CW-complexes are constructed from n-dimensional disks and attaching maps. As mentioned above, a simplicial complex is often referred to as a polyhedron. Because CW-complexes use regular cells and a much more flexible method of attaching these cells to one another (as we shall see shortly), they are sometimes referred to as 'second generation polyhedra' (Jänich 1980 pg. 93).

Following Jänich (1980) with some modifications, a normal CW-complex is a pair (**X**, ξ) (where ξ is a cellular subdivision of a topological space **X**) which satisfies the following properties:

cw1: ("Characteristic Maps") For each n-cell $c \in \xi$ there is a continuous map $f: D^n \rightarrow \mathbf{X}$ taking the interior of D^n homeomorphically onto the cell c and S^{n-1} into the union of cells of dimension at most $n-1$.

cw2: ("Closure Finiteness") The closure of each cell $c \in \xi$ intersects only a finite number of other cells.

cw3: ("Weak Topology") $\mathbf{A} \subset \mathbf{X}$ is closed if and only if every $\mathbf{A} \cap$ closure of c is.

Conditions **cw2** and **cw3** were intended to cover cases under which infinitely many cells may be included in a complex (Jänich 1980). Both conditions are trivially satisfied by the finite cell complexes we wish to deal with in this research.

The mathematician J.H.C Whitehead showed that all the topological properties that could be investigated using simplicial complexes could also be investigated using normal CW complexes. However the main advantage to topologists is that the normal CW complex is 'a minimal representation' of the topological properties of a space. The excess of unimportant geometric structure captured by simplicial complexes is almost entirely removed and the notion of a regular cell complex is made rigorous. These advantages result from the generality of the cell attaching maps specified by condition **cw1**. In particular, note that continuity is the only restriction on the transformation of the boundary of the characteristic map. This means that the boundary of an n-cell in a CW complex need not be homeomorphic to S^{n-1}, as it must be in the case of the simplex and the regular cell.

As an example, figure 4.4 (see Jänich (1980) pg. 102) shows a CW complex which subdivides a 2-sphere formed by attaching a 0-cell to the empty space, then attaching a 2-cell to this 0-cell, by continuously deforming S^1 (the boundary of the 2-cell) such that it forms a 0-cell.

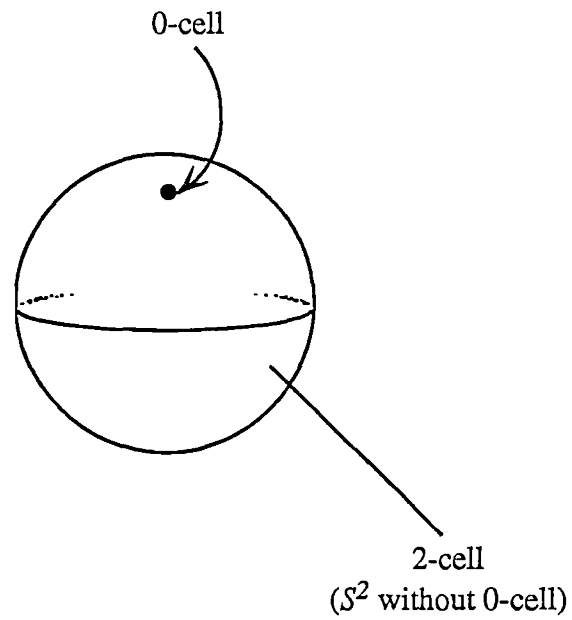


Figure 4.4 - the subdivision of a 2-sphere by a CW complex - the boundary of the 2-cell is a 0-cell (see the continuous condition of **cw1** above).

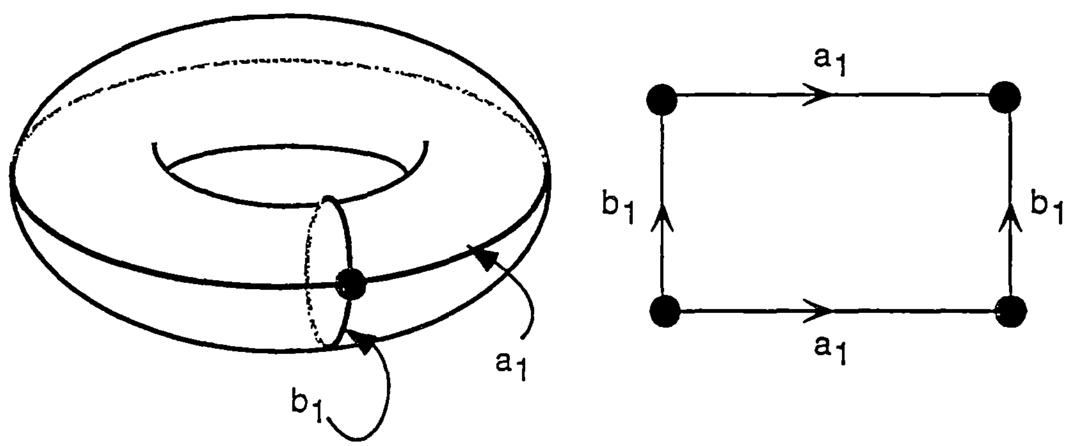


Figure 4.5 - the subdivision of a torus by a CW complex - the boundary of the 2-cell consists of two 1-cells each of which are loops or rings

Naturally the topological properties of each of the 2-manifolds shown in figures 4.4 and 4.5 can still be obtained from the CW complexes. In particular note that in figure 4.4, the Euler characteristic of the sphere is still valid (ie. the alternate sum of the number of cells in each dimension is equal to 2) even though there are no 1-cells! To further illustrate the effectiveness of the normal CW complex in representing the critical topological properties, it will be seen (see sections 4.4 and section 6.3.2) that in both figure 4.4 and 4.5 above, the 0-cells and 1-cells of the CW complexes form the generators of the fundamental group of the 2-manifolds they represent.

The cell attaching process gives rise to an alternate definition of a CW complex as an adjunction space (Brown 1988). An adjunction space is created by adjoining or attaching 0-cells, then 1-cells, then 2-cells and so on (figure 4.6).

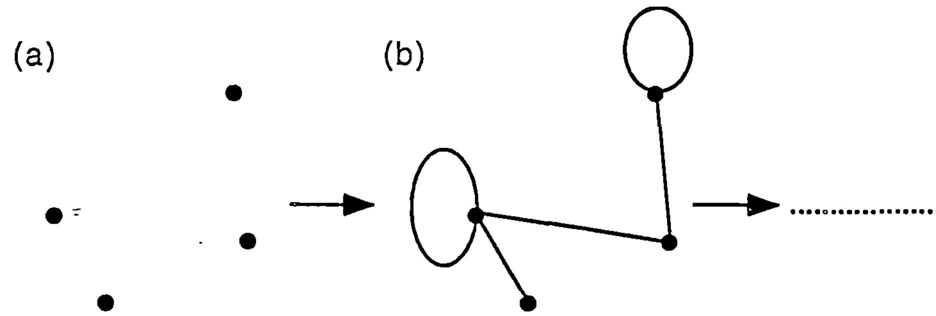


Figure 4.6 - the normal CW complex as an adjunction space (an example of the generic process shown in figure 4.3) (a) attaching 0-cells (b) attaching 1-cells and so on....

The notion of cell attaching will be mentioned again in the next chapter and in detail during the discussion of the topological construction operators in chapter 6.

Some researchers have defined topological models for spatial information systems which have aspects similar to some of those defined for normal CW complexes. For example, Weiler (1986) and Mäntylä (1988) permit the boundary of a face to be a 0-cell and Corbett (1985) takes the very general step of permitting the characteristic map of a cell to be continuous with respect to the boundary. However the only direct reference to CW complexes appears to have been in computational geometry by Brisson (1990). Brisson uses a variation of the normal CW complex known as a regular CW complex. A regular CW complex is simply a normal CW complex for which the characteristic map f in condition **cw1** is restricted to be a homeomorphism with respect to both the interior *and* the boundary of the prototypical cell - see Lundell and Weingram (1969).

In their most general form normal CW complexes are not suitable by themselves for use in topological spatial models (as is also noted by Rossignac and O'Connor 1991 and Pigot 1992). The main reason is (as noted above) that they were designed by topologists to simplify the calculation of the topological properties of (very abstract) spaces whilst designers of spatial information systems are interested in both the topological properties and the geometric structure (or topography) of a spatial object. However if a different cell complex is used to capture the geometric structure, the effectiveness of the normal CW complex in representing topological properties may be useful. This idea will be taken up in chapter 5.

In the next section, the basis of the regular CW approach (ie. conditions **cw1-3** with the restriction on f to be a homeomorphism in **cw1**) will be used in defining the generalized regular cell complex to optimize the representation of the geometric structure (or topography) whilst retaining the ability to calculate topological properties.

4.3 Generalized Regular Cell Complexes

The class of spaces suitable to form boundary cycles of cells must be sufficiently general to minimize fragmentation of spatial objects but sufficiently well-known such that their topological properties can be used to distinguish them. We initially focus on manifolds and manifolds with boundary(s) from topology as described in section 3.3 because by definition they are very general (the term manifold literally means 'having varying form') and for low-dimensional applications (≤ 3) they are well known (ie. a classification system exists).

A *generalized regular n*-cell is the homeomorph of a Euclidean n-manifold with subdivided (n-1)-manifold boundaries ($0 \leq n \leq 3$). The properties of both the Euclidean n-manifold and of the (n-1)-manifold boundaries have been defined in section 3.3. A spatial object is a set of one or more generalized regular cells with the additional requirement that the cells intersect along their boundaries. The result is the *generalized regular cell complex. Generalized* because it extends and incorporates the simplicial and regular CW complexes used in topology and *regular* because the subdivided manifold boundary cycles form the (n-1)-skeleton.

Following the approach given in the last section for CW-complexes, we redefine the prototypical n-cell in condition **cw1**, as a Euclidean n-manifold with orientable (n-1)-manifold boundary cycles. A generalized regular cell complex shares properties **cw2** and **cw3** with a CW-complex, only property **cw1** needs to be changed.

grc1: ("Characteristic Maps") For each n-cell (now a Euclidean n-manifold with connected, orientable, two-sided, subdivided manifold boundary cycles) $c \in \xi$ there is a homeomorphism f into $\mathbf{X}$ taking the interior Euclidean n-manifold of the prototypical cell onto the interior of c *and* each orientable (n-1)-manifold boundary cycle of the prototypical cell into the union of cells of dimension n-1.

The homeomorphic nature of the characteristic map in **grc1** leads directly to the combinatorial definition of a generalized regular n-cell. A generalized regular n-cell

c is an n-manifold with i subdivided (n-1)-manifold boundary cycles ($i \geq 1$). Figure 4.7 shows some examples of generalized regular cells.

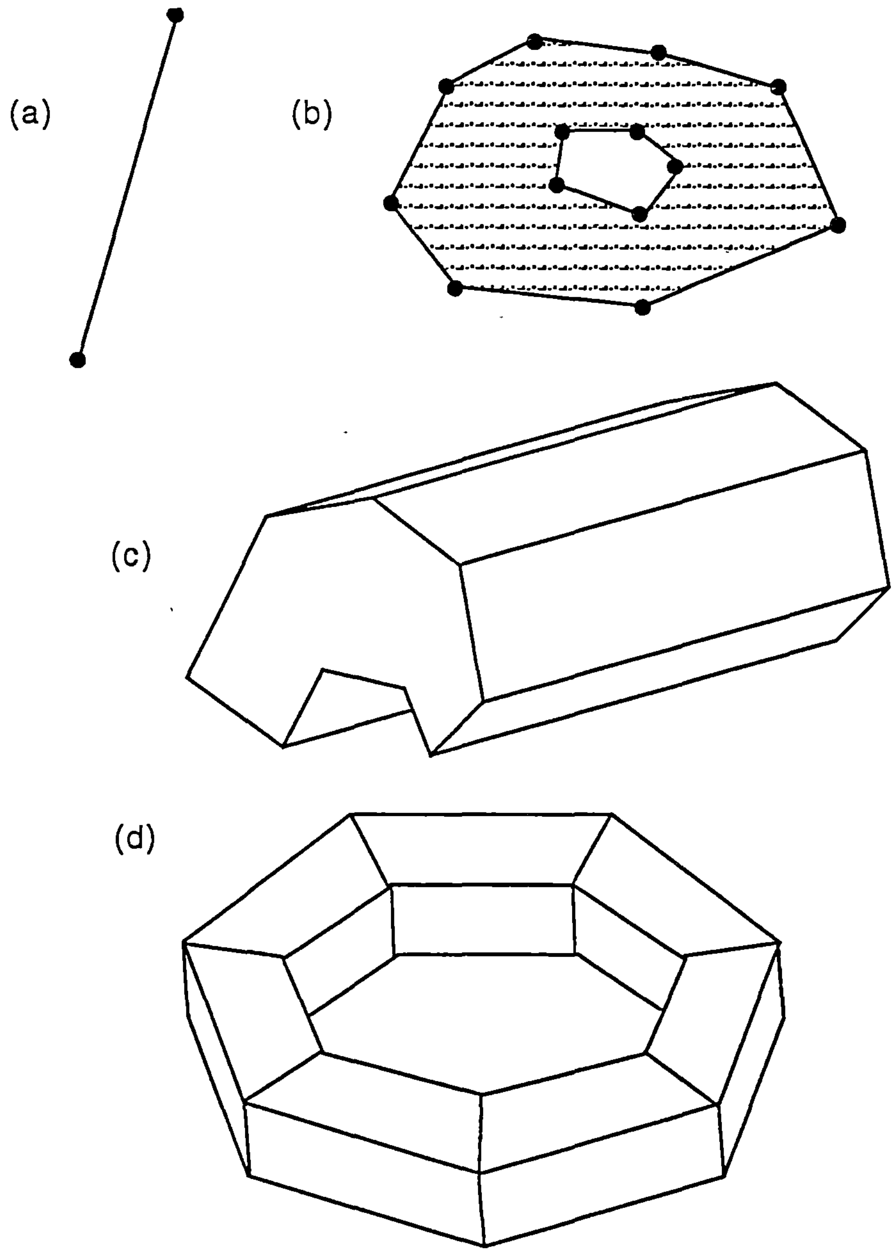


Figure 4.7 - examples of generalized regular cells (a) a GR 1-cell (b) a GR 2-cell with two subdivided 1-manifold boundaries (c) a GR 3-cell with simply connected subdivided 2-manifold boundary (d) a GR 3-cell with non-simply connected 2-manifold boundary

The cells of the boundary cycles of an n-cell form its (n-1)-skeleton which is itself a generalized regular cell complex (hence the use of the term subdivided). There are three important points to note:

1. the insistence on the characteristic map being both a homeomorphism of the interior *and* the boundary of the prototypical cell specifically excludes identification of points in the boundary cycles of generalized regular cells; ie. situations such as those given for normal CW complexes in figures 4.4 and 4.5 cannot occur.

2. the definition of the interior of the prototypical n-cell as a Euclidean n-manifold and the homeomorphic characteristic map ensure that each cell has trivial topological properties (ie. avoiding situations such as a handle in a 2-cell shown in figure 4.8 - see also Weiler (1988) for a similar restriction but defined only for faces or 2-cells). Thus, the topological properties of the space carrying the cell complex are recoverable from the relationships between the cells.

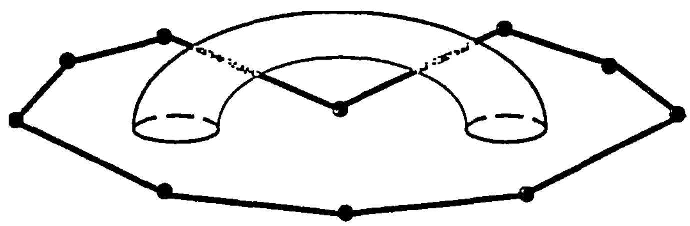


Figure 4.8 - a 2-cell which is homeomorphic to a 2-manifold with boundary but not homeomorphic to a Euclidean 2-manifold with boundary (the (n-1)-skeleton of the 2-cell is shown with thick black lines and dots). Another example of an illegal cell is the 'basket shaped thingy' in figure 1.9 on pg. 14 of Scott Carter (1993)

3. the definition of the boundary cycles as orientable (n-1)-manifolds minimizes fragmentation of the spatial object without sacrificing a well defined nature. This is particularly true for 3-cells which because they now have 2-manifold boundary cycles are much 'closer' to many 3-dimensional spatial objects - see figure 4.7 (c) and (d) above.

Having defined generalized regular cells and the generalized regular cell complex (ie. a spatial object) we now consider how the methods used for calculating topological properties and representing a cell complex in a computer, can be extended from simplicial and CW complexes to generalized regular cell complexes. The major methodologies, their advantages and the topological properties they deliver, are given in the next three sections.

4.4 Fundamental group (1-dimensional homotopy group)

In chapter 3 we gave a brief account of the notion of homotopies between paths with the same two end points. However the most useful paths to investigate are closed paths or loops because the behaviour of homotopies of simple loops (ie. non-fractal and tame) is sufficient to indicate the connectivity of a general topological space and thus classify low dimensional manifolds (≤ 3) - Stillwell (1980). The equivalence classes of 1-dimensional loops in a space, defines the first homotopy group or *fundamental group* - a well known concept in topology originally derived by H. Poincaré. Loosely speaking, the fundamental group of any space is an algebraic record of the difficulty in 'shrinking' a set of embedded loops to a point without leaving the space (Francis 1987). In this section we give a point-set and combinatorial overview of the fundamental group and how it may be extended to generalized regular cell complexes, because in chapter 6 the methodologies described here will be used to define topological construction operators.

Formally, the fundamental group is defined by denoting ($\mathbf{X}$,x_0) as a pair consisting of a space $\mathbf{X}$ and a point x_0 in $\mathbf{X}$. Let Ω denote the set of all maps of S^1 into $\mathbf{X}$, such that the first point of S^1 is mapped to the point x_0 of $\mathbf{X}$. Essentially, we assume that Ω denotes the set of all loops in $\mathbf{X}$ with the given base point x_0 ie. $\Omega = \{ f: \mathrm{I} \rightarrow \mathbf{X} \mid f(0) = x_0 = f(1) \}$. Two loops f and g are said to be equivalent ($f \sim g$) if there exists a homotopy h_t: $\mathbf{I} \rightarrow \mathbf{X}$ ($0 \leq t \leq 1$) such that $h_0 = f$ and $h_1 = g$ and $h_t(0) = x_0 = h_t(1)$ for every t. This equivalence relation is reflexive, symmetric and transitive, hence the loops in Ω are divided into disjoint equivalence classes. A multiplication of representative loops from disjoint equivalence classes under which Ω forms a group, can be defined as follows.

For any two loops $f, g \in \Omega$ with $f(1) = g(0)$ their product $f.g$ is defined by:

$(f.g)(t) = f(2t)$ for $0 \leq t \leq 1/2$ *or* $g(2t-1)$ for $1/2 \leq t \leq 1$

Intuitively, $f.g$ is the loop traced by moving along the loops f and g in succession. Three conditions ensure that Ω forms a group:

1. That the multiplication is associative (ie. $<f.g>.h = f.<g.h>$)

2. That there exists an identity loop or constant loop k at x_0 defined by $k(t) = x_0$ where $0 \leq t \leq 1$ and thus $k.g = g$

3. That the inverse of any loop f (known as f^{-1}) is defined as $f^{-1}(t) = f(1 - t)$ where $0 \le t \le 1$ and thus $f.f^{-1}$ = the identity loop.

The disjoint equivalence classes of Ω form a group known as the first homotopy group or just the fundamental group of **X** at x_0, which is denoted by $\pi_1(\mathbf{X},x_0)$. It is important to note that the fundamental group is independent of the choice of the base point x_0; ie. for x_0 and $y_0 \in \mathbf{X}$, $\pi_1(\mathbf{X},x_0)$ has the same form as (or is isomorphic to) $\pi_1(\mathbf{X},y_0)$.

From each equivalence class of loops it is possible to choose a representative element; ie. a loop which represents or *generates* all other loops in that class. Loosely speaking, these representative elements are known as generators of the fundamental group. The difficulty now is to determine which generators are not loops that can be collapsed via a homotopy to the base point of the fundamental group without leaving the space. For example, consider the annulus in figure 4.9. There is an equivalence class of loops generated by the loop f which cannot be collapsed to a point. There is also an equivalence class of loops generated by element g which can be collapsed to a point. As another example consider the 2-sphere. Any loop in the boundary of a 2-sphere can be collapsed to its base point without leaving the 2-sphere. Therefore the fundamental group of the 2-sphere is trivial since all generators reduce to a point ie. the identity loop. Loops which can be collapsed to a point are known as null-homotopic.

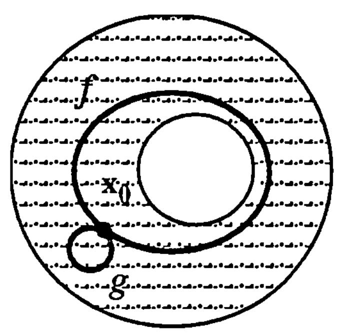


Figure 4.9 - equivalence classes of loops in the annulus

Following the traditional finitization methodology used in topology to solve a problem expressed in terms of point-set topology, a finite presentation of the fundamental group of any space **X** can be derived from any finite simplicial or CW complex it carries, because all possible loops can be deformed onto the 1-skeleton of the complex. Briefly, a set of loops is calculated by constructing a spanning tree of the vertices (or 0-cells) and then forming an approach path p_i from the base vertex V to each vertex v_i in the1-skeleton. Each edge v_iv_j is associated with the loop $p_i(v_iv_j)p_j^{-1}$. If the edge is not in the spanning tree then the loop is a generator of the fundamental group of a graph formed by the 1-cell complex. This method can then be extended to determine the generators of a 2-cell complex by 'attaching' 2-cells to the appropriate 1-cycles in the 1-skeleton and removing those fundamental loops which form the boundaries of one or more 2-cells. The entire methodology is known as the Tietze method (after H. Tietze - see Stillwell 1980).

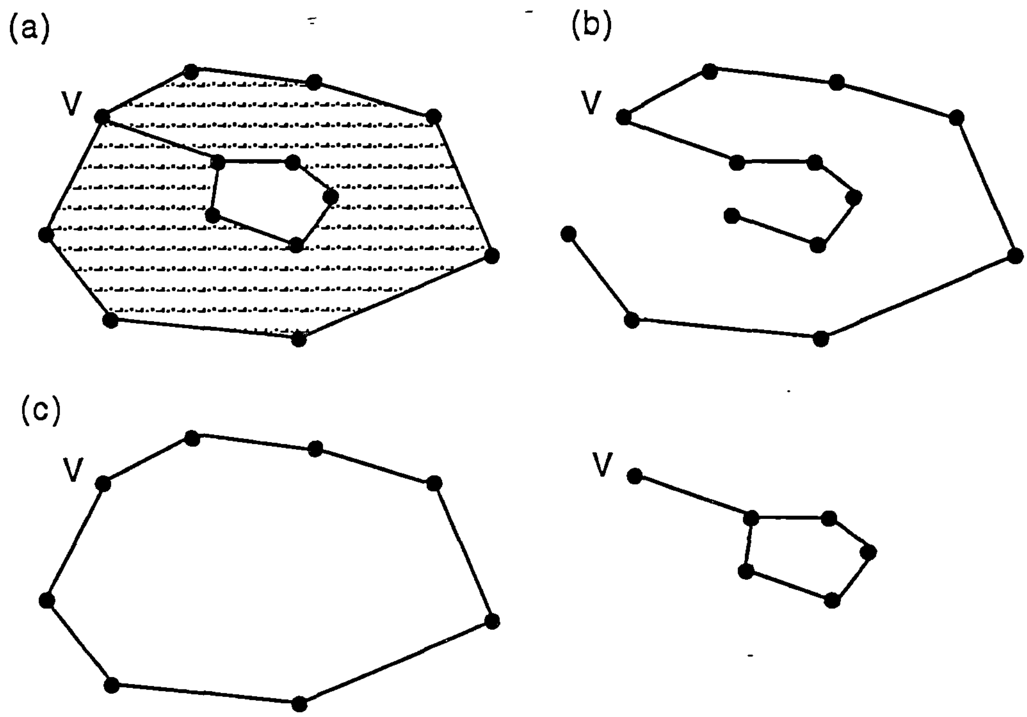


Figure 4.10 - the Tietze method applied to a 2-cell from a normal CW-complex (a) the 2-cell from the normal CW complex (b) a spanning tree of the 0-cells or vertices (c) the first fundamental 1-cycle (d) the second fundamental 1-cycle

As an example, figure 4.10(a) shows a 2-cell from a normal CW-complex. Figure 4.10(b) shows a spanning tree of the vertices whilst figure 4.10(c) shows the two 1-cycles which form the fundamental 1-cycles. The second 1-cycle in figure 4.10(c) has a spur (ie. a 1-cell traversed twice, once in each direction) effectively connecting the 1-cycle to the vertex V. This spur can be removed, which in homotopy terms means that the loops share the vertex V and thus the fundamental group of the 1-skeleton of this 2-cell consists of a 'bouquet' of two loops. However when the 2-cell itself is added, either loop may be deformed across the 2-cell; ie. both loops become members of the same equivalence class of loops (see figure 4.9). Thus the fundamental group (and the strong deformation retract) of this 2-cell from a normal CW complex, consists of a single loop.

The key points to note about the application of the Tietze method for calculation of the fundamental group are:

1. its dependence upon the 2-skeleton of the cell complex

2. the path-connectivity of the 2-skeleton which is necessary for the spanning tree process to determine the basis loops (figure 4.11).

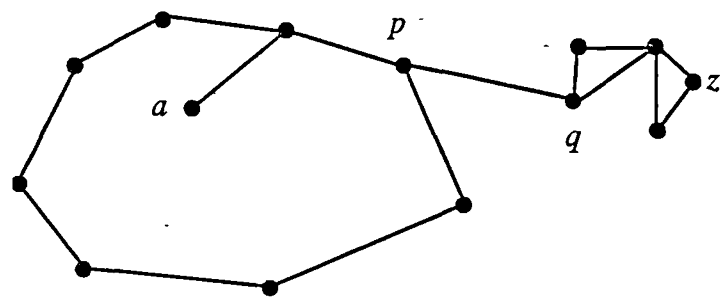
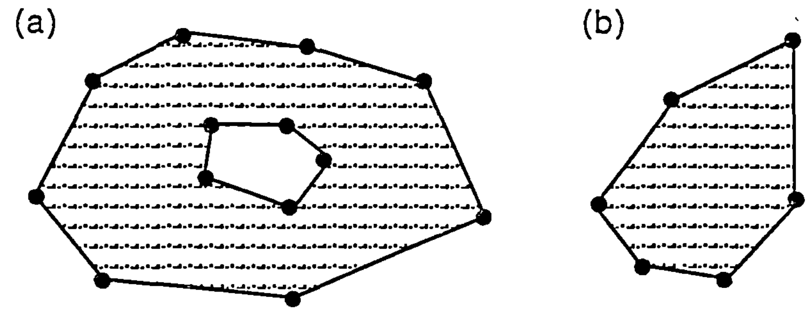


Figure 4.11 - path connectivity in a 1-cell complex - the complex is path connected but by removing the 1-cell between the 0-cells p and q (for example), then no path exists between a and z and the 1-cell complex is no longer path-connected.

Since the calculation of the fundamental group is dependent on the 2-skeleton of the complex carried by the space, then in order to extend the Tietze method to generalized regular cell complexes we need to consider the differences between generalized regular k-cells and k-cells from a regular CW-complex ($1 \leq k \leq 2$), since cells in a regular CW complex are most similar to those of the generalized regular cell complex. By definition **grc1**, generalized regular 1-cells and the 1-cells of the regular CW-complex agree since a Euclidean 1-manifold with boundary is always homeomorphic to a 1-sphere with 0-sphere boundary (two 0-cells) ie. a 1-cell in a regular CW complex. However a generalized regular 2-cell is the homeomorph of a Euclidean 2-manifold with one or more 1-manifold boundary cycles, whilst a 2-cell in a regular CW complex is a 2-sphere with a single 1-sphere as its boundary cycle. In other words a regular 2-cell corresponds to the special case of a generalized regular 2-cell with a single 1-manifold boundary cycle (figure 4.12(b)).



- Figure 4.12 - difference between a generalized regular and regular cell (a) a generalized regular 2-cell (b) a 2-cell in a regular CW complex

However if we insist on rigorous path connectivity, then one 'cut-line' (Massey 1967, Scott Carter 1993 and section 3.3.1) would be required to connect the two boundary 1-cycles of the generalized regular 2-cell in figure 4.12(a) - the 2-cell from a normal CW complex shown in figure 4.10(a), would then result. There are two difficulties involved in explicitly representing such cut-lines as 1-cells:

1. They do not conform with the combinatorial definition of the generalized regular cell because with the addition of the cut-line, the (n-1)-skeleton would not be composed of the 1-cells and 0-cells of the boundary 1-cycles; and,

2. They are a clumsy complication when representing and manipulating the cell complex because they are an artificial construct required to maintain path-connectivity. They are not part of the 'real' data - see also Corbett (1975), White (1983) and most implementations involving 2-cell complexes described in chapter 2.

Fortunately, it is well known that for a Euclidean 2-manifold with n boundary 1-cycles (ie. a generalized regular 2-cell), the number of cut-lines required is always n-1 - see section 3.3.1 and theorem 3, pg. 10 of Scott Carter (1993). Consequently the number of cut-lines required to ensure path-connectivity in the 2-skeleton of any generalized regular cell complex can always be easily determined and thus it is not necessary to explicitly represent the cut-lines as 1-cells. Assuming the presence of these 'virtual' cut-lines, the Tietze method immediately extends to the 2-skeleton of any generalized regular cell complex, without losing the flexibility obtained by the definition of the generalized regular cell.

As an example of these ideas consider the torus shown in figure 4.13.

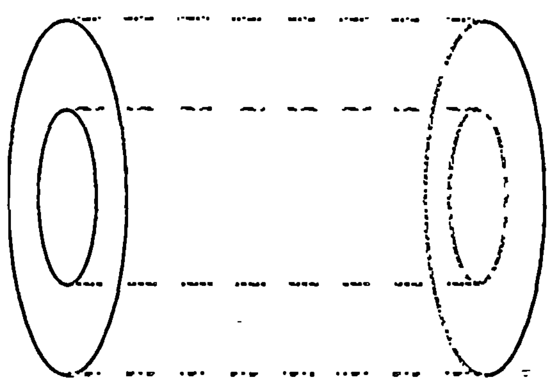


Figure 4.13 - 2-skeleton of a torus constructed from four generalized regular 2-cells each of which has two boundary 1-cycles (the 0-cells and 1-cells in the boundary 1-cycles are not shown)

This torus is constructed from four generalized regular 2-cells, each of which has two boundary 1-cycles (ie. four annuli); the 1-cells and 0-cells in these boundary cycles are not shown in any of the figures. There are two annuli at each end of the torus (which are easily recognizable in figure 4.14(a)) and two annuli for the outer and inner cylindrical walls (shown in figure 4.14(b) looking down on the cylinders to emphasize that they are also annuli).

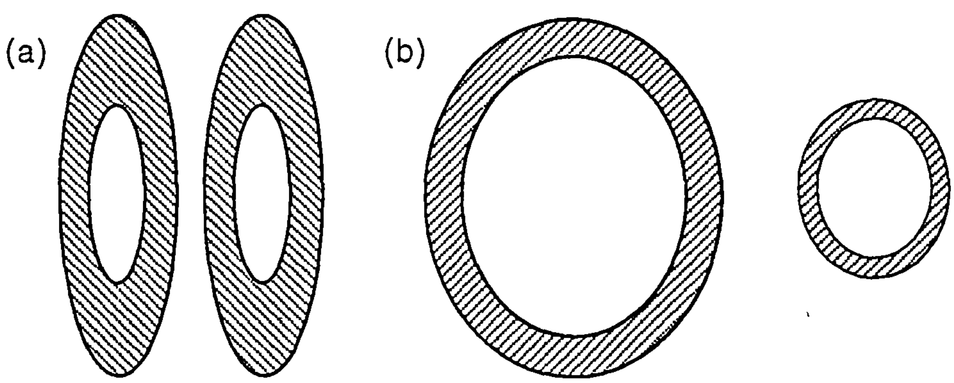


Figure 4.14 - the four generalized regular 2-cells of the torus in figure 4.13 (each of which is an annulus) (a) the annuli at either 'end' of the torus (b) the outer and inner cylinders viewed from above are also annuli (Note that the dashed lines in 4.13 are *not* 1-cells and the 0-cells and 1-cells in the boundary 1-cycles of the annuli are not shown).

Such cases arise quite frequently in CAD applications (see Mäntylä 1988 for example) where curved surfaces are permitted. Using the simple result given above, four virtual cut-edges (figure 4.15) are required, since each annuli has one boundary 1-cycle. When the Tietze method is applied to the resulting path-connected complex, two 1-cycles result. In this case one of the 1-cycles is equivalent to each of the boundary 1-cycles of the generalized regular 2-cells (since they all share a common 1-cycle) and the other is a 1-cycle formed from the virtual cut-lines. These two cycles are the generators of the fundamental group.

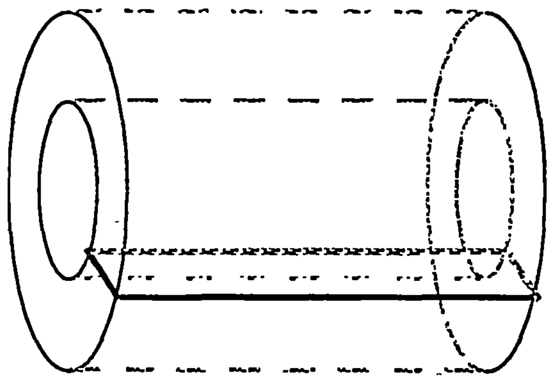


Figure 4.15 - the cycle of virtual cut-lines in the torus of figure 4.13.

This example also helps to illustrate how the generalized regular cell complex represents spatial objects with a minimum of fragmentation, without losing the ability to calculate topological properties. Together with the fact that the definition of the generalized regular cell encapsulates the regular and simplicial cells, this is the main advantage of the generalized regular cell over the regular and simplicial cells most often used in spatial information systems.

4.5 Homology groups

The generators of the n-dimensional homology groups are the n-cycles which do not 'bound' any part of the cell complex. These cycles are a finitization of those cycles which do not bound any part of the carrier space; ie. they bound holes in the space. Clearly homology theory is another tool (besides homotopy) for analyzing the connectivity or the presence of 'holes' in topological spaces. However, this research focuses on homotopy theory in preference to homology, for two primary reasons:

1. Homotopy theory is much more easily defined and conceptually simpler than homology theory. Homotopy based strategies that are used to calculate the fundamental group (see section 4.4), will be shown in chapter 6 to be very useful for defining and controlling topological construction operators.

2. The fundamental group is far more discriminating than the first homology group, and for topological spaces of dimension ≤ 3, contains all the information available from homology (Stillwell 1980 pg. 171).

Given that the first homology group is essentially an abelianization of the fundamental group, future research could be directed toward extending the work of Saalfeld (1989) to analyze boolean set operations on generalized regular cell complexes using the Mayer-Vietoris homology sequence.

4.6 Ordering and Representation of Cell Neighborhoods

The last two sections have covered the extension of the theory required to calculate connectivity from a space which is subdivided by a simplicial or CW complex to a space which is subdivided by a generalized regular cell complex. This section will focus on the ordering of the cell neighborhoods in the generalized regular cell complex and on suitable data structures which represent both the cells of the generalized regular cell complex and these ordering relationships.

Recall from chapter 1 that ordering results were given for the neighborhoods of cells in a k-dimensional regular CW complex forming a subdivided k-manifold. These ordering results can be divided into two classes:

1. Circular Orderings A circular ordering is defined when the $(j+1)$-cells and $(j+2)$-cells returned by the repeated application of the coboundary operation to a j-cell (ie. 'the coboundary of the coboundary of a j-cell' - White 1978) may be placed in circular order 'about' the j-cell ($k-2 \leq j \leq k$).

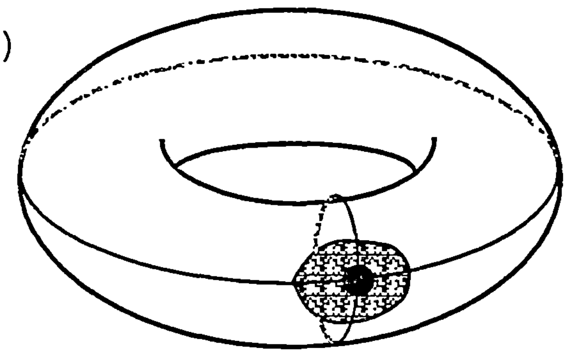
   Examples include the circular ordering of the 'umbrella' (Lefschetz 1975) of 1-cells and 2-cells about a 0-cell in a subdivided 2-manifold and the circular ordering of 2-cells and 3-cells about a 1-cell in a subdivided 3-manifold.

2. Two-Sided Orderings A two sided ordering is defined by the application of the coboundary relation to find the two $(j+1)$-cells cobounding a j-cell. Brisson (1990) describes two-sided orderings as special cases of the circular orderings by assuming the presence of non-existent $(j+2)$-cells (see section 1.4) so that a circular ordering of $(j+1)$-cells and $(j+2)$-cells about a j-cell can be defined. To avoid confusion in the discussion below we separate two-sided orderings from circular orderings.

Examples include the left and right cobounding 2-cells of a 1-cell in a subdivided 2-manifold or the inside/outside cobounding 3-cells of a 2-cell in a subdivided 3-manifold.

These orderings are based on the fact that if a k-cell complex subdivides a k-manifold, then the cells that intersect a 'small' k-dimensional neighborhood of any j-cell (k-2 $\leq j \leq k$-1) may be ordered 'about' the j-cell. These orderings are consistent because every point of the underlying k-manifold is guaranteed to have a k-dimensional 'disk' neighbourhood (see figure 4.16).

(a)



(b)

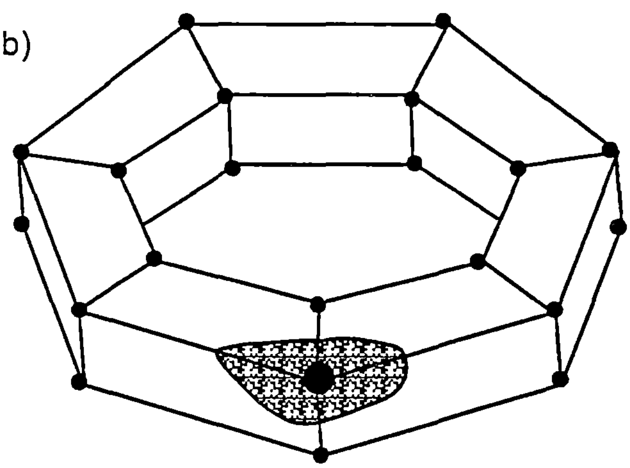


Figure 4.16 - subdivision of a 2-manifold (the torus) by a 2-dimensional regular CW complex (a) the torus (b) a subdivision of the torus (notice the subdivision of the shaded neighborhood)

The existence of both the circular ordering and the two-sided ordering results in subdivided manifolds has been proven in a very concise manner by Brisson (1990).

However, as mention earlier in section 4.3, a generalized regular cell complex (ie. a spatial object) need not be a subdivided manifold or even a subdivided manifold with boundary. How can the circular ordering results for subdivided manifolds be consistently applied to all cells in such complexes?

The solution to this problem is to 'embed' the spatial object in a manifold (of the same or higher dimension) and represent the manifold as a 'world' cell - see White (1978) and the similar solution for the restricted domain of subdivided manifolds with boundary in Brisson (1990). For 3-dimensional applications of this research the Euclidean 3-manifold will be used, but regardless of the type of 3-manifold chosen, the cells of the generalized regular 3-cell complex will assume the circular ordering results that result from the 3-dimensional disk neighborhoods of the points in a 3-manifold.

In most topological theory (including Brisson 1990), the term 'subdivision' implicitly assumes a 'complete' partition of the n-manifold by the n-cells of the complex similar to that shown in figure 4.16 above; ie. the space is completely 'replaced' by a 'finitization' of its point-sets. In a spatial information system, the spatial object need not form a 'complete' partition of the Euclidean manifold it is embedded in. For example, in mining applications we may wish to represent a network of pipes carrying compressed air and water services in mine tunnels. The resulting 1-cell complex(es) do not form a 'complete' partition of the Euclidean 3-manifold in which they are embedded. Figure 4.17(a) & 4.17(b) show some more abstract examples.

As a result, special cases of the circular and two-sided ordering results occur. For example, in figure 4.17(a) each 1-cell has a circular ordering which consists of a single 3-cell (the 'world' 3-cell) instead of a set of 2-cells and 3-cells. In figure 4.17(b), each 1-cell has a two-sided ordering which consists of the same 2-cell (the 'world' 2-cell) on both sides, instead of two distinct 2-cells as would occur in a 'complete' partition.

With these issues in mind we now describe how the cell-tuple of Brisson (1990) can be extended from a 'complete' partition of a manifold (ie. a subdivided manifold) to include the special cases that result from the fact that a spatial object is simply embedded in a Euclidean manifold. In particular, we focus on extending the cell-tuple to the representation of a k-dimensional spatial object as a k-dimensional generalized regular cell complex in an n-dimensional Euclidean manifold ($0 \leq k \leq n \leq 3$).

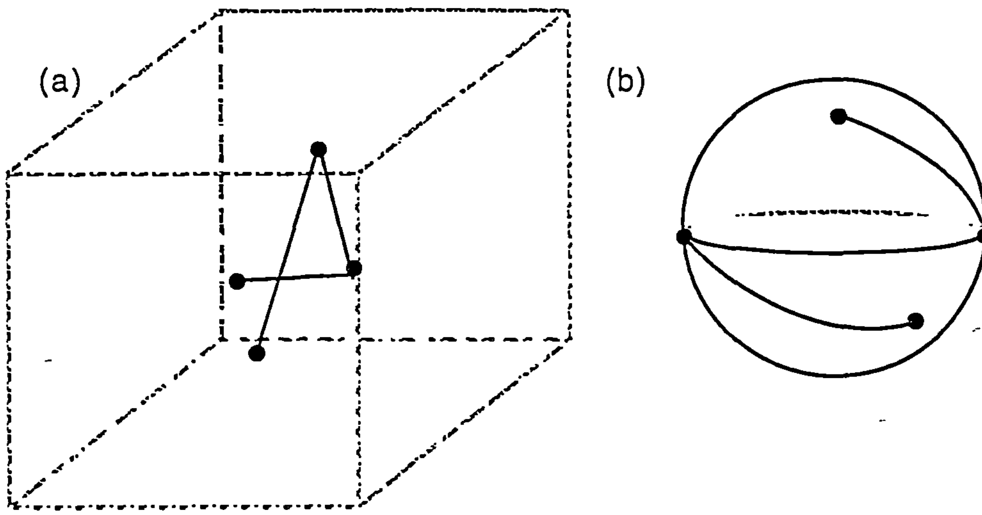


Figure 4.17 - two spatial objects which do not form 'complete' partitions of the manifold that they are embedded in (a) a 1-cell complex contained within but not forming a 'complete' partition of a Euclidean 3-manifold (b) a 1-cell complex contained within but not forming a 'complete' partition of a 2-manifold (in this case a sphere) - the 1-dimensional object in both cases is a 1-cell complex consisting of four 0-cells and three 1-cells.

Recall from section 2.5.1, that given an n-manifold M^n and K a regular cell subdivision of M^n, a *cell-tuple* t is a set of cells $\{c_{\alpha 0},\ldots\ldots,c_{\alpha n}\}$ where any k-cell $c_{\alpha k}$ is incident to (k+*1*)-cell $c_{\alpha k+1}$ (ie. is a part of a boundary of $c_{\alpha k+1}$), $0 \leq k \leq n$ and there are $n+1$ cells in each tuple. The cells within the tuple are ordered according to dimension and the set of tuples describing the subdivided manifold is represented by the symbol T_M. The k-dimensional component of any tuple t is referred to as t_k where $t_k = c_{\alpha k}$.

The basic operator on tuples which also defines the circular ordering results and thus traversal of T_M, is the *switch* operator. For $t \in T_M$, the *switch* operator essentially swaps a single i-cell in t to obtain another 'unique' cell-tuple t', where 'unique' implies both:

1. $t_i' \neq t_i$ and $t_j' = t_j$ for all $i \neq j$ and $0 \leq j \leq n$. ie. $switch_i(t) \neq t$, and,

2. for each dimension i, $switch_i(t)$ returns only one tuple t'.

The 'complete' partition (ie. subdivided manifold) condition and the consequent circular ordering relationships are encapsulated in five conditions on the result of the *switch* operator given in Brisson (1990) pg. 50:

ct1: $\text{switch}_i(t) \neq t$

ct2: $\text{switch}_{ij}(t) \neq t$

ct3: $\text{switch}_{i2}(t) = t$

ct4: if $j = i \pm 1$, $\exists$ m ≥ 2 such that $switch_{(ij)m}(t) = t$ (figure 4.18)

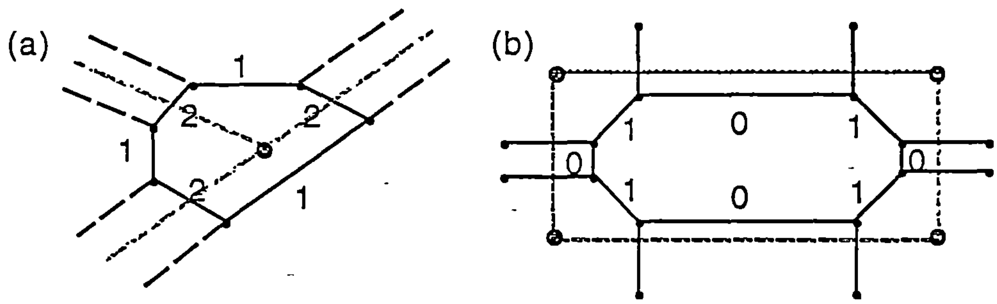


Figure 4.18 - illustration of **ct4** - (a) $m = 3$, $i = 1$ and $j = 2$; the circular ordering of 1-cells and 2-cells about a 0-cell in the 2-manifold ie. cobounding 1-cells and their cobounding 2-cells (b) Circular ordering of 1-cells and 0-cells about an abstract c_{-1} cell - forms the boundary cycle of a 2-cell ie. $m = 3$, $i = 0$, $j = 1$.

ct5: if $j \neq i \pm 1$, then $switch_{(ij)}2 = t$ (figure 4.19)

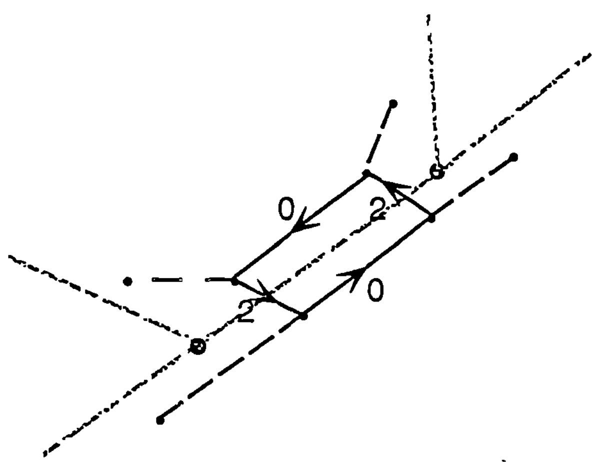


Figure 4.19 - illustration of **ct5** - in this case $i = 0$ and $j = 2$ showing that each 1-cell in a subdivided 2-manifold has two cobounding 2-cells

The undirected path-connected (see section 4.4) graph whose nodes are the cell-tuples and whose edges are represented by *switch* operations is known as $\mathbf{G_{T_M}}$ (see figure 2.9 in section 2.5.1 for an example).

The last aspect of Brisson's work that we use forms the link between the cell tuples and the cells themselves. From Brisson (1990), if $c_{\alpha i}$ is an *i*-cell of the regular CW-complex and $\mathbf{T_M}$ the set of cell tuples, then the subset of cell tuples for which $t_i = c_{\alpha i}$; is known as the set of cell tuples associated with $c_{\alpha i}$ or $assoc(c_{\alpha i})$ (figure 4.20(a)). Notice that by ignoring all $switch_i$ operations, $\mathbf{G_{T_M}}$ is separated into a set of path-connected components. Each component is a path-connected graph formed by the associated set of tuples for any i-cell $c_{\alpha i}$ and the switch operations between them (figure 4.20(b)). We will refer to this graph as $G_{T_{C_{\alpha i}}}$ and note that it is only path-connected in a subdivided manifold.

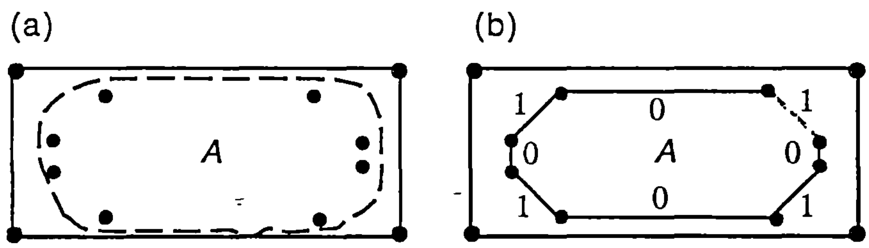


Figure 4.20 - the associated set of cell tuples for a 2-cell (A) in a regular CW-complex. (a) *assoc*(*A*) (ie. the tuples inside the dotted line) (b) G_{T_A}

The main advantages of Brisson's cell tuple are:

1. Correspondence with the notions of *vertex use* (Weiler 1986) and *cusp* (Gursoz et. al. 1991) as well as the fact that it is a simple and effective compression of the method of representing boundary and coboundary cells with integer identifiers as is common practice in the 2-dimensional map models currently used in GIS (see chapter 2).

2. Simplicity - the tuples and their properties are extremely simple to implement. For example, using the fact that *switch* always returns one tuple for each dimension, a very simple implementation would consist of two lists of tuples one containing the tuples themselves, the other containing the tuples that result from each *switch* operation (Brisson 1990).

3. Access to information about both the boundary-coboundary relationships of a cell (via the associated set of tuples) and to the ways in which these cells may be ordered 'about' the cell (via the *switch* operator) if such an ordering exists.

The situation we want to extend the cell-tuple to, is that of a generalized regular k-cell complex (representing a k-dimensional spatial object) which is contained within a Euclidean n-manifold ($0 \leq k \leq 3$ and $k \leq n \leq 3$). Each special case is now examined individually, after which general rules describing the extension will be derived. Note that throughout these examples, T_{R^n} is defined as the set of cell-tuples of the k-cell complex in the Euclidean manifold R^n and $G_{T_{R^n}}$ is the graph whose nodes are the cell-tuples of T_{R^n} and whose edges are represented by switch operations.

4.6.1 0-dimensional Spatial Object in R^n ($1 \leq n \leq 3$)

R^1

For a 0-dimensional spatial object in R^1, there is only one tuple t in T_{R1} for which t_0 = the 0-cell and t_1 = the world 1-cell representing R^1. A special case of the two-sided ordering of the 0-cell exists in which the world 1-cell is the same on either side ie. $switch_1(t) = t$ (figure 4.21(a)).

It would appear that the result of $switch_0(t)$ should be undefined (= $\varnothing$) since there are no further 0-cells in the world 1-cell. However the alternative (shown in figure 4.21(a)) is to set $switch_0(t) = t$. This alternative has the advantage that a special case of the circular ordering of 1-cells and 0-cells about a (-1)-cell (given by Brisson 1990 in **ct4** above) can be applied; ie. $m = 1, i = 0, j = 1$.

If there is more than one 0-cell in R^1, then another special case can be defined based on the fact that a 0-cell 'separates' R^1 into two 'pieces'. Looking at figure 4.21(b), $switch_0(t)$ can be defined to make $G_{T_{R1}}$ connected. The situation is somewhat similar to that of the 1-dimensional spatial object in R^1. However, it should be emphasized that this (ie. $n = 1$) is the only instance for which $G_{T_{R^n}}$ is path-connected when more than one spatial object exists in R^n. Higher dimensional analogues do not exist.

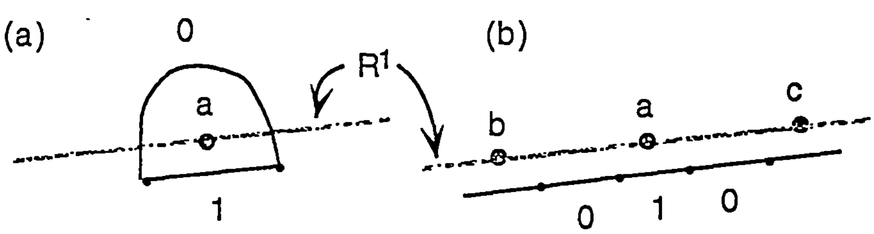


Figure 4.21 - 0-dimensional spatial object(s) in R^1 (a) a single 0-cell, a single tuple $t = (a, R^1)$ - $switch_1(t) = t$ and $switch_0(t) = t$ (b) more than one 0-cell, tuples are $t^1 = (a, R^1)$, $t^2 = (b, R^1)$, $t^3 = (c, R^1)$ - for $t = t^1, t^2, t^3$, $switch_1(t) = t$ as in figure 4.21(a), however $switch_0(t^2) = t^1$ and $switch_0(t^1) = t^3$.

R^2

For a 0-dimensional spatial object in R^2, there is only one tuple t in T_{R2} in which t_0 = the 0-cell, t_2 is the world 2-cell representing R^2, $switch_2(t) = t$, and since there are no incident 1-cells, $t_1 = \varnothing$ (thus $switch_1(t) = \varnothing$) and $switch_0(t) = \varnothing$. A special case of the circular ordering of 1-cells and 2-cells given by **ct4** exists and consists of one 2-cell (ie. that representing R^2) defined by $switch_2(t) = t$ (figure 4.22).

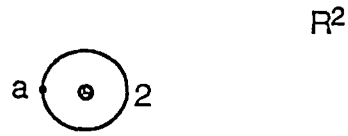


Figure 4.22 - 0-dimensional spatial object in R^2 (= plane of page) - if $t = (a, \varnothing, R^2)$ $switch_0(t)$, $switch_1(t) = \varnothing$ and $switch_2(t) = t$.

R^3

For a 0-dimensional spatial object in R^3, there is only one tuple t in T_{R3} in which t_0 is defined, t_3 is the 3-cell representing R^3. Since $t_1, t_2 = \varnothing$ (because there are no 1-cells and 2-cells) and the 0-cell is contained within the world 3-cell (R^3), $switch_i(t) = \varnothing, 0 \le i \le 3$).

4.6.2 1-dimensional Spatial Object in R^n ($1 \le n \le 3$)

R^1

For a 1-dimensional spatial object in R^1, the representation of R^1 by a world 1-cell together with the 1-cells of the 1-dimensional spatial object forms a 'complete' partition of R^1 (ie. the same as a subdivided 1-manifold) and thus the results of Brisson (1990) given above by conditions **ct1-5** (see above) apply without modification.

R^2

For a 1-dimensional spatial object in R^2, $\forall\, t \in T_{R2}$, t_0 and t_1 are defined, t_2 is the world 2-cell representing R^2 and $switch_2(t)$ is always t. A special case of the circular ordering of 1-cells and 2-cells about a 0-cell (given by **ct4**) exists because $switch_2(t)$ is always t (figure 4.23).

Note that as in the case of a 0-cell in R^1, there are two alternatives for $switch_1(t)$ when there is a single 1-cell in the coboundary of a 0-cell (figure 4.24(a) and (b)). Once again, we choose to set $switch_1(t) = t$ (figure 4.24(b)) so that **ct4** becomes valid for $m = 1, i = 1, j = 2$ and remains valid for $m \ge 2, i = 0, j = 1$; ie. the circular ordering of 0-cells and 1-cells about a (-1)-cell. As a result, the 1-cycles in the 1-dimensional spatial object can also be derived from the $switch_0$ and $switch_1$

operations; ie. the circular ordering of 1-cells and 0-cells about an (abstract) (-1)-cell.

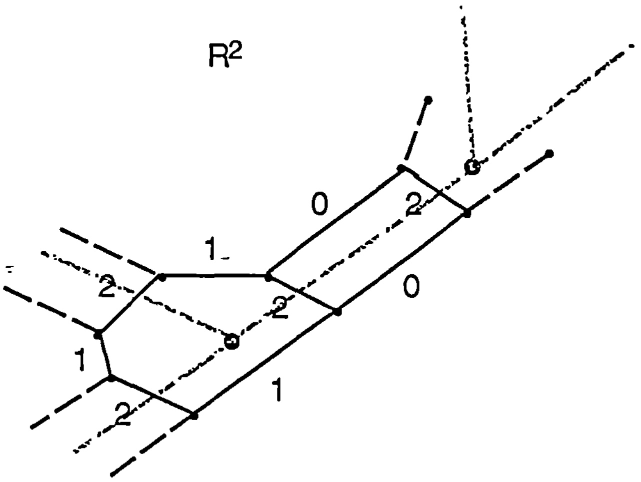


Figure 4.23 - 1-dimensional spatial object in R^2 (= plane of page)

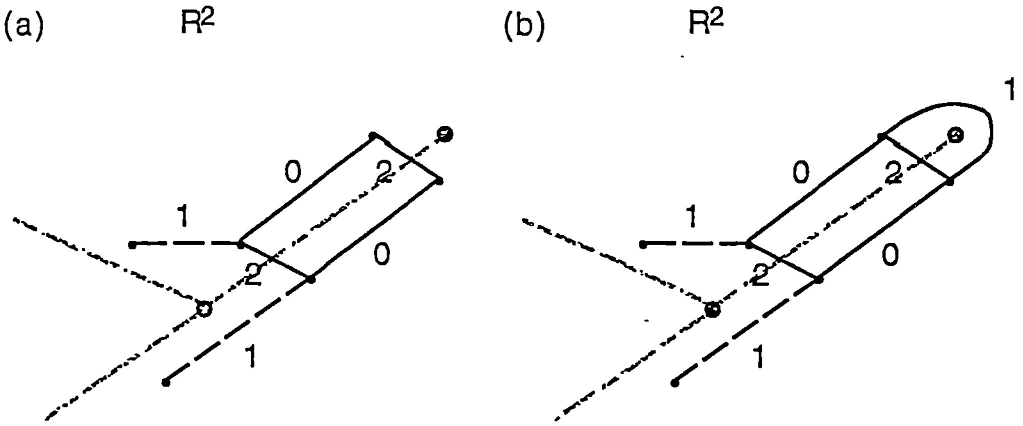


Figure 4.24 - the alternatives for tuples at a 0-cell which has a single cobounding 1-cell (dangling 1-cell) for a 1-dimensional spatial object in R^2 (= plane of page) **(a)** $switch_1(t) = \varnothing$ for the tuples associated with the 0-cell. The circular ordering of 1-cells and 2-cells about the 0-cell is not defined since $switch_1(t) = \varnothing$ **(b)** $switch_1(t) = t$ for all tuples associated with the 0-cell. The circular ordering of 1-cells and 2-cells about the 0-cell is defined by $switch_1(t) = t$ and $switch_2(t) = t$ ie. $m = 1, i = 1, j = 2$ in a special case of **ct4**.

$\mathbf{R^3}$

For a 1-dimensional spatial object in R^3, $\forall\, t \in \mathbf{T}_{R3}$, t_0 and t_1 are defined, t_3 is the world 3-cell representing R^3, and since there are no 2-cells, $t_2 = \varnothing$ (and thus $switch_2(t) = \varnothing$). A special case of the circular ordering of 2-cells and 3-cells about a 1-cell (given by **ct4**) exists because of the world 3-cell (ie. that representing R^3) and the fact that there are no 2-cells. It is defined by the fact that for all tuples, $switch_3(t) = t$ (figure 4.25(a)).

Note that there is no ordering of cobounding 1-cells in the 3-dimensional neighborhood of a 0-cell in R^3. However if a 0-cell has just two cobounding 1-cells then they can be put in a two-sided order in the 1-dimensional subspace they form. This will be known as a 1-dimensional subspace ordering (see the 0-cell *b* in figure 4.25(b)). It should also be noted that $G_{T_R 3}$ is not path-connected ($switch_1(t) = \varnothing$) when three or more 1-cells cobound a 0-cell (see the 0-cell *a* in figure 4.25(a)).

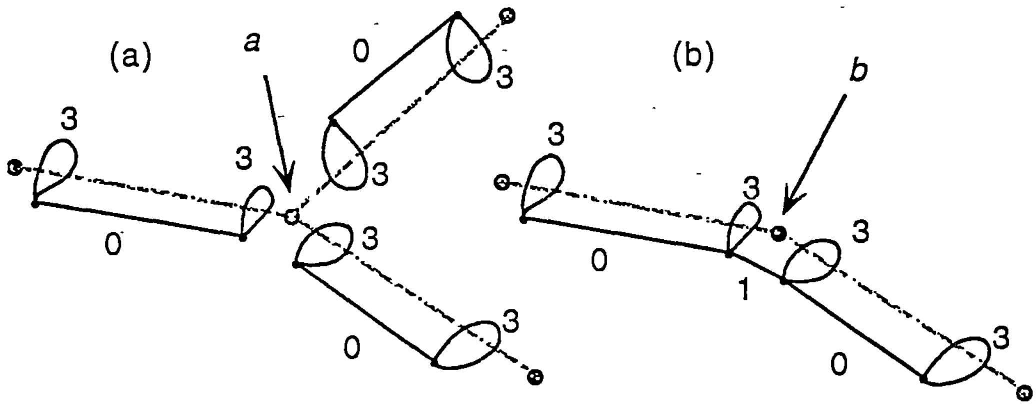


Figure 4.25 - a 1-dimensional spatial object in R^3 **(a)** The circular ordering of 2-cells and 3-cells about a 1-cell consists of the world 3-cell only. Notice that $G_{T_R 3}$ is not path-connected at *a* ie. all tuples in *assoc*(*a*) have $switch_1 = \varnothing$. The reason for this is that there is no definable ordering of three or more cobounding 1-cells about a 0-cell in R^3. **(b)** a 1-dimensional subspace ordering formed around the 0-cell *a* since there are just two cobounding 1-cells.

4.6.3 2-dimensional Spatial Object in R^n ($2 \le n \le 3$)

R^2

For a 2-dimensional spatial object in R^2, $\forall\, t \in T_{R2}$, each component of the tuple is defined (ie. $\neq \varnothing$) since R^2 is represented by a world 2-cell. Thus the results of Brisson (1990) given by **ct1-5** (see above) apply with the only difference being that $G_{T_R 2}$ will not be path-connected when any generalized regular 2-cell has more than one boundary 1-cycle (figure 4.26). In addition $G_{Tc_{\alpha 2}}$ consists of two distinct components.

R^3

For a 2-dimensional spatial object in R^3 $\forall\, t \in T_{R3}$, t_0, t_1 and t_2 are defined, t_3 is the world 3-cell representing R^3 and $switch_3(t)$ is always t; ie. the world 3-cell which represents R^3. A special case of the circular ordering of 2-cells and 3-cells about a 1-cell (given by **ct4**) exists because $switch_3(t)$ is always t (figure 4.27).

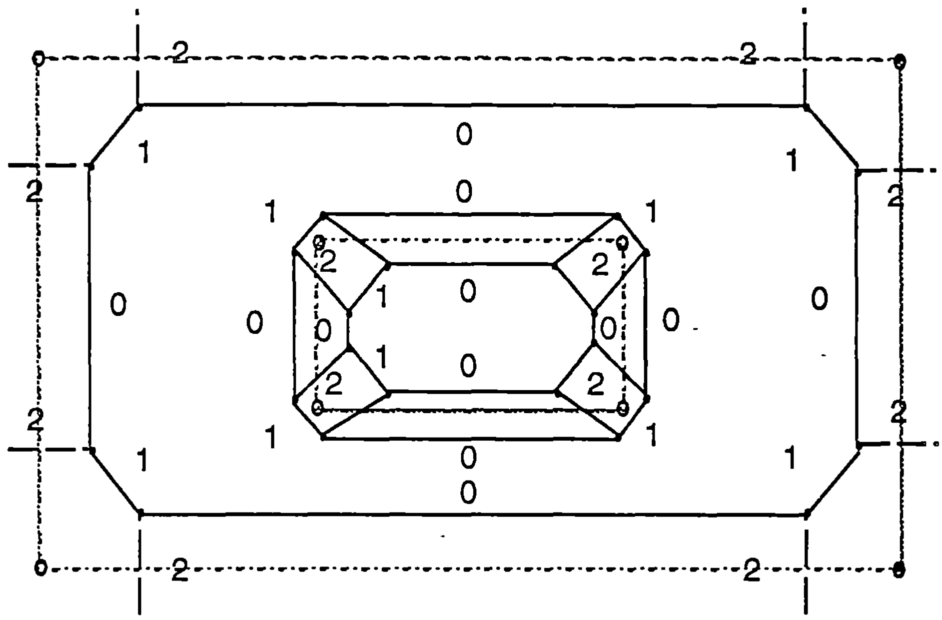


Figure 4.26 - $G_{Tc_{\alpha 2}}$ is not path connected when a 2-cell which has more than one boundary 1-cycle

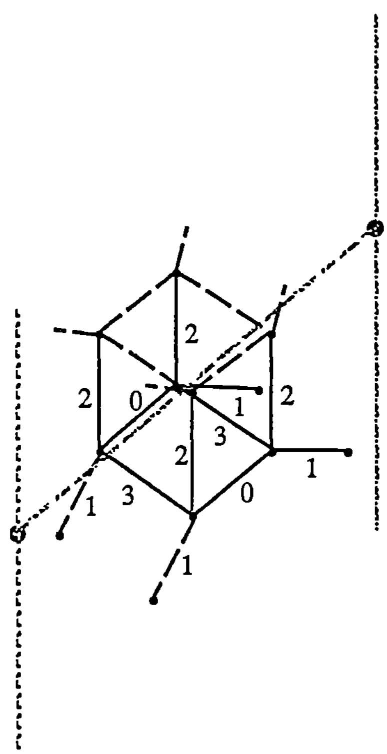


Figure 4.27 - a portion of $G_{T_R}3$ for a 2-dimensional spatial object in R^3.

Analogous to the case of a 1-dimensional spatial object in R^2, there are two alternatives for the special case of $switch_2(t)$ when there is only one 2-cell in the coboundary of a 1-cell (figure 4.28(a) and (b)). Once again we choose to set $switch_2(t) = t$ (figure 4.28(b)) and **ct4** becomes valid for $m = 1$, $i = 2$ and $j = 3$. Note also that this choice makes **ct4** valid for $m \geq 2$, $i = 1$ and $j = 2$ ie. circular orderings of 1-cells and 2-cells exist about the 0-cell a (see figure 4.28(b)).

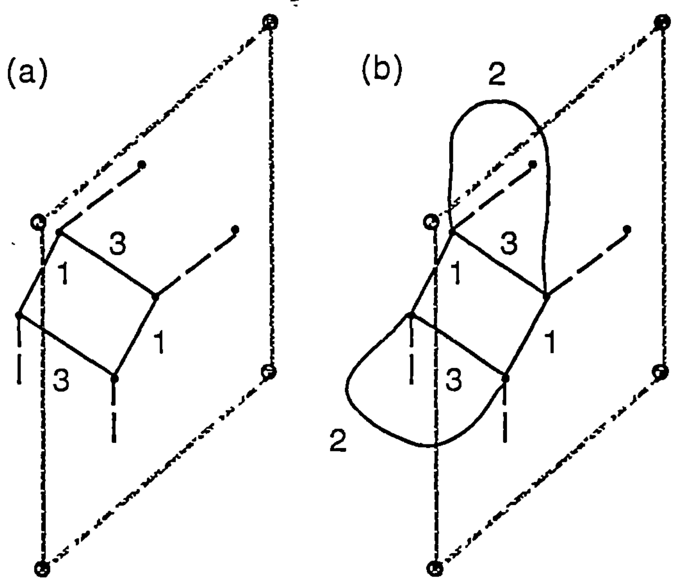


Figure 4.28 - the alternatives for tuples at a 1-cell which has only one 2-cell in its coboundary **(a)** $switch_2(t) = \varnothing$ for the tuples associated with the 1-cell. Thus the circular ordering of cobounding 1-cells and 2-cells is not defined since $switch_2(t) = \varnothing$. **(b)** $switch_2(t) = t$ for tuples associated with the 0-cell. The circular ordering of cobounding 2-cells and 3-cells is defined by $switch_2(t) = t$ and $switch_3(t) = t$ ie. $m = 1$, $i = 2$, $j = 3$ (a special case of **ct4**) Also circular orderings exist for $m \geq 2$, $i = 1$, $j = 2$ (ie. the circular ordering of 1-cells and 2-cells about a 0-cell).

Since generalized regular 2-cells may have more than one boundary cycle, $\mathbf{G_{Tc_{\alpha 2}}}$ may not be path connected.

Lastly, it should be noted (once again) that there is no ordering of cobounding 1-cells in the 3-dimensional neighborhood of a 0-cell in R^3. However, the *switch* operator does capture circular orderings in the subspace topology of any incident 2-cells. eg. when two or more 2-cells share a 0-cell as in figure 4.29. Analysis of these 2-dimensional subspace circular orderings can be derived from the set of tuples associated with the 0-cell (eg. number of 2-cells sharing etc). It should be noted that in figure 4.29, $\mathbf{G_{Ta}}$ is not path-connected - there are two distinct components.

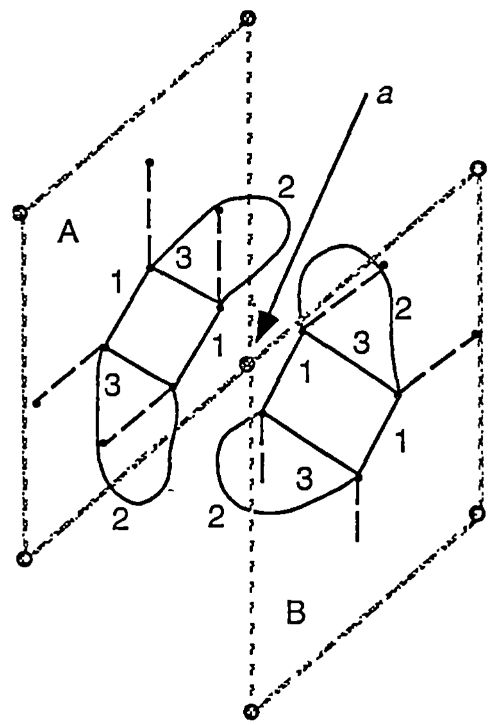


Figure 4.29 - a 2-dimensional spatial object in R^3 in which two 2-cells (A and B) share a common 0-cell a. There is no known ordering of cobounding 1-cells about the 0-cell a in R^3 however two 2-dimensional subspace orderings exist. In general, $G_{Tc_{\alpha 0}}$ may not be path-connected at a 0-cell. In this case G_{Ta} has two distinct components.

4.6.4 3-dimensional Spatial Object in R^3

R^3

For a 3-dimensional spatial object in R^3, all components of the tuples are defined and the results of Brisson (1990) (ie. **ct1-5**) apply since R^3 is represented as a world 3-cell (the result is similar to a subdivided 3-manifold). However if a generalized regular i-cell (i=2 or 3) has more than one boundary cycle, $\mathbf{G_{Tc_{\alpha i}}}$ will not be path-connected.

Lastly, it should be noted (once again) that there is no ordering of cobounding 1-cells in the 3-dimensional neighborhood of a 0-cell in R^3. However, the *switch* operator does capture circular orderings definable in the subspace topology of the 2-manifold boundary cycles of any incident 3-cells; eg. figure 4.30(a) shows two 3-cells which share a common 0-cell. Analysis of these 2-dimensional subspace circular orderings can be derived from the set of tuples associated with the 0-cell (eg. number of 3-cells sharing etc). However as shown in figure 4.30, $\mathbf{G_{Tc_{\alpha 0}}}$ may not be path-connected, in this case there are two distinct components of $G_{Tc_{\alpha 0}}$, one for each 3-cell.

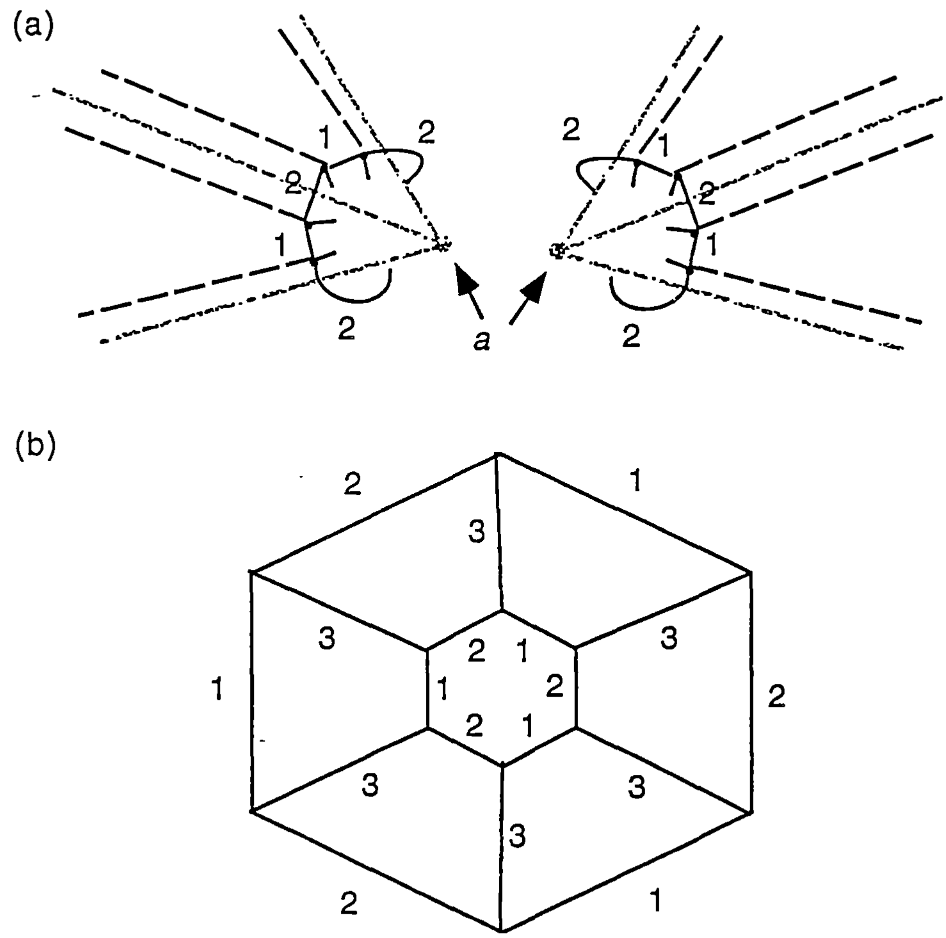


Figure 4.30 - subspace orderings in the neighborhood of a 0-cell in a 3-dimensional spatial object (a) A 3-dimensional spatial object in R^3 in which two 3-cells share the 0-cell a (*switch* operations are shown) (b) the subspace circular orderings for each 3-cell defined by *switch* operators ie. there are two circular subspace orderings of 1-cells and 2-cells about the 0-cell a for each 3-cell. G_{Ta} is not path-connected because no switch operation links the subspace orderings.

4.6.5 Summary

The differences between this extension (outlined by the examples above) and Brisson's original description (as given by **ct1-5** and the definition of switch) for regular CW complexes can be summarised as follows, where **grct** stands for generalized regular cell tuple.

For a k-dimensional spatial object contained within R^n ($0 \le k \le n \le 3$), if $t \in T_{R^n}$ and $t = \{c_{\alpha 0},\ldots,c_{\alpha n}\}$ then:

grct1 Since the Euclidean manifold is always represented as an n-cell then $\forall\, t \in T_{R^n}$, $t_n \neq \varnothing$.

grct2 If $k = n$ then **ct1-5** apply without modification since the effect of the world cell is to make the partition complete (ie. equivalent to a subdivided manifold).

grct3. If $k \neq n$ then:

1. If $k = n - 1$ then $t_0,\ldots,t_n \neq \varnothing$, $switch_0(t),\ldots,switch_n(t) \neq \varnothing$.

   Each tuple is duplicated so that the circular orderings specified by **ct4** are valid for all cells of dimension $\ge n - 2$ with the following special cases/modifications:

   (a) $switch_n(t) = t$ (**ct1** is no longer valid).

   (b) If an (n-2)-cell a has just one cobounding (n-1)-cell then we set $switch_{n-1}(t) = t$ for all $t \in assoc(a)$ (see figure 4.24(b) and 4.28(b)). In addition, **ct4** must be trivially modified to allow $m = 1$, for $i = n$-1 and $j = n$ defined by $switch_{n-1}(t) = t$ and $switch_n(t) = t$. In this case, **ct2** is no longer valid.

2. If $k = n - 2$ then $t_{n-1} = \varnothing$, $switch_{n-1}(t) = \varnothing$ and $switch_n(t) = t$ defines R^n (the world cell) as the only cell in the circular ordering about an (n-2)-cell. (see figure 4.22 and 4.25).

3. If $k = 0$ and $n = 3$ then $c_1,\ldots,c_{n-1} = \varnothing$ and $switch_0(t),\ldots,switch_n(t) = \varnothing$ (see figure 4.23).

As a result of **grct3.1** and **grct3.2**, **ct1** and **ct2** are relaxed and **ct5** only applies when the necessary *switch* operations are defined (ie. $\neq \varnothing$). See for example figure 4.25 above where **ct5** for $i = 0$ and $j = 2$ does not apply because $switch_2(t) = \varnothing$.

grct4 Path Connectivity of $G_{Tc_{\alpha i}}$ for any i-cell $c_{\alpha i}$ ($0 \le i \le k$)

1. When $k \geq 2$ and $i \geq 2$, $c_{\alpha i}$ may have more than one boundary cycle, $G_{Tc_{\alpha i}}$ is not path-connected. However *assoc*($c_{\alpha i}$) provides an implicit link between the cells of the different boundary cycles. Note that the definition of the associated set of tuples is another reason why cut-lines between boundary cycles (see section 4.4) are unnecessary.

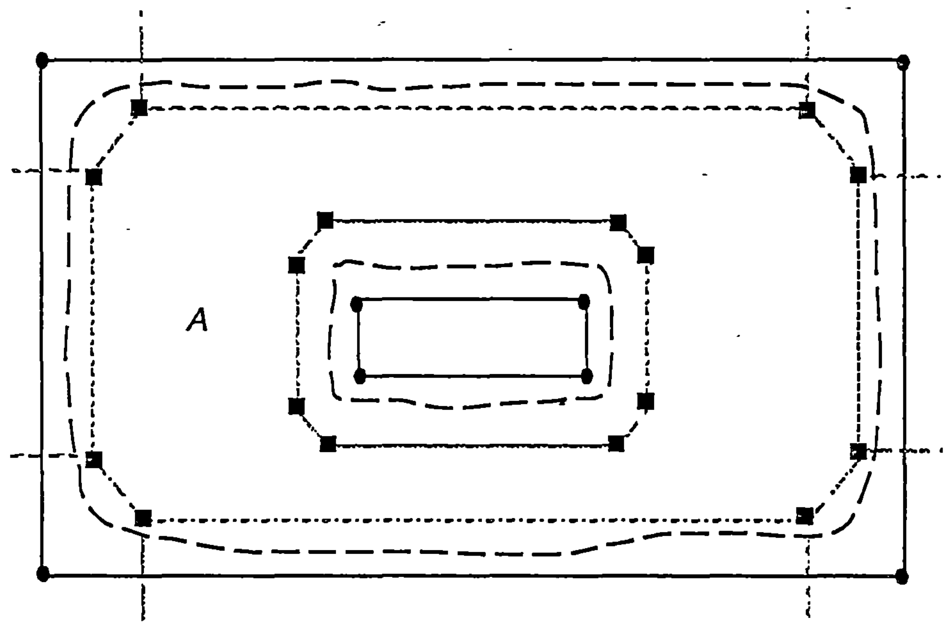


Figure 4.31 - the associated set of tuples for the generalized regular 2-cell (shown as highlighted squares) A. *assoc*(A) is contained within the two dashed lines and provides an implicit link between the cells in the two boundary 1-cycles.

2. When $i < n - 2$ and $k > 0$, $G_{Tc_{\alpha i}}$ *may not* be path connected because although there is no ordering of all cobounding (i+1)-cells in the n-dimensional neighborhood about $c_{\alpha i}$, the switch operator can capture orderings in either (i+2) or (i+1)-dimensional subspaces. For example, when $i = 0$, $k = 2$ (or 3) and $n = 3$ (ie. 3-dimensional neighborhood of a 0-cell in a 2-dimensional or 3-dimensional spatial object in R^3 - see figure 4.29 and 4.30 above), the switch operator captures circular orderings about the 0-cell within 2-dimensional subspace(s). Whilst when $i = 0$, $k = 1$ and $n = 3$ (see figure 4.25(b)), the switch operator can capture a two-sided ordering in the 1-dimensional subspace, if the subspace exists. See figure 4.25(a) for a case where the subspace does not exist.

3. When $k = 0$, $G_{Tc_{\alpha 0}}$ *is not* path connected since a 0-dimensional spatial object is a discrete space; ie. the 0-cells are not connected in any way.

Notice that in these three cases, the neighborhoods of such cells must be analyzed using any available switch information (eg. subspace orderings) *together* with the associated set of tuples.

4.6.6 Discussion

The most flexible aspect of this extension of the cell-tuple is that for each cell neighborhood, it is possible to use any existing cell coboundary ordering information (via *switch* operations) as well as an unordered list of boundary-coboundary cells (via the set of cell tuples associated with the cell). This is an advantage over representations which specify unordered lists of boundary and coboundary cells eg. the models of Corbett (1985) and Rossignac and O'Connor (1991) - see chapter 2.

It is particularly interesting and important to note that, $G_{T_{R^n}}$ as defined by the *switch* operator with the extensions that we have defined above (particularly **grct3.1 (b)**), turns out to be both a direct formalization and encapsulation of the three standard cycles making up the tri-cyclic cusp of Gursoz et. al. (1991); ie. a cusp is roughly equivalent to a pair of cell-tuples, disk cycles are circular orderings formed by $switch_1$ and $switch_2$ (figure 4.32), loop cycles are circular orderings formed by $switch_0$ and $switch_1$ (figure 4.33), and edge-orientation cycles are circular orderings formed by $switch_2$ and $switch_3$ - see section 2.6.5 for a description of these cycles. The pseudomanifold classification in the next chapter encapsulates all the remaining variations in the cusp entity described by Gursoz et. al., with some additional situations not mentioned in their description of the tri-cyclic cusp.

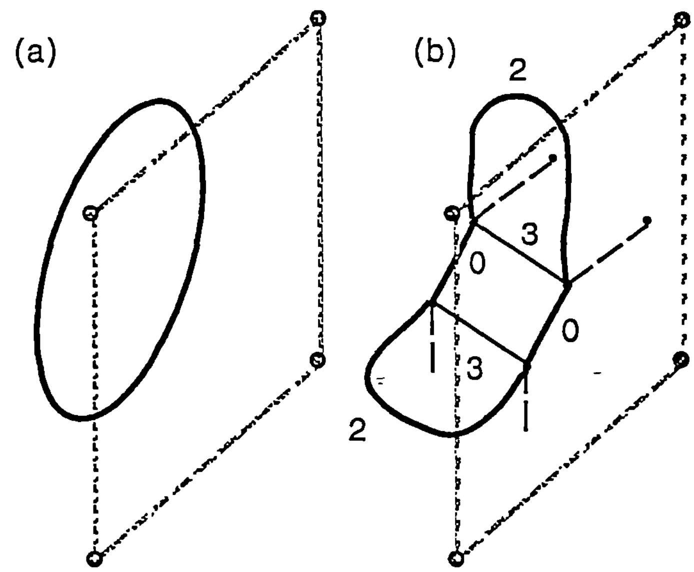


Figure 4.32 - An equivalence between a disk cycle of Gursoz et. al. (1991) and a circular ordering of $switch_1$ and $switch_2$ operations (a) The disk cycle of Gursoz et. al. (b) The equivalent circular ordering of $switch_1$ and $switch_2$ operations.

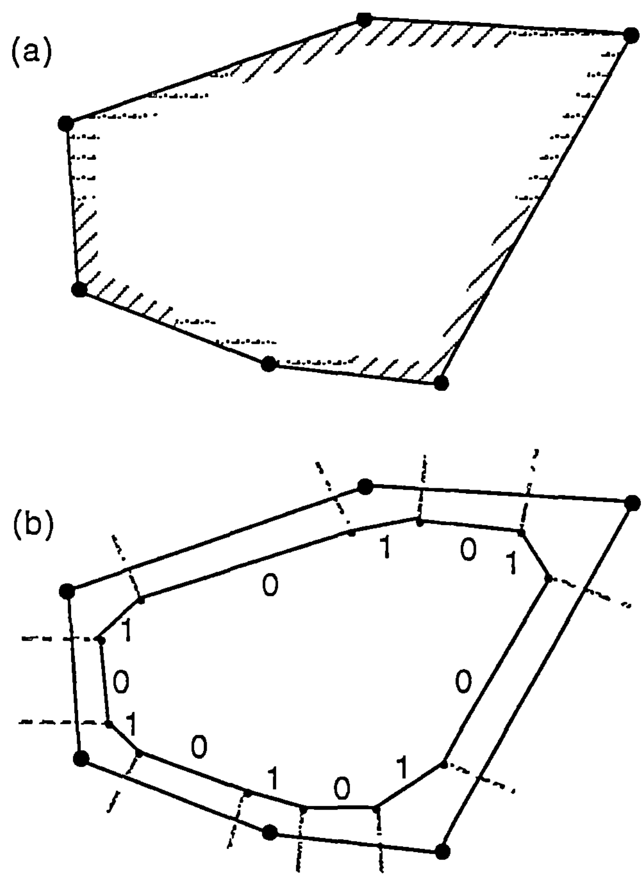


Figure 4.33 - An equivalence between a loop cycle of Gursoz et. al. (1991) and a circular ordering of $switch_0$ and $switch_1$ operations (a) The loop cycle of Gursoz et. al. (b) The equivalent circular of $switch_0$ and $switch_1$ operations.

The implicit nature of the cell-tuple also makes it much easier to manipulate and maintain than the tri-cyclic cusp because the tri-cyclic cusp (along with the radial edge of Weiler 1988) requires the definition and maintenance of redundant global topological elements (eg. shells, loops, walls, regions etc) and their degenerate forms - see section 2.1 and 2.6.

A disadvantage of the cell-tuple in this form and in Brisson's original form is the rapid growth in the number of tuples as the complexity of the geometric structure increases (see also Paoluzzi et. al. 1993). The large number of tuples may make analysis of spatial relationships between spatial objects quite slow since many tuples would have to be searched. This problem will be addressed later in chapter 5.

Chapter 5

Generalized Singular Cell Complexes

5.1 Introduction

The last chapter introduced the generalized regular cell complex as a method for representing both the geometric and the topological structure of a spatial object contained within a Euclidean manifold. One of the aims of this research is the development of a topological model in which it is possible to represent and analyze spatial relationships between multiple spatial objects of different dimensions (a multi-dimensional domain). The approach we take in this chapter is based on the formation of a new cell complex representing the union of one or more generalized regular cell complexes by geometrically intersecting their cells. The resulting cell complex is then 'embedded' within a Euclidean manifold in order to obtain the circular orderings and two-sided ordering results (as in section 4.6). The rest of this chapter focuses on the representation of this new cell complex within a topological model. The details of the union process (by geometrically intersecting the cells) and the modification of existing boolean set operations are a subject for future research. The differences between this new k-cell complex and a generalized regular k-cell complex are:

1. the boundary cycle(s) of each k-cell ($k = 2$ or 3) may no longer be subdivided (k-1)-manifold(s) (figure 5.1).

2. k-cells may have (k-1)-cell complexes which are internal to their boundary cycle(s). In other words, the cells of the boundary cycles of the k-cell are now a subset of its (k-1)-skeleton (figure 5.2).

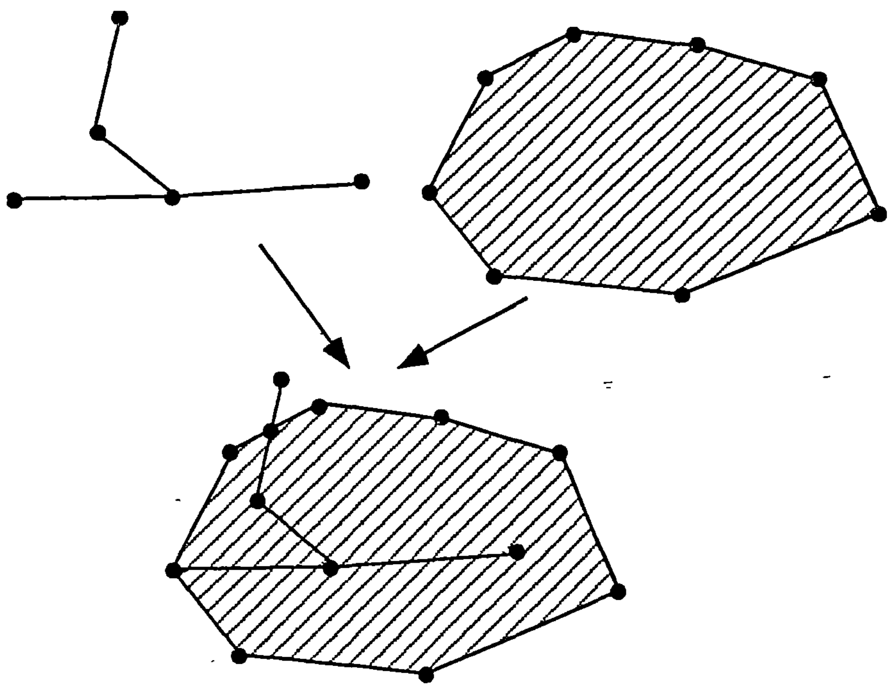


Figure 5.1 - the boundary cycle of a 2-cell is no longer a subdivided 1-manifold as a result of the union of the 1-dimensional spatial and 2-dimensional objects.

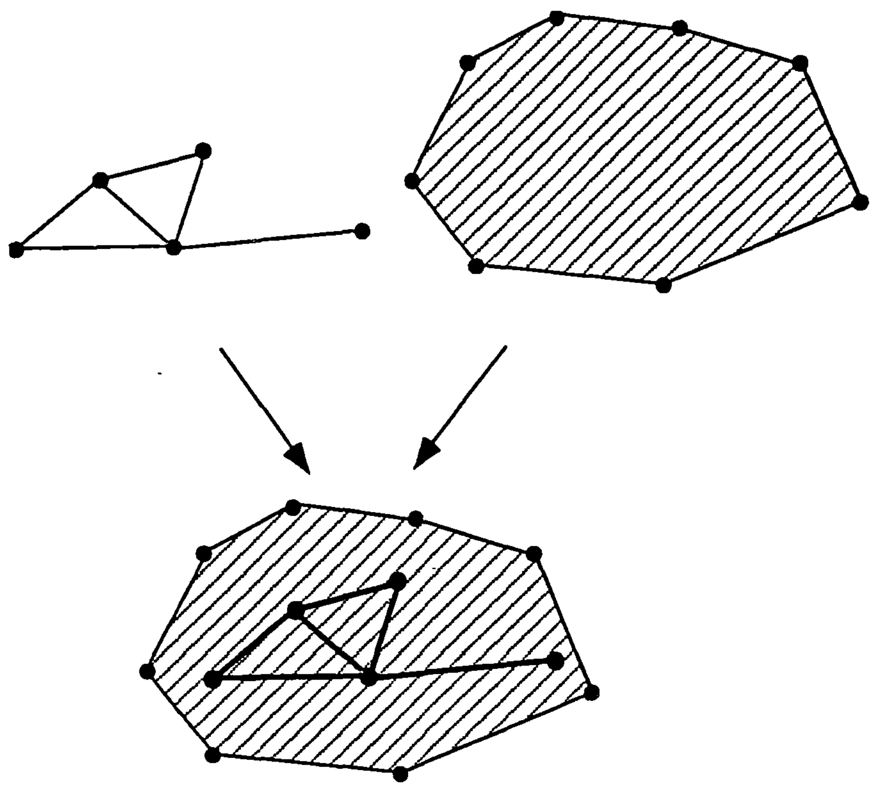


Figure 5.2 - a cell complex (shown in heavy black lines) which is internal to the boundary cycles of a 2-cell.

3. all generalized regular cell complexes representing the spatial objects are subcomplexes.

Many of the topological models reviewed in chapter 2 do not deal with such possibilities and/or do not make any formal attempt at describing and modelling them. The aim of this chapter is to define the properties of such a cell complex and apply the results developed in the previous chapter to represent it as an implicit topological model.

In investigating the properties of such a complex it will become clear that it may also be used as a replacement for the generalized regular cell complex, further extending the domain of spatial objects that can be represented. Examples where this extended domain may be useful are when:

1. Spatial objects are multidimensional eg. a river system which consists of lakes and streams some of which are 1-dimensional some of which are 2-dimensional.

2. Spatial objects evolve or change with time due to the action of some physical process. For example, a surface representing a rock layer may collapse in on itself to create singularities - see also the applications of the surface evolver of Brakke (1993).

After discussing and comparing the advantages and disadvantages of an alternative approach to the representation of spatial relationships between spatial objects in section 5.2, section 5.3 describes the new cell complex which we refer to as the generalized singular cell complex. Section 5.4 gives the classification of the boundary cycles of a generalized singular cell. Section 5.5 describes the extension of the ordering results given in chapter 4 for generalized regular cell complexes, to generalized singular cell complexes. Section 5.6 introduces a 2-dimensional extension of the 1-dimensional arc (as used in 2-dimensional GIS) in order to provide a compressed topological representation of the generalized singular cell complex.

5.2 Modelling Spatial Relationships between Spatial Objects

It is a fundamental requirement of many applications to access the relationships between spatial objects of different dimensions; eg. a geoscientist is definitely interested in the spatial relationships between a borehole (1-dimensional) and a set of rock layers (3-dimensional); ie. does the borehole pass through a particular rock layer or 'touch' or 'glance' the boundary of one particular rock layer and pass through another rock layer? What is the primitive basis for such spatial relationships and how can they be determined?

The most primitive *basis* for answering such questions is provided by topology. Other types of spatial relationships exist (eg. those defined using metric concepts such as distance and direction) but topological relationships are the most primitive and/or fundamental because as is noted in chapter 1, topology deals with properties of spaces (and relationships between spaces) which do not change under very general transformations (ie. transformations with few restrictions). This is the basis for the claim that topology is the 'most general' geometry (Meserve 1955). For this reason and the fact that the analytical power of most current 2-dimensional GIS is based on topological relationships, we confine our discussion to topological relationships.

The neighborhood boundary-coboundary relationships *between cells* (intra cell complex) in the generalized regular cell complex, capture two types of topological relationship: adjacency and containment (Pigot 1991). Considerable research exists on efforts to expand these basic topological relationships. A formal approach has been developed by Max Egenhofer - see for example Egenhofer and Franzosa (1991), the original paper of Pullar and Egenhofer (1988) and the extensions in Pigot (1991).

The topological relationships *between spatial objects* (inter cell complex) can be calculated by examining the relationships between the constituent cells of their generalized regular cell complexes. The advantage of using their cells is that they are much simpler spaces than the spatial objects themselves (see chapter 1) and thus topological relationships are easier to formalize, calculate and/or store.

There are two basic approaches to modelling topological relationships between spatial objects which may be distinguished by whether they store individual spatial objects and calculate topological relationships as required (ie. an object-based approach) or whether another cell complex is formed by taking the union of the spatial objects. Both approaches are now examined in more detail.

5.2.1 Object Based

In this approach, spatial objects are independently represented as generalized regular cell complexes. The most efficient implementations calculate topological relationships between the spatial objects as required. For example, in 2-dimensional GIS, the latest implementations of this approach are: the Spatial DataBase Engine (SDBE - Geographic Technologies Inc. 1992) and a similar solution independently suggested in Clementini et. al. (1993), which is planned for implementation using GEO++ (Vijlbrief & van Oosterom 1992). Other implementations proposed for computer-aided design applications explicitly store a set of topological relationships eg. the modular boundary model of De Floriani et. al. (1991) and the integration of n-G-maps of different dimensions described briefly in Lienhardt (1991).

Advantages:

i) When the spatial objects do not have a complex geometric structure and the relationships between them are simple (ie. just one type of relationship such as adjacency and a small number of cells involved in the relationship) then calculations are simple and retrieval is fast.

ii) Access to all topological relationships between the spatial objects (including overlap) eases the design, and improves the domain of spatial query languages. The large number of different permutations of topological relationships created by distinguishing between different dimensions of the cells involved in the spatial relationships can be avoided by 'overloading' a number of representative or fundamental relationships. eg. 'equal', 'overlap', 'touch', 'in', 'cross' and 'disjoint' in Clementini et. al. (1993) and a slightly larger and more expressive set of relationships in the implementation of the Spatial Database Engine of Geographic Technologies Inc. (1992).

Disadvantages:

i) Where spatial objects 'share' cells, these cells must be duplicated. Whilst this is not such a problem for man-made objects such as those used in computer-aided design or for land-parcels in a land information system, this requirement can cause very large overheads in storage space (and thus retrieval times) particularly in 3-dimensional applications such as the geosciences where spatial objects may have many facets in their boundary cycles.

ii) Spatial relationships must be calculated using metric information. In practice the problems of inconsistent topology and geometry caused by the limited precision of computer floating point arithmetic may be alleviated by scaling the coordinates to integers and using a false origin (see for example, Franklin and Kankanhalli 1993). However the underlying assumption is that the spatial relationships will be simple; eg. two solids may share a face. If they are not and/or the spatial objects have a complex geometric structure then the overheads involved in recalculating these relationships each time a query is initiated will become very large.

We have explored some of the issues relating to 3-dimensional applications of an object-based approach by classifying topological relationships between simplexes of the *same* dimension in a Euclidean 3-manifold - see Pigot (1991) and appendix A of this thesis. To our knowledge no classification scheme has been given for topological relationships between cells of *different* dimensions in a Euclidean 3-manifold.

5.2.2 Cell Complex

In the second approach, another cell complex is formed by taking the union of the generalized regular cell complexes which represent the spatial objects. For 3-dimensional applications, this cell complex is contained within a world 3-cell representing the Euclidean 3-manifold in order to ensure the existence of the circular ordering results as described in sections 1.4 and 4.6. The idea behind this approach is that the relationships between the spatial objects are carried within the cell neighborhoods of this complex. These relationships are then reconstructed by examining the cell neighborhoods ie. using generic operators based on the boundary-coboundary relations. For 3-dimensional applications, this approach is an extension of the underlying concepts defined for 2-dimensional GIS in White (1978) and Corbett (1979).

<u>Advantages:</u>

1. Spatial relationships are implicitly confined to adjacency and containment by the cell complex and thus all spatial relationships between spatial objects can be determined without reference to metric information by inspecting the neighborhood boundary-coboundary relationships of cells.

2. Shared cells are represented only once.

Disadvantages:

1. Unable to model overlapping cells in a simple manner - such relationships are expressly forbidden in the definition of the cell complex. Cells which overlap must be intersected and new cells created.

2. The computational cost of building the generalized singular cell complex by intersecting the individual generalized regular cell complexes is high. However the build process is very similar to that used in boolean set analysis (or overlay processing) and this powerful analysis technique must be available regardless of whether this approach or the object based approach is used.

Neither approach has any specific advantages when methods for maintaining topological consistency (ie. ensuring that spatial objects are within the domain) are considered since, for 3-dimensional applications, consistency can only be realistically maintained by ensuring that all cell boundary cycles are valid (see section 6.2.4 for more discussion of this issue).

This research focuses on the cell complex approach in preference to the object based approach. The main reasons for not adopting the object based approach are the storage and processing overheads created by:

1. the redundancy involved in the representation of shared cells

2. the need to calculate spatial relationships every time a query is initiated.

These overheads are expected to be particularly severe for complex spatial objects in 3-dimensional applications such as the geosciences. However it should be noted that the generalized regular cell complex (developed in the last chapter) and the construction operators described in chapter 6 could be applied to the object based approach.

The next section gives a detailed description of both the cells and the cell complex which result from the union of the generalized regular cell complexes.

5.3 Generalized Singular Cell Complexes

Recall from section 5.1 and figure 5.1, that the boundary cycles of the k-cells in the cell complex formed by taking the union of the generalized regular cell complexes are no longer subdivided (k-1)-manifolds. The underlying reason is that the neighborhoods of some points of the (k-1)-manifolds do not have (k-1)-dimensional neighborhoods (figure 5.3). Since almost all points *do* obey the manifold condition, topologists refer to such boundary cycles as "pseudomanifolds" (see for example, Dewdney 1972 and White 1971). We adopt this terminology.

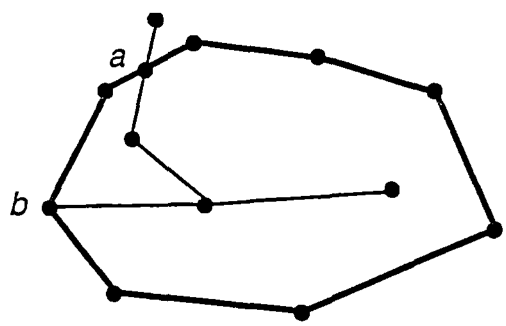


Figure 5.3 - 1-manifold (highlighted) with two points (a and b) whose neighborhoods are no longer homeomorphic to an (n-1)-dimensional disk - a 1-pseudomanifold

Corbett (1979) (see also White 1984) gives an equivalent definition by describing such situations as singular cells. Following Corbett (1979), each boundary cycle of a singular 2-cell is the image of a continuous map applied to a 1-manifold (or 1-sphere). This is in contrast to the regular 2-cell (see section 4.2), where each boundary cycle is the image of a homeomorphism applied to a 1-manifold. The basic difference is that for singular cells the one-to-one nature of the homeomorphism (usually used for regular cells) is relaxed, to allow a subset of points in the 1-manifold boundary cycle to be identified forming a singularity. The singularity is the set of points whose neighborhoods are no longer homeomorphic to 1-dimensional disks. Thus equivalent definitions for a pseudomanifold would be 'a manifold with singularities' or 'a singular manifold'. It is important to note that the set of manifolds is a subset of the set of pseudomanifolds, since a manifold is simply a pseudomanifold with no singularities. As a result, we refer to the cells that result from the union of generalized regular cell complexes as generalized *singular* cells. We say that their boundary cycle(s) are subdivided *pseudomanifolds*.

Recall from chapter 4 that the (k-1)-skeleton of a generalized regular k-cell is composed of the cells in the subdivided (k-1)-manifold boundary cycles only. In a generalized singular k-cell, the cells of the subdivided (k-1)-pseudomanifold boundary cycles are a *subset* of its (k-1)-skeleton. The reason is that cell complexes of dimension $\leq k$-1 may exist in the interior of a generalized singular k-cell.

Combining the results of these discussions gives us the simple combinatorial definition of a generalized singular cell.

gsc: A generalized singular k-cell is a Euclidean k-manifold with i subdivided (k-1)-pseudomanifold boundary cycles ($i \geq 1$). The j-cells ($j \leq k$-1) in the subdivided (k-1)-pseudomanifold boundary cycles are a subset of the (k-1)-skeleton.

With the usual requirement that the generalized singular cells intersect along their boundaries, the result is the *generalized singular* cell complex. *Generalized* because it extends and incorporates simplicial, regular CW and generalized regular cell complexes mentioned previously and *singular* because the boundary cycles are subdivided pseudomanifolds.

Since the definition of a pseudomanifold implies a very large number of possibilities (particularly in higher dimensions), the notion of an identification space is used to classify pseudomanifolds into three simple or primitive types in the next section of this chapter. The classification will then be used in the extension of the ordering results from generalized regular cells (given in section 4.6) to generalized singular cells.

5.4 Classification of Pseudomanifold Boundary Cycles

This section gives a classification of pseudomanifolds forming the boundary cycles of generalized singular cells using identification, in a manner similar to that given in Corbett (1979) for 2-dimensional GIS.

The process of identification is a very general way of either gluing a space to itself or to another space. The resulting space is often called an identification space. Identification spaces are examples of a much wider class of spaces known in topology as quotient spaces (see Chapter III in Jänich 1980 for a good description).

However, since we do not need the wider implications of quotient spaces at present, we can ignore the unnecessary terminology that goes with them and use a much simpler definition of an identification space, which is based on that given in Armstrong (1982) pg. 71.

Ident1 Let $\mathbf{X}$ be a topological space, $\mathbf{A} \subset \mathbf{X}$ a subspace and f: $\mathbf{A} \rightarrow \mathbf{X}$ a continuous map. The aim is to glue/attach $\mathbf{X}$ to itself by identifying points $x \in \mathbf{A}$ with their images $f(x) \in \mathbf{X}$. We define a partition so that two points lie in the same partition if and only if they are identified under f. The subsets of this partition are:

1. pairs of points $(x, f(x))$;
2. points in $\mathbf{X} - \mathbf{A} - f(\mathbf{A})$

The identification space associated with this partition is $\mathbf{X} \cup_f \mathbf{X}$.

However it is often easier and more desirable (particularly with the type of cells we describe here) to create the singularity by gluing two spaces together. This process can also be described using identification. That is, to glue the two spaces together, points in homeomorphic subspaces are identified using a continuous transformation. The process is known as attaching.

Ident2 Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces, $\mathbf{A} \subset \mathbf{X}$ a subspace and f: $\mathbf{A} \rightarrow \mathbf{Y}$ a continuous map. The aim is to attach/glue $\mathbf{Y}$ to $\mathbf{X}$ by identifying points $x \in \mathbf{X}_0$ with their images $f(x) \in \mathbf{Y}$. The resulting identification space is denoted by $\mathbf{Y} \cup_f \mathbf{X}$. Commencing with the disjoint union $\mathbf{X} + \mathbf{Y}$, we define a partition so that two points lie in the same partition if and only if they are identified under f. The subsets of this partition are:

1. pairs of points $(x, f(x))$;
2. points in $\mathbf{Y} - \mathbf{A}$;
3. points in $\mathbf{X} - f(\mathbf{A})$.

The identification space associated with this partition is $\mathbf{Y} \cup_f \mathbf{X}$ and the map f is known as an attaching map (figure 5.4).

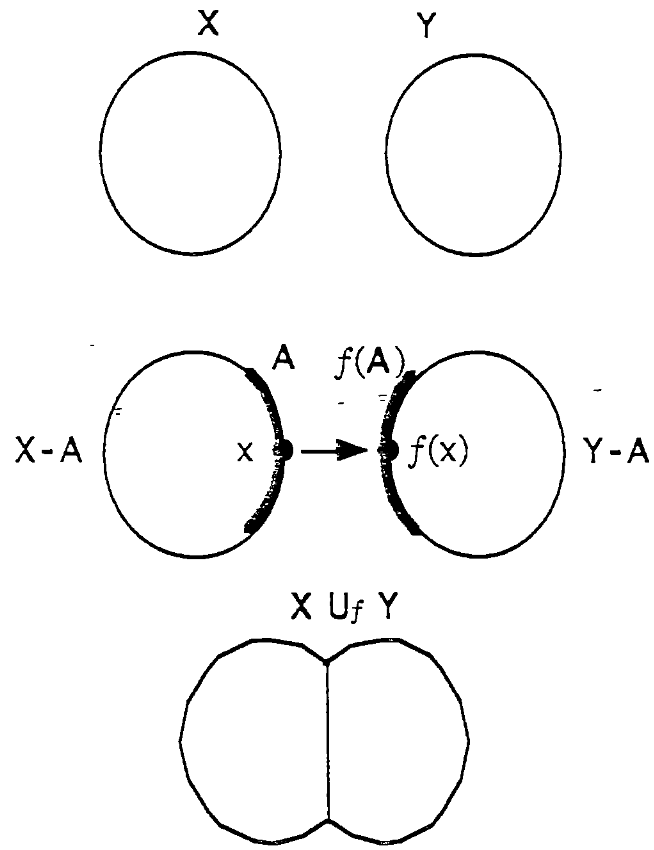


Figure 5.4 - gluing two spaces together forms an identification space (a) the disjoint union of two spaces **X** and **Y** ie. **X** + **Y** (b) the subsets of the partition (c) **Y** $\cup_f$ **X**.

We will now give an overview of the classification of a (k-1)-pseudomanifold boundary cycle of a generalized singular k-cell formed by applying **Ident1** and **Ident2** to the subdivided (k-1)-manifold boundary cycle of a generalized regular k-cell. Each pseudomanifold boundary cycle will be classified as one of three distinct types:

<u>Type 1</u> Type 1 pseudomanifolds are formed by identifying homeomorphic cells in a manifold using **Ident1**. As an example, consider identifying two antipodal 0-cells of a sphere. The 'pinched' sphere that results is a type 1 2-pseudomanifold.

<u>Type 2</u> Type 2 pseudomanifolds result from identifications of a manifold that create interior 'holes' (eg. the 2-cell in figure 2.3(b)) or 'cavities' in a generalized singular cell. Since 'holes' must have a 1-cycle (1-manifold) boundary and 'cavities' must have a 2-cycle (2-manifold) boundary, we follow Corbett (1979) pg. 12 in labelling type 2 pseudomanifolds *cyclic*. Corbett's approach to describing these pseudomanifolds used **Ident1**

Ident1 (see figure 2.3(b) for example). We will use the constructive approach of **Ident2** to identify homeomorphic cells in two distinct manifolds. The advantage of using **Ident2** is that it is much simpler in higher dimensions. As an example, consider the 2-cell shown in figure 2.3(b). Using **Ident2,** the type 2 1-pseudomanifold boundary cycle of this 2-cell can be formed by identifying two 0-cells in a pair of 1-manifolds.

Type 3 Type 3 pseudomanifolds result from identifications of a manifold that do not form interior 'holes' or 'cavities' in a generalized singular cell. In other words, they do not form a proper part of the boundary of the generalized singular cell (Corbett 1979 pg. 12). We will follow Corbett (1979) pg. 12 in labelling type 3 pseudomanifolds *acyclic.* As for type 2 pseudomanifolds, Corbett described these pseudomanifolds using **Ident1** (see figure 2.2) whereas we will use the simpler constructive approach of **Ident2** to identify homeomorphic cells from a manifold and a cell. As an example, a type 3 1-pseudomanifold can be formed by identifying a 0-cell of the 1-manifold with one of 0-cells forming the boundary of a 1-cell.

As mentioned above, the classification system used in this thesis is based on that given in Corbett (1979). However, Corbett, who was interested in the 1-pseudomanifold boundary cycles of 2-cells, did not need to consider type 1 pseudomanifolds. The reason is that two 0-cells in the 1-manifold boundary cycle of a 2-cell form a 0-sphere. The Schoenflies theorem (see Munkres 1975, pg. 385) states that the 0-sphere separates a 1-manifold (a 1-sphere) boundary cycle into two components. If the two 0-cells of the 0-sphere are identified, then the result is two distinct 2-cells whose boundary cycles 'share' a point. This situation is a *cyclic* (or what we call a type 2) 1-pseudomanifold (see figures 2.3 (a) and (b)). By contrast, in an orientable 2-manifold, pairs of 0-cells and pairs of 1-cells that do not form a 1-sphere, do not separate the 2-manifold and when identified they form neither cyclic nor acyclic 2-pseudomanifolds. The classification of such pseudomanifolds (ie. the type 1 2-pseudomanifolds) used in this thesis was derived by Whitney (1944) (but see also Francis 1987 pg. 8 and Scott Carter 1993 pg. 41).

The ultimate aim of our classification is to enable analysis of the orderings (eg. circular etc - see section 4.6) of cells in the neighborhood of those cells involved in the singularity of the (k-1)-pseudomanifold boundary cycle of a generalized singular k-cell. The reason that we focus only on this neighborhood is that, loosely speaking, it is the only neighborhood of the pseudomanifold that is 'not a manifold'. Given this aim and the generality of the pseudomanifold concept, there are a number of simplifications we will make to the classification:

1. We only consider identifications of pairs of homeomorphic j-cells where $j \leq k$-2. Although it is to identify more than two j-cells (eg. a type 1 2-pseudomanifold can be formed by identifying more than two 0-cells in a 2-manifold), a pair of cells is sufficient to determine the general case. Notice that we do not consider identifications of (k-1)-cells. The reasons, given in terms of the three types of pseudomanifold, are as follows:

    (a) In the classification of type 1 pseudomanifolds, the theory which we use (see Whitney 1944 and Scott Carter 1993 pg. 41 in particular), does *not* cover such cases. Whilst it is possible to visualize simple cases (eg. identifying two distinct 2-cells in a 2-manifold), further identifications soon give rise to spaces that conflict with the definition of a k-dimensional generalized singular cell. For example, the identification of three distinct 1-cells in a 1-manifold may result in a space that cannot be a generalized singular 2-cell. An extreme example of 'bad behaviour' under such identifications is given by Stillwell (1980) pg. 251 (also quoted in Brisson 1990).

    (b) In the classification of type 2 and type 3 pseudomanifolds under **Ident2**, identifying a pair of (k-1)-cells would not produce a pseudomanifold boundary cycle. For type 2 pseudomanifolds, such an identification produces two distinct generalized regular k-cells. For type 3 pseudomanifolds the result is a single generalized regular k-cell because a (k-1)-cell will be exactly identified with a (k-1)-cell in the (k-1)-manifold boundary cycle.

2. Although identifications can have either an exterior or interior form, we only consider the interior forms. As an example, the type 2 (ie. cyclic) 1-pseudomanifolds shown in figures 5.5(a) and 5.5(b) represent exterior and interior forms that result from identifying a pair of 0-cells

from the 1-manifold boundary cycles of two generalized regular 2-cells using **Ident2.**

In our classifications we will consider interior forms only (eg figure 5.5(a)) because although the cells involved in the neighborhood are different, the main characteristics of the neighborhood of both interior and exterior forms are the same. For example, in the exterior form shown in figure 5.5(b) the world 2-cell would appear twice in the circular ordering of 2-cells and 1-cells about the singular 0-cell whilst for the interior form in figure 5.5(a) it is the 'larger' 2-cell that appears twice. It is obvious that the 2-dimensional neighborhood of the identified 0-cells has the same characteristics in both figure 5.5(a) and (b), but the cells involved are different.

(a)

(b)

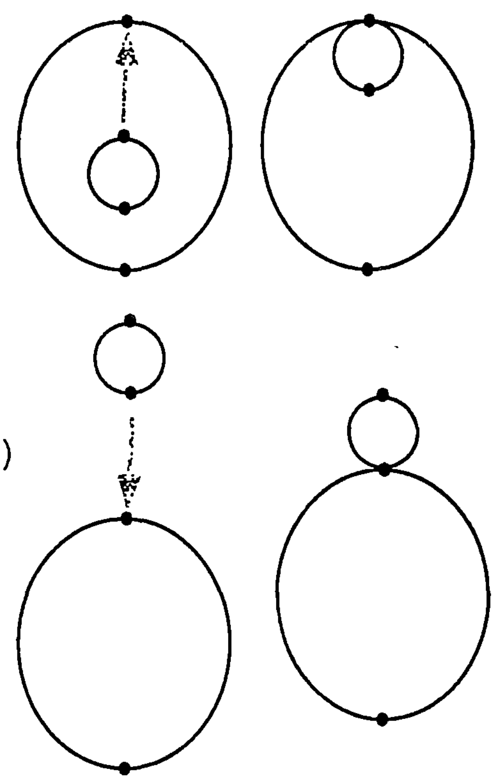


Figure 5.5 - exterior and interior forms of identification **(a)** interior **(b)** exterior

5.4.1 Type 1 Pseudomanifolds

In their simplest form, type 1 pseudomanifolds are identification spaces formed by gluing together certain points of an orientable manifold forming the boundary cycle of a generalized regular cell (ie. definition **Ident1**).

We need a theory to classify the identification spaces that result from the identification maps in a well-defined way. Whitney (1944) provides such a theory through differential topology by considering smooth, stable mappings of differentiable manifolds. A differentiable n-manifold is an n-manifold with a differential structure. A differential structure consists of a collection of charts (known as an atlas) in which each chart is a homeomorphism from an n-dimensional neighborhood of M^n to R^n. It is simply a method for transferring the local processes which may be carried out in R^n to M^n (ie. differentiation). Since a differential structure (or smoothing) can be found for all manifolds up to dimension three (see Gauld 1982 pg. 62), there is no loss in generality involved in applying the results developed by Whitney for diffentiable manifolds, to the wider domain of (topological) manifolds that we refer to in this research.

The cases put forward in Whitney's theory (see Francis 1987 pg. 8) to classify type 1 pseudomanifolds are individually described as follows.

A. *n*-point

If we identify n ($n \geq 2$) 0-cells of the smooth 2-cell complex the result is an n-point where n is known as the degree; eg. a double point results from identifying two 0-cells (figure 5.6). Loosely speaking, the neighborhood of a double point in a 2-pseudomanifold looks like two cones or dunce hats with their tips identified.

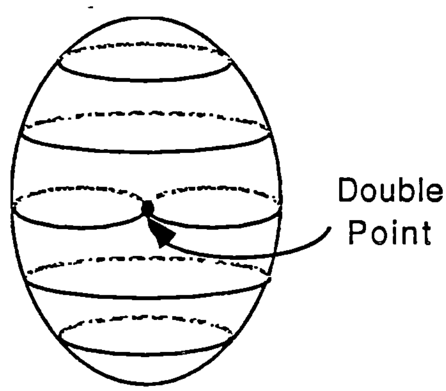


Figure 5.6 - a 2-pseudomanifold with a double point

As indicated earlier, we have only described the situation for 2-pseudomanifolds because the situation for 1-pseudomanifolds is covered in a much simpler way by the type 2 (cyclic) singularities. For an example of a double point in a type 1 pseudomanifold, see section 2.2 in chapter 2 and White (1984) pg. 21.

B. *n*-line/*n*-line with pinchpoint

There are two cases for an n-line, with the distinction based on whether the boundary 0-cells of the 1-cells are coincident prior to identification:

1. <u>n-line</u>: Identification of n 1-cells whose boundary 0-cells are not coincident where n is the degree; eg. a double line results from identifying two such 1-cells (figure 5.7). Loosely speaking, the neighborhood of a double line looks like two planes crossing at a line.

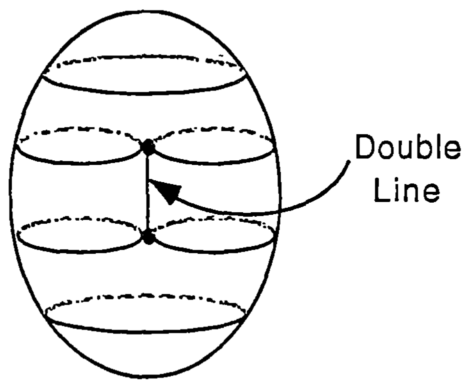


Figure 5.7 - a 2-pseudomanifold with a double line

2. <u>n-line plus pinch point</u>: Identification of n 1-cells all of which have single coincident boundary 0-cell, where n is the degree, For example, a 2 line plus pinch point (also known as a Whitney umbrella) is shown in figure 5.8. For additional intuition see figure 3(12) on pg. 7 of Francis (1987).

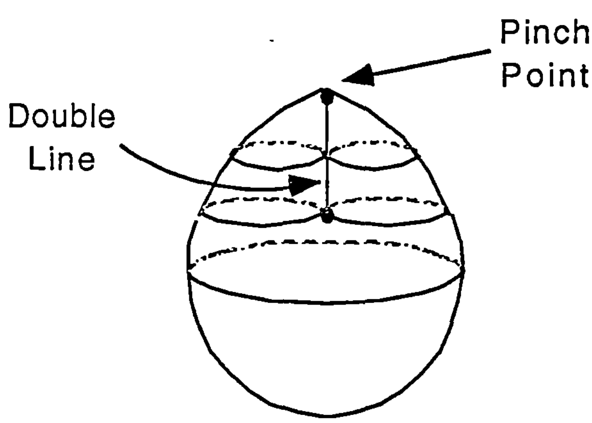


Figure 5.8 - a 2-pseudomanifold with a double line - the neighborhood of the double line is shown as two planes intersecting at a line.

It is interesting to note that unlike the other two types of pseudomanifolds, type 1 pseudomanifolds do not result from the process of taking the union of generalized regular cell complexes.

5.4.2 Type 2 Pseudomanifolds

Type 2 k-pseudomanifolds ($k = 1,2$) are identification spaces whose identified points (or singularities) result in *cyclic* structures. In general they are described (using the notation of **Ident2**) as the result of attaching a k-manifold (boundary cycle of a (k+1)-cell c_a) to another k-manifold (the boundary cycle of a second (k+1)-cell c_b). As an example, for subdivided 2-manifold boundary cycles, let $D^1 \subset M_a^2$ and $f: D^1 \rightarrow M_b^2$ be continuous. $M_a^2 \cup_f M_b^2$ is obtained by attaching M_a^2 to M_b^2 along a 1-cell (D^1) using the attaching map f.

1. For 1-manifolds: Attach a subdivided 1-manifold boundary cycle M_a^1 by identifying a 0-cell ($D^0 \subset M_a^1$) with a 0-cell in another subdivided 1-manifold boundary cycle M_b^1 ($f: D^0 \rightarrow M_b^1$) to form a 1-pseudomanifold boundary cycle ($M_a^1 \cup_f M_b^1$) (figure 5.9).

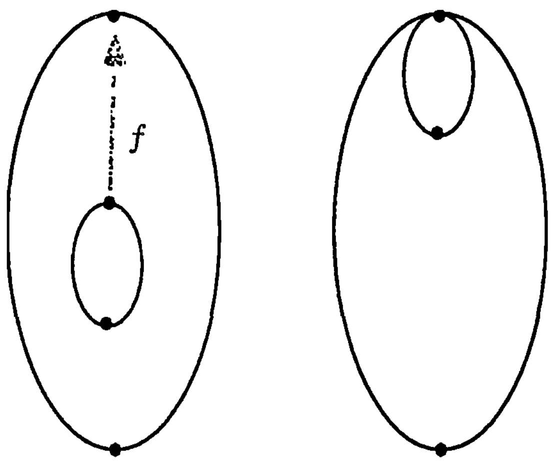


Figure 5.9 - a 1-pseudomanifold boundary cycle formed by attaching two 1-manifolds along a 0-cell using the attaching map f

2. For 2-manifolds: Attach a 2-manifold M_a^2 by identifying a 0-cell ($D^0 \subset M_a^2$) or a 1-cell ($D^1 \subset M_a^2$) with a 0-cell or a 1-cell in another subdivided 2-manifold boundary cycle M_b^2 ($f: D^0 \rightarrow M_b^2$ or $f: D^1 \rightarrow M_b^2$ respectively) to form a 2-pseudomanifold boundary cycle ($M_a^2 \cup_f M_b^2$).

5.4.3 Type 3 Pseudomanifolds

Type 3 pseudomanifolds are identification spaces whose identified points (or singularities) result in *acyclic* structures. In general they are described (using the notation of **Ident2**) as the result of attaching j-cells to k-manifolds ($0 < j < k$). As an example, for subdivided 2-manifolds, let $D^1 \subset D^2$ and $f: D^1 \rightarrow M^2$ be continuous. $M^2 \cup_f D^2$ is obtained by attaching a 2-cell (D^2) along a 1-cell (D^1) using the attaching map f.

<u>1. For 1-manifolds:</u> Attach a 1-cell by identifying a 0-cell in its boundary ($D^0 \subset D^1$) with a 0-cell in a subdivided 1-manifold boundary cycle ($f: D^0 \rightarrow M^1$) to form a 1-pseudomanifold boundary cycle ($M^1 \cup_f D^1$) (figure 5.10).

<u>2. For 2-manifolds:</u> There are two cases:

(a) Attach a 1-cell by identifying a 0-cell ($D^0 \subset D^1$) in its boundary with a 0-cell in a subdivided 2-manifold boundary cycle ($f: D^0 \rightarrow M^2$) to form a 2-pseudomanifold boundary cycle ($M^2 \cup_f D^1$).

(b) Attach a 2-cell by identifying a 0-cell ($D^0 \subset D^2$) or a 1-cell ($D^1 \subset D^2$) in its boundary with a 0-cell or 1-cell in a subdivided 2-manifold boundary cycle ($f: D^0 \rightarrow M^2$ or $f: D^1 \rightarrow M^2$ respectively) to form a 2-pseudomanifold boundary cycle ($M^2 \cup_f D^2$).

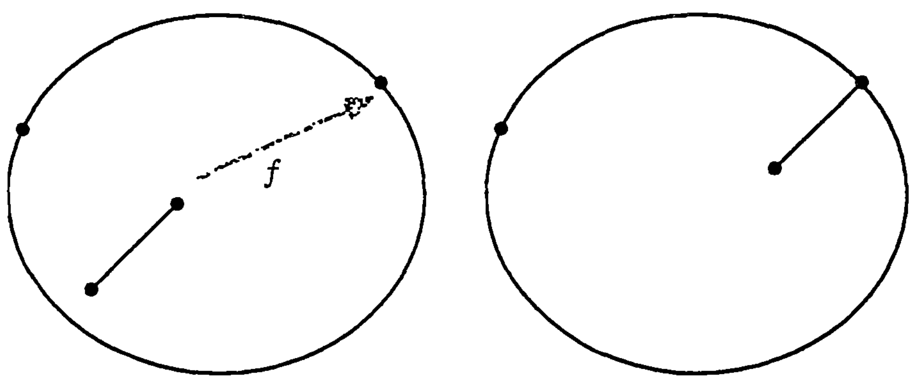


Figure 5.10 - a 1-pseudomanifold boundary cycle formed by attaching a 1-cell to a 1-manifold along a 0-cell using the attaching map f.

5.4.4 Summary

The important points to reiterate about these primitive (ie. elementary form) pseudomanifolds are:

1. We have only considered pseudomanifolds resulting from identifications of pairs of homeomorphic cells in the manifold boundary of a generalized regular cell. Whilst it is possible to identify more than two cells, a pair of cells suffices for the purpose of analyzing orderings

(eg. circular etc - see section 4.6) of cells in the neighborhood of the singularity of the pseudomanifold.

2. All pseudomanifolds may have both an exterior, as well as an interior form. Figures 5.6-5.10 show interior forms only. When it comes to analyzing the orderings of cells in the neighborhood of the singularity of a pseudomanifold, both forms have the same basic characteristics only the cells involved differ.

3. The identification maps (**Ident1** and **Ident2**) trivially carry the essential connectedness and compactness properties (see section 3.3) of the manifold to the pseudomanifold (see Jänich 1980 pg. 33).

4. The cell attaching maps in type 2 and 3 pseudomanifolds attach cells along subsets of their boundary cycles.

Type 1 pseudomanifolds are intended for use in temporal applications where it is often necessary to model the changes in the boundary of a spatial object using the less restrictive notion of homotopy, instead of homeomorphism. For example, consider a rock layer collapsing under the weight of the layers above it. Over time, the model may eventually show singularities where boundaries of the non-competent layer have collapsed onto themselves, in which case the boundary becomes a pseudomanifold. See Braake (1993) and Kunii & Shinagawa (1990) for other examples.

Type 2 and 3 pseudomanifolds are expected to represent the relationships formed by the intersection of 1, 2 and 3-dimensional generalized regular cell complexes within the same topological model.

The only other research which attempts any similar form of classification is the non-manifold model of Weiler (1986). As mentioned in chapter 2, Weiler's approach is an explicit one, based on generalizing manifold solids (ie. solids with 2-manifold boundaries) to non-manifold solids. In this research we have taken the opposite approach, by starting with a cell, generalizing it from its simplicial and regular forms and defining a spatial object as a collection of these generalized regular cells. Weiler's adjacency relationships reflect the difference between his approach and the one we have taken, since they include explicit elements (such as loops and shells) whilst all elements of our relationships are implicitly defined cells. In general, an 'adjacency relationship' is similar to only one class of the identification spaces examined above ie. those produced by **Ident2.** In other words, the general concept of an 'adjacency relationship' cannot be used to describe other identification spaces eg. those produced by **Ident1** for example.

In the next section, the classification of the pseudomanifold boundary cycles presented in this section will be used to analyze circular and subspace ordering results in a generalized singular cell complex.

5.5 Analysis of Ordering Results

We now use the pseudomanifold classification scheme given in the last section to analyze ordering results (see section 4.6) for a j-cell ($j \leq k$-2) involved in any singularity of a (k-1)-pseudomanifold boundary cycle of a generalized singular k-cell. Since $j \leq$ (k-2) and the generalized singular k-cell exists within a Euclidean k-manifold, the particular ordering results we will be interested in are the circular and subspace orderings.

It is important to reiterate that we focus on the cells involved in the singularity because they are the only cells in a subdivided pseudomanifold that do not have neighborhoods like those of a subdivided manifold.

Furthermore, since the definition of the generalized singular k-cell (see **gsc** in section 5.3) also permits j-cell complexes ($j \leq k$-1) internal to a (k-1)-pseudomanifold boundary cycle, we will analyze orderings for cells in these complexes as well.

The analysis of the ordering results will be described using the cell-tuple, since it has been chosen in section 4.6 as a suitable representation for the cell complexes described in this thesis.

We will analyze internal cell complexes first.

5.5.1 Internal Cell Complexes

The definition of the generalized singular cell makes it possible for k-cells ($2 \leq k \leq 3$) to have cell complexes of dimension $\leq k - 1$ internal to their boundary cycles. As is the case when we embed a generalized regular cell complex in a Euclidean manifold (see section 4.6), these internal cell complexes take on the circular and two-sided ordering results applicable to the dimension of the cell that contains them.

A. Internal to a Generalized Singular 2-Cell

For a j-cell ($j < 2$) internal to a generalized singular 2-cell A, there are two basic cases to consider:

1. $j = 0$, the 0-cell has the circular ordering results applicable to a 0-cell in R^2 (section 4.6.1). This is a special case of **ct4** where the circular ordering of 1-cells and 2-cells about a 0-cell consists of just one 2-cell (figure 5.11(a)). If the 'world' cell which contains A is not a 2-cell (eg. when A belongs to a generalized singular 3-cell complex) then the tuples take on the two-sided ordering results appropriate to A (see section 4.6.3) ie. $switch_3(t)$ and t_3 are defined for each tuple. For clarity, figure 5.11(b) shows $switch_3(t)$ for the tuples associated with the internal 0-cell only.

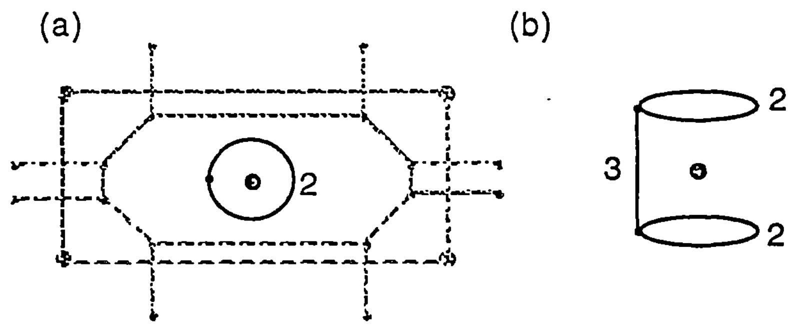


Figure 5.11 - internal 0-cell for a generalized singular 2-cell A (a) the circular ordering it obtains from the interior of A ie. $switch_2(t) = t$ and (b) with the two sided ordering propagated from A (A not shown)

2. $j = 1$, the 1-cell has the two-sided and circular ordering applicable to a 1-cell in R^2 via duplication of the tuples (section 4.6.3 and figure 5.12(a)). Once again, if the 'world' cell which contains A is not a 2-cell (eg. when A belongs to a generalized singular 3-cell complex) then the tuples take on the two-sided ordering results appropriate to A (section 4.6.3) ie. $switch_3(t)$ and t_3 are defined for each tuple (figure 5.12(b)).

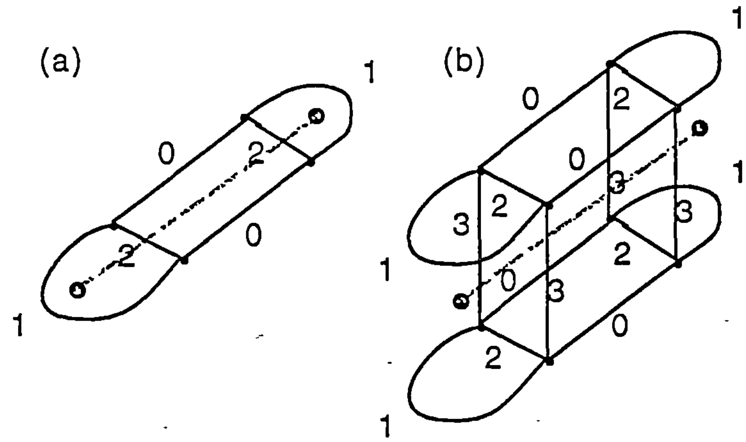


Figure 5.12 - internal 1-cell for a generalized singular 2-cell A (a) the two sided and circular orderings obtained from the interior of A (A is not shown for the sake of clarity) and (b) with the two-sided ordering propagated from A.

B. Internal to a Generalized Singular 3-Cell

For a j-cell ($j < 3$) internal to the 2-pseudomanifold boundary cycles of a 3-cell, the results are exactly the same as those described for 0-cells, 1-cells and 2-cells in R^3 (as given in sections 4.6.1-4.6.4), since the interior of a generalized singular 3-cell is by definition a Euclidean 3-manifold (see **gsc** in section 5.3).

In general it should be noted that the presence of internal cell complexes in both generalized singular 2-cells and 3-cells indicates an additional case where $G_{Tc_{\alpha i}}$ ($2 \leq i \leq 3$) is not path connected. This is analogous to the case where a generalized regular 2-cell or 3-cell has more than one boundary cycle (see **grct4.1** in section 4.6.5).

5.5.2 Analysis of Circular Orderings

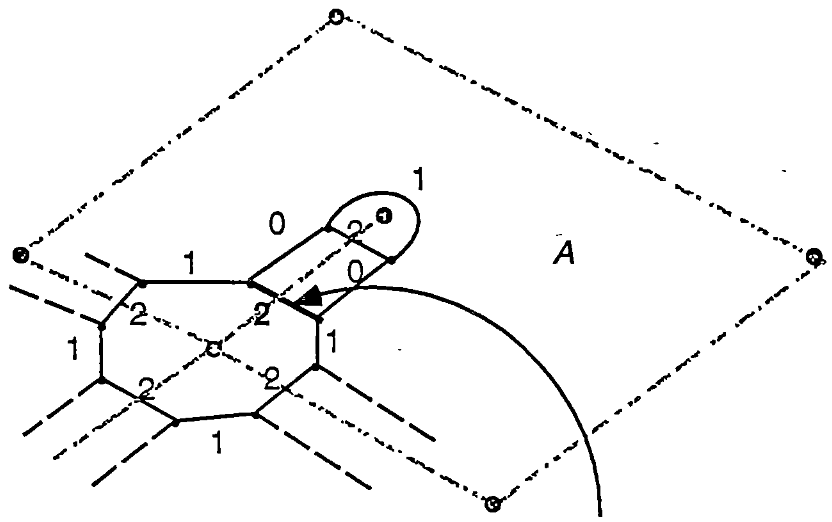
Now we analyze the circular orderings indicated by the pseudomanifold classifications.

A. Generalized Singular 2-Cells

For a generalized singular 2-cell (A), the circular orderings of 1-cells and 2-cells about each 0-cell and the circular orderings of 1-cells and 0-cells about a cell of dimension -1 (see **ct4** in section 4.6 with $i = 1, j = 2$ and $i = 0, j = 1$ respectively) now incorporate the following situations:

1. for $i = 1, j = 2$ (ie. the circular ordering of 1-cells and 2-cells about a 0-cell) there are two tuples for which $switch_2(t) = t$ if A has a type 3 1-

pseudomanifold boundary cycle formed by identifying a 1-manifold and at least one 1-cell along a 0-cell. In general, the results for a 1-dimensional spatial object in R^2 (given in section 4.6.2) apply in the 2-dimensional space formed by A (figure 5.13).



$switch_2(t) = t$

Figure 5.13 - modification of the circular ordering of 1-cells and 2-cells about a 0-cell to include $switch_2(t) = t$ if A has a type 3 1-pseudomanifold boundary cycle

Notice that there will be two extra tuples for which $switch_2(t) = t$, for each additional 1-cell identified at the 0-cell. Also, these 1-cells can be removed from the circular ordering of 1-cells and 2-cells at the 0-cell by ignoring all duplicate tuples (ie. where $\mathrm{switch}_2(t) = t$).

Lastly for $i = 0, j = 1$ (ie. the circular ordering of 1-cells and 0-cells about a cell of dimension -1) there will be a set of duplicate tuples for each 1-cell identified at the 0-cell. The duplicate set of tuples is also recognizable by the fact that $switch_2(t) = t$.

2. For $i = 1, j = 2$ (ie. the circular ordering of 1-cells and 2-cells about a 0-cell), there will be two additional tuples for which $t_2 = A$, if A has a type 2 1-pseudomanifold boundary cycle formed by identifying two 1-manifolds along a 0-cell (figure 5.14). Note that there will be two additional tuples with this condition for each 1-manifold identified at that 0-cell. This condition will also appear in the circular ordering of 1-cells and 0-cells about a cell of dimension -1 (ie. $i = 0, j = 1$).

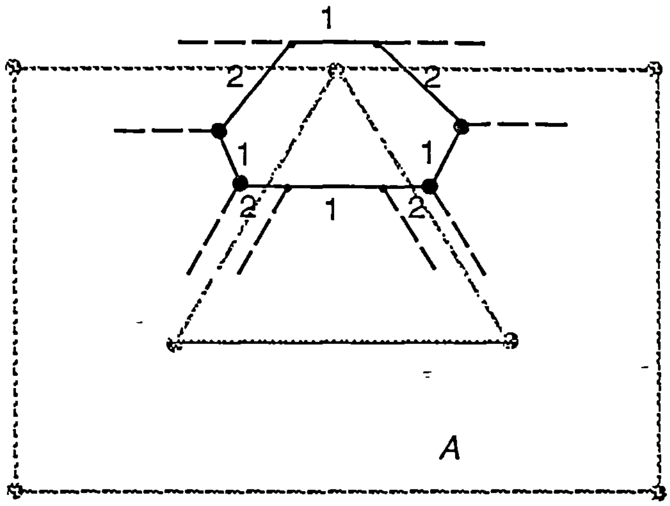


Figure 5.14 - repetition of $t_2 = A$ in the circular ordering of 1-cells and 2-cells about a 0-cell if A has a type 2 1-pseudomanifold boundary cycle (the tuples for which $t_2 = A$ are shown as shaded black dots)

If A is a part of a generalized singular 3-cell complex then each tuple takes on the *switch*$_3$ operators applicable to A.

B. Generalized Singular 3-Cells

For a generalized singular 3-cell (A), the circular ordering of 2-cells and 3-cells about a 1-cell (see **ct4** in section 4.6.3 with $i = 2, j = 3$) now incorporates conditions analogus to those given for the circular orderings of 1-cells and 2-cells about a 0-cell in a generalized singular 2-cell as well as those derived from an n-line (with/without pinch point) and a type 3 2-pseudomanifold formed by identifying two 2-manifolds along a 1-cell. These cases are now examined:

1. There will be two tuples for which *switch*$_3(t) = t$ in the circular ordering of 2-cells and 3-cells about a 1-cell if A has a type 3 2-pseudomanifold boundary cycle formed by identifying a 2-manifold and at least one 2-cell along a 1-cell. In general, the results for a 2-dimensional spatial object in R^3 (given in section 4.6.3) apply in the 3-dimensional subspace formed by A (figure 5.15). Note that the 1-cell along which the identification takes place may be internal to a 2-cell, as a result of section 5.5.1.A. This is the situation shown in figure 5.15.

   Such 2-cells can be removed from the circular ordering of 2-cells and 3-cells about a 1-cell by ignoring all duplicate tuples (ie. where switch$_3(t) = t$).

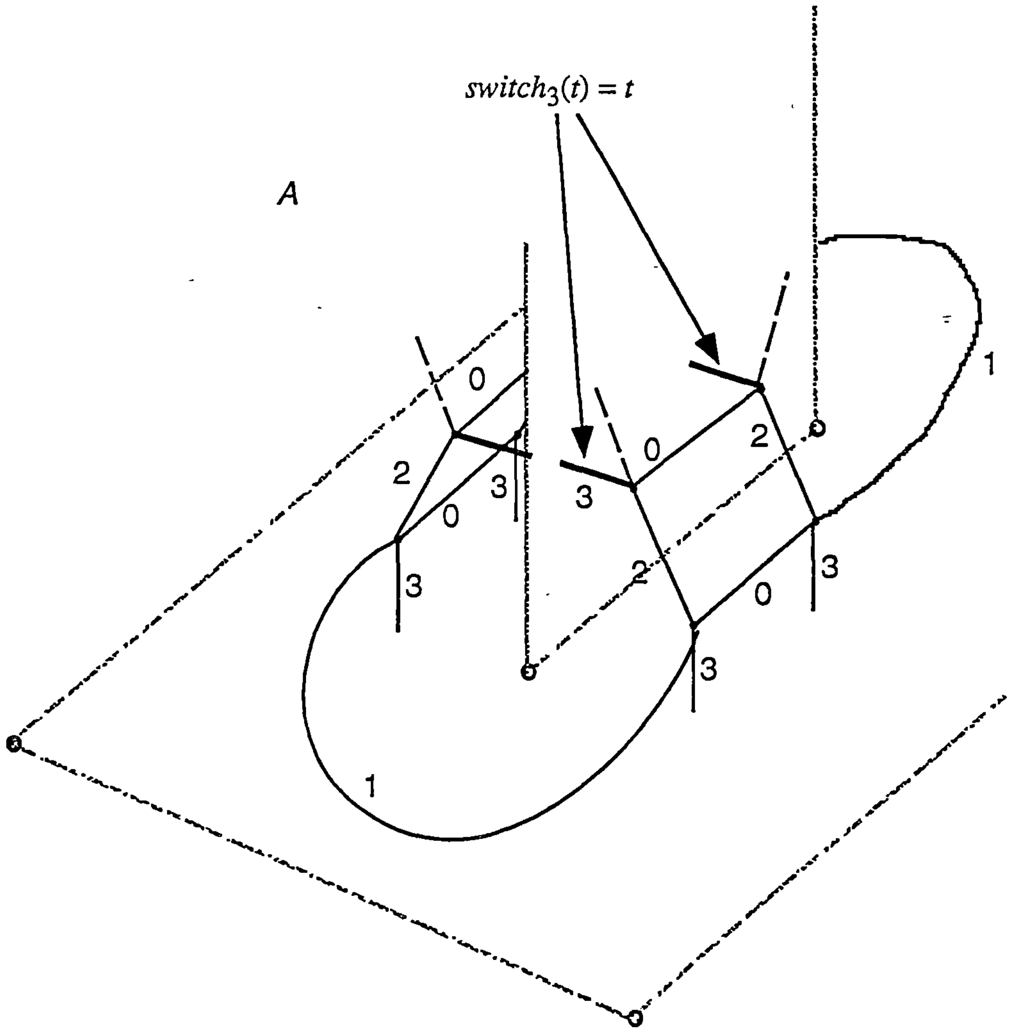


Figure 5.15 - the circular ordering of 2-cells and 3-cells about a 1-cell includes $switch_3(t) = t$ if A has a type 3 2-pseudomanifold boundary cycle. Notice that in this diagram we have shown the situation where the 1-cell along which identification occurs is internal to the boundary cycle of a 2-cell (see section 5.5.1.A).

2. If A has either:

(a) a type 1 2-pseudomanifold boundary cycle formed by identifying 2 1-cells in a 2-manifold as a 2-line or double line (figure 5.16) or double line with pinch point (figure 5.17) - see section 5.3.1.A; or,

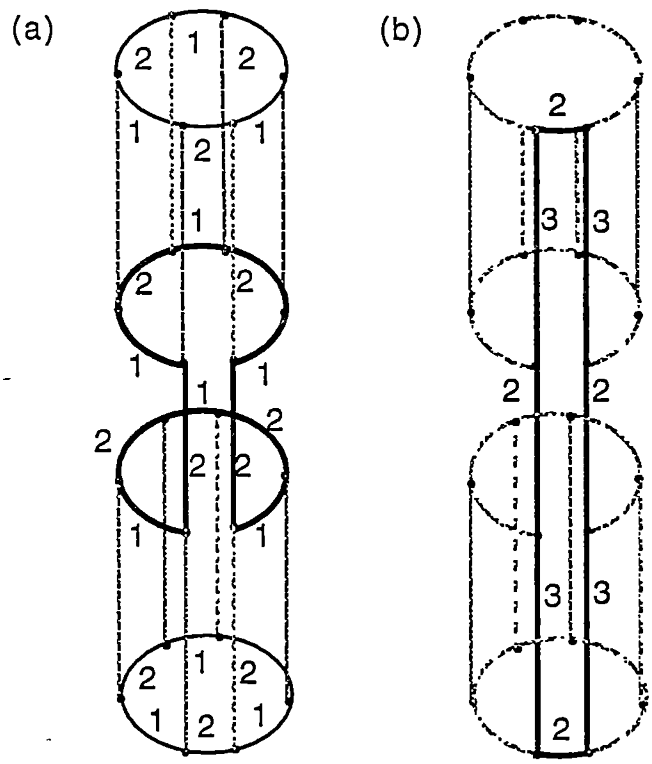


Figure 5.16 - circular orderings in the set of tuples associated with a 0-cell boundary in a double line (figure 5.7) **(a)** The inner circular ordering (highlighted) results from the identification that formed the double line. The dotted lines indicate *switch*$_3$. **(b)** the circular ordering of 2-cells and 3-cells about the double line (highlighted) - it contains four tuples for which $t_3 = A$.

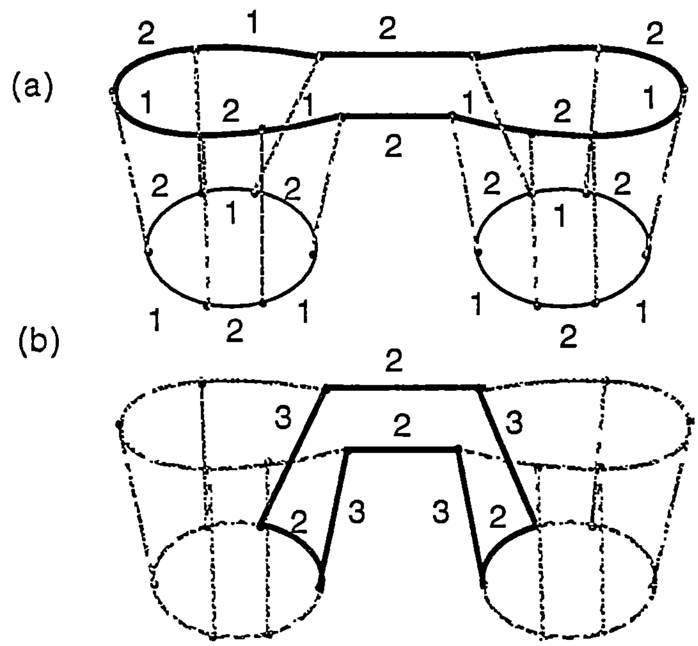


Figure 5.17 - circular orderings in the set of tuples associated with the pinch point (figure 5.8) **(a)** The outer circular ordering (highlighted) results from the identification. The dotted lines indicate *switch*$_3$. **(b)** the circular ordering of 2-cells and 3-cells about the double line - it contains four tuples for which $t_3 = A$.

(**b**) a type 2 2-pseudomanifold boundary cycle formed by identifying two 2-manifolds along a 1-cell. This particular case is indistinguishable from that of the double line in figure 5.16.

In each case the ordering of 2-cells and 3-cells about the n-line created by the identification will have four tuples for which $t_3 = A$. In general there will be $2n$ such tuples in the circular ordering of 2-cells and 3-cells about an n-line.

5.5.3 Analysis of Subspace Orderings

Recall from section 4.6.5 that for all spatial objects in R^3 there is no ordering of cobounding 1-cells (and thus 2-cells and 3-cells) in the 3-dimensional neighborhood of a 0-cell. However, the *switch* operator does capture circular orderings about the 0-cell which may be defined in any 2-dimensional subspaces (eg. the 2-manifold boundary cycles of incident 3-cells in a 3-dimensional spatial object in R^3) or two-sided orderings in a 1-dimensional subspace (eg. two incident 1-cells from a 1-dimensional spatial object in R^3). Variations on these situations given by the classification of pseudomanifold boundary cycles are as follows:

1. Cells of different dimensions may share the 0-cell, as is indicated by the type 3 2-pseudomanifold formed by identifying a 2-manifold and a k-cell (k = 1,2) along a 0-cell. Consequently, there may be circular orderings of 1-cells and 2-cells about the 0-cell in distinct 2-dimensional subspaces *and* two-sided orderings about the 0-cell between two incident 1-cells. For example, when a 0-cell belongs to the interior of a 2-cell (ie. a circular ordering) and has two cobounding 1-cells in a generalized singular 3-cell complex (figure 5.18), the two-sided ordering in the 1-dimensional subspace is not held by a $switch_1$ operation. The reason is that the uniqueness of the switch operation would be violated since some tuples would have more than one $switch_1$ operation. Instead, the two-sided ordering must be reconstructed by examining the set of tuples associated with the 0-cell - in this case $assoc(a)$.

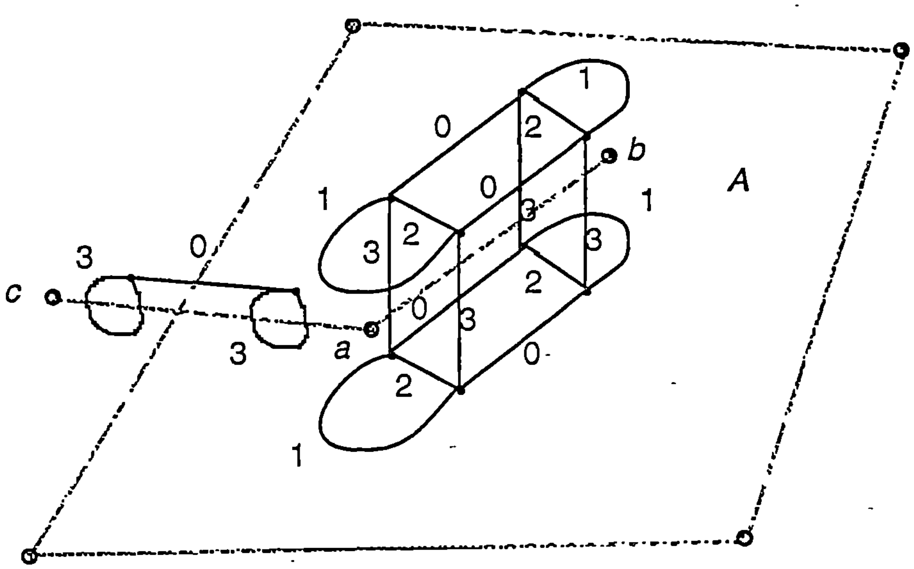


Figure 5.18 - mixture of 2-dimensional and 1-dimensional subspace orderings about a 0-cell in R^3. The 1-cell *a-b* is interior to the 2-cell *A*. The 0-cell *a* has two circular orderings (one for each side of *A*) and a potential two-sided ordering defined by the 1-cells *c*-a and *a-b*. The two-sided ordering is ignored.

2. Another variation in subspace orderings indicated by the pseudomanifold classification relates to 2-dimensional subspace orderings about a 0-cell in a 2-pseudomanifold. Given a generalized singular 3-cell (*A*), with:

   (a) a type 1 2-pseudomanifold boundary cycle formed by identifying two distinct 0-cells in a 2-manifold (double point); or,

   (b) a type 2 2-pseudomanifold boundary cycle formed by identifying two 2-manifolds along a 0-cell (indistinguishable from a double point).

   The set of cell tuples associated with the 0-cell $c_{\alpha 0}$ at the double point will have four circular orderings of 1-cells and 2-cells, instead of the usual two (figure 5.19). Notice also that all tuples in the inner circular orderings of 1-cells and 2-cells (highlighted in figure 5.19) will have $t_3 = A$ and that the graph formed by the switch operations between the set of cell-tuples associated with $c_{\alpha 0}$ (ie. $G_{Tc_{\alpha 0}}$) is not path-connected. For any *n*-point there will be $2n$ circular orderings, *n* of which have $t_3 = A$. In general, the degree of the *n*-point can be obtained from the number of circular orderings of 1-cells and 2-cells for which $t_3 = A$.

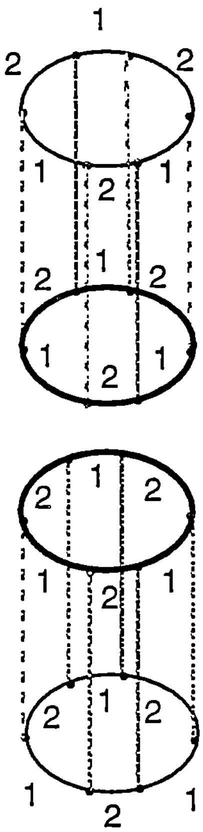


Figure 5.19 - the circular orderings of 1-cells and 2-cells about a double point in a type 1 2-pseudomanifold boundary (of a generalized singular 3-cell A) formed by identifying two distinct 0-cells in a 2-manifold (the tuples in the highlighted inner cycles all have $t_3 = A$)

5.5.4 Summary

Using the pseudomanifold classification given in section 5.4, the ordering results for the neighborhoods of cells involved in the singularities of the pseudomanifold boundaries of generalized singular 2-cells and 3-cells have been analyzed. The following table is a map between the classified pseudomanifolds and the subsections of sections 5.5.2 and 5.5.3 in which the ordering results are analyzed:

Type 1 Pseudomanifolds (from section 5.4.1)

double point	section 5.5.3 (2)
double line/double line with pinch-point	section 5.5.2.B (2(a))

Type 2 Pseudomanifolds (from section 5.4.2)

A pair of 1-manifolds identified along a 0-cell	section 5.5.2.A (2)
A pair of 2-manifolds identified along:	
(a) a 0-cell	section 5.5.3 (2)

(b) a 1-cell section 5.5.2.B (2(b))

Type 3 Pseudomanifolds (from section 5.4.3)

A 1-manifold and a 1-cell identified along a 0-cell		section 5.5.2.A (1)
A 2-manifold and a 1-cell identified along a 0-cell		section 5.5.3 (1)
A 2-manifold and a 2-cell identified along:		
	(a) a 0-cell	section 5.5.3 (1)
	(b) a 1-cell	section 5.5.2.B (1)

Edelsbrunner and Mücke (1990) have devised a symbolic technique for removing degenerate conditions from input data to geometric algorithms. The advantage of such a technique is that it removes the need to process special cases in algorithms. Loosely speaking, a pseudomanifold is a manifold with a number of special neighborhoods. Thus it would be useful to explore a topological equivalent to the simulation of simplicity (see Pigot 1994) which symbolically removes singularities from pseudomanifolds. Such a technique would be especially useful in common operations (such as display) that require traversal of the cell complex forming a spatial object.

An additional benefit of such a technique is that it may allow us to distinguish the neighborhoods of singularities of different pseudomanifolds which are not distinguishable from analysis of the associated set of tuples eg. the neighborhood of the double line/double line with pinch point is very difficult to distinguish from the neighborhood formed by identifying two 2-manifolds along a 1-cell.

Much of the overall effect on the ordering results when moving from generalized regular to generalized singular cell complexes, reflects the way in which the generalized singular cell complex is formed from the union of a number of different generalized regular cell complexes (see section 5.1). The following situations specifically highlight this point:

1. Circular orderings in a generalized singular k-cell complex incorporate the circular orderings defined for (k-1) and (k-2)-dimensional generalized regular cell complexes in R^n (**grct3.1** and **grct3.2** in section 4.6.5).

2. Cells for which only subspace orderings are possible (eg. a 0-cell of any cell complex in R^3) may be shared by cells of different dimensions. However only the subspace orderings of highest dimension are

captured by the switch operations. Lower dimensional subspace orderings must be reconstructed using the associated set of tuples. It should be noted that the lack of ordering results (and thus the failure of switch) arises from the dimension of the cell involved in the singularity, not the pseudomanifold classification. As an example, in R^3, the cells in the neighborhood of a 0-cell cannot be ordered about it regardless of whether the 0-cell is a part of generalized regular 1-cell complex such as that described in section 4.6.2 or a double point in the 2-pseudomanifold boundary cycle of a generalized singular 3-cell.

3. Cell complexes internal to the pseudomanifold cycles of a generalized singular cell take on both the circular orderings relevant to the Euclidean interior of the cell and those of the cell itself if the cell does not have the same dimension as the world cell. For example, a 0-cell internal to the boundary cycles of a 2-cell in a generalized singular 3-cell complex firstly obtains the circular ordering applicable to a 0-cell in R^2 (section 4.6.1) and then the *switch*$_3$ operations applicable to the 2-cell (see section 5.5.1.A).

The other variations are much less obvious. For example, the ways in which the subspace circular orderings about 0-cells change in the neighborhoods of pinch points etc. Such variations are much more likely to result from applications where spatial objects are deformed (eg. time based deformation of a 3-dimensional object) or abstract visualizations of non-spatial variables.

Lastly, the pseudomanifold classification described in section 5.4, which forms the basis for the analysis of the ordering results described in this section, ensures that $G_{T_{R^n}}$ (as defined by the *switch* operator between cell tuples) encapsulates all the remaining variations in the cusp entity described by Gursoz et. al. (1991) (see section 4.6.6) and some additional situations not mentioned in their description of the tri-cyclic cusp (eg. *n*-lines and *n*-lines with pinch point in section 5.3.1). As an example consider the three 'disk cycles' (ie. one of the cycles in the tri-cyclic cusp) shown in figure 5.20. The situation may be described as a type 2 pseudomanifold formed by identifying a 2-manifold and a 1-cell along a common 0-cell labelled *a*. The disk cycles shown for *a* are encapsulated by *switch*$_1$ and *switch*$_2$ operations between cell-tuples associated with *a* (ie. $t_0 = a$) and *switch*$_3$ for all tuples in *assoc*(*a*) which are also associated with the 1-cell (ie. t_1 = the 1-cell).

5.6 Arcs: Optimizing Representation of Topological Relationships

In describing the generalized regular and generalized singular cell complexes we have concentrated on optimizing both the representation of geometric structure and the topological relationships and properties, by defining the generalized regular (singular) n-cell as a Euclidean n-manifold with (n-1)-(pseudo)manifold boundary cycles. In this section we concentrate on optimizing the representation of the topological structure and relationships. To do this we examine the notion of an *arc* - see for example Hocking and Young (1961) and Takala (1991).

The 1-dimensional arc or 1-arc has long been used in 2-dimensional GIS. For examples, see the DIME segment in Corbett (1979) and the arc-node model itself (Aronoff 1989). Higher dimensional equivalents to an arc are not so well-known. Takala (1991) gives a theoretical discussion of simplicial n-chains and n-arcs and Corbett (1985) describes a cell which at first glance appears to be a generalization of the notion of the 1-arc, but the essential idea is lost when maps for describing their geometry are introduced. To gain a better perspective on an appropriate generalization, we re-examine the notion of a 1-arc from a topological point of view.

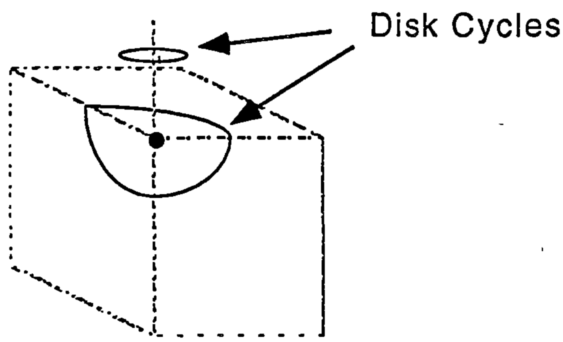


Figure 5.20 - three disk cycles of the tri-cyclic cusp (the disk cycle interior to the boundary of the subdivided 2-manifold is not shown).

In a 2-dimensional GIS, a 1-arc effectively 'replaces' a 'string' of 1-cells, all of which have the same coboundary information (ie. left and right polygon), by a 1-manifold with boundary. This 'replacement' strategy turns out to be very flexible because as is indicated in section 3.3, the boundary of a 1-manifold may be a single 0-cell (in which case an arc replaces a ring of 1-cells) or two 0-cells (the arc replaces a chain of 1-cells). 1-arcs have proven to be very useful because:

1. 1-arcs are a 'compression' of both the boundary and coboundary relationships which would normally have to be represented by each of

their component 1-cells. Figure 5.21(a)-(d) shows an example of a 1-arc and the compression it provides.

An important consequence of this compression is faster access to topological relationships between cells (ie. information held in the cell neighborhoods) since there will be fewer cells (figure 5.22).

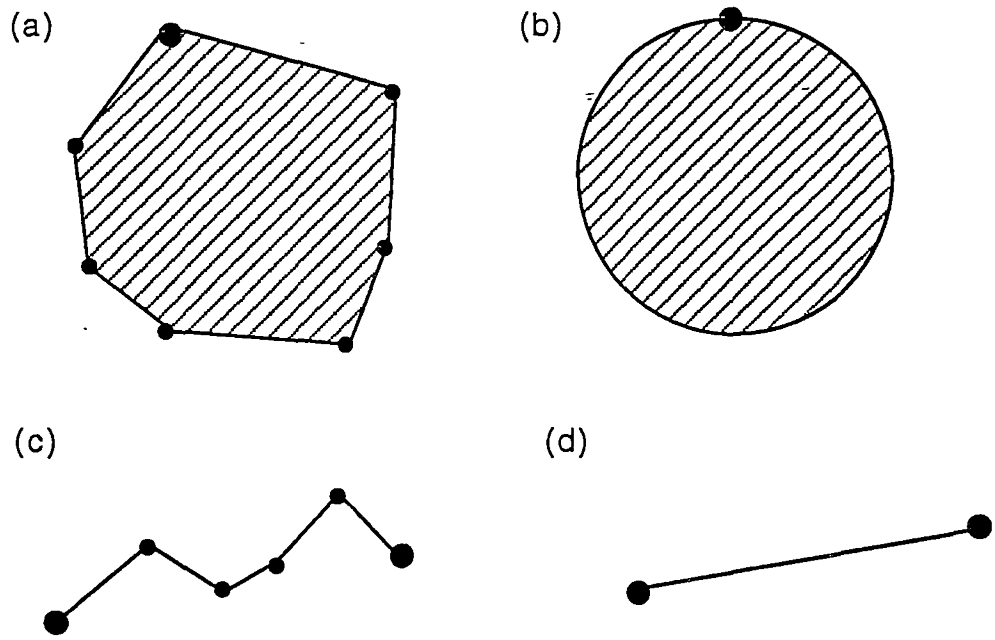


Figure 5.21 - the compression of topological information provided by the 1-arc in a 2-dimensional GIS (a) a polygon (b) the polygon boundary replaced by a 1-arc (c) a chain of 1-cells (d) the chain of 1-cells replaced by a 1-arc

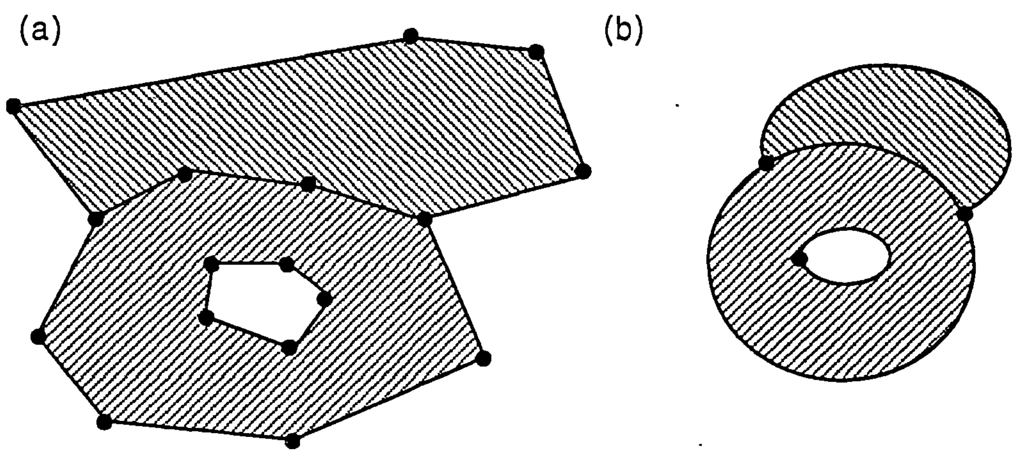


Figure 5.22 - comparing the geometric and topological representations (a) the cells of a geometric representation of a 2-cell complex in GIS (b) the topological representation. The key point of this example (and figure 5.21) is that the arcs really do not carry any geometric structure at all ie. they are a topological representation only

Note that geometric structure is unimportant in 5.22(b); the only purpose of the topological representation is to capture the topological relationships between cells. In fact we can think of each arc as member of an 'abstract' arc complex because if we require a physical realization of an arc or the arc complex, then we refer to the geometric structure.

2. Arcs can be represented using the same data structures as a cell (see Corbett 1979 and Corbett 1985 for example).

In conventional topological terms, a 1-arc may be represented by a 1-cell from a normal CW complex (see section 4.2). Recall that the characteristic map given by condition **cw1** in section 4.2 ensures that the interior of such a 1-cell is the homeomorphic image of the interior of a 1-disk but its boundary need only be the continuous image of the boundary of the 1-disk; ie. the two 0-cells in the boundary of the 1-disk may be identified by the characteristic map. Figure 5.23(a) shows the 1-cell which results from a continuous image whilst figure 5.23(b) shows a result of the homeomorphic image.

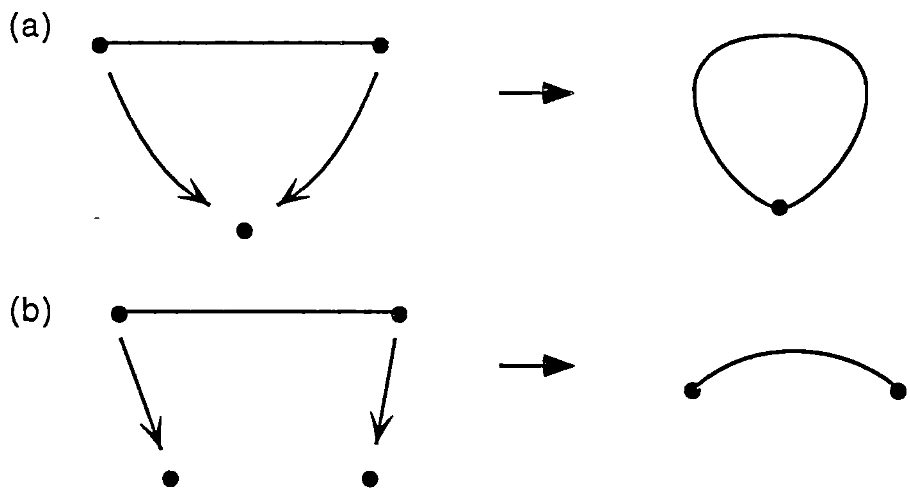


Figure 5.23 - two cases of a 1-cell in a normal CW complex (a) when the characteristic map is continuous with respect to the boundary of the prototypical 1-cell (b) when the characteristic map is homeomorphic with respect to the boundary of the prototypical 1-cell.

With these facts about 1-arcs in mind we can now determine the desirable properties of a 2-dimensional arc or 2-arc:

1. Replacement of 'walls' of 2-cells each of which has the same coboundary or two-sided ordering information. For maximum

flexibility (particularly when dealing with triangulated surfaces) it would be preferable if a 2-arc could replace the entire boundary of a 3-cell.

2. Trivial (or at least easily determined) topological properties.

From these desirable properties it is clear that orientable 2-manifolds with boundaries are natural candidates for use as 2-arcs since they are both well-known (see the classification in section 3.3.1) and sufficiently general to provide effective compression of the coboundary information held by the 2-cells of any cell complex.

As was the case for the 1-arc, a 2-arc may also be represented by a 2-cell from a normal CW-complex. The characteristic map in condition **cw1** (section 4.2) ensures that the interior of such a 2-cell is the homeomorphic image of the interior of a 2-disk, but its boundary need only be the continuous image of the boundary of the 2-disk; ie. the 1-cycle forming the boundary of the 2-disk may be identified by the characteristic map.

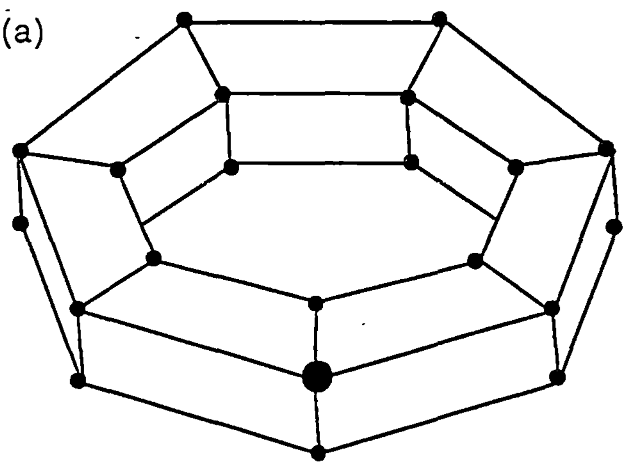
As an example, the 2-manifold boundary cycle of a generalized regular 3-cell may be completely replaced by a single 2-arc if there are no other adjacent 3-cells (figure 5.24).

Naturally we would also like to use 1-arcs and 2-arcs in the generalized singular 3-cell complex. To do this it is sufficient to modify the definition of the 2-arc and 2-arc complex to take into account interior 1-arc and 0-arc complexes and permit 2-arcs to have type 1 pseudomanifold identifications. The result is the singular 2-arc and singular 2-arc complex.

1. A singular 2-arc may have a 1-arc complex or 0-arc complex internal to its boundary cycles (see the same requirement for generalized singular cells and the results of section 5.5.1). This permits other 2-arcs and 1-arcs to intersect it along 0-arcs and 1-arcs (figure 5.25).

2. A 2-arc may have type 1 pseudomanifold identifications (section 5.4.1). To represent such cases the cells involved in the identification would need to be duplicated according to the degree of the identification. For example, a double point would be represented by two identical 0-cells internal to the boundary cycle(s) of the 2-arc.

(a)



(b)

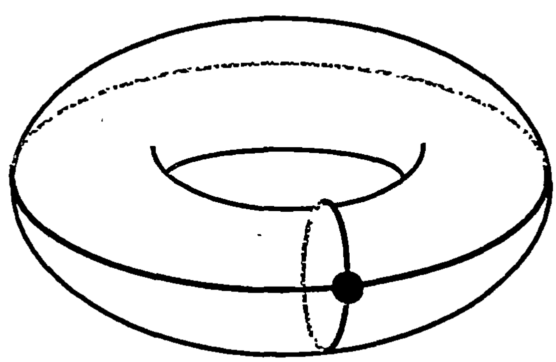


Figure 5.24 - a 2-arc representing a torus. (a) the generalized regular 3-cell complex (ie. the representation of both geometric and topological structure) and (b) the 2-arc representing only the topological structure. The 2-arc is equivalent to the 2-dimensional normal CW complex representation of the torus given in figure 4.5 in section 4.2.

(a) (b)

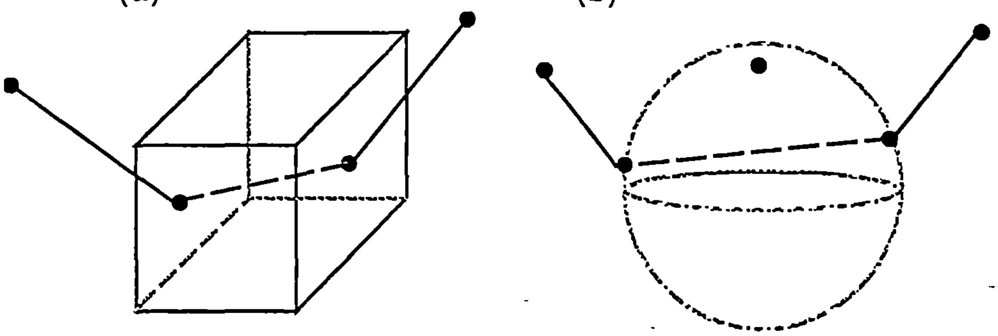


Figure 5.25 - a simple generalized singular 3-cell complex and the 2-arcs and 1-arcs that result (**a**) the generalized singular 3-cell complex which consists of one 3-cell and three 1-cells (**b**) the singular 2-arc and three 1-arcs - in addition to one 2-cell and its boundary (a 0-cell), the singular 2-arc has two interior 0-cells that result from intersections with the 1-arcs

There are three main advantages that would result from the application of arcs to generalized regular and singular cell complexes:

1. The boundary-coboundary information for the 2-skeleton of the subdivided 2-manifold boundary cycles of the 3-cells in the generalized regular 3-cell complex is effectively held twice by the cell-tuple - on both sides of the 2-cell (see section 4.6.4). If 2-arcs are introduced, then they capture the information about cobounding 3-cells and thus it is no longer necessary to represent this information in the cell tuples. The effect is that the number of tuples is halved and the 3-cell component of each tuple (t_3) is no longer required.

2. In line with normal CW-complexes, 2-arcs capture a minimum of geometric structure. This ensures that their topological properties are easily calculated since the fundamental group is often held directly within the boundary of the 2-arc (see section 4.2). Together with fast access to topological relationships between arcs, this allows quicker calculation of important topological properties (such as connectivity) particularly for spatial objects whose boundaries are subdivided manifolds (many solid objects have 2-manifold boundaries).

3. Arcs would facilitate rapid reconstruction (based on their compression) of many of the global topological elements of the explicit models described in chapter 2 eg. shells in the tri-cyclic cusp of Gursoz et. al. (1991) and the radial edge of Weiler (1986).

The only disadvantage of using 2-arcs is that two distinct cell complexes will be required: the generalized regular (or singular cell) complex which represents the geometric structure, and the arc complex which represents the fundamental topological structure and some topological relationships. In a 2-dimensional GIS, only the arc complex is necessary - generalized regular and/or singular cell complexes are not required because the boundary-coboundary relationships of the 1-cells 'replaced' by the 1-arc can be trivially represented as a string of coordinates. See the 1-cells and the 1-arcs that replace them in figures 5.21 and 5.22 above. The reason that the 2-arc complex cannot replace the generalized singular and regular cell complexes in 3-dimensional applications is that the 2-cells represented by the 2-arc do not have trivial boundary-coboundary relationships.

Arcs are introduced as a possible solution to the expected problems in querying topological relationships when dealing with a large number of cell-tuples eg. when representing spatial objects with complex geometric structures. There are obviously many issues involved with their implementation that are not covered here. Some of the most important issues to be investigated in future research are:

1. A data structure for representation of arcs and the arc complex. At present a simple model which holds the unordered boundary and coboundary information for a cell may be sufficient (eg. Corbett 1985 or Rossignac & O'Connor 1991). Given that the arc complex is a restricted form of the normal CW complex, more research on the efficient representation of normal CW complexes (as also noted in Lienhardt 1991) will be of benefit.

2. Maintaining the connection between the cell tuples representing the generalized regular (singular) 2-cells and the (singular) 2-arc that replaces them in the arc complex. This connection may be amenable to an inheritance mechanism. One possible direction for the implementation and maintenance of this connection could be an extension of the principles underlying the object-oriented database models of Milton et. al. (1993) or Vijlbrief & van Oosterom (1992).

Arcs and the arc complex encapsulate the important topological relationships and properties only. The main focus of this research is much more general, since the generalized regular and singular cell complex encapsulate both the representation of the geometric structure and the topological relationships and properties.

Chapter 6
Topological Operators

6.1 Introduction

So far this research has focused on the definition and representation of spatial objects in an attempt to define a topological model. The last part of this research deals with a set of topological operators for constructing these spatial objects.

As mentioned in chapter 1, a minimum, generic set of topological construction operators must be able to construct each k-cell (ie. a Euclidean k-manifold with (k-1)-manifold boundary cycles) and to join these k-cells along subsets of their boundaries to build a generalized regular k-cell complex (representing a k-dimensional spatial object) in a Euclidean n-manifold where $0 \leq k \leq n \leq 3$.

The generic cell complex construction operators described by Corbett (1985) and Brisson (1990) typify this approach and its advantages; ie. they constitute a small set of operators and are both data structure and application independent. The major problem in the construction of spatial objects with such topological operators is *consistency*. An object is consistent if it falls within the domain of the topological model and has the required topological properties eg. the correct number of 'holes' and/or the dimension of neighborhoods. In this research, the problem is compounded by the fact that the domain of spatial objects that we have chosen has very few restrictions that could be used for consistency purposes. Thus, the best we can achieve is to maintain consistency for a subset of spaces in this domain: the subdivided k-manifolds and k-manifolds with boundary ($k \leq 2$). This subset (which includes generalized regular k-cells and the boundary cycles of generalized regular

3-cells) can then be used as 'building blocks'; ie. they are joined to construct a spatial object.

To achieve consistency when constructing subdivided manifolds and manifolds with boundary, we restrict the generic cell complex construction operators such that they preserve a very general topological invariant known as the homotopy type (see section 3.3 for a definition). The theory that underlies the modification is a combinatorial version of homotopy equivalence originally put forward by the mathematician J.H.C. Whitehead for CW-complexes. We will use these combinatorial homotopy operators to show that any subdivided manifold with boundary can be constructed from a very simple space that encapsulates all its topological information: its strong deformation retract (see section 3.3.2). To construct a subdivided manifold consistently, we firstly construct a subdivided manifold with boundary and then add a 'closing' face. Lastly, we show that the local Euler operators (Mäntylä 1988) are in fact a very simple form of combinatorial homotopy.

Next, we describe a link between the generating 1-cycles of the fundamental group, combinatorial homotopy and an algorithm for reconstruction of the topology of a subdivided 2-sphere from a 'wire-frame' (or 1-skeleton) originally specified by Ganter (1981). This link is based on the fact that the first stage of Ganter's algorithm is actually the Tietze method (see section 4.4 and Stillwell 1980 pg. 137) for finding the generating cycles of the fundamental group of a graph (sometimes known as the fundamental cycles).

After establishing this link we use it to extend the domain of Ganter's algorithm to reconstruction of the topology of any subdivided 2-manifold or 2-manifold with boundary, from its 1-skeleton. The extension is based on the fact that the second part of Ganter's algorithm reduces the fundamental cycles returned by the Tietze method to a minimum length basis set of 1-cycles, where 'minimum length' implies minimum number of edges in each 1-cycle. Such 1-cycles usually match the boundary cycles of the faces or 2-cells in a 2-sphere embedding. If the embedding is not in a 2-sphere, but in a torus for example, then 1-cycles that generate the fundamental group (and/or those homotopic or 'equivalent' to them) will be present in the minimum length basis 1-cycles. Prior knowledge of the generating 1-cycles (not null homotopic) allows us to distinguish them from those that form the boundaries of faces (null homotopic). The face boundary cycles returned by the algorithm can then be joined to a 1-skeleton containing the generating 1-cycles (and to each other) to form a 2-manifold with boundary, using the combinatorial homotopy operators outlined above. If the space is actually a subdivided 2-manifold, then we need only add a 'closing' face to the subdivided 2-manifold with boundary.

In summary, the rest of this chapter is laid out as follows. Section 6.2 gives a review of existing topological operators, their extension to generalized regular cell complexes (ie. spatial objects) and an overview of the combinatorial homotopy operators. Section 6.3 describes a combinatorial notion of homotopy equivalence as a strategy for constructing a subdivided k-manifold with boundary ($k \leq 2$) from a 1-cell complex containing the deformation retract (ie. an appropriate bouquet of loops). The results of this investigation are the combinatorial homotopy operators. Section 6.4 puts forward the link between the topology reconstruction algorithm of Ganter (1981) for 2-spheres, the generators of the fundamental group and the combinatorial homotopy operators. This link is then used to extend this algorithm to subdivided 2-manifolds and 2-manifolds with boundary.

6.2 Review of Existing Topological Operators

6.2.1 The Euler Operators for Subdivisions of 2-Manifolds

For computer-aided design (CAD) systems such as the geometric work bench (GWB) of Mäntylä (1988), the emphasis is on 2-manifold boundary models. Since 2-manifolds are well known spaces in topology there are many simple invariants that can be used to control topological operators. The invariant usually used for this domain is a modification of the Euler-Poincaré equation (originally given in Baumgart 1974, but see Mäntylä 1988):

$$v - e + f = 2(s - h) + r \tag{6.1}$$

where v = number of vertices (0-cells), e = number of edges (1-cells), f = number of faces (2-cells), s = number of shells (2-cycles), h = genus of shell (+1 for every torus) and r = number of non-simply connected faces or rings (ring = annulus or 2-cell with more than one boundary cycle).

By equation 6.1, all operators are constrained to the 2-manifold domain.

The most primitive operator, *make-vertex-face-shell* (or MVFS), creates a skeletal 2-sphere (the simplest 2-manifold) consisting of a single vertex, face and shell.

Operators which add additional structure to the skeletal 2-sphere without changing its global topological properties (ie. they preserve the homeomorphism type of the 2-sphere) are known as local operators. To avoid changes to global topological properties, local operators may only change the terms in the left hand side of equation 6.1. The useful permutations are *make-edge-vertex* (MEV) and *make-edge-face* (MEF). The specific effects of the local operators can be summarised as follows:

1. Application to the skeletal 2-sphere can only construct a complex homeomorphic to a 2-sphere (figure 6.1). In particular, recall that MVFS always creates a skeletal 2-sphere with one vertex and one face (a normal CW complex - see section 4.2). As a result Mäntylä (1988) (and others) define the MEV and MEF operators as vertex and face 'splitting' operators (respectively).

2. Application to any existing subdivided 2-manifold preserves its homeomorphism type.

Global operators are defined firstly to create non simply connected 2-manifolds by changing the homeomorphism type of the 2-manifold (eg. create a torus from a sphere) and secondly to glue two 2-manifolds together to form another 2-manifold. Both operations are analogous to a process known in topology as the connected sum (see Mäntylä 1988 pg. 143). To perform the connected sum whilst preserving equation 6.1, it is necessary to create a 'hole' in a face; ie. turn a simply connected face into an annulus or *ring* and perform the connected sum along the boundary cycle of the 'hole' - the inner boundary cycle of the ring.

To construct a ring whilst preserving equation 6.1, an internal edge in the interior of the face is created using MEV. The edge between the two vertices is then removed (KEMR) and the ring can be constructed on the interior vertex (figure 6.2). As is stated in Mäntylä (1988), the *kill-edge-make-ring* (KEMR) operator is a 'convenience' operator to facilitate the formation of the connected sum.

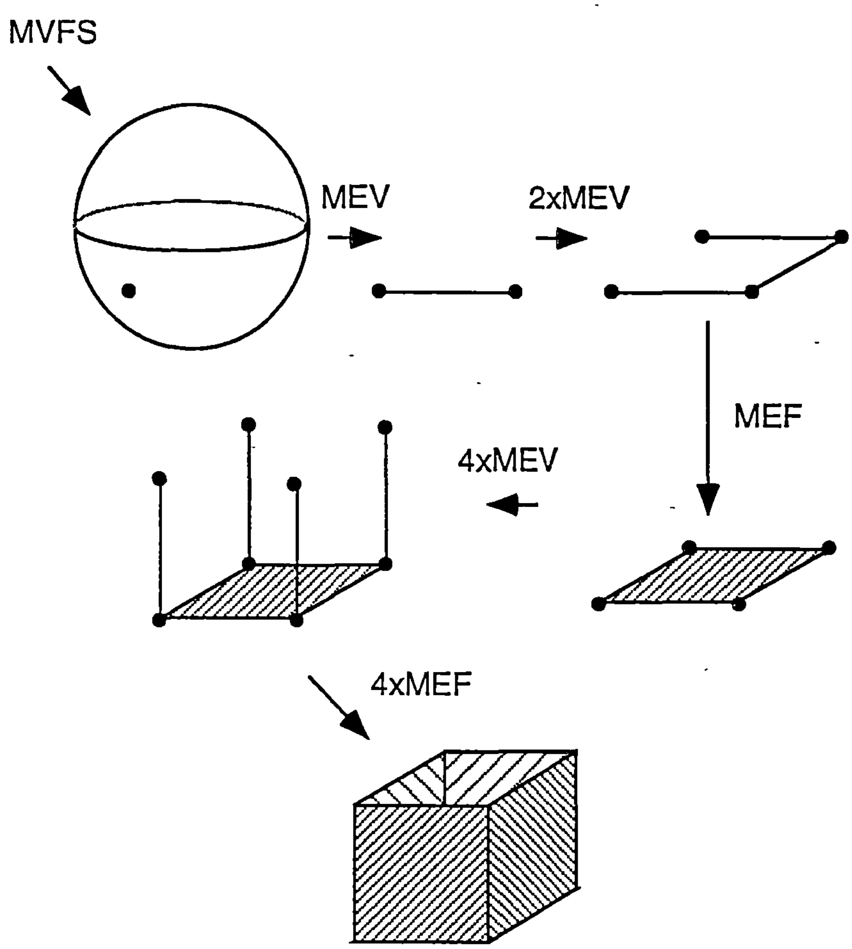


Figure 6.1 - construction of a cube or box using the local Euler operators. A skeletal 2-sphere is created using MVFS. Notice how the effect of the MEV operator is to 'split' the initial vertex (then others) into two vertices by creating an edge whilst the MEF operator 'splits' faces (in this example only the initial face is split). The face that seems to be missing from the box is actually the initial face created by MVFS since the whole process has taken place 'on' the 2-sphere

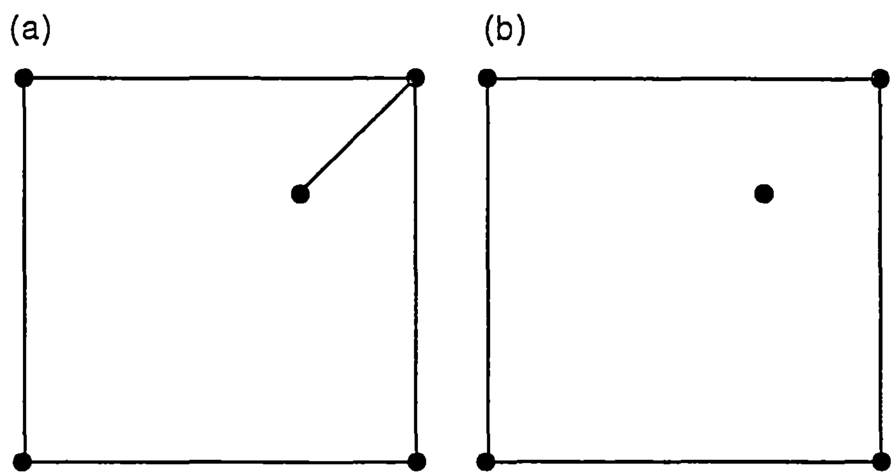


Figure 6.2 - preparing to change a face into an annulus (KEMR)

With the mechanism for changing a face into an annulus (or ring) in place, a 2-sphere can now have its genus changed ie. become a torus. In figure 6.3, a ring is constructed in a face of the cube from figure 6.1, then a square 'cylinder' is constructed on the boundary cycle of the hole using a sequence of MEV and MEF operators. Lastly, the face at the end of the new structure is removed and a ring and hole added via the *kill-face-make-ring-hole* (KFMRH) operator. Loosely speaking figure 6.3 represents an internal form of the connected sum operation; ie. between two faces of the same shell. An external form of the connected sum would be defined in a similar way except the two faces would belong to different shells.

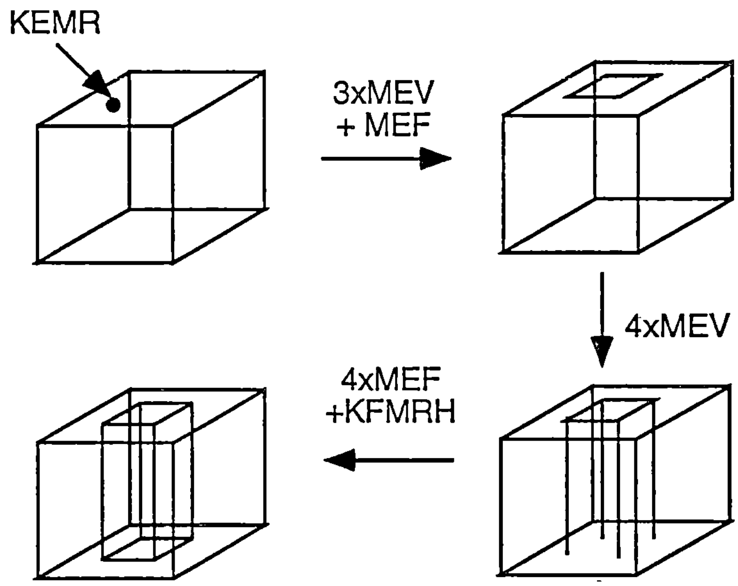


Figure 6.3 - the internal connected sum. The Euler-Poincaré equation is preserved by forming a ring in a face, then adding the shell bounding the hole to the boundary 1-cycle of the ring. Lastly the face that the emerging hole meets is replaced by a ring with KFMRH

Hanrahan (1985) extended the local Euler operators to the construction of subdivided 3-manifolds using the 3-dimensional form of the Euler-Poincaré equation. He implemented a third local operator: *make-face-cell* (MFC) and its inverse *kill-face-cell* (KFC) where a 'cell' is defined as a regular 3-cell (see section 4.2). However the 3-dimensional form of the Euler-Poincaré characteristic does *not* define an integer invariant for subdivisions of 3-manifolds (and for higher dimensions) so no global operators could be defined - see the discussion of the unsolved Poincaré conjecture in Stillwell (1980) pg. 246.

It is important to remember that the Euler operators are topological operators only; ie. they construct the cells and the relationships between the cells. Geometry (eg. the assignment of coordinates to the 0-cells) is not considered..

6.2.2 Generic Cell Complex Construction Operators

For n-dimensional geometric modelling (ie. confined to spatial objects of dimension n) topological invariants cannot be relied upon because of the difficulty in classifying manifolds of dimension ≥ 3. Consequently generic cell complex construction operators are simple methods for creating cells and then attaching them to other cells in order to form the cell complex.

Corbett (1985) specifies two operators: *identify* which identifies all the cells in two homeomorphic structures (either from the same cell or from two distinct cells) and *create* which when given a set of (n-1)-boundary cycles, creates an n-cell by 'filling in the interior' of these boundary cycles. These two operators are independently described in later research by Brisson (1990), who called them *join* and *lift* respectively. Brisson uses the term *lift* because the process of creating an n-cell involves building the (n-1)-boundary cycles and raising the dimension to fill in the interior. Examples of the operators described by Corbett (1985) and Brisson (1990) are shown in figures 6.4 and 6.5.

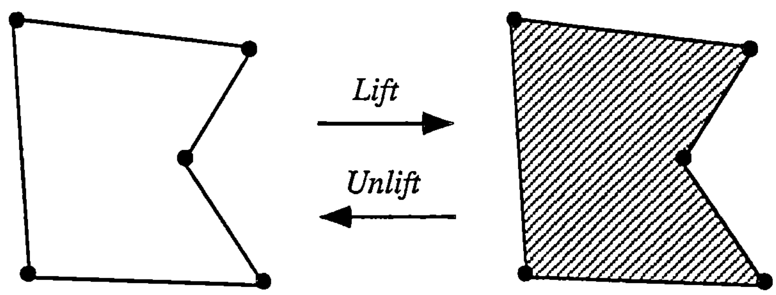


Figure 6.4 - the *lift* (and *unlift*) operator (or *create/erase* in Corbett 1985)

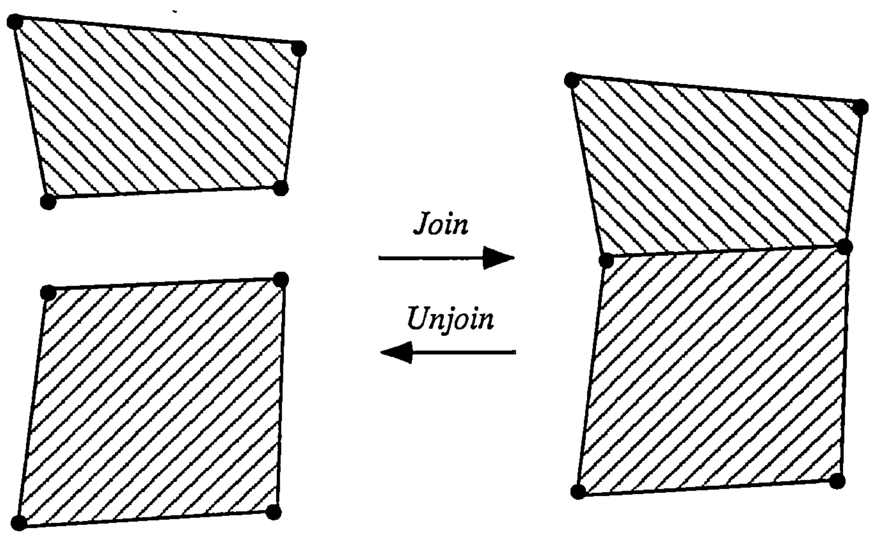


Figure 6.5 - the *join* (and *unjoin*) operator (or *identify* in Corbett 1985)

Brisson's *join* operator is restricted to joining a pair of n-cells along two (n-1)-cells in their boundary cycles in order to ensure that the result is a strict partition of an n-manifold or n-manifold with boundary (see the result of Lundell & Weingram 1969 pg. 82). Corbett's equivalent *identify* operator has no such restrictions. As an example, he describes the construction of a cylinder from a rectangular 2-cell by applying the *identify* operator to opposite edges of the 2-cell - see Corbett (1985) pg. 16. Despite the additional generality of Corbett's operators, we use the names and operators given by Brisson since these operators and the cell-tuple are extended in this research. It is surprising that the simplicity and elegance of Corbett's research (both on topological operators and topological models) was not realized except in independent research carried out some five years later.

In summary, the generic topological operators necessary to construct any k-cell complex are:

1. Cell creation by performing a *lift* on their boundary cycles (eg. *lift* on a (-1)-cell creates a 0-cell, *lift* on two 0-cells creates a 1-cell, *lift* on a set of 1-cycles creates a 2-cell etc.)

2. Joining or attaching k-cells along (subsets of) their boundary cycles eg. (k-1)-cells etc.

Despite the restrictions on Brisson's domain, we will describe his implementation of the *lift* and *join* operators, since our intention is to extend the cell-tuple to both the representation and construction of generalized regular cell complexes (construction of generalized singular cell complexes is left for future research). The implementation of the *join* and *lift* operators using the cell-tuple structure is dependent upon the following facts about G_T (ie. the graph formed by the tuples and switch operations - see section 4.6):

1. That each node of the graph G_T has a single edge corresponding to an i-dimensional switch operation where $i \leq k$.

2. Excluding all i-dimensional *switch* operations separates G_T into connected components each of which may be associated with an i-dimensional cell.

Based on these properties, Brisson defines a function $trav(t,\{0,.....,k\})$. Given a tuple t, *trav* traverses all tuples obtainable from t by repeated applications of $switch_i$ operations where $i \in \{0,....,k\}$.

Following Brisson (1990), the *lift* operation, for creating a k-cell A, takes as input a cell tuple t in the (k-1)-skeleton (t has dimension k-1). Then for each $t' \in trav(t,\{0,....,k-1\})$, the k-dimensional component of t', t_k', is set to A and $switch_k(t')$ is set to a special value indicating the boundary of the cell (figure 6.6).

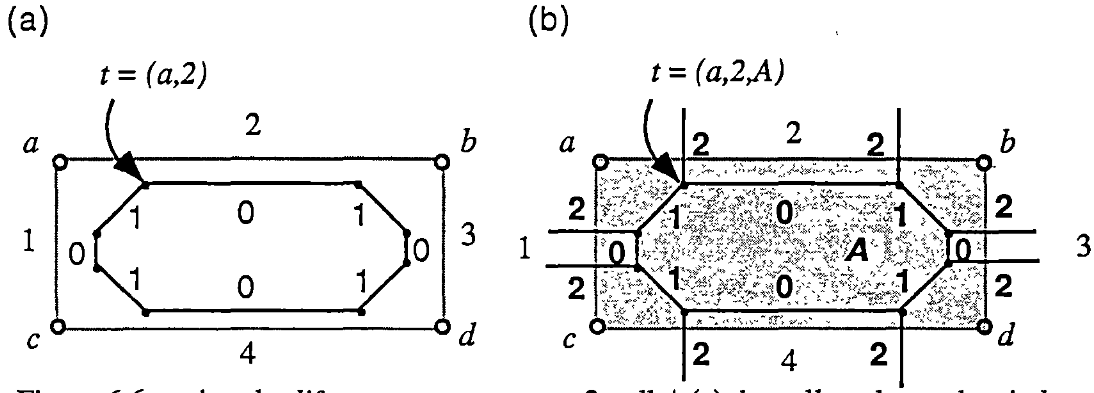


Figure 6.6 - using the *lift* operator to create a 2-cell A (a) the cell-tuples and *switch* operations *before* the *lift* operator is applied - $trav(t,\{0,1\})$ returns all tuples (including t) in the 1-skeleton (b) *after* the *lift* operator is applied

Notice that the *lift* operator does not create all the cell-tuples that would normally be associated with a k-cell; ie. the 'exterior' tuples are not created since $switch_k$ is set to $\varnothing$ (indicating the boundary of the cell) in all cell-tuples (figure 6.6(b)). The underlying reason for the partially created cell-tuple structure is the restriction of the domain to subdivided manifolds and manifolds with boundary. In a subdivided k-manifold or k-manifold with boundary, pairs of k-cells may only be joined along (k-1)-cells - there is no need to create any more tuples, since they would unnecessarily complicate the definition and usage of the *join* operation.

Following Brisson (1990), the effect of the *join* operation is to identify two (k-1)-cells a and b in the (k-1)-skeletons of two k-cells A and B. The *join* operator takes as input two cell-tuples t^1 and t^2 where $t^1 \in assoc(a)$ and $t^2 \in assoc(b)$. If we let list $l_1 = trav(t^1,\{0,....,k-2\})$ and list $l_2 = trav(t^2,\{0,....,k-2\})$ then for each tuple t in l_1 and t' in l_2, the *join* operator sets $switch_k(t) = t'$ and $switch_k(t') = t$ (figure 6.7). The last step of the *join* operation makes the m-dimensional components t_m ($m \leq k$-1) of all tuples associated with the identified (k-1)-cells equivalent.

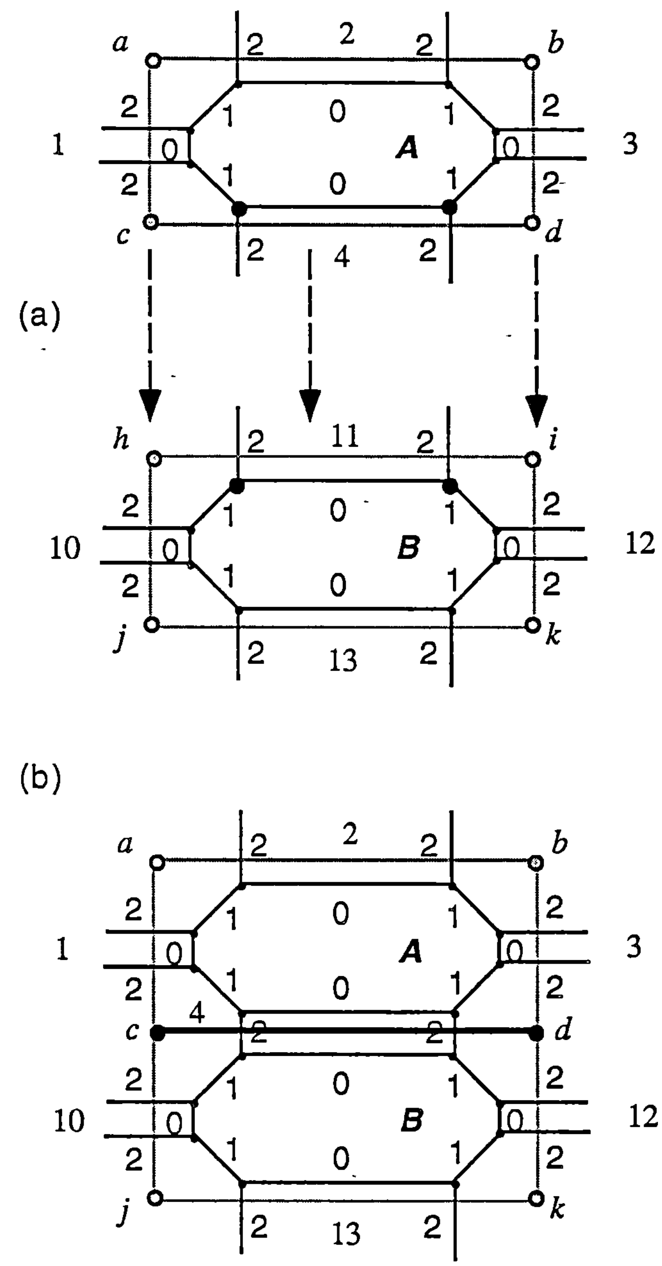


Figure 6.7 - implementing the *join* operator on a pair of 2-cells (**a**) the cell-tuples and *switch* operations representing the two 2-cells before the *join* operator is applied ($trav(t, \{0\})$ returns the highlighted tuples) (**b**) after the *join* operator is applied - note that the 0-dimensional components of the tuples originally associated with the 0-cells *h* and *i* have been changed to *c* and *d* respectively and the 1-dimensional components of the cell-tuples originally associated with the1-cell 11 are now equal to 4.

Brisson introduces one additional operator specific to regular CW-complexes: *split* which together with its inverse, *unsplit*, is a useful convenience operator for editing and manipulating regular CW-complexes. We ignore such operators in this research, since the focus is a set of basic operators for construction of cell complexes. Manipulation and editing operations are left as a subject for future research.

For GIS applications, an operation similar to *lift* (and its inverse, *unlift*) has been defined for insertion and editing operations on spaces subdivided by simplicial complexes in Egenhofer et. al. (1989). In such cases the domain is restricted to either a 2-sphere or Euclidean 2-manifold with one or more boundary cycles (a map) and the effect of the operators on the underlying space is the same as that of the local Euler operators of Mäntylä (1988) ie. they cannot change its topological properties.

Egenhofer's operators are designed to preserve the entire simplicial structure of the complex instead of invariants like the Euler-Poincaré characteristic. The advantages of Egenhofer's operators are simplicity and the formal demonstration of their 'correctness' based on the simplicial axioms (based on axioms originally given in Frank and Kuhn 1986). We do not consider an extension of the global enforcement of a simplicial complex from digital terrain and map models to 3-dimensional applications in this research (see section 4.2 for some reasons).

A different approach to the representation and manipulation of triangulations in 2-dimensional GIS applications which yields a dynamic approach to the construction of a 2-cell complex from digitized lines, is the Voronoi diagram (a 2-cell complex): the dual construction of a triangulation in which the Delauney criterion is enforced (see Saalfeld 1987, Gold 1992 & 1994). Operations on the cells of the Voronoi diagram described in Chris Gold's papers appear to be similar to the 'splitting' effects of the local Euler operators as defined in Mäntylä (1988) and reviewed above. The relationship between these operators and the extension of this Voronoi-based methodology to 3-dimensional applications could be investigated in future research.

Once again, it is important to note that all the generic cell complex construction operators are topological operators only; ie. they construct the cells and the relationships between the cells. Geometry (eg. the assignment of coordinates to the 0-cells) is not considered.

6.2.3 Extending the Generic Cell Complex Construction Operators

In this section we extend the generic cell complex construction operators (*lift* and *join*) reviewed in the last section the problem of constructing a generalized regular

k-cell complex (ie. a k-dimensional spatial object) embedded within a Euclidean n-manifold ($0 \leq n \leq k \leq 3$). There are three important points/steps we need to consider:

1. The boundary cycles of generalized regular k-cells are orientable, subdivided (k-1)-manifolds. Since Brisson's *lift* and *join* operators are specifically designed for constructing subdivided manifolds (described above) we apply them to the construction of the boundary cycles of generalized regular k-cells. The only modification required deals with generalized regular j-cells ($2 \leq j \leq k$-1) that have more than boundary cycle.

2. Generalized regular k-cell complexes are always embedded within a Euclidean n-manifold ($k \leq n$) which is represented as a world n-cell. To create individual k-cells in the Euclidean n-manifold from their boundary cycles (created in step 1) we use a modification of Brisson's *lift* operator known as *Elift* (after *Embedded lift*). The effect of the *Elift* operator is to 'fill in the interior' of the cell (as for Brisson's *lift* operator), then create the embedding using the rules described in section 4.6.5.

3. Generalized regular k-cell complexes are formed by joining or attaching generalized regular k-cells to one another within the Euclidean n-manifold. To join the embedded generalized k-cells (created in step 2) along subsets of their boundary cycles, we use a modification of Brisson's *join* operator which we call *Ejoin* (after *Embedded join*). The effect of the *Ejoin* operator is to identify j-cells ($0 \leq j \leq k$-1) in a pair of k-cells (created via *Elift* in step 2). For example, a pair of 2-cells can be joined together by identifying two 0-cells from their respective boundary cycles.

Before describing the *Elift* and *Ejoin* operators, it is important to note that the extension of the cell-tuple implementation of the *join* and *lift* operators to generalized regular cell complexes is simplified by the fact that:

1. Each tuple in the graph G_{TM^n} (formed from the cell-tuples and switch operations) has one and only one $switch_i$ operation for all $0 \leq i \leq n$; and,

2. G_{TM^n} may be split into connected components by ignoring i-dimensional switch operations (see figure 4.20(b)).

These are the same critical properties of G_T that Brisson's original implementation of *join* and *lift* is dependent upon (as mentioned in the last section). It should also be noted that these properties are maintained in the subsequent extension of the cell-tuple to generalized singular cell complexes (described in chapter 5), but we don't discuss construction-operators for generalized singular cell complexes here (this is a subject for future research).

A. Generalized Regular Cells with more than one Boundary Cycle

The *lift* operation described in section 6.2.2 must be modified to create a generalized regular k-cell A with i boundary cycles ($i \geq 1$). The modified *lift* operator takes as input a list of cell tuples $t^1.....t^i$ of dimension k-1 each of which 'belongs' to a boundary cycle. Then for each $t' \in trav(t^i,\{0,....,k\text{-}1\})$, t_k is set to A and $switch_k(t')$ is set to $\emptyset$, indicating the boundary of the cell, just as is done for a regular cell (figure 6.8). Notice that in the application of the *lift* operator to regular cells given in the previous section, it is assumed that the cell-tuples representing the (k-1)-skeleton returned by the *trav* function will be in the 'interior' of the k-cell. When a generalized regular k-cell with i boundary cycles ($i \geq 1$) is constructed tuples belonging to all boundary cycles will be treated as though they are in the 'interior'. This is equivalent to assuming that the relative orderings of the inner boundary cycles are opposite to that of the outer boundary cycle. For example, if the outer boundary cycle is oriented in a counter-clockwise direction, then the inner boundary cycles will be oriented in a clockwise direction - see Corbett (1985).

B. *Elift*

Generalized regular k-cell complexes are always embedded within a Euclidean n-manifold ($k \leq n$) which is represented as a world n-cell. To create an individual k-cell in the Euclidean n-manifold we assume that its subdivided (k-1)-manifold boundary cycles have been formed using the conventional *lift* and *join* operators of Brisson (1990) given in section 6.2.2. Furthermore we assume that the *lift* operator (with the modifications described in 6.2.3.A, if required) has been applied to create the k-cell. To create the embedding of the k-cell in the world n-cell, the following additional steps (derived from the rules given in section 4.6.5) are necessary:

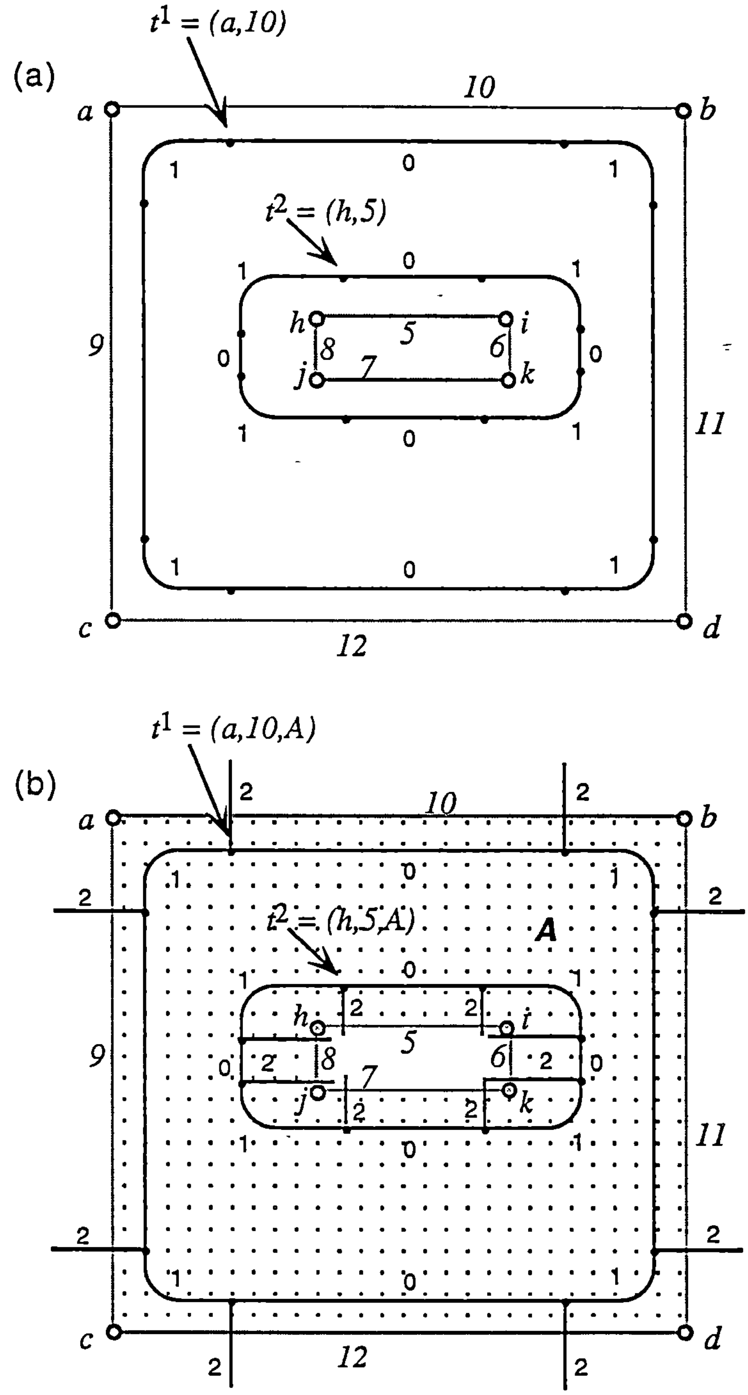


Figure 6.8 - constructing a generalized regular 2-cell A with more than one boundary 1-cycle **(a)** The cell-tuples of the outer boundary cycle and the cell-tuples of the inner boundary cycle **(b)** the cell-tuples after the *lift* operation

1. When $k = n$, duplicate all cell-tuples and switch operations, set t_n to the world cell (R^n) in the duplicate tuples and connect corresponding tuples via $switch_n$ operations.

2. When $k = n - 1$, duplicate all cell-tuples and switch operations (as per the case of $k = n$) and connect corresponding tuples via $switch_n$ operations *and* $switch_{n-1}$ operations. For *all* tuples, set t_n to the world cell R^n.

3. When $k = n - 2$, for all tuples, set t_n to the world cell R^n, $switch_n(t) = t$ and t_{n-1} (along with $switch_{n-1}(t)$) to $\varnothing$.

4. When $n = 3$ and $k = 0$, for all tuples, set t_n to the world cell R^n. Also set $t_1 \ldots t_{n-1}$ and $switch_0(t) \ldots switch_n(t)$ to $\varnothing$.

The *duplicate* operation may be implemented as follows. For each tuple t, create a duplicate tuple t' and set $switch_n(t) = t'$ and $switch_n(t') = t$. Then for $j = 0,\ldots,n-1$ set $switch_j(t') = switch_n(switch_j(switch_n(t')))$.

As an example of the *Elift* operator, consider the case of $k = 2$ and $n = 3$. To create an embedded 2-cell A (which will later form part of a 2-dimensional spatial object in R^3) using the *Elift* operator, set t_2 to A and $switch_2(t) = \varnothing$ for all tuples in the 1-cycle (subdivided 1-manifold), in the same way as the conventional *lift* operator of Brisson (1990) (figure 6.9(a)). Next, duplicate all tuples and switch operations setting t_3 to R^3, $switch_3(t') = t$ and $switch_3(t) = t'$ (figure 6.9(b)). Lastly, set $switch_2(t) = t'$ and $switch_2(t') = t$ for all tuples (figure 6.9(c)).

C. *Ejoin*

Generalized regular k-cell complexes are formed by joining or attaching generalized regular k-cells to one another within the Euclidean n-manifold. The effect of the *Ejoin* operator is to identify j-cells ($0 \leq j \leq k-1$) (a and b) in a pair of k-cells K and L, where K and L were previously created using *Elift*. In terms of the ordering results given in section 1.4 and extended in section 4.6.5, the identification effectively 'merges' the ordering about the j-cell a, in K, with the ordering about the j-cell b, in L. Therefore, given that the *Ejoin* operator takes as input two cell-tuples t^1 and t^2 where $t^1 \in assoc(a)$ and $t^2 \in assoc(b)$, there are three situations that the *Ejoin* operator must deal with:

1. <u>Two-sided orderings</u> - merging of two-sided orderings occurs when $k = n$ and $j = n - 1$.

   If we let list $l_1 = trav(t^1, \{0,\ldots,j-1\})$ and list $l_2 = trav(t^2, \{0,\ldots,j-1\})$ then for each tuple t in l_1 and t' in l_2:

1.1 Set $t^z = switch_j(switch_{j+1}(t))$ and $t^{z'} = switch_j(switch_{j+1}(t'))$. Then set $switch_j(t^z) = t^{z'}$ and set $switch_j(t^{z'}) = t^z$.

1.2 Delete the tuples returned by $switch_{j+1}(t)$ and $switch_{j+1}(t')$. Set $switch_{j+1}(t) = t'$ and $switch_{j+1}(t') = t$.

Figure 6.10 shows an example where two 2-cells are joined along a 1-cell in R^2. The effect of steps **1.1** and **1.2** is to merge the circular ordering about the 0-cell and the two-sided ordering about the 1-cell, for each pair of tuples associated with the 1-cell.

Note that Brisson's *join* operator effectively merges two-sided orderings except that it is restricted to the simple case where the *k*-cells belong to a subdivided *k*-manifold.

2. Circular orderings - merging of circular orderings occurs when $k \geq n - 1$ and $j = n - 2$.

If we assume that t^1 and t^2 would be adjacent in the new circular ordering about the *j*-cell and let list $l_1 = trav(t^1, \{0,....,j\text{-}1\})$ and list $l_2 = trav(t^2, \{0,....,j\text{-}1\})$ then for each tuple t in l_1 and t' in l_2:

2.1 Set $t^z = switch_{j+1}(t)$ and $t^{z'} = switch_{j+1}(t')$. Then set $switch_{j+1}(t^z) = t^{z'}$ and set $switch_{j+1}(t^{z'}) = t^z$.

2.2 Set $switch_{j+1}(t) = t'$ and $switch_{j+1}(t') = t$.

Figure 6.11 shows an example where two 2-cells are joined along a 0-cell in R^2. The effect of steps **2.1** and **2.2** is to merge the circular orderings about the 0-cell.

3. No ordering - there are no orderings to merge when $j = n - 3$.

To simplify the description of the *Ejoin* operator, any subspace orderings, such as a two-sided 1-dimensional subspace ordering about a 0-cell in 1-dimensional spatial object in R^3, are created after the complex has been constructed.

The last step of the *Ejoin* operation (in all three cases) is to equate the *m*-dimensional components t_m $(m \leq j)$ of all tuples associated with the identified *j*-cells and their boundary cells (ie. boundary cells of dimension 0,...*j*-1).

Lastly, it should be noted that the definition of the *Ejoin* operator in terms of the orderings simplifies its description and indicates its generality.

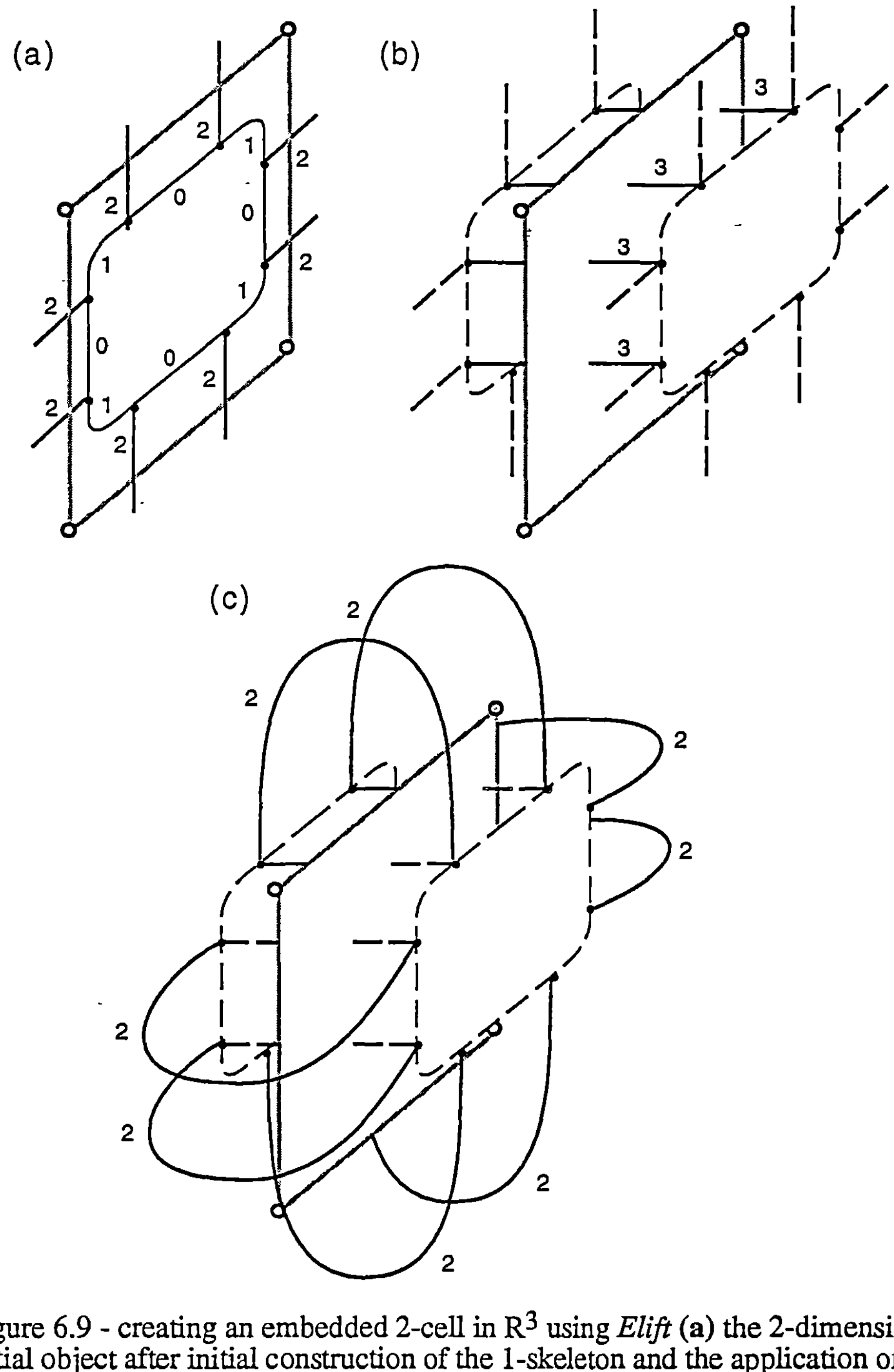


Figure 6.9 - creating an embedded 2-cell in R^3 using *Elift* (**a**) the 2-dimensional spatial object after initial construction of the 1-skeleton and the application of a 2-cell *lift* (**b**) duplication of cell-tuples, setting t_3 to R^3 and $switch_3(t) = t$ (**c**) setting $switch_2(t) = t$. The 2-cell can now be joined to other 2-cells in R^3

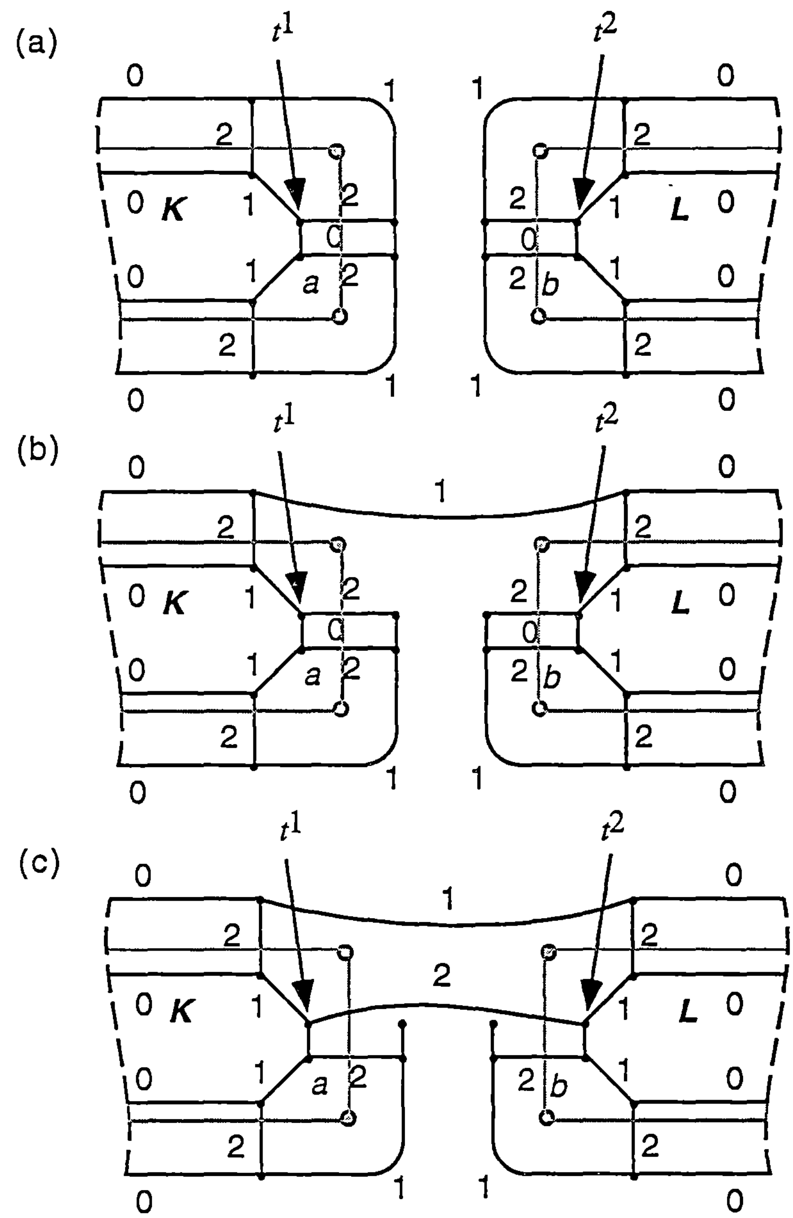


Figure 6.10 - joining two 2-cells along a 1-cell in R^2 using the two-sided version of the *Ejoin* operator **(a)** the two 2-cells *K* and *L* created using *Elift*. **(b)** Resetting the $switch_1$ operations (step **1.1**) **(c)** Resetting the $switch_2$ operations (step **1.2**). Processing of the next tuples returned by the *trav* functions, the deletion of the left over tuples and the effective identification of *a* and *b* (by resetting the components of their tuples) are not shown.

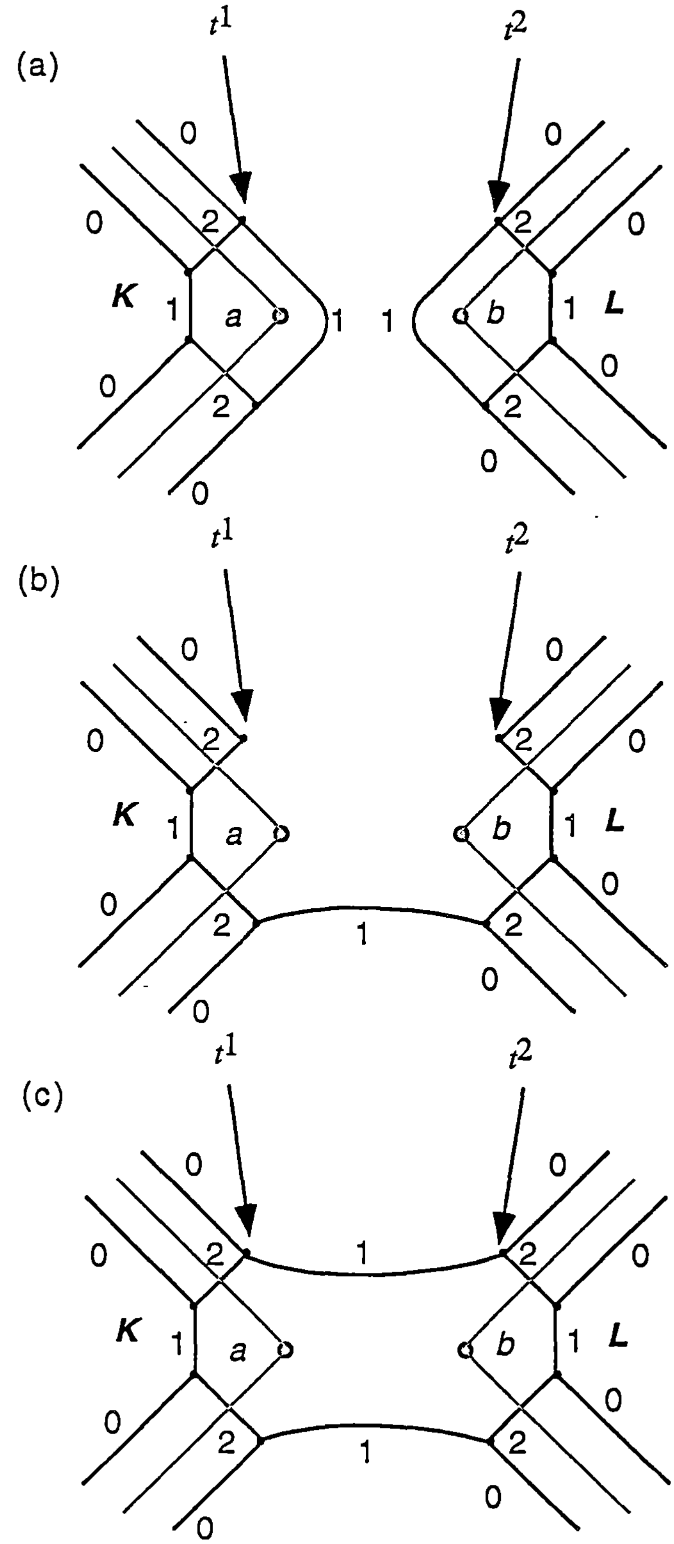


Figure 6.11 - joining two 2-cells along a 0-cell in R^2 using the circular ordering version of the *Ejoin* operator. **(a)** the two 2-cells *K* and *L* created using *Elift*. **(b)** Resetting the $switch_1$ operations as per step **2.1** **(c)** Resetting the $switch_1$ operations as per step **2.2.** Note that effective identification of *a* and *b* (by resetting the components of their tuples) is not shown. Note that any number of additional 2-cells can be joined along this 0-cell by following the same procedure.

6.2.4 Overview of the Combinatorial Homotopy Operators

A major problem with the construction of cell complexes which represent the spatial objects in any topological model is *consistency* or *soundness* (Mäntylä 1988). In topological terms *consistency* is knowing that the constructed cell complex actually realizes what was expected (ie. topological properties such as genus are correct) and perhaps more importantly, that it subdivides a space that is contained within the domain. Sections 6.2.1 and 6.2.2, describe a number of elegant solutions that have been employed for topological models with limited domains. For example, the Euler operators ensure that subdivisions of orientable 2-manifolds (ie. surfaces of 3-dimensional spatial objects) are always created because they are based on a topological invariant (the Euler-Poincaré equation) which distinguishes between orientable 2-manifolds. In the wider domain described by Brisson (1990) (ie. subdivisions of k-manifolds and k-manifolds with boundary), pairs of k-cells must be constructed with the *lift* operator and joined (using the *join* operator) along (k-1)-cells in their boundary cycles in order to ensure that the cell complex forms a subdivided k-manifold or k-manifold with boundary when $k \leq 3$ - see also **ct1-5** in section 4.6 and pg. 50-51 of Brisson (1990). If $k > 3$, the unsolved Poincaré conjecture (see Stillwell 1980 pg. 246-7) indicates that despite the restriction on the *join* operator, the resulting k-cell complex may *not* be a subdivided k-manifold or k-manifold with boundary. It is also important to note that although Brisson's consistency checks are sufficient to determine that the result is a subdivided manifold or manifold with boundary, individual homeomorphism types are not distinguished. For example, a subdivided 2-manifold may be a 2-sphere or a torus or any other 2-manifold.

The domain of the spatial information system described in this research is much wider than any of those described in sections 6.2.1 and 6.2.2 (with the exception of Corbett 1985). Spatial objects may be k-dimensional ($0 \leq k \leq 3$) and are not necessarily subdivided k-manifolds or k-manifolds with boundary as in Brisson (1990). Instead a k-dimensional spatial object is a collection of generalized regular k-cells joined together along their boundaries (ie. a generalized regular k-cell complex) and embedded within a Euclidean n-manifold (ie. R^n) where $n \geq k$. Topological consistency, in terms of ensuring that a k-dimensional spatial object is within the domain, could be enforced by maintaining the following conditions:

1. All generalized regular k-cells are Euclidean k-manifolds with subdivided (k-1)-manifold boundary cycles; and,

2. Cells may only be joined along their boundary cycles.

The 'local' approach to consistency described here is necessary because there are no enforceable global restrictions on the space that a spatial object subdivides. In this respect the local approach is similar to that described by topologists for CW-complexes (see section 4.2). Fortunately, we can do a little better and ensure consistency in the construction of a slightly larger set of spaces which still includes the generalized regular k-cells ($k \leq 2$) and the boundary cycles of generalized regular 3-cells: the subdivided k-manifolds and k-manifolds with boundary ($k \leq 2$).

From the point of view of topological *consistency* or *soundness* it would be extremely useful to be able to find a very 'simple' cell complex which encapsulates the topological properties of a subdivided manifold or manifold with boundary and a simple process which constructs the remainder of the subdivided manifold or manifold with boundary using the *lift*/*Elift* and *join*/*Ejoin* operators or some simple restriction of these operators. Finding a 'simple' cell complex which encapsulates the topological properties of a spatial object is equivalent to finding a simple but widely applicable topological invariant. Such an invariant is the homotopy type and the 'simple' cell complex should be a strong deformation retract of the space (see sections 3.2, 3.3.2). Constructing the remainder of the cell complex in a consistent manner requires us to find a combinatorial method which enacts the point-set notion of strong deformation retraction, thereby preserving the homotopy type. Fortunately such a method (known as combinatorial homotopy) was developed by the mathematician J.H.C. Whitehead (see Cohen 1972 pg. 2 for example).

The main advantages in using the deformation retract, homotopy type and combinatorial homotopy are:

1. The deformation retracts of many useful spaces, particularly the subdivided manifolds with boundary, are easy to identify.

2. More spaces are of the same homotopy type than are of the same homeomorphism type (Brown 1988 pg. 238). In effect, the homotopy type is a 'coarser' topological invariant than the homeomorphism type. This 'coarseness' can be seen in the table of homotopy types for some manifolds and manifolds with boundary in section 3.3.3. From this table it is clear that two spaces may have the same homotopy type yet not be homeomorphic. For example, the 2-sphere with two boundary cycles (a topological cylinder - see Corbett 1985) and the circle both

have the same homotopy type. However the homotopy type is not so coarse as to be impractical because within each class of manifold and each class of manifold with boundary, the homotopy type is unique.

3. They can be used in algorithms that reconstruct the topology of any 2-manifold or 2-manifold with boundary from a wire-frame or 1-skeleton (see section 6.4). The 2-cells produced by such algorithms can be attached to the strong deformation retract of a 2-manifold with boundary using combinatorial homotopy in order to ensure consistency and correctness. A subdivided 2-manifold is handled indirectly by constructing a subdivided 2-manifold with a single boundary to which the *lift/Elift* operator from sections 6.2.2 and 6.2.3 must be applied as the final step in the reconstruction process.

4. The cell attaching operations that preserve the homotopy type (known as combinatorial homotopy) may be described using simple restrictions on the *join/Ejoin* operator given in sections 6.2.2 and 6.2.3.

6.3 Development of the Combinatorial Homotopy Operators

We turn now to the theory necessary to define the deformation retract and combinatorial homotopy; the basis of what we will refer to as the *combinatorial homotopy operators.*

6.3.1 Homotopy Equivalence and the Strong Deformation Retract

Recall from section 3.2 in chapter 3 that given two topological spaces **X** and **Y**, a map f: $\mathbf{X} \rightarrow \mathbf{Y}$ is called a homotopy equivalence between **X** and **Y** if it possesses a "homotopy inverse" g: $\mathbf{Y} \rightarrow \mathbf{X}$ and $g \circ f$ and $f \circ g$ are homotopic to the identity maps on **X** and **Y** respectively. If **X** and **Y** are homotopy equivalent, then they are said to have the same *homotopy type*. Intuitively, we may think of a homotopy equivalence as a continuous transformation which 'shrinks' or 'expands' a space in such a way that its homotopy type is always preserved.

Section 3.2 then gave a definition of the *strong deformation retract* of a space **X** as a subspace **A** which may be physically realized using a particular homotopy equivalence known as a *strong deformation retraction.* Intuitively, a strong deformation retraction takes each point of the space **X** along a continuous path into the space **A** with the only proviso being that the points of **A** do not move (see Jänich 1980 pg. 62).

The key point about the (strong) deformation retraction is that it induces an isomorphism between the fundamental group of **X** and the fundamental group of **A** (Stillwell 1980 pg. 122). Thus, apart from having the same homotopy type, the fundamental groups of **X** and **A** are 'equivalent'. This 'equivalence' permits topologists to use a strong deformation retract in place of the original space when calculating the fundamental group.

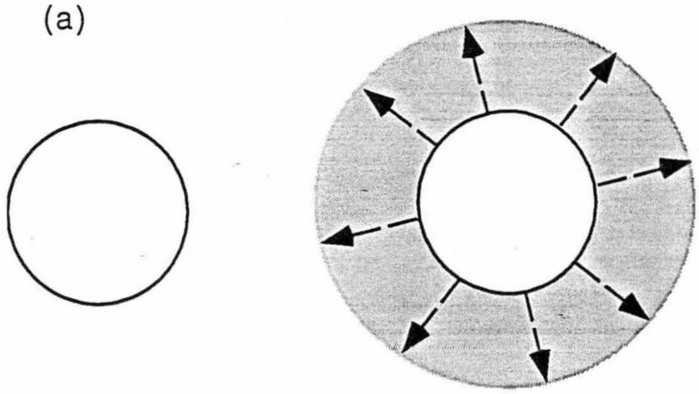
We will follow this precedent by trying to find a cell attaching (ie. combinatorial) process which effectively 'reverses' the strong deformation retraction on any strong deformation retract of a subset of spaces in the domain of our model: the subdivided k-manifolds with boundary ($1 \le k \le 2$). In an intuitive, geometric setting, this is equivalent to finding a continuous function whose effect is to 'thicken' the strong deformation retract until the desired manifold with boundary is obtained (figure 6.12(a)). As a consequence of the isomorphism induced by a strong deformation retraction, the fundamental group of any subdivided k-manifold ($k \le 2$) with boundary has the same properties (ie. it is a free group) as the fundamental group of a graph (Stillwell 1980 pg. 141).

We choose the subdivided k-manifolds with boundary because each one has a graph as its strong deformation retract (see section 3.3.2). Furthermore, the strong deformation retract, as a subspace of the 1-skeleton of any subdivided manifold with boundary, is both simple and easy to recognize.

By contrast, the subdivided k-manifolds do not have such a subspace which could form a (strong) deformation retract (see section 3.3.2.B). However, they can still be constructed consistently, by constructing a subdivided k-manifold with a single boundary first, and then 'closing' this boundary with a k-cell.

In order to show that the cell attaching process to be described in the next section does not change the homotopy type (and thus the fundamental group) of the subdivided manifold with boundary, we will need one additional construct from homotopy theory: the *mapping cylinder*. Following Cohen (1972) pg. 1, if $f\colon \mathbf{X} \to \mathbf{Y}$ is a map, then the mapping cylinder Mf is obtained by taking the disjoint union of $\mathbf{X} \times \mathbf{I}$ and $\mathbf{Y}$ and identifying $(x,1)$ with $f(x)$ for all $x \in \mathbf{X}$.

(a)



(b)

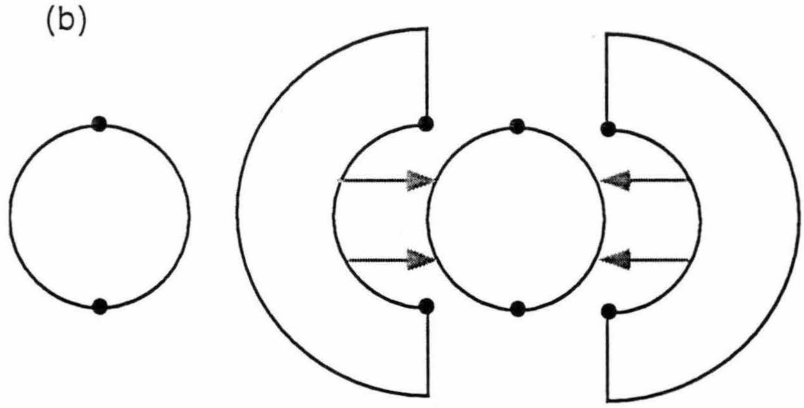


Figure 6.12 - 'thickening' the deformation retract of an annulus **(a)** geometrically using a continuous function **(b)** combinatorially ie. by attaching cells

It turns out that **Y** is actually a strong deformation retract of Mf.

mc-sdr There exists a map p: $Mf \rightarrow \mathbf{Y}$, given by:

$$p[x,t] = [x,1] = [f(x)] \text{ when } t < 1$$

$$p[y] = [y] \text{ for all } y \in \mathbf{Y}$$

Cohen (1972) pg. 2 states that the proof that **Y** is a strong deformation retract of Mf consists of 'sliding along the rays of Mf'. The proof can be found as proposition 12.1 on pg. 19 of Hu (1959). The idea of 'sliding along the rays of Mf' can be seen intuitively in figure 6.13(b), where 'rays' extending from **X** (the annulus) to **Y** (the circle and strong deformation retract of Mf) in the mapping cylinder Mf, are shown.

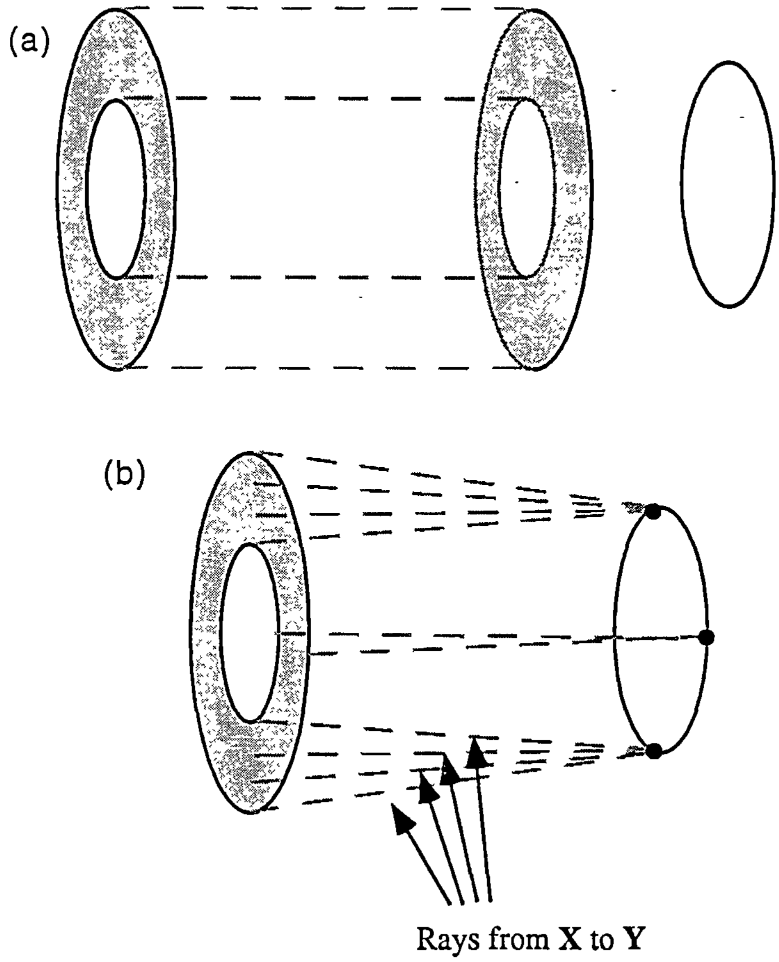


Figure 6.13 - forming the mapping cylinder Mf of f when $\mathbf{X}$ = annulus and $\mathbf{Y}$ = circle **(a)** $\mathbf{X} \times \mathbf{I}$ and $\mathbf{Y}$ **(b)** the mapping cylinder Mf and the 'rays' between $\mathbf{X}$ and $\mathbf{Y}$

After slightly restricting the CW attaching maps of Cohen (1972) pg. 14 to suit our goal of constructing a subdivided manifold with boundary from its strong deformation retract, we will rely (as does Cohen) on the mapping cylinder and this result to show that attaching a generalized regular cell with a single boundary cycle, to a generalized regular cell complex, does not change the homotopy type.

6.3.2 Combinatorial Homotopy

In the 1930's methods for defining a homeomorphism between spaces carrying simplicial complexes were based on combinatorial 'moves'. A combinatorial definition of homeomorphism (Alexander 1930) between two spaces **K** and **L** (both carrying simplicial complexes) was based on whether it was possible to get from **K** to **L** in a finite series of such 'moves' (Cohen 1972). The mathematician J.H.C Whitehead adopted a similar approach when attempting to give a combinatorial definition of homotopy equivalence between spaces carrying simplicial complexes. Essentially two spaces **K** and **L** are homotopy equivalent if it is possible to get from **K** to **L** using a series of simplicial *collapses* or *expansions* which were also called 'moves'.

As an example of these moves, figure 6.14 shows how a 2-simplex (*abc*) collapses across itself to a cone made of two 1-simplexes (*ab* and *ac*). Then each 1-simplex collapses to the shared 0-simplex (*a*) (figure 6.14). Expansion is essentially the reverse: we can create a 2-simplex from a 0-simplex via two 1-simplex expansions from the 0-simplex and a 2-simplex expansion on the cone of 1-simplexes.

Simplicial collapses (and expansions) enact homotopy equivalence in the form of strong deformation retractions on a local or cell based level, since each 2-simplex has a point as its strong deformation retract and is incident to other 2-simplexes via a 1-simplex or 0-simplex.

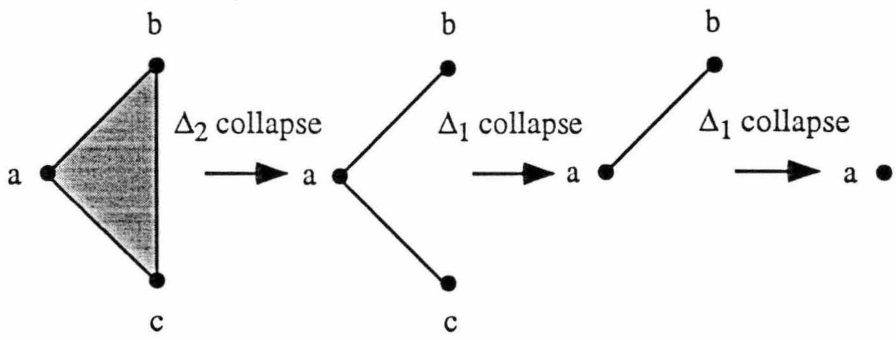


Figure 6.14 - a 2-simplex collapses to a cone of two 1-simplexes, each 1-simplex then collapses to a 0-cell

The local Euler operators of Mäntylä (1988) (see section 6.2.1) are a cellular extension of this simplicial form of combinatorial homotopy applied to subdivisions of a 2-manifold. Although they were originally defined such that they preserve the Euler-Poincaré characteristic, the underlying topological invariant is actually the homotopy type. Seen in this light, the significance of the local Euler operators can be described as follows:

1. The skeletal plane model operator MVFS(KVFS) creates(deletes) a vertex from which cellular expansions can take place and a world 2-cell to ensure that the expansion operations occur in the 2-sphere.

2. MEV(KEV) are 1-cell expansions(collapses)

3. MEF(KEF) are 2-cell expansions(collapses)

Figure 6.15 shows how a 2-cell collapses to a point via an initial KEF and a sequence of KEVs.

A proof given in Mäntylä (1988) pg. 141 indicates that the underlying topological invariant is the homotopy type and that the local Euler operators enact an extension of the simplicial form of combinatorial homotopy, because it demonstrates that the local Euler operators collapse the subdivision of a 2-sphere to the vertex (ie. a point) of the skeletal plane model. However the underlying notions of homotopy type and combinatorial homotopy are not mentioned by Mäntylä.

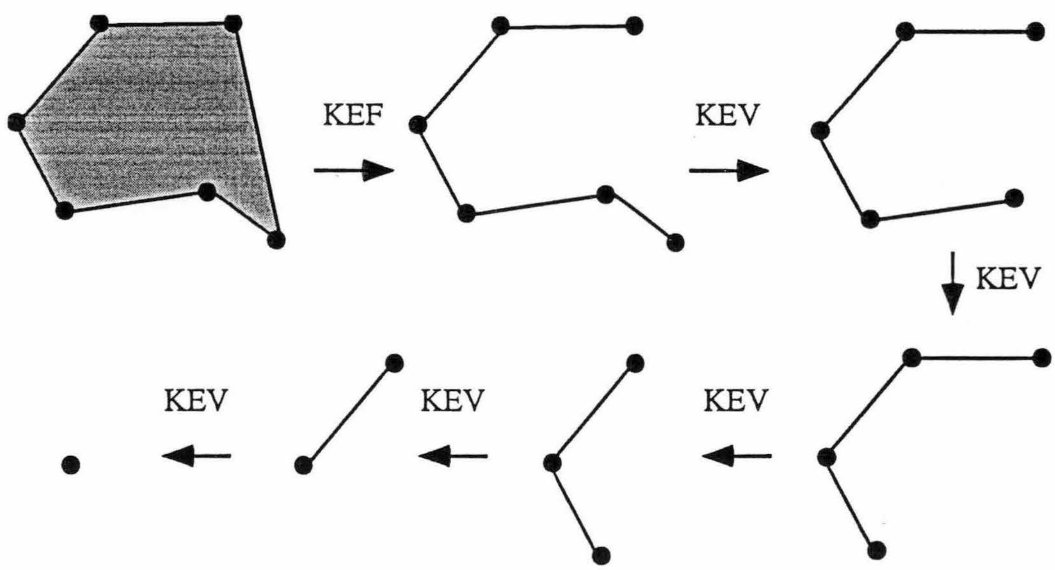


Figure 6.15 - collapsing a 2-cell to a point or 0-cell (its strong deformation retract) using the local Euler operators. Applying KEF to a 2-cell collapses it to an string of 1-cells, then the 1-cells are removed from the string via repeated application of KEV

Analogues of the simplicial collapse and expansion operators (which also include the local Euler operators) were developed in the more general setting of CW-complexes (see section 4.2) by the mathematician, J.H.C Whitehead. These analogues are expressed in terms of CW-complex attaching maps somewhat similar to **Ident2** in section 5.3. Following Cohen (1972) pg. 14, given two CW complexes **K** and **L**, **K** is a CW *expansion* of **L** if and only if:

cwa1 $\mathbf{K} = \mathbf{L} \cup c_{\alpha k\text{-}1} \cup c_{\alpha k}$, where $c_{\alpha k\text{-}1}$ and $c_{\alpha k}$ are not in **L**.

cwa2 there exists a disk pair $(D^k, D^{k\text{-}1})$ homeomorphic to $(\mathbf{I}^k, \mathbf{I}^{k\text{-}1})$ and a map $\varphi: D^k \rightarrow \mathbf{K}$ such that:

- **(a)** φ is the characteristic map for $c_{\alpha k}$ (see **cw1** in section 4.2)
- **(b)** $\varphi|D^{k\text{-}1}$ is a characteristic map for $c_{\alpha k\text{-}1}$ (see **cw1** in section 4.2)
- **(c)** $\varphi(P^{k\text{-}1})$ is a subset of the $(k\text{-}1)$-skeleton of **L** where $P^{k\text{-}1}$ is the closure of $(\partial D^k - D^{k\text{-}1})$

Geometrically, the elementary expansions of the CW complex **L** correspond to the attachings of a k-dimensional disk D^k along a simply connected subspace (or face) in its boundary by a continuous map φ. Conditions **cwa2(a)** and **cwa2(b)** describe how the disk and the subset of its boundary not involved in the attaching are the CW cells $c_{\alpha k}$ and $c_{\alpha k\text{-}1}$ in **K**, whilst **cwa2(c)** describes how $P^{k\text{-}1}$, a simply connected subspace of $D^{k\text{-}1}$, is attached to **L**.

To prove that the homotopy type of **K** is the same as that of **L**, Cohen (1972) pg. 15 uses the fact that the ball pair $(D^k, D^{k\text{-}1})$ in **cwa2** is homeomorphic to $(\mathbf{I}^k, \mathbf{I}^{k\text{-}1})$ to show that there exists a cellular strong deformation retraction $d: \mathbf{K} \rightarrow \mathbf{L}$. By **cwa2(c)**, there exists a continuous function $\varphi_o: \mathbf{I}^{k\text{-}1} \rightarrow \mathbf{L}^{k\text{-}1}$ (the $(k\text{-}1)$-skeleton of **L**) such that the result is homeomorphic to $\mathbf{L} \cup_{\varphi_o} \mathbf{I}^k$ (and thus **K**). But $\mathbf{L} \cup_{\varphi_o} \mathbf{I}^k$ is the mapping cylinder of φ_o. As a consequence, there is a cellular strong deformation retraction $d: \mathbf{K} \rightarrow \mathbf{L}$ because **L** is the strong deformation retract of the mapping cylinder, $\mathbf{L} \cup_{\varphi_o} \mathbf{I}^k$ (see **mc-sdr** at the end of section 6.3.1 and Cohen (1972) pg. 1).

To adapt the notion of CW attaching to generalized regular cells, we will use:

gra1 A restricted form of the CW attaching map φ_o described above. In the CW case, the only restriction on φ_o is continuity. As an example, figure 6.16(a) shows how a continuous attaching map φ_o permits identifications in $P^{k\text{-}1}$ (the face of the disk) when it is attached to **L**. The result of such an attaching is not permitted in a generalized regular cell complex (see **grc1** in section 4.4). To achieve an attaching which is compatible with **grc1** we restrict φ_o to be a homeomorphism. That is, φ_o is not just continuous, but also one-to-one, onto and with a continuous inverse.

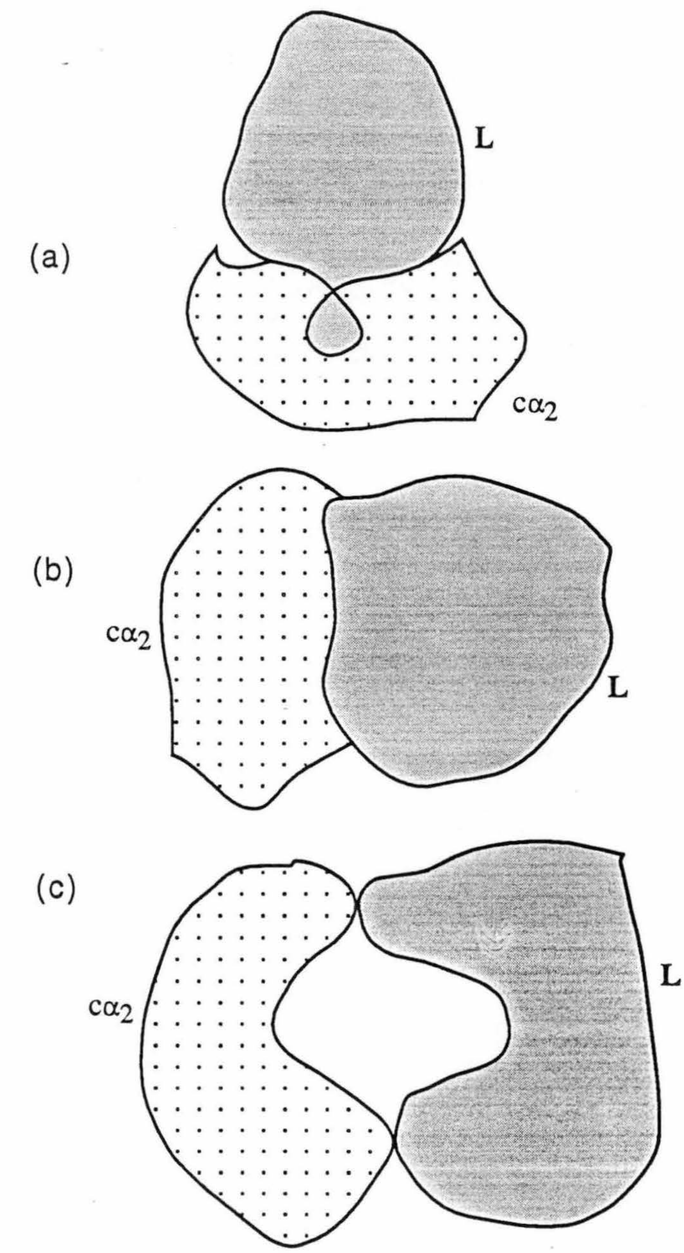


Figure 6.16 - valid and invalid CW attaching maps (a) with an identification (b) without an identification (c) invalid CW attaching map because the attaching does not occur along a simply connected subset of $c\alpha_2$ and **L** - an extra 'hole' is added.

gra2 Generalized regular cells of dimension ≤ 2, with *one* boundary cycle. Such cells form a subset of the class of CW cells since they can be obtained by restricting the characteristic map $f: D^k \rightarrow c_{\alpha k}$ ($k \leq 2$), of a CW cell (given by **cw1** in section 4.2) to be a homeomorphism of the boundary of D^k as well as its interior. They actually correspond to regular CW cells as defined in section 4.2 and Lundell and Weingram (1969) pg. 78. Note that we cannot use generalized regular cells with more than one boundary cycle because they do not have the point as their strong deformation retract.

Noting that the class of homeomorphisms is a subset of the class of continuous maps and that both the CW cell $c_{\alpha k-1}$ in condition **cwa2(b)** and P^{k-1} in condition **cwa2(c)** will correspond to one or more cells from the (k-1)-skeleton of the generalized regular cell in **gra2**, then the proof that CW attaching preserves homotopy type (see above and Cohen 1972, pg. 15) extends without modification to generalized regular k-cells ($k \leq 2$) with one boundary cycle.

The adaptation of the CW attaching theorem to generalized regular k-cells ($k \leq 2$) shows how such a k-cell may be attached to any generalized regular k-cell complex without changing the homotopy type of this complex. The mechanisms used to carry out these attaching maps on a generalized regular k-cell complex represented by the extended cell-tuple are the *join*/*Ejoin* operators from sections 6.2.2 and 6.2.3.

Assume that a generalized regular k-cell $c_{\alpha k}$ is created with one boundary cycle in order to satisfy **gra2**. Our aim is to attach a simply-connected subspace G of the boundary cycle of $c_{\alpha k}$ to a simply connected subspace H of the (k-1)-skeleton of the generalized regular k-cell complex **L**, using the *join*/*Ejoin* operator. Firstly, we will assume that the number of (k-1)-cells in G and H is the same. If this is not so, then an operator such as Brisson's *split* (see section 6.2.1 and Brisson 1990) can be applied to G to make it so. Let $\mathbf{T} = (t^1....t^n)$ be the list of tuples in G for which $switch_k(t) = \emptyset$ (when created with the *lift* operator) or $t_k = \emptyset$ (when created with the *Elift* operation) $t \in G$; and $\mathbf{T'} = (t^{1'}....t^{n'})$ be the list of tuples in H where $switch_k(t') = \emptyset$ (when created with the *lift* operator) or $t_k' = \emptyset$ (when created with the *Elift* operation), $t' \in H$. The following condition must be met to successfully attach $c_{\alpha k}$ to **L**:

cta <u>G and H are subdivided (k-1)-disks</u>: If this is so, then it should be possible to remove from **T** and **T'** all tuples with the following condition (equation 6.2):

$t_{k-2}^{i+1} = t_{k-2}^{i}$ and $t_{k-2}^{i+1}{}' = t_{k-2}^{i}{}'$ $\forall\, t^i \in G, t^{i'} \in H$ and $i = 2,...,n$-1 (6.2)

The result should be a (k-2)-sphere. As an example, when k=2, G and H are subdivided 1-disks if just two 0-cells (the 0-sphere) remain after all other 0-cells meeting this condition are removed. Unfortunately, this is not a sufficient condition, as H may still have identifications.

If identification(s) are required in the proposed attaching map from G to H (eg. figure 6.16(a)), then at least one (k-2)-cell in H will have more than two cobounding (k-1)-cells. This can be detected by analyzing the number of tuples in H which meet equation 6.2.

If G and H are subdivided (k-1)-disks without identifications, then the attaching map can be completed by repeated application of the *join/Ejoin* operator to successive (k-1)-cells in G and H.

However, our aim is to directly construct any subdivided k-manifold with boundary ($k \leq 2$) from its strong deformation retract. A subdivided k-manifold can be indirectly constructed by first constructing it as a k-manifold with boundary, and then applying a 'closing' face using a *lift/Elift* operator. Since the subdivided manifolds and manifolds with boundary form a subset of the class of generalized regular cell complexes, we can use two of their specific characteristics developed in section 3.3 and section 4.6, together with **cta**, to ensure consistency:

sm1 The difference between 'perforation' and 'handle' cycles in the strong deformation retracts of the 2-manifolds with boundary. Those 1-cycles that form the boundary cycles of 2-manifolds with boundary are known as 'perforation' 1-cycles (figure 6.17(a)), whilst those that result from the addition of a 'handle' to a 2-sphere are known as handle 1-cycles (figure 6.17(b)). In the case of a perforation 1-cycle, 2-cells may only be attached along the outside, whilst for a handle 1-cycle, two 2-cells may be attached to each 1-cell in the cycle.

In terms of the extended cell tuple structure, we will hold the strong deformation retract of the subdivided 2-manifold as a 1-dimensional generalized regular cell complex ie. a 1-dimensional cell-tuple structure.

Although this seems restrictive, it actually simplifies the attaching process. Generalized regular 2-cells will be created from one boundary cycle using the *lift* operator described in section 6.2.2. If a 2-cell is to be attached to the strong deformation retract then that part of its boundary cycle is created by copying the required 1-cells and 0-cells from the cell tuple structure representing the strong deformation retract. The *lift* operator is then applied and the resultant 2-cell automatically incorporates the part of the strong deformation retract that it is attached to. This procedure avoids the need to store and manipulate unnecessary embedding information for both the strong deformation retract and the 2-cells themselves.

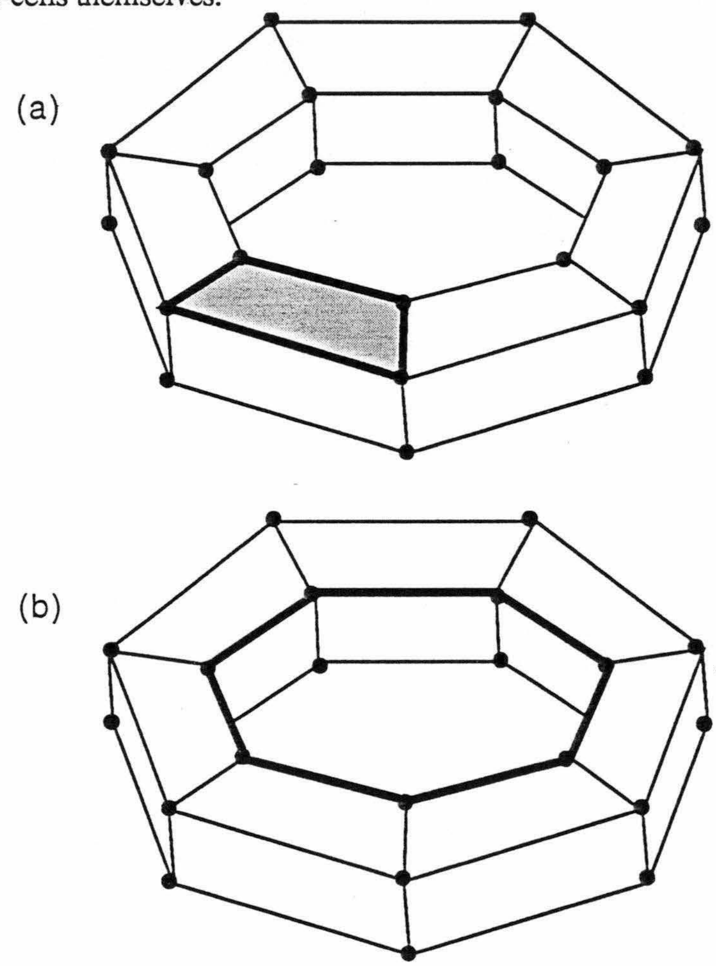


Figure 6.17 - perforation and handle 1-cycles in a subdivided 2-manifold (ie. the torus) **(a)** a 'perforation' 1-cycle (interior is shaded) **(b)** a 'handle' 1-cycle (highlighted)

sm2 Each j-cell ($1 \leq j \leq 2$) in the subdivided k-manifold or k-manifold with boundary has a k-dimensional neighborhood. Following Brisson (1990) pg. 80 and Lundell and Weingram (1969), pg. 82 (see also section 6.2.2), we need only ensure that there are just two cobounding k-cells for each (k-1)-cell. This is the 'two-sided' ordering described in sections 1.4.1 and 4.6.

In terms of the extended cell-tuple, the assumption that each k-cell is created with the *lift* operator of section 6.2.2, automatically ensures that no more than two k-cells can possibly cobound any (k-1)-cell. However to ensure that each (k-1)-cell has no less than two cobounding k-cells, we need only check that each tuple t has $switch_2(t) \neq \varnothing$, where t does not belong to any 'perforation' cycle .

For arbitrary k-cell complexes ($k \leq 2$), it is *not* known whether a combinatorial homotopy equivalence enacted by a finite sequence of cellular expansions and collapses (also known as a *simple* homotopy equivalence) is also a point-set homotopy equivalence (eg. such as those given for 2-manifolds with boundary in section 3.3.2.B). In other words, it is not known whether cellular homotopy type (also known as *simple* homotopy type) equals homotopy type (Cohen 1972 pg. 82). However, if we let **K** be a graph or 1-skeleton representing the strong deformation retract of a k-manifold with boundary ($k \leq 2$) from section 3.3.2.A, we can show that repeated attaching of generalized regular k-cells to **K** under **gra1** and **gra2** preserves the homotopy type of **K**. This, together with conditions **sm1** and **sm2** which enforce k-dimensional neighborhoods of j-cells ($j \leq k$-1), ensures that the constructed generalized regular k-cell complex will be a subdivided k-manifold with boundary. To show that repeated attaching of generalized regular k-cells preserves homotopy type, we will use a simple inductive argument.

Step 1: Attaching *one* generalized regular k-cell, $c_{\alpha k}$, does not change the homotopy type of **L**. The proof (described above) is based on **L** being the cellular strong deformation retract of $\mathbf{L} \cup_{\varphi_0} \mathbf{I}^k$ (the mapping cylinder of the attaching map φ_o), and $c_{\alpha k}$ being homeomorphic to $\mathbf{I}^k$ (see **mc-sdr** in section 6.3.1 and **cwa1**, **cwa2** with the modifications in **gra1** and **gra2**, above).

Step n+1: Assume inductively that **M** has been constructed as a generalized regular k-cell complex by attaching n generalized regular k-cells to **L** such that the homotopy type of **M** is the same as **L**. Attaching *one* additional generalized regular k-cell to **M** cannot change the homotopy type of **M** by the same argument used in step 1.

Now that we have specified the way in which the generalized regular cell attachings carry out combinatorial homotopy and thus preserve the homotopy type, we will give an example of its application to the construction of a subdivided 2-manifold with boundary from its strong deformation retract. In this example, we assume that both the strong deformation retract and the generalized regular 2-cells to be attached to it, have been created as described earlier in this section. We will also give an example showing how the technique is applied to a subdivided 2-manifold. In all cases, we will use the term 'cell' to refer to generalized regular cells.

A. Constructing a Subdivided 2-Manifold with Boundary

Figure 6.18(a) shows a strong deformation retract of a 2-sphere with two boundaries. The remainder of this space is constructed from the strong deformation retract by repeatedly attaching 2-cells along 1-cells (or 0-cells) in their boundary cycles (figure 6.18(b)).

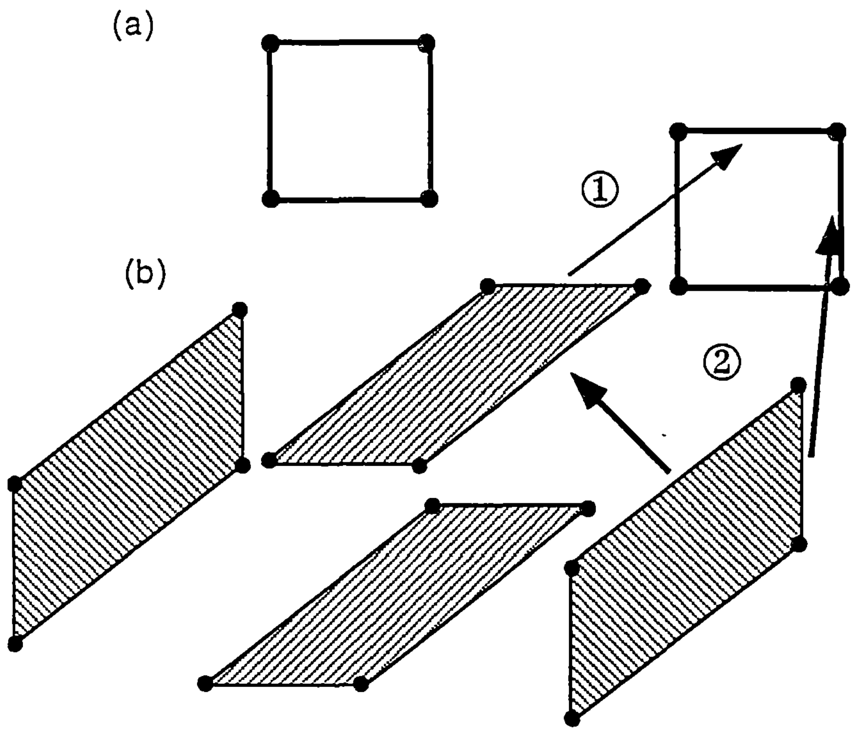


Figure 6.18 - constructing a subdivided 2-sphere with two boundary cycles (ie. a cylinder) using the combinatorial homotopy operators **(a)** the strong deformation retract of the 2-sphere with two boundary 1-cycles (a perforation loop) **(b)** the first two stages in adding the 2-cells to create the remainder of the cylinder

As described in the previous section (see **gral**), to ensure that the homotopy type is preserved each 2-cell may only be attached to the complex along a simply connected subspace of its boundary cycle. For example, in figure 6.18(b), stage ②, the 2-cell is attached to both the strong deformation retract and the adjacent 2-cell. In terms of the cell-tuple, this 2-cell would be created (via the *lift* operator) with the 1-cells and 0-cells of the strong deformation retract in its boundary cycle and then joined to the adjacent 2-cell via a *join* operation. To check that the subspace along which the 2-cell is attached is simply connected, we need only ensure that the *join* operation is applied to adjacent 0-cells or 1-cells. In terms of the cell tuple, **cta** (as discussed in the previous section) enforces this condition.

To ensure that the resulting cell complex is a 2-manifold with boundary, we must check that each 1-cell which is not part of a perforation cycle, has two cobounding 2-cells. In terms of the cell-tuple, we need only ensure that each tuple t in such 1-cells has $switch_2(t) \neq \varnothing$ (see **sm2** in the previous section).

In the development of the generalized regular cell attaching map, the only 2-cells that can be attached are those with just one boundary cycle. The reason for this is that 2-cells with more than one boundary cycle are not collapsible spaces (ie. they do not have the point as their strong deformation retract). For example, a 2-cell with two boundary cycles, such as that shown in figure 6.19(a), is actually homotopy equivalent to the bouquet of two loops. Since such 2-cells cannot be attached, they can only be created by dissolving the 'interior' 1-cells of a generalized regular 2-cell complex previously created with the combinatorial homotopy operators (see figure 6.19(b)).

B. Constructing a Subdivided 2-Manifold

To consistently construct any subdivided 2-manifold we firstly construct it as a 2-manifold with boundary (as described in the previous section) and then add one additional 2-cell to 'close' the 2-manifold. Finding this 2-cell can be automated by gathering the 1-cells that have just one cobounding 2-cell and applying the *lift* operator, *or* a new 2-cell can be created and joined along the appropriate 1-cells using the *join* operator.

Figure 6.20(a) shows the strong deformation retract of the subdivided double torus with a single boundary. Notice that the strong deformation retract shown in figure 6.20(a) is not the bouquet of four 'handle' 1-cycles as predicted in section 3.3.2. However, it is homotopy equivalent to the bouquet of four 1-cycles since the 1-cell between the two pairs of loops may be collapsed to a point.

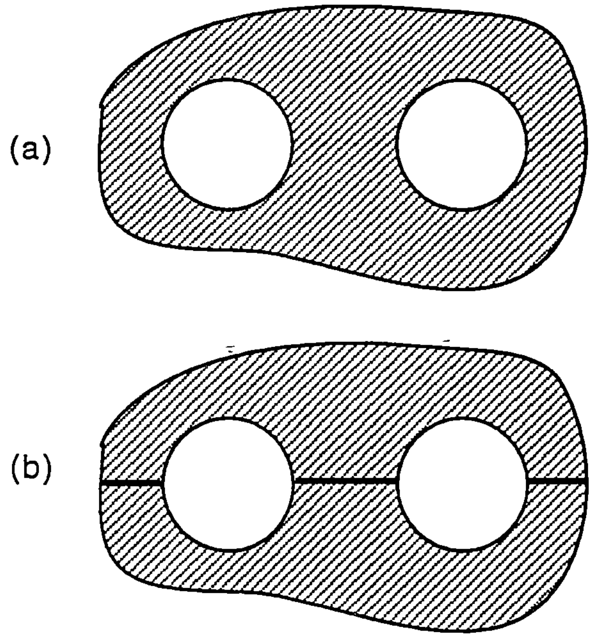


Figure 6.19 - forming a generalized regular 2-cell with more than one boundary 1-cycle (**a**) the required 2-cell (**b**) the homotopy equivalent space constructed by combinatorial homotopy - the 'interior' 1-cells (highlighted) must be dissolved.

To check that the strong deformation retract of any subdivided 2-manifold with boundary (as specified by the user) is actually homotopy equivalent to that predicted in section 3.3.2.A, the Tietze method (see section 4.4 and the algorithm to be discussed in section 6.4) can be applied. The Tietze method should return the same number of fundamental cycles (in this case four cycles) as the number of cycles in the bouquet of loops predicted in section 3.3.2.A

As required by the generalized regular attaching map, the strong deformation retract is a path-connected 1-skeleton. Subdivided 2-manifolds such as the torus shown in figure 4.13, may be constructed using a two stage process. The first stage uses a strong deformation retract and cell attaching process like that shown in figure 6.20(a). The second stage involves "dissolving" unwanted 2-cells to form generalized regular 2-cells with more than one boundary cycle, similar to those described at the end of section 6.3.2.A.

Lastly, this example shows how we can construct the subdivided 2-manifold boundary cycle of any generalized regular 3-cell. To physically create the generalized regular 3-cell, the last step would be to apply the *lift/Elift* operator to the subdivided 2-manifold.

(a)

(b)

(c)

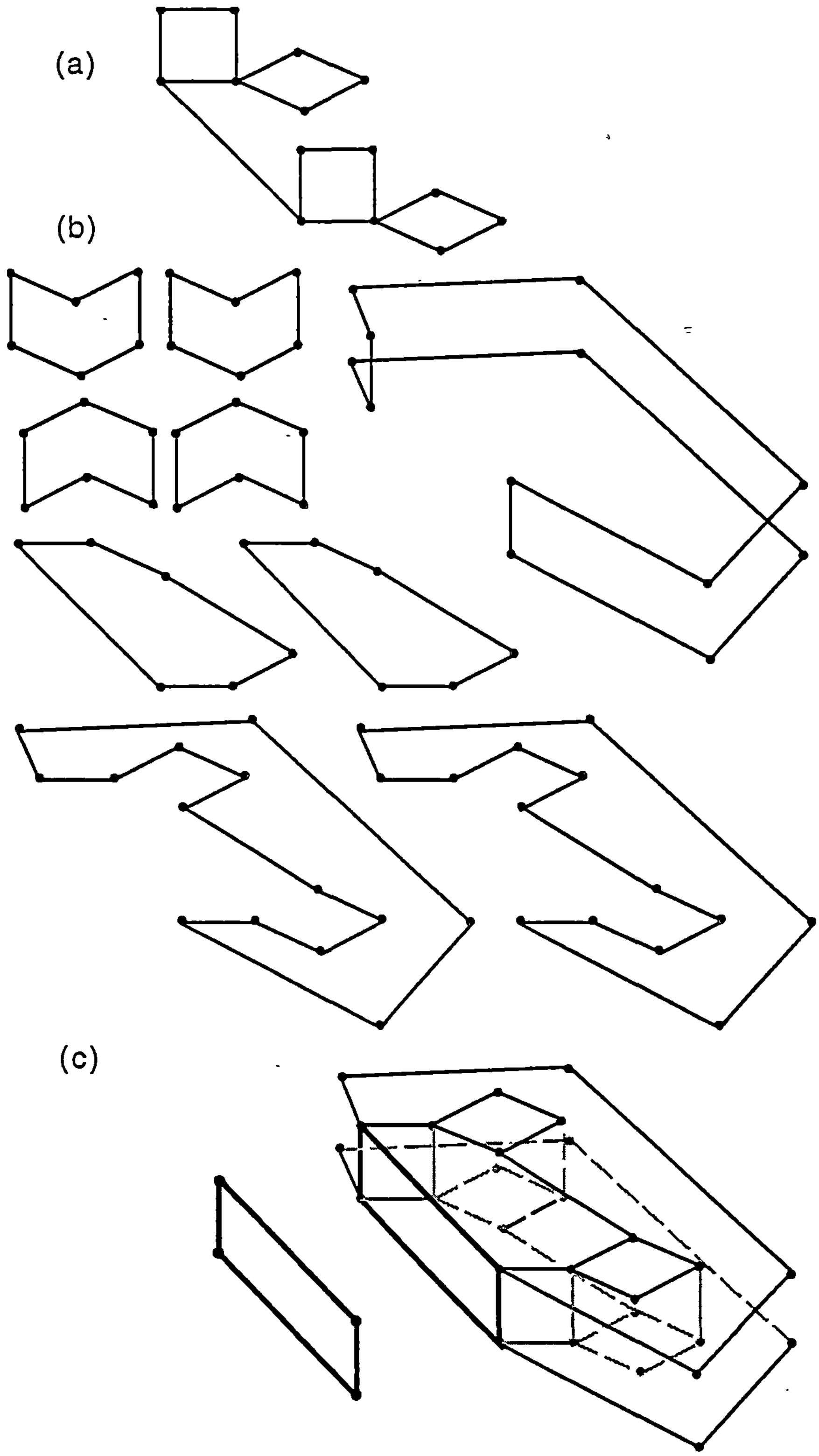


Figure 6.20 - using the combinatorial homotopy operators to construct a double torus with boundary **(a)** a 1-skeleton forming the strong deformation retract of the double torus with boundary - all four loops are 'handle' loops **(b)** the 2-cells that will be attached to create the double torus with boundary **(c)** adding the 'closing' 2-cell to form the double torus

6.4 Building the Topology of a Subdivided Manifold/Manifold with Boundary from a 1-Skeleton

In the last two sections we described the low-level, generic, topological operators suitable for the construction of a generalized regular cell complex (representing a spatial object) on a 'cell by cell' basis. In many applications, the 1-skeleton (a graph) of the cell complex is already provided and it is necessary to find the remaining cells of the complex. This process is sometimes known as 'building the topology' (in GIS applications) or the 'wire-frame reconstruction' problem (in CAD applications). For 3-dimensional applications, there is insufficient information in the 1-skeleton to uniquely determine the homeomorphism type of the space subdivided by the reconstructed cell complex. In other words, the problem is under-constrained and any solution is inherently *ambiguous*, since there may be more than one valid solution. In this section we show how the Tietze method (for calculating the fundamental group - see section 4.4), the strong deformation retract and the combinatorial homotopy operators may be used to solve the ambiguity problem for a subset of the domain of spatial objects consisting of the subdivided 2-manifolds and 2-manifolds with boundary. Despite the restriction, this subset of the domain is large enough to ensure that the algorithm can be used to construct the boundary cycles of generalized regular 3-cells and spatial objects corresponding to subdivided 2-manifolds and 2-manifolds with boundaries. These complexes can be used as 'building blocks' to construct all 2-dimensional and 3-dimensional spatial objects in much the same way as was mentioned in the last section.

A general solution to the problem of constructing a spatial object from a 1-skeleton is important because it is one of the fundamental methods for getting data into a spatial information system. Other methods which attempt to construct spatial objects from a 0-skeleton (eg. the alpha shape techniques of Edelsbrunner 1987, pg. 310) may also benefit from this approach.

6.4.1 Review of Existing Topology Reconstruction Algorithms

A. Planar Sweep

In 2-dimensional GIS, the technique that has been used to construct the topology of a map model from a 1-skeleton (or 'spaghetti') which forms a planar graph (ie. may be embedded in the plane without self-crossings) is known as planar or plane sweep. Constructing the topology of the map model can be compared to the plane sweep algorithms given in computational geometry for constructing the faces of an arrangement of lines in a surface which is topologically equivalent to the Euclidean plane (Edelsbrunner 1987). However, as van Roessel (1991) notes, these algorithms are often specified for "simple objects such as rectangles, Voronoi diagrams or convex polygons" and often contain exceptions eg. no line may be vertical - see Edelsbrunner (1987). Although such exceptions may be removed using symbolic perturbation schemes such as the simulation of simplicity (see Edelsbrunner and Mücke 1990), van Roessel (1991) gives a methodology for this technique which avoids these exceptions and can be applied to the construction of both generalized regular and singular 2-cell complexes; ie. arrangements that are not path-connected, may have 'dangling' lines and internal cell complexes. Following van Roessel (1991), the technique works through the following process:

1. Constructing planar circular orderings of edges about each vertex by examination of the coordinates of the co-vertices. Firstly, the co-vertices of each vertex are classified into one of two hemi-disks known as half-nodes, which are labelled **above** and **below** (according to the plane sweep line). Then, within each half-node, co-vertices are ordered according to whether they are to the **right of** another co-vertex in the hemidisk circular ordering. These relationships are then combined to give a total sequential circular ordering of co-vertices around a vertex which corresponds to the circular ordering of 1-cells in the coboundary of a 0-cell given in chapter 1.

2. Form monotone polygonal segments. That is, sequences of line segments in which each internal vertex has alternating above/below half-nodes and each vertex has two cobounding line segments.

3. Form lobes (ie. monotone polygons adjacent to a left and right monotone polygonal segment). Lobes can be created, joined, split and terminated. When a lobe splits into or joins two other lobes, the lobes are said to be connected. Polygons, rings and chains (1-arcs in the terminology of this research) are then output by tracking the connected lobe data structure.

The planar sweep can be generalized to a hyperplane sweep (Edelsbrunner 1987) which together with modifications for singular cells similar to van Roessel's, would seem to form the basis of a possible technique for constructing both generalized regular and singular cell complexes in R^3. However van Roessel's modifications and the algorithm itself are dependent upon analysis and classification of the circular ordering of monotone polygonal segments and monotone polygons about each vertex in a planar graph. Unfortunately there is no corresponding ordering of edges, faces and 3-cells about a vertex in a Euclidean 3-manifold (see section 1.4 and 4.6). Further investigation of this issue, the hyperplane sweep algorithm of Edelsbrunner (1987) and local methods such as those proposed in Franklin and Kankanhalli (1993) are areas for future research.

B. Wire Frame Reconstruction Algorithms

Ganter (1981) (see also Ganter and Uicker 1983) considers the problem of finding and constructing the 1-cycles that form the boundaries of faces or 2-cells from a 1-skeleton. The 1-skeleton must be a planar graph, because the only surfaces that Ganter considers are those that are homeomorphic to a 2-sphere, though *rings* (see the Euler operators in section 6.2.1) may be used to 'separate' other 2-manifolds (such as the torus) into distinct planar graphs which can then be individually analyzed in order to find faces (see figure 6.3 in section 6.2.1).

The basis of Ganter's method is the construction of a spanning tree of the vertices of the planar graph. A spanning tree of any graph is a tree contained within the graph which includes all its vertices. Since a tree cannot contain any 1-cycles (ie. it is acyclic) some edges of the graph will not be included in the spanning tree (figure 6.21(a) and (b)).

Using this property, a set of 1-cycles known as the fundamental cycles can be calculated. Each fundamental cycle is formed by tracing a path from the root through the spanning tree, through each edge or 1-cell that is not included in the spanning tree, and back to the root, removing any edges that are traversed twice (figure 6.22).

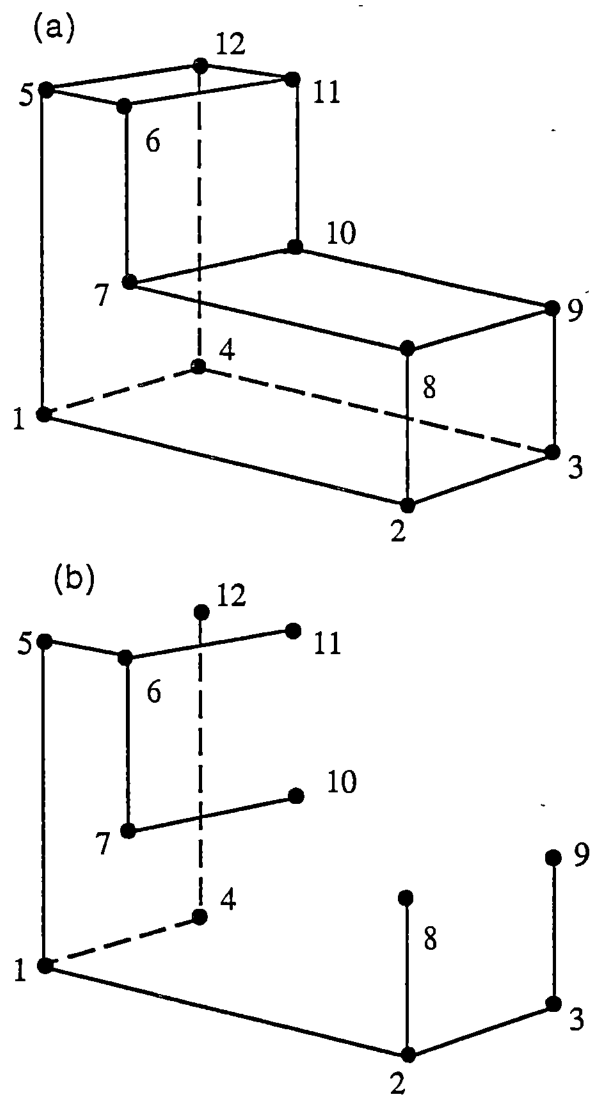


Figure 6.21 - an example 1-skeleton and spanning tree (a) the 1-skeleton of a subdivision which embeds in a space homeomorphic to a 2-sphere (1-skeleton is a planar graph) and (b) a spanning tree of the 1-skeleton.

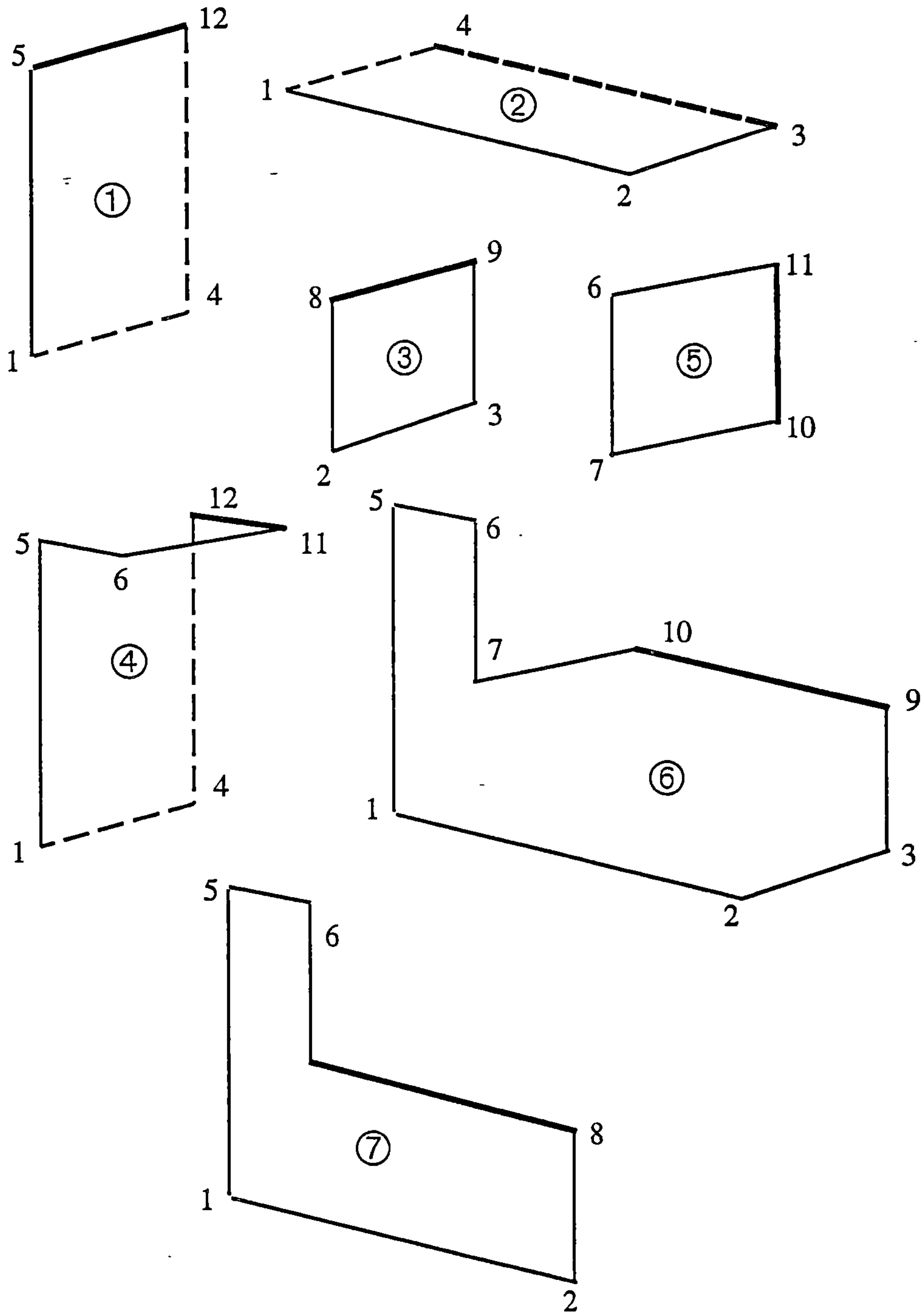


Figure 6.22 - the fundamental cycle set of the planar graph in figure 6.21 - the edges of the graph that are not in the spanning tree are highlighted.

Ganter recognised that the 1-cycles forming the boundaries of the 2-cells in a planar graph *usually* correspond with the minimum length basis set of 1-cycles of the 1-skeleton. This is a set from which every other 1-cycle of the graph can be generated. Since each fundamental cycle consists of one or more basis 1-cycles of the graph, Ganter used a reduction process which performed a series of exclusive OR (XOR) operations between any two fundamental 1-cycles (figure 6.23).

The decision about when to XOR two fundamental cycles is based on two heuristics originating in Deo (1974) (see also Deo et. al. 1982) which seek to find the minimum length basis set of 1-cycles of any graph (ie. the graph need not be planar):

1. Each 1-cycle contained in the graph has a minimum number of edges in common with any other 1-cycle.

2. When the sum of all edges in all 1-cycles of a graph is a minimum then the 1-cycles are the basis 1-cycles of the graph.

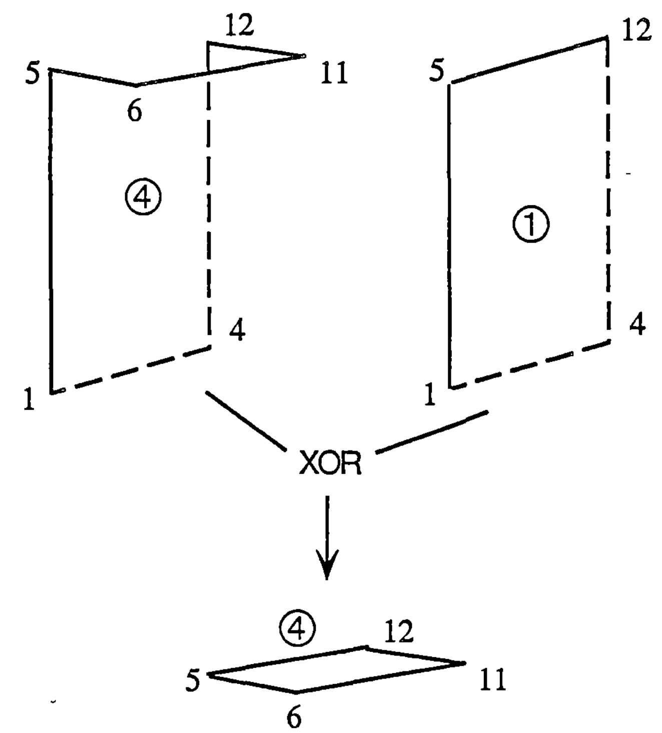


Figure 6.23 - reducing a fundamental 1-cycle to a basis 1-cycle. From the fundamental cycle set shown in figure 6.22, cycle 4 can be reduced XOR'ing it with cycle 1. The new cycle replaces cycle 4 in the fundamental cycle set. Further XOR operations are then performed on other cycles.

After all minimum length basis 1-cycles have been found, there will always be one missing face. Courter and Brewer (1986) (whilst describing another implementation of Ganter's algorithm) call this face the 'closing face' and give the reason for its existence in terms of the difference between the number of fundamental cycles in a planar graph ($v - e + 1$ from the Euler equation - see equation 6.1 in section 6.2.1 and Massey 1967) and the number of faces in a 2-sphere ($v - e + 2$). The closing face actually exists within the set of minimum length basis 1-cycles and may be found by XOR'ing all the 1-cycles in this set. If we go on with the XOR process for the fundamental cycles in figure 6.22 above, the closing face is bounded by the 1-cycle with vertices 3,4,12,11,10,9,3. It is interesting to notice a similar situation with spaces constructed using the local Euler operators in section 6.2.1 (figure 6.1) except that in that case the initial operator MVFS creates this 'closing' face by default.

Courter and Brewer (1986) improve upon Ganter's algorithm by removing the need for operator intervention to remove what they (following Ganter 1981) term are 'interior' faces. 'Interior' face is the name given to a minimum length basis 1-cycle that does *not* correspond with the boundary of a face in the 2-sphere, because if it did, some of its edges would have more than two cobounding faces (figure 6.24). This is the situation that causes the only mismatch between the minimum length basis set of 1-cycles achieved by the heuristics of Deo (1974) and the boundary 1-cycles of faces in a subdivision of the 2-sphere.

Courter and Brewer state that such faces may be detected by the fact that some of their edges have three cobounding faces. Due to the application of the second heuristic given above, such 1-cycles replace those that we are trying to find in the spanning tree (see figure 6.24(b)). Courter and Brewer remove the offending 'interior' faces and form one or more subgraphs from the collections of edges which have either zero or one cobounding face. Each subgraph is then analyzed to find fundamental 1-cycles which are then reduced to minimum length basis 1-cycles via the XOR process. Figure 6.25 shows the subgraph from figure 6.24(b) that needs to be analyzed after the removal of the 'interior' face).

(a)

(b)

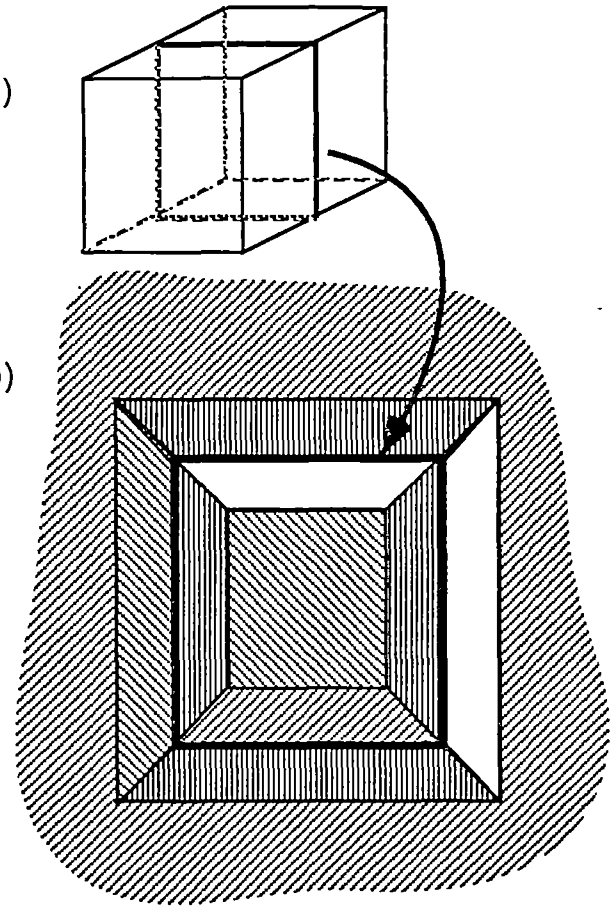


Figure 6.24 - an 'interior' face (a) An 'interior' face (highlighted lines) that would be found by the application of the cycle reduction process (b) The planar graph showing the 1-cycle which forms the boundary of the 'interior' face (highlighted) and a set of shaded faces that would be returned by the minimum length/least interaction heuristics. Notice that two faces are missing, one of these is the usual 'closing' face whilst the other has been 'replaced' by the 'interior' face.

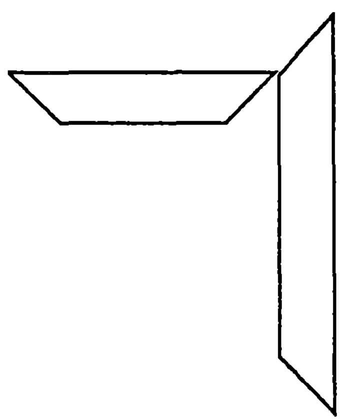


Figure 6.25 - the subgraph of edges from figure 6.24(b) that have zero or one cobounding faces after the removal of the 'interior' faces. It is processed in the same way as the original graph in order to determine the remaining two faces.

Courter and Brewer (1986) also note that no further 'interior' faces can be generated, because for each subgraph (ie. formed from the sets of edges with zero or one cobounding faces) we are no longer seeking an embedding in a 2-sphere, but in the 2-sphere with boundary (ie. the homeomorph of a polygon).

Hanrahan (1982) & (1985) expresses the wire-frame reconstruction problem in terms of graph theory. The aim is not only to find the surface or surfaces in which the graph embeds but also whether the graph embeds uniquely within that surface (ie. avoiding problems like the 'interior' faces in Ganter 1981 and Courter and Brewer 1986). Hanrahan uses the Edmonds permutation technique (Edmonds 1960) to show that although it is possible to find all embeddings of a general graph in a surface, it is also computationally expensive, since for a graph with n vertices there may be as many as $n!^n$ different embeddings. To overcome the problem of which surface the graph embeds in, Hanrahan reduces the domain of spaces from general 2-manifolds to 2-spheres (ie. planar graphs). To ensure that the graph embeds uniquely in a 2-sphere, Hanrahan applies a result about graph edge-vertex connectedness which is specific to planar graphs. That is, every planar graph which is triply connected (each vertex is connected by exactly three edges) embeds uniquely in the 2-sphere.

Wesley and Markowsky (1980) give a remarkable algorithm which discovers all possible embeddings of an arbitrary graph or wire-frame in any 2-manifold (and more complex spaces as well). The algorithm permits some singularities (ie. some pseudomanifolds) and cells with more than one boundary cycle (although they must be connected to the 'outside' boundary cycle with a 'bridge' edge). Unfortunately their algorithm is dependent upon the faces of the objects being geometrically planar, which is both an unnecessary reference to geometric information (as is shown by the efficacy of Ganter's algorithm) and an inconvenient restriction when constructing or altering objects interactively, particularly since planarity is not guaranteed when a face has more than three vertices. Lastly, although the process of calculating every possible embedding is very useful, it is also very costly particularly with more complex objects than those encountered in CAD applications - see the comments of Hanrahan 1982 about the complexity of such a task, given in the last paragraph. Removing these disadvantages, possibly by integrating this approach with some of the others described above, would be an interesting subject for future research.

This research proposes an extension of Ganter's algorithm (Ganter 1981) to all subdivided 2-manifolds (ie. including the sphere with *k*-handles) and subdivided 2-manifolds with boundary. The reasons for choosing Ganter's algorithm (in preference to the others reviewed above) are:

1. Despite Ganter's formulation for planar graphs (ie. embeddings in the 2-sphere) the algorithm *can* be extended to the 1-skeleton of other 2-manifolds (such as the torus) and 2-manifolds with boundary. Of the other alternatives, the plane sweep algorithm probably has a tractable higher-dimensional analogue. However the construction of the boundary cycles of generalized regular cells would still require a solution similar to that we propose here, whilst Hanrahan's method is confined to planar graphs by the condition on the edge-vertex connectivity of the graph.

   Problems with ambiguous embeddings in the case of general 2-manifolds and 2-manifolds with boundary are resolved by apriori knowledge of the generating 1-cycles of the fundamental group. This is similar to the requirement that a spatial object be built cell by cell from its strong deformation retract using the combinatorial homotopy operators as described in section 6.3.2.

   The claim that this extension can be made, is based on the fact that Ganter's algorithm calculates the fundamental cycles of a graph. Topologists use the fundamental cycles to find the generators of the fundamental group of spaces such as graphs, 2-manifolds, 2-manifolds with boundary and more general spaces. If, however, the generating 1-cycles of the fundamental group of the space are already known, then the algorithms used by topologists may be 'reversed' such that (in this case) they return the 1-cycles that form the boundary cycles of 2-cells, instead.

2. Unlike the method of Wesley and Markowsky (1980), Ganter's method is a topological method; ie. it does not impose any initial geometric restrictions on faces. Although the ultimate geometry of the faces in the spatial information system described in this research will most likely be planar, all of the results of chapters 4 and 5 are not dependent on this. The reason is that in many applications there is a need to obtain and

integrate data from a variety of sources including engineering/CAD systems where faces may not be geometrically planar. Thus a topology reconstruction algorithm must be able to deal with such faces even if they are eventually converted to geometrically planar faces.

The next section takes up the first of these points in more detail, explaining the underlying topological basis of Ganter's algorithm and why the generating 1-cycles of the fundamental group must be provided in order to find the 1-cycles that form the boundary cycles of the 2-cells. It also makes clear, why both the generating 1-cycles of the fundamental group *and* the 1-cycles that form the boundaries of 2-cells cannot be calculated from the 1-skeleton.

6.4.2 The Extended Topology Reconstruction Algorithm

As mentioned in the last section, Ganter's method of detecting the boundary cycles of faces in a planar graph actually corresponds with that used by topologists to find the generating 1-cycles (often just called the *generators*) of the fundamental group of any graph. The basis of the method used by topologists, is that the spanning tree of a graph is a subspace of the graph which does not contribute generators to its fundamental group, since a tree has the homotopy type of a point and thus, a trivial fundamental group. This in turn, implies that the spanning tree of the graph can be collapsed to a point using a strong deformation retraction. In point-set terms, the result of this collapsing process is a bouquet of n 1-cycles (figure 6.26) where n is the number of edges in the graph which are not in the spanning tree.

The mathematician H. Tietze devised a process which returns an equivalent result, but which uses a combinatorial process based on the spanning tree. It is the Tietze method (see also section 4.4) that Ganter unknowingly used as the basis of his algorithm. The set of fundamental cycles of a graph is calculated by constructing a spanning tree of the vertices (or 0-cells) and then forming an approach path p_i from the base vertex V (or root) to each vertex v_i in the graph. Each edge v_iv_j is associated with the loop $p_i(v_iv_j)p_j^{-1}$. If the edge is not in the spanning tree, then the loop is a fundamental cycle and a generator of the fundamental group of the graph (figure 6.27).

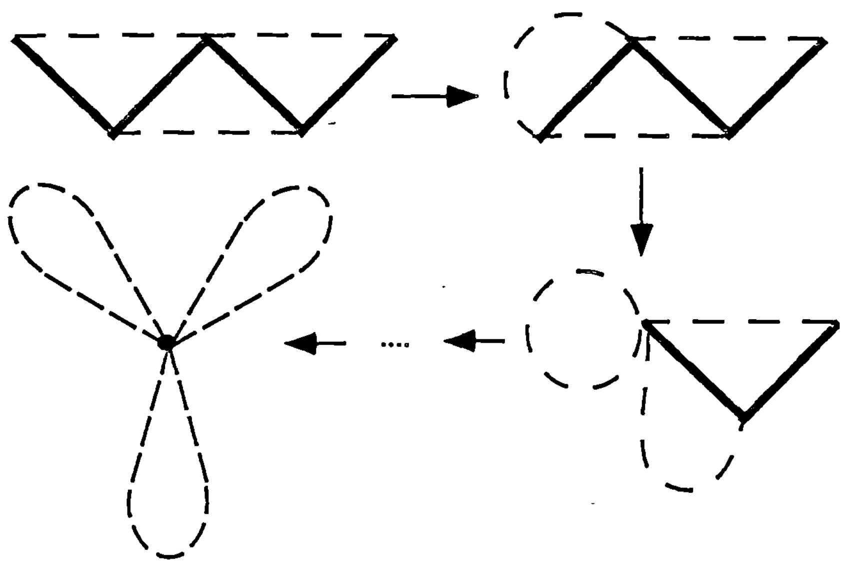


Figure 6.26 - collapsing the spanning tree of a simple graph to form a bouquet of three circles - the generators of its fundamental group. The spanning tree of the graph is shown by the thick black lines. Each step in the process represents the collapse of one edge of the spanning tree to a point.

If the simple graph we have given in figure 6.27 had a 2-cell embedding, then the fundamental cycles resulting from the Tietze method would correspond with the boundary 1-cycles of these faces or 2-cells. However, this correspondence does not always occur - see the example given in figure 6.21 and 6.22. As mentioned in the last section, Ganter realized that the minimum length basis set of 1-cycles (calculated from the fundamental cycle set using the heuristics described in Deo 1974) is very *close* to the set of 1-cycles that form the boundaries of 2-cells in an embedding of a planar graph, ie. a subdivision of the 2-sphere. Differences occur because the minimum length criterion returns 'interior' faces, which (as mentioned above) 'replace' some of the 1-cycles that form the boundaries of 2-cells.

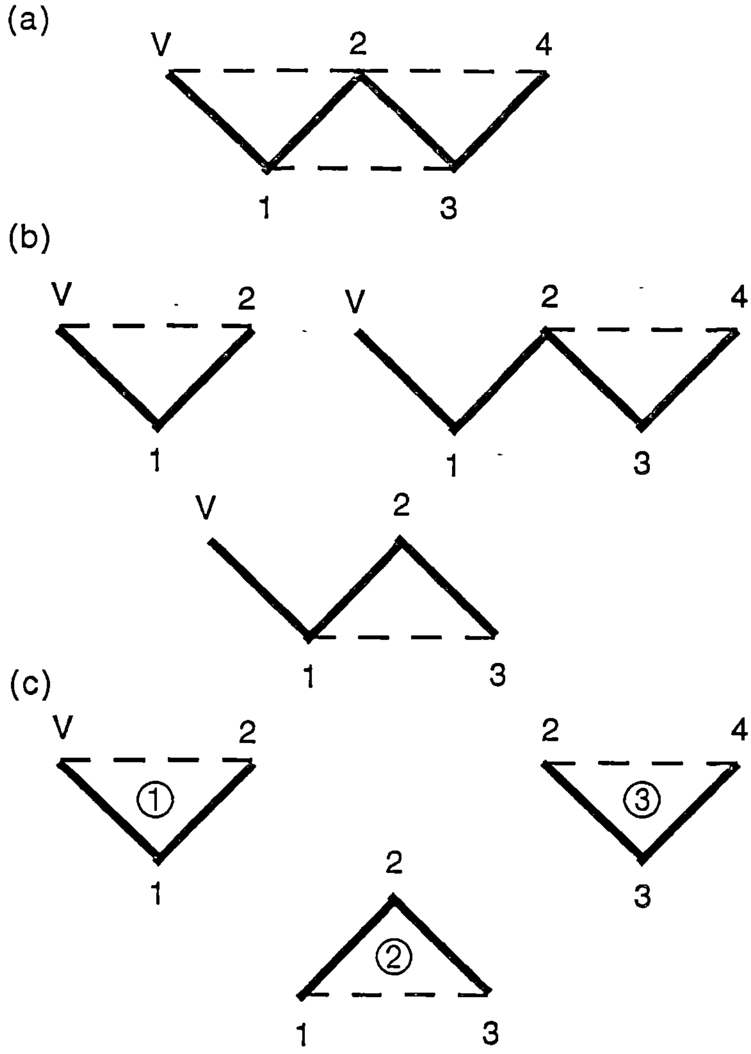


Figure 6.27 - the Tietze method applied to the graph in figure 6.26. **(a)** the graph **(b)** the paths traced from V through the spanning tree to each edge of the graph that is not in the spanning tree and returning to V **(c)** After removing the edges in the paths that are traversed twice (spurs) the result is a set of three fundamental cycles - a result which is equivalent (in topological terms) to the intuitive point-set method given in figure 6.26 above.

In general, the set of 1-cycles that form the boundaries of 2-cells in a subdivided 2-manifold or 2-manifold with boundary, are a basis set of a more general criterion - homotopies of 1-cycles or loops (see section 4.4). The boundary cycles of 2-cells are the 'smallest' set of null-homotopic 1-cycles, in that they cannot be deformed onto any other null-homotopic 1-cycle, except the trivial 1-cycle (ie. a point). The modification to Ganter's algorithm, suggested by Courter and Brewer (1986) for dealing with 1-cycles that bound 'interior' faces (ie. removal of 'interior' faces and processing of the graph(s) formed by the edges which have zero or one face) achieves the required reduction from the minimum length basis cycle set to the basis set of null-homotopic 1-cycles. The reasoning becomes clearer if we ignore (for the moment) the 'interior' face of these 1-cycles and think in terms of homotopies of 1-cycles in the 2-sphere and the 'smallest' set of null-homotopic 1-cycles. What we find is that these 1-cycles actually 'contain' other 1-cycles; ie. they may be deformed homotopically onto any of the 1-cycles they 'contain' (figure 6.28).

Whilst referring to homotopies of 1-cycles, it is interesting to note that the XOR reduction process devised by Ganter (1981) actually performs homotopies of the 1-cycles in the set of fundamental cycles.

Now we examine what happens when we try and extend this approach to graphs which are not planar, but embed in general 2-manifolds (such as the torus) and in the 2-manifolds with boundary; ie. the graph is the 1-skeleton of a subdivided 2-manifold or subdivided 2-manifold with boundary. In terms of homotopies of 1-cycles, we find that because the fundamental group of such surfaces contains generating 1-cycles, the assumption that the 1-cycles in the minimum length basis set will (eventually) form the boundaries of 2-cells is no longer correct. The reason is that 2-manifolds may have generating 1-cycles in their fundamental group, unlike the sphere which has none. For example, the torus has two generating 1-cycles ('handle' 1-cycles) in its fundamental group. If Ganter's approach is applied to the 1-skeleton of a subdivided torus, then the generating 1-cycles (and any 1-cycles homotopic to them) appear in the set of minimum length basis 1-cycles after the XOR reduction process is completed. The same is true of 2-manifolds with boundary, since 'perforation' 1-cycles will also be present.

It turns out that there is insufficient topological information in the 1-skeleton to unambiguously distinguish between those 1-cycles that represent the generators of the fundamental group (including those 1-cycles homotopic to them) and those 1-cycles that are null-homotopic (ie. those that form the boundaries of faces or 2-cells). These ambiguities are the underlying topological reason for the exhaustive calculation of the different embeddings of a wire-frame in a general surface given in Wesley and Markowsky (1980). The existence of these ambiguities also indicates why the other topology reconstruction algorithms mentioned in section 6.4.1.B do not venture beyond planar graphs.

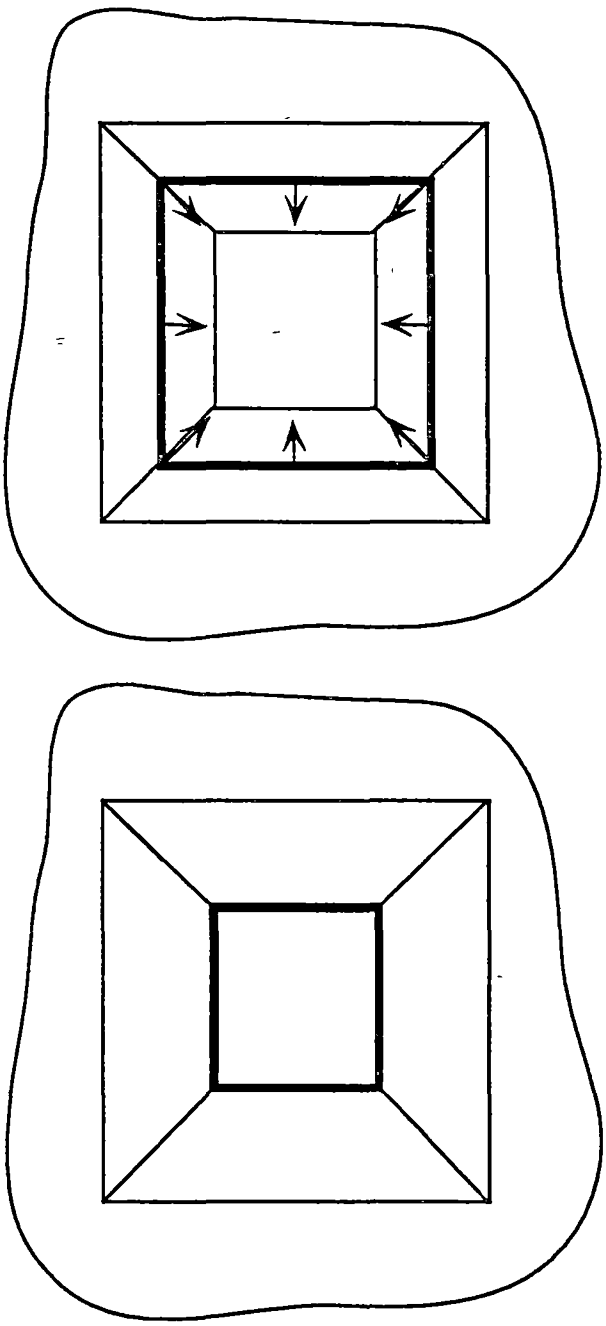


Figure 6.28 - a 1-cycle forming the boundary cycle of an 'interior' face of the 2-sphere may be deformed homotopically onto another 1-cycle in the 2-sphere (see figure 6.24).

The statement that there is insufficient information in the 1-skeleton alone to distinguish between the minimum length basis 1-cycles that belong to the null-homotopic basis set and those that represent or are homotopic to the generators of the fundamental group can be verified by examining the information used by topologists to calculate the generators of the fundamental group of a 2-manifold or 2-manifold with boundary. Stillwell (1980) pg. 138 describes this process as a continuation of the Tietze method. Using the set of fundamental cycles derived from a spanning tree of the 1-skeleton, the aim is to study the effect on the generators of the fundamental group of the graph of attaching disks to the graph; ie. 'filling in the interior' of the 1-cycles that will form boundary cycles of faces or 2-cells in the subdivided 2-manifold or 2-manifold with boundary. We might expect that the effect will be that some or all of the fundamental cycles of the graph will be made null-homotopic. For example, if the graph is the 1-skeleton of a subdivided torus then some fundamental cycles should remain after the disk attaching process because we know that the fundamental group of the torus has two generating 1-cycles. Thus, it should be possible to group the remaining fundamental cycles into two equivalent categories from which two representative 1-cycles may be chosen. These representative 1-cycles will be the generators of the fundamental group of the torus. Alternatively, if the graph is the 1-skeleton of a subdivided 2-sphere, then every fundamental cycle should be made null-homotopic by the disk attaching process because we know that the fundamental group of the sphere is trivial ie. all 1-cycles are null-homotopic. It is clear that this process presupposes knowledge about which 1-cycles form the boundary cycles of 2-cells. Unfortunately, this is knowledge which is not directly available to us as we only have the 1-skeleton of the subdivided 2-manifold.

To solve this problem, we essentially reverse the process described above. That is, instead of assuming knowledge of the basis set of 1-cycles that form the boundary cycles of 2-cells in the subdivided 2-manifold and then trying to find the 1-cycles that generate its fundamental group, we assume that the generating 1-cycles have been indicated in the 1-skeleton and then attempt to find those 1-cycles that form the boundaries of 2-cells (ie. are null-homotopic).

Apart from being relatively easy to identify in subdivided 2-manifolds and 2-manifolds with boundary, we can think of the generating 1-cycles of the fundamental group as the boundary cycle(s) of the canonical polygon of the underlying 2-manifold or 2-manifold with boundary (see section 3.3.1). This effectively clears up the traditional ambiguities about which 1-cycles of a wire-frame form the boundaries of 'holes' (ie. homotopic to the generating 1-cycles of the fundamental group) and which form the boundary cycles of faces. It also removes the need to exhaustively calculate and distinguish between the all possible 2-manifold embeddings of a wire-frame as is done by Wesley and Markowsky (1980), for example.

There are two remaining issues:

1. <u>Distinguishing 1-cycles homotopic to the generators of the fundamental group</u>

For *subdivided 2-manifolds*, any handle 1-cycles that are specified as generators of the fundamental group before the topology reconstruction algorithm is run are not unique. In fact as mentioned in section 4.4, the generating 1-cycles are just representatives of an equivalence class of such 1-cycles. In terms of the algorithm, this means that the XOR process may return minimum length basis 1-cycles that are homotopic to the generating 1-cycles of the fundamental group (as mentioned above). Topologists use the combinatorial homotopy operators (ie. cellular collapses and expansions) to show that such 1-cycles are homotopic to one another. However the application of the minimum length heuristics and the fact that the strong deformation retract is homotopy equivalent to a bouquet of 1-cycles (ie. a set of 1-cycles sharing a vertex) ensures that such 1-cycles will have the following two distinguishing properties:

(a) Edges with three or more faces incident to them. That is, they meet the definition of an 'interior' face as described by Courter and Brewer (1986).

(b) They share a single vertex/0-cell with at least one of the indicated generating 1-cycles of the fundamental group. This result comes from our knowledge of strong deformation retracts in section 3.3.2 and the Poincaré Duality theorem, which loosely speaking, states that each generating 1-cycle in a 2-manifold is self-dual and must thus intersect another generating 1-cycle at a point (see Scott Carter 1993, pg. 40 and pg. 251).

In a *subdivided 2-manifold with boundary*, the discussion of their construction in sections 6.3.1 and 6.3.2 indicates that the perforations and their boundary cycles (specified as generating 1-cycles of the fundamental group) may be thought of as 'empty' faces; ie. faces which have had their interior point-set removed. Consequently, the boundary 1-cycles of these perforations (perforation 1-cycles) will appear in the minimum length basis 1-cycles, directly matching those specified (apriori) as generators of the fundamental group. Handle 1-cycles will also appear in the minimum length basis 1-cycles, and can be distinguished as described above for subdivided 2-manifolds.

2. The 'closing face' problem for subdivided 2-manifolds

Courter and Brewer (1986) state that in the reconstruction of the topology of a subdivision of a 2-sphere from a planar graph, the boundary 1-cycle of a single face known as the 'closing face' is always missing (see the discussion on the closing face in section 6.4.1 above). They state that the reason for this is the difference between the Euler equation for the number of fundamental 1-cycles in a graph and the Euler equation for the number of faces in a subdivided 2-sphere. This statement also implies that Ganter's technique for subdivided 2-spheres actually returns a subdivided 2-sphere with boundary.

Whilst Courter and Brewer's statement is correct, recasting it in terms of the fundamental group (a more indicative topological invariant than the Euler equation) provides a much clearer meaning which also extends to any 2-manifold, not just the 2-sphere. Loosely speaking, the 'reversed' Tietze method used in our extension of Ganter's algorithm (described earlier in this section) involves calculation of the fundamental cycles of the 1-skeleton (a graph), removal of any 1-cycles that are equivalent to the generators of the fundamental group of the 2-manifold, and retaining the remaining 1-cycles (after various reductions) as 'faces' or 2-cells.

The closing face problem arises exactly because the 1-skeleton is a graph and the fundamental group of a graph is a free group (Stillwell 1980 pg. 97) generated by the fundamental cycles returned by the Tietze method. Our extended algorithm simply removes some of these fundamental cycles by 'filling them in' (ie. attaching faces) but it *cannot* change the freeness of the generators because (by **grc1**) the 2-cells in a generalized regular cell complex *cannot* be attached such that they create relations between the generators eg. such as those formed by the 2-cell in the CW complex of the torus in figure 4.5 (see section 4.2). As a consequence, the only spaces that can be reconstructed are subdivided

2-manifolds with boundary since they also have freely generated fundamental groups (Massey 1967 pg. 131).

Futhermore, the generators of the fundamental group of the 2-manifold we are trying to obtain, correspond to both the strong deformation retract (section 3.3.2) *and* the generators of the fundamental group of the 2-manifold minus the closing face. This result is shown by Stillwell (1980) pg. 141 and Massey (1967) pg. 131, theorem 5.3, who indicate that the fundamental groups of both the sphere with *n* handles and the sphere with *n* handles plus one boundary cycle are generated by 2*n* 1-cycles ie. $a_1, b_1, \ldots\ldots, a_n, b_n$. The same relationship holds for the sphere and the sphere plus one boundary cycle - both have a trivial fundamental group since all 1-cycles in both spaces are null-homotopic.

Lastly, it is important to note that the closing face is not unique. Ganter's algorithm (and the extension we describe) cannot determine apriori *which* face will be the closing face. For example, in the case of a 2-sphere embedding, Courter and Brewer (1986) correctly state that the closing face may be viewed as the infinite face (world 2-cell) in a planar representation of the subdivided 2-sphere such as that shown in figure 6.24(b). However it is possible to construct many planar representations of a subdivided 2-sphere, each with a different face as the infinite face. Whilst the non-deterministic nature of the closing face is not a problem when constructing a 2-manifold, it *is* a problem when attempting to construct a 2-manifold with more than one boundary. The simplest solution to this problem is to ensure that the user specifies the additional perforation 1-cycle together with the actual perforation cycles and any handle 1-cycles that generate the fundamental group. The algorithm can now proceed as for the 2-manifold (ie. including the calculation and addition of the non-deterministic closing face to the list of valid faces), with the final step being the removal of all faces corresponding to perforation cycles.

From these discussions it can now be seen that the application of Ganter's algorithm (with its underlying basis in the Tietze method and the extensions of Courter and Brewer 1986) to the problem of finding an embedding of a planar graph (ie. an embedding in the 2-sphere) as described in section 6.4.1, is a restricted case of the extended approach described above, since the fundamental group of the 2-sphere is trivial. In other words, since all 1-cycles in the 2-sphere are null-homotopic, Ganter's algorithm did not require the user to indicate the generators of the fundamental group.

We can now give a complete overview of the extended topology reconstruction algorithm.

A. Overview

Given the generators of the fundamental group of a subdivided 2-manifold or 2-manifold with boundary, the method of Ganter (1981) with the extensions of Courter and Brewer (1986) and those described above can be applied to the 1-skeleton using the following steps.

1. Construct a spanning tree of the vertices in the 1-skeleton using a breadth-first search technique, with the root of the spanning tree at any vertex contained within the generators. The spanning tree is then used to produce a set of fundamental cycles. This process can be done using any of the algorithms described in Deo et. al. (1982).

2. The fundamental cycles are then reduced to the set of minimum length basis 1-cycles using the XOR homotopy process governed by the two heuristics of Deo (1974); ie. two 1-cycles c_1 and c_2 are reduced to form a new 1-cycle c_1' if:

   (a) the total number of edges in c_1' is less than *or* equal to the total number of edges in c_1, and;

   (b) the total number of edges that c_1' shares with all other 1-cycles is less than the number of edges that c_1 shares with all other 1-cycles.

   The following statements can now be made about the set of minimum length basis 1-cycles:

   For the 2-sphere: every 1-cycle is null-homotopic because the 2-sphere has trivial fundamental group. Some 1-cycles may form the boundaries of 'interior' faces. As noted above, the 2-sphere is the only 2-manifold considered by Ganter (1981) and Courter and Brewer (1986).

For the 2-sphere with $m \geq 1$ boundary 1-cycles (eg. the cylinder): Since the perforations in a 2-manifold with boundary 1-cycles can be thought of as 'empty' faces, some of the 1-cycles returned by the algorithm will correspond with these perforation 1-cycles which are specified (apriori). In addition, there may be 'interior' faces as for the 2-sphere.

For the 2-sphere with k-handles ($k \geq 1$) (eg. the torus): Some 1-cycles found by the reduction process will match or are homotopic to the handle 1-cycles forming the generators of the fundamental group (specified apriori). If they are homotopic to a handle 1-cycle then they must share a single vertex with one of the other handle 1-cycles in the generators of the fundamental group, since the fundamental group of the sphere with k-handles has $2k$ generators (ie. $2k$ loops sharing a vertex). In addition, homotopic 1-cycles will also display the same criteria as interior faces (ie. some of the edges have three or more cobounding faces). Lastly, conventional interior faces (as described above for 2-spheres) may also exist.

For the 2-sphere with k-handles ($k \geq 1$) and m boundary 1-cycles ($m \geq 1$) (eg. the perforated torus): Once again, since the perforations in a 2-manifold with boundary 1-cycles can be thought of as 'empty' faces, some 1-cycles returned by the algorithm will correspond with these perforation 1-cycles (which are specified apriori). The remaining 1-cycles obey the conditions set down for the 2-sphere with k-handles above.

3. Ignoring all 'interior' faces formed by null-homotopic 1-cycles and/or all 'interior' faces formed by 1-cycles homotopic to one of the generators of the fundamental group, the set of edges with zero or one cobounding faces are formed into a subgraph and resubmitted to the cycle finding process in step 1.

4. Assuming that the 1-skeleton specified as input to the algorithm actually has the fundamental group generated by the specified 1-cycles, the end of the process will be reached when no further reductions can be made at step 3. At this stage all 1-cycles should be either null-homotopic (ie. the boundary 1-cycles of faces) or homotopic ('equivalent') to at least one generator of the fundamental group.

A global check can then be made on the subdivided 2-manifold/2-manifold with boundary by using a version of the Euler-Poincaré equation for a 2-sphere with k handles and m boundary cycles ($k,m \geq 1$):

$$v - e + f = 2 - 2k - m$$

B. Implementation

A prototype of this algorithm has been constructed using PRO Matlab version 4.1 by MathWorks Inc. For the present we have adopted the approach of Ganter (1981), where the input to the algorithm consists of a list of edges expressed in terms of their begin and end vertices which form the 1-skeleton of the 2-manifold or 2-manifold with boundary. The following processing steps are required:

1. The list of edges is formed into an adjacency matrix **X** where $\mathbf{X}(i,j) = 1$ if there is an edge between vertex i and vertex j, or 0 otherwise.

2. The adjacency matrix is processed by an algorithm described in Paton (1969) which has the advantage of finding the spanning tree and the fundamental cycle set at the same time (but see Deo et. al. 1982 for discussions on its efficiency). The output is a cycle matrix **C** where $\mathbf{C}(i,j) = 1$ if edge j is in cycle i, or 0 otherwise.

3. An interactions matrix **Z** is formed in order to support the application of the heuristics which reduce the fundamental cycles to the set of minimum length basis 1-cycles. $\mathbf{Z}(i,j)$ = the number of edges that cycle i has in common with cycle j. This matrix is formed by taking the vector dot product of row i with row j in **C**.

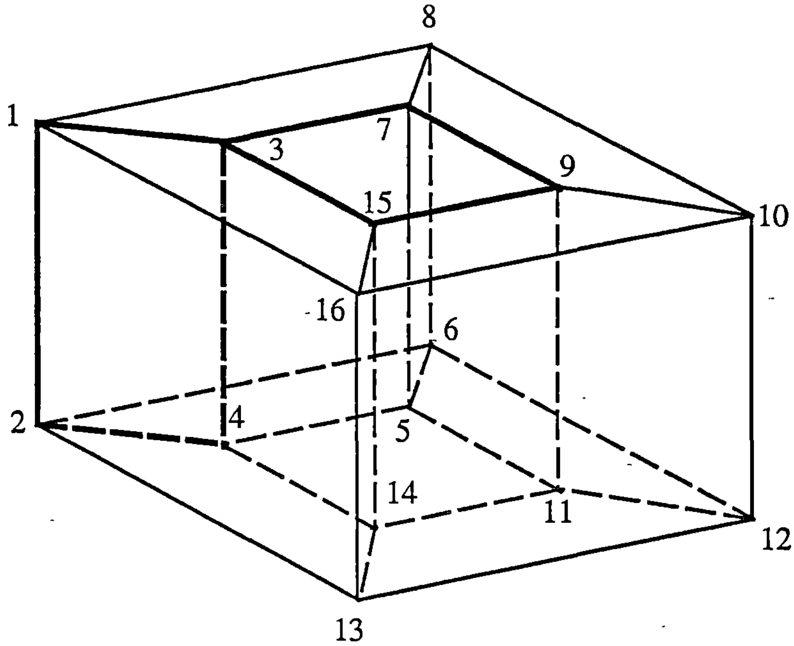
4. After each application of the XOR process the **C** is modified and **Z** is recalculated. Note that the modification to **C** may take the form of changes within a row (ie. when a cycle is reduced) or the addition and deletion of cycles.

We might expect the matrices in this version of the reconstruction algorithm to become very large (possibly unmanageable) if, for example, the input was the triangulated 2-manifold surface of a 3-dimensional spatial object. However one of the specified advantages of the reconstruction algorithm is that it is not dependent upon the geometry of the faces. Thus we may use a strategy similar to that which underlies the regular 2-arc in chapter 5; ie. split the triangulated data into a set of simple 2-arcs then submit the 1-skeleton formed by the boundaries of these 2-arcs to the topology reconstruction algorithm. After the topology of this 2-arc complex has been constructed, the topology of the triangles formed by the points and edges both 'within' the 2-arc (a 2-manifold with boundary) and forming the boundary of the 2-arc can be reconstructed. If necessary, the points alone may be inserted into a constrained Delauney triangulation algorithm (where the constraints are the boundaries of the 2-arc) or alternatively, some other method which can take advantage of the existing triangulated nature of the data, may be used.

C. Example

As an example of the algorithm consider the 1-skeleton of the torus shown in figure 6.29(a) and the spanning tree in figure 6.29(b). The chosen generators of its fundamental group are the two 1-cycles 1-2-3-4 and 3-7-9-15 (highlighted in figure 6.29(a)). Figures 6.29-6.33 show the remaining steps as per the algorithm overview given above.

(a)



(b)

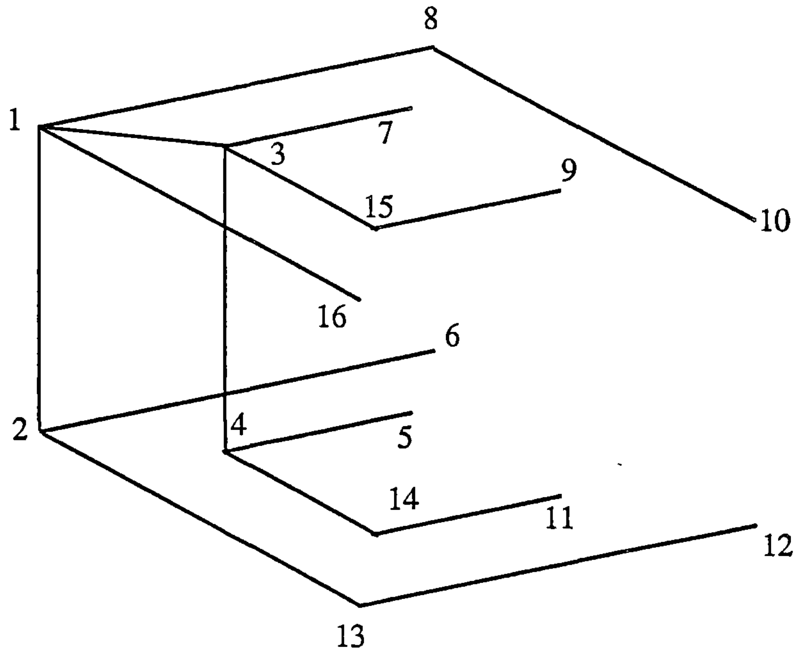


Figure 6.29 - the 1-skeleton of the torus and a spanning tree (a) the 1-skeleton of a subdivided torus (generators are highlighted) (b) A spanning tree of the 1-skeleton

The first stage of the algorithm produces seventeen fundamental cycles as shown in figure 6.30.

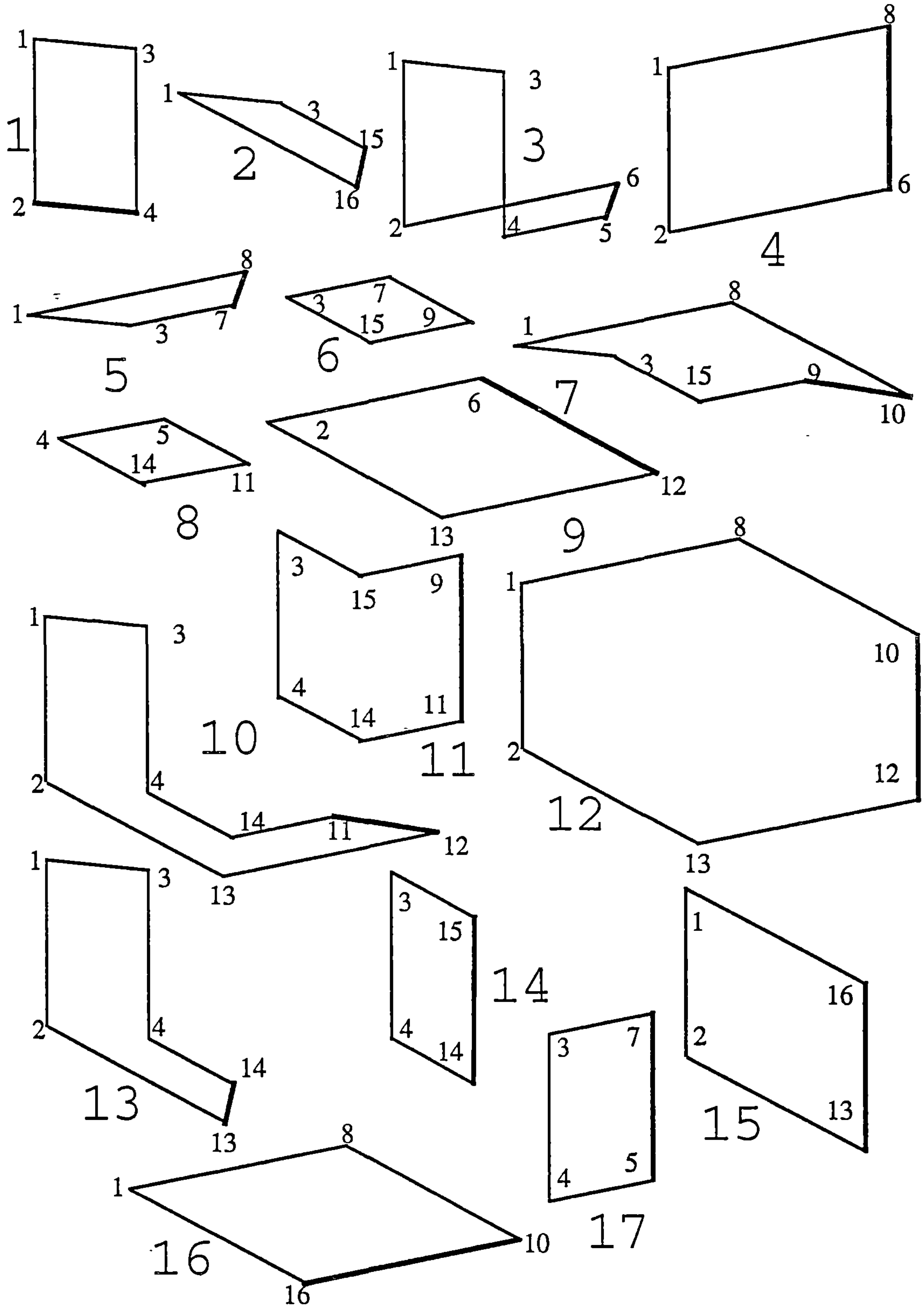


Figure 6.30 - **Step 1** - The result of the application of the Tietze method to the 1-skeleton in figure 6.29 is seventeen fundamental cycles (highlighted edges represent the edges missing from the spanning tree in figure 6.29(b))

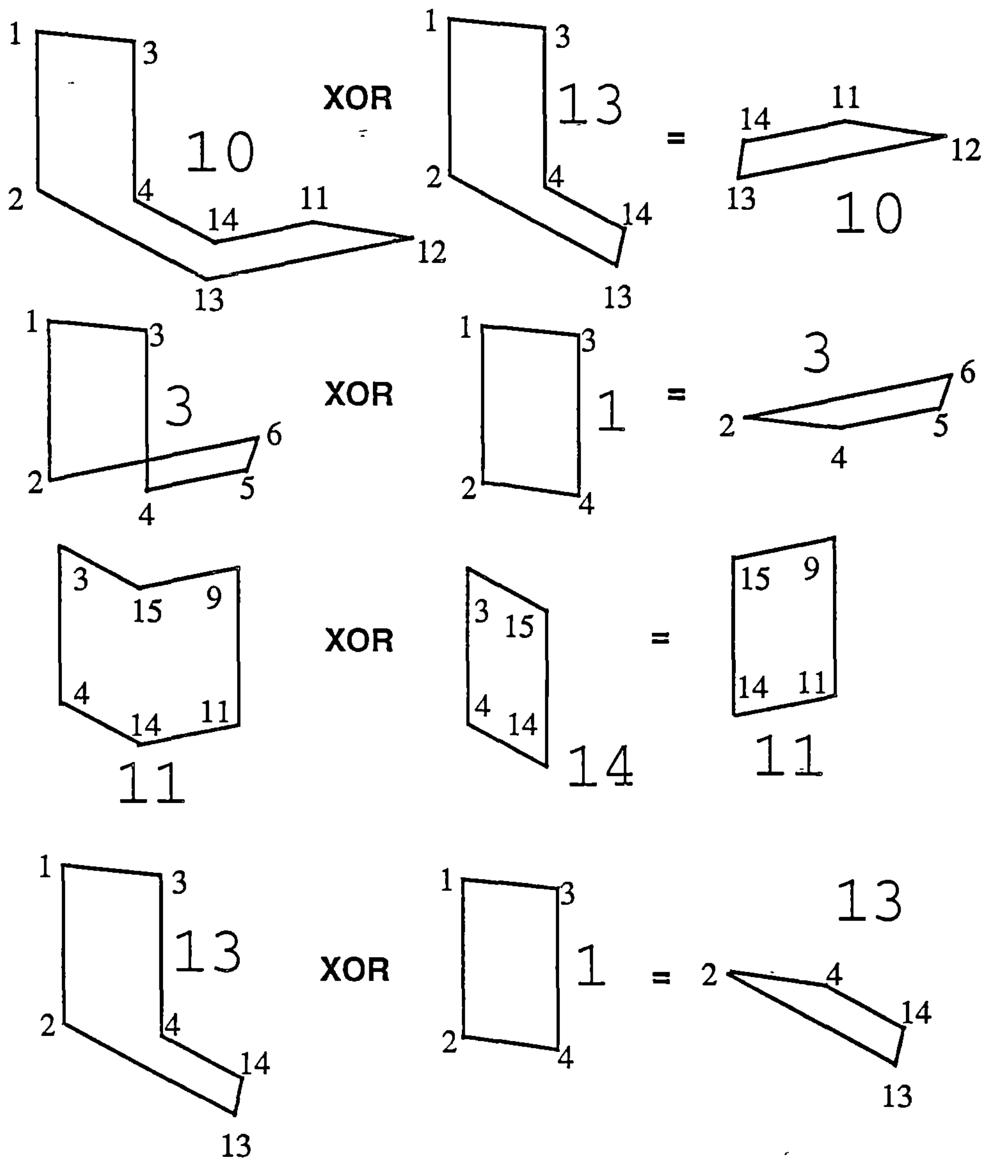


Figure 6.31 - **Step 2** - Reducing the fundamental cycles in figure 6.29 to the minimum length basis 1-cycles. Note that 1-cycles that are equivalent (homotopic) to one of the generators of the fundamental group (highlighted) share a single point with the other generator (continued overleaf).

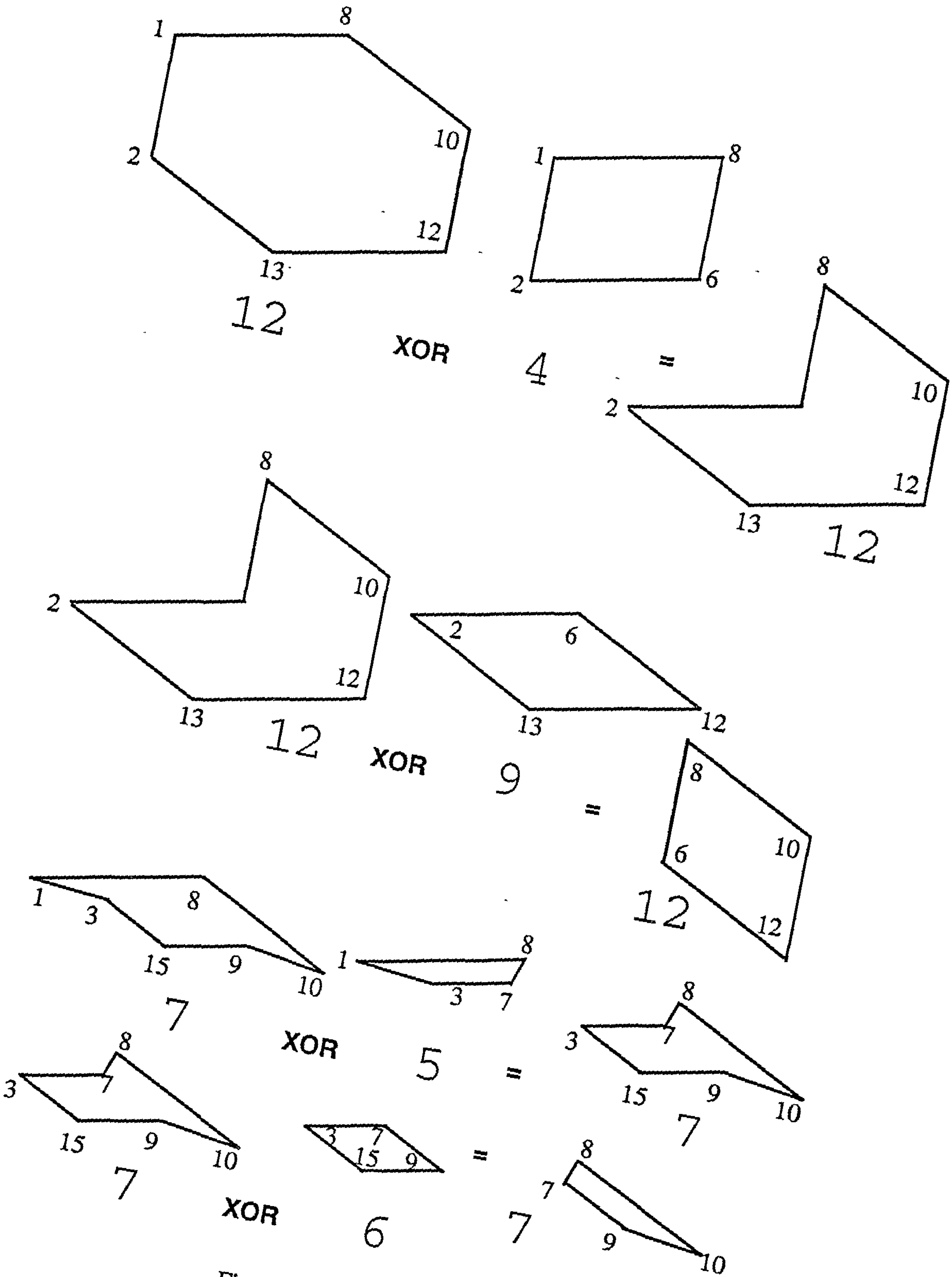


Figure 6.31 (continued)

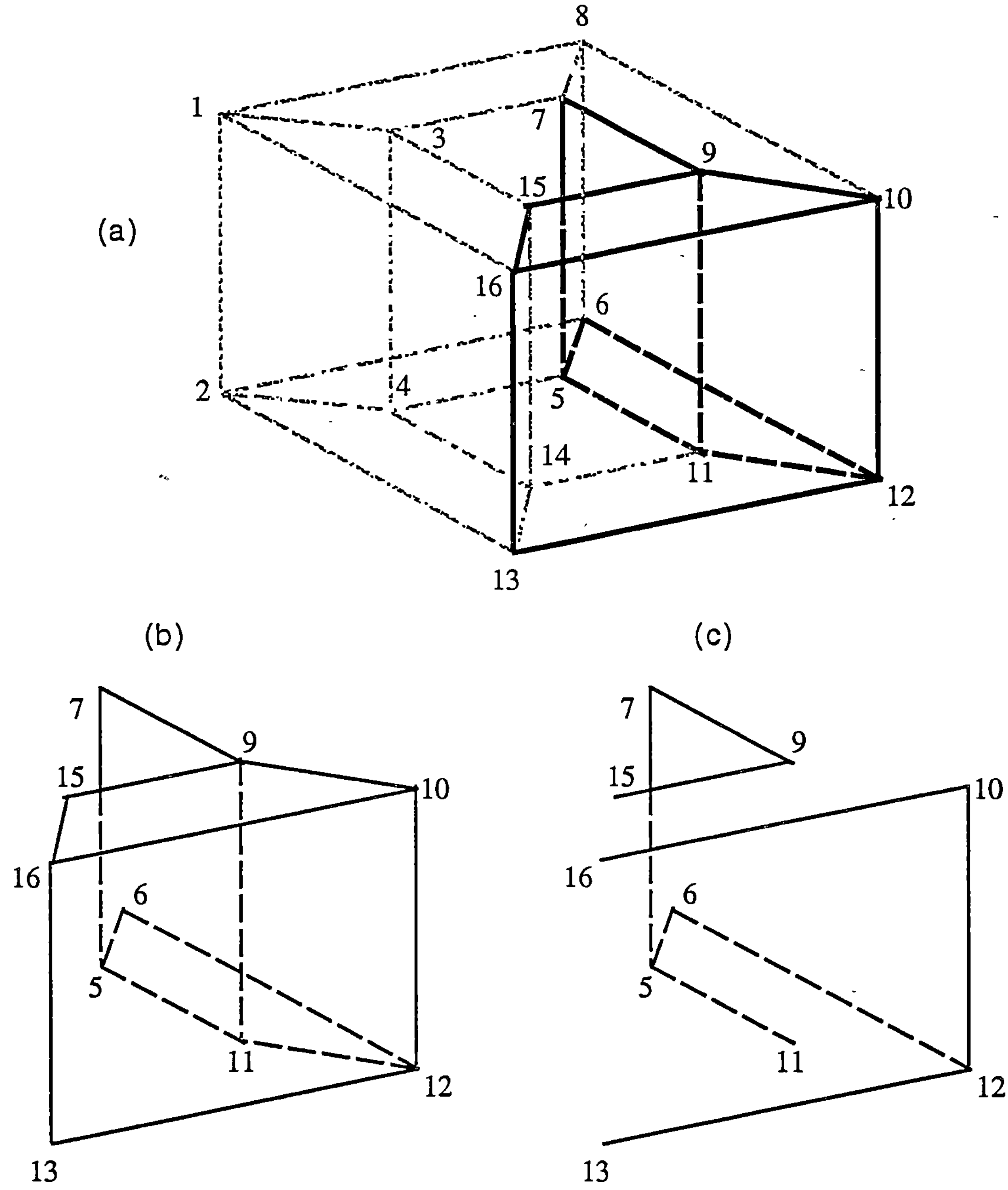


Figure 6.32 - **Step 3** - the graph formed by edges with zero or one cobounding faces. **(a)** and **(b)** The graph formed from those edges of the 1-skeleton which have either zero or one cobounding face after the XOR process (ignoring the 1-cycles that are equivalent (homotopic) to the generators of the fundamental group ie. cycles 1,6,8,9,16) **(c)** Spanning tree of this graph.

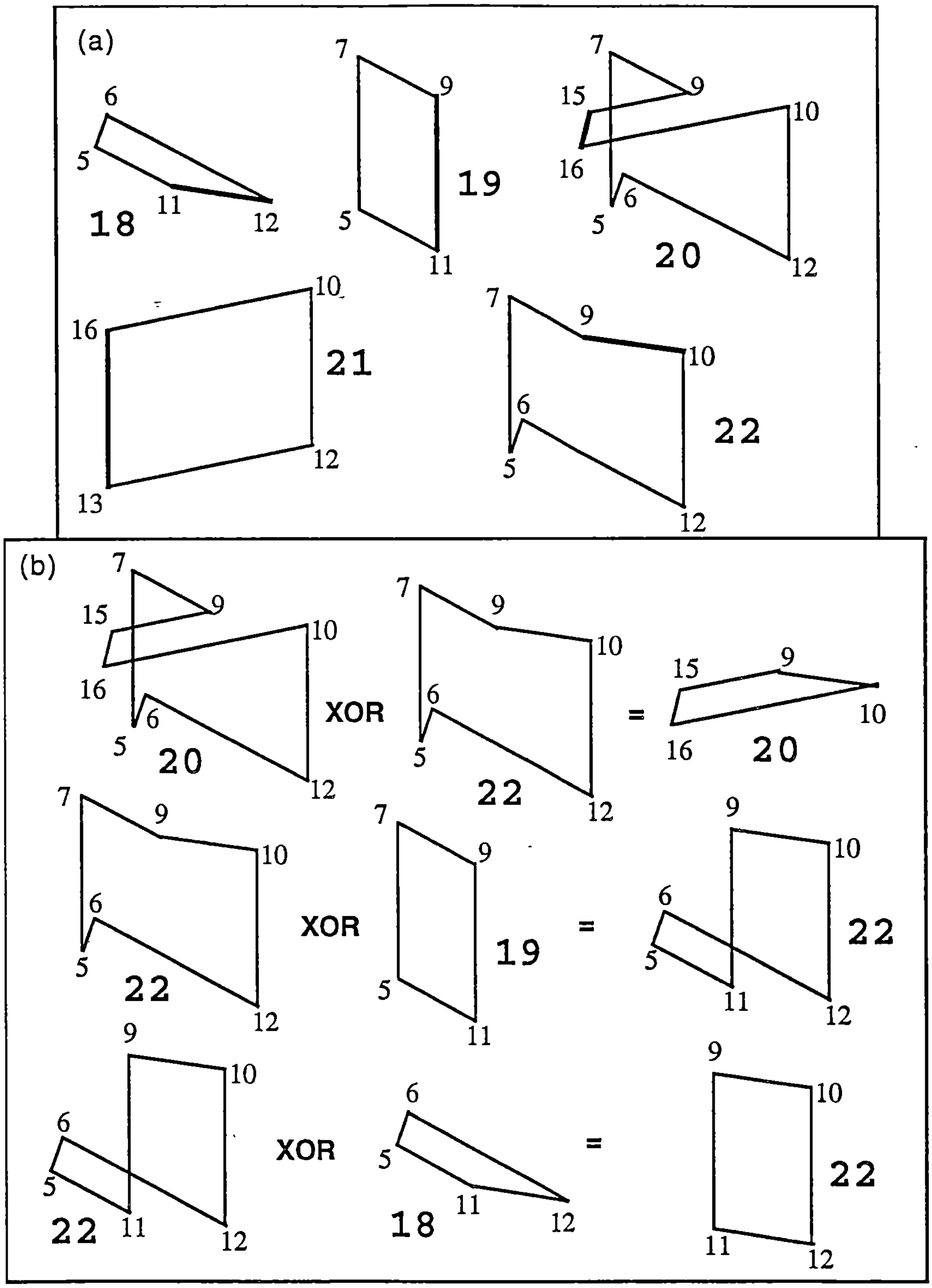


Figure 6.33 - **Step 1 and 2 (2nd Iteration)** - Fundamental cycles and XOR reductions **(a)** The fundamental cycles of the graph in figure 6.32(b) (derived from the spanning tree in figure 6.32(c)). **(b)** XOR reductions produce four faces plus one other 1-cycle (22) which, since it shares a vertex with one of the generators of the fundamental group (ie. homotopic to the other 1-cycle), can be ignored

Since there are no more edges with zero or one cobounding faces to process at step 3 (2nd Iteration), the process is complete. Ignoring the six not null-homotopic 1-cycles, the Euler equation $v - e + f = 2 - 2k$ shows that the process is complete since $16 - 32 + 16 = 2 - 2 = 0$.

D. Assembling the Subdivided Manifold/Manifold With Boundary

Not surprisingly, the *lift* and *join* operators (see section 6.2.3) and the combinatorial homotopy methodology described in section 6.3.2, can be used to construct the subdivided manifold returned by the topology reconstruction algorithm.

The generators of the fundamental group and any other 1-cells of the 1-skeleton necessary to form a path-connected 1-dimensional cell complex homotopy equivalent to the strong deformation retract, are assembled and the underlying 1-dimensional cell-tuple structure is created as described in section 6.3.2. As each 1-cycle is returned, the cell-tuple structure underlying its 1-skeleton is created and a *lift* operation is performed in order to create the 2-cell forming the face. If the 2-cell is to be attached to the strong deformation retract, then it is created with the necessary 1-cells and 0-cells from the 1-dimensional cell-tuple structure. When a 2-cell is to be joined to another then the *join* operator, with the combinatorial homotopy restriction described in section 6.3.2 is used.

Successful completion of the algorithm and/or the correct result is based on the assumption that the 1-skeleton of the 2-manifold forming the input data actually has a fundamental group containing the specified generators. Using the combinatorial homotopy operators provides a further check on this aspect of consistency, because when an anomaly is detected, the complex may already be partially constructed. A suite of elegant editing methods to fix any anomalies that occur during the algorithm is an area for future research.

Lastly, the *Elift* operator can be used to construct embedded 3-cells from reconstructed subdivided 2-manifolds, or embedded 2-cell complexes from reconstructed subdivided 2-manifolds with boundary. The *Ejoin* operator could then be used to join these spaces together to form a generalized regular k-cell complex ($k \leq 3$) in the Euclidean 3-manifold.

Chapter 7

Conclusions

7.1 Synopsis

The five major contributions of this thesis are:

1. The introduction of the generalized regular cell complex to reduce the number of cells required to represent an individual spatial object (particularly in 3-dimensional applications) whilst maintaining:

   (a) the ability to calculate important topological properties such as connectivity

   (b) 'backwards-compatibility' with simplicial and regular cell complexes ie. simplexes and regular cells are special cases of generalized regular cells.

   The reduction is achieved by generalizing the traditional regular n-cell in topology (ie. n-disks with (n-1)-sphere boundary cycle) to a Euclidean n-manifold with one or more (n-1)-manifold boundary cycles ($n \leq 3$). This optimization and the advantages are somewhat similar to those made in topology when simplexes were generalized to regular cells in a CW complex.

2. The introduction of the generalized singular cell complex to provide an integrated model for the representation of spatial objects (represented by generalized regular cell complexes) of different dimensions (multi-dimensional domain) within one cell complex. When generalized regular cell complexes are combined, cell boundaries may no longer be manifolds and a cell may have other cell complexes internal to its boundary cycles. We have

shown that the cell boundaries are now pseudomanifolds (via a primitive classification), which together with any internal cell complexes gives the definition of a generalized singular n-cell as a Euclidean n-manifold with (n-1)-pseudomanifold boundary cycles and internal structure ($n \leq 3$). The fact that the class of pseudomanifolds includes the class of manifolds shows that the generalized singular cell is 'backwards compatible' with generalized regular, regular and simplicial cells.

3. The introduction of topological construction operators based on the concept of 'expanding' a space from its strong deformation retract on a 'cell by cell' basis. The expansion process is the combinatorial equivalent of the point-set notion of strong deformation retraction (a homotopy equivalence) and is shown to be useful for constructing subdivided manifolds and manifolds with boundary, which can then be joined to form any generalized regular cell complex representing a spatial object.

4. The extension of an algorithm by Ganter (1981) that reconstructs the topology of a 2-sphere from its 1-skeleton, to any subdivided 2-manifold or 2-manifold with boundary. The basis of this extension is a previously unrealized relationship between Ganter's method and the Tietze method for calculating the generators of the fundamental group of a graph.

5. The extension of the implicit cell-tuple structure of Brisson (1990) to the representation of both generalized singular and regular cell complexes along with the discovery that the three distinct cycles of cusps defined intuitively for the tri-cyclic cusp of Gursoz et. al. (1991) can be encapsulated within the modified graph of coboundary orderings formed by the *switch* operations and the cell-tuples.

The generalized regular and singular cell complexes, the construction/reconstruction operators and the extended cell-tuple representation form the foundation of a simple, implicit topological model suitable for representing and analyzing spatial relationships between spatial objects of different dimensions in a 3-dimensional space.

7.2 Future Research

This research sets the foundation for a 3-dimensional topological model by attempting to answer the fundamental questions about the representation and construction of a topological model to support multi-dimensional domains. Some of the important questions and issues that remain to be answered are discussed in the following subsections.

7.2.1 Arcs

An efficient representation for the 2-dimensional arcs (2-arcs) introduced in chapter 5 must be developed to reduce the topological neighborhood information and thus the number of cell-tuples for each cell. At present the unordered neighborhood listings used by models such as those described in Corbett (1985) and Rossignac and O'Connor (1991) could be applied, particularly Corbett (1985) where the actual cells are very similar to the concept of the 2-arc defined in this research. However ordering information does exist and should be captured in a similar manner to the extension of the cell-tuple proposed in this research.

The main advantage of a 2-arc is that, loosely speaking, it is a 'super 2-cell'. A 2-arc removes the need to duplicate the same two-sided ordering information (ie. inside and outside 3-cell) for the generalized regular 2-cells in a 2-dimensional spatial object in R^3 or a 3-dimensional spatial object in R^3. This will lead to large savings in the number of tuples that need to be stored (particularly if triangulated surfaces are being used) and in the reconstruction of spatial objects from their cells, without the redundancy and maintenance overheads incurred by introducing additional global elements as is done in the radial-edge of Weiler (1986) and other explicit models. As explained in chapter 5, this concept is a direct (but necessarily more complex) generalization of the 1-arc used in 2-dimensional GIS and is largely inspired by the work of Corbett (1985).

7.2.2 Implementation

The main advantages of the cell-tuple (see also Brisson 1990) which are highlighted by the extensions proposed in this research, are:

1. *Uniformity* - unlike the quad-edge of Guibas and Stolfi (1985) or the facet-edge of Dobkin and Laszlo (1989) which can only be applied to subdivided 2-manifolds and 3-manifolds respectively, the cell-tuple can be applied to cell-complexes subdividing manifolds of any dimension. This property is essential to its application to the generalized regular and singular cell complexes given in this research.

2. *Simplicity* - the cell tuple is inherently simple and does not require complex data structures.

Both these advantages directly affect the implementation of the cell-tuple. As mentioned in section 2.5.1, section 4.4 and Brisson (1990), a simple implementation of the cell-tuple could be designed around a relational database where two lists of cell-tuples are maintained. The first list represents the tuples themselves whilst the second represents the results of the switch operations. An alternative implementation (also suggested by Brisson 1990), draws on the idea that the cell-tuples and switch operations form a graph. In the graph representation, the tuples are represented as nodes with the switch operations as pointers between the nodes.

The graph implementation is the main focus of our implementation effort since the main advantage is that graphs are well-known structures in computational geometry and computer science. One particular issue we have analyzed is the lack of path-connectivity of the graph in the set of tuples associated with a cell. An efficient implementation of the associated tuples concept is needed as the analysis of both generalized regular and singular cell complexes shows that only some switch information is available in many important situations (eg. in the neighborhoods of 0-cells in 3-dimensional spatial objects) and sometimes there is none at all (eg. when k-cells ($k \geq 2$) have more than one boundary cycle).

Regardless of the implementation strategy, a disadvantage of the cell-tuple (that arises from its inherent simplicity) is the combinatorial explosion in the number of cell-tuples (see also Paoluzzi et. al. 1993). Apart from the arc complex (see section 7.2.2), one possible way of reducing the number of cell-tuples is to use the cusp of Franklin and Kankanhalli (1993) in place of the cell-tuple (the similarity between the cell-tuple and the cusp of Gursoz et. al. 1991 is described in sections 2.6.5, 4.6.6 and 5.5.4 of this research). The switch operations could be applied to the vectors in Franklin and Kankanhalli's cusp. The number of cusps required would be less than the number of cell-tuples, yet the advantages of the cell-tuple remain.

Lastly, it would be interesting to examine the relationship between the graph formed by the cell-tuples and the switch operations and the proximity preserving orderings described in Saalfeld (1990).

7.2.3 Higher-Dimensional Applications

Given the difficulties topologists have experienced in classifying topological manifolds of dimensions ≥ 3, it is natural to ask whether the use of a cell with manifold (and thus pseudomanifolds) boundary cycle(s) is feasible for 4-dimensional temporal applications (space-time topology) and higher dimensional visualization applications. However it is essential to note that the difficulty in distinguishing higher-dimensional manifolds does not exclude the traditional definition of a cell. That is, it makes little difference whether an n-cell is an n-disk with (n-1)-sphere boundary (regular cell) or a Euclidean n-manifold with (n-1)-(pseudo)manifold boundary (generalized regular (singular) cell) since the unsolved Poincaré conjecture (see for example Stillwell 1980) indicates that neither the (n-

1)-sphere or an (n-1)-manifold can be recognized from their cell complexes when $n \geq 4$. Fortunately the full generality and complexity of these higher dimensional manifolds can be avoided by modelling higher dimensional spatial objects as the topological product of a lower-dimensional space and some simple space such as the interval. There are two areas where the topological product has been or can be successfully applied:

1. In visualization applications it is often a requirement to visualize attributes of spatial objects in a 3D SIS. For example a volume of ocean (represented by a 3-dimensional solid) may have a number of different attributes associated with its spatial representation by some type of sampling process (eg. temperature, salinity, oxygen, chlorophyll, nutrients). Roughly speaking, these attributes are visualized by attaching a cartesian space representing the attributes to the cell complex representing the spatial object. The visualization space is the topological product of the cells in the complex and the cartesian attribute space and is formed using the techniques of differential geometry (fibre bundles) eg. Butler & Pendley (1989). The topological product propagates the important topological properties of the spatial object to the visualization.

2. Simple spatio-temporal applications where time is linear and thus may be modelled as an additional dimension eg. geological applications. Spatio-temporal objects can be formed using the homeomorphic (or topological) product of a spatial pseudomanifold with an interval representing time (see Pigot and Hazelton 1992 for an introduction). The topological product allows the important topological properties (eg. connectivity etc) of the spatio-temporal objects to be derived from the topological properties of the original spatial objects.

7.2.4 Topological Operators

Future research on topological operators will focus on the following areas:

1. Extending the combinatorial homotopy theory. The modifications to the generic cell complex construction operators and the combinatorial homotopy theory described in this research are sufficient to: (a) directly construct subdivided k-manifolds with boundary that have strong deformation retracts of a point or a bouquet of circles, and; (b) indirectly construct a subdivided k-manifold by firstly constructing it as a k-manifold with boundary and then applying a 'closing' face. However, the adaptation of the CW attaching map, described in section 6.3.2 describes how to attach a generalized regular k-cell

($k \leq 2$) with one boundary cycle to *any* generalized regular cell complex, without changing the homotopy type. It also appears that the attaching map need not apply to cells of the same dimension eg. 1-cells could be attached to 2-cell complexes. This, together with a wider investigation of retracts and the deformation retracts of the pseudomanifolds described in chapter 5 (using theory described by Borsuk 1966), could provide the basis of an extension of the combinatorial homotopy operators to generalized singular cell complexes.

2. The construction of arcs and the arc complex. Having developed a representation for arcs (see section 7.2.1), the most natural method for constructing a generalized regular cell complex would be to construct the arcs using the combinatorial homotopy operators and then the generalized regular cells 'within' each arc by 'subdividing' the arc. As an example, consider the construction of a triangulated solid object such as an ore body in a mine information system. Firstly, the 2-arcs or faces of the solid could be attached to the strong deformation retract using combinatorial homotopy operators. Next, the solid itself is created by applying the *lift* operator to the cell-tuples of the 2-arcs as indicated in chapter 6. Lastly, coordinates are assigned to the 0-cells and each 2-arc can then be triangulated using an appropriate set of sampled coordinates. The triangulation captures the geometric structure and its topology is represented as a generalized regular 2-cell complex whilst the arc complex captures the topology between the individual arcs (and thus the generalized regular 2-cell complexes) as well as the topology of the generalized regular 3-cell representing the ore body.

3. Manipulating and editing generalized regular cell complexes. A general theory for developing topological operators to edit and manipulate existing cell complexes could be based on a modification of the theory of surgery (see Gauld 1982 and the suggestion in Takala 1991). Surgery is a technique by which two manifolds may be cut and joined along their boundaries. The modification of surgery is based on the idea that any manifold may be turned into a manifold with boundary and then joined with itself or other manifolds along that boundary. The *join/Ejoin* operators and the internal and external connected sums of 2-manifolds described by the global Euler operator KFMRH (see chapter 6) are, in fact, restricted forms of surgery on what we have described as generalized regular cells. Extending the surgery process to generalized regular cell complexes (ie. spatial objects that are not necessarily manifolds) and the division of surgery operations into two types - those that

change the homotopy type and those that do not - are subjects for future research.

4. Geometric consistency and an integrated method for manipulating both geometric and topological information. In this research we have focused only on maintaining topological consistency by developing construction operators based on combinatorial homotopy to preserve the homotopy type of the strong deformation retract. Geometric consistency is the process of ensuring that the geometry assigned to the vertices agrees with the topology. Problems with geometric consistency usually arise when the geometry of the 0-cells (ie. coordinates) of a cell complex does not agree with the topological information. Such situations may result from round-off error in geometric calculations (due to the limited precision arithmetic used in computer implementations of the real field) or in the assignment of coordinates to the 0-cells of a newly constructed cell complex. For example, two distinct 0-cells may have the same coordinates yet not be identified within the topology. Four alternative solutions to this problem are discussed and compared in Hoffman (1989) and there is much active research on robust algorithms that avoid or minimise the possibility of inconsistencies.

   The issue of manipulating both the topology and the geometry in a consistent manner must be addressed if existing spatial objects are to be manipulated and edited (as mentioned in above) and higher-dimensional applications developed (see section 7.2.3 and some of the applications of the surface evolver of Brakke 1993).

5. Full implementation of the extended topology reconstruction algorithm. A prototype of the matrix based approach has been constructed in PRO MatLab version 4.1.

   Of particular interest to this implementation would be a re-examination of the geometric dependencies and the decision trees in the method used by Wesley and Markowsky (1980) to calculate different embeddings of a 1-skeleton. It may be that their geometric approach could be combined with our extension of Ganter's method.

   The application of the modified Tietze method to other topology reconstruction problems such as those relating to reconstruction of a solid object from cross-sections or contours (Meyers et. al. 1992) and extrusion processes, will have the same benefits as those described in chapter 6. Both problems are under-constrained and many potential ambiguities could be resolved using the generators of the fundamental group of the required surface. For example, underground mine planning, three-dimensional workings are usually developed by extruding the 'floor' or level plan of existing or planned new work. Since the tunnels often contain pillars (eg. for roof support) many existing reconstruction algorithms struggle with the non-simply connected 2-manifolds formed by the extrusion. However, if the floor plan is path-connected and the generators of the fundamental group of the 2-manifold are indicated, then this information can be used in the modified Tietze method defined in chapter 6, to reconstruct the topology of the 2-manifold formed by the extrusion.

It would also be interesting to look at whether the modified Tietze method could be integrated within the 3-dimensional alpha shape reconstruction method put forward in Edelsbrunner (1987) and implemented in the *Alvis* package (Edelsbrunner and Mücke 1994).

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