SOME STUDIES IN ELASTIC SHELL STRUCTURES

by

R.F. Rish B.Sc., M.I.C.E., F.I.E. Aust.

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University of Tasmania

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SUMMARY

SECTION A

CHAPTER ONE introduces differential equations applied to the bending of a beam on simple supports. The solution is taken in the form of a Fourier series, each term of which satisfies the boundary conditions of the beam. It is shown that a solution of this form produces no constants of integration. The more advanced problem of the beam on elastic supports is then studied and it is shown how the solution to the differential equation is obtained and a table of derivatives drawn up. Particular problems are then solved by consideration of the boundary conditions.

CHAPTER TWO considers the stresses and deformations of a complete cylindrical shell with axi-symmetric loading. The differential equation is derived and shown to be of the same form as that for the beam on an elastic foundation. The solution is used to explain the anticlastic bending of a plate.

CHAPTER THREE derives the simplest form of the shell roof equation, that due to Schorer, and introduces an improved method for obtaining the derivatives of the solution. A direct design approach is introduced which is suitable for teaching an undergraduate class the design of a roof with post tensioned edge beams.

CHAPTER FOUR develops the membrane theory of cooling towers built up of a number of conical sections. Published results of tests on a model cooling tower are reanalysed to give better agreement with the theory than was obtained at the time. The theory is extended in the form of a computer program to deal with hyperboloid shells. The failures at Ferrybridge are considered and attributed to the analysis of cone-toroid shells as hyperboloids.

CHAPTER FIVE applies Schorer's equation to the deformation of a complete cylindrical shell with unsymmetrical loadings. A fourth order differential equation is derived similar to that of the axi-symmetrically loaded case but with smaller roots. The inextensional bending solution for open tanks is developed and used in place of the particular integral in a number of problems of practical interest.

CHAPTER SIX describes a method by which the roots of Flugge's equation for complete cylindrical shells can be extracted. It is shown that two sets of roots are obtained, the first identical to the shell with an axi-symmetrical load, the second identical to the shell with an unsymmetric load.

SECTION B

The method of chapter six is applied to Flugge's shell roof equation and gives rise to a new characteristic equation with explicit roots.

A shell roof with post tensioned edge beams is analysed using a Fourier series for the post-tension which converges rapidly. An edge correction is applied to retain compatibility at the ends of the edge beam.

Acknowledgments

Many people have helped me in the course of this project and it is not possible to mention them all by name.

Mr. Ian Baldwin and Mr. Alan Christian assisted in constructing the models tested. Miss. N. Lazenby has carried out the duplication of the paper and typed those pages that are not full of typing errors. Mr. J. Boothroyd taught me what I know of computing, and the encouragement, guidance and criticism of my supervisor, Professor A.R. Oliver is acknowledged with gratitude.

PREFACE

Since coming to the University of Tasmania in 1961 I have been lecturing to final year students in the theory of plates and shells. The thesis has been written in a form suitable for use in the course and has been derived from first principles, either in the body of the text or in the appendices at the end of each chapter. This means that the reader is not being referred constantly to other texts in order to follow the reasoning.

My interest in shell structures began eighteen years ago when I was engaged as a cooling tower engineer by W.V. Zinn Consulting Engineers of London. The only published material on cooling tower shells was by A. Fischer who had developed the membrane theory of single cones subjected to a wind load expressed as a Fourier series. The considerable stiffening effect of the upper part of the cone was neglected and another error was introduced by the differentiating of the Fourier series. I was able to overcome both of these problems and so entered the general study of shells feeling that I had already made a small contribution to the subject. On the other hand I had taken a short war time degree course and found the mathematics involved in the theory of cylindrical shells extremely daunting. The only writer in the field that I was able to follow with any facility was S. Timoshenko whose "Theory of Plates and Shells" I have taken as a model in exposition. This was because Timoshenko has assumed that his readers either did not know or had forgotten all but the simplest mathematics and so was prepared to develop his mathematical tools whenever the need for them arose.

Following Timoshenko I have introduced fourth order differential equations by considering the behaviour of beams on elastic foundations. The study of circular axi-symmetric shells follows naturally from this.

The easiest shell roof equation, that of Schorer is then developed together with a novel and simple method of obtaining the derivative of the solution. The edge beam problem is normally beyond the capabilities of an undergraduate class but a direct design approach is adopted which enables a class to design a shell roof with a post tensioned edge beam in a one hours problem class. This approach is probably not appropriate for practical design work as it does not yield the most economical solution, but the method is accurate and shows clearly the principles involved.

Another look has been taken at the cooling tower problem and the test results have been reanalysed with more satisfactory correlation between the tests and the theory. The failures of the towers at Ferrybridge in 1965 have been studied and a suggestion for the cause of the failures advanced.

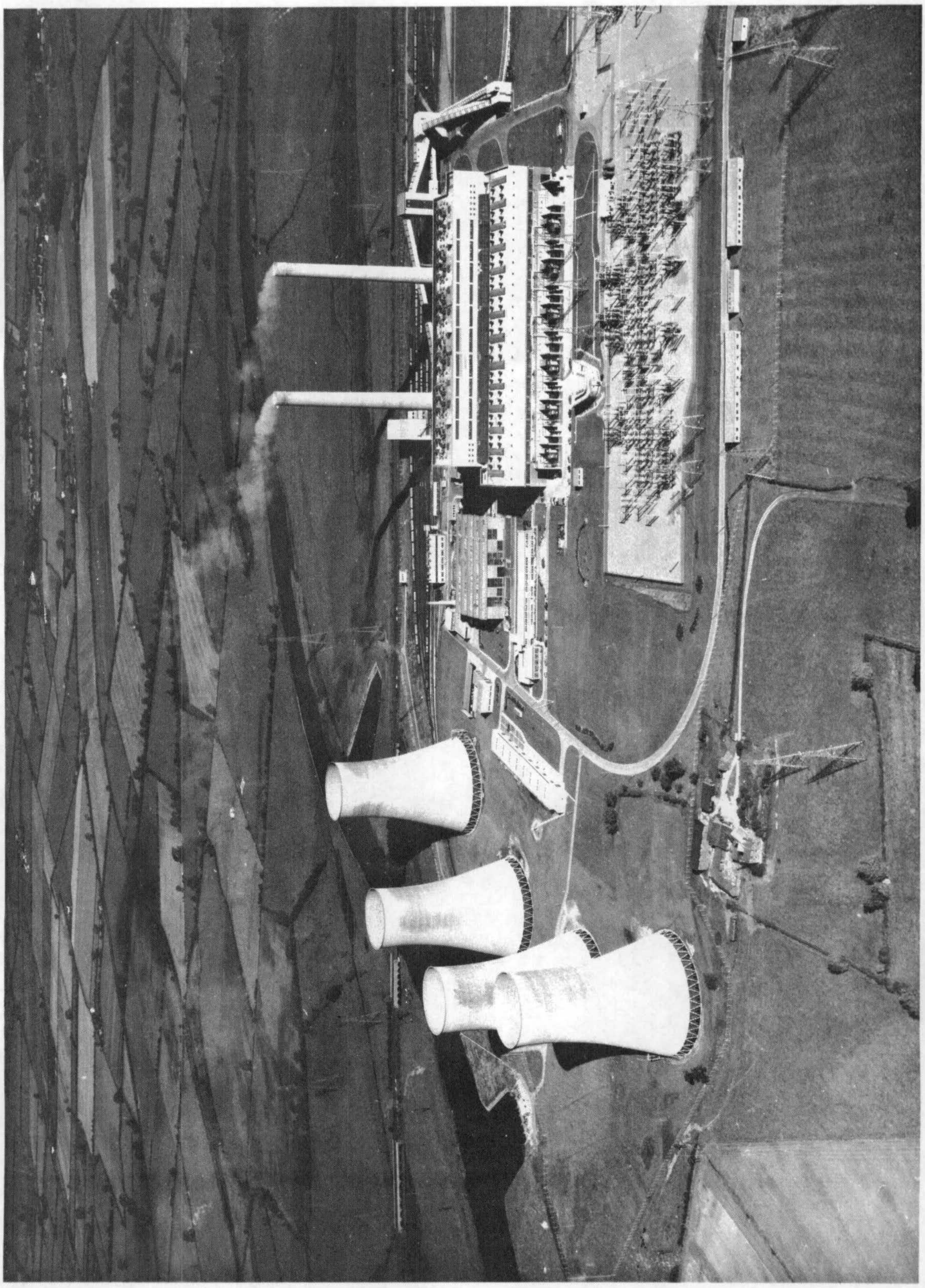
A similar equation to Schorer's has been derived for complete cylindrical shells with unsymmetrical loading. This equation is a fourth order one, similar to the axi-symmetric equation but with smaller roots. Edge effects therefore do not die away rapidly. A cylindrical shell with one end open and the other closed can bend without stretching under defined boundary conditions. This inextensional solution can serve as part of the solution in a number of problems of practical interest in the same way as the membrane theory acts as part of the solution in the case of the shell roof.

One paper that I found of great interest was by N.J. Hoff which compared Flügge's shell equation with Donnell's. This suggested to me that the roots of Flügge's equation could be extracted and those terms that did not contribute to the result could be dropped to produce a simplified equation. This was first tried in the case of the complete cylindrical shell and resulted in two fourth order equations, the first identical with the axi-symmetric equation, the second with the unsymmetric equation. These could be combined in the general case.

When the same method was applied to the shell roof equation a new eighth order equation arose. This had explicit roots and was more accurate than the Donnell equation for the longer shells. It was found that bending in the x direction could be neglected but not the twisting moments. The effect of the twisting moment on the membrane shear stresses was however negligible and the orthogonal membrane shears have been taken as equal throughout.

This simplified Flügge equation has been applied to the case of a shell roof with post tensioned edge beams. An improved method of handling the post tension is introduced which agrees well with tests on a model shell and with a finite element analysis carried out by a post graduate student Pham Lam. This work has been presented in a form suitable for independent publication in section B.

The author's publications on shell structures are included in the bibliography.



Castle Donnington Power Station

A group of cooling towers designed by the author when working with W.V. Zinn and Associates, London. Built by the Mitchell Construction Company Ltd, Peterborough.

Chapter 1.

THE BENDING OF BEAMS

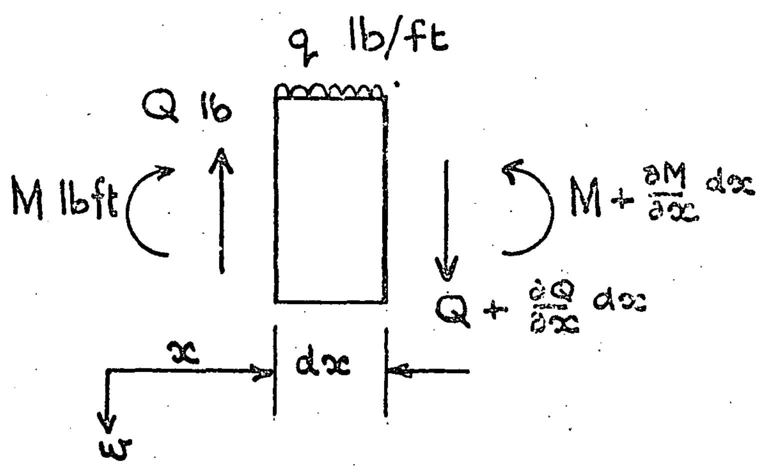
The study of elastic shells is considered to be a difficult one for an engineer. The subject will be introduced by first discussing a problem which is more familiar.

A uniformly loaded simply supported beam will be analysed by solving its general differential equation, a method that is also of value in the study of shells.

The general differential equation is obtained by combining the statics of an element with the moment-deformation relation. Its solution is the complete answer to the problem when the boundary conditions are included. The choice of a form of solution which includes the boundary conditions may sometimes avoid having to evaluate constants of integration. This form of solution is known as an eigenfunction.

SIMPLY SUPPORTED BEAM

Statics of Element



Resolving Vertically

$$\frac{\partial Q}{\partial x}dx + q\,dx = 0$$

$$\partial Q/\partial x = -q$$

Taking Moments

$$\frac{\partial M}{\partial x}dx - Q\,dx = 0$$

$$\partial M/dx = Q$$

Moment Deformation Relation

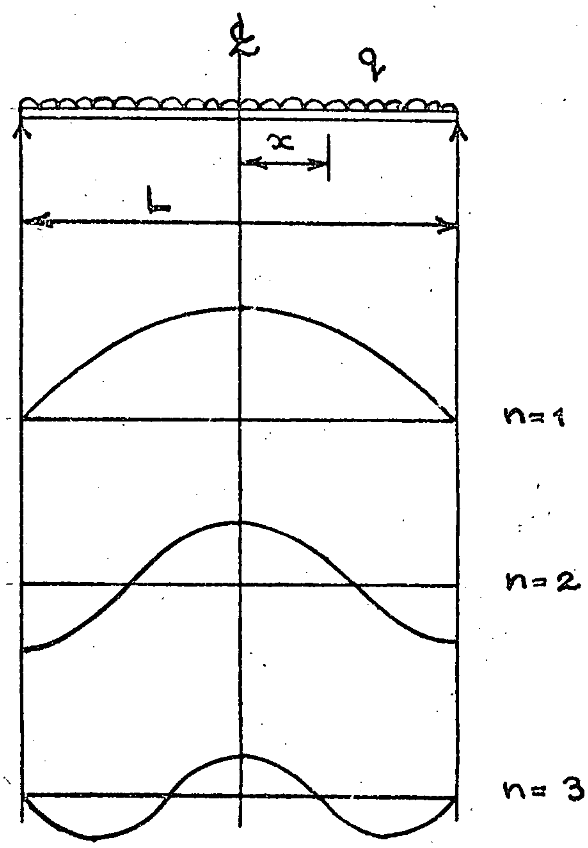
$$\left.\begin{aligned} M &= -EI\ \partial^2 w/\partial x^2 \\ Q &= \partial M/\partial x = -EI\ \partial^3 w/\partial x^3 \\ q &= -\partial Q/\partial x = EI\ \partial^4 w/\partial x^4 \end{aligned}\right\} \quad \ldots (1-1)$$

The differential equation of the beam is:

$$\partial^4 w/\partial x^4 = q/EI \quad \ldots \quad (1-2)$$

Some approximations and assumptions have been made in the derivation of this equation. The deformation due to shear has been neglected and the curvature taken as $+\frac{\partial^2 w}{\partial x^2}$ which is only correct if the deflections are not too large. It is assumed that there are no longitudinal forces in the beam with a vertical resultant. Otherwise equation (1-2) applies to any elastic beam with any loading and any condition of support. The particular case of the simply supported uniformly loaded beam of constant flexural rigidity will now be discussed.

The solution will be taken in the form of a Fourier series where each term satisfies the boundary conditions which are
(a) Symmetry about the ℄
(b) Zero deflection at the ends.



Take $\omega = \sum_{n=1,3,5\ldots}^{\infty} \omega_n \cos \frac{n\pi x}{L}$

Condition (a) is satisfied by taking only the cosine terms of the Fourier series, condition (b) by taking only the odd values of n.

Substitution in (1-1) gives:

$$\left.\begin{aligned} M &= EI(\pi/L)^2 \sum \omega_n n^2 \cos n\pi x/L \\ Q &= -EI(\pi/L)^3 \sum \omega_n n^3 \sin n\pi x/L \\ q &= EI(\pi/L)^4 \sum \omega_n n^4 \cos n\pi x/L \end{aligned}\right\} \ldots (1\text{-}3)$$

A constant value of q can be expressed as the Fourier series: (Appendix 1)

$$\sum_{n=1,3,5\ldots}^{\infty} q_n \cos n\pi x/L$$

Where

$$q_1 = 4q/\pi \qquad q_3 = -4q/3\pi \qquad q_5 = 4q/5\pi \ldots$$

Substituting in the last equation of (1-3) and taking a term by term correspondence of the series gives:

$$\omega_n = \pm 4qL^4/EI\pi^5 n^5$$

Which obviously decreases very rapidly with n.

For n = 1 $\qquad w = 0.01307\ qL^4/EI \qquad$ at x = 0

The accurate numerical coefficient is 5/384 or 0.01302

The higher derivatives are less accurate if only the first term is used.

Thus $\qquad M = 4qL^2/\pi^3 \qquad = 0.129\ qL^2 \qquad$ at $x = 0$

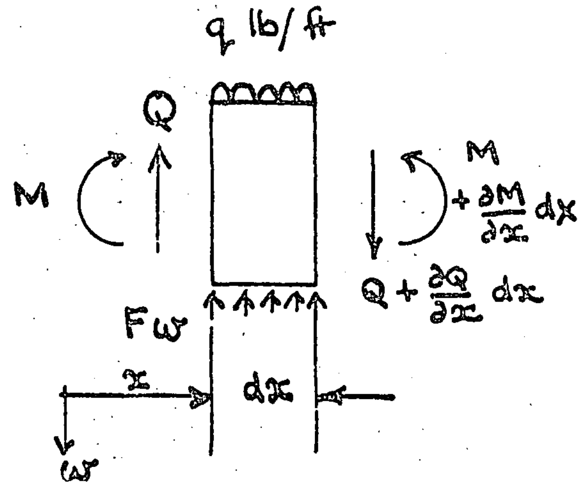
Whereas the correct coefficient is 1/8 or 0.125

And $\qquad Q = 4qL/\pi^2 = 0.403\ qL \qquad$ at $x = L/2$

Whereas the correct coefficient is 0.5

In shell analysis it will be found that terms are integrated with respect to x and the error involved in using only the first term in the Fourier series for w is acceptable.

BEAMS ON ELASTIC FOUNDATIONS



The reaction at the base of the beam is taken as varying linearly with the deflection, the elastic constant being F lb/sq ft.

Statics of Element

Resolve vertically $\quad \partial Q/\partial x = F\omega - q$

Take Moments $\quad \partial M/\partial x = Q$

Moment-deformation $\quad M = -EI\, \partial^2\omega/\partial x^2$

Then $\quad Q = -EI\, \partial^3\omega/\partial x^3$

And the general differential equation is:

$$\frac{\partial^4\omega}{\partial x^4} + \frac{F}{EI}\omega = \frac{q}{EI} \quad \ldots \quad (1-4)$$

This has a particular solution

$$\omega = q/F$$

which is the deflection of the beam under the distributed load q without any reactions or other point loads.
The complementary function is the solution of the characteristic equation

$$\partial^4\omega/\partial x^4 + 4b^4\omega = 0 \quad \ldots(1-5) \text{ where } 4b^4 = F/EI$$

Taking the solution in the form $\quad \omega = Ae^{px}$
and substituting in (1-5)

$$Ap^4e^{px} + 4b^4Ae^{px} = 0$$

giving the auxiliary equation $\quad p^4 + 4b^4 = 0$

then $p = b\sqrt[4]{-4} = b(\pm 1 \pm i)$
where i is the imaginary $\sqrt{-1}$
This can be checked by direct multiplication

$$(\pm 1 \pm i)^2 = 1 \pm 2i - 1 = \pm 2i \ ; \quad (\pm 2i)^2 = -4$$

A has four complex values corresponding to the four complex roots of the auxiliary equation

and $$\omega = A_1e^{b(1+i)x} + A_2e^{b(1-i)x} + A_3e^{b(-1+i)x} + A_4e^{b(-1-i)x}$$

Combining the imaginary parts into cosine and sine terms to get a real solution (Appendix 1) gives:

$$\begin{aligned} w = \; & e^{bx}\,(C1 \cos bx + C2 \sin bx) \\ & + e^{-bx}\,(C3 \cos bx + C4 \sin bx) \end{aligned}$$

Differentating this expression gives:

$$\begin{aligned} \partial w/\partial x = w' = \; & b\,e^{bx}\,\{(C1 + C2)\cos bx + (-C1 + C2)\sin bx\} \\ & + b\,e^{-bx}\,\{(-C3 + C4)\cos bx + (-C3 - C4)\sin bx\} \end{aligned}$$

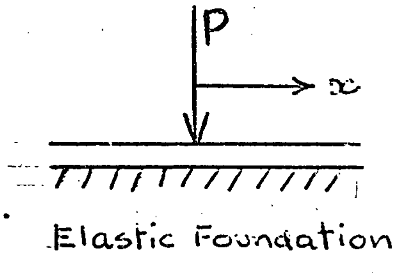
which is the same expression as before but with a multiplier and different coefficients. Tabulating the result and repeating the manipulation enables the other derivatives to be obtained rapidly and reduces the opportunity for error.

$f(w)$	Mult	e^{bx}		e^{-bx}	
		$\cos bx$	$\sin bx$	$\cos bx$	$\sin bx$
w	1	$C1$	$C2$	$C3$	$C4$
w'	b	$C1 + C2$	$-C1 + C2$	$-C3 + C4$	$-C3 - C4$
w''	$2b^2$	$C2$	$-C1$	$-C4$	$C3$
w'''	$2b^3$	$-C1 + C2$	$-C1 - C2$	$C3 + C4$	$-C3 + C4$

TABLE 1-1

The values of the integration constants C are found from the boundary conditions.

Point Load on a Long Beam



Now e^{bx} increases rapidly with increase of x. As the effects of a point load decrease with distance on a long beam $C1$ and $C2$ must be zero.

At x = 0 $\quad e^{-bx} = 1,\ \cos bx = 1,\ \sin bx = 0,\ w' = 0 \quad \therefore\ -C3 + C4 = 0$

At x = 0 $\quad Q = -P/2 = -EI\,w'''$

$$\therefore\quad 2b^3\,(C3 + C4) = P/2EI$$

$$\therefore\quad C3 = C4 = P/8EIb^3 = Pb/2F$$

Maximum deflection

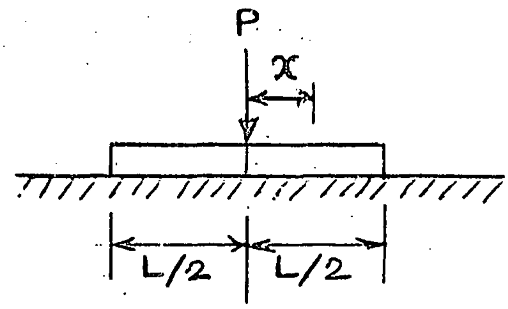
$$w_0 = C3 = Pb/2F$$

Maximum bending moment

$$M_0 = 2b^2\,EI\,C4 = P/4b$$

A similar fourth order equation arises in the study of cylindrical shells with axi-symmetric loading.

Short Beam with Central Load



It is assumed that the beam does not lift off the foundation.

If the beam is short it is not possible to take C1 = C2 = 0
Four boundary conditions are required to solve for the four ~~xxxxx~~ integration constants. They are:

(1) At $x = 0$ $\quad Q = -P/2$

(2) At $x = 0$ $\quad \partial w/\partial x = 0$

(3) At $x = L/2$ $\quad M = 0$

(4) At $x = L/2$ $\quad Q = 0$

The boundary conditions can be expressed as simultaneous equations that can be solved to find the constants. Putting $bL/2 = k$

$$(1) \quad -C1 + C2 + C3 + C4 = -\frac{Q}{2b^3EI} = \frac{Pb}{F}$$

$$(2) \quad C1 + C2 - C3 + C4 = 0$$

$$(3) \quad -e^{k}\sin k\, C1 + e^{k}\cos k\, C2 + e^{-k}\sin k\, C3 - e^{-k}\cos k\, C4 = 0$$

$$\text{or} \quad a_{31}C1 + a_{32}C2 + a_{33}C_3 + a_{34}C4 = 0$$

$$(4) \quad -e^{k}(\cos k + \sin k)C1 + e^{k}(\cos k - \sin k)C2$$
$$+ e^{-k}(\cos k - \sin k)C3 + e^{-k}(\cos k + \sin k)C4 = 0$$

$$\text{or} \quad a_{41}C1 + a_{42}C2 + a_{43}C3 + a_{44}C4 = 0$$

In matrix form this can be expressed as:

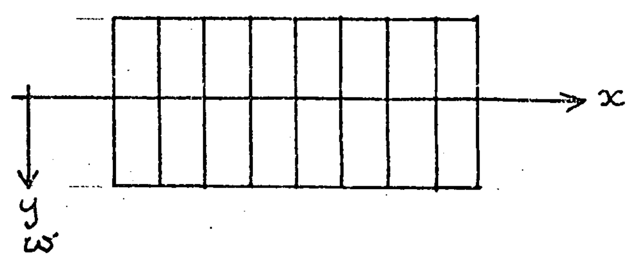
$$\begin{bmatrix} -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} C1 \\ C2 \\ C3 \\ C4 \end{bmatrix} = \begin{bmatrix} Pb/F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This can be readily solved in a particular case and the deflection, moments and shears calculated from equations (1-1) and Table (1-1).

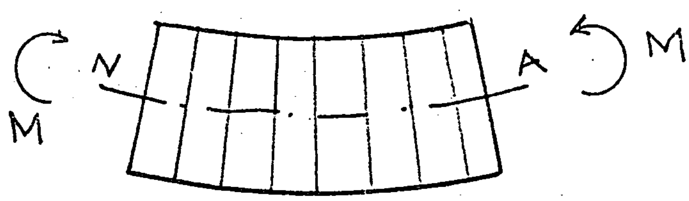
APPENDIX 1

Bending of beam

Consider a short length of beam subjected to a bending moment M and divided into plane sections normal to the axis.

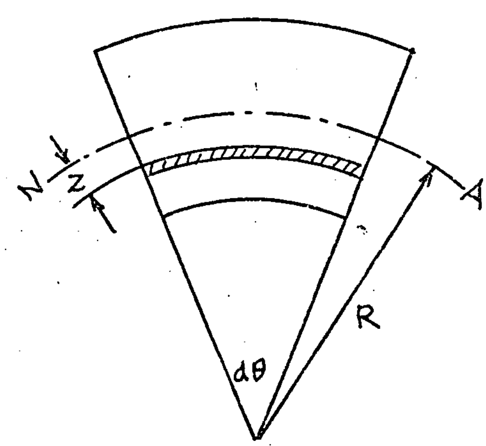


After bending the curvature must be constant as the bending moment is constant. The only way the elements can still fit together is for plane sections to remain plane.



It will be assumed that there is a horizontal plane NA which does not change its length during bending. This is known as the neutral plane or the neutral axis. The radius of curvature of the neutral axis will be called R.

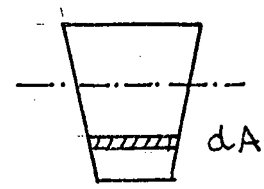
Consider an elemental area dA distance z below the NA.



The old length is the same as the neutral axis which has not changed = $R \, d\theta$

The new length $= (R - z)\, d\theta$

Strain $\epsilon = \dfrac{(R-z)d\theta - R\,d\theta}{R\,d\theta} = -\dfrac{z}{R}$



Stress $\sigma = -\dfrac{Ez}{R}$

Force $= -\dfrac{Ez}{R}\, dA$

The net horizontal force on the section is zero $\therefore \dfrac{E}{R}\int z\, dA = 0$

and z must be measured from the centroid of the section, ie the NA must pass through the centroid.

Taking moments $M = -\dfrac{E}{R}\int z^2\, dA = -\dfrac{EI}{R}$

Giving the familiar engineers' formula

$$\frac{M}{I} = -\frac{E}{R} = \frac{\sigma}{z}$$

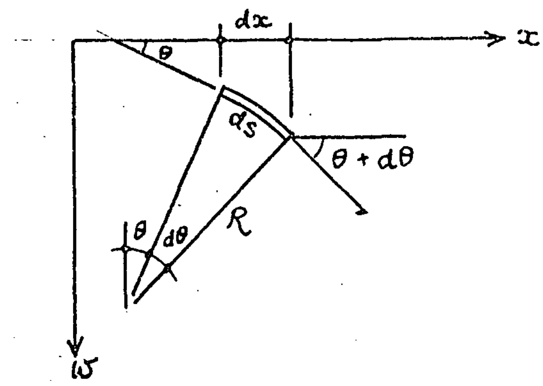
Now $1/R = d\theta/ds$

If slopes are small

$\theta = dw/dx$ and $ds = dx$

$$\therefore 1/R = \frac{d}{dx}\left(\frac{dw}{dx}\right) = \frac{d^2w}{dx^2}$$

And $M = -EI\, d^2w/dx^2$



Fourier Series for Constant Load

Let $q \equiv \sum q_n \cos n\pi x/L$ where n has valus 1 to ∞

Multiply both sides by a particular value of $\cos n'\pi x/L$ and integrate from 0 to $\pm L/2$

$$\int_{-L/2}^{+L/2} q \cos\frac{n'\pi x}{L}\, dx \equiv \sum \int_{-L/2}^{+L/2} q_n \cos\frac{n\pi x}{L} \cos\frac{n'\pi x}{L}\, dx$$

If q is a constant the left hand side of the equation is

$$q\frac{L}{n'\pi}\left\{\sin\frac{n'\pi x}{L}\right\}_{-L/2}^{+L/2} = \frac{2qL}{n'\pi}\sin\frac{n'\pi}{2}$$

$= 2qL/n'\pi$ if n' is 1, 5, 9 ...

$= -2qL/n'\pi$ if n' is 3, 7, 11 ...

Considering the right hand side of the equation

$$\int_{-L/2}^{+L/2} q_n \cos\frac{n\pi x}{L} \cos\frac{n'\pi x}{L} = 0 \quad \text{unless} \quad n = n'$$

$$= q_n L/2 \quad \text{if} \quad n = n'$$

$\therefore q_n = +4q/n\pi$ if $n = 1, 5, 9 \ldots$

$\qquad -4q/n\pi$ if $n = 3, 7, 11 \ldots$

Manipulation of complex roots

In general $e^{ibx} = \cos bx + i \sin bx$

$$A_1 e^{b(1+i)x} = e^{bx}(A_1 \cos bx + A_1 i \sin bx)$$

$$A_2 e^{b(1-i)x} = e^{bx}(A_2 \cos bx - A_2 i \sin bx)$$

Putting $C_1 = A_1 + A_2$ $\qquad C_2 = (A_1 - A_2)i$

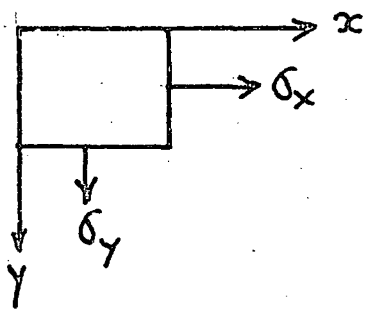
$$w = e^{bx}(C_1 \cos bx + C_2 \sin bx) \ldots$$

As the deflection is real the constants must be real.

Chapter 2.

BENDING IN SHELLS

Moments and forces will be taken per unit width of surface.
Consider a lamina stretched in its own plane by stresses in two directions at right angles x and y.



Taking Hooke's Law and Poisson's ratio into account:

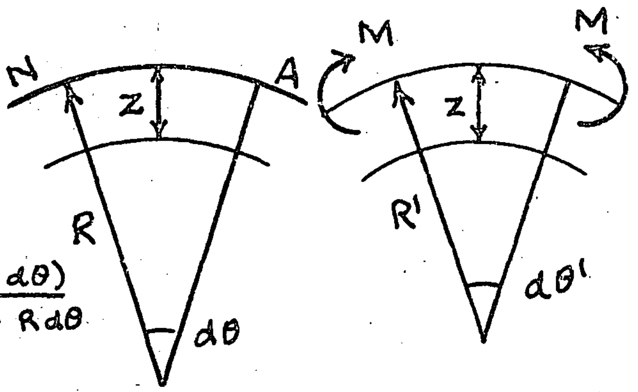
$$E\epsilon_x = \sigma_x - \nu\sigma_y$$
$$E\epsilon_y = \sigma_y - \nu\sigma_x$$
$$\nu E\epsilon_y = \nu\sigma_y - \nu^2\sigma_x$$

$$E(\epsilon_x + \nu\epsilon_y) = (1-\nu^2)\sigma_x$$

or $$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \quad \ldots (2\text{-}1)$$

Similarly $$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x)$$

Let the lamina be part of a thin shell of thickness h, with original radius of curvature R measured to the middle surface of the shell.
The shell is bent by a distributed bending moment M lb in/in to radius R'.
The middle surface of the plate corresponding to the neutral axis of a rectangular beam does not stretch



$$\therefore R\,d\theta = R'\,d\theta'$$

The strain of the lamina ϵ

$$= \frac{(R'-z)d\theta' - (R-z)d\theta}{(R-z)d\theta} = \frac{-z(d\theta' - d\theta)}{R'd\theta' \text{ or } Rd\theta}$$

if z is small compared with R.

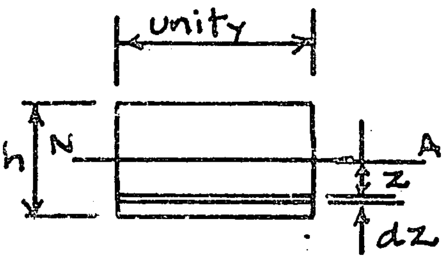
Hence $$\epsilon = -z\left(\frac{1}{R'} - \frac{1}{R}\right) = -z\chi$$

where χ is change of curvature $\frac{1}{R'} - \frac{1}{R}$

Thus from ~~(1-2)~~ equation (2-1)

$$\sigma_x = -\frac{Ez}{(1-\nu^2)}(\chi_x + \nu\chi_y)$$
$$\sigma_y = -\frac{Ez}{(1-\nu^2)}(\chi_y + \nu\chi_x)$$

Taking moments



$$M_x = \int \sigma_x \, z \, dz$$

$$= -\frac{E}{(1-\nu^2)}(\chi_x + \nu\chi_y)\int_{-h/2}^{+h/2} z^2\,dz$$

$$= -\frac{Eh^3}{12(1-\nu^2)}(\chi_x + \nu\chi_y)$$

$$= -D(\chi_x + \nu\chi_y) \qquad \ldots \quad (2\text{-}2)$$

Where D the flexural rigidity = $Eh^3/12(1-\nu^2)$

Similarly $\quad M_y = -D(\chi_y + \nu\chi_x)$

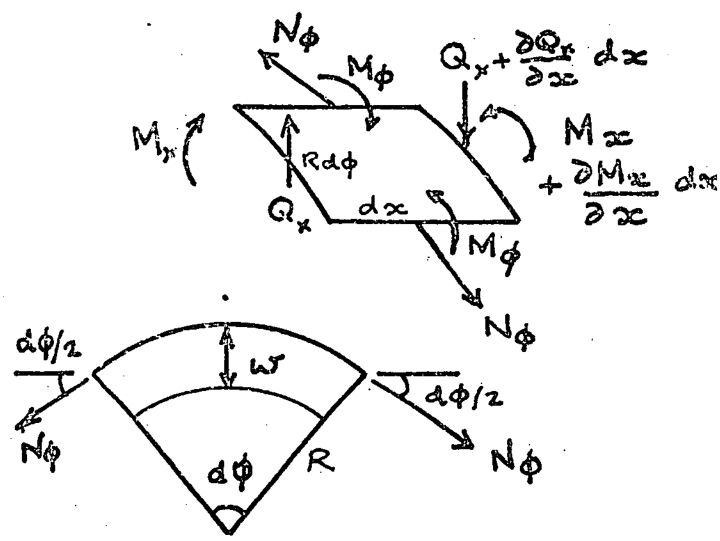
The maximimum bending stress is on the outside of the plate.

$$\sigma = \sigma_{max}\, z/0.5h \quad \text{and} \quad M = \frac{2\sigma_{max}}{h}\int z^2\,dz = \sigma h^2/6$$

$$\therefore \quad \sigma_{max} = 6M/h^2$$

<u>Axi-Symmetrical Bending of Circular Cylinder by Edge Loads</u>

The statics of the element will be considered first:



From the symmetry of the loading N_ϕ and M_ϕ are constant and there will be no membrane shears (in the middle surface) nor normal shears Q_ϕ . N_x and the normal load q will be taken as zero.

Resolving radially and multiplying the distributed forces per unit length by the distance over which they are acting:

$$\frac{\partial Q_x}{\partial x}dx \; R\,d\phi + N_\phi \, dx \, d\phi = 0$$

$$\therefore \quad \partial Q_x/\partial x = -N_\phi/R$$

Taking moments in the x direction:

$$\frac{\partial M_x}{\partial x}.dx\,R\,d\phi = Q_x\,R\,d\phi\,dx \quad \therefore \quad \partial M_x/\partial x = Q_x \cdots (2\text{-}3)$$

And $$\frac{\partial^2 M_x}{\partial x^2} = \frac{\partial Q_x}{\partial x} = -\frac{N_\phi}{R} \qquad \ldots \quad (2\text{-}4)$$

The force-displacement relations are obtained as follows:

$$\epsilon_\phi = ((R-w)\,d\phi - R\,d\phi)/R\,d\phi = -w/R$$

$$\sigma_\phi = E\,\epsilon_\phi$$ as there is no stress in the x direction.

$$= -Ew/R$$

$N_\phi = \sigma_\phi h$ lb/ unit length acting on the middle surface

$= -Ehw/R$

$M_x = -D(\chi_x + \nu\chi_\phi)$

$\chi_x = \partial^2 w/\partial x^2$ as shell was straight in the x direction

$\chi_\phi = \partial^2 w/R^2\partial\phi^2 = 0$ as w does not vary with ϕ

$+(1/(R-w) - 1/R) = \frac{w}{R^2}$ if w is small compared with R

$$\therefore \quad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu w/R^2\right)$$

It will be shown that $\frac{\nu w}{R^2}$ is very small compared with the first term in the bracket.

$$\therefore \quad M_x = -D\,\partial^2 w/\partial x^2$$

And $\quad Q_x = -D\,\partial^3 w/\partial x^3$ from equation (2-3)

Substituting in equation (2-4)

$$-D\,\partial^4 w/\partial x^4 = Ehw/R^2$$

or $$\partial^4 w/\partial x^4 + \frac{Eh}{R^2 D} w = 0$$

Putting this in the form

$$\partial^4 w/\partial x^4 + 4\beta^4 w = 0 \quad \ldots\ldots (2\text{-}5)$$

where $\quad 4\beta^4 = Eh/R^2 D = 12(1-\nu^2)/R^2h^2$

It will be seen that we have the same characteristic equation as the beam on an elastic foundation. The solution will therefore be the same with β substituted for b. The assumption as to the relative magnitudes of the curvatures can now be checked.

If the constants of integration are taken as being of the same order the ratio $\quad \partial^2 w/\partial x^2 \; : \; \nu w/R^2$

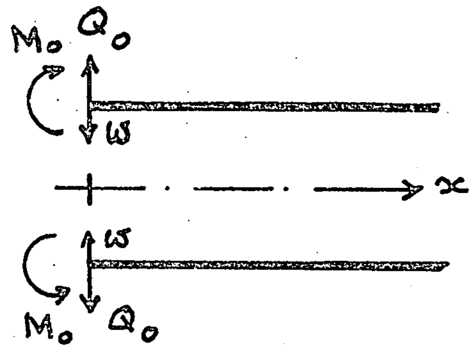
is $\quad 2\beta^2 \; : \; \nu/R^2$

or $\quad \sqrt{12(1-\nu^2)}/Rh \; : \; \nu/R^2$

Taking Poisson's ratio ν as 0.3 the ratio becomes 11R/ h : 1
A typical value of R/ h would be 100/1 so rato is 1100:1
And neglect of the second term is obviously justified.

As the complexity of the loading increases it will be found that the expressions become too difficult to handle if all the terms are employed. It then becomes of critical importance to know what terms may be safely dropped. A general method of working this out will be introduced in a later chapter.

Long Cylinder with edge loads



$$M_o = -D\,w'' \quad \text{at } x=0$$
$$Q_o = -D\,w''' \quad \text{at } x=0$$

The complete solution to the characteristic equation is:

$$w = e^{\beta x}\,(C1\cos\beta x + C2\sin\beta x) + e^{-\beta x}\,(C3\cos\beta x + C4\sin\beta x) \quad \ldots\ (2\text{-}6)$$

As the deflections die away with increase of x **C1** and **C2** are zero. Referring to Table 1-1 we find that

$$M_o = -D\,2\beta^2(-C4) \quad \therefore \quad C4 = M_o/2\beta^2 D$$

$$Q_o = -D\,2\beta^3(C3 + C4)$$

$$\therefore\ C3 = -Q_o/2\beta^3 D - C4 = -\frac{1}{2\beta^3 D}(\beta M_o + Q_o)$$

Hence $\quad w_o = C_3 = -\dfrac{1}{2\beta^3 D}(\beta M_o + Q_o) \quad \cdots \quad (2\text{-}7)$

And $\quad w'_o = \beta(-C3 + C4) = \dfrac{1}{2\beta^2 D}(2\beta M_o + Q_o) \cdots (2\text{-}8$

These equations enable most problems in the axi-symmetric bending of shells to be readily solved.

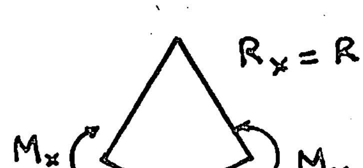
$e^{-\beta x}$ becomes small compared with 1 when $\beta x > 3$ ie < 0.05
So edge effects die away when $x > 3/\beta > 2.3\sqrt{Rh}$
This means that for axi-symmetric loading on a complete cylindrical shell almost all shells can be taken as long shells.

This is not the case when unsymmetrical edge loads are applied, the effects will then extend a considerable distance from the edgg.

Anti-Clastic Bending of a Wide Plate

When a plate is bent by a moment exerted in one direction only it will be found that the edges curl up in the opposite direction. This effect can be studied by the shell theory just worked out.

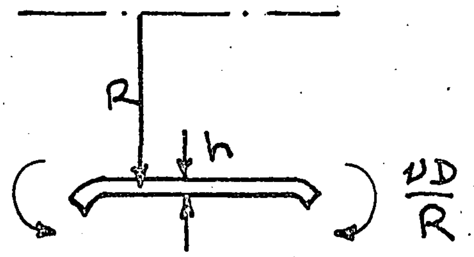
First assume that the plate is bent to form part of a circular cylinder radius R with no curvature in the y direction.



$$M_x = -D(\chi_x + \nu\chi_y) = -D/R$$

$$M_y = -D(\chi_y + \nu\chi_x) = -\nu D/R$$

Then M_y can be cancelled out by applying end moment $\frac{\nu D}{R}$ to the cylinder



$$w_0 = -\frac{1}{2\beta^3 D}(\beta M_0 + Q_0)$$

$$= -\frac{1}{2\beta^2 D}\cdot\frac{\nu D}{R}$$

$$= -\frac{Rh}{\sqrt{12(1-\nu^2)}}\frac{\nu}{R}$$

$$= -\frac{\nu}{\sqrt{12(1-\nu^2)}}h$$

If $\nu = 0{\cdot}3$ $\qquad w_0 = -h/11$

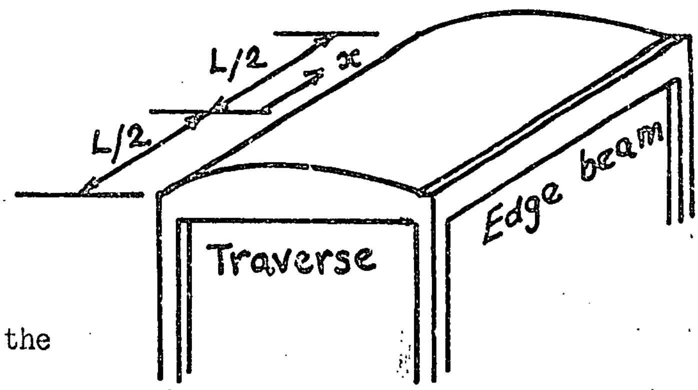
The shape of the bent plate can be found from equation (2-6).

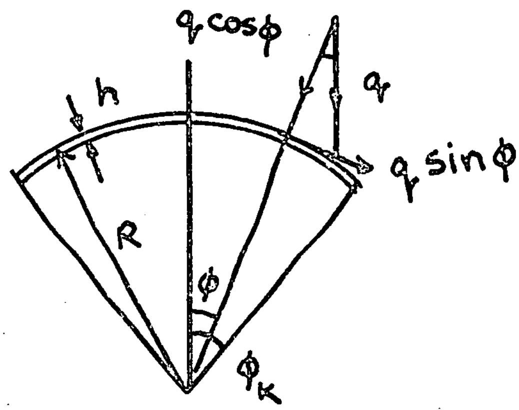
Chapter 3

SHELL ROOFS

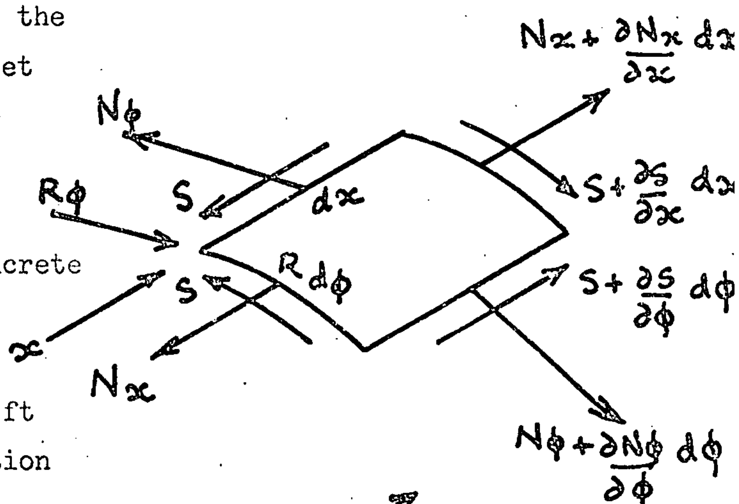
Membrane Analysis of Shell Roofs

When the loading and edge beam conditions are appropriate a cylindrical circular shell roof can deflect without changing its radius of curvature. Thus no bending moment in the ϕ direction will occur and hence no shear normal to the shell. As the curvature in the longitudinal direction is usually negligible the shell can resist the applied forces with stresses only in the direction of the shell. This is known as membrane action and is statically determinate ie the stresses can be calculated by statics alone.





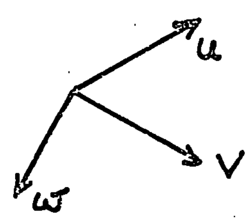
A shell roof with uniform vertical loading will be subjected to a membrane analysis and the corresponding edge loading and deformations calculated. If the edge conditions are not the same as the membrane analysis bending will be set up. This may be handled by the edge load theory to be developed in the next section.



Shell roofs are normally of concrete and Poisson's ratio is usually taken as zero.

Taking a vertical loading q' lb/sq ft as a Fourier series in the x direction

$$q = \pm \frac{4q'}{n\pi} \cos \frac{n\pi x}{L} = q_n \cos mx$$



Resolving in the z, ϕ , x directions

$$N\phi / R = -q \cos \phi$$

$$\partial S/\partial x + \partial N\phi / R\partial\phi = -q \sin\phi$$

$$\partial N_x/\partial x + \partial S/R\partial\phi = 0$$

Hence

$$N\phi = -q_n R \cos\phi \cos mx$$

$$\partial S/\partial x = -2q_n \sin\phi \cos mx$$

$$\partial^2 N_x/\partial x^2 = (2q_n/R) \cos\phi \cos mx$$

The traverses are considered to be rigid in the z and ϕ directions and to be flexible without restraint in the x direction.

Integrating and putting in boundary conditions S =0 at x = 0 and $N_x = 0$ at x =±L /2

$$S = -(2q_n/m)\sin\phi\cos mx$$
$$N_x = -(2q_n/Rm^2)\cos\phi\cos mx$$

The force deformation relations are

$$\epsilon_x = \partial u/\partial x \qquad \text{Appendix 3}$$
$$= \sigma_x/E \quad \text{if } \nu = 0$$
$$= N_x/Eh \qquad \therefore \quad \partial^2 u/R\partial\phi\partial x = (1/Eh)\,\partial N_x/R\,d\phi$$

$$\epsilon_\phi = (\partial v/\partial\phi - w)/R = N_\phi/Eh$$

$$\gamma_{x\phi} = \partial v/\partial x + \partial u/R\partial\phi$$
$$= S/Gh$$
$$= 2S/Eh \quad \text{if } \nu = 0$$

Then $\partial^2 v/\partial x^2 + \partial^2 u/R\partial\phi\partial x = (2/Eh)\,\partial S/\partial x$

or $$\partial^2 v/\partial x^2 = (1/Eh)(2\,\partial S/\partial x - \partial N_x/R\partial\phi)$$
$$= (q_n/Eh(-4 - 2/R^2m^2)\sin\phi\cos mx$$

Integrating and putting in boundary conditions v =0 at x == ± L /2

$$v = (q_n/Eh)(4/m^2 + 2/R^2m^4)\sin\phi\cos mx$$

Now $$w = \partial v/\partial\phi - R N_\phi/Eh$$
$$= \partial v/\partial\phi + (q_n/Eh)R^2\cos\phi\cos mx$$
$$= (q_n/Eh)(4/m^2 + 2/R^2m^4 + R^2)\cos\phi\cos mx$$

The rotation $\theta = \partial w/R\partial\phi + v/R$

In membrane action the shape remains circular, the rotation will be due only to stretching in the ϕ direction. This is negligible.

It is convenient to keep all quantities in terms of cos mx so that they can be coordinated at the centre of the shell where x =0 and cos mx is unity. Gathering together the forces and displacements worked out at x =0 :

$$\left.\begin{aligned} N_\phi &= -q_n R\cos\phi \\ \partial S/\partial x &= -2q_n\sin\phi \\ N_x &= -(2q_n/Rm^2)\cos\phi \\ v &= (q_n/Eh)(4/m^2 + 2/R^2m^4)\sin\phi \\ w &= (q_n/Eh)(4/m^2 + 2/R^2m^4 + R^2)\cos\phi \\ \theta &= 0 \end{aligned}\right\} \quad \cdot\cdot(3\text{-}$$

Edge Load Theory for Shell Roofs

Schorer's method provides the simplest form of characteristic equation for the shell. It is accurate only for long shells ie when $\pi R / L < 0.7$

The approximations are:

(1) M_x, Q_x, $M_{x\phi}$, ν are neglected.

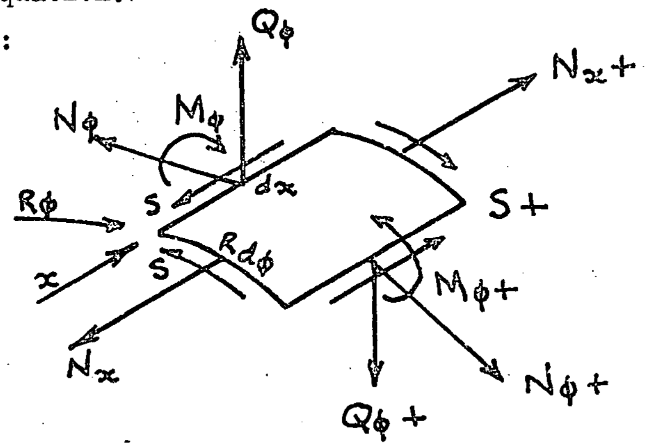
(2) ϵ_ϕ, $\gamma_{x\phi}$ are condidered small .

(3) Component of Q in ϕ direction is neglected.

(4) M_ϕ is taken as $-D\,\partial^2 w/R^2\partial\phi^2$ and the $\frac{w}{R^2}$ curvature is neglected.

The last two are the more serious errors which are shared by the much more complicated Donnell-Jenkins equation..

Resolving and taking moments on element:



$$Q_\phi = \partial M_\phi / R\partial\phi = -\frac{D}{R^3}\frac{\partial^3 w}{\partial\phi^3}$$

$$N_\phi = -\frac{\partial Q}{\partial\phi} = \frac{D}{R^3}\frac{\partial^4 w}{\partial\phi^4}$$

$$\frac{\partial S}{\partial x} = -\frac{\partial N_\phi}{R\partial\phi} = -\frac{D}{R^4}\frac{\partial^5 w}{\partial\phi^5}$$

$$\frac{\partial N_x}{\partial x} = -\frac{\partial S}{R\partial\phi}$$

$$\frac{\partial^2 N_x}{\partial x^2} = -\frac{\partial^2 S}{R\partial\phi\partial x} = \frac{D}{R^5}\frac{\partial^6 w}{\partial\phi^6} \quad \ldots \quad (3\text{-}2)$$

Integrating and putting in boundary conditions $S = 0$ at $x = 0$ and $N_x = 0$ at $x = \pm L/2$

$$S = -\frac{D}{R^4 m}\frac{\partial^5 w}{\partial\phi^5}\sin mx$$

$$N_x = -\frac{D}{R^5 m^2}\frac{\partial^6 w}{\partial\phi^6}\cos mx$$

Taking ϵ_ϕ as small compared with the bending displacements

$$\epsilon_\phi = \partial v / R\partial\phi - w/R$$

$$\therefore \partial v/\partial\phi = w$$

and $$\partial^5 v/\partial\phi\,\partial x^4 = \partial^4 w/\partial x^4 \quad \ldots \quad (3\text{-}3)$$

Now $\gamma_{x\phi} = \partial u / R\partial\phi + \partial v/\partial x$

Taking $\gamma_{x\phi}$ as small compared with the bending displacements

$$\partial v/\partial x = -\partial u / R\partial\phi$$

and $$\partial^5 v/\partial\phi\,\partial x^4 = -\partial^5 u / R\partial\phi^2\partial x^3 \quad \ldots(3\text{-}4)$$

$$\epsilon_x = \partial u/\partial x = N_x / Eh$$

and $$-\partial^5 u / R\partial\phi^2\partial x^3 = -\frac{1}{EhR}\frac{\partial^4 N_x}{\partial\phi^2\partial x^2} \quad \ldots(3\text{-}5)$$

Combining equations (3-2) to (3-5) gives

$$\frac{\partial^8 \omega}{\partial\phi^8} + \frac{12R^6}{h^2}\frac{\partial^4\omega}{\partial x^4} = 0 \quad \ldots(3\text{-}6)$$

This is Schorer's characteristic equation. It will be noted that this is not dependent on the expression used for the deflection in the x direction.

A solution to Schorer's equation is $\omega = A e^{p\phi}\cos mx$

the indicial equation being $p^8 + 12R^6m^4/h^2 = 0$

or $p^8 + k^8 = 0$ where $k = 1.364\sqrt[4]{R/h}\sqrt{Rm} \quad \ldots(3\text{-}7)$

p has eight values from De Moivre's theorem

$$k\begin{bmatrix} \pm\cos\theta_1 \pm i\sin\theta_1 \\ \pm\cos\theta_2 \pm i\sin\theta_2 \end{bmatrix} = \begin{matrix} \pm\beta_1 \pm i\alpha_1 \\ \pm\beta_2 \pm i\alpha_2 \end{matrix}$$

where $\theta_1 = 22.5°$ and $\theta_2 = 67.5°$. Therefore in this case

$$\beta_1 = \alpha_2 = 0.924\,k\,; \qquad \beta_2 = \alpha_1 = 0.383\,k\,;$$

Combining the imaginary parts to produce trig terms and throwing away the positive real parts as the edge effects die away from the edge

$$\omega = \big(e^{-\beta_1\phi}(C1\cos\alpha_1\phi - C2\sin\alpha_1\phi) + e^{-\beta_2\phi}(C3\cos\alpha_2\phi - C4\sin\alpha_2\phi)\big)\cos mx \quad \ldots(3\text{-}8)$$

This expression has to be differentiated successively to obtain the stresses as laid out in equations (3-1). The general form of the n th derivative is:

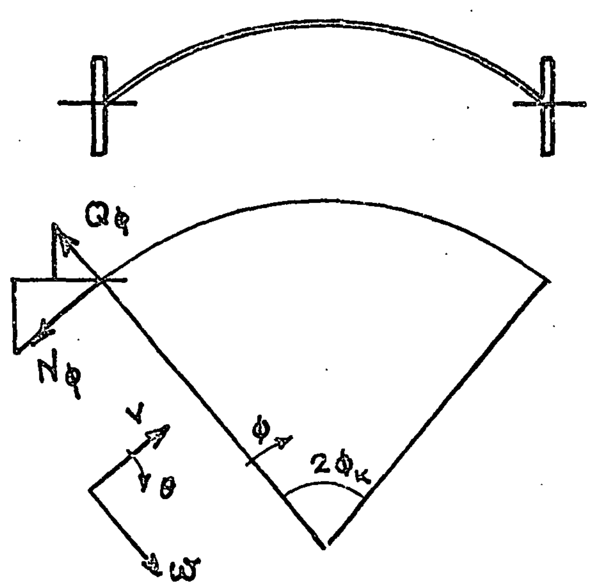
$$(-k)^n\Big\{e^{-\beta_1\phi}\big((C1\cos n\theta_1 + C2\sin n\theta_1)\cos\alpha_1\phi + (C1\sin n\theta_1 - C2\cos n\theta_1)\sin\alpha_1\phi\big) + e^{-\beta_2\phi}\big((C3\cos n\theta_2 + C4\sin n\theta_2)\cos\alpha_2\phi + (C3\sin n\theta_2 - C4\cos n\theta_2)\sin\alpha_2\phi\Big\}\cos mx \quad \ldots(3\text{-}9)$$

(see Appendix 3)

The coefficients of the shell functions are shown in the following Table (3-1).

$f(\omega)$	n	$\bar{R}$	$\bar{K}$ $(-k)^n$	$\cos n\theta_1$	$\sin n\theta_1$	$\cos n\theta_2$	$\sin n\theta_2$
v	-1	1	$-1/k$	0.924	-0.383	0.383	-0.924
ω	0	1	1	1	0	1	0
$\partial\omega/\partial\phi$	1	1	$-k$	0.924	0.383	0.383	0.924
$M\phi$	2	$-D/R^2$	k^2	0.707	0.707	-0.707	0.707
$Q\phi$	3	$-D/R^3$	$-k^3$	0.383	0.924	-0.924	-0.383
$N\phi$	4	D/R^3	k^4	0	1	0	-1
$\partial s/\partial x$	5	$-D/R^4$	$-k^5$	-0.383	0.924	0.924	-0.383
N_x	6	$-D/R^5m^2$	k^6	-0.707	0.707	0.707	0.707

Direct Design of Shell with Post-tensioned Edge Beams



The method to be outlined does not lead to the most economic design of shell or edge beam. It is however a good introduction to the use of edge load theory and reduces the calculations to the simplest possible level.

$e^{-0.924k\phi}$ decreases rapidly with ϕ and is negligible at the opposite side of the shell. The same can not be said for $e^{-0.383k\phi}$ unless the shell is very thin.

To prevent the edge effects from from one edge beam being carried over to the other side and complicating the calculations we will choose the boundary conditions to make C_3 and C_4 zero. The use of a post tensioned edge beam gives us enough control to do this.

If C_3 and C_4 are made zero then at $\phi = 0$ $x = 0$

$$f(\omega) = \bar{R}\bar{K}\left(C_1 \cos n\theta_1 + C_2 \sin n\theta_1\right) \quad + \text{ membrane effects.}$$

The edge beam is made narrow enough to be flexible without restraint in the horizontal direction. It will however be fairly stiff against torsion and will be taken as having no rotation [illegible]

Then
$$\underbrace{\left(Q_\phi \sin\phi_K + N\phi\cos\phi_K\right)}_{\text{edge load}} + \underbrace{N_\phi\cos\phi_K}_{\text{membrane}} = 0$$

And $R\theta = (\partial w/\partial\phi + v)$ edge load $= 0$

These boundary conditions enable C1 and C2 to be determined, and hence the deflections, forces and moments in the shell. The edge beams can then be designed to agree with the forces and deflections at the edge of the shell.

Example

L = 120 ft.	$E = 3 \times 10^6$ psi $= 432 \times 10^6$ lb / sq ft.
R = 30 ft.	$\nu = 0$
h = 0.25 ft;	q' = 50 lb / sq ft. (mostly dead load).
$\phi_k = 40^\circ$	Taking the first term in the Fourier series
$\cos 40^\circ = 0.766$	$q = 4q'/\pi$ = 63.6 lb / ft sq.
$\sin 40^\circ = 0.643$	$m^2 = (\pi/L)^2$ = 1/1460

Membrane forces and displacements (Equations (3-1)).
at L.H.) edge $\phi = -40^\circ$ $x = 0$:

$$N\phi = -qR\cos\phi = -1460 \text{ lb/ft}$$
$$\partial S/\partial x = -2q\sin\phi = +81.8 \text{ lb/ft}^2$$
$$N_x = -(2q/Rm^2)\cos\phi = -4730 \text{ lb/ft}$$
$$v = (q/Eh)(4/m^2 + 2/R^2m^4)\sin\phi = -0.0040 \text{ ft}$$
$$w = (q/Eh)(4/m^2 + 2/R^2m^4 + R^2)\cos\phi = +0.0052 \text{ ft}$$

Edge Conditions

$$k = 1.364\sqrt[4]{R/h} \qquad \sqrt{Rm} = 4$$
$$D/R^3 = Eh^3/12R^3 = 20.8 \text{ lb/ft}^2$$

From Table 3-1 with $C_3 = C_4 = 0$ the boundary conditions are:

Condition 1 No horizontal reaction

$$\begin{aligned} Q_\phi \sin\phi_k &= 328\,C1 + 792\,C2 \\ N_\phi \cos\phi_k &= \underline{ + 4080\,C2 - 1118} \\ & 328\,C1 + 4872\,C2 - 1118 = 0 \end{aligned}$$

Condition 2 No rotation

$$\begin{aligned} \partial w/\partial\phi &= -3.696\,C1 - 1.532\,C2 \\ v &= \underline{-0.231\,C1 + 0.096\,C2} \\ R\theta &= -3.927\,C1 - 1.436\,C2 = 0 \end{aligned}$$

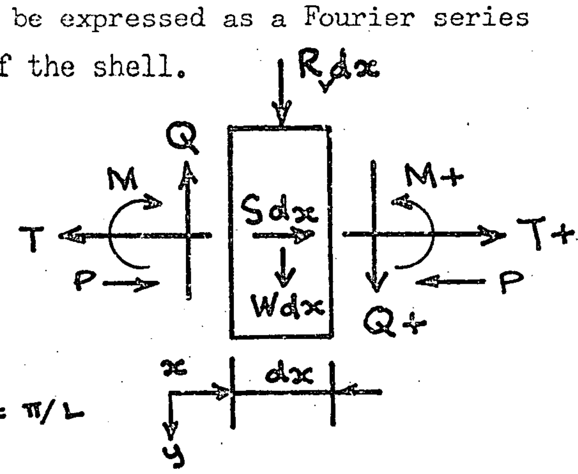
Then $C1 = -0.086$ and $C2 = +0.235$

The forces and displacements at the edge of the shell can now be calculated and are set out in Table (3-2):

$f(\omega)$	$\bar{R}\bar{K}$	$C_1 \cos n\theta$,	$C_2 \sin n\theta$,	Edge Load	Membrane	Total
v	-0.25	-0.079	-0.090	0.042	-0.004	0.038
ω	1	-0.086	0	-0.086	0.005	-0.081
$\partial\omega/\partial\phi$	-4	-0.079	0.090	-0.044	0.004	-0.040
M_ϕ	-10000	-0.061	0.166	-1050	0	-1050
Q_ϕ	1330	-0.033	0.217	245	0	245
N_ϕ	5320	0	0.235	1250	-1460	-210
$\partial S/\partial x$	710	0.033	0.217	178	81.8	260
N_x	-138000	0.061	0.166	-31400	-4730	-36130

Design of Edge Beam weight W lb / ft.

The self weight of the edge beam must be expressed as a Fourier series term in the same way as the loading of the shell.



Resolving horizontally:

$$\partial T/\partial x = -S$$

$$T = -\iint \partial S/\partial x \, dx \, dx$$

$$= (L/\pi)^2 \, \partial S/\partial x - P$$

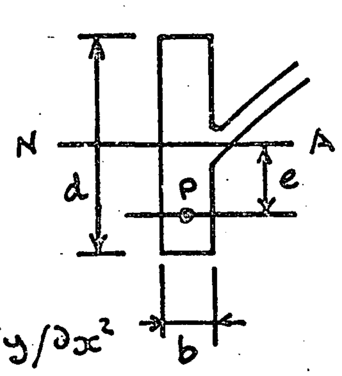
Allowing for post tension with $m = \pi/L$

Resolving vertically :

$$\partial Q/\partial x = -(R_v + W)$$

Taking moments: $\partial M/\partial x = Q$

$$M = \iint \partial Q/\partial x \, dx\, dx = (L/\pi)^2 (R_v + W)$$



Allowing for post tension

$$M = (L/\pi)^2 (R_v + W) - Pe = -EI \, \partial^2 y/\partial x^2$$

Stress at springing taken at the neutral axis of the edge beam in this case $\sigma_x = N_x / h = T/A$ where $A = bd$

In our example at x = 0 on the L.H. edge beam

$$R_v = Q_\phi \cos 40° - N_\phi \sin 40° = 323 \text{ lb/ft}$$

$$y = \omega \cos 40° - v \sin 40° = -0.086 \text{ ft}$$

$$(L/\pi)^2 \, \partial S/\partial x = 380 \text{ kips}$$

$\sigma_x = N_x / h = -144$ kips/ft^2 or 1000 psi compression

The edge beam must be designed to satisfy these edge conditions if the analysis is to be correct.

Take width $b = 0.5$ ft.

depth $d = 5$ ft.

Tendon force is P at eccentricity e ft.

Area $A = 2.5$ ft.

$I = 5.2$ ft.^{4}

$W = 2.5 \times 150 \times 4/\pi = 478$ lb / ft~~sq~~.

(taking first term of Fourier series).

$R_v + W = 0.8$ kips/ft.

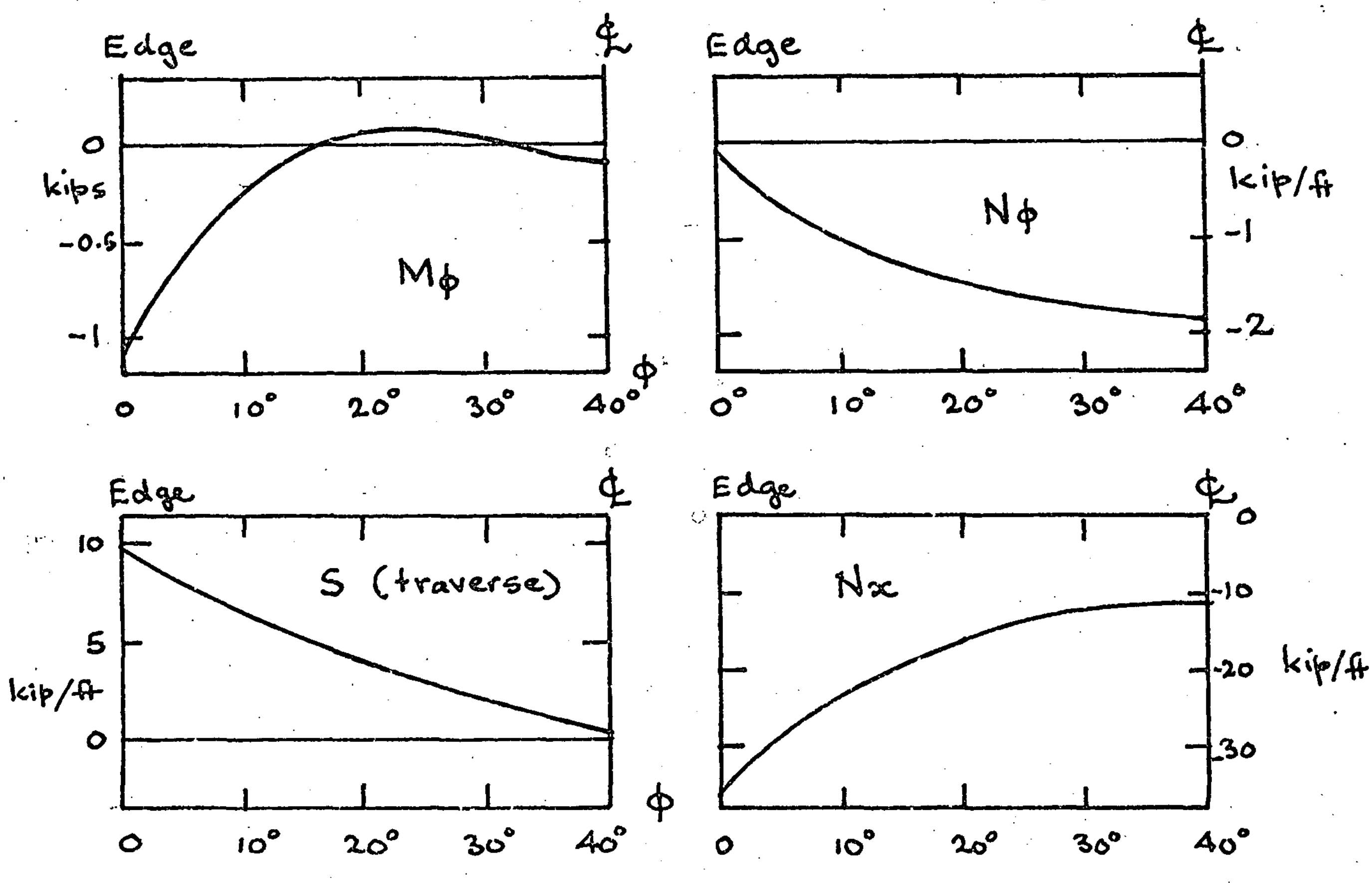
$$M = (\pi/L)^2 EIy = (L/\pi)^2 (R_v + W) - Pe$$
$$-132 \text{ kip ft} = 1170 - Pe$$

$$\sigma_x = (380 - P)/2.5 = -144 \text{ kip/ft}^2$$
$$\therefore \quad P = 740 \text{ kips}$$
$$\therefore \quad e = 1.76 \text{ ft}$$

Maximum bending stress = $Md/2I = \pm 63.5$ kip/ft^2

Maximum stress = -207.5 kip/ft^2 = -1440 psi

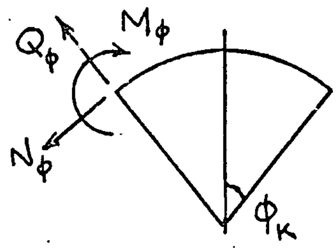
The post tension should become smaller towards the end of the beam. The variation of forces around the shell at x = o are shown below:



ANALYSIS OF SHELL WITHOUT EDGE BEAMS

The boundary conditions on the left hand edge are:

(a) $M_\phi = 0$
(b) $Q_\phi = 0$
(c) $N_\phi = 0$
(d) $\partial S/\partial x = 0$



Each of these stresses will be due to the edge load effects from both edges ($\phi = 0$ & $2\phi_K$) and the membrane stresses.

The effect of the edge load on the right hand side will be added to that of the left hand side when n is even and subtracted when it is odd.

These calculations are tedious to attempt without a computer. To facilitate computer programming the equations will be put in matrix form.

Expressing Table 3-1 and equations 3-1 in matrix form the stress and deflection functions f(w) can be written:

$$A.B.C + F \quad .... \qquad (3\text{-}10)$$

where A[8:4] ie eight rows, four columns corresponding to the eight values of n (-1 to 6) is:

$$\bar{R}\,\bar{K}\,[\cos n\theta_1 \quad \sin n\theta_1 \quad \cos n\theta_2 \quad \sin n\theta_2]$$

Matrix B is made up of the sum or difference of the effects from either edge.

Putting $\quad e^{-\beta_1\phi}\cos\alpha_1\phi = ec1 \quad , \quad e^{-\beta_1\phi}\sin\alpha_1\phi = es1 \quad$ etc

$$B1 = \begin{bmatrix} ec1 & -es1 & 0 & 0 \\ es1 & ec1 & 0 & 0 \\ 0 & 0 & ec2 & -es2 \\ 0 & 0 & es2 & ec2 \end{bmatrix}$$

with $\phi = 0$
(L.H. edge)
$ec1 = ec2 = 1$
$es1 = es2 = 0$

B2 is of the same form with $\phi = 2\phi_K$ when considering the L.H. edge or $2\phi_K - \phi$ elsewhere. A.B2 is added to A.B1 for even values of n subtracted for odd values of n.

C is the column vector $\{C1 \quad C2 \quad C3 \quad C4\}$

And F is the column vector containing the eight membrane functions corresponding to the edge load functions. F[2] and F[3] will of course be zero as no bending or normal shears are produced in membrane action.

The four boundary conditions can then be expressed as:

(a) A[2].(B1+B2).C = E[2].C = -F[2]

(b) A[3].(B1-B2).C = E[3].C = -F[3]

(c) A[4].(B1+B2).C = E[4].C = -F[4]

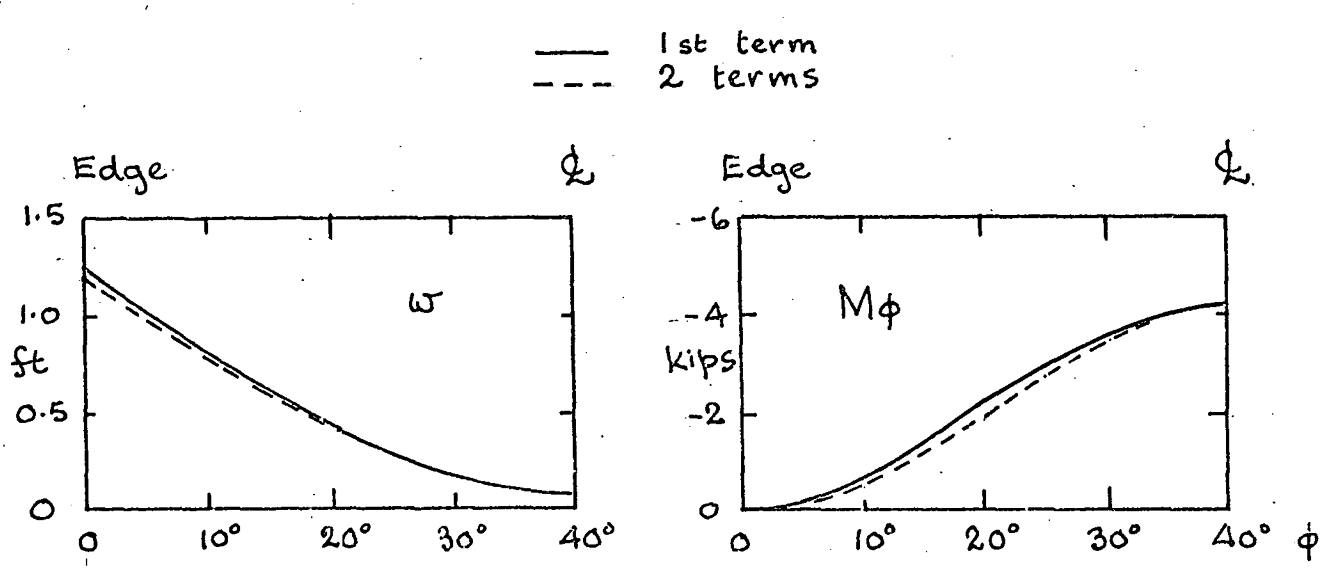
(d) A[5].(B1-B2).C = E[5].C = -F[5]

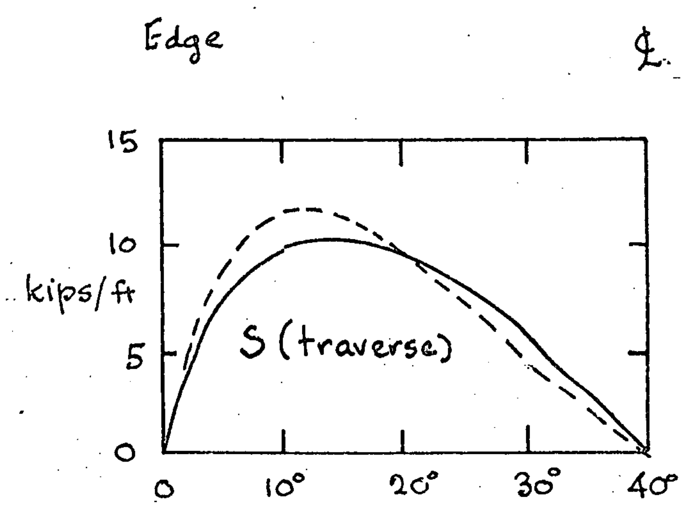
This set of four simultaneous equations can be solved to find the values of C1, C2, C3, C4. Then any stresses or deflections in the shell at x=0 can be found from (3-10), by putting the appropriate value of in the C matrix. At other values of x multiply by cos mx.

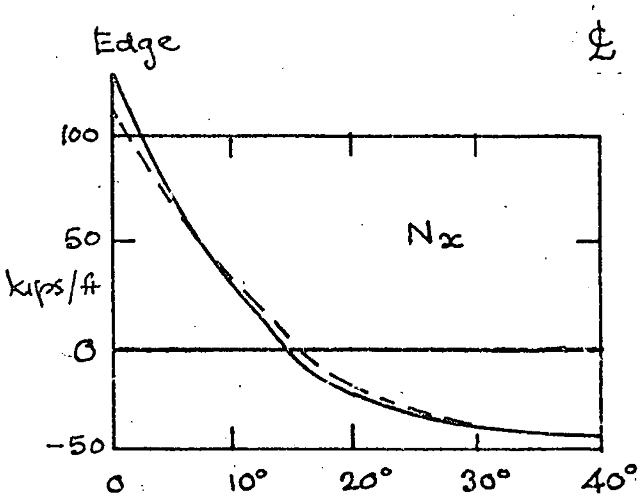
Some ingenuity is required to program this in an efficient manner. It is suggested that the reader writes this program as an exercise. It took me a full working day.

The results of the analysis of the shell of the same dimensions used in the previous section ie. q= 50 lb/sq ft, L= 120 ft, R= 30 ft, $\phi_k = 40^{\circ}$ are shown below. Both the first andsecond term in the Fourier series for loading and deformations have been taken. It will be seen that little difference is made by taking the second term into account.

The stresses calculated for the shell without edge beams do not fully demonstrate the advantages of and edge beam over a free edge. Playing about with a model will show that a free edge is extremely flexible to live loads and would tend to flap about in a gusty wind. This could rapidly destroy the shell. A shell with an edge beam however is very stiff and can safely be designed for uniform loads.







APPENDIX 3

<u>To obtain the derivatives of a solution to a linear differential equation</u>

The solution is $$y = A_1 e^{p_1 x} + A_2 e^{p_2 x} \ldots$$

where $$p_1 = -(\beta + \alpha i) \qquad p_2 = -(\beta - \alpha i) \ldots$$

then $$\partial^n y / \partial x^n = A_1 p_1^n e^{p_1 x} + A_2 p_2^n e^{p_2 x} \ldots$$

Let $$\beta = k \cos\theta \qquad \alpha = k \sin\theta$$

Now $$e^{i\alpha x} = \cos\alpha x + i \sin\alpha x$$

and $$(k\cos\theta + k i \sin\theta)^n = k^n(\cos n\theta + i \sin n\theta)$$

from de Moivre's theorem.

$$\therefore \partial^n y/\partial x^n = A_1(-k)^n(\cos n\theta + i\sin n\theta)e^{-\beta x}(\cos\alpha x - i\sin\alpha x) + A2(-k)^n(\cos n\theta - i\sin n\theta)e^{-\beta x}(\cos\alpha x + i\sin\alpha x)$$

$$= (-k)^n e^{-\beta x}\left\{\begin{array}{l}\{(A1+A2)\cos n\theta + (A1-A2)i\sin n\theta\}\cos\alpha x \\ \{(A1+A2)\sin n\theta - (A1-A2)i\cos n\theta\}\sin\alpha x\end{array}\right\}$$

putting $$A1 + A2 = C1 \qquad (A1-A2)i = C2$$

$$\partial^n y/\partial x^n = (-k)^n e^{-\beta x}\left\{\begin{array}{l}(C1\cos n\theta + C2\sin n\theta)\cos\alpha x \\ (C1\sin n\theta - C2\cos n\theta)\sin\alpha x\end{array}\right\}$$

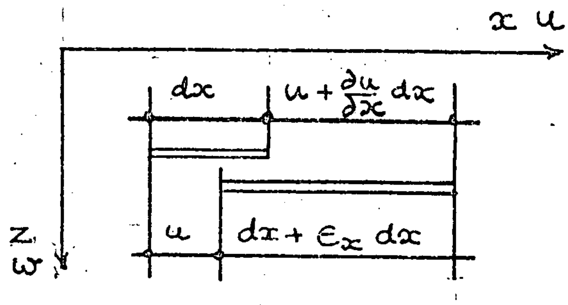
$$= (-k)^n \begin{bmatrix}\cos n\theta & \sin n\theta\end{bmatrix} e^{-\beta x}\begin{bmatrix}\cos\alpha x & -\sin\alpha x \\ \sin\alpha x & \cos\alpha x\end{bmatrix}\begin{bmatrix}C1 \\ C2\end{bmatrix}$$

Force Deformation Relations for Shell

Forces in shell are expressed per unit of length.

Thus $N_x = \sigma_x h$ lb/ ft.

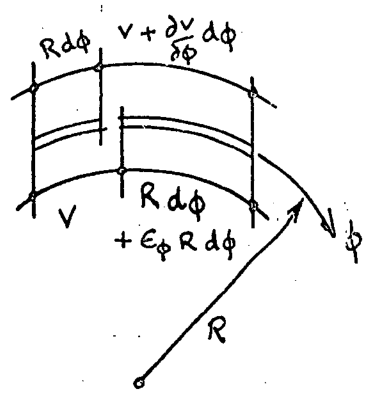
M_x lb ft/ ft or lb.



(a) Stretching of shell

$$dx + u + \frac{\partial u}{\partial x} dx = u + dx + \epsilon_x dx$$

$\therefore \epsilon_x = \partial u / \partial x$ if change of slope is small.



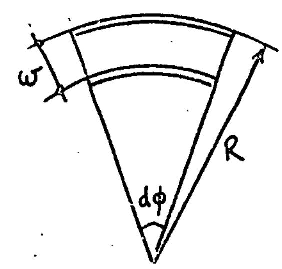
Similarly $\epsilon_\phi = \partial v / R \partial \phi$

However if there is a radial displacement there will be an additional strain

$$\{(R - w) d\phi - R d\phi\} / R d\phi = -w/R$$

$$\therefore \epsilon_\phi = \frac{1}{R}(\partial v / \partial \phi - w)$$

as for lamina in equation (2-1)

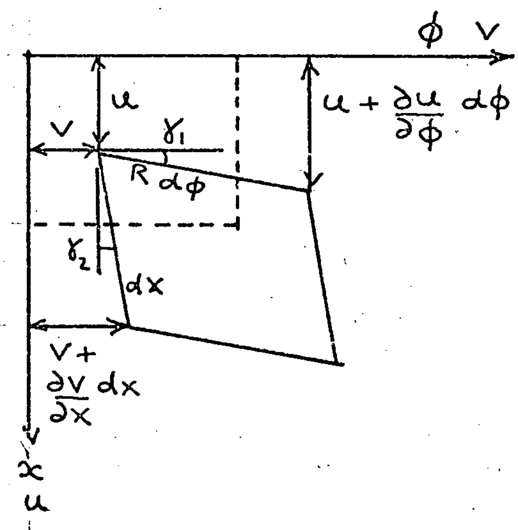


$$N_x = \frac{Eh}{1 - \nu^2}(\epsilon_x + \nu \epsilon_\phi)$$

$$N_\phi = \frac{Eh}{1 - \nu^2}(\epsilon_\phi + \nu \epsilon_x)$$

When the shell is deformed by shear the shearing strain $\gamma_{x\phi} = \gamma_1 + \gamma_2$

$$\gamma_{x\phi} = \frac{\partial u}{R \partial \phi} + \frac{\partial v}{\partial x}$$



The membrane shear S

$$= \tau h = G \gamma h = \frac{Eh}{2(1+\nu)} \cdot \gamma_{x\phi}$$

The sign convention for shear makes a positive shear extend the diagonal in the positive direction.

(b) Bending of shell

χ is change of curvature.

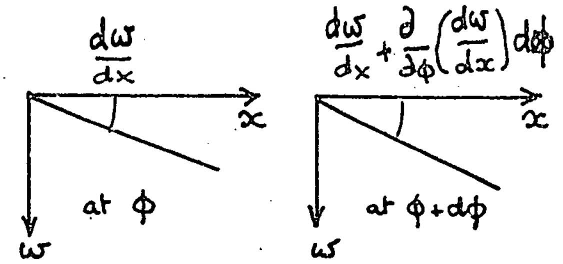
$$\chi_x = \partial^2 w / \partial x^2 \qquad \text{as for beam.}$$

χ_ϕ is similarly $\partial^2 w / R^2 \partial \phi^2$ but in addition there will be a change of curvature $\frac{1}{(R - w)} - \frac{1}{R} = \frac{w}{R^2}$ if w is small compared with R.

Then as in equation (2-2)

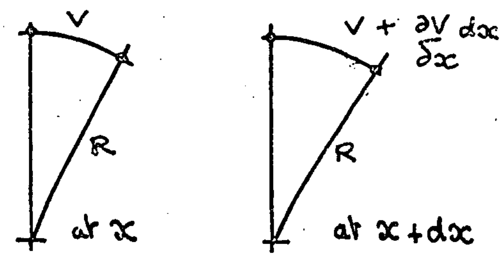
$$M_x = -D(\chi_x + \nu\chi_\phi)$$

and $$M_\phi = -D(\chi_\phi + \nu\chi_x)$$



Twist is change of angle in unit length

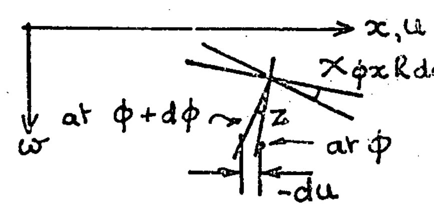
$$= \frac{\partial}{\partial\phi}\left(\frac{dw}{dx}\right)d\phi \Big/ R\,d\phi = \frac{\partial^2 w}{R\,\partial\phi\,\partial x}$$



But there will be in addition in a cylindrical shell a twist due to the movement in the ϕ direction. $= \partial v/R\,dx$

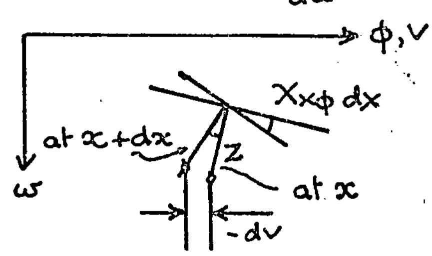
$$\therefore \chi_{x\phi} = \chi_{\phi x} = \frac{1}{R}\left(\frac{\partial^2 w}{\partial\phi\,\partial x} + \frac{\partial v}{\partial x}\right)$$

Consider the displacement of a point distance z below the middle surface.



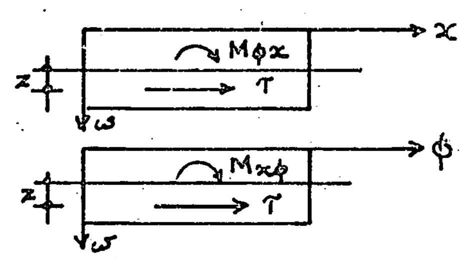
$$\partial u/R\,\partial\phi = -z\,\chi_{\phi x} \qquad \partial v/\partial x = -z\,\chi_{x\phi}$$

$$\therefore \tau = -2Gz\,\chi_{x\phi} = -\frac{Eh}{1+\nu}\,z\,\chi_{x\phi}$$



The directions of the twisting moments $M_{x\phi}$ and $M_{\phi x}$ will be taken as those causing a positive deflection across the positive diagonal. (this is different from that used by Timoshenko).

Integrating across the thickness of the shell:



$$M_{x\phi} = M_{\phi x} = -\int_{-h/2}^{h/2} \tau z\,dz = \frac{Eh^3}{12(1+\nu)}\chi_{x\phi}$$
$$= D(1-\nu)\chi_{x\phi}$$

Collecting together the relations we have derived for a cylindrical shell with x taken in the direction of the generator and putting $N = Eh/(1-\nu^2)$.

$$\epsilon_x = \partial u/\partial x \qquad \epsilon_\phi = \frac{1}{R}(\partial v/\partial\phi - w)$$

$$N_x = N(\epsilon_x + \nu\epsilon_\phi) \qquad N_\phi = N(\epsilon_\phi + \nu\epsilon_x)$$

$$\gamma_{x\phi} = \gamma_{\phi x} = \partial u/R\,\partial\phi + \partial v/\partial x \qquad S = \frac{N}{2}(1-\nu)\gamma_{x\phi}$$

$$\chi_x = \partial^2 w/\partial x^2 \qquad \chi_\phi = \frac{1}{R^2}(\partial^2 w/\partial\phi^2 + w)$$

$$M_x = -D(\chi_x + \nu\chi_\phi) \qquad M_\phi = -D(\chi_\phi + \nu\chi_x)$$

$$\chi_{x\phi} = \chi_{\phi x} = \frac{1}{R}(\partial^2 w/\partial x\,\partial\phi + \partial v/\partial x)$$

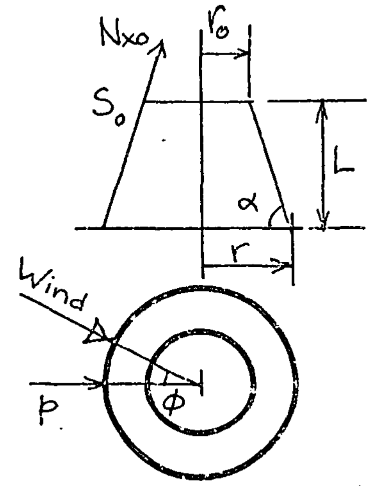
$$M_{x\phi} = M_{\phi x} = D(1-\nu)\chi_{x\phi} \qquad \ldots \quad (3\text{-}11)$$

Chapter 4.

COOLING TOWERS

Natural draught cooling towers form the most numerous and economically important class of civil engineering shell structures. The largest towers are over 400 feet high and three hundred feet base diameter with walls usually 6 inches thick. They are built in groups at steam powered electricity generating stations and large chemical plants and have to sustain their own weight and the forces set up by the wind.

The analysis will commence by considering the stresses set up by wind in a shell forming the frustrum of a cone carrying on its upper edge a known meridian force N_{xo} and shear S_o both known functions of angle ϕ.



When the shell is thin compared with the radius and the tower is made up of two or more cones of different angles, the deflections are small and a good approximation for the distributed forces is obtained by neglecting the bending and twisting moments in the shell and considering only the three membrane forces N_x, N_ϕ and S.

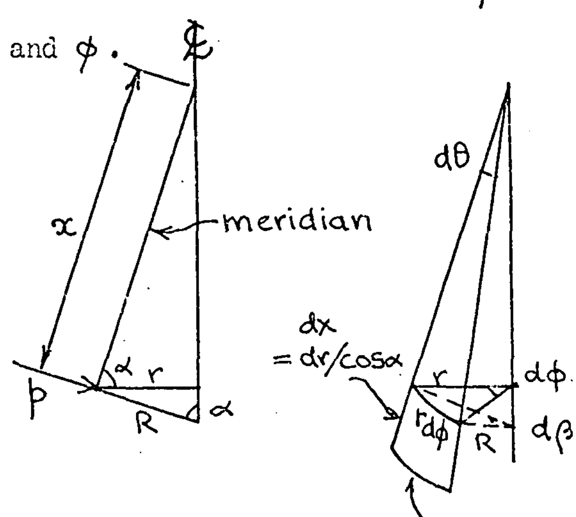
It is assumed that the wind pressure is a known function of ϕ. The shell variables will be taken as r and ϕ.

From the geometry of the shell

$$x = r/\cos\alpha \quad \therefore \quad dx = dr/\cos\alpha$$

$$x\,d\theta = r\,d\phi$$

$$\therefore \quad d\theta = r\,d\phi/x = d\phi\cos\alpha$$

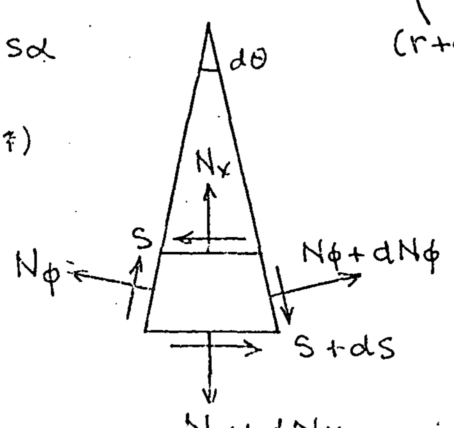


<u>Resolving normal to the shell</u>

$$N_\phi\,d\beta\,dr/\cos\alpha = -p\,R\,d\beta\,dr/\cos\alpha$$

$$\therefore N_\phi = -pR = -pr/\sin\alpha \quad \ldots(4\text{-}1)$$

The other equations of equilibrium differ from a cylinder in that



(a) N_ϕ has a component in the meridian direction

$$N_\phi \, d\theta \, dr / \cos\alpha \quad = \quad N_\phi \, dr \, d\phi$$

(b) The meridian S has a component in the ϕ direction

$$S \, d\theta \, dr / \cos\alpha \quad = \quad S \, dr \, d\phi$$

(c) The lower boundary of the element is larger than the upper boundary.

Resolving in the ϕ direction

$$dN_\phi \, dr/\cos\alpha + (S+dS)(r+dr)d\phi + S\,dr\,d\phi = S\,r\,d\phi$$

$$\therefore \frac{1}{\cos\alpha}\frac{dN_\phi}{d\phi} + r\frac{dS}{dr} + 2S = 0$$

$$\therefore r^2\frac{dS}{dr} + 2rS = \frac{d}{dr}(r^2 S) = -\frac{r}{\cos\alpha}\frac{dN_\phi}{d\phi}$$

Substituting for N_ϕ from (4-1), putting $p' = dp/d\phi$ and integrating with respect to r

$$r^2 S = \frac{p' r^3}{3\sin\alpha\cos\alpha} + C$$

where C varies with ϕ.

Now $S = S_o$ when $r = r_o$

$$\therefore C = r_o^2 S - p' r_o^3 / 3\sin\alpha\cos\alpha$$

$$\therefore S = \frac{p'}{3\sin\alpha\cos\alpha}(r - r_o^3/r^2) + S_o (r_o/r)^2 \ldots (4\text{-}2$$

and putting $r_o/r = n$

$$S = \frac{p' r(1-n^3)}{3\sin\alpha\cos\alpha} + S_o n^2 \quad \ldots\ldots \quad (4\text{-}2)$$

Resolving in the meridian direction

$$(N_x + dN_x)(r+dr)d\phi + dS\,dr/\cos\alpha = N_x\,r\,d\phi + N_\phi\,dr\,d\phi$$

or $$r\frac{dN_x}{dr} + N_x = \frac{d}{dr}(rN_x) = N_\phi - \frac{1}{\cos\alpha}\frac{dS}{d\phi}$$

Substituting from (4-1) and (4-2) and integrating with respect to r

$$rN_x = -pr^2/2\sin\alpha$$
$$- \frac{p''}{3\sin\alpha\cos^2\alpha}\left(\frac{r^2}{2} + \frac{r_o^3}{r}\right)$$
$$+ \quad S_o' \, r_o^2 / r\cos\alpha \quad + D$$

Where D varies with ϕ.

Now $N_x = N_{xo}$ when $r = r_o$

$$\therefore \quad D = r_o N_{xo} + \frac{p r_o^2}{2 \sin\alpha} + \frac{p''}{3 \sin\alpha \cos^2\alpha}\left(\frac{r_o^2}{2} + r_o^2\right) - \frac{S_o' r_o}{\cos\alpha}$$

$$\therefore \quad N_x = N_{xo}\frac{r_o}{r} - \frac{p}{2 \sin\alpha}\left(r - \frac{r_o^2}{r}\right)$$

$$- \frac{p''}{3 \sin\alpha \cos^2\alpha}\left\{\frac{1}{2}\left(r - \frac{3r_o^2}{r}\right) + \left(\frac{r_o^3}{r^2}\right)\right\}$$

$$+ \frac{S_o'}{\cos\alpha}\left(\frac{r_o^2}{r^2} - \frac{r_o}{r}\right)$$

substituting

$$n = r_o/r$$

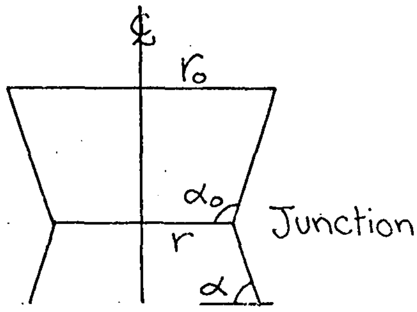
$$\left.\begin{aligned} N_x = {} & N_{xo}.n - \frac{pr(1-n^2)}{2 \sin\alpha} \\ & - \frac{p'' r}{6 \sin\alpha \cos^2\alpha}(1 - 3n^2 + 2n^3) \\ & - S_o'(n - n^2)/\cos\alpha \end{aligned}\right\} \quad \ldots(4\text{-}3)$$

For the special case of a cylindrical segment it can be shown from the membrane theory for a cylindrical shell that:

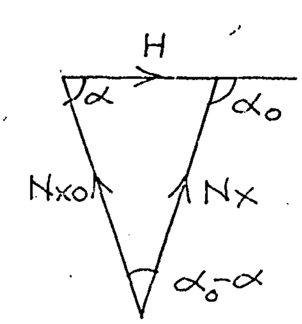
$$\left.\begin{aligned} N_\phi &= -pr \\ S &= p'L + S_o \\ N_x &= N_{xo} - p''L^2/2r - S_o'L/r \end{aligned}\right\} \quad \ldots.(4\text{-}4)$$

This will be left as an exercise for the reader.

<u>The calculation of N_{xo} and S_o</u>



For the topmost segment of the cooling tower N_{xo} and S_o are zero as there is a free edge. The membrane stresses at the base can then be calculated from the equations derived above.

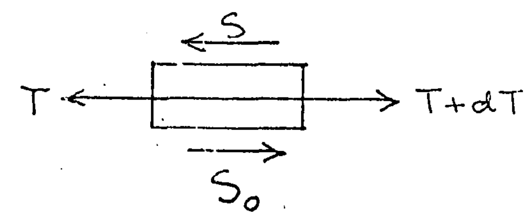


The membrane force N_x at the junction is then resolved into

(a) N_{xo} in the direction of the lower shell.

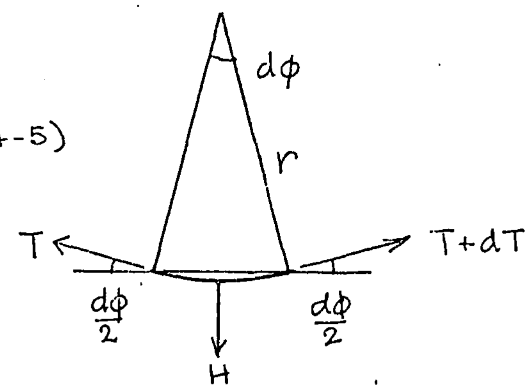
(b) H in the radial direction.

The membrane shear just above the junction is S and just below the junction S_o. H will produce a localised radial force T at the junction.



$$N_{xo} = N_x \sin(\pi - \alpha_o)/\sin\alpha$$
$$= N_x \sin\alpha_o/\sin\alpha \quad \ldots (4\text{-}5)$$

$$H = N_x \sin(\alpha_o - \alpha)/\sin\alpha$$



<u>Resolving radially</u>

$$T\,d\phi = H r\,d\phi \quad \therefore \quad T = Hr$$

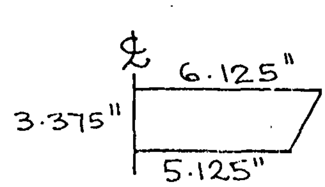
<u>Resolving in the ϕ direction</u>

$$(S - S_o) r\,d\phi = dT \quad \therefore \quad S - S_o = \frac{1}{r}\frac{dT}{d\phi} = H'$$

$$\therefore \quad S_o = S - H' = S - N_x' \sin(\alpha_o - \alpha)/\sin\alpha \quad \ldots (4\text{-}6)$$

In 1956 two model cooling towers of the dimensions shown below were tested in the pressurised wind tunnel at the National Physical Laboratories in Teddington England at Reynold's numbers approaching those of full scale towers in moderate winds. Pressure distributions were obtained inside and outside the first tower which was robustly constructed of sheet steel. A second similar but more lightly constructed brass tower was fitted with strain gauge rosettes and used to measure strains near the base. The discussion of the experimental results will follow the analysis of the tower. The following has been machine worked but is given to only three decimal places for conciseness.

<u>Top Section</u>

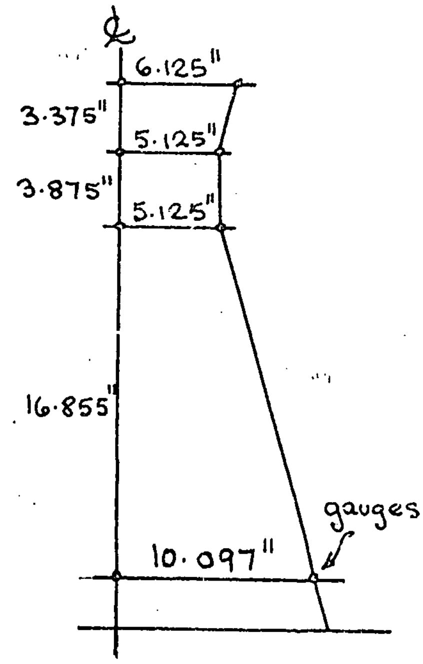


$n = 1.195 \qquad n^2 = 1.428 \qquad n^3 = 1.706$

$1 - n^3 = -0.706 \qquad 1 - 3n^2 + 2n^3 = 0.0128$

$\alpha = 180° - 73°30' \qquad \sin\alpha = 0.958 \qquad \cos\alpha = -0.284$

$\cos^2\alpha = 0.0807 \qquad r = 5.125$



$$S = p'\frac{r(1-n^3)}{3\sin\alpha\cos\alpha} = \underline{\underline{4.434\ p'}}$$

$$N_x = -p\frac{r(1-n^2)}{2\sin\alpha} - p''\frac{r(1-3n^2+2r^3)}{6\sin\alpha\cos^2\alpha}$$

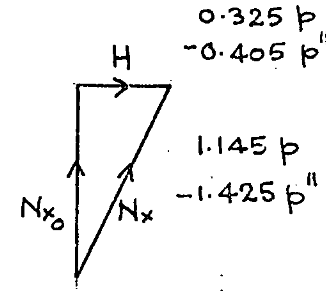
$$= \underline{\underline{1.145\ p - 1.425\ p''}}$$

Second Section

$$S_o = 4.434\ p' - 0.325\ p' + 0.405\ p'''$$

$$= \underline{\underline{4.109\ p' + 0.405\ p'''}}$$

$$N_{xo} = \underline{\underline{1.097\ p - 1.366\ p''}}$$

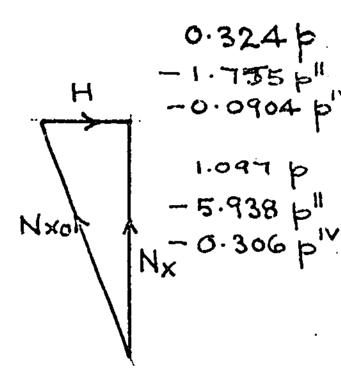


$$S = p'L + S_o = \underline{\underline{7.984\ p' + 0.405\ p'''}}$$

$$N_x = N_{xo} - p''L^2/2r - S_o'\,L/r$$

$$= \underline{\underline{1.097\ p - 5.938\ p'' - 0.306\ p^{IV}}}$$

Third Section



$n = 0.512 \qquad n^2 = 0.2621 \qquad n^3 = 0.1341$

$1-n^3 = 0.8659 \qquad 1-3n^2+2n^3 = 0.7440$

$1-n^2 = 0.7379 \qquad n-n^2 = 0.2499 \qquad r = 10.097$

$r_o = 5.125 \qquad \alpha = 73°32' \qquad \sin\alpha = 0.959$

$\cos\alpha = 0.2834 \qquad \cos^2\alpha = 0.0803$

$$H' = 0.324\ p' - 1.755\ p''' - 0.0904\ p^V$$

$$S_o = S - H' = \underline{\underline{7.660\ p' + 2.159\ p''' + 0.0904\ p^V}}$$

$$N_{xo} = \underline{\underline{1.144\ p - 6.192\ p'' - 0.319\ p^{IV}}}$$

The membrane stresses at the strain gauge level recorded more accurately are:

$$\left.\begin{aligned} S &= \underline{\underline{12.7577\ p' + 0.53713\ p''' + 0.022309\ p^V}} \\ N_x &= \underline{\underline{-3.32629\ p - 20.63168\ p''}} \\ &\quad \underline{\underline{-2.003685\ p^{IV} - 0.0764932\ p^{VI}}} \\ N_\phi &= \underline{\underline{-10.55\ p}} \end{aligned}\right\} \quad \ldots(4\text{-}7)$$

Model Tests

The Reynold's Number of a cooling tower throat diameter one hundred feet in a sixty mile per hour wind is:

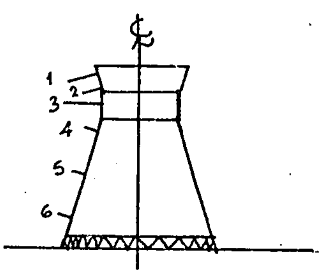
$$R_e = Vd/\nu = 88 \times 100 / 1.58 \times 10^{-4} = 56 \times 10^{6}$$

At 1/100 scale $R_e = 5.6 \times 10^{5}$

This lies in the region where the boundary layer is changing from a laminar to a turbulent one and the pressure distribution obtained would not necessarily be applicable to higher Reynold's numbers. Furthermore the stresses set up in the model tower would have been too small to measure accurately.

For these reasons it was arranged for the model to be tested in the pressurised wind tunnel at the National Physical Laboratories at Reynold's Numbers of 6.8×10^{6} and 11.9×10^{6}. No significant differences in pressure and stress distribution (measured in terms of the reference velocity head) were obtained. between the two tests.

Pressure readings were taken both inside and outside the shell at the levels shown and have been averaged for a given value of ϕ in Table 4-1.



To obtain the membrane stresses in the shell from the analysis it is necessary to find a curve to fit the empirical pressure readings around the shell. As successive derivatives of p with ϕ are required in the analysis it is not possible to choose a Fourier Series to fit the pressure readings. A limited Fourier Series becomes inaccurate on differentiating and an infinite one may diverge.

A simple expression in the form:

$$p = p_o \cos B\phi + C$$

may be chosen to fit the pressure curve over the front portion of the tower. (where p_o is pressure when $\phi = o$ and c is constant)

Then

$$\left.\begin{aligned} p^{II} &= -p_o B^2 \cos B\phi \\ p^{IV} &= +p_o B^4 \cos B\phi \\ p^{VI} &= -p_o B^6 \cos B\phi \end{aligned}\right\} \quad \ldots\ldots (4-8)$$

And
$$\left.\begin{aligned} p' &= -p_0 \; B \; \sin B\phi \\ p''' &= +p_0 \; B^3 \; \sin B\phi \\ p^{v} &= -p_0 \; B^5 \; \sin B\phi \end{aligned}\right\} \qquad \ldots\ldots(4\text{-}9)$$

Taking p as $(0.916 \cos 2.52\phi + 0.637)p_r$ a good fit for the pressure curve is obtained over the front $\pm 40^\circ$. Substituting these values in (4-7) N_x, N_ϕ and S are obtained and are compared with the measured stresses in Fig. 4-1 below. The strain values and their reduction to membrane stresses are given in Appendix 4.

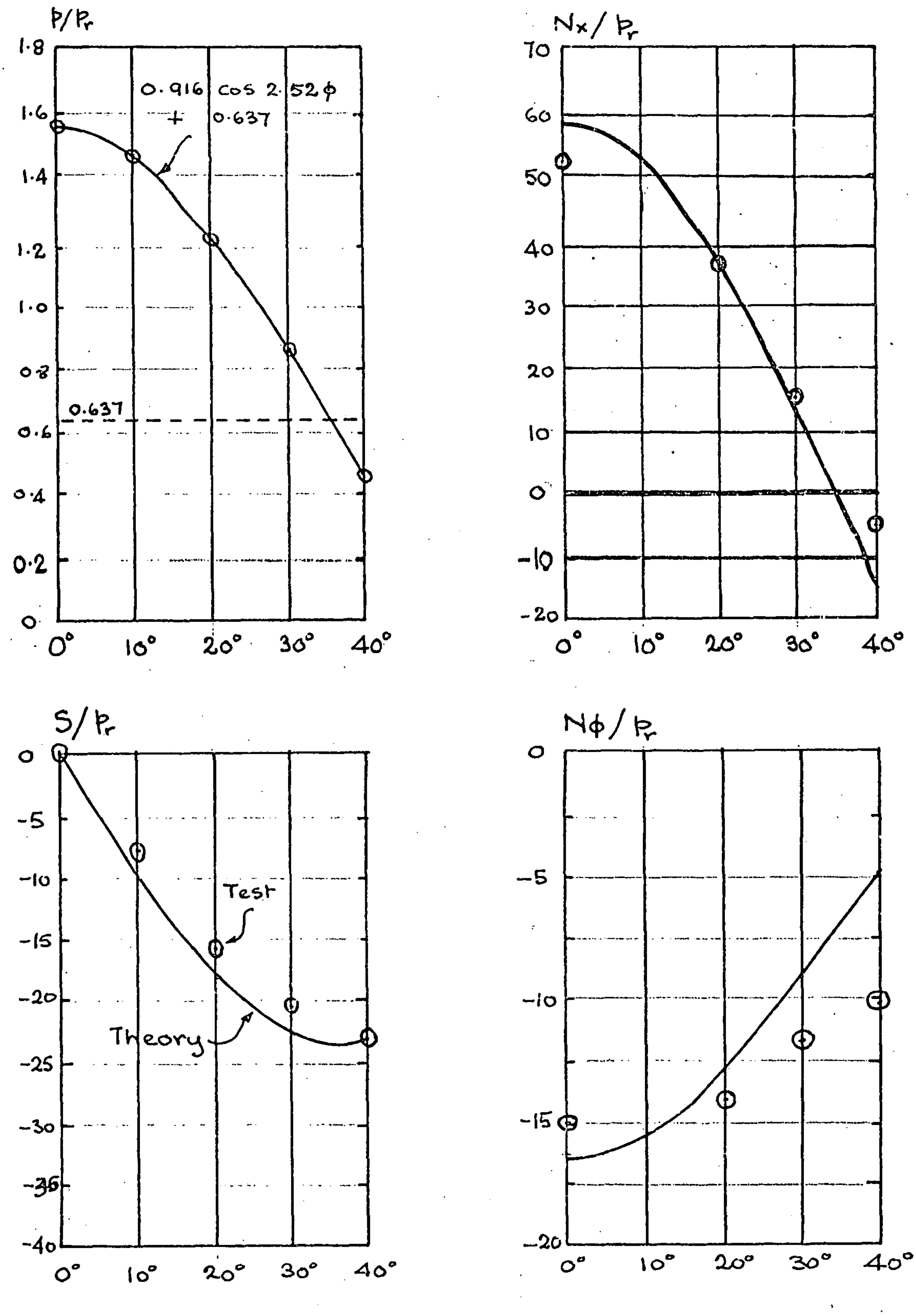


FIG 4-1

Fourier Analysis of Model Shell

It was felt that it would be of interest to fit the theory to the experimental results around the whole circumference of the shell. The method adopted was as follows:

The curves of N_x, S and p can be expressed respectively as:

$$\Sigma\, a_n \cos n\phi$$
$$\Sigma\, b_n \sin n\phi$$
$$\Sigma\, c_n \cos n\phi$$

Substituting in equations (4-7),(4-8) and (4-9)

$$\left. \begin{aligned} a_n = &-c_n(3.32629 - n^2 \times 20.63168 \\ &+ n^4_{\times} 2.003685 - n^6_{\times} 0.0764932) \end{aligned} \right\} \qquad \ldots(4\text{-}10)$$

and

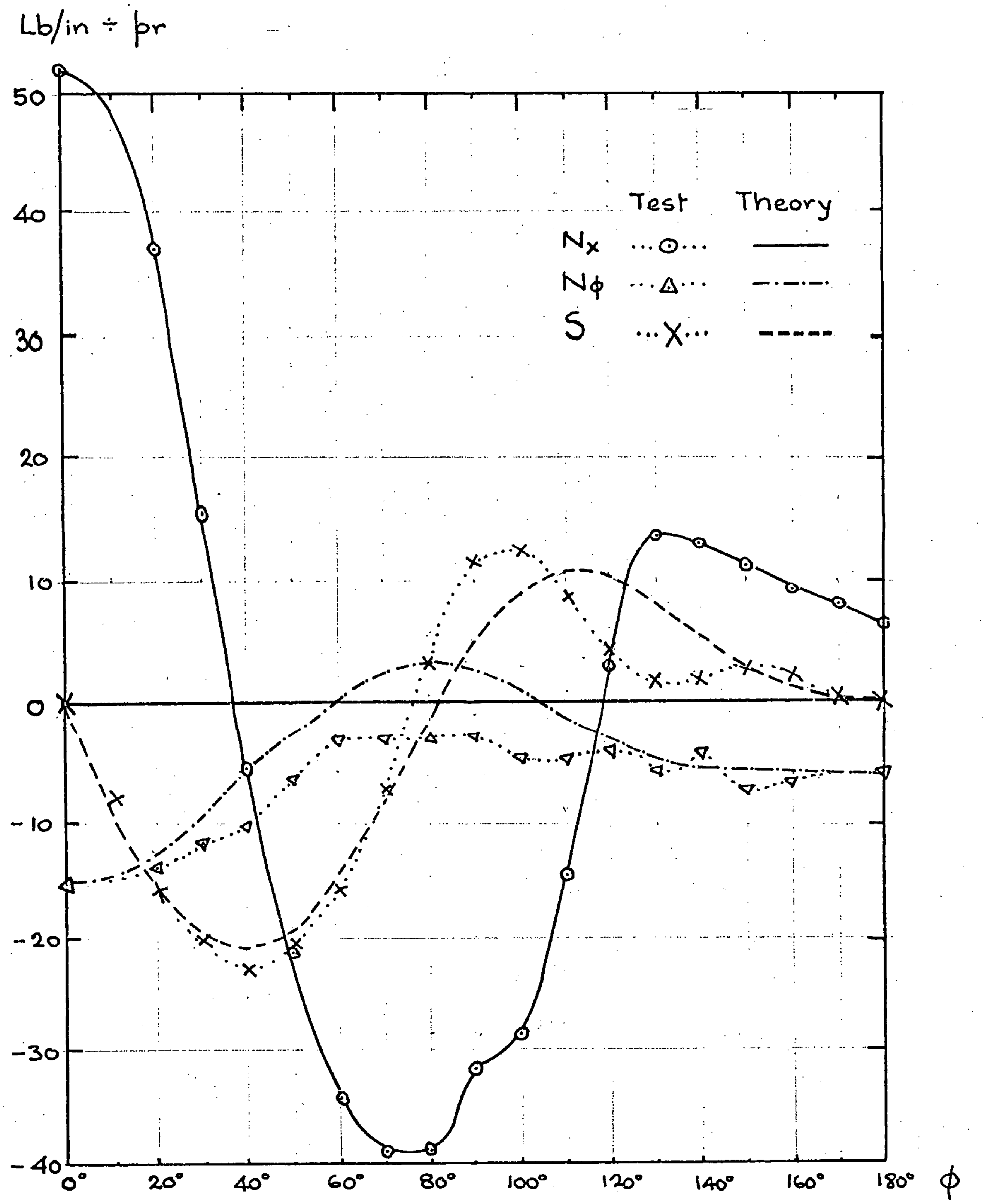
$$b_n = -c_n(n \times 12.7577 - n^3 \times 0.53713 + n^5 \times 0.022309) \qquad \ldots(4\text{-}11)$$

A Fourier series of 19 terms (every ten degrees) was fitted to the experimental curve of N_x by computer to find a_n. c_n was then calculated from (4-10) and b_n from (4-11). The curves for p and S were then computed from the Fourier series and are plotted in Figure 4-2 against the empirical values. The process involves the integration of the Fourier series and is hence permissible.

The Fourier coefficients for the wind pressure are given in Table 4-2 and may be used in design. As however the important membrane stresses lie between $\phi = 0°$ and $40°$ it is much easier to use the single term expression

$$p = p_o \cos 2.52\phi + 0.46\, p_{top}$$

where p_o is the wind pressure on the front face of the tower at the level under consideration.


Lb/in ÷ pr
50
40
30
20
10
0
-10
-20
-30
-40
Test
Theory
Nx
Nφ
S
0°
20°
40°
60°
80°
100°
120°
140°
160°
180°
φ

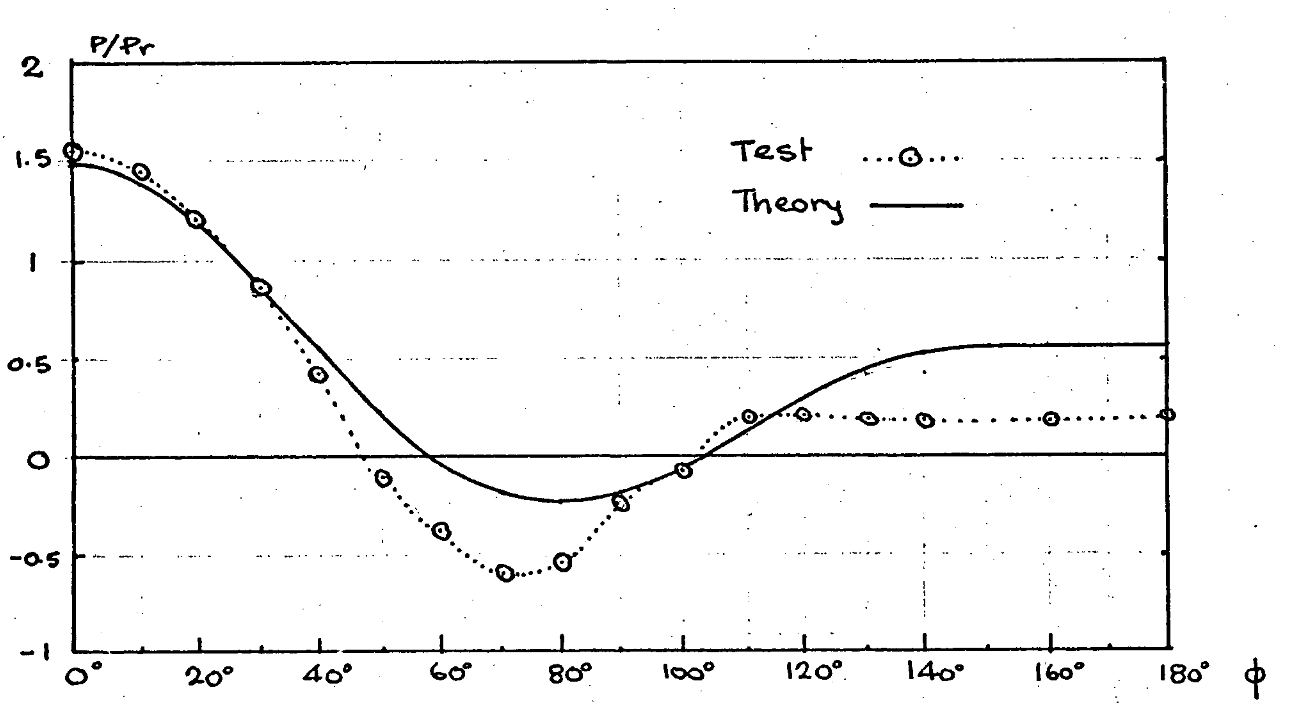
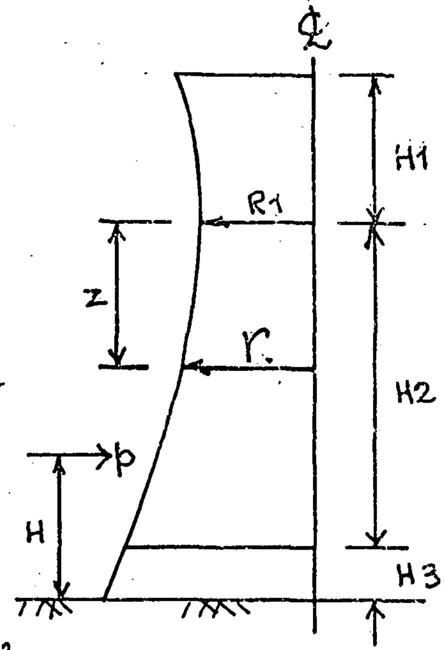

P/Pr
2
1.5
1
0.5
0
-0.5
-1
Test
Theory
0°
20°
40°
60°
80°
100°
120°
140°
160°
180°
φ

Fig 4-2

<u>Computer Analysis of Hyperboloid Shells</u>



Cooling towers are not built today with two or three conical sections although such towers were sometimes made before the Second World War. The most usual and the most satisfactory shape is a hyperboloid or hyperbola of revolution although other similar shapes are sometimes employed.

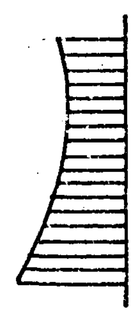
The equation of a hyperbola is $x^2/a^2 - y^2/b^2 = 1$

or $$r^2 = Az^2 + C$$

If R_1 is the throat radius and R_2 the radius at $z = H_2$ then

$$C = R_1^2 \quad \text{and} \quad A = (R_2^2 - C)/H_2^2$$

Cooling tower shells are constructed with formwork that is straight in the meridian direction. It is therefore arguable that it is correct to take the tower as being made up of a large number of conical segments. Thus the analysis in the previous section can be applied to the "hyperboloid" cooling tower without loss of accuracy.



The velocity of a strong wind is normally taken as varying with height in an exponential form $$V = V_{40}\,(H/40)^m$$

m is taken in the British code of practice CP3-Chapter V as 0.13.

It would be expected that the pressure on the front face of the tower would be $\rho V^2/2g$ where ρ is the density of the air in lb/cu ft and V is measured in ft/sec. This is known as a velocity pressure. Then

$$p = p_{40}\,(H/40)^{2m} \cos B\phi$$

Where p_{40} is the velocity pressure at forty feet above the ground.

The constant suction inside the tower has a small effect which is taken into account in the working program but is omitted here in the interests of simplicity.

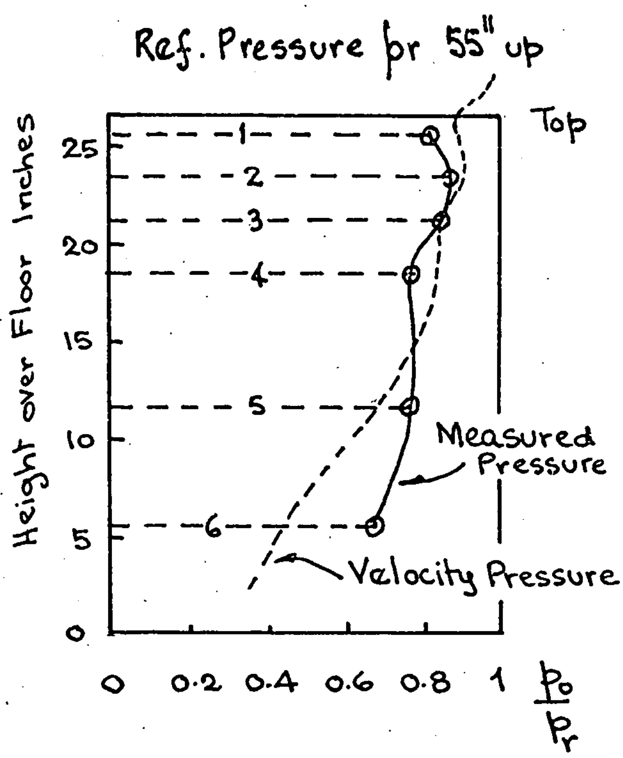
The vertical depth L of each segment is taken as H1 divided by a suitable integer J . This eliminates the possibility of any segment degenerating into a cylinder, as the computation commences at the top. The calculation is carried out as follows.

```
read  R1, R2, H1, H2, H3, p40 ; m, B, J

C := R1^2
A := (R2^2 - C)/ H2^2
L := H1/J
Nx := S := 0
H := H1 + H2 + H3 + L/2
Z := -H1
r0 := sqrt (A Z^2 + C)

Z := Z + L
H := H - L
r := sqrt (A Z^2 + C)
n := r0/r
α := arctan ( L/(r - r0))
if r < r0 then α := π + α
Nxo := Nx sin α0 / sin α
So := S + Nx B sin(α0 - α)/ sin α
p := p40 (H/40)^(2m)
S := -p B r (1 - n^3)/ 3 sin α cos α
       + So n^2
Nx := Nxo.n - p r (1 - n^2)/2 sin α
       + p B^2 r (1 - 3n^2 + 2n^3)/6 sin α cos α
       + So B (n - n^2)/ cos α
Nφ := -p r/ sin α
α0 := α ;   r0 := r ;
print  Z, p, r, Nx, Nφ, S.

Until
Z > H2
```

The analysis of the truly hyperbolic tower has been carried out by Martin and Scriven. Little difference in results is obtained compared with the method just outlined whereas the theory and programming are considerably more difficult. Furthermore Martin and Scriven's method is not applicable to the cone-toroid towers that are often built. In this case the lower part of the tower is a cone while the upper part is formed of an arc of a circle or a hyperbola rotated about the centre line. Only a minor modification in our program is necessary to make it apply to any shape of tower. It should be noted that the membrane stresses are considerably larger in a cone-toroid tower than in a hyperboloid of similar dimensions. It was the failure on the part of the designers to appreciate this that was the basic cause of the collapse of the cooling towers in Ferrybridge in 1965.

<u>Further Discussion on Wind Pressure</u>

The results of the tests on the first model cooling tower indicates that the wind pressures on the front of the tower in a wind that increases from the ground up are not exactly the velocity pressures but are higher near the base of the tower and lower closer to the top. Also the distribution of pressure around the front face is different from that in a uniform wind being slightly less acute ie $B = 2.45$ compared with 2.52 for a uniform wind. These effects can be shown to cancel out and the velocity pressure taken together with a B value of 2.52 gives a realistic value for the membrane stresses in an isolated cooling tower.



There is a further factor to be considered. Towers are usually built in groups and the upstream towers modify the pressure distribution around the downstream ones giving rise to a worst B value of 3·27. This result was obtained by further tests carried out at the National Physical Laboratories in 1962.

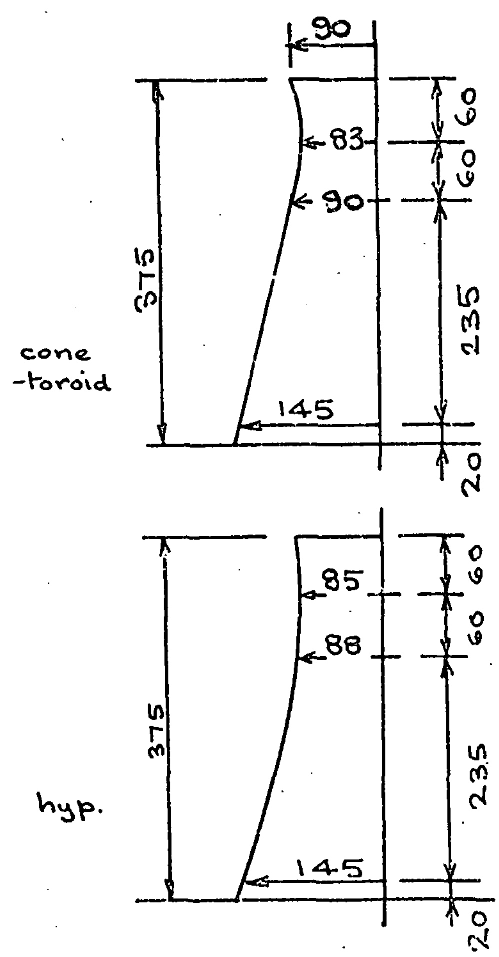
This change in the B value has very little effect on a hyperboloid cooling tower, but greatly increases the membrane stresses in a cone-toroid.

Ferrybridge

On 1st November 1965 a moderate gale blew in the English Midlands. At Ferrybridge in Yorkshire the Meteorological Office estimated that the maximum wind speed lasting for one minute was between 49 and 54 mph at 40 ft above the ground and 68-74 mph at 375 ft above the ground. There were eight cooling towers newly constructed at the power station each costing £290 000, standing 375 ft high with a base diameter of 290 ft and walls five inches thick. The towers had been designed for a steady wind velocity of 63 mph at 40 ft rising to 84.3 mph at 375 ft in accordance with an exponential value of 0.13. Nevertheless three of the eight towers collapsed in the wind. It is the opinion of the author that these failures were caused by designing the towers as hyperboloids and constructing them as cone-toroids.

Reinforcement was provided in the towers to carry the difference between the meridian stresses due to the wind and the dead weight of the shell.

The meridian stresses due to the design wind on the tower as designed and the tower as constructed are shown in Figure 4-3. It will be seen that for the cone-toroid the reinforcement will be stressed to 450% of the design for an isolated tower ie in the front rank, and 900% for the shielded towers.



However for the one minute velocities experienced at the site at the time of the collapse, the unshielded towers would have been stressed to about 100% of design and the shielded ones to 400%. This is in agreement with the failure of the towers in the rear rank but not the front rank of the installation.

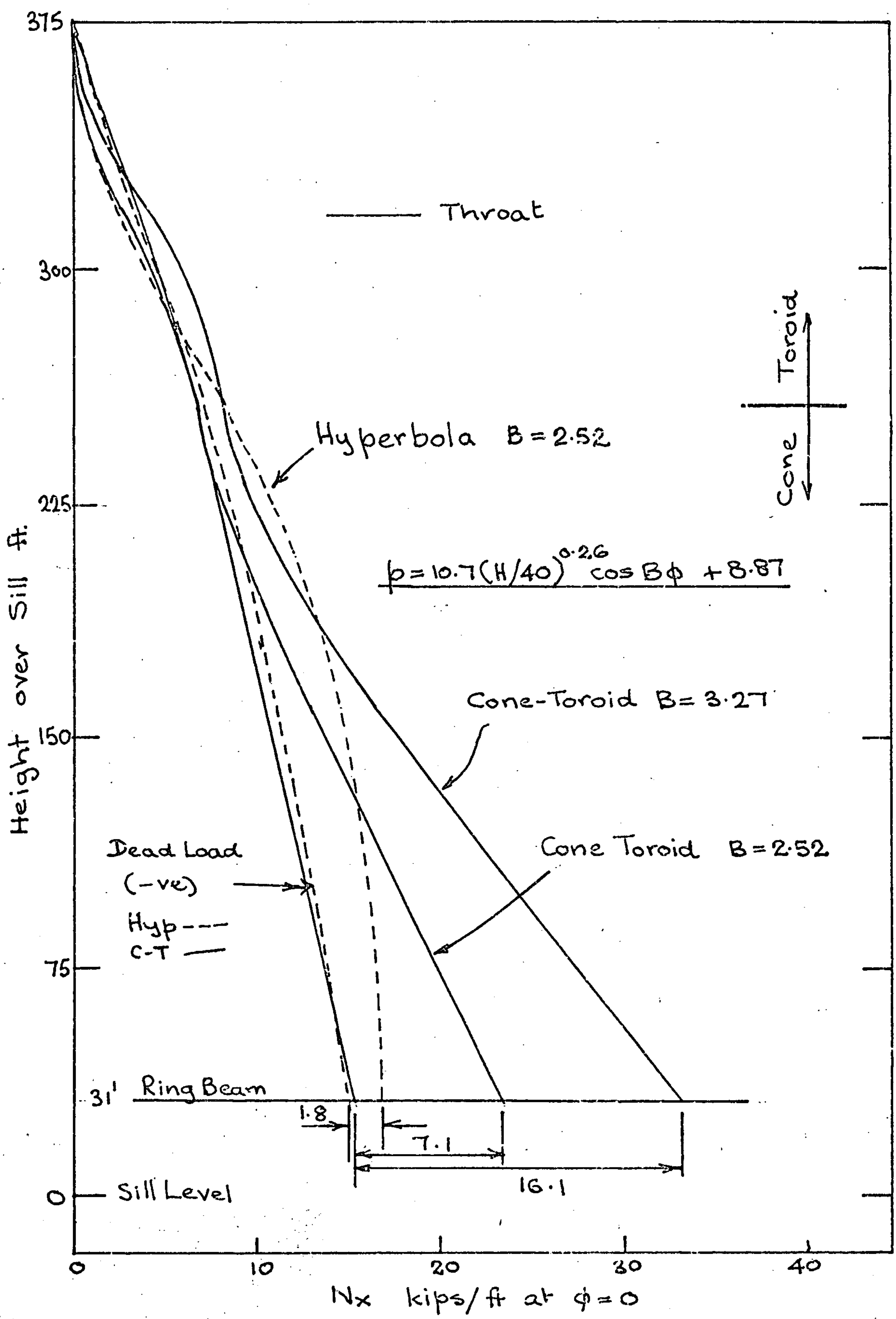


FIG 4-3

APPENDIX 4

Stress-Strain Values for Model Tower

Strips were cut out of the model shell after the tests in the wind tunnel and subjected to a tensile loading test in a Hounsfield Tensometer. These tests gave an E value of 20×10^6 psi for the inside gauge and 13.06×10^6 for the outside gauge, over the strain range experienced in the model tests. This difference in E value was most likely due to the straightening of a strip with inital curvature rather than a difference in the gauge factors of matched gauges and an average value of 16.53×10^6 has been taken for the conversion of strain into stress. A poisson's ratio of 0.33 has been taken. The shell was made of 24 gauge brass sheet (0.025 inches thick). Then from equations (3-11)

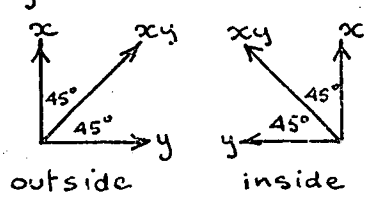
$$N_x = Eh/(1-\nu^2)\,\{\epsilon_x + \nu\,\epsilon_y\} \quad \text{etc}$$

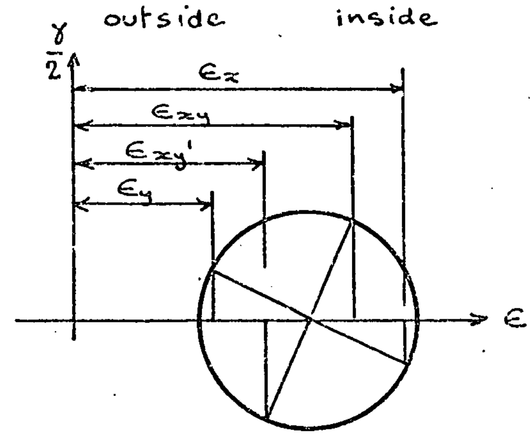
The original strain readings were recorded in NPL/Aero/316 as $\% \times 10^6 \div \rho v^2/2g$ sq ft /lb. To obtain membrane stresses as N_x/p_r etc where p_r is measured in lb/sq in.

$$N = 16.53 \times 10^6 \times 0.025 \times 10^{-8} \times 144/0.891 = 0.668$$

$$N(1-\nu)/2 = 0.224$$

The strain rosettes consisted of three gauges at 45° x, xy, y. The strains for the inside and outside gauges in the x and y directions at $\pm\phi$ have been averaged and recorded as e_x and e_y. The strains for the inside xy gauge at $-\phi$ and the outside at $+\phi$ have been averaged and recorded as e_{xy}. The strains for the inside gauge at ϕ and the outside at $-\phi$ have been averaged and recorded as e_{xy}'.





From the Mohr Circle

$$\epsilon_x + \epsilon_y = \epsilon_{xy} + \epsilon_{xy}'$$

$$\therefore\; e_x + e_y = e_{xy} + e_{xy}' + 4E'$$

Where E' is the error assumed equal for the four gauges.

Then $\epsilon_x = e_x - E'$ $\quad \epsilon_y = e_y - E'$

$$\epsilon_{xy} = e_{xy} + E' \qquad \epsilon_{xy'} = e_{xy'} + E'$$

$$\gamma_{xy} = \epsilon_{xy} - \epsilon_{xy'} = e_{xy} - e_{xy'}$$

Average pressures and strains $\div \rho v^2/2g$

$\phi°$	p	e_x	e_y	e_{xy}	$e_{xy'}$	ϵ_x	ϵ_y
0	1.553	100	-51	17.5	17.5	96.5	-54.5
10	1.456	–	–	-2	33.5	–	–
20	1.218	74	-41.5	-26.5	46	70.7	-44.7
30	0.862	35	-27.5	-45.5	47.5	33.6	-28.9
40	0.427	-1.5	-13	-62	42	-2.9	-14.4
50	-0.004	-34.5	1.5	-64	31	-34.5	1.5
60	-0.385	-58	12	-54.5	16.5	-56	14
70	-0.609	-65.5	16	-40	-6.5	-64.8	16.8
80	-0.531	-66.5	15	-16	-27.5	-64.5	17
90	-0.235	-53.5	12	7.5	-44.5	-52.4	13.1
100	0.071	-46	7.5	10	-46.5	-45.8	8
110	0.194	-20	2.5	5.5	-31.5	-22.1	0.4
120	0.204	9	-6	5	-11.5	6.6	-8.4
130	0.197	30	-14.5	4	-1	26.9	-17.6
140	0.189	28.5	-11	5.5	-2.5	24.9	-14.6
150	–	25	-16.5	5	-5.5	22.8	-18.8
160	0.183	21	-15.5	5	-5	19.6	-16.9
170	–	–	–	-3	-5	–	–
180	0.181	13	-14	1	1	13.8	-13.3

$$N_x = 0.668\ (\epsilon_x + 0.33\,\epsilon_y)$$

$$N_\phi = 0.668\ (\epsilon_y + 0.33\,\epsilon_x)$$

$$S = 0.224\ (e_{xy} - e_{xy'})$$

The test data were processed on an Olivetti Programma desk computer.

TABLE 4-2

Fourier Coefficients

n	a_n	C_n
0	-1.438	$+4.321 \times 10^{-1}$
1	$+3.057$	$+1.988 \times 10^{-1}$
2	$+32.131$	$+6.175 \times 10^{-1}$
3	$+19.714$	$+2.599 \times 10^{-1}$
4	-1.073	-8.438×10^{-3}
5	$+2.138$	$+4.694 \times 10^{-3}$
6	$+0.708$	$+4.138 \times 10^{-4}$
7	-2.295	-4.416×10^{-4}
8	$+0.503$	$+3.821 \times 10^{-5}$
9	$+0.325$	$+1.1125 \times 10^{-5}$
10	-1.422	-2.431×10^{-5}
11	$+0.040$	-3.665×10^{-7}
12	$+0.212$	$+1.117 \times 10^{-6}$
13	-0.109	-3.441×10^{-7}
14	-0.407	-8.100×10^{-7}
15	-0.057	-7.336×10^{-8}
16	$+0.042$	$+3.648 \times 10^{-7}$
17	$+0.265$	$+1.572 \times 10^{-7}$
18	-0.285	-1.188×10^{-7}

N_x at strain gauge level $= \sum_{0}^{18} a_n \cos n\phi$

$$P/P_r = \sum_{0}^{18} C_n \cos n\phi$$

Chapter 5.

COMPLETE CYLINDRICAL SHELLS

Schorer's equation $\frac{\partial^8 w}{\partial \phi^8} + \frac{12 R^6}{h^2} \frac{\partial^4 w}{\partial x^4} = 0$ for the deformation of a cylindrical shell under the action of edge loads was a partial differential equation with variables in both the ϕ and x directions. This was handled in the case of the shell roof by taking the deformation in the x direction as a trigonometric function. The function chosen was a term of a Fourier series which satisfied the boundary conditions at the traverses. This reduced the partial to an ordinary eighth order differential equation with boundaries at the edges of the shell.

The complete shell has no boundaries in the ϕ direction so that it is appropriate to take the deformation as a trigonometric function in the ϕ direction when the loading is not uniform around the circumference.

Putting $w = \bar{w} \cos m\phi$ Schorer's equation becomes:

$$\frac{\partial^4 \bar{w}}{\partial x^4} + \frac{h^2 m^8}{12 R^6} \bar{w} = 0$$

Which is a fourth order ordinary differential equation in the x direction and can be handled by the methods derived for the beam on an elastic foundation. A considerable increase in accuracy without increase in difficulty in this case is obtained by taking the curvature in the ϕ direction as $\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \phi^2} + w\right)$ instead of $\frac{\partial^2 w}{R^2 \partial \phi^2}$ as in the Schorer method. Also the component of Q_ϕ in the ϕ direction which was previously neglected due to the necessity of simplifying the equation can now be taken into account.

Finally the boundary conditions in the x direction are more easily expressed if the equation is derived in terms of u instead of w. Surprisingly enough it is found that the equation is not changed in any way by this change in variable.

The simplifying assumptions to be made, with the exceptions mentioned above are similar to Schorer's. They will be justified later.

They are:

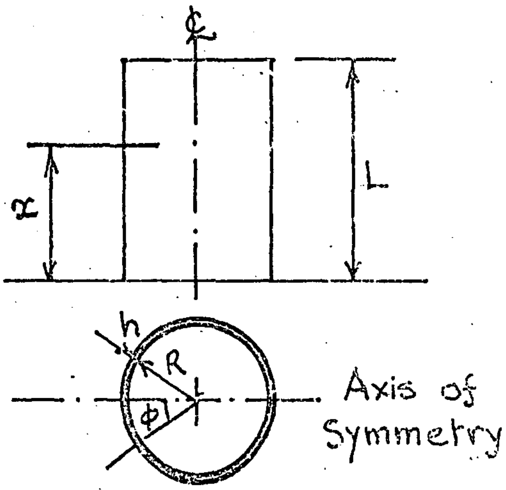
(a) νN_ϕ is small compared with N_x.

(b) M_x Q_x $M_{x\phi}$ are negligible.

(c) ϵ_ϕ is small compared with deformations due to bending.

(d) $\gamma_{x\phi}$ is small compared with deformations due to bending.

Taking moments and resolving in the ω, ϕ and x directions the forces on the edges of the element, the following relations are obtained.

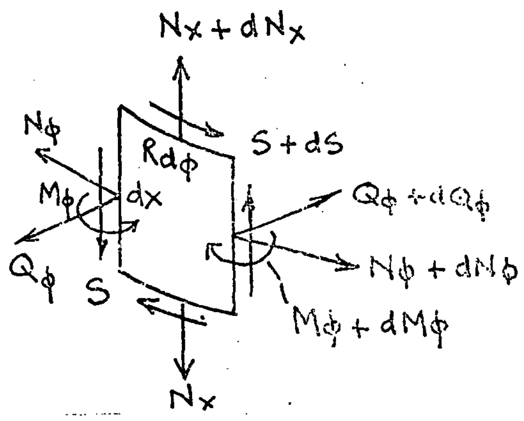


$$Q_\phi = \frac{\partial M_\phi}{R\,\partial\phi}$$

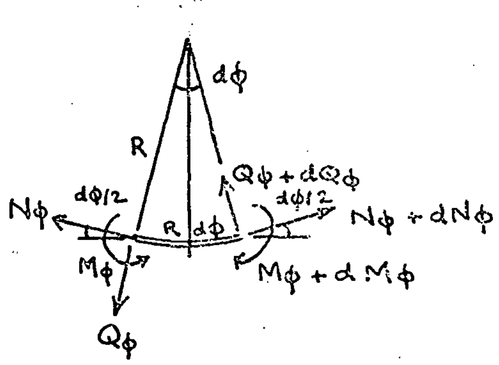
$$N_\phi = -\frac{\partial Q_\phi}{\partial\phi} = -\frac{\partial^2 M_\phi}{R\,\partial\phi^2}$$

$$\frac{\partial S}{\partial x} = \frac{Q_\phi}{R} - \frac{\partial N_\phi}{R\partial\phi} = \frac{1}{R^2}\left(\frac{\partial M_\phi}{\partial\phi} + \frac{\partial^3 M_\phi}{\partial\phi^3}\right)$$

$$\frac{\partial N_x}{\partial x} = -\frac{\partial S}{R\partial\phi} \qquad \ldots \quad (5\text{-}1)$$



To simplify the diagram $\frac{\partial N_x}{\partial x}dx$ has been written dN_x etc.



Taking the radial and shear strains as being small and putting

$$M_\phi = -\frac{D}{R^2}\left(\frac{\partial^2\omega}{\partial\phi^2} + \omega\right)$$

$$\epsilon_\phi = \partial v/R\partial\phi - \omega/R \qquad\qquad \gamma_{x\phi} = \partial v/\partial x + \partial u/R\partial\phi$$

$$\therefore \quad \partial v/\partial\phi = \omega \qquad\qquad \partial v/\partial x = -\partial u/R\partial\phi$$

$$\therefore \quad \partial^2 v/\partial\phi\partial x = \partial\omega/\partial x \qquad\qquad \partial^2 v/\partial\phi\partial x = -\partial^2 u/R\partial\phi^2$$

$$\text{or} \quad \partial\omega/\partial x = -\partial^2 u/R\partial\phi^2 \quad \ldots \quad (5\text{-}2)$$

We now take the final equation of statics (5-1) in the form:

$$\frac{\partial^3 N_x}{\partial x^3} = -\frac{\partial^3 S}{R\,\partial\phi\,\partial x^2} = -\frac{1}{R^3}\left(\frac{\partial^3 M_\phi}{\partial\phi^2\,\partial x} + \frac{\partial^5 M_\phi}{\partial\phi^4\,\partial x}\right)$$

This includes all the four equations of statics.

Simplifying condition (a) leads to:

$$\epsilon_x = N_x / Eh$$

or

$$N_x = Eh\,\partial u/\partial x$$

Combining all these relations with the equations of geometry (5-2) the differential equation of the shell is obtained.

$$Eh\,\frac{\partial^4 u}{\partial x^4} = \frac{D}{R^5}\left(\frac{\partial^3 w}{\partial\phi^2\partial x} + 2\,\frac{\partial^5 w}{\partial\phi^4\partial x} + \frac{\partial^7 w}{\partial\phi^5\partial x}\right)$$

$$= -\frac{D}{R^6}\left(\frac{\partial^4 u}{\partial\phi^4} + 2\,\frac{\partial^6 u}{\partial\phi^6} + \frac{\partial^8 u}{\partial\phi^8}\right)$$

or

$$\frac{\partial^4 u}{\partial x^4} + \frac{D}{EhR^6}\left(\frac{\partial^4 u}{\partial\phi^4} + 2\,\frac{\partial^6 u}{\partial\phi^6} + \frac{\partial^8 u}{\partial\phi^8}\right) = 0 \quad \cdots (5\text{-}3)$$

This partial differential equation is now turned into an ordinary one by putting $u = \bar{u}\cos m\phi$

Then

$$\frac{\partial^4 \bar{u}}{\partial x^4}\cos m\phi + \frac{D}{EhR^6}(m^4 - 2m^6 + m^8)\,\bar{u}\cos m\phi = 0$$

Dividing through by $\cos m\phi$ and putting $D = Eh^3/12(1-\nu^2)$

$$\frac{\partial^4 \bar{u}}{\partial x^4} + 4b^4\,\bar{u} = 0$$

Where

$$4b^4 = \begin{cases} h^2/12(1-\nu^2)R^6 \\ \times\ m^4(m^2-1)^2 \end{cases} \quad \text{or} \quad b = \frac{m}{2R}\sqrt{\frac{h(m^2-1)}{R\sqrt{3(1-\nu^2)}}}$$

It should be noted that this root b is small compared with β for axi-symmetric loading, which is $\sqrt{\frac{\sqrt{3(1-\nu^2)}}{Rh}}$

Therefore e^{bx} remains close to unity for comparatively large values of x and the effects of an unsymmetrical edge load will not die away rapidly from the edge.

The problem remains of expressing the forces, moments and deflections of the shell in terms of the longitudinal displacements u.

N_x is already known. The rest are obtained from the equations of statics.

$$N_x = Eh \frac{\partial \bar{u}}{\partial x} \cos m\phi \qquad \ldots (5\text{-}4)$$

$$-\frac{\partial S}{R\partial\phi} = \frac{\partial N_x}{\partial x} = Eh \frac{\partial^2 \bar{u}}{\partial x^2} \cos m\phi$$

$$\therefore \; S = -\frac{EhR}{m} \frac{\partial^2 \bar{u}}{\partial x^2} \sin m\phi \qquad \ldots (5\text{-}5)$$

$$\frac{\partial S}{\partial x} = -\frac{EhR}{m} \frac{\partial^3 \bar{u}}{\partial x^3} \sin m\phi$$

$$= \frac{m(m^2-1)}{R^2} \bar{M}_\phi \sin m\phi$$

$$\therefore \; \bar{M}_\phi = -\frac{EhR^3}{m^2(m^2-1)} \frac{\partial^3 \bar{u}}{\partial x^3} \qquad \ldots (5\text{-}6)$$

$$= \frac{D}{R^2} (m^2-1)\, \bar{w}$$

$$\therefore \; \bar{w} = -\frac{EhR^5}{D\, m^2(m^2-1)^2} \frac{\partial^3 \bar{u}}{\partial x^3} \qquad \ldots (5\text{-}7)$$

$$\text{or } -\frac{m^2}{4 b^4 R} \frac{\partial^3 \bar{u}}{\partial x^3}$$

$$Q_\phi = +\frac{m \bar{M}_\phi}{R} \sin m\phi$$

$$= -\frac{EhR^2}{m(m^2-1)} \frac{\partial^3 \bar{u}}{\partial x^3} \qquad \ldots (5\text{-}8)$$

$$N_\phi = \frac{m^2}{R} \bar{M}_\phi \cos m\phi$$

$$= -\frac{EhR^2}{m^2-1} \frac{\partial^3 \bar{u}}{\partial x^3} \cos m\phi \qquad \ldots (5\text{-}9)$$

$$M_\phi = -\frac{EhR^3}{m^2(m^2-1)} \frac{\partial^3 \bar{u}}{\partial x^3} \cos m\phi \qquad \ldots (5\text{-}10)$$

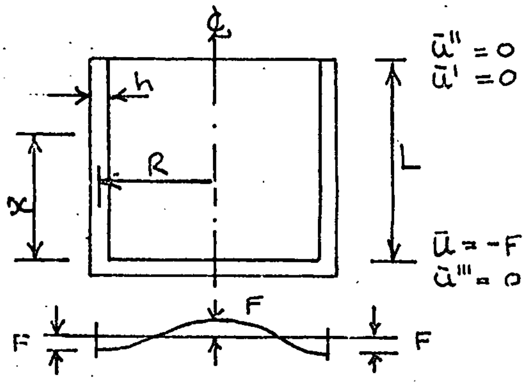
It will be noted that integration with respect to ϕ produces no constant of integration as this is taken care of by the eigenfunction.

Tank with uneven Settlement

Cylindrical tanks with open ends are often constructed upon ground that settles unevenly under the weight. It is of interest to be able to calculate the stresses that are set up in the tank walls due to this settlement. This can be done by thebtheory just developed.

The settlement will be taken as $u_0 = -F\cos 2\phi$ which means that a point at the circumference at $x = 0$ and $\phi = 0$ settles $4F$ relative to the other three quarter points.

The lower end of the tank wall is prevented from moving radially by the floor. At the open end of the tank there is a free edge and there can be no shearing or longitudinal stresses.



These four boundary conditions are sufficient to enable the constants of integration C1, C2, C3, C4 to be evaluated.

(1) At $x = 0$ $\quad \bar{u} = -F$

(2) At $x = 0$ $\quad w = 0 \quad \therefore \quad \bar{u}'''$ or $\dfrac{\partial^3 \bar{u}}{\partial x^3} = 0$

(3) At $x = L$ $\quad S = 0 \quad \therefore \quad \bar{u}''$ or $\dfrac{\partial^2 \bar{u}}{\partial x^2} = 0$

(4) At $x = L$ $\quad N_x = 0 \quad \therefore \quad \bar{u}'$ or $\dfrac{\partial u}{\partial x} = 0$

The table of derivatives of the solution to the shell equation is given below. It is identical in form to Table 1-1 for the beam on an elastic foundation with $\bar{u}$ substituted for w.

$$m = 2 \quad \therefore \quad b = \frac{1}{R}\sqrt{\frac{3h}{R\sqrt{3(1-\nu^2)}}}$$

TABLE 5-1

$f(\bar{u})$	Mult.	e^{bx}		e^{-bx}	
		$\cos bx$	$\sin bx$	$\cos bx$	$\sin bx$
$\bar{u}$	1	C1	C2	C3	C4
$\bar{u}'$	b	C1 + C2	−C1 + C2	−C3 + C4	−C3 − C4
$\bar{u}''$	$2b^2$	C2	− C1	− C4	C3
$\bar{u}'''$	$2b^3$	−C1 + C2	−C1 − C2	C3 + C4	−C3 + C4

When $x = 0$: $e^{bx} = e^{-bx} = \cos bx = 1$; $\sin bx = 0$;

Putting $E = e^{bL}$, $C = \cos bL$, $S = \sin bL$

the set of four simultaneous equations to be solved to find the constants of integration can be expressed in matrix form as:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ -SE & CE & S/E & -C/E \\ (C-S)E & (C+S)E & -(C+S)/E & (C-S)/E \end{bmatrix} \begin{bmatrix} C1 \\ C2 \\ C3 \\ C4 \end{bmatrix} = \begin{bmatrix} -F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

When the solution is carried out by computer the matrix can be left in this form and handled by any available procedure. It is difficult to get a satisfactory solution by slide-rule as the constants can not be obtained with sufficient accuracy.

A solution using a desk top computer such as the Olivetti Programma 101 can be attempted as follows:

a_{32} and a_{41} are made unity by dividing the third line by CE and the fourth line by $(C-S)E$.

Putting $T = S/C$, $U = (C+S)/(C-S)$, $V = e^{-2bL}$

the matrix becomes:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ -T & 1 & TV & -V \\ 1 & U & -UV & V \end{bmatrix} \begin{bmatrix} C1 \\ C2 \\ C3 \\ C4 \end{bmatrix} = \begin{bmatrix} -F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A Gauss-Seidel method was at first attempted. This is done by first setting all C 's to zero, then calculating in turn

$$C3 := -F - C1 ;$$
$$C4 := C1 - C2 - C3 ;$$
$$C2 := C1.T - C3.T.V + C4.V ;$$
$$C1 := -C2.U + C3.U.V - C4V ;$$

The process was to have been repeated until the constants converged to a steady value. Unfortunately the method which was simple to program and did not require much storage did not work, the solution oscillating and diverging rapidly.

An attempt was then made to damp down the oscillation by letting the constants take up a value half way between their original value and the value obtained above.

Thus:

$$C3 := (-F - C1 + C3)/2 \;;$$
$$C4 := (C1 - C2 - C3 + C4)/2 \;;$$
$$C2 := (C1.T - C3.T.V + C4V + C2)/2 \;;$$
$$C1 := (-C2.U + C3.U.V - C4V + C1)/2 \;;$$

The solution then converged in a satisfactory fashion to a steady value for the examples attempted.

Example

A reinforced concrete tank 100 feet diameter, with walls one foot thick and 50 feet high undergoes a differential settlement of 9.6 inches. Taking E as 3 million lb/sq in and Poisson's ratio as zero find the maximum vertical and bending stresses and the maximum radial deflection.

$F = -0.2$ ft	$\nu = 0$
$R = 50$ ft	$E = 432 \times 10^6$ lb/sq ft.
$L = 50$ ft	$D = 36 \times 10^6$ lb ft.
$h = 1$ ft	$m = 2$

$$b = \frac{1}{R}\sqrt{\frac{h\sqrt{3}}{R}} = 0.00372 \text{ ft}^{-1}$$
$$k = bL = 0.186$$
$$V = e^{-2k} = 0.689354$$
$$T = \tan k = 0.188175$$
$$U = \frac{\cos k + \sin k}{\cos k - \sin k} = 1.463585$$

The damped Gauss Seidel method gave the following values for the constants of integration:

$$C1 = -0.0995712$$
$$C2 = -0.0030296$$
$$C3 = -0.1004288$$
$$C4 = +0.0038872$$

$$N_x \text{ at } x = 0,\ \phi = 0 = Ehb(C1 + C2 - C3 + C4)$$
$$= \underline{\underline{2.756 \text{ kips / ft}}}$$

$$w \text{ at } x = L,\ \phi = 0$$
$$= \frac{2}{bR}\left\{ e^{k}\left((-C1+C2)\cos k + (-C1-C2)\sin k\right) + e^{-k}\left((C3+C4)\cos k + (-C3+C4)\sin k\right)\right\}$$
$$= \underline{\underline{-0.7996 \text{ ft}}}$$

$$M_\phi \text{ at } x = L,\ \phi = 0 = \frac{3D}{R^2} w = \underline{\underline{33\,200 \text{ lb}}}$$

The vertical stress at the base of the tank is not in this instance very important amounting to only 19.1 lbs/sq in. However the bending stress at the top of the wall $6M_\phi/h^2$ has the very large value of 1330 lb/sq in. and would certainly crack the wall.

It will be shown in the next section that the neglect of the twisting moment leads in this example to an underestimation of N_x. The calculated values for deformation and M_ϕ are however reasonably accurate.

Inextensional Bending of Tank

The problem worked out above showed that the stresses and strains in the middle surface of the shell produced by the differential settlement are small and the bending moments and changes of curvature are large. It is therefore practicable to solve the problem using an energy method and assuming that no strain occurs in the middle surface.

The conditions of inextensional bending which must apply throughout the shell can be expressed as:

$$\left.\begin{aligned} \epsilon_x &= \partial u/\partial x = 0 \\ \epsilon_\phi &= \frac{1}{R}\left(\partial v/\partial\phi - w\right) = 0 \\ \gamma_{x\phi} &= \partial u/R\partial\phi + \partial v/\partial x = 0 \end{aligned}\right\} \quad \ldots(5\text{-}11)$$

As the curvature takes place in the ϕ direction and the wall is initially straight in the vertical direction, the wall must remain straight after deformation if stretching is not to take place.

Taking w as linear with x and zero at $x = 0$, and choosing a symmetrical Fourier series for the deformation in the ϕ direction:

$$w = x \sum B_m m \cos m\phi$$

$$\partial v/\partial\phi = w \qquad \therefore \qquad v = x \sum B_m \sin m\phi$$

There is no constant of integration as $v = 0$ when $x = 0$, that is it is taken care of by the eigenfunction.

$$\partial u/\partial\phi = -R\, \partial v/\partial x = -R \sum B_m \sin m\phi$$

$$\therefore \quad u = R \sum \frac{B_m}{m} \cos m\phi$$

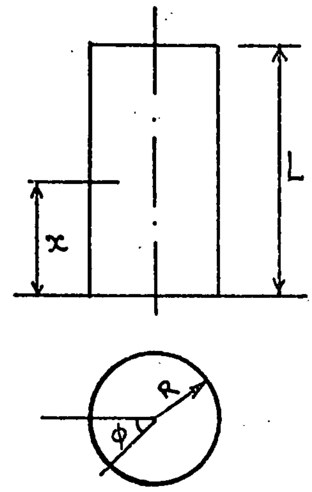
$$\epsilon_x = \partial u/\partial x = 0$$

thus satisfying all of the equations of inextensional bending (5-11).

Collecting together the expressions for the deformations:

$$\left.\begin{aligned} w &= x \sum B_m m \cos m\phi \\ v &= x \sum B_m \sin m\phi \\ u &= R \sum \frac{B_m}{m} \cos m\phi \end{aligned}\right\} \quad \ldots(5\text{-}12)$$

The corresponding curvatures using equations (3-11) are:



$$\chi_x = \partial^2 w / \partial x^2 = 0$$

$$\chi_\phi = \frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \phi^2} + w\right)$$
$$= -\frac{x}{R^2} \sum B_m m(m^2-1) \cos m\phi$$

$$\chi_{x\phi} = \frac{1}{R}\left(\frac{\partial^2 w}{\partial x \partial \phi} + \frac{\partial v}{\partial x}\right)$$
$$= -\frac{1}{R} \sum B_m (m^2-1) \sin m\phi$$

For an element of a shell the elemental strain energy

$$dV = \left\{ \frac{D}{2}\left(\chi_x^2 + \chi_\phi^2 + 2\chi_x\chi_\phi\right) + D(1-\nu)\chi_{x\phi}^2 \right\} R\,d\phi\,dx$$

This is derived in Appendix 5. As $\chi_x = 0$

$$dV = \frac{DR}{2}\left\{ \chi_\phi^2 + 2(1-\nu)\chi_{x\phi}^2 \right\} d\phi\,dx$$

Integrating over the whole area of the shell to get the total strain energy V

$$V = \frac{DR}{2} \int_0^{L}\!\!\int_0^{2\pi} \left(\frac{x}{R} \sum B_m m(m^2-1) \cos m\phi\right)^2 + 2(1-\nu)\left(\frac{1}{R} \sum B_m (m^2-1) \sin m\phi\right)^2 d\phi\,dx$$

$$= \frac{D\pi L}{2R} \sum B_m^2 (m^2-1)^2 \left\{ \frac{L^2 m^2}{3R^2} + 2(1-\nu) \right\} \quad \cdots (5\text{-}13)$$

The second term in the brakket is due to the strain energy of twisting. It will be seen that unless $L^2m^2/3R^2$ is large compared with $2(1-\nu)$ the effect of the twisting moment can not be neglected.

In the problem worked out in the previous section

$L/R = 1, m = 2$, and $\nu = 0$.

$$V = \frac{D\pi}{2} B_2^2 \, 9 \left\{ \frac{4}{3} + 2 \right\} = 15 D B_2^2 \pi$$

If $N_x = N_o \cos 2\phi$ and $u = \frac{R}{2} B_2 \cos 2\phi$ at base of shell

the external energy is

$$\frac{1}{2} \int_0^{2\pi} N_x u R d\phi = \frac{N_o R^2}{4} B_2 \int_0^{2\pi} \cos^2 2\phi \cdot d\phi$$

$$= N_o R^2 B_2 \pi / 4$$

Equating the internal and external energies

$$B_2 = R^2 N_o / 60 D$$

But $\quad u_o = R B_2 / 2$

$$\therefore \quad N_o = 120 u_o D / R^3 = 6.92 \text{ kips/ft}$$

The value obtained before 2.756 kips/ft may be compared with 6.92 x 1.33 / 3.33 = 2.77 kips/ft obtained by using the strain energy of bending only.

$$\omega \text{ at } L / u \text{ at } o = \frac{L \, 2 B_2 \cos 2\phi}{R \frac{B_2}{2} \cos 2\phi} = 4L/R = 0.4 \text{ ft}$$

Thus the deflection and bending moment at the top of the shell obtained by a characteristic equation that neglected the twisting moment is still accurate, whereas the value of N_x obtained is not satisfactory when the shell is short.

For a long shell such as a cylindrical chimney with differential settlement the characteristic equation provides a solution that is more accurate as the strain energy of twisting is of less importance. Furthermore the combination of inextensional bending for carrying loads normal to the surface and charasteristic equation to handle the end loads provides a satisfactory solution for wind loads on a cylindrical chimney and some other problems of a complete cylindrical shell. This is analogous to the use of membrane theory combined with a characteristic equation in the case of a shell roof.

When $L^2 m^2 / 4R^2$ is large compared with $2(1-\nu)$ the strain energy of twisting can be neglected and

$$V = \frac{D \pi L^3}{6 R^3} m^2 (m^2-1)^2 B_m^2 \qquad \text{....(5-14).}$$

Wind Pressure on Cylindrical Chimney

The wind pressure around a circular chimney will be represented by a Fourier series with seven terms as follows:

$$p/p_r = \sum_{m=0}^{6} a_m \cos m\phi$$

or $$100\, p/p_r = 22 + 33.8 \cos\phi + 53.3 \cos 2\phi$$

$$+ 47.1 \cos 3\phi + 16.6 \cos 4\phi - 6.6 \cos 5\phi - 5.5 \cos 6\phi$$

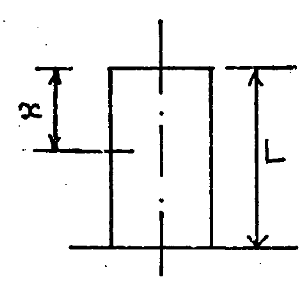
This is derived from the test results in NPL/ Aero/ 316a.

The wind pressure is assumed constant over the height of the tower and p_r will be taken as the velocity pressure = (velocity in miles per hour)^2/ 400 lb/sq ft.

The first two terms do not cause deformations of the shell out of the circle in the plan view, and hence will produce only membrane stresses. The second term is the only one that produces a net overturning moment on the shell.

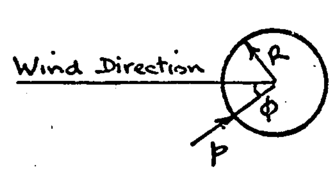
The membrane stresses will now be derived for the benefit of those who declined my invitation in chapter 4 to work it out for themselves.

It is convenient in this case to measure x from the top.



Resolving radially forces upon the element:

$$N_\phi \, d\phi \, dx + p R \, d\phi \, dx$$



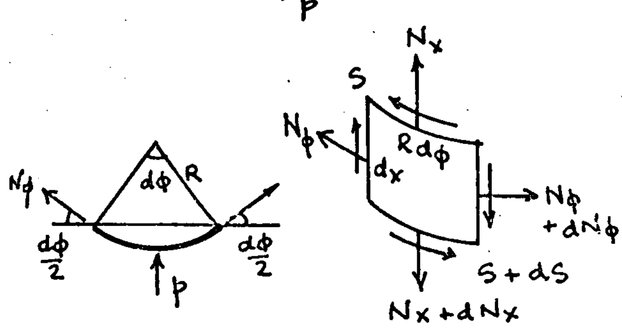
$$\therefore \quad N_\phi = -pR$$

Resolving in the ϕ direction

$$dS\, R\, d\phi + dN_\phi \, dx = 0$$

$$\therefore \quad \frac{dS}{dx} = \frac{dN_\phi}{R\, d\phi} = \frac{dp}{d\phi}$$

$$\therefore \quad S = x \frac{dp}{d\phi} + \text{const.}$$



At the top of the shell there is a free surface and hence no shear stress. ie. $S = 0$ when $x = 0$

$$\therefore \quad S = x \, dp/d\phi$$

Resolving in the x direction

$$dN_x \, R \, d\phi + dS \, dx = 0$$

$$\therefore \quad \frac{dN_x}{dx} = -\frac{dS}{R\,d\phi} = -\frac{x}{R}\frac{d^2p}{d\phi^2}$$

$$\therefore \quad N_x = -\frac{x^2}{2R}\frac{d^2p}{d\phi^2} + \text{constant}$$

Again there is no longitudinal stress at the top of the shell so that the constant of integration is zero.

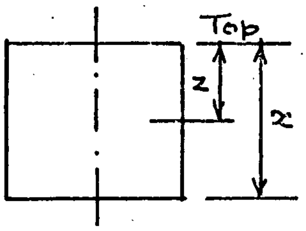
For the first two terms of the Fourier series:

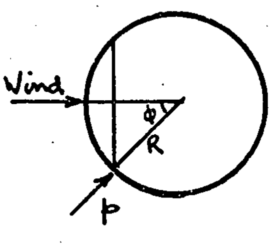
$$p = (a_0 + a_1 \cos\phi)$$
$$dp/d\phi = -a_1 \sin\phi$$
$$d^2p/d\phi^2 = -a_1 \cos\phi$$

Hence the membrane stresses are:

$$\left.\begin{aligned} N\phi &= -(a_0 + a_1\cos\phi)R \\ S &= -x\, a_1 \sin\phi \\ N_x &= \frac{x^2}{2R}\, a_1 \cos\phi \end{aligned}\right\} \ldots (5\text{-}15)$$

It is of interest to observe the relation between the membrane stresses just derived and those obtained by the normal engineering formulae for bending and shear stresses.





The overturning moment on the chimney is:

$$\int_0^x\int_0^{2\pi} z\, p \cos\phi \; R\, d\phi\, dz$$

$$= R\int_0^x\int_0^{2\pi} z(a_0\cos\phi + a_1\cos^2\phi)\, d\phi\, dz$$

$$= R\pi a_1 x^2/2$$

The I of a ring section is $\pi R^3 h$ ft.^{4}

$$\therefore N_x = \sigma_x h = \frac{M}{I} R\cos\phi\, h = \frac{x^2}{2R} a_1 \cos\phi$$

The net shear force on the section is

$$\int_0^x\int_0^{2\pi} p\cos\phi \; R\, d\phi\, dz$$

$$= R\int_0^x\int_0^{2\pi} (a_0\cos\phi + a_1\cos^2\phi)\, d\phi\, dz = R\pi a_1 x$$

$$S = Th = \frac{R\pi a_1 x}{I \; 2h} \int_{-\phi}^{\phi} R^2 h \cos\phi \, d\phi$$
$$= a_1 x \sin\phi$$

Which apart from a change in sign in the expression for shear is the same result as obtained by the membrane theory.

The other terms in the Fourier series will be dealt with as follows.

It will be assumed that inextensional bending occurs and that the base lifts in accordance with this assumption.

End forces are then applied to cancel out this base lift. The effects of this end force are calculated using the modified Schorer equation (5-3).

The effects of the membrane equations for the first two terms and inextensional bending plus end loads for the higher terms are added to get the final stresses and deformations in the shell.

Inextensional Bending of Long Shell

The external energy is half the product of the wind pressure and the radial deflection integrated over the whole shell. The radial deflection will be taken as the same shape as the loading for each term in the series. If Fourier terms are cross multiplied the integration around the shell is zero. This property of the series is known as orthogonality. For each term considered

$$V = \frac{1}{2} \int_0^L \int_0^{2\pi} a_m \cos m\phi \, . \, x \;\; B_m \, m \cos m\phi \;\; R \, d\phi \, dx$$
$$= \frac{L^2}{4} \, a_m B_m \, m \pi R \quad \text{lb ft}$$

Equating this to the strain energy for inextensional bending of a long shell given in equation (5-14):

$$V = \frac{D\pi L^3}{6R^3} \, m^2 (m^2-1)^2 \, B_m^2$$

$$B_m = \frac{3 \, a_m \, R^4}{2D \, L \, m(m^2-1)^2}$$

Substituting back in equations (5-12) to obtain the deformations:

$$w = \frac{3 \, a_m \, R^4 \, x \cos m\phi}{2 \, D L \, (m^2-1)^2} \quad \dots \quad (5\text{-}16)$$

$$\left.\begin{aligned} v &= \frac{3\,a_m\,R^4\,x\,\sin m\phi}{2\,D\,L\,m\,(m^2-1)^2} \\ u &= \frac{3\,a_m\,R^5\,\cos m\phi}{2\,D\,L\,m^2\,(m^2-1)^2} \end{aligned}\right\} \quad \ldots(5\text{-}16)$$

If the base does not in fact lift the value

$$F = \frac{3\,a_m\,R^5}{2\,D\,L\,m^2\,(m^2-1)^2}$$

can be substituted in the equations for the tank settlement and the stresses and deflections added to the inextensional bending and membrane cases.

Example

A concrete cylinder is 50 ft. in diameter and has walls 1 ft. thick. The top is open and the base rigidly fixed. Taking E as $3\text{x}10^6$ p.s.i and ν as 0.2 what are the maximum stresses set up by a 75 mph wind? The length of the cylinder is 100 ft.

$p_r = 14.1$ lb/ft^2 $\qquad D = 37.5\times10^6$ lb ft^2

The deflections at the top due to inextensional bending is:

$$\omega_o = \frac{3\,R^4}{2\,D}\sum_2^6 \frac{a_m}{(m^2-1)^2}$$

$$= 0.10074\ p_r\,R^4/D = 0.0148 \text{ ft}$$

$$F = u_o = \frac{3\,R^5}{2\,DL}\sum_2^6 \frac{a_m}{m^2(m^2-1)^2} = 0.02349\ p_r R^5/DL$$

$$= 8.626\times10^{-4} \text{ ft}$$

The membrane stresses at the base at ϕ =0 are:

$$N_x = L^2 a_1/2R = (p_r L^2/50)\times 0.338 = 951 \text{ lb/ft}$$

$$S = -a_1 L = -p_r L \times 0.338 = 477 \text{ lb/ft at } \phi = 90^\circ$$

The other stresses are calculated from edge load theory, have been evaluated by computer and are tabulated below.

m	F ft	M_ϕ top	N_x base	S base at $\pm\phi^\circ$	
2	8.06×10^{-4}	776	4290	830	45
3	4.45×10^{-5}	791	994	213	30
4	2.51×10^{-6}	146	104	31	22.5
5	-2.50×10^{-7}	−36	−16	−6	18
6	-6.79×10^{-8}	−21	−6	−3	15

It will be observed that the lift at the base due to inextensional bending is extremely small and that any tendency for the base not to be pulled back completely will decrease the N_x and S values due to the edge loads. However the values for M_ϕ which are caused by the difference between the deflections due to inextensional bending and those due to the edge loads will increase if there is any net base lift with a maximum value equal to the inextensional bending case.

Chimney with an elastic foundation

When the boundary values of a differential equation depend on a combination of derivatives such as $u \propto N_x = Eh\, du/dx$ it is convenient to express the table of derivatives (5-1) in a matrix form as follows:

$$\begin{bmatrix} u \\ u' \\ u'' \\ u''' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ b & b & -b & b \\ 0 & 2b^2 & 0 & -2b^2 \\ -2b^3 & 2b^3 & 2b^3 & 2b^3 \end{bmatrix} \begin{bmatrix} ce & se & 0 & 0 \\ -se & ce & 0 & 0 \\ 0 & 0 & c/e & s/e \\ 0 & 0 & -s/e & c/e \end{bmatrix} \begin{bmatrix} C1 \\ C2 \\ C3 \\ C4 \end{bmatrix}$$

Where $c = \cos bx$, $s = \sin bx$, $e = \exp bx$ as before.

or U = A.B.C.

The boundary condition $u - K.Eh.u' = -F$ at x=0

may then be expressed as:

$$D \,.\, B_o \,.\, C = -F$$

where for j = 1,2,3,4,

$$D[1,j] = A[1,j] - K.Eh.A[2,j]$$

And B_o is the matrix B filled with numbers corresponding to x = 0. The other equations to be used to find the vector C are:

$$D \,.\, B_o \,.\, C = 0$$

$$D \,.\, B_L \,.\, C = 0$$

$$D \,.\, B_L \,.\, C = 0$$

where for j = 1,2,3,4

$$D[2,j] = A[4,j], \quad D[3,j] = A[3,j], \quad D[4,j] = A[1,j].$$

These last three boundary conditions are the same as those on page 50. B_L is the matrix B filled with numbers corresponding to x = L.

This is simple to program when there are standard computer procedures for the multiplication and solution of matrices.

Experimental Verification of Edge Load Theory

It is difficult to test the theory for the effect of wind loads on cylindrical chimneys, especially when a large pressurised wind tunnel such as the one at Teddington is not available. However the basic ideas could be tried out by loading a horizontal steel cylinder near the open end with a point load, measuring the changes of diameter with a large micrometer and the lifting of the base with dial gauges. Strains were measured near the fixed ends with E.R.S. gauges.

For inextensional bending from equation (5-12)

At $x = Z$, $\phi = 0$ $\qquad w = z \sum_2^{\infty} B_m m$

For a longish shell from equation (5-14)

$$\frac{\partial V}{\partial B_m} = \frac{D \pi L^3}{3 R^3} m^2 (m^2-1)^2 B_m = P \frac{\partial w}{\partial B_m} = P z m$$

$$\therefore B_m = 3 P z R^3 / D \pi L^3 m (m^2-1)^2$$

Substituting in equation (5-12)

$$\left.\begin{aligned} w &= \frac{3 P z R^3 x}{D \pi L^3} \sum_2^{\infty} \frac{\cos m\phi}{(m^2-1)^2} \\ u &= \frac{3 P z R^4}{D \pi L^3} \sum_2^{\infty} \frac{\cos m\phi}{m^2 (m^2-1)^2} \end{aligned}\right\} \qquad (5\text{-}17)$$

The edge loads required to reduce u at the base to zero are handled as before.

The dimensions of the steel shell used for the experiment are as shown.

E was taken as 30×10^6 psi

ν as 0.3.

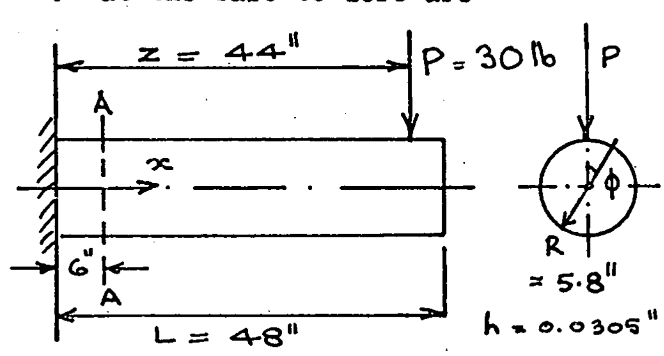


FIG 5-1

Strains were measured at four positions around the shell at A-A spaced at 90° intervals, using Saunders-Roe foil gauges and a Bruel and Kjoer strain meter.

Measurements of deflections in the u direction made at the back of the base plate at four positions showed that the "pull back" of the lift due to inextensional bending at m = 2 was only 93.5% effective due to the flexing of the plate. The same percentage "pull back" was assumed for the other values of m greater than 1. This affects the change of radii and that part of the stresses due to edge load action.

The change in horizontal radius (at ϕ =90°) is mainly due to the m = 2 term and is shown in Fig 5-2 together with the measured values. The shell was very sensitive to small pressures near the open end and great care had to be taken with the measurements. Contact between the micrometer and the stuck on buttons was judged by sound.

Stresses were measured by averaging the strain gauge readings on the inside and outside of the shell in both the x and ϕ directions and inserting in the appropriate equation (3-11).

The computed values of N_x at six inches from the base are as follows:

m	N_x pull back 100%	N_x pull back 93.5%	$\cos m\phi$ 0°	30°	60°	90°	120°	150°	180° (ϕ)
1	10.79	10.79	1	0.866	0.5	0	-0.5	-0.866	-1
2	35.71	33.41	1	0.5	-0.5	-1	-0.5	0.5	1
3	16.05	15.00	1	0	-1	0	1	0	-1
4	3.61	3.28	1	-0.5	-0.5	1	-0.5	-0.5	1
5	0.92	0.86	1	-0.866	0.5	0	-0.5	0.866	-1
	Total N_x		63.6	23.6	-27.5	-30.0	-9.2	6.4	10.1

These membrane stresses are plotted in Figure 5-3 together with the measured values. It will be observed that good agreement is obtained.

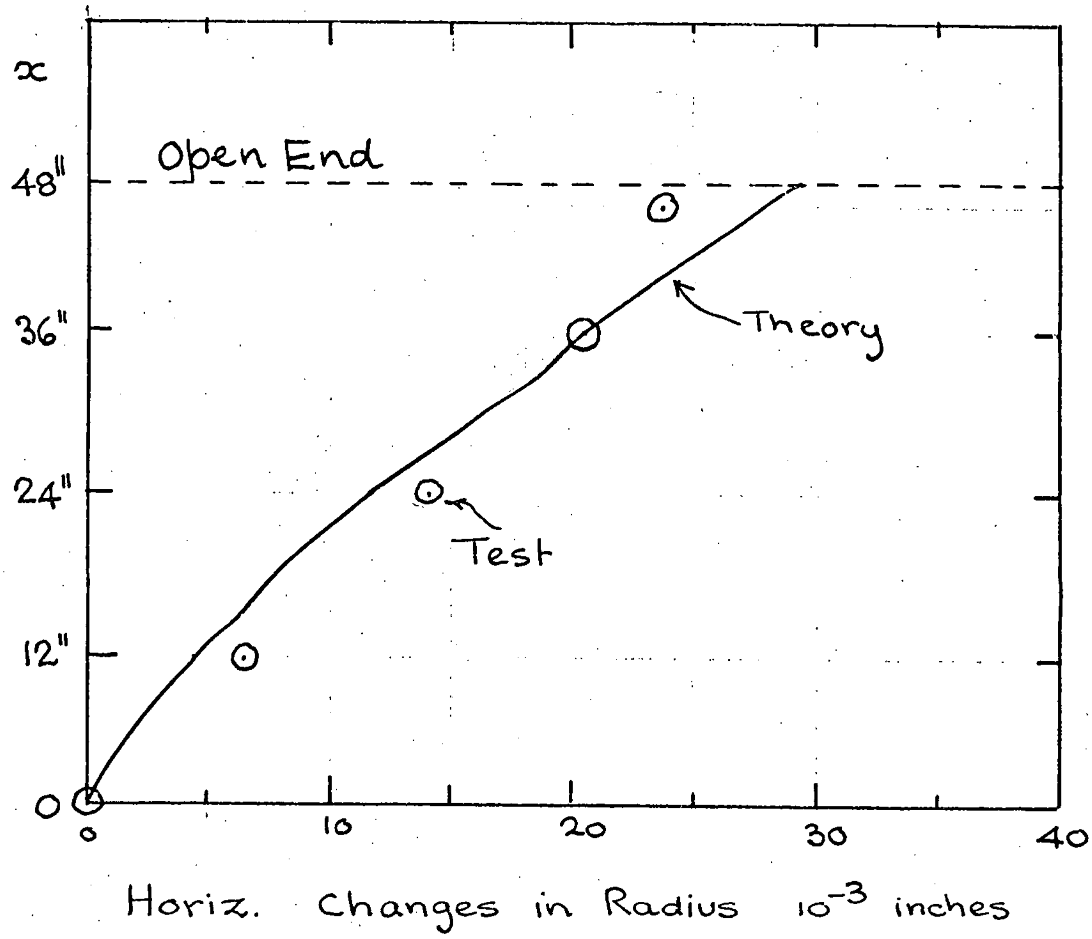

x
Open End
48"
36"
24"
12"
0
Theory
Test
0
10
20
30
40
Horiz. Changes in Radius 10^{-3} inches

FIG 5-2

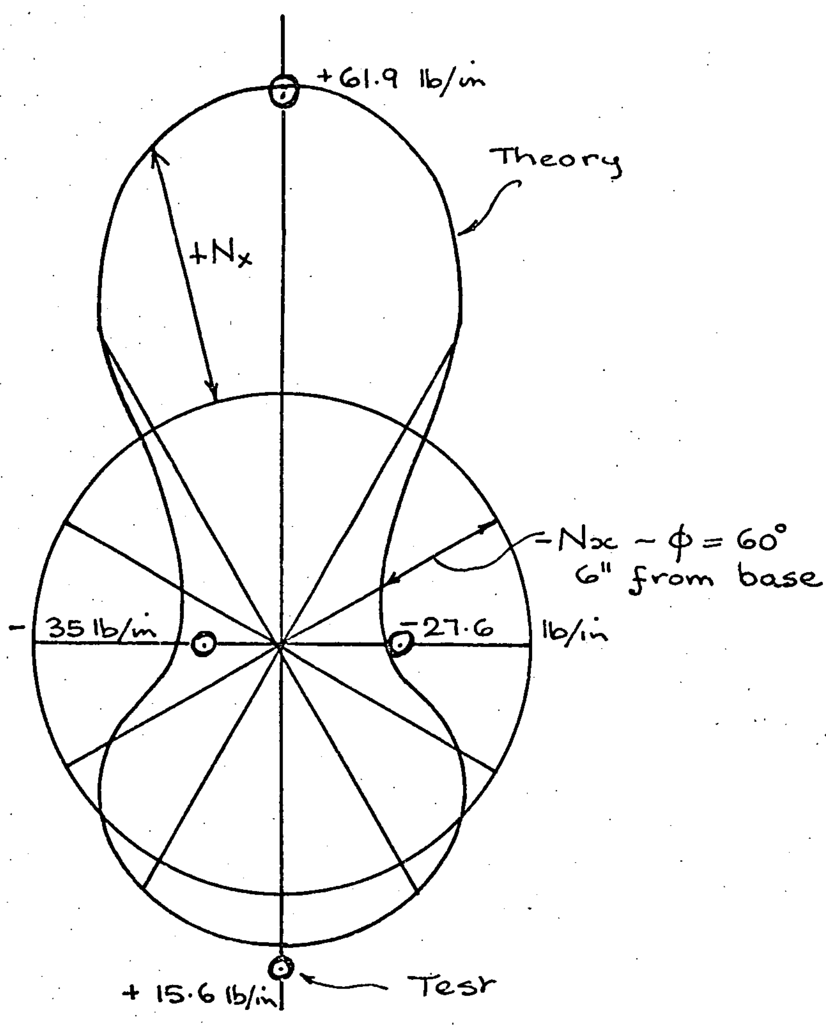
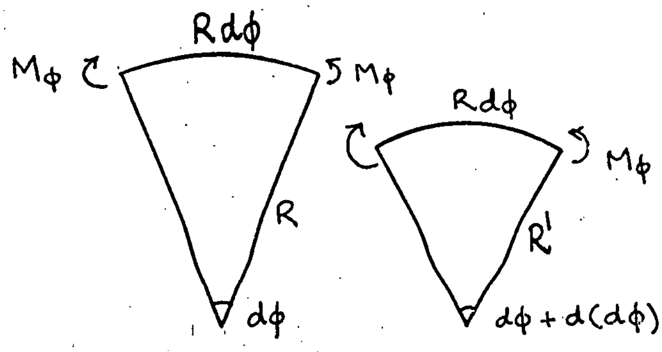

+61.9 lb/in
Theory
$+N_x$
$-N_x \sim \phi = 60°$
6" from base
-35 lb/in
-27.6 lb/in
+15.6 lb/in
Test

FIG 5-3

APPENDIX 5

Strain Energy of Shells

(a) Pure Bending



The bending moment acting on the side of an element lengths dx and Rdϕ is $M_\phi dx$ lb in.

The change of angle in the same direction is $R d\phi \left\{ \frac{1}{R} - \frac{1}{R'} \right\} = -\chi_\phi R d\phi$

The work done which is stored in the form of elastic strain energy is $-\frac{1}{2} M_\phi \chi_\phi R d\phi\, dx$

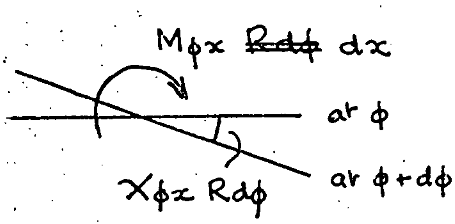
Similarly the strain energy due to bending in the x direction is $-\frac{1}{2} M_x \chi_x R d\phi\, dx$

But $M_x = -D(\chi_x + \nu \chi_\phi)$ from equations (3-11)

And $M_\phi = -D(\chi_\phi + \nu \chi_x)$

$\therefore \quad dV = \frac{D}{2} \left\{ \chi_x^2 + \chi_\phi^2 + 2\nu \chi_x \chi_\phi \right\} R d\phi\, dx$

(b) Pure Twisting



The twisting moment acting on the side of the element in the ϕx direction is $M_{\phi x} dx$. The change of angle in the same direction is $\chi_{\phi x} R d\phi$.

The work done which is stored in the form of elastic strain energy is $\frac{1}{2} M_{\phi x} \chi_{\phi x} R d\phi\, dx$

Similarly the strain energy due to twisting in the $x\phi$ direction is $\frac{1}{2} M_{x\phi} \chi_{x\phi} R d\phi\, dx$

As $M_{\phi x} = M_{x\phi}$ and $\chi_{\phi x} = \chi_{x\phi}$ $\quad dV = M_{x\phi} \chi_{x\phi} R d\phi\, dx$

But $M_{x\phi} = D(1-\nu) \chi_{x\phi}$ from equations (3-11)

$\therefore \quad dV = D(1-\nu) \chi_{x\phi}^2 R d\phi\, dx$

Chapter 6.

FLUGGE'S EQUATION FOR COMPLETE CYLINDRICAL SHELLS

In the previous chapters fairly drastic simplifications and restrictions have been made to produce equations that are easy to handle. Complete cylindrical shells have been assumed to be subjected to purely axi-symmetric loading with bending only in the x direction, or to deformations that produce bending only in the ϕ direction.

The general case has been handled by Flügge without simplification and the characteristic equation is therefore rather lengthy. He used a non-dimensional form of the eigenfunction

$$w/R = A\, e^{px/R} \cos m\phi$$

and obtained the auxiliary equation

$$(p^2 - m^2)^4 + 4K^4 p^4 + 2\nu p^6 - 6m^2 p^4 + 2F p^2 + m^4(1-2m^2) = 0 \quad \cdots (6\text{-}1)$$

where $\quad 4K^4 = 12(1-\nu^2)(R/h)^2$

and $\quad F = m^2[(4-\nu)m^2 - 2 + \nu]$

This is a quartic equation in p^2 and as $4K^4$ is very large compared with the other coefficients it can be solved quite readily by the algebraic method of Ferrari.

Putting $P = p^2$ and $M = m^2$ the coefficients of the quartic can be tabulated as follows:

P^4	P^3	P^2	P	1
1	$-4M$ 2ν	$6M^2$ $-6M$ $4K^4$	$-4M^3$ $2F$	M^4 $M^2(1-2M)$
1	$2a$	b	$2c$	d

In a typical case $R = 3$in. $h = 0.03$ in. $\nu = 0.3$ $m = 2$, $M = 4$. The tabulated values are then:

P^4	P^3	P^2	P	1
1	-16 0.6	96 -24 109 000	-256 104.8	256 -112
1	-15.4	109 072	-151.2	144

In order to solve the quartic equation

$$P^4 + 2a P^3 + b P^2 + 2c P + d = 0$$

it will be assumed possible to express it as the difference of two squares:

$$(P^2 + qP + r)^2 - (sP + t)^2 \qquad \ldots(6\text{-}2)$$

or $$P^4 + 2q P^3 + (q^2 + 2r - s^2) P^2 + 2(qr - st)P + (r^2 - t^2)$$

Then on comparing coefficients

$$\left.\begin{aligned} q &= a; & (q^2 + 2r - s^2) &= b; \\ (qr - st) &= c; & (r^2 - t^2) &= d; \end{aligned}\right\} \qquad \ldots(6\text{-}3)$$

Eliminating s and t and substituting for q we obtain the cubic equation

$$r^3 - \frac{b}{2} r^2 + (ac - d)r + \frac{1}{2}[d(b - a^2) - c^2] = 0$$

Putting in the figures for our typical case

$$r^3 - 54536 r^2 + 400.32 r - 3\,921\,868 = 0$$

This has the real root $r = 54535.994$ which is almost exactly the value of $b/2$ (54536) and practically $2K^4$ (545000).

From equations (6-3) $t = \sqrt{(r^2 - d)} = r(1 - \frac{d}{2r^2})$ binomial

and $s = (ar - c)/t = a - c/r$ as $t \simeq r$

Now equation (6-2) can be put in the form:

$$P^2 + aP + r = \pm(sP + t)$$

which can be expressed as the two quadratic equations:

$$P^2 + (a + s)P + (r + t) = 0$$

$$P^2 + (a - s)P + (r - t) = 0$$

Now $\quad r + t = 2r \quad$ and $\quad a + s = 2a$

as $\quad 2r \gg d/2r \quad$ and $\quad 2a \gg c/r$

$$\therefore \quad P^2 + 2aP + 2r = 0 \qquad (6\text{-}4)$$

$$\text{and} \quad P^2 + \frac{c}{r}P + \frac{d}{2r} = 0 \qquad (6\text{-}5)$$

Equation (6-4) can be solved to give one set of roots:

$$p^2 = P = -a \pm \sqrt{(a^2 - 2r)} = -a \pm i\sqrt{2r} \qquad \text{as } 2r \gg a^2$$

The imaginary part of this is much larger than the real,(46:1) in our typical case, so that p can be taken as $(2r)^{1/4}(\pm 1 \pm i)/\sqrt{2}$

or $\quad K(\pm 1 \pm i)$

The first part of the solution will then be:

$$\begin{aligned} \omega_1 &= A\,e^{\frac{K}{R}(\pm 1 \pm i)x} \cos m\phi \\ &= A\,e^{\beta(\pm 1 \pm i)x} \cos m\phi \end{aligned}$$

where $\quad 4\beta^4 = 4K^4/R^4 = 12(1-\nu^2)/R^2h^2$

This will be recognisable as the solution for the case of axi-symmetric bending dealt with in Chapter 2.

The second part of the solution is obtained from equation (6-5).

$$p^2 = P = -\frac{c}{2r} \pm \sqrt{\left(\frac{c}{2r}\right)^2 - \frac{d}{2r}} = -\frac{c}{2r} \pm i\sqrt{\frac{d}{2r}} \qquad \text{as } \left(\frac{c}{2r}\right)^2 \ll \frac{d}{2r}$$

The imaginary part of this is again much larger than the real,(50:1) in our typical case, so that p can be taken as $\left(\frac{d}{2r}\right)^{1/4}(\pm 1 \pm i)/\sqrt{2}$

Therefore $\quad \omega_2 = A\,e^{b(\pm 1 \pm i)x} \cos m\phi$

where $\quad 4b^4 = \dfrac{M^2(M-1)^2}{4K^4\,R^4} = \dfrac{m^4(m^2-1)^2h^2}{12(1-\nu^2)R^6}$

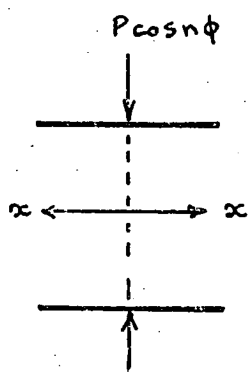
This will be recognisable as being the same as that obtained in the case dealt with in Chapter 5 where the bending in the x direction was not considered.

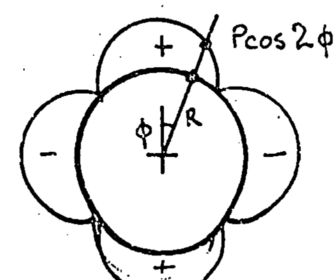
Neither of the characteristic equations corresponding to the two sets of roots contain a term for twisting moment. This suggests that the twisting moments can be neglected unless the shell is very short. In this case an analysis by inextensional bending will normally become possible.

Most problems that involve both sets of roots can be handled by assuming that bending moments, slopes and normal shears in the x direction are properties of the large roots, while longitudinal stresses and displacements, radial deflections and membrane shears are properties of the small roots.

The integration constants of that part of the solution with large roots will be taken as C1, C2, C3, C4 as before with displacements in w. The integration constants for the large roots will be taken as B1, B2, B3, B4 with displacements in u.

Example





Consider a varying ring load $P\cos m\phi$ about the centre of an infinitely long cylinder where:

$P = 30$ lb/in $\quad R = 3$ in $\quad h = 0.03$ in

$E = 30\times10^6$ psi $\quad \nu = 0.3 \quad m = 2 \quad D = 74.175$ lb in

The edge effects will die away in a very long cylinder

Hence $\quad C1 = C2 = B1 = B2 = 0$

The next boundary condition for the large roots is zero slope at

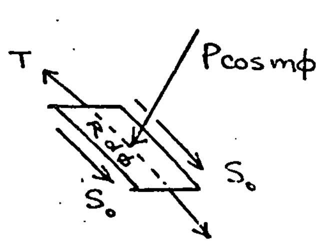
$x = 0$ giving: $\quad dw/dx = 0$ ...(a)

It will be assumed that the ring load is first carried in normal shear Q_x by the large roots giving the boundary condition

$$Q_x = -\frac{P}{2}\cos m\phi \qquad \text{...(b)}$$

A boundary condition for the small roots iszero longitudinal displacement at $x=0$ giving $\quad u = 0$ ...(c)

The effect of the large root part of the solution dies away rapidly from the boundary as $e^{-\beta x}$ diminishes rapidly with x. When Q_x is zero the equilibrium of the narrow ring containing the edge load yields a boundary condition for the small roots:



Resolving radially

$$T = -PR\cos m\phi$$

Resolving tangentally

$$2S_0 R\,d\phi = -dT$$

$$\therefore\ S_0 = -\frac{1}{2R}\frac{dT}{d\phi} = -\frac{Pm}{2}\sin m\phi \quad \ldots (d)$$

Large roots

$$\beta^2 = \sqrt{3(1-\nu^2)} \Big/ Rh = 18.359 \text{ in}^{-2}$$

$$\beta = 4.285 \text{ in}^{-1} \qquad 2\beta^3 = 157.321 \text{ in}^{-3}$$

Small roots

$$b^2 = \frac{m^2(m^2-1)\,h}{4\sqrt{3(1-\nu^2)}\;R^3} = 20.174 \times 10^{-4} \text{ in}^{-1}$$

$$b = 4.492 \times 10^{-2} \text{ in}^{-1} \qquad 2b^3 = 181.228 \times 10^{-6} \text{ in}^{-3}$$

Then at $x = 0$

(a) $\partial w/\partial x = 0 \qquad \therefore C3 = C4$

(b) $Q_x = -D \, . \, 2\beta^3 (C3 + C4) = -P/2$

$\therefore C3 = C4 = P/8\beta^3 D = 6.427 \times 10^{-4}$ in

(c) $u = 0 \qquad \therefore \quad B3 = 0$

(d) $S = -\dfrac{EhR}{m}\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{EhR}{m} 2b^2 (-B4) = -\dfrac{Pm}{2}$

$\therefore B4 = -55.076 \times 10^{-4}$ in

Deflection w at $x = 0$ is $C3 - m^2 B4/2bR$

$= 0.000643 + 0.081746$

$= 0.0824$ in

The effect of the large roots dies away when

$e^{-\beta x} < 0.1$ ie $\beta x = 2.3$ or $x = 0.54$ in

The effect of the small roots dies away when

$e^{-bx} < 0.1$ ie $bx = 2.3$ or $x = 51$ in

The longitudinal stress N_x at $x = 0$ is $Eh\,\partial u/\partial x$

or $Ehb\,B4 = -222.6$ (lb/in) $\cos 2\phi$

More Accurate Analysis of Case Dealt with Above

It is recognised that the method employed above savours somewhat of sleight of hand, and a more thorough analysis is required in order to justify it. It is inconvenient in this case to employ a different variable for both sets of roots and the deflection in the radial direction w will be used. The integration constants for the small roots will of course have different values from above and will be taken as K1, K2, K3, K4.

Consideration of the moment deformation relations for the shell together with the statics of the element, while ignoring the twisting moments leads to the following relations:

$$w = \bar{w} \cos m\phi$$

$$M\phi = -D(\chi_\phi + \nu \chi_x)$$
$$= -D\left\{ \frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \phi^2} + w \right) + \nu \frac{\partial^2 w}{\partial x^2} \right\}$$
$$= D\{ (m^2-1) w / R^2 - \nu w'' \}$$

$$M_x = -D(\chi_x + \nu \chi_\phi)$$
$$= D\{ -w'' + \nu (m^2-1) w / R^2 \}$$

$$Q_\phi = \partial M\phi / R\partial\phi = D\{-m(m^2-1)\bar{w}/R^3 + \frac{\nu m \bar{w}''}{R}\} \sin m\phi$$

$$Q_x = \partial M_x / \partial x = D\{ -w''' + \nu (m^2-1) w' / R^2 \}$$

$$N\phi = -\partial Q_\phi / \partial\phi - R\, \partial Q_x / \partial x$$
$$= D\{ m^2(m^2-1) \frac{w}{R^3} - \nu m^2 \frac{w'}{R} + w^{IV} R - \nu (m^2-1) \frac{w''}{R} \}$$
$$= D\{ R w^{IV} - \nu (2m^2-1) w''/R + m^2(m^2-1) w / R^3 \}$$

$$\partial S/\partial x = Q\phi / R - \partial N\phi / R\partial\phi$$
$$= D\{ -m(m^2-1)\bar{w}/R^4 + \nu m \bar{w}''/R^2$$
$$+ m\bar{w}^{IV} - \nu m (2m^2-1)\bar{w}''/R^2 + m^3(m^2-1)\bar{w}/R^4 \} \sin m\phi$$
$$= D\{ m\bar{w}^{IV} - 2\nu m(m^2-1)\bar{w}''/R^2 + m(m^2-1)^2\bar{w}/R^4 \} \sin m\phi$$

$$\therefore \quad S = D\{ m\bar{w}''' - 2\nu m(m^2-1)\bar{w}'/R^2 + m(m^2-1)^2 \int \bar{w}/R^4 \} \sin m\phi$$

$$\partial N_x / \partial x = -\partial S / R\partial\phi$$
$$= D\{ -m^2 w'''/R + 2\nu m^2 (m^2-1) w'/R^3 - m^2(m^2-1)^2 \int w / R^5 \}$$

$$\therefore \quad N_x = D\{ -m^2 w''/R + 2\nu m^2(m^2-1) w/R^3 - m^2(m^2-1)^2 \iint w / R^5 \}$$

$$\epsilon_x = \partial u / \partial x = 1/Eh \{ N_x - \nu N\phi \}$$

$$\therefore \quad u = \frac{1}{Eh}\{ \int N_x - \nu \int N\phi \}$$

$$= \frac{D}{Eh}\{ -\nu R w''' + [\nu^2(2m^2-1) - m^2] w'/R$$
$$+ \nu m^2(m^2-1) \int w / R^3 - m^2(m^2-1)^2 \iiint w / R^5$$

Putting in values of ν, m, R for the problem the following equations are obtained:

$$Q_x = D(-\omega''' + 0.1\,\omega)$$
$$u = D/Eh\left(-0.9\,\omega''' - 1.1233\,\omega' + 0.1333\int\omega - 0.14815\iiint\omega\right)$$
$$S = D\left(2\bar{\omega}''' - 0.4\,\bar{\omega}' + 0.2222\int\omega\right)\sin 2\phi$$

As both sets of roots belong to the solution of fourth order equations of the form

$$\partial^4\omega/\partial x^4 + 4\beta^4\omega = 0$$

and

$$\partial^4\omega/\partial x^4 + 4b^4\omega = 0$$

Then

$$\omega = -\omega^{IV}/4\beta^4 \quad \text{or} \quad -\omega^{IV}/4b^4$$
$$\int\omega = -\omega'''/4\beta^4 \quad \text{or} \quad -\omega'''/4b^4$$
$$\iint\omega = -\omega''/4\beta^4 \quad \text{or} \quad -\omega''/4b^4$$
$$\iiint\omega = -\omega'/4\beta^4 \quad \text{or} \quad -\omega'/4b^4$$

Table 51, page 50 can then be used for the integrals as well as the derivatives.

The coefficients for the matrix to be solved for the problem can now be filled in. The calculations are given in full so that the relative importance of each factor can be assessed. In general the high derivatives stress the large roots and the high integrals stress the small roots.

	C3	C4	K3	K4
(a) ω'	-4.2847	+4.2847	-0.0449	+0.0449
(b) $-\omega'''$	-157.3210	-157.3210	-0.0002	-0.0002
$0.1\,\omega'$	- 0.4285	+ 0.4285	-0.0045	+0.0045
Q_x/D	-157.7495	-156.8925	-0.0047	+0.0043
(c) $-0.9\,\omega'''$	-141.5889	-141.5889	-0.0002	-0.0002
$-1.1233\,\omega'$	+ 4.8131	- 4.8131	+0.0505	-0.0505
$+0.1333\int\omega$	- 0.0156	- 0.0156	-1.4843	-1.4843
$-0.14815\iiint\omega$	- 0.0005	+ 0.0005	-408.7574	+408.7574
$u/(D/Eh)$	-136.7918	-146.4171	-410.1913	+407.2224
(d) $2\omega'''$	+314.6420	+314.6420	+0.0004	+0.0004
$-0.5\,\omega'$	+1.7139	- 1.7139	+0.0180	-0.0180
$0.2222\int\omega$	- 0.0259	- 0.0259	- 2.4738	- 2.4738
S/D	+ 316.3300	+312.9022	- 2.4554	-2.4414

Putting in the boundary conditions

$$w' = u = S = 0 \quad , \quad Q_x / D = 0.2022$$

and solving, the values of the integration constants are obtained

$$C3 = 6.4809 \times 10^{-4} \text{ in}$$
$$C4 = 6.3720 \times 10^{-4} \text{ in}$$
$$K3 = 0.08123 \text{ in}$$
$$K4 = 0.08227 \text{ in}$$
$$\therefore \; w_o = C3 + K3 = 0.0813 \text{ in}$$

which can be compared with 0.0824" obtained by the approximate method.

An examination of the matrix suggests that although the slope cannot serve as a boundary condition for the small root part of the solution, ~~it~~ the small root constants can affect the large root constants. This should be taken into account when the approximate method is being used and when u_o is not zero.

NOTATION

Symbols are defined when they first appear in the text. The general notation is as follows

BEAMS

A	sqft	Cross-sectional area
E	lb/sqft	Young's modulus
I	ft^4	Second moment of area of section
J	ft^4	Polar moment of area
M	lb ft	Bending moment
P	lb	Concentrated force
q	lb/ft	Distributed loading
Q	lb	Shearing force
w	ft	Vertical deflection
x	ft	Longitudinal co-ordinate

SHELLS

D	lb ft.	Flexual rigidity $Eh^3/12(1-\nu^2)$
R, L, h	ft	Radius, length, thickness
u, v, w	ft	Longitudinal, circumferential, radial displacements
N_x, N_ϕ, S	lb/ft	Longitudinal, circumferential, ~~radial~~ shear forces
ϵ_x, ϵ_ϕ, γ		Longitudinal, circumferential, ~~radial~~ shear strains
M_x, M_ϕ, $M_{x\phi}$	lb	Longitudinal, circumferential, twisting moments
Q_x, Q_ϕ	lb/ft	Normal shearing forces
x(ft), ϕ		Longitudinal, radial co-ordinates
ν		Poisson's ratio

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SECTION B

THE ANALYSIS OF CYLINDRICAL SHELL ROOFS WITH POST TENSIONED EDGE BEAMS

by

Raphael Foner Rish B.S.C., M.I.C.E., F.I.E. Aust.

Senior Lecturer, University of Tasmania

SYNOPSIS

A new characteristic equation for cylindrical shell roofs is developed, together with a method for obtaining the solution and its derivatives.

Post tension is introduced into the edge beam by shearing forces varying linearly from a maximum at the traverse to zero at the quarter points. The Fourier series for this converges rapidly. An edge correction is then made to restore the post tension to the end of the edge beam and obtain compatability of strain with the shell edge.

The method is compared with experimental results on a model shell and with the results of a finite element program.

NOTATION

u, v, w (ft)	longitudinal, circumferential, radial displacements
N_x, N_ϕ, S (lb/ft)	" , " , shearing forces
$\epsilon_x, \epsilon_\phi, \gamma_{x\phi}$	" , " , " strains
$M_x, M_\phi, M_{x\phi}$ (lb)	" , " , twisting moments
Q_x, Q_ϕ (lb/ft)	normal shearing forces
V_ϕ (lb/ft)	normal force at edge
x (ft), ϕ (rad)	longitudinal, radial coordinates
a (rad)	angular rotation of shell
R, L, h (ft)	radius, length, thickness of shell
ϕ_k (rad)	half angle of shell
E (lb/ft^2)	Young's modulus
ν	Poisson's ratio
D (lb ft)	flexural rigidity of shell $Eh^3/12(1-\nu^2)$
N	constant in Fourier series term $F_N \cos N\pi x/L$
m	$N\pi R/L$
k, θ	parametric constants
$4K^4$	$12(1-\nu^2) R^2/h^2$
p	a root of the characteristic equation
A (sq ft), I (ft^4)	area, moment of inertia of edge beam
P (lb)	post tension load in edge beam

INTRODUCTION

A digital computer enables the gravity loading of a cylindrical shell roof to be handled by using a sufficient number of terms of the Fourier series for the loading.

When a similar attempt is made to determine the stresses due to post tension[5] the errors increase with the number of terms employed.

This difficulty is basically due to the assumption that the traverse of the shell does not resist normal movements, and cannot transmit any of the post tension directly to the shell. When this assumption is abandoned it becomes possible to handle the post tension in an economical manner.

THE CHARACTERISTIC EQUATION

Flügge[1] has developed the differential equation for the cylindrical shell with the minimum of approximations. As he points out the mathematical analysis of such a system is far from simple. The roots of the auxiliary equation arising from his equation are difficult to extract accurately and the force deformation relations are very complicated.

The following section shows how Flügge's equation can be simplified without serious loss of accuracy using Ferrari's method for the solution of a quartic. The simplified equation has explicit roots which can be readily employed in the design of shell roofs.

The eigenfunction for the radial deflection will be used in the non dimensional form

$$\omega/R = A\,e^{p\phi}\cos mx/R$$

where

$$m = N\pi R/L$$

and all forces and deformations in the shell appear in the form of terms in a Fourier series in x.

The auxiliary to Flügge's equation then becomes

$$(p^2 - m^2)^4 + 4m^2K^4 + 2p^6 + Fp^4 + Gp^2 - 2\nu m^2 = 0 \quad \ldots (1)$$

where

$$4K^4 = 12(1-\nu^2)R^2/h^2$$
$$F = 1 - 2(4-\nu)m^2$$
$$G = 6m^4 - 2(2-\nu)m^2$$

The auxiliary is thus a quartic in p^2 and the coefficients can be tabulated as follows

Table 1

p^8	p^6	p^4	p^2	1
1	$-4m^2$ 2	$+6m^4$ $+F$	$-4m^6$ $+G$	m^8 $+4m^4K^4$ $-2m^6\nu$
1	$2a$	b	$2c$	d

What happens next can best be shown by putting in numbers for a typical case. Using the dimensions of Gibson's long shell[2], R = 30 ft, h = 0.25 ft, L = 120 ft and $\nu = 0.15$, and taking the first term of the Fourier series N = 1, m = $\pi/4$ and $4K^4 = 168912$

The table then becomes

p^8	p^6	p^4	p^2	1
1	-2.4674 $+2$	$+2.2830$ -3.7497	-0.9388 $+0.0006$	$+0.1448$ $+64271.1849$ -0.0704
1	-0.4674	-1.4667	-0.9382	$+64271.259$

To solve the quartic

$$p^8 + 2ap^6 + bp^4 + 2cp^2 + d = 0$$

it will be assumed possible to express it as the difference of two squares

$$(p^4 + qp^2 + r)^2 - (sp^2 + t)^2 = 0 \quad \ldots\ldots (2)$$

or $$p^8 + 2qp^6 + (q^2 + 2r + s^2)p^4 + 2(qr - st)p^2 + (r^2 - t^2) = 0$$

On comparing coefficients

$$q = a\ ; \qquad q^2 - 2r - s^2 = b\ ;$$
$$-qr - st = c\ ; \qquad r^2 - t^2 = d\ ; \qquad \ldots (3)$$

Eliminating s and t and substituting for q we obtain the cubic equation

$$r^3 - \frac{b}{2}r^2 + (ac - d)r + \tfrac{1}{2}[d(b - a^2) - c^2] = 0 \qquad \ldots (4)$$

Putting in the numbers for our typical case the cubic becomes

$$r^3 + 0{\cdot}7335\, r^2 - 64271{\cdot}1497\, r - 48889{\cdot}2214 = 0 \quad \ldots (5)$$

d is much larger than the other numbers in equation (4) and assuming that r has a small real value it is evident from inspection that r will be very nearly equal to $\frac{1}{2}(b - a^2)$ or -0.760668. Evaluating (5) on a desk computer for trial values of r shows that the correct value of r is -0.760671.

Now (2) can be expressed as:

$$(p^4 + qp^2 + r^2) + (sp^2 + t) = 0$$
and $$(p^4 + qp^2 + r^2) - (sp^2 + t) = 0$$

or $$p^4 + (q + s)p^2 + (r^2 + t) = 0$$
and $$p^4 + (q - s)p^2 + (r^2 - t) = 0 \quad \ldots (6)$$

Transposing (3) it is found that $t = \sqrt{(r^2 - d)} = 2m^2K^2i$

as r^2 is negligible compared with d which is very nearly equal to $4m^4K^4$.

$s = (qr - c)/t$ which is very small compared with q, and r^2 is very small compared with t.

The two quadratic equations in p^2 (6) then reduce to:

$$p^4 + ap^2 + t = 0$$
$$p^4 + ap^2 - t = 0$$

hence

$$p^2 = -\frac{a}{2} \pm \sqrt{(\frac{a^2}{4} \pm t)}$$

$$= -\frac{a}{2} \pm \sqrt{(\frac{a^2}{4} \pm 2m^2K^2i)}$$

The imaginary part under the square root is much larger than the real so that

$$p^2 = -\frac{a}{2} \pm mK(\sqrt{2i}) = -\frac{1}{2} + m^2 + mK(\pm 1 \pm i) \quad \ldots (7)$$

This can be compared with the roots for Schorer's equation:

$$p^2 = mK(\pm 1 \pm i)$$

and with those of the widely used D.K.J. equation:

$$p^2 = m^2 + mK(\pm 1 \pm i)$$

It will be seen that for long shells, i.e. where $m^2 < \frac{1}{2}$ the roots of Schorer's equation are more accurate than those of the D.K.J. equation.

If the factors in Table 1 that do not contribute towards (7) are eliminated we are left with the auxiliary equation:

$$p^8 - 4m^2p^6 + 2p^6 + 4m^4K^4 = 0$$

which corresponds to the dimensional characteristic equation

$$\underline{\underline{\frac{\partial^8 w}{\partial \phi^8} + 4R^2 \frac{\partial^8 w}{\partial \phi^6 \partial x^2} + 2\frac{\partial^6 w}{\partial \phi^6} + 4K^4R^4\frac{\partial^4 w}{\partial x^4} = 0}} \quad \ldots (8)$$

Before (8) can be used in design it is necessary to find the force-deformation relations corresponding to it. This was done by working back through Flügge's calculations and leaving out any terms that did not lead to the desired characteristic equation.

It appeared that M_x could be neglected as in Schorer's equation but not the twisting moment. That ϵ_ϕ could be considered small compared with w/R and $\partial v/R\partial\phi$ but that $\gamma_{x\phi}$ could not be neglected when compared with $\partial u/R\partial\phi$ and $\partial v/\partial x$. Other approximations leading to the desired result were the ignoring of νN_ϕ compared with N_x in the calculation of ϵ_x, the taking of $M_{x\phi}$ as $\frac{D(1-\nu)}{R}\cdot\frac{\partial^2 w}{\partial\phi\,\partial x}$ and M_ϕ as $-D\chi_\phi$. In general the method leads to the inclusion of first and second order terms and the neglect of higher order terms.

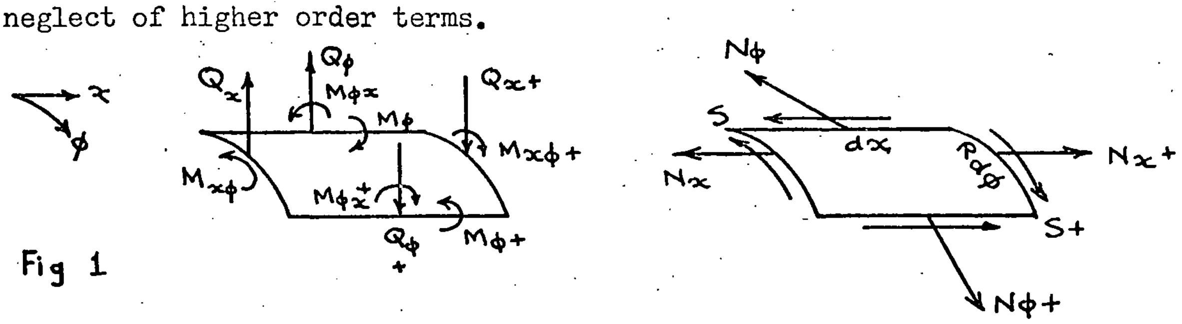


Fig 1

From the statics of the element in Fig. 1.

$$Q_\phi = \frac{\partial M_\phi}{R\partial\phi} - \frac{\partial M_{x\phi}}{\partial x}$$

$$Q_x = -\frac{\partial M_{x\phi}}{R\partial\phi}$$

$$N_\phi = -\frac{\partial Q_\phi}{\partial\phi} - R\frac{\partial Q_x}{\partial x}$$

$$= -\frac{\partial^2 M_\phi}{R\,\partial\phi^2} + 2\frac{\partial^2 M_{x\phi}}{\partial\phi\,\partial x}$$

$$\frac{\partial S}{\partial x} = -\frac{\partial N_\phi}{R\,\partial\phi} + \frac{Q_\phi}{R}$$

$$= \frac{\partial^3 M_\phi}{R^2\,\partial\phi^3} - \frac{2}{R}\frac{\partial^3 M_{x\phi}}{\partial\phi^2\,\partial x} + \frac{\partial M_\phi}{R^2\,\partial\phi}$$

$$\frac{\partial N_x}{\partial x} = -\frac{1}{R}\frac{\partial S}{\partial\phi}$$

$$\frac{\partial^2 N_x}{\partial x^2} = -\frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{\partial S}{\partial x}\right)$$

$$= -\frac{1}{R^3}\frac{\partial^4 M_\phi}{\partial\phi^4} + \frac{2}{R^2}\frac{\partial^4 M_{x\phi}}{\partial\phi^3\,\partial x} - \frac{1}{R^3}\frac{\partial^2 M_\phi}{\partial x^2}$$

The force-deformation relations are:

$$M_\phi = -\frac{D}{R^2}\left(\frac{\partial^2 \omega}{\partial\phi^2} + \omega\right)$$

$$M_{x\phi} = \frac{D(1-\nu)}{R}\frac{\partial^2\omega}{\partial\phi\,\partial x}$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{N_x}{Eh}$$

$\epsilon_\phi = \frac{1}{R}\left(\frac{\partial v}{\partial\phi} - \omega\right)$ this is considered small compared with the separate components on the R.H. side which are due mainly to bending $\therefore\ \omega = \frac{\partial v}{\partial\phi}$

$$\gamma_{x\phi} = \frac{\partial u}{R\partial\phi} + \frac{\partial v}{\partial x} = \frac{S}{Gh} = \frac{2(1+\nu)}{Eh}S$$

$$\frac{\partial^4\omega}{\partial x^4} = \frac{\partial^6 v}{\partial\phi\,\partial x^4} = \frac{2(1+\nu)}{Eh}\frac{\partial^4 S}{\partial\phi\,\partial x^4} - \frac{\partial^5 u}{R\partial\phi^2\,\partial x^3}$$

$$-\frac{\partial^5 u}{R\partial\phi^2\partial x^3} = -\frac{1}{EhR}\frac{\partial^4 N_x}{\partial\phi^2\,\partial x^2}$$

$$= -\frac{1}{EhR}\left\{-\frac{\partial^6 M_\phi}{R^3\,\partial\phi^6} + \frac{2\,\partial^6 M_{x\phi}}{R^2\partial\phi^5\,\partial x} - \frac{\partial^4 M_\phi}{R^3\,\partial\phi^4}\right\}$$

$$= -\frac{D}{EhR}\left\{\frac{1}{R^5}\left(\frac{\partial^8\omega}{\partial\phi^8} + 2\frac{\partial^6\omega}{\partial\phi^6}\right) + \frac{2(1-\nu)}{R^3}\frac{\partial^8\omega}{\partial\phi^6\partial x^2}\right\}$$

$$\therefore\quad \underline{\underline{\frac{\partial^8\omega}{\partial\phi^8} + 2\frac{\partial^6\omega}{\partial\phi^6} + 4R^2\frac{\partial^8\omega}{\partial\phi^6\partial x^2} + \frac{12(1-\nu^2)R^6}{h^2}\frac{\partial^4\omega}{\partial x^4} = 0}}$$

This is the same equation as (8)

The radial deflection in the shell will now be taken in the form

$$\omega = W\cos mx/R = W\cos N\pi x/L$$

where W is a function only of ϕ

The deformations and forces at the centre of the shell where x = 0 can be expressed as derivatives of W with respect to ϕ. The values elsewhere can be obtained by multiplying by $\cos mx/R$

$$\begin{aligned}
v &= \int W\, d\phi \\
a &= \frac{1}{R}\left(\frac{dW}{d\phi} + v\right) \\
M\phi &= -\frac{D}{R^2}\left(\frac{d^2W}{d\phi^2} + W\right) \\
V\phi &= Q_\phi - \frac{dM_{x\phi}}{dx} = -\frac{D}{R^3}\left\{\frac{d^3W}{d\phi^3} + [1-2(1-\nu)m^2]\frac{dW}{d\phi}\right\} \\
N\phi &= \frac{D}{R^3}\left\{\frac{d^4W}{d\phi^4} + [1-2(1-\nu)m^2]\frac{d^2W}{d\phi^2}\right\} \\
\frac{dS}{dx} &= -\frac{D}{R^4}\left\{\frac{d^5W}{d\phi^5} + [2-2(1-\nu)m^2]\frac{d^3W}{d\phi^3}\right\} \\
Nx &= -\frac{D}{R^3m^2}\left\{\frac{d^6W}{d\phi^6} + [2-2(1-\nu)m^2]\frac{d^4W}{d\phi^4}\right\}
\end{aligned}$$

Now $W = Ae^{p\phi}$ where A and p have eight complex values. Before we can tackle the design of the shell roof we have to be able to extract W and its derivatives in terms of real constants and quantities. A simple method of doing this will now be outlined.

TO OBTAIN THE DERIVATES OF A SOLUTION TO A LINEAR DIFFERENTIAL EQUATION

Two terms of the solution will be taken in the form:

$$W = A_1 e^{p_1\phi} + A_2 e^{p_2\phi}$$

Then $$d^nW/d\phi^n = A_1 p_1^n e^{p_1\phi} + A_2 p_2^n e^{p_2\phi}$$

where $p_1 = \beta + \alpha i \qquad p_2 = \beta - \alpha i$

It is always possible to express β as $k\cos\theta$ and α as $k\sin\theta$

Now $$e^{i\alpha\phi} = \cos\alpha\phi + i\sin\alpha\phi$$

and $$(k\cos\theta + i\sin\theta)^n = k^n(\cos n\theta + i\sin\theta)$$

from DeMoivre's theorem.

$$\begin{aligned}
\therefore\ d^nW/d\phi^n &= A_1(\beta+\alpha i)^n e^{\beta\phi}e^{\alpha i\phi} + A_2(\beta-\alpha i)^n e^{\beta\phi}e^{-\alpha i\phi} \\
&= A_1k^n(\cos n\theta + i\sin n\theta)e^{\beta\phi}(\cos\alpha\phi + i\sin\alpha\phi) \\
&\quad + A_2k^n(\cos n\theta - i\sin n\theta)e^{\beta\phi}(\cos\alpha\phi - i\sin\alpha\phi) \\
&= k^n e^{\beta\phi}\left\{\begin{matrix}[(A_1+A_2)\cos n\theta + (A_1-A_2)i\sin n\theta]\cos\alpha\phi \\ [-(A_1+A_2)\sin n\theta + (A_1-A_2)i\cos n\theta]\sin\alpha\phi\end{matrix}\right\}
\end{aligned}$$

Putting $A_1 + A_2 = C1$ and $(A_1 - A_2)i = C2$

$$d^nW/d\phi^n = k^n e^{\beta\phi}\left\{\begin{matrix}(C1\cos n\theta + C2\sin n\theta)\cos\alpha\phi \\ (-C1\sin n\theta + C2\cos n\theta)\sin\alpha\phi\end{matrix}\right\}$$

In matrix form this is conveniently put as

$$k^n[\cos n\theta \quad \sin n\theta]\, e^{\beta\phi}\begin{bmatrix}\cos\alpha\phi & \sin\alpha\phi \\ -\sin\alpha\phi & \cos\alpha\phi\end{bmatrix}\begin{bmatrix}C1 \\ C2\end{bmatrix}$$

or A.B.C.

where A depends on the order of the derivative
B the angular position in the shell and C is the vector containing the constants of integration.

DEVELOPING THE PROGRAM

The roots of the simplified Flügge equation form two sets

1) $p^2 = -\frac{1}{2} + m^2 + mK \pm mKi = R1 \pm mKi$

2) $p^2 = -\frac{1}{2} + m^2 - mK \pm mKi = -R2 \pm mKi$

We shall take first the first set with the positive imaginary part

$$k_1^2 (\cos 2\theta_1 + i \sin 2\theta_1) = R1 + mKi$$

$$\tan 2\theta_1 = mK/R1 \quad \text{or} \quad \theta_1 = \tfrac{1}{2} \arctan(mK/R1)$$

$$k_1^4 \cos^2 2\theta_1 = R_1^2$$

$$k_1^4 \sin^2 2\theta_1 = m^2 K^2$$

$$k_1^4 (\cos^2 2\theta_1 + \sin^2 2\theta_1) = k_1^4 = R_1^2 + m^2 K^2$$

$$\therefore \quad k_1 = \sqrt[4]{(R_1^2 + m^2 K^2)}$$

Taking now the second set with the positive imaginary part

$$\tan 2\theta_2 = -mK/R2 \quad \therefore \theta_2 = \tfrac{1}{2}(\pi - \arctan mK/R2)$$

$$k_2 = \sqrt[4]{(R_2^2 + m^2 K^2)}$$

Then $\beta_1 = k_1 \cos \theta_1 \qquad \alpha_1 = k_1 \sin \theta_1$

$\beta_2 = k_2 \cos \theta_2 \qquad \alpha_2 = k_2 \sin \theta_2$

The matrices for the derivations can now be set out as follows:

A will be an 8 x 8 matrix with rows corresponding to the n values required $(-1 \text{ to } 6)$ and columns having the values

$k_1^n \cos n\theta_1 \quad k_1^n \sin n\theta_1 \quad (-k)^n \cos n\theta_1 \quad (-k_1)^n \sin n\theta_1$
$k_2^n \cos n\theta_2$ etc.

B will also be an 8 x 8 matrix as follows

B1	0	0	0
0	B2	0	0
0	0	B3	0
0	0	0	B4

$$B1 = e^{\beta_1\phi}\begin{bmatrix} \cos\alpha_1\phi & \sin\alpha_1\phi \\ -\sin\alpha_1\phi & \cos\alpha_1\phi \end{bmatrix}$$

$$B2 = e^{-\beta_1\phi}\begin{bmatrix} \cos\alpha_1\phi & -\sin\alpha_1\phi \\ \sin\alpha_1\phi & \cos\alpha_1\phi \end{bmatrix} \quad \text{etc.}$$

C will be a 1 x 8 column matrix containing the eight constants of integration C1..........C8.

A new 8 x 8 matrix D is now produced, the row number corresponding to the order of the highest derivative in the expression for the shell displacement or stress. This is done by employing equations (9).

For j having the values 1 to 8

$$D_{-1j} = A_{-1j} \quad \ldots\ldots\ldots \quad v$$

$$D_{0j} = A_{0j} \quad \ldots\ldots\ldots \quad w$$

$$D_{1j} = \frac{1}{R}(A_{1j} + A_{-1j}) \quad \ldots\ldots \quad a$$

$$D_{2j} = -\frac{D}{R^2}(A_{2j} + A_{0j}) \quad \ldots\ldots \quad M\phi$$

$$D_{3j} = -\frac{D}{R^3}(A_{3j} + K1\,A_{1j}) \quad \ldots\ldots \quad V\phi$$

$$D_{4j} = +\frac{D}{R^3}(A_{4j} + K1\,A_{2j}) \quad \ldots\ldots \quad N\phi$$

$$D_{5j} = -\frac{D}{R^4}(A_{5j} + K2\,A_{3j}) \quad \ldots\ldots \quad dS/dx$$

$$D_{6j} = -\frac{D}{R^3 m^2}(A_{6j} + K2\,A_{4j}) \quad \ldots\ldots \quad Nx$$

Where $K1 = 1 - 2(1-\nu)m^2$

And $K2 = 2 - 2(1-\nu)m^2$

STRESSES DUE TO POST TENSIONING

The handling of the gravity loading of the shell is too well known to require repetition [2, 3]. However the usual method of replacing the post tension by the Fourier series

$$\frac{4}{\pi} P \sum_{N=1,3,5..}^{\infty} \frac{1}{N} \sin \frac{N\pi}{2} \cos \frac{N\pi x}{L} \qquad \dots . \ (10)$$

leads to serious difficulties.

This is due to the shear at the edge of the shell being proportional to the rate of change of the force in the edge beam, or the differential of (10) which can be seen to oscillate with increasing number of terms near the centre of the shell and diverge near the traverses.

A more satisfactory series is obtained by assuming that the past tension is fed into the edge beam by shearing forces decreasing linearly from the traverse to the quarter points. This produces the parabolic distribution of post tension shown in Fig. 2(b).

The Fourier series for this is $P \equiv \sum_{1,3,5\dots} P_N \cos \frac{N\pi x}{L} =$

$$\sum \frac{128P}{(N\pi)^3}\left(\sin \frac{N\pi}{2} - \sin \frac{N\pi}{4}\right)\cos \frac{N\pi x}{L} \qquad \dots . (11)$$

Four terms of this series summed up on a desk computer showed almost perfect agreement with the curve chosen and can be differentiated without much loss of accuracy.

The theory was tested by the construction of a small steel model shell shown in Fig. 8.. Tubular edge beams were soldered on to the edges of the shell. A steel rod was passed through one of the edge beams and stressed by means of nuts screwed on the ends. Buckling was avoided by fixing the tendon at the centre by set screws. The variation of strain was measured with eight pairs of Huggenberger tensometers.

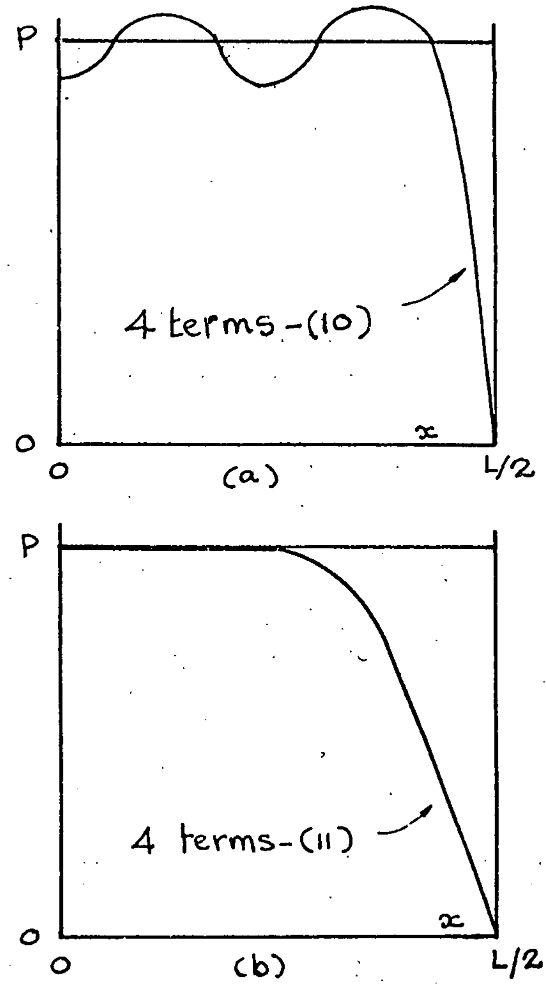


Fig 2

The boundary conditions for the model shell can be expressed fairly simply.

The B matrix is calculated for the right hand edge. This is multiplied by the D matrix to give matrix E (8 x 8) which relates to the shell displacements, moments and forces for the right hand edge when multiplied by the C vector.

The four boundary conditions for the right hand edge can then be put into the first four rows of matrix F (8 x 8) and matrix G (8 x 1) as follows:

1) The rotations of the shell edge and the edge beam are equal

Reference to Fig. 3(a) shows that

$$\frac{dT}{dx} = (M_\phi + bN_\phi)\cos\frac{N\pi x}{L}$$

$$\frac{da}{dx} = -\frac{T}{GJ}$$

$$a = \frac{1}{GJ}\iint (M_\phi + bN_\phi)\cos\frac{N\pi x}{L}\,dx$$

$$= \left(\frac{L}{N\pi}\right)^2 \frac{1}{GJ}(M_\phi + bN_\phi)$$

at $x = 0$.

Then for j having values from 1 to 8

$$F_{1j} = E_{1j} - \left(\frac{L}{N\pi}\right)^2 \frac{1}{GJ}(E_{2j} + bE_{4j})$$

$$G_1 = 0$$

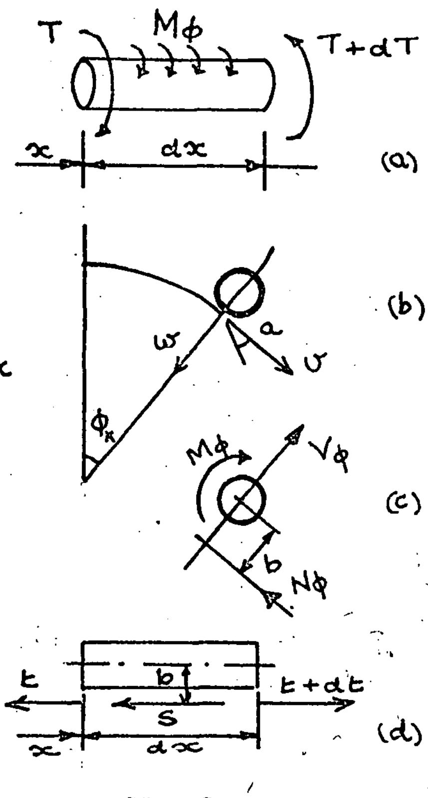


Fig 3

2) The radial displacements w of the shell and the edge beam are equal.

Reference to Fig. 3(d) shows that

$$dt = S\,dx$$

$$t = \iint \frac{dS}{dx}\,dx$$

$$= -\left(\frac{L}{N\pi}\right)^2 \frac{dS}{dx}$$

at $x = 0$

For the edge beam

$$-EI\frac{d^4w}{dx^4} = V_\phi \cos\frac{N\pi x}{L}$$

$$M = -EI\frac{d^2w}{dx^2} = \left\{-\left(\frac{L}{N\pi}\right)^2 V_\phi + tb\right\}\cos\frac{N\pi x}{L}$$

$$\therefore \quad w = -\frac{1}{EI}\left(\frac{L}{N\pi}\right)^2\left\{\left(\frac{L}{N\pi}\right)^2 V_\phi + b\left(\frac{L}{N\pi}\right)^2\frac{dS}{dx}\right\}$$

$$F_{2j} = E_{0j} + \frac{1}{EI}\left(\frac{L}{N\pi}\right)^4 \{ E_{3j} + b.E_{5j} \}$$

$$G_2 = 0$$

3) The tangental displacements v of the shell and the edge beam are equal.

$$EI \frac{d^4 v}{dx^4} = -N\phi \cos \frac{N\pi x}{L}$$

$$\therefore \quad v = -\frac{1}{EI}\left(\frac{L}{N\pi}\right)^4 N\phi \quad at \quad x = 0$$

$$F_{3j} = E_{-1j} + \frac{1}{EI}\left(\frac{L}{N\pi}\right)^4 E_{4j}$$

$$G_3 = 0$$

4) The longitudinal strains of the shell and the edge beam are equal. The strain in the edge beam is $\frac{t - P_N}{EA}$ due to the longitudinal force. The strain in the edge beam at the springing due to bending is

$$-b\frac{d^2 w}{dx^2} = b\left(\frac{N\pi}{L}\right)^2 w \quad at \quad x = 0$$

The strain in the shell at the springing is $\frac{1}{Eh}(N_x - \nu N\phi)$

$$F_{4j} = \frac{1}{Eh}(E_{6j} - \nu E_{4j}) + \frac{1}{EA}\left(\frac{L}{N\pi}\right)^2 E_{5j} - b\left(\frac{N\pi}{L}\right)^2 E_{0j}$$

$$G_4 = -\frac{P_N}{EA}$$

The B matrix is then recalculated for the left hand edge. A new E matrix is produced by multiplying B by D. The second half of the F and G matrices can then be filled in a similar manner to the first, making allowance for some sign differences.

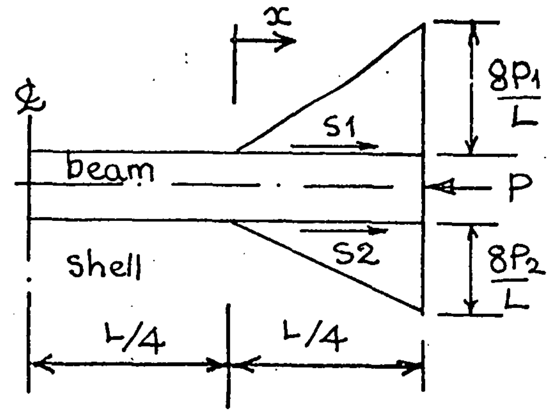
The eight simultaneous equations represented by $F \times C = G$ are solved to find the integration constants C. The shell stresses and displacements can then be calculated from D.B.C., the change in angular position modifying only B.

Correction at Corners of Prestressed Shell

At regions remote from the corners the method outlined gave excellent agreement with tests on the model, and with the results of a finite element program developed by Pham Lam[4]. The boundary conditions assumed however imply that N_x is zero at the traverses. This means that compatability of strain cannot apply at the ends of the edge beam where the strain is the greatest.

It is evident that the traverses can transmit some of the post tension and this is allowed for in the following analysis.

It will be supposed that the post tension is returned to the corners of the shell by applying shear forces S1 to the edge beam and S2 to the shell edge, both varying linearly from the quarter points to the traverse. These will produce the parabolic variation of longitudinal stress shown in Fig. 4 and will be apportioned to retain compatability of strain.



The characteristic equation of the shell will be taken in its simplest form:

$$\frac{d^8u}{d\phi^8} + 4K^4R^4 \frac{d^4u}{dx^4} = 0 \quad \cdots (12)$$

The longitudinal strain $\frac{du}{dx}$ will vary with x^2

$\therefore \; d^4u/dx^2 = 0$

Equation (12) then reduces to

$d^8u/d\phi^8 = 0$

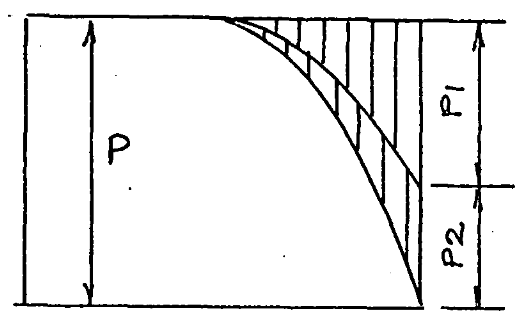


Fig 4

Then $\frac{du}{dx} = (c1\phi^7 + c2\phi^6 + c3\phi^5 \ldots c8)\, x^2$

Assuming symmetry about the ℄ $\frac{du}{dx}(\phi) = \frac{du}{dx}(-\phi)$ and C1, C3, C5, C7 are zero. If the traverse is fairly flexible in the x direction the forces produced by the end correction will die away rapidly from the edge. It is also clear that only compatability of strain with the edge beam is of importance. The solution to (12) will then be taken as

$$\frac{du}{dx} = c2\, \phi^6\, x^2 \quad \cdots (13)$$

If ϵ is the longitudinal strain at the corners of the shell

$$N_x = Eh\phi^6 \epsilon\, 16x^2/L^2 = Eh\epsilon\phi^6 \quad \text{at } x = L/4$$

Then $P2 = \int_0^{\phi_K} N_x \, R\, d\phi = REh\epsilon\phi_K^7/7$

$P1 = EA\epsilon$

Then $P = E\epsilon(A + Rh\,.\,\phi_K^7/7) \quad \ldots (14)$

From which ϵ, P1 and P2 can be calculated.

The edge correction has been added to the simplified Flugge solution using four terms of the Fourier series and is compared in Fig. 5 with the measured strains along the edge beam of the model shell.

A long shell with post tension loading on a retangular edge beam has also been analysed and is compared with the results of Pham Lam's finite element program in Fig. 6.

FUTURE WORK

The edge correction method implies a departure from the usual assumptions that the traverses do not resist longitudinal movements. A rough analysis suggests that a reasonable design of traverse can be obtained to satisfy the assumptions of the edge correction. This will be the subject of further study.

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APPENDIX

Analysis of shell with post tensioned rectangular edge beams (Figs. 6, 7)

$P = 500$ kips
$L = 120$ ft
$R = 30$ ft
$h = 0.25$ ft
$\phi_k = 40° = 0.69813$ rad

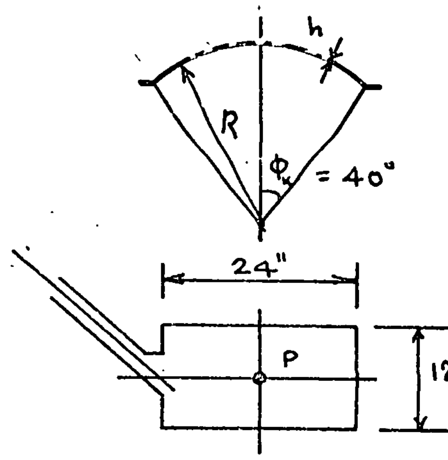


Fig 7

$$P = E\epsilon\,(A + hR\phi_k^7/7)$$
$$= E\epsilon\,(2 + 0.08666)$$
$$= E\epsilon \times 2.08666 = 500 \text{ k.}$$

At corner

$$N_x = Eh\epsilon = 500 \times 0.25/2.08666 = 60 \text{ kips/ft.}$$

This edge correction will diminish with ϕ to the 6th power of ϕ/ϕ_k as follows

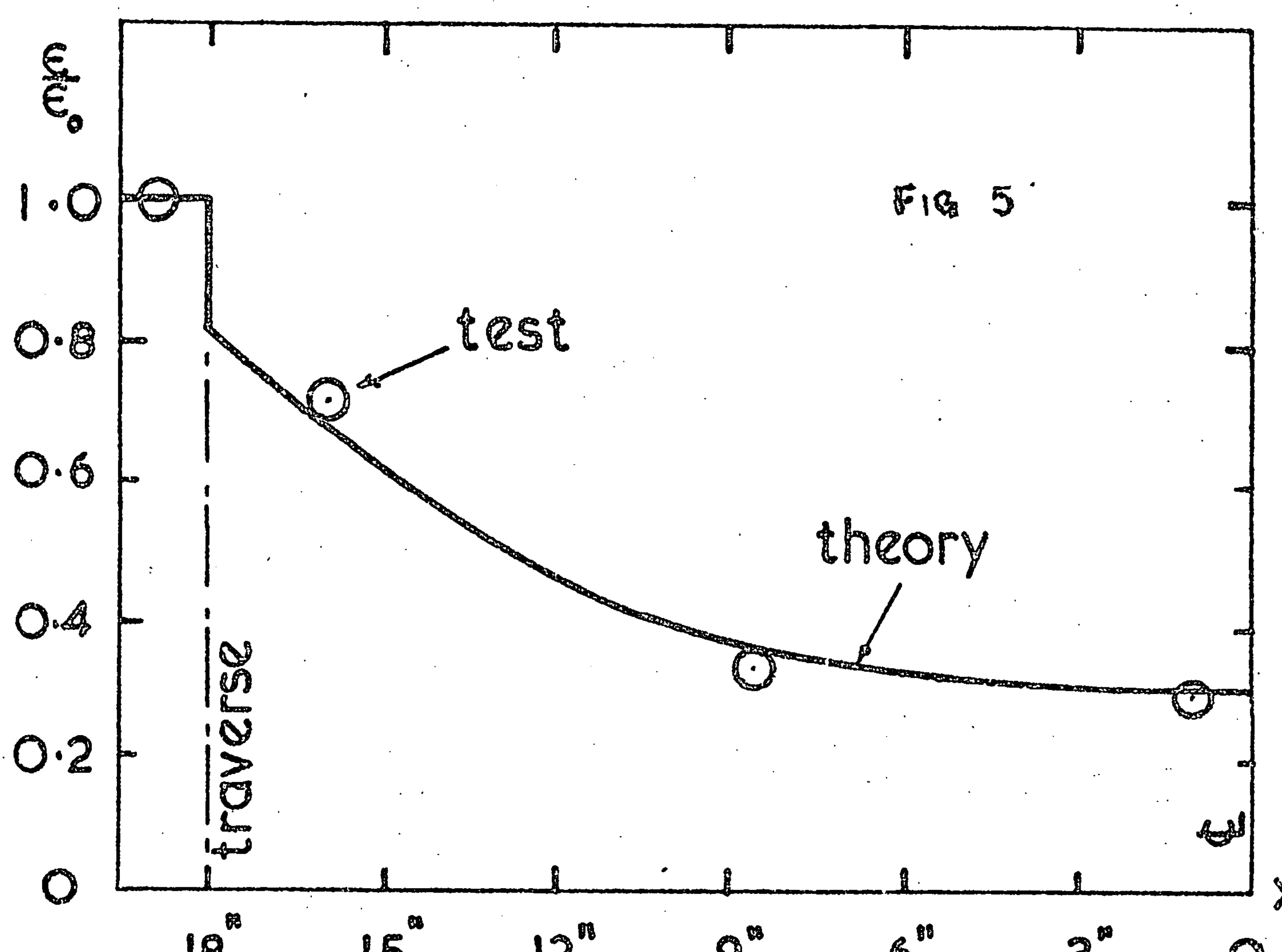
ϕ/ϕ_k	1	0.9	0.8	0.7	0.6	0.5	0.4
Factor	1	0.531	0.262	0.118	0.047	0.016	0.004

It will also diminish parabolically back to zero at the $\frac{1}{4}$ points of the shell.

The N_x values obtained from (8) are as follows

N	P_N kips	N_x ϕ 40°	N_x ϕ 36°
1	604.56	-52.292	-39.773
3	-130.50	10.931	7.572
5	28.19	-2.243	-13.321
7	-1.76	0.141	0.069

N_x kips/ft	Sum Fourier Series	Edge Correction	Total $\phi = 36°$	Sum Fourier Series	Edge Correction	Total $\phi = 40°$
Traverse	0	31.860	31.860	0	60.000	60.00
	17.002	14.160	31.162	23.566	26.666	50.23
	28.090	3.540	31.630	38.127	6.666	44.79
1/4 Pt.	32.487	0	32.487	43.020	0	43.02
	33.351	0	33.351	43.466	0	43.46
	33.427	0	33.427	43.398	0	43.39
℄	33.464	0	33.464	43.463	0	43.46



$\frac{\epsilon_m}{\epsilon_o}$
1.0
0.8
0.6
0.4
0.2
0
Fig 5
test
theory
traverse
18"
15"
12"
9"
6"
3"
0

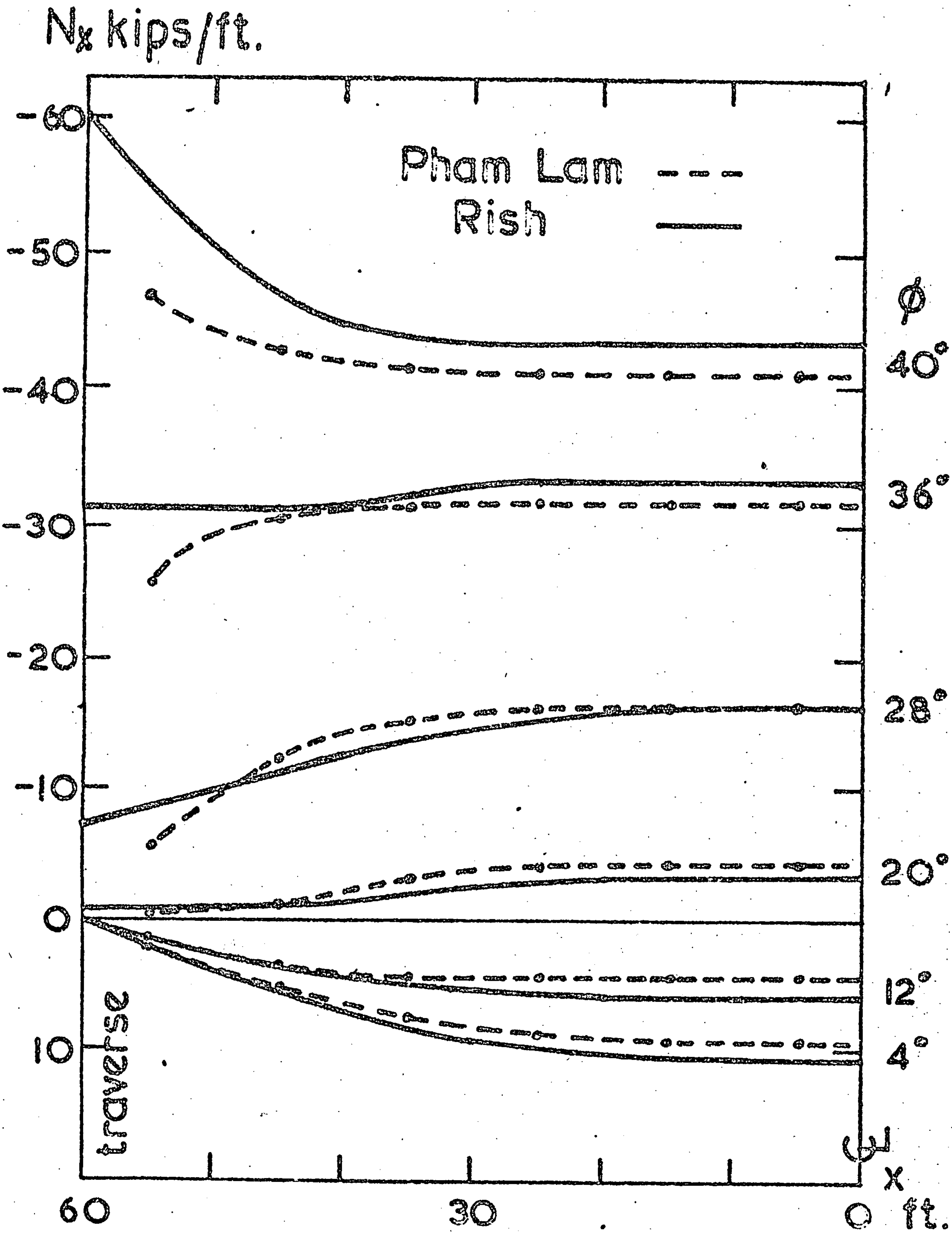

Nx kips/ft.
-60
-50
-40
-30
-20
-10
0
10
Pham Lam
Rish
φ
40°
36°
28°
20°
12°
4°
traverse
ω
x
60
30
0 ft.

FIG 6

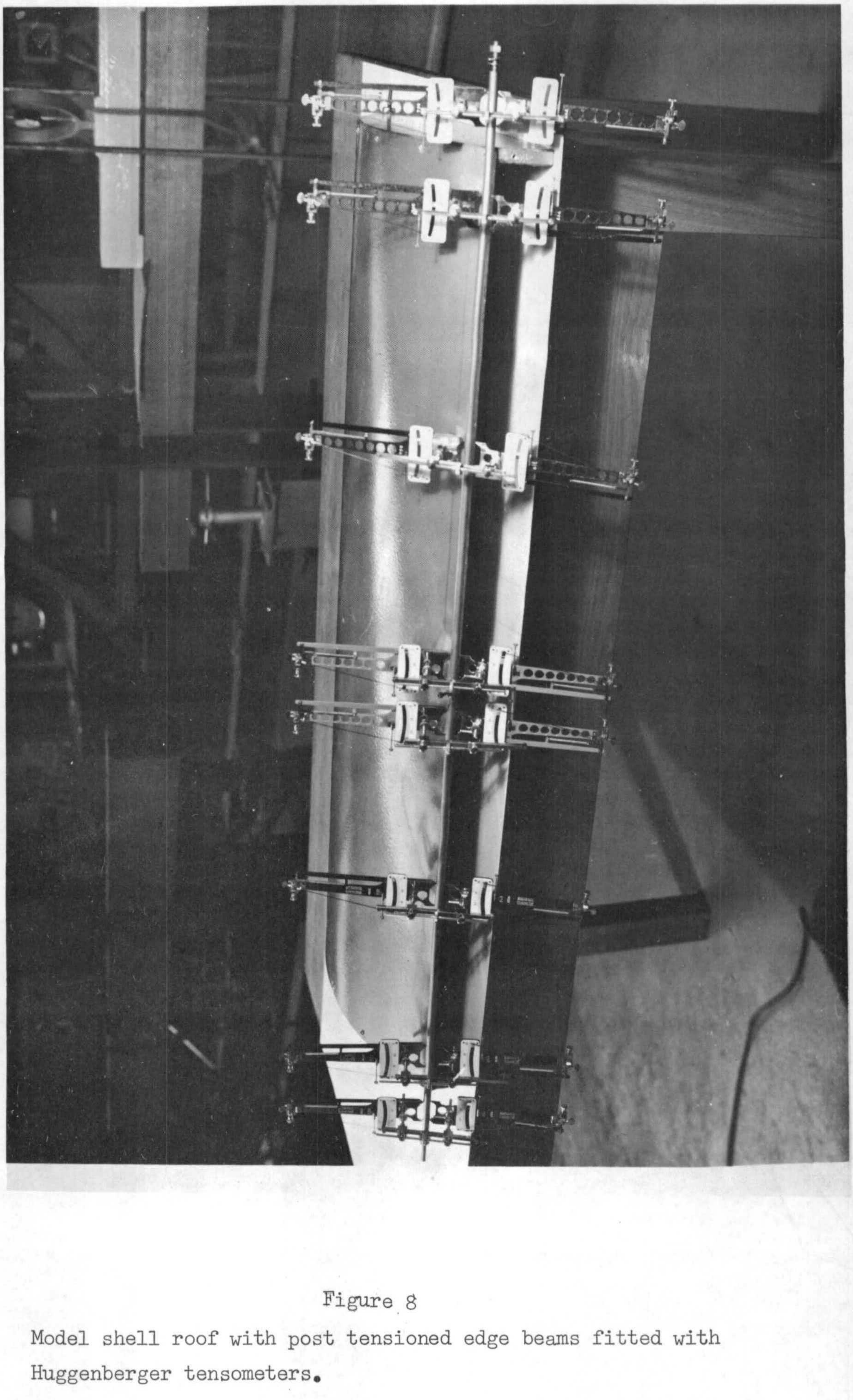


Figure 8

Model shell roof with post tensioned edge beams fitted with Huggenberger tensometers.