MATROIDS ON DOMAINS OF FUNCTIONS

BY

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K. Prendergast.

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CONTENTS

SUMMARY

Any integer-valued function with finite domain E defines, by means of an associated submodular function on 2^E, a matroid $M(E)$.

The class $\mathcal{M}$ of matroids so obtained is closed under restriction, contraction, and is self dual. We show it consists precisely of those transversal matroids having a presentation in which the sets of the presentation are nested.

We give an excluded minor characterisation of $\mathcal{M}$.

We count the members of $\mathcal{M}$ on an n-set and exhibit explicitly those on a 6-set.

We extend the above investigation, using Rado's Selection Principle, and permitting E to be infinite, to pregeometries.

Finally, by examining some integer-valued functions on E^r, with r possibly greater than 1, we discuss some of the properties of the class of matroids so obtained.

INTRODUCTION

A matroid is essentially a set with an independence structure defined on its subsets. The term *matroid* arose from the generalisation of the columns of a matrix following consideration of the independence of those columns. This thesis is concerned with a class of matroids $M(E)$ which can be defined in a certain way from functions defined on the ground set E.

It is well known that a matroid $M(E)$ can be obtained from submodular increasing functions defined on 2^E, but in practice such functions are rather rare. The motivation for this thesis initially was the hope that from the more prolific functions on domain E, it would be possible in some way to build up to submodular functions which define some well known matroids. This hope was partially fulfilled, but the investigation uncovered a simply defined and interesting class, firmly located in the usual hierarchies of matroids.

Matroid theory began in 1935 with Whitney's basic paper [27]. He had been working in graph theory for some years and had noticed similarities between the ideas of independence and rank in graph theory, and the ideas of linear independence and dimension in vector spaces, and in this paper he used the concept of matroid to abstract and formalise these similarities.

At about the same time van der Waerden [24] was approaching the ideas of linear and algebraic dependence axiomatically, so he too was instrumental in the birth of matroid theory.

After this bright beginning the study lapsed for about twenty years, with the important exception of papers by Birkhoff [1], MacLane [12],[13], and Dilworth [6],[7],[8] on lattice theoretic and geometric aspects of matroid theory, and two papers by Rado [18],[19] on combinatorial applications of matroids and infinite matroids. Tutte [21],[22] revived the study in 1958 with a characterisation of those matroids which arise from graphs and at about the same time Rado [20] returned to the field with a study of the representability problem of matroids.

Since then interest in matroids and their applications has grown rapidly and there is now a sizable body of literature on the subject. Of particular importance has been the applications of matroids to transversal theory and the associated investigation of transversal matroids. This work was pioneered by Edmonds and Fulkerson [10] and Mirsky and Perfect [17], and has produced many new results as well as elegant proofs of earlier results in transversal theory.

Many other aspects of combinatorial theory have been subsumed in matroid theory over the past 15 years and the result has been a firmer linking of combinatorics to the mainstream of mathematics.

Matroids have been used for engineering applications recently, for example Weinberg's work on electrical network synthesis [25].

The theme of this thesis, matroids defined by submodular functions, had its beginning with a paper of Dilworth [8], in which seemed to be implicit the fact that a matroid can be defined by its submodular rank function. The first explicit derivation of matroids from submodular functions is thought to be due to Edmonds and Rota [11] in 1966, and a generalisation of this result was produced by McDiarmid [14]. Further work on the relationship between submodular functions and matroids was done by Edmonds [9] and Pym and Perfect [17].

In this thesis Chapter 1 is simply a restatement of the many different axiomatic ways of defining a matroid together with some well known results necessary for the development of the thesis. A similar resume can be found in a paper of Wilson [28].

Chapter 2 contains the basic "arithmetic" of the thesis. It establishes that a submodular function on 2^E can be derived from any integer valued increasing function defined on E and characterises the matroids so formed by standardising the defining functions. The class $\mathcal{M}$ of matroids so obtained is shown to be self dual and closed under taking restrictions.

Whereas the treatment of the matroids of $\mathcal{M}$ was in arithmetical terms in Chapter 2, in Chapter 3 the approach is more in the mainstream of matroid theory. The class $\mathcal{M}$ is characterised in terms of the unique minimal non-trivial flats of its minors and also by its excluded minors.

Chapter 4 shows that $\mathcal{M}$ contains exactly 2^n pairwise non-isomorphic members on an n-set. The number of matroids in the class is compared to earlier lower bounds established by Crapo [4] and Bollobás [3] for the class of all matroids on an n-set. We then examine those on a 6-set and by use of the excluded minor property the matroids not in $\mathcal{M}$ are identified.

Chapter 5 establishes that $\mathcal{M}$ is properly contained in the class of transversal matroids and obtains a necessary and sufficient condition for a transversal matroid to belong to $\mathcal{M}$. Another condition in terms of circuits is produced for a matroid to belong to $\mathcal{M}$.

In Chapter 6 the results of the earlier chapters are extended to the class of pregeometries, which are defined on possibly infinite ground sets S by integer valued functions on S . The principal tools in this investigation are the results on submodular functions (semimodular in [5]) of Crapo and Rota, and Rado's Selection Principle.

The final chapter, Chapter 7, deals with functions defined on E^r , from which submodular functions and ensuing matroids are obtained.

The value of the function on each r-tuple is the sum of the values of functions (defined on E) on the components of the r-tuple, and the matroids so obtained are the union of matroids in $\mathcal{M}$. As Welsh [26] pointed out, this result is implicit in the more general results of Pym and Perfect [17] for sums of arbitrary submodular functions. The matroids so obtained are shown to constitute exactly the class of transversal matroids. They are characterised in terms of flats and circuits. We see that more graphic matroids are included in this class of matroids than in $\mathcal{M}$. Finally there is a failed conjecture. It had been hoped that it would be possible to obtain submodular functions defining some well known matroids from the functions on E^r , by allowing r to increase. However a counter example is provided.

With the exception of the abovementioned result implicit in work by Pym and Perfect, the work in Chapters 2 to 7 inclusive is not in the literature.

The author gratefully acknowledges the help of James Oxley and Don Row in obtaining theorem 3.6, and also particularly James Oxley for obtaining the excluded minor characterisation of theorem 3.9, and suggesting a detailed examination of the members of $\mathcal{M}$ on 6-point ground sets.

NOTATION

For most of the thesis we consider structures on a finite set, and this set is designated E . When we deal with an infinite set it is designated S . Elements of E or S are denoted by lower case letters and subsets by upper case. The empty set is denoted by ϕ , and $A \backslash B$ is the set consisting of elements which are in A but not in B . $A \dot{\cup} B$ denotes the disjoint union of A and B .

Where the meaning is clear, we abbreviate $\{a\}$ to a. For example $A \cup a$ means $A \cup \{a\}$ and $A \backslash a$ means $A \backslash \{a\}$.

A function from the set E to the set F is denoted by $f:E \to F$, and a function from the power set 2^E to the power set 2^F by $\theta:2^E \to 2^F$.

If T is a subset of E we denote the restriction of f to T by $f_{|T}$. The matroid M on E restricted to T is denoted by $M|T$.

A family or collection of subsets of E is denoted by $(E_i \subseteq E : E_i$ has the required property), or, if it is possible to list the subsets, by $(E_1, E_2, \ldots, E_m)$.

The set of integers is denoted by Z .

CHAPTER 1

There are several equivalent ways in which matroids may be defined. Proof of their equivalence is in Whitney's original paper [27]. Some are listed below.

1. Independent Sets

This is the set of axioms favoured by many because of its obvious relationship to linear algebra, which makes it easily recognized. It is the set most commonly used in this thesis.

A *matroid* $M(E, \mathcal{I})$ consists of a finite set E, together with a non-empty collection $\mathcal{I}$ of subsets which are called *independent* sets of E, satisfy the following properties:

(i) $I \in \mathcal{I}$ and $J \subseteq I \Rightarrow J \in \mathcal{I}$;

(ii) if $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists $a \in J \backslash I$ such that $I \cup a \in \mathcal{I}$.

Any set not in $\mathcal{I}$ is *dependent*.

It follows from the above that all maximal independent subsets of any subset A of E have the same number of elements. The maximal independent sets are known as *bases* and their size is the *rank* of the matroid. This brings us to the next two axiomatic descriptions.

2. Bases

A *matroid* $M(E, \mathcal{B})$ consists of a non-empty finite set E, together with a non-empty collection $\mathcal{B}$ of subsets of E, which are called *bases*, satisfying the following property:

if $B_1, B_2 \in \mathcal{B}$ and $a \in B_1 \backslash B_2$ there exists $b \in B_2 \backslash B_1$ such that $(B_1 \cup b) \backslash a \in \mathcal{B}$.

3. Rank Function

A *matroid* $M(E, \rho)$ consists of a non-empty finite set E, together with an integer valued function $\rho: 2^E \to Z$, called the *rank function*, which satisfies the following properties:

(i) for each $A \subseteq E$, $0 \leq \rho(A) \leq |A|$;

(ii) if $A \subseteq B \subseteq E$ then $\rho(A) \leq \rho(B)$;

(iii) for any $A, B \subseteq E$, $\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$.

If $\rho(A \cup a) = \rho(A)$, then a is said to depend on A, or to be in the *closure* of A, and the set $\sigma(A) = \{a \in E: \rho(A \cup a) = \rho(A)\}$ is said to be the closure of A. This leads us to the next axiomatic description.

4. Closure

A *matroid* $M(E,\sigma)$ consists of a non-empty finite set E, together with a function $\sigma: 2^E \to 2^E$, called the closure operator, which satisfies the following properties:

(i) for each $A \subseteq E$, $A \subseteq \sigma(A)$;

(ii) if $A \subseteq \sigma(B)$, then $\sigma(A) \subseteq \sigma(B)$;

(iii) if $a \in \sigma(A \cup b)$, $a \notin \sigma(A)$, then $b \in \sigma(A \cup a)$.

This is the set of axioms adopted by Crapo and Rota [5], and they use the term *pregeometry* rather than matroid allowing E to be infinite. The closures are also known as *flats*, and this term will sometimes be used in this thesis.

The final set of axioms we consider is somewhat different, in that it is not inspired by linear algebra but rather by graph theory. It is in terms of circuits, which in graph theory are finite sequences of distinct edges defined in terms of vertices as follows: $\{v_0,v_1\}$, $\{v_1,v_2\}$, ..., $\{v_m,v_0\}$, i.e. they are polygons. In a matroid a *circuit* is defined as a minimal dependent set.

5. Circuits

A *matroid* $M(E, \mathcal{C})$ consists of a non-empty finite set E, together with a collection $\mathcal{C}$ of non-empty subsets of E, called circuits, satisfying the following properties:

(i) no circuit properly contains another circuit ;

(ii) if $a \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$ are distinct, then there exists $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) \backslash a$.

This set of axioms was favoured by Tutte [23].

Throughout this thesis we do not distinguish between the sets of axioms defining the matroid and merely represent it as $M(E)$.

As well as the matroid entities mentioned above, i.e. independent sets, rank function, bases, closures and circuits, there are others which are frequently used. Those which are used in this thesis are as follows.

A *cobase* of the matroid $M(E)$ is any set $E \backslash B$, where B is a base of $M(E)$. It can easily be shown that the collection of cobases of $M(E)$ is the collection of bases of a matroid, and this matroid is designated $M^*(E)$ and is called the *dual matroid* of $M(E)$. This result was first established by Whitney [27].

Following from the above, the *corank* $\rho^*: 2^E \to Z$ of the matroid $M(E)$ is the rank function of $M^*(E)$.

A *cocircuit* of $M(E)$ is a circuit of $M^*(E)$.

A *hyperplane* is a maximal proper flat of $M(E)$. It can be shown that a hyperplane is the set complement of a cocircuit.

We now consider a few types of matroids which will be referred to later. They are graphic matroids, transversal matroids and matroids representable in Euclidean space. First we need a definition of isomorphism of matroids.

Two matroids $M_1(E_1)$ and $M_2(E_2)$ are isomorphic if there exists a bijection $\theta:E_1 \to E_2$ which preserves independence.

A *graphic matroid* is one which is isomorphic to a matroid defined on edges of a graph by letting the circuits of the matroid be the edge sets of polygons of the graph.

A *transversal* of a finite family $\mathcal{U} = (E_1,E_2,\ldots, E_m)$ of subsets of E is a set of m distinct elements of E, one chosen from each of the subsets E_i ; a *partial transversal* of $\mathcal{U}$ is a transversal of some subfamily of $\mathcal{U}$. It is easily shown that the partial transversals of $\mathcal{U}$ satisfy the properties specified above for independent sets of a matroid. The bases of the matroid are the maximal partial transversals of $\mathcal{U}$. We call a matroid $M(E)$ a *transversal matroid* if there exists some family $\mathcal{U}$ of subsets of E such that the family of independent sets of $M(E)$ is precisely the family of partial transversals of $\mathcal{U}$.

Euclidean representation of a matroid is possible if it is isomorphic to the matroid induced on a set of points in R^n by the usual affine closure.

As we saw above, a function $\rho:2^E \to Z$ having certain properties defines a matroid whose rank function is ρ. One of those properties was that for any $A,B \subseteq E$, $\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$, and a function having this property is known as a *submodular function*. A function having the property that $A \subseteq B \subseteq E \Rightarrow \rho(A) \leq \rho(B)$ is an *increasing function*. The following observation is used throughout this thesis.

<u>THEOREM 1.1</u> *A submodular increasing function* $f:2^E \to Z$ *defines a matroid on the set* E.

<u>Proof</u>: Let $\mathcal{C} = (\phi \neq C \in 2^E: f(C) < |C|,\ f(K) \geq |K|$ for all $K \subset C)$. We proceed to prove that $\mathcal{C}$ is the collection of circuits of a matroid.

(i) Obviously no member of $\mathcal{C}$ properly contains another member of $\mathcal{C}$.

(ii) For any $C \in \mathcal{C}$, $f(C) = |C| - 1$. We consider now any distinct $C_1, C_2 \in \mathcal{C}$ whose intersection is non-empty, containing say the element a. Now applying the submodularity of f, and the fact that it is increasing, we have

$$\begin{aligned} f((C_1 \cup C_2)\backslash a) \leq f(C_1 \cup C_2) &\leq f(C_1) + f(C_2) - f(C_1 \cap C_2) \\ &\leq |C_1| - 1 + |C_2| - 1 - |C_1 \cap C_2| \\ &< |(C_1 \cup C_2)\backslash a| \ . \end{aligned}$$

Furthermore we know that $(C_1 \cup C_2)\backslash a$ contains a set K such that $f(J) \geq |J|$ for all $J \subseteq K$, since at least $C_1\backslash a$ and $C_2\backslash a$ have this property. Therefore $(C_1 \cup C_2)\backslash a$ contains a member of the collection $\mathcal{C}$, whence $\mathcal{C}$ is the collection of circuits of a matroid. //

<u>COROLLARY 1.2</u> *The collection* $\mathcal{I}$ *of independent sets of a matroid* $M(E)$ *defined by a submodular increasing function* $f:2^E \to Z$, *is given by*

$$\mathcal{I} = \{I \in E: f(J) \geq |J| \text{ for all } J \subseteq I\} \cup \{\phi\} \quad . \quad //$$

The above corollary appeared in a paper by Pym and Perfect [17] in 1970. As remarked in that paper, Edmonds and Rota had already proved a more comprehensive result.

It is necessary to point out, as did Pym and Perfect [17], that if $M(E)$ is a matroid on E, it may be possible to find a function $f:2^E \to Z$ which is not submodular, but for which the set I is independent if and only if $f(J) \geq |J|$ for all $J \subseteq I$. Their example was as follows. Let $M(E)$ be the free matroid on E, and define $f:2^E \to Z$ by the equations (i) $f(A) = |E|$ for $A \subset E$, and (ii) $f(E) = 2|E|$.

We recall from graph theory that a graph can have a loop, i.e. an edge whose two vertices are identical, and multiple edges, i.e. edges having the same two vertices. A graph having no loops or multiple edges is called a *simple graph*. Analogously a matroid can have elements of rank zero, i.e. they are in the closure of the empty set, and it can have a set A such that $|A| > 1$, $\rho(a) = 1$ for all $a \in A$, and $\rho(A) = 1$. A matroid having neither of the above is a *simple matroid*. Obviously a graphic matroid is simple if and only if it is isomorphic to a matroid defined in the abovementioned manner on a simple graph.

CHAPTER 2

In the previous chapter we saw that an integer-valued increasing submodular function on a set E defines a matroid. Examples of such functions are the dimension function on subspaces of a vector space (in which the submodular inequality becomes an equality and the function is modular), and the rank function of a matroid. In this chapter we construct a submodular function on arbitrary sized subsets from a function defined on singletons, and in this manner generate a particular class of matroids.

We obtain the function $f : 2^E \to Z$ from a function $\mu : E \to Z$ as follows. Let

$$f(A) = \max\{\mu(a) : a \in A\} \text{ for all } \phi \neq A \in 2^E ,$$

and $$f(A) = \min\{\mu(a) : a \in E\} \text{ for } A = \phi .$$

LEMMA 2.1 *Let* f *be a function as defined above. Then* f *is increasing and submodular.*

Proof: It is obvious that f is increasing. For any subsets $A,B \subseteq E$ let $a \in A$, $b \in B$ be such that $\mu(a) \geq \mu(x)$ for all $x \in A$, $\mu(b) \geq \mu(x)$ for all $x \in B$. If $\mu(a) \geq \mu(b)$ then $f(A) = f(A \cup B)$ and $f(B) \geq f(A \cap B)$, whence $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. The same result holds if $\mu(b) \geq \mu(a)$. //

We call matroids induced by the function as defined above *matroids of the class* M, and similarly f is called a *function of the class* F.

The geometric structure of the matroid is characterised by its independent set, closures, circuits, bases, and we now examine some of these for the matroids of class $\mathcal{M}$.

The independent sets of a matroid obtained from a submodular function f are the family $\mathcal{I}$ of sets given by

$$\mathcal{I} = (I : f(J) \geq |J| \text{ for all } J \subseteq I) .$$

In terms of μ , $\mathcal{I}$ is given by the following lemma.

<u>LEMMA 2.2</u> *If* $M(E) \in \mathcal{M}$ *is obtained from the function* $\mu : E \to Z$, *then the collection of independent sets of* $M(E)$ *is precisely the family*

$$\mathcal{I} = \phi \cup (I \subseteq E : \phi \neq J \subseteq I \Rightarrow \exists\, a \in J \text{ such that } \mu(a) \geq |J|).$$

<u>Proof</u>: Suppose I is independent, i.e. $f(J) \geq |J|$ for all $J \subseteq I$. Then on any $J \subseteq I$ the maximum value μ takes is at least $|J|$, so there exists $a \in J$ such that $\mu(a) \geq |J|$.

Conversely suppose $I \in \mathcal{I}$; then obviously for every $J \subseteq I$, $f(J) \geq |J|$, whence I is independent. //

A description of the circuits of a matroid of the class is given in the following lemma.

<u>LEMMA 2.3</u> *If* $M(E) \in \mathcal{M}$ *is obtained from the function* $\mu : E \to Z$, *a subset* $C \subseteq E$ *is a circuit of* $M(E)$ *if and only if its elements can be labelled* $a_1, \ldots, a_r$ *(where* $|C| = r$*) such that*

$$r - 1 \geq \mu(a_i) \geq i \quad \textit{for all} \quad 1 \leq i \leq r\text{-}1, \quad \textit{and} \quad \mu(a_r) = r\text{-}1.$$

<u>Proof</u>: Suppose the elements of C can be so labelled. Then obviously $f(D) \geq |D|$ for all $D \subseteq C\backslash\{a_r\}$. Further $f(D) \geq |D|$ for all $D \subseteq C\backslash\{a_i\}$ where $i \neq r$, since this set is obtained from

$C - \{a_r\}$ by substituting a_r for a_i, and $\mu(a_r) \geq \mu(a_i)$. Therefore $f(D) \geq |D|$ for all $D \subset C$ and $f(C) < |C|$ so C is a circuit.

Conversely suppose C is a circuit; then $\mu(c) < r = |C|$ for all $c \in C$. Let a_r be an element of C on which μ takes its maximum value, which necessarily is $r - 1$. Let a_{r-1} be an element of $C \backslash \{a_r\}$ on which μ takes its maximum value, which is $r - 1$. By continuing this process we obtain $a_{r-2}, \ldots, a_1$ so that $\mu(a_i) \geq i$ for $r - 1 \geq i \geq 1$. //

The *closure* $\sigma(A)$ of a subset $A \subseteq E$ is given in the following lemma.

LEMMA 2.4 *The closure* $\sigma(A)$ *of a subset* $A \subseteq E$ *in the matroid* $M(E) \in \mathbb{M}$ *is given by*

$$\sigma(A) = A \cup \{a \in E : \mu(a) \leq |J|\},$$

where J *is a subset of* A *maximum with respect to*

(i) J *is independent, and*

(ii) $\max\{\mu(a) : a \in J\} = |J|$.

Proof: We show that $\sigma(A)$ as defined above is precisely the closure of A in the matroid M(E). Obviously any $b \in A$ is in both the closure and $\sigma(A)$. Consider $b \notin A$ but in the closure of A. Then the joining of b to A does not increase the size of any maximal independent set in A, whence $\mu(b) \leq |J|$ and $b \in \sigma(A)$. Conversely suppose $b \notin A$, $b \in \sigma(A)$. Then $\mu(b) \leq |J|$, whence b does not increase the size of any maximal independent set in A, and b is in the closure of A. Therefore $\sigma(A)$ and the closure of A in M(E) are identical. //

Information about the matroids under consideration is more accessible if some ordering exists on the elements of E. Since this only involves relabelling the elements, no generality is lost. We say that E is μ-*ordered* when the elements of E are arranged and identified by the symbols $a_1, \ldots, a_n$ (where $|E| = n$) such that

$$\mu(a_{i+1}) \geq \mu(a_i) \quad \text{for} \quad i = 1, \ldots, n-1.$$

Similarly a set $A \subseteq E$ is μ-*ordered* if E is μ-ordered, and we identify the elements of A as $a_{i1}, \ldots, a_{im}$ where $i1 < \ldots < im$.

We now move on to a consideration of bases and we recall that a basis is a maximal independent set.

LEMMA 2.5 *Any maximal set* $\{a_{ij}: \mu(a_{ij}) \geq j,\ j = 1, \ldots, r\}$ *of the* μ-*ordered set* E *is a basis of the matroid* $M(E) \in \mathcal{M}$.

Proof: The set is obviously independent and also maximal. //

LEMMA 2.6 *If there exist* $a_n, a_{n-1}, \ldots, a_{n-s} \in E$ *such that*

$$\mu(a_n) > \ldots > \mu(a_{n-s}) > \mu(b) \quad \textit{for all} \quad b \in E\backslash\{a_n, \ldots, a_{n-s}\},$$

then $a_n, \ldots, a_{n-s}$ *are in every basis.*

Proof: Suppose the result is true for $a_n, \ldots, a_{n-j}$, where $j < s$. We proceed by induction on j. Suppose a_{n-j-1} is not a member of every basis and let B be a basis such that $a_{n-j-1} \notin B$. Then

$$f((B\backslash\{a_n,\ldots, a_{n-j}\}) \cup \{a_{n-j-1}\}) > f(B\backslash\{a_n,\ldots, a_{n-j}\}) \geq |B| - j - 1,$$

whence $f(B \cup \{a_{n-j-1}\}) \geq |B| + 1$, which contradicts the maximality of B with respect to independence. Therefore if the result is true for $n, \ldots, n-j$, it is also true for $n - j - 1$.

We now consider the case of $j = 0$. Suppose $a_n \notin B$, where B is some basis. Then $f(B \cup a_n) > f(B) \geq |B|$, whence $B \cup a_n$ is independent, which contradicts the maximality of B. //

It is obvious that there are many functions μ which induce the same matroid, so we now find upper and lower bounds for all such functions and establish their uniqueness. We begin with a standardised function obtained from μ. This *standardised function* l is defined as follows on a μ-ordered set E:

$$l(a_k) = \min(\mu(a_k), l(a_{k-1}) + 1), \text{ where}$$

$$l(a_1) = \max(\min(1, \mu(a_1)), 0) .$$

We sometimes say that a_k is a member of *level* $l(a_k)$, or has *level* $l(a_k)$.

<u>LEMMA 2.7</u> *If* $l(a_j) = l(a_k)$ *for elements* a_j, a_k *of a* μ*-ordered set* E *and* $j < k$, *then* $\mu(a_j) = \mu(a_k) = l(a_j) = l(a_k)$.

<u>Proof</u>: Since $j < k$ we have $l(a_j) \leq l(a_{k-1})$ whence $l(a_j) < l(a_{k-1}) + 1$. Therefore $l(a_j) = l(a_k)$ implies that $l(a_k) = \mu(a_k)$, and since $l(a_j) \leq \mu(a_j) \leq \mu(a_k)$ we have the result. //

<u>LEMMA 2.8</u> *The rank* $\rho(M)$ *of the matroid* $M(E) \in \mathbb{M}$ *on a* μ*-ordered set* E *is given by* $\rho(M) = \max\{l(a) : a \in E\}$.

<u>Proof</u>: If $\max\{l(a) : a \in E\} = r$ then l takes all values from 1 to r (and possibly also 0), and only those values. Hence we can choose

a set $B = \{b_1, \ldots, b_r\} \subseteq B$ so that $l(b_i) = i$ for all $1 \le i \le r$. Then $\mu(b_i) \ge i$ and hence B is independent. The function l maps any other $a \in E$ to say k, and this property is shared by $b_k \in B$. By Lemma 2.7 therefore $\mu(a) = \mu(b_k)$ whence the set $\{b_1, \ldots, b_k, a\}$ is not independent. B is therefore a basis. //

LEMMA 2.9 *A set comprised of single representatives of any number of distinct levels is independent.* //

LEMMA 2.10 *A μ-ordered set $B = \{a_{i1}, \ldots, a_{i\rho}\}$ is a basis of a rank ρ matroid $M \in \mathbb{M}$ if and only if $l(a_{i\rho}) = \rho$ and $l(a_{ij}) \ge j$ for all $1 \le j \le \rho-1$.*

Proof: Suppose $l(a_{i\rho}) = \rho$ and $l(a_{ij}) \ge j$ for $1 \le j \le \rho-1$; then obviously $\{a_{ij}: 1 \le j \le \rho\}$ is independent and being of size ρ must be a basis. Conversely suppose B is a basis; then $l(a_{i1}) \ge 1$ and also $l(a_{ij}) \ge j$ for all $2 \le j \le \rho$, otherwise there exists j such that $l(a_{ij}) = j - 1$ and $l(a_{ij}) = l(a_{ij-1})$. The latter implies that $\mu(a_{ij}) = j - 1$ which contradicts the independence of B. Lemma 2.8 establishes that $l(a_{i\rho}) = \rho$. //

The function l is defined on all elements of E, and E is l-ordered (in the same sense as it is μ-ordered), so it is natural to enquire what standardised function is obtained from l. It turns out that the process of standardising the function is an idempotent process as can be seen from the following definition and lemma. We define

$$l^2(a_k) = \min(l(a_k), l^2(a_{k-1})+1) \text{ and}$$
$$l^2(a_1) = \max(\min(1, l(a_1)), 0) ,$$

i.e. l^2 is the standardised standardised function obtained from μ.

LEMMA 2.11 $l^2 = l$.

Proof: We proceed by induction on k. Suppose $l^2(a_k) = l(a_k)$; then

$$l^2(a_{k+1}) = \min(l(a_{k+1}), l^2(a_k)+1)$$
$$= \min(l(a_{k+1}), l(a_k)+1)$$
$$= l(a_{k+1}) .$$

It is obvious that $l^2(a_1) = l(a_1)$. //

LEMMA 2.12 l *induces the same matroid* $M(E) \in \mathcal{M}$ *as does* μ .

Proof: l induces a matroid of the class by Lemma 2.1 and according to Lemmas 2.10 and 2.11 the matroid has the same bases as that induced by μ . //

LEMMA 2.13 *If* $l, h: E \to Z$ *are standardised functions obtained from* $\mu, \nu: E \to Z$ *respectively, and* μ, ν *induce the same matroid* $M(E) \in \mathcal{M}$ *, then* $l = h$.

Proof: Let E be μ-ordered and let a_i be the first element of E for which $l(a_i) \neq h(a_i)$. Suppose $l(a_i) > h(a_i)$ and $l(a_i) = k$. Then single representatives from each of the l-levels $1, \ldots, k-1$, with a_i, constitute an independent set in the matroid induced by μ , but a dependent set in the matroid induced by ν . A similar result follows if $h(a_i) > l(a_i)$. Since both matroids are the same we conclude $l = h$. //

In summary, all functions $\mu: E \to Z$ which induce the same matroid $M \in \mathcal{M}$ effect the same standardised function $l: E \to Z$, and l induces the matroid M also.

We move now to another function obtained from μ. It is apparent that any value of $\mu(a)$ in excess of $\rho(M)$ has no effect on the structure of the matroid. Therefore we *normalise* μ in the following way:

$$\hat{\mu}(a) \begin{cases} = \mu(a), & \text{if } \mu(a) < \rho(M) \\ = \rho(M), & \text{if } \mu(a) \geq \rho(M) \end{cases}.$$

We also define a further function $\mu': E \to Z$, which is obtained from μ, as follows:

$$\mu'(a) \begin{cases} = \max\{\mu(b): b \in C\}, \text{ where } C \text{ is a circuit of maximum cardinality containing } a, \\ = \rho(M), \text{ if } a \text{ is not in any circuit.} \end{cases}$$

<u>LEMMA 2.14</u> *The matroid induced by* μ' *is precisely the matroid induced by* μ.

<u>Proof</u>: Let I be independent in $M(E)$, the matroid induced by μ. Suppose there exists $a \in I$ such that a is not a member of any circuit; then

$$f'(I) = \max\{\mu'(x): x \in I\} = \rho(M) \geq |I| .$$

For those subsets $J \subseteq I$ whose elements are all members of some circuit, $f(J) \geq |J|$ implies that some $b \in J$ is in a circuit of cardinality at least $|J| + 1$, whence $\mu'(b) \geq |J|$ and $f'(J) \geq |J|$. Therefore $f'(J) \geq |J|$ for all $J \subseteq I$ and I is independent in the matroid induced by μ'.

Conversely let I be independent in the matroid induced by μ' and suppose that for some $J \subseteq I$, $f(J) < |J|$. If J is not minimal with respect to this property we choose J_1 which is. Then J_1 is a circuit of $M(E)$, which implies that for all $b \in J_1$, $\mu'(b) < |J_1|$, whence I is not independent in the matroid induced by μ' and we have a contradiction. Therefore there cannot be any $J \subseteq I$ for which $f(J) < |J|$ and so I is independent in $M(E)$. //

LEMMA 2.15 *If* μ, ν *induce the same matroid* $M \in \mathcal{M}$, *then* $\mu' = \nu'$. //

It has therefore been established that for a matroid of class $\mathcal{M}$, both 1 and μ' are unique. The following lemma shows that they are lower and upper bounds of all the normalised functions which induce the same matroid, i.e. they are unique lower and upper bounds.

LEMMA 2.16. $1 \leq \hat{\mu} \leq \mu'$.

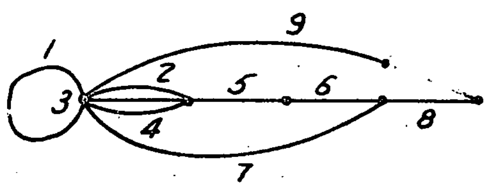
Proof: The first part is obvious from the definitions of 1 and $\hat{\mu}$. For the second, $\hat{\mu}(a) = \mu'(a)$ for all a such that $\mu(a) \geq \rho(M)$ and $\hat{\mu}(a) = \mu(a) \leq \mu'(a)$ for all a such that $\mu(a) < \rho(M)$. //

The uniqueness of μ' may be expressed in terms of the automorphisms of the matroid. We define an *automorphism* of the matroid M to be a bijection $\theta: E \rightarrow E$ such that I is independent in M if and only if θI is independent in M. (Here θI means $\{\theta(a): a \in I\}$.)

LEMMA 2.17 *The automorphisms of a matroid* $M(E) \in \mathcal{M}$ *are precisely the* μ'*-preserving permutations of* E.

Proof: This follows immediately from the uniqueness of μ'. //

We return now to a study of the standardised functions 1. A graph of 1 against the elements of the μ-ordered set E is revealing because it pictorially conveys information about the structure of the induced matroid. Bases, circuits and closures are more easily discerned. An example of a graphical matroid which belongs to the class $\mathcal{M}$ is depicted below.



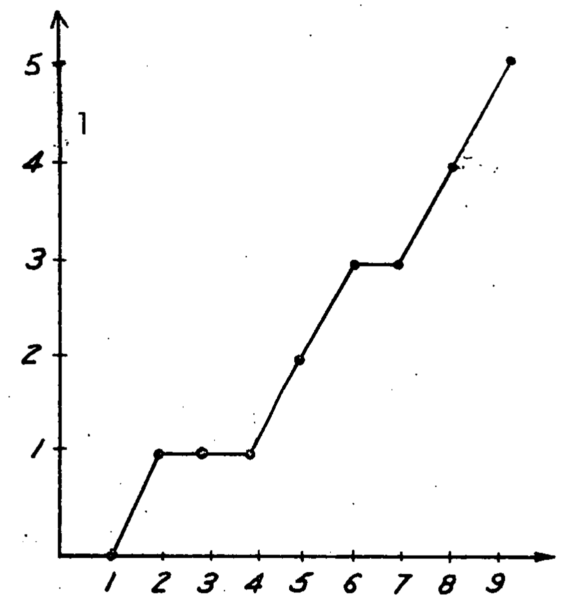


Fig. 1. Graph of function l for a graphical matroid.

Another way in which matroids may be characterised is by means of cocircuits, and we use this characterisation to move towards duals of matroids of class $\mathbb{M}$. A cocircuit is the set complement of a hyperplane and therefore the cocircuit can be described in terms of a basis and a single element of that basis. If B_i is a basis and $a_{ij} \in B_i$ then we denote the associated cocircuit as D_{ij}. This description need not be unique, but every cocircuit can be so described.

<u>LEMMA 2.18</u> *If* $B_i = \{a_{i1}, \ldots, a_{i\rho}\}$ *is a* μ*-ordered basis of the matroid* $M(E) \in \mathbb{M}$, *and* $a_{ij} \in B_i$ *then the cocircuit* D_{ij} *is given by*

$$D_{ij} = (\{a: l(a) > m(k < j: \mu(a_{ik}) = k)\}\backslash B_i) \cup \{a_{ij}\}$$

where

$$\begin{aligned} m(k < j: \mu(a_{ik}) = k) &= \max(k < j: \mu(a_{ik}) = k) \text{ if } k \text{ exists,} \\ &= 0 \text{ if no such } k \text{ exists.} \end{aligned}$$

<u>Proof</u>: The hyperplane obtained from B_i and a_{ij} falls into one of two classes, namely (i) those for which there exists $k < j$ such

that $\mu(a_{ik}) = k$, and (ii) those for which $\mu(a_{ik}) > k$ for all $1 \le k \le j$. For (i), Lemmas 2.4 and 2.10 establish that

$$\sigma(B_i\backslash\{a_{ij}\}) = (B_i\backslash\{a_{ij}\}) \cup \{a: l(a) \le \max(k < j: \mu(a_{ik}) = k\} \quad ,$$

and the complementary cocircuit is as required.

For (ii), $\sigma(B_i\backslash\{a_{ij}\}) = B_i\backslash\{a_{ij}\}$ and therefore $D_{ij} = (E\backslash B_i) \cup \{a_{ij}\}$, which can be rewritten $D_{ij} = (\{a: l(a) > 0\}\backslash B_i) \cup \{a_{ij}\}$. //

We now introduce another function derived from l , which will be necessary in obtaining the dual matroid. We define $l^*: E \to Z$, where $|E| = n$ and ρ is the rank of the matroid induced by l , as follows:

$$l^*(a_i) = l(a_{i-1}) + n - i + 1 - \rho$$

and

$$l^*(a_1) = n - \rho$$

The following results are necessary in establishing duality.

<u>LEMMA 2.19</u> (i) $l^*(a_i) = l^*(a_{i+1})$ *if* $l(a_i) = l(a_{i-1}) + 1$ *and* $l^*(a_i) = l^*(a_{i+1}) + 1$ *if* $l(a_i) = l(a_{i-1})$.

(ii) $i > j \Rightarrow l^*(a_i) \le l^*(a_j)$ *and* $i > j \Leftarrow l^*(a_i) < l^*(a_j)$. //

It is obvious that l^* induces a matroid of class $\mathbb{M}$ on E, and we denote this matroid by M_1. By the reasoning of Lemma 2.8, $\rho(M_1) = l^*(a_1) = n - \rho$.

<u>LEMMA 2.20</u> $l^{**} = l$.

<u>Proof</u>: We l^*-order the elements of E by reversing the μ-order. This is consistent with Lemma 2.19. Then for any $a_i \in E$ (i being the position in the μ-ordering), we have

$$1^{**}(a_i) = 1^*(a_{i+1}) + n - (n-i+1) + 1 - (n-\rho)$$
$$= 1^*(a_{i+1}) + i - n + \rho,$$

and by substituting for $1^*(a_{i+1})$ we complete the proof. //

LEMMA 2.21 *If* $M_1(E)$ *is the matroid induced by* 1^* *, then the levels of the elements of* E *in* $M_1(E)$ *are the values of* 1^* *on the elements.*

Proof: Let the standardised function obtained from 1^* be L, and let the 1^*-ordering be the reverse of the μ-ordering. Then for all $a_i \in E$, $L(a_i) = \min(1^*(a_i), 1^*(a_{i+1}) + 1) = 1^*(a_i)$ by Lemma 2.19. //

We come now to the most important result of this chapter, namely that the class M is closed under taking duals. We use the fact that one matroid is the dual of the other if and only if the circuits of one are precisely the cocircuits of the other.

THEOREM 2.22 $M^* = M_1$.

Proof: We can assume, without loss of generality, that E is μ-ordered. We refer throughout to levels in M and M_1 and to avoid confusion we call them 1-levels and 1^*-levels respectively.

Lemma 2.19 implies that the 1^*-levels have the following structure. Elements on a particular 1-level in M occupy, in reverse μ-order, successive 1^*-levels, except for the first element in that 1-level, which occupies the same 1^*-level as the second element in the

1-level. All elements in successive single element non-zero 1-levels occupy the same 1*-level, and this 1*-level is that of the first element of the multi element 1-level immediately greater than them. If there is no multi element 1-level greater than the abovementioned single element 1-levels, then $1(a_n) = \rho$ and $1(a_{n-1}) = \rho - 1$, whence $1^*(a_n) = 0$ and the elements of all those single element 1-levels occupy 1*-level 0.

Consider a cocircuit D_{ij} determined by the basis $B_i = \{a_{i1}, \ldots, a_{i\rho}\}$ and the element a_{ij}. Let $m(k < j: \mu(a_{ik}) = k) = h$ and let m be such that $\mu(a_m) = h$ and $\mu(a_{m+1}) > h$. Then

$$D_{ij} = (\{a_{m+1}, \ldots, a_n\} \backslash \{a_{i(h+1)}, \ldots, a_{i\rho}\}) \cup \{a_{ij}\} ,$$

and it has $n + h + 1 - m - r$ elements. Further, the maximum value of 1^* on $\{a_{m+1}, \ldots, a_n\}$ is $1^*(a_{m+1}) = h + n - m - r$, and also by the reasoning above on the 1*-levels, $1^*(a_{ij}) = h + n - m - r$. All we now require for D_{ij} to be a circuit of M_1 , is for the value of 1^* on the $n + h - m - r$ elements of $D_{ij} \backslash a_{ij}$ arranged in reverse μ-order to be at least $1, 2, \ldots, n+h-m-r$ respectively.

It is obvious that representatives of each of the 1*- levels $1, 2, \ldots, n+h-m-r$ have the required property. If therefore a_{ik} $(k \neq j)$ is the lone, first or second element of the k-th 1-level its removal from $\{a_m, \ldots, a_n\}$ still leaves a representative of its 1*-level. If $1(a_{ik}) = k$ but a_{ik} is not the lone, first or second element of the k-th 1-level then its removal from $\{a_{m+1}, \ldots, a_n\}$ also removes an 1*-level, but this is compensated for by the first or second element of the k-th 1-level, on which the value of 1^* is higher than on a_{ik}. Finally for each a_{ik} such that $l(a_{ik}) > k$, there exists a lesser l-level (but greater than h) which does not have a representative in B_i , and therefore on the lone, first or second element of that l-level the value of l^* is at least $l^*(a_{ik})$, and therefore compensates for the removal of a_{ik}. This establishes that the removal of a_{ik}, $k \neq j$, $h+1 \leq k \leq \rho$ from $\{a_{m+1}, \ldots, a_n\}$ leaves $n+h+1-m-r$ elements which constitute a circuit in $M_1(E)$.

We now show that every circuit in $M_1(E)$ is a cocircuit in $M(E)$. Let $L_i = \{a \in E: l(a) = i\}$ for all $1 \leq i \leq \rho$ and $|L_i| = n_i$. Further, let C* be a circuit in M_1 and let L_p be the l-level containing the first member of C* , i.e. the element having the lowest subscript. Since the value of l^* on the first and second elements of C* is the same, it follows that the first element is the first element in the l-level p or is a single element l-level. The rank of C* in M_1 is then $\sum_{i=p}^{\rho} n_i - (\rho-p+1)$ and the number of elements in C* is $\sum_{i=p}^{\rho} n_i - (\rho-p)$.

Let the h-th l-level be the greatest multi element l-level less than p (we take $h = 0$ if no such level exists). Then the number of elements of l-level greater than h is $\sum_{i=h+1}^{\rho} n_i$ and the number of those not in C* is $\sum_{i=h+1}^{\rho} n_i - \sum_{i=p}^{\rho} n_i + \rho - p$, which equals $\rho - h - 1$. C* is a cocircuit if the above $\rho - h - 1$ elements, together with an element of C* , belong to a basis of M. This element of C* must not have a multi element l-level between its l-level and l-level h.

Let the g-th l-level be the lowest l-level greater than h which has more than one element. If no such l-level exists then $l^*(a) = 0$ for all a such that $l(a) > h$ and all such a are single element circuits of M_1 . It is obvious that they are single element cocircuits of M , and for this case the proof is complete.

If such an 1-level does exist we consider the 1-levels $q+1, \ldots, \rho$. Suppose that for any j , $q+1 \le j \le \rho$, less than $\rho - j + 1$ elements of $L_j \cup \ldots \cup L_\rho$ are excluded from C^* , i.e. at least $\sum_{i=j}^{\rho} n_i - (\rho-j)$ members of C^* are in $L_j \cup \ldots \cup L_\rho$. Since the maximum value of 1^* on this union is $\sum_{i=j}^{\rho} n_i - (\rho-j+1)$, this implies that C^* properly contains another circuit of M^* , which is impossible, so we conclude that at least $\rho - j + 1$ elements of $L_j \cup \ldots \cup L_\rho$ are excluded from C^* for $q + 1 \le j \le \rho$.

We now have that the number of elements of C^* is $n_q + (n_{q+1}-1) + \ldots + (n_\rho-1)$. Further, 1-levels q to ρ inclusive contain an independent set in M , disjoint with C^* , of size at least $\rho-q$, and there are $q-h-1$ single element 1-levels between 1-level h and 1-level q. There are three possibilities for the composition of C^* , namely:

(i) C^* contains all of 1-level q and none of the elements from the single element 1-levels between h and q.

(ii) C^* contains all of 1-level q and some of the elements from the single element 1-levels between h and q.

(iii) C^* does not contain all of the elements from 1-level q, which implies that it must contain some elements from the single element 1-levels between h and q.

If possibility (i) applies then the $\rho-q$ elements of 1-levels $q+1$ to ρ inclusive which were omitted from C^* , together with the $q-h-1$ elements between 1-levels h and q, and any element from 1-level 1, form part of a basis, and hence C^* is a cocircuit of M.

If possibility (ii) applies then ρ-q elements of l-levels q+1 to ρ inclusive are not in C* and for every element of l-levels between h and q which is in C* there is an additional element from l-levels q+1 to r inclusive not in C*. These elements not in C*, together with the elements between l-levels h and q not in C* , and any element from l-level q , form part of a basis and hence C* is a cocircuit of M.

Finally if possibility (iii) applied then ρ-q+1 elements of l-levels q to ρ inclusive are not in C* , and for each element of l-levels between h and q which is in C* there is an element from l-levels q to ρ inclusive not in C* . These elements not in C* , and one element between l-levels h and q which is in C* , form part of a basis and hence C* is a cocircuit of M. //

It is informative to look at the graphs of l and l* and the figure below is an example.

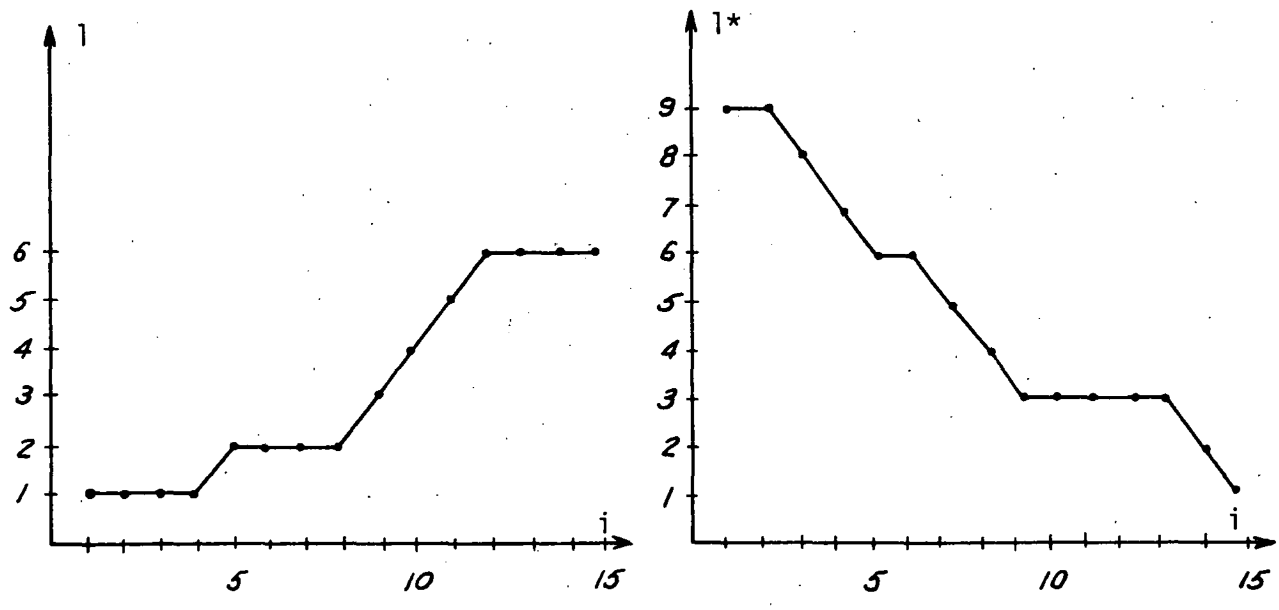


Fig. 2. Graphs of l and l* on a 15 element set.

It will be noted that there is a relationship between the gradients of the two graphs. For example the gradient of the function l for $1 \le i \le 4$ is 0, whereas that for l^* for $2 \le i \le 5$ is -1, and the gradient of l for $8 \le i \le 12$ is 1 whereas that for l^* for $9 \le i \le 13$ is 0. Inspection of the relationship between l and l^* shows that this is general, i.e.

$l'(a_j) = 1$ for $i \le j \le k \Rightarrow l^{*\prime}(a_j) = 0$ for $i+1 \le j \le k+1$

and $l'(a_j) = 0$ for $i \le j \le k \Rightarrow l^{*\prime}(a_j) = -1$ for $i+1 \le j \le k+1$.

Another way of viewing the above is to represent the set as in the figure below.

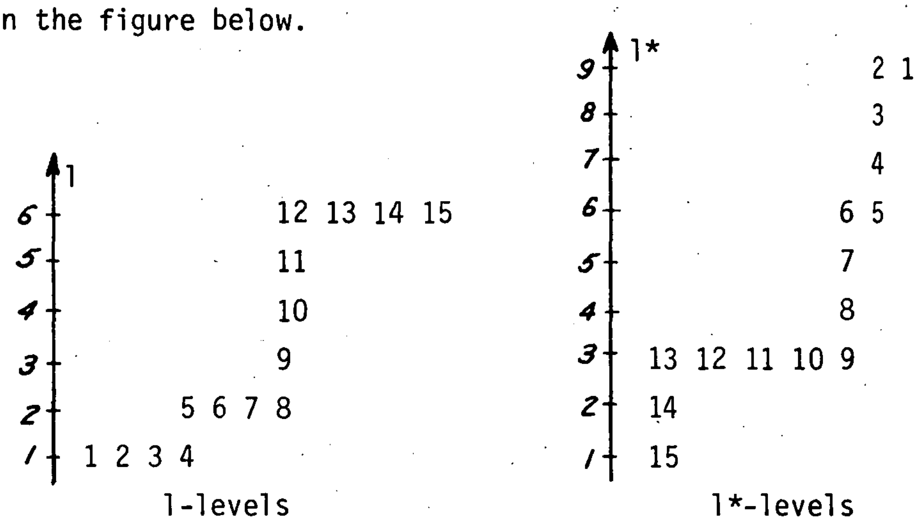


Fig. 3. Levels of l and l^*

In the above figure rows can be regarded as comprising elements for which there is no increase in level over the preceding element, while columns comprise those for which there is an increase in level over the preceding element. With this classification rows in l representation are columns in l^* representation and vice versa. Again, because of the relationship between l and l^*, this result is general.

We conclude this chapter with a lemma concerning restriction.

LEMMA 2.23. $M_\mu|T = M_{\mu|T}$ *, where the subscripts refer to the inducing functions of the matroid of class* $\mathbb{M}$.

Proof: We define $f_{|T}(A) = \max\{\mu_{|T}(a) : a \in A\}$ for all $A \subseteq T$. It is immediately obvious that on all $A \subseteq T$, $f_{|T} = f$. It follows then that for any $I \subseteq T$ which is independent in M , $f_{|T}(J) \geq |J|$ for all $J \subseteq I$, whence I independent in $M_\mu|T \Rightarrow I$ independent in $M_{\mu|T}$.

Conversely if I is independent in $M_{\mu|T}$ then $f(J) = f_{|T}(J) \geq |J|$ for all $J \subseteq I$ and also $I \subseteq T$, whence I is independent in $M_\mu|T$. //

CHAPTER 3

Chapter 2 was concerned with the "arithmetic" of matroids of $\mathbb{M}$. We now establish that a characterisation in more general terms is available. Firstly we show that $\mathbb{M}$ consists exactly of matroids, all of whose minors are free or have unique minimal non-trivial flats. Secondly we give an excluded minor characterisation of $\mathbb{M}$. Again in this chapter E is finite. The term flat rather than closure in used so that we can conveniently speak of it without reference to the sets of which it is the closure. A flat F of M is *non-trivial* if it is the closure of a proper subset. It is a *non-trivial extension* of a flat H if it is the closure of $H \cup P$ for some proper subset P of $F \setminus H$. Otherwise F is a *free extension* of H.

Consider a matroid each of whose minors is either free or has a unique minimal non-trivial flat. We denote the class of matroids having this property by $\mathbb{M}'$.

<u>LEMMA 3.1</u> *Each* $M' \in \mathbb{M}'$ *on a ground set* E, *has a finite chain* $\sigma(\phi) = F_0 \subset F_1 \ldots \subset F_k \subseteq E$, *where* F_{i+1} *is the unique minimal non-trivial extension of* F_i *for* $0 \le i < k$ *and* F_k *has no non-trivial extension. Each flat in* M' *is a direct sum of some* F_i *and a free matroid.*

<u>Proof</u>. Let $\sigma(\phi) = F_0$ and suppose there exists a chain $F_0 \subset F_1 \subset \ldots F_i$ such that F_{j+1} is the unique minimal non-trivial extension of F_j for $0 \le j < i$. Then either E is a free extension of F_i in which case $k = i$, or there exists a minimal non-trivial extension of F_i. Suppose there exist two such extensions H and H'. Then we consider the minor $M' \circ (E \setminus (H \cap H'))$.

Applying a standard result of matroid theory we have $x \in \sigma_{cont}(H\backslash H')$ $\Leftrightarrow$ $x \in \sigma((H\backslash H') \cup (H \cap H')) = H$. Since the minor is a matroid only on $E\backslash(H \cap H')$ we conclude $x \in \sigma_{cont}(H\backslash H')$ and $x \in E\backslash(H \cap H')$ $\Leftrightarrow$ $x \in H\backslash H'$, whence $H\backslash H'$ is a flat in the contraction.

As H is a non-trivial extension of F_i in M' it contains a circuit C which is not contained in F_i. Furthermore $H \cap H'$ either is F_i or is a free extension of it, whence $C \nsubseteq H \cap H'$, so $H\backslash H'$ contains a circuit in the contraction. Therefore $H\backslash H'$ is a non-trivial flat in the contraction, and by the same reasoning so is $H'\backslash H$. It follows that both contain minimal non-trivial flats which must be disjoint. This is impossible since $M' \in \mathbb{M}'$ and we conclude that there exists F_{i+1} which is a unique minimal non-trivial extension of F_i. By induction we obtain the required chain of flats.

Any flat either (i) is free, or (ii) is an F_i, or (iii) is a free extension of an F_i. Therefore a flat F is the direct sum of F_i, for some $0 \le i \le k$, and the free matroid $M'|(F\backslash F_i)$. //

We prove $\mathbb{M}' \subseteq \mathbb{M}$ by characterising the circuits of members of $\mathbb{M}'$.

<u>LEMMA 3.2</u> *For any* $M' \in \mathbb{M}'$, *having flats as specified in Lemma 3.1, the circuits contained in* F_i *but not in* F_{i-1} *are exactly* C *satisfying* $|C| = \rho(F_i) + 1$, $|C \cap F_j| \le \rho(F_j)$ *for* $j \le i$. *These, for all* i, *are the circuits of* M'.

<u>Proof</u>: We proceed by induction. Either $F_0 = \phi$ whence the circuits in F_1 have the required properties, or each element of F_0 is a loop C satisfying $|C| = 1 = \rho(F_0) + 1$. Now suppose the circuits contained in F_j but not in F_{j-1} are as prescribed for all $j < i$.

If C is a circuit contained in F_i, $C \not\subseteq F_{i-1}$, then $\sigma(C)$ is a flat which by Lemma 3.1 is F_j, some j and obviously $j = i$; therefore $|C| = \rho(F_i) + 1$. For $j < i$, $C \cap F_i \neq C \Rightarrow C \cap F_j$ is independent and $|C \cap F_j| = \rho(C \cap F_j) \leq \rho(F_j)$.

Conversely let C satisfy $C \subseteq F_i$, $C \not\subseteq F_{i-1}$, $|C| = \rho(F_i) + 1$, $|C \cap F_j| \leq \rho(F_j)$ for all $j < i$. From this prescription C is dependent and so contains a circuit C' . If $C' \subseteq F_j$ for some $j < i$, $|C'| = |C' \cap F_j| \leq |C \cap F_j| \leq \rho(F_j)$, which implies that $|C'| \neq \rho(F_j) + 1$, contradicting the proven property of any such circuit. Hence $C' \subseteq F_i$, $C' \not\subseteq F_{i-1}$, so $|C'| = \rho(F_i) + 1 = |C|$, and $C = C'$.

We have inductively characterised all circuits contained in some F_i. But every flat is the direct sum of some F_i and a free matroid, hence all circuits have been characterised. //

<u>LEMMA 3.3</u> $\mathcal{M}' \subseteq \mathcal{M}$.

<u>Proof</u>: Consider any $M' \in \mathcal{M}'$ with a chain of non-trivial extensions as specified in Lemma 3.1. We define an appropriate function on the ground set E of M' as follows:

$$\mu(e) = \begin{cases} \rho(F_i), \text{ if } e \in F_i \backslash F_{i-1}, \text{ with } F_{-1} = \phi \\ \rho(E), \text{ if } e \in E \backslash F_k . \end{cases}$$

The function μ induces a matroid $M_\mu \in \mathcal{M}$ and we prove $M_\mu = M'$ by considering the circuits in both.

If C is a circuit in M' then for some i, $C \subseteq F_i$, $C \not\subseteq F_{i-1}$, $|C| = \rho(F_i)+1$ and $|C \cap F_j| \leq \rho(F_j)$ for all $j < i$. Let $C = \{c_1, \ldots, c_s\}$ where $\mu(c_1) \leq \ldots \leq \mu(c_s) = \rho(F_i) = |C| - 1 = s - 1$.

For all $r < s$, either $c_r \in F_i \backslash F_{i-1}$, or $c_r \in F_j \backslash F_{j-1}$ for some $j < i$. In the first case, $\mu(c_r) = \rho(F_i) = s - 1$, and in the second case $\mu(c_r) = \rho(F_j) \geq |C \cap F_j| \geq r$. We conclude that $s - 1 \geq \mu(c_r) \geq r$ for $1 \leq i \leq s-1$, and $\mu(c_s) = s - 1$, and so by Lemma 2.3 C is a circuit in M_μ.

Conversely if C is a circuit in M_μ, $s-1 \geq \mu(c_r) \geq \min\{r,s-1\}$ for $1 \leq r \leq s = |C|$ and so $\mu(c_s) = s-1 = \rho(F_i)$, say. Then for all $j < i$, $C \cap F_j = \{c_r: \mu(c_r) \leq \rho(F_j)\} \subseteq \{c_r: r \leq \rho(F_j)\}$, giving $|C \cap F_j| \leq \rho(F_j)$. But $s = |C| = \rho(F_i) + 1$. Hence C is a circuit in M'. //

To prove $\mathcal{M} \subseteq \mathcal{M}'$ it suffices if we prove that $M \in \mathcal{M}$ is either free or has a unique minimal non-trivial flat, and that $\mathcal{M}$ is closed with respect to taking minors.

<u>LEMMA 3.4</u> *Each* $M \in \mathcal{M}$ *is a free matroid or has a unique minimal non-trivial flat.*

<u>Proof</u>: Let F and F' be minimal non-trivial flats in M with $\rho(F) \leq \rho(F')$. Lemma 2.4 implies that $\mu(a) \leq \rho(F)$ for all $a \in F$ and $\mu(a) \leq \rho(F')$ for all $a \in F'$. It follows also from Lemma 2.4 that $F \subseteq F'$ and since both are minimal, $F = F'$. //

We know from Lemma 2.23 that $\mathcal{M}$ is closed with respect to restrictions and it remains to show that the same applies for contractions.

<u>LEMMA 3.5</u> *Any contraction of a member of* $\mathcal{M}$ *is in* $\mathcal{M}$.

<u>Proof</u>: If $M(E) \in \mathcal{M}$, then for any $T \subseteq E$, $M \circ T = (M^*|T)^*$ and we know from Theorem 2.22 that $M^* \in \mathcal{M}$. //

THEOREM 3.6 $\mathcal{M} = \mathcal{M}'$. //

We move on now to the second part of this chapter, namely the excluded minor characterisation. We characterise $\mathcal{M}'$ and hence $\mathcal{M}$, by its excluded minors. For k = 2, 3, ... consider a set E, $|E| = 2k$, $E = E_1 \dot{\cup} E_2$ with $|E_1| = |E_2| = k$ and put

$\mathcal{C} = \{E_1, E_2\} \cup \{C: C \not\supset E_1, C \not\supset E_2, C \subset E, |C| = k+1\}$

LEMMA 3.7 *$\mathcal{C}$ is the collection of circuits of a matroid ^kM with underlying set E, for each $k = 2, 3, \ldots$.*

Proof: Consider any two distinct members C_1, C_2 of $\mathcal{C}$ with a common element e. Then $|(C_1 \cup C_2)\backslash e| \geq k+1$ and so $(C_1 \cup C_2)\backslash e$ contains a member of $\mathcal{C}$. //

LEMMA 3.8 $^kM \notin \mathcal{M}'$.

Proof: Both E_1 and E_2 are minimal non-trivial flats. //

THEOREM 3.9 *$\mathcal{M}'$ is characterised by the family ^kM , $k = 2, 3, \ldots$, of excluded minors.*

Proof: We consider any matroid which is not in $\mathcal{M}'$; it has at least one minor which has two minimal non-trivial flats. We choose a minor M which satisfies this condition but whose own proper minors are in $\mathcal{M}'$. This is possible since, if not, the matroid has no minor which has a unique minimal non-trivial flat or is free, and minors of rank 1 obviously have this property.

The chosen minor has two minimal non-trivial flats, say E_1 and E_2, of minimal rank. If $E \neq E_1 \cup E_2$ we choose $e \in E\backslash(E_1 \cup E_2)$ and obtain the restriction $M|E\backslash e$. Since I independent in $M \Leftrightarrow$ I independent in $M|E\backslash e$ for $I \subset E_i$, $i = 1$ or 2 , it follows that E_1 and E_2 are minimal non-trivial flats in $M|E\backslash e$. But this is a contradiction of our choice of minor. Thus $E = E_1 \cup E_2$.

We now prove that E_1 and E_2 are circuits of M. Suppose E_1 is not a circuit; then M has a circuit C which is properly contained in E_1 and we consider the contraction $M \circ (E\backslash e)$ where $e \in E_1\backslash C$. This contraction has non-trivial flats $E_1\backslash e$ and E_2 (or $E_2\backslash e$ if $e \in E_2$) and these are minimal, which is a contradiction. Therefore E_1 is a circuit, and similarly E_2 is a circuit.

This paragraph shows that E_1 and E_2 are disjoint. We assume to the contrary that $e \in E_1 \cap E_2$, and consider the contraction $M\circ(E\backslash e)$. In this $E_1\backslash e$ and $E_2\backslash e$ are both circuits and flats, and hence minimal non-trivial flats. Therefore $E_1\backslash e = E_2\backslash e$, whence $E_1 = E_2$, which contradicts our choice of M. Hence E_1 and E_2 are disjoint.

We show that $|E_1| = |E_2|$. Choose any element in E , say $e \in E_2$, and form the minor $M\circ(E\backslash e)$. In this contraction $E_2\backslash e$ is a circuit and a flat, and hence a minimal non-trivial flat. Also $\sigma_{cont}(E_1) = \sigma(E_1 \cup e)\backslash e$ is a non-trivial flat, and by the choice of M , cannot be minimal. Therefore $\rho_{cont}(E_2\backslash e) < \rho_{cont}(E_1)$, whence $\rho(E_2)-1 < \rho(E_1 \cup e)-1 = \rho(E_1)$, since $e \notin E_1 = \sigma(E_1)$, and it follows that $\rho(E_2) \leq \rho(E_1)$. Choice of any $a \in E_1$ similarly leads to $\rho(E_1) \leq \rho(E_2)$, and we conclude that $|E_1| = |E_2| = k$, say, for some $k > 1$.

It only remains to prove that the circuits other than E_1 and E_2 in M are exactly the subsets of E of size $k+1$ which contain neither E_1 nor E_2 . Since E_1 and E_2 are minimal non-trivial flats of M it follows that all circuits have at least k elements. Suppose C is a third circuit of M and $|C| = k$; then $C \cap E_1 \neq \emptyset$

and $C \cap E_2 \neq \emptyset$. The flat $\sigma(C)$ is non-trivial with rank k-1, and it has a subset F which is a minimal non-trivial flat. Considering the minimal non-trivial flats E_1 and F as above, we have $E = E_1 \dot{\cup} F$, F is a circuit and $|F| = k$. Therefore $F = C$ and $C \cap E_1 = \emptyset$ which contradicts the necessary properties of C, and so $|C| \geq k+1$. We need only to show that $\rho(M) = k$ to prove that all circuits other than E_1 and E_2 have size k+1. Choosing $e \in E_2$ and considering the contraction $M \circ (E \backslash e)$ as above, we have $E_2 \backslash e \subset \sigma_{cont}(E_1) = \sigma(E_1 \cup e) \backslash e$, whence $E_2 \subset \sigma(E_1 \cup e)$ and so $E_1 \cup e$ spans M , giving $\rho(M) = k$. Consequently $M = {}^kM$, for some $k > 1$. //

CHAPTER 4

The numbers of simple matroids on ground sets of small sizes are well known [2], and using this information it is easy to find the numbers of matroids on those sets. It is natural to enquire how many of these belong to the class $\mathcal{M}$. In this chapter we list all matroids on sets up to size 6, and by making use of the excluded minor property we identify those which are not in $\mathcal{M}$.

It is necessary first of all to establish a method of counting matroids on small sets. The following definition and lemmas are to that end.

For any $T \subset E$, the restriction $M|T$ of a matroid M on a ground set E is a *simple matroid associated with* M , or a *canonical matroid* of M if

$$T \cap \sigma(\phi) = \phi \ ,|T \cap \sigma(a)| = 1 \text{ for all } a \in E\backslash\sigma(\phi).$$

<u>LEMMA 4.1</u> $M|T$ *is a simple matroid.*

<u>Proof</u>: $\sigma_{rest.}(\phi) = \sigma(\phi) \cap T = \phi$ and $\sigma_{rest.}(a) = \sigma(a) \cap T = a$, since $|T \cap \sigma(a)| = 1$. //

<u>LEMMA 4.2</u> *All simple matroids associated with* M *are isomorphic, and maximal simple restrictions of* M . *Any restriction of* M *isomorphic to these simple matroids associated with* M *is itself associated with* M.

<u>Proof</u>: Let $M|T$ and $M|T'$ be simple matroids associated with M. Then there exist bijections $\alpha:\sigma(a) \to a \in T$, $\sigma(\phi) \to \phi$ and $\beta:\sigma(a) \to a \in T'$, $\sigma(\phi) \to \phi$, whence there also exists a bijection $\beta\alpha^{-1} = \theta:T \cup \sigma(\phi) \to T' \cup \sigma(\phi)$.

If $I = \{a_1, \ldots, a_r\}$ is independent in M then $\sigma(I) = \sigma(\bigcup_{i=1}^{r} \sigma(a_i))$. Suppose $I \subseteq T$ and $\theta(I)$ is dependent, i.e. there exists $a \in T'$

such that $a \in \sigma(\theta(I)\backslash a)$. Then $a \in \sigma(\cup\, \sigma(b):b \in \theta(I)\backslash a)$, whence $a \in \sigma(\{\theta^{-1}(b):b \in \theta(I)\backslash a)$, and it follows that $\sigma(a)$ and hence $\theta^{-1}(a)$ is a member of the same closure. This contradicts the independence of I, and we conclude that $\theta(I)$ is independent. Therefore θ is an isomorphism from $M|T$ to $M|T'$.

For $a \notin T$, $a \in \sigma(b)$ for some $b \in T$, so $M|T \cup a$ is not simple. Therefore $M|T$ is a maximal simple restriction.

Suppose $M|T' \cong M|T$ and $M|T$ is associated with M. Then obviously T' does not contain two elements, one of which is in the closure of the other, whence $|T' \cap \sigma(a)| = 1$ for all $a \in T'$, and also $T' \cap \sigma(\phi) = \phi$. Therefore $M|T'$ is associated with M. //

<u>LEMMA 4.3</u> *Two matroids M and M' are isomorphic exactly when there is a mapping $\theta:E \to E'$ such that $\theta_{|T}$ is an isomorphism of associated simple matroids $M|T$ and $M'|T'$ and $|\sigma(a)\backslash\sigma(\phi)| = |\sigma'(\theta(a))\backslash\sigma'(\phi)|$ for all $a \in E\backslash\sigma(\phi)$ and $|\sigma(\phi)| = |\sigma'(\phi)|$.*

<u>Proof:</u> M, M' isomorphic implies that there exists $\theta:E \to E'$, whence $\theta_{|T}$ is a bijection of T onto T' and $\theta_{|T}(I)$ is independent in $M'|T'$ for I independent in $M|T$. Also θ being an isomorphism guarantees $|\sigma(a)\backslash\sigma(\phi)| = |\sigma'(\theta(a))\backslash\sigma'(\phi)|$ and $|\sigma(\phi)| = |\sigma'(\phi)|$.

Conversely suppose there exists $\theta:E \to E'$ such that $\theta_{|T}:T \to T'$ is an isomorphism of $M|T$ and $M'|T'$, and $|\sigma(a)\backslash\sigma(\phi)| = |\sigma'(\theta(a))\backslash\sigma'(\phi)|$ and $|\sigma(\phi)| = |\sigma'(\phi)|$. Suppose $\theta(I)$ is dependent in M', while I is independent in M. Then there exists $J \subset I$ such that $\theta(J)$ is independent in M' and $a \in \sigma'(\theta(J))$ for $a \in I\backslash J$. We take $J' \subset T'$ such that J' consists exactly of single representatives of the closures of all elements of $\theta(J)$. Then

$a \in \sigma'(J')$, whence $a \cup \theta^{-1}(J')$ is dependent in M. There exists $K \subset I$, $|K| = |J'| = |J|$, such that K consists exactly of single representatives of closures of all members of $\theta^{-1}(J')$ and $K \cup a$ is dependent. The latter is impossible whence $\theta(I)$ is independent in M' if I is independent in M. The conditions upon the size of the closures of the empty set and of singletons ensure that θ is a bijection and hence an isomorphism. //

Every member of a set of pairwise non-isomorphic matroids on a ground set of size 6 has a canonical simple matroid, and of course a number have the same canonical simple matroid. On the other hand every simple matroid on a ground set of size up to 6 can be extended to a matroid on a ground set of size 6 by the inclusion of additional elements in the closure of the empty set or of one or more of the elements of the simple matroid. Therefore the matroids on a ground set of size 6 partition naturally into classes, each class being the non-isomorphic matroids having the same canonical simple matroid. Lemma 4.2 says that the sameness is only to isomorphism, i.e. the classes are distinguished by having associated pairwise non-isomorphic simple matroids. It is easy to list all the non-isomorphic simple matroids up to size 6. We do this by taking the set $E = \{1,2,3,4,5,6\}$ and listing the simple matroids $M|T$ for some $T = \{r \in E : r < m+1\}$, $m = 0, 1, 2, \ldots, 6$.

Associated with each matroid M having M' as a canonical simple matroid we have the partition $\{E_i : 0 \leq i \leq m\}$, where E_i is the closure of i and E_0 is the closure of the empty set. (E_0 of course may be empty). It follows from Lemma 4.3 that two such matroids M_1, M_2 having partitions $\{E_i^1\}$, $\{E_i^2\}$ are isomorphic exactly when there exists an automorphism θ of M' such that $|E_i^2| = |E_{\theta i}^1|$, for $i = 0, 1, 2, \ldots m$.

Therefore for each M' we count the number of partitions which pairwise do not have this property.

We first list the simple matroids on a set of size at most 6. We know that there are 43, and as all are sub-matroids of ordinary euclidean space we so represent them. Where possible they are also shown as graphs underneath.

TABLE 1

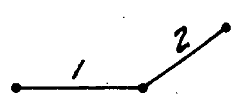
Simple Matroids on $T = \phi$

0. rank 0

Simple Matroids on $T = \{1\}$

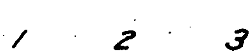
1. rank 1

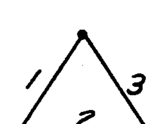
Simple Matroids on $T = \{1,2\}$



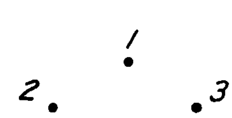
2. rank 2

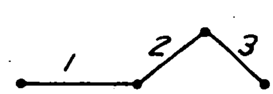
Simple Matroids on $T = \{1,2,3\}$





3. rank 2

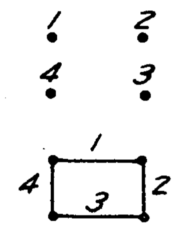




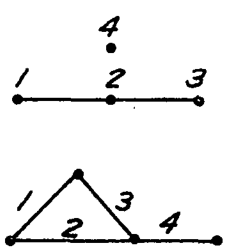
4. rank 3

Simple Matroids on T = {1,2,3,4}

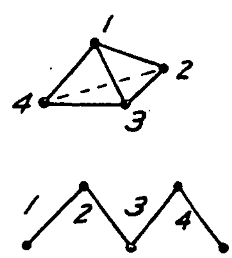
5. rank 2



6. rank 3

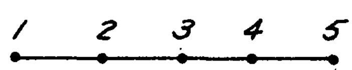


7. rank 3

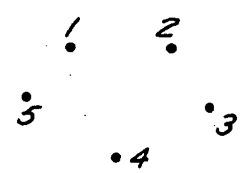


8. rank 4

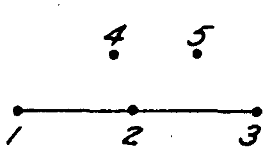
Simple Matroid on T = {1,2,3,4,5}



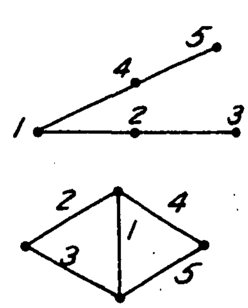
9. rank 2



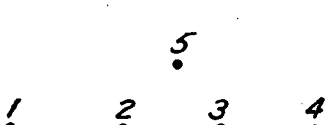
10. rank 3



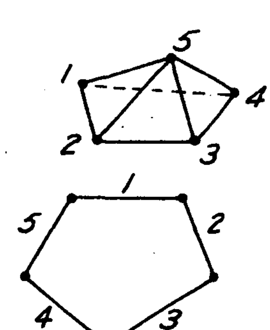
11. rank 3



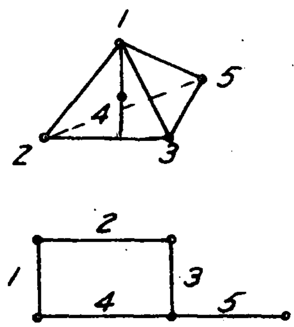
12. rank 3



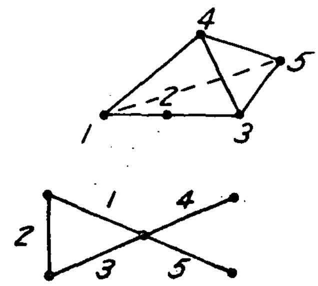
13. rank 3



14. rank 4

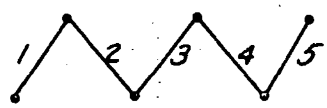


15. rank 4



16. rank 4

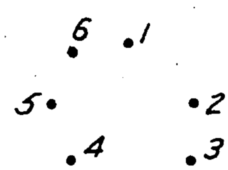
5 points in general position in E^4



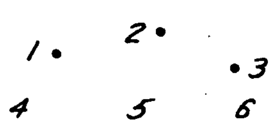
17. rank 5

Simple Matroids on T = {1,2,3,4,5,6}

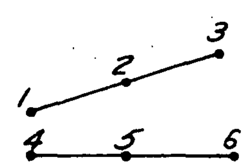
18. rank 2



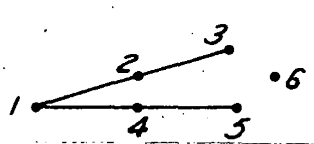
19. rank 3



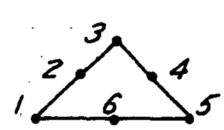
20. rank 3



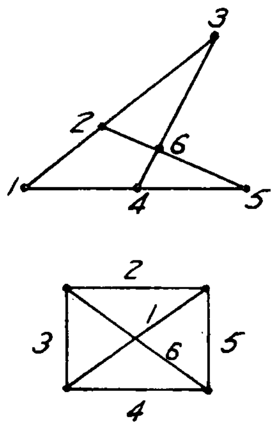
21. rank 3



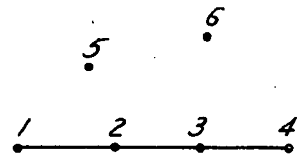
22. rank 3



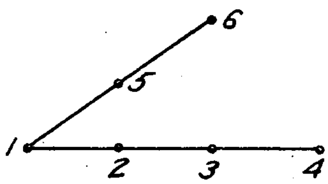
23. rank 3



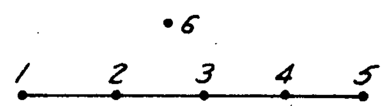
24. rank 3



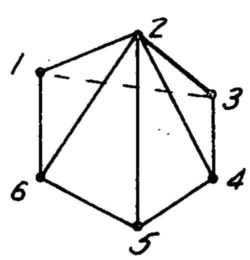
25. rank 3



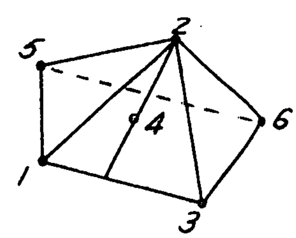
26. rank 3



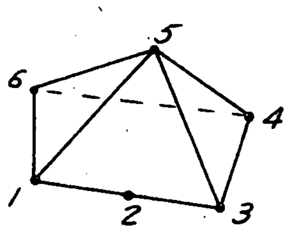
27. rank 3



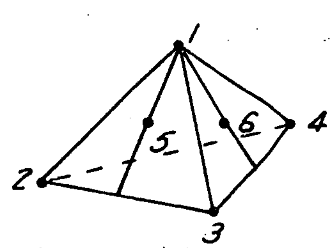
28. rank 4



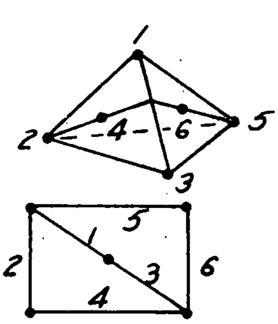
29. rank 4



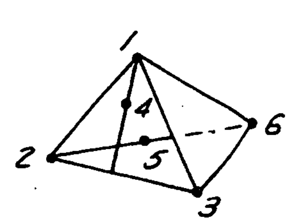
30. rank 4



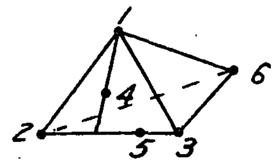
31. rank 4



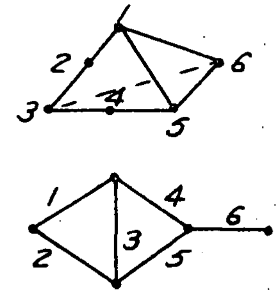
32. rank 4



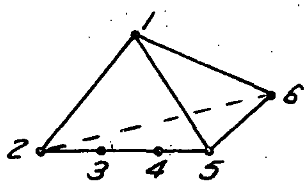
33. rank 4



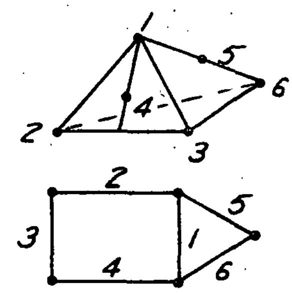
34. rank 4



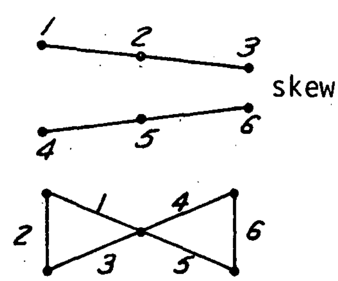
35. rank 4



36. rank 4

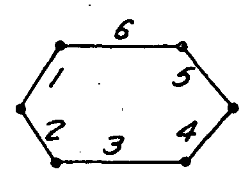


37. rank 4



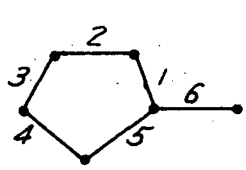
38. rank 4

6 Pts. in general posn. in E^4



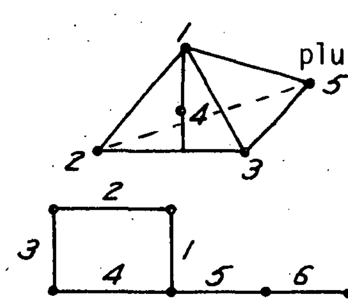
39. rank 5

5 Pts. in general posn. in E^3
plus one pt. in 4th dimension



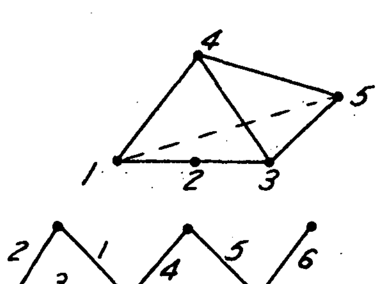
40. rank 5

plus one pt. in 4th dimension



41. rank 5

plus one pt. in 4th dimension



42. rank 5

6 Pts. in general posn. in E^5

43. rank 6

The table below lists all the matroids on a ground set of 6 elements in terms of the simple matroids with which they are associated and the partitions described above. The column $M|T$ lists the simple matroids as numbered above.

TABLE 2

$\|T\|$	$M\|T$ number as in Table 1	$\|E_0\|, \|E_1\|, \ldots, \|E_m\|$	Notation for M	Cumulative Total
0	0	6	0.1	1
1	1	5 1	1.1	
		4 2	1.2	
		3 3	1.3	
		2 4	1.4	
		1 5	1.5	
		0 6	1.6	7
2	2	4 1 1	2.1	
		3 2 1	2.2	
		2 3 1	2.3	
		2 2 2	2.4	
		1 4 1	2.5	
		1 3 2	2.6	
		0 5 1	2.7	
		0 4 2	2.8	
		0 3 3	2.9	16
3	3	3 1 1 1	3.1	
		2 2 1 1	3.2	
		1 3 1 1	3.3	
		1 2 2 1	3.4	
		0 2 2 2	3.5	
		0 3 2 1	3.6	
		0 4 1 1	3.7	
	4	3 1 1 1	3.8	
		2 2 1 1	3.9	
		1 3 1 1	3.10	
		1 2 2 1	3.11	

		0 2 2 2	3.12	
		0 3 2 1	3.13	
		0 4 1 1	3.14	30
4	5	2 1 1 1 1	4.1	
		1 2 1 1 1	4.2	
		0 2 2 1 1	4.3	
		0 3 1 1 1	4.4	
	6	2 1 1 1 1	4.5	
		1 2 1 1 1	4.6	
		0 2 2 1 1	4.7	
		0 3 1 1 1	4.8	
	7	2 1 1 1 1	4.9	
		1 1 1 1 2	4.10	
		1 2 1 1 1	4.11	
		0 2 1 1 2	4.12	
		0 2 2 1 1	4.13	
		0 3 1 1 1	4.14	
		0 1 1 1 3	4.15	
	8	2 1 1 1 1	4.16	
		1 2 1 1 1	4.17	
		0 2 2 1 1	4.18	
		0 3 1 1 1	4.19	49
5	9	1 1 1 1 1 1	5.1	
		0 2 1 1 1 1	5.2	
	10	1 1 1 1 1 1	5.3	
		0 2 1 1 1 1	5.4	
	11	1 1 1 1 1 1	5.5	
		0 2 1 1 1 1	5.6	
		0 1 1 1 2 1	5.7	
	12	1 1 1 1 1 1	5.8	
		0 2 1 1 1 1	5.9	
		0 1 2 1 1 1	5.10	
	13	1 1 1 1 1 1	5.11	
		0 2 1 1 1 1	5.12	
		0 1 1 1 1 2	5.13	
	14	1 1 1 1 1 1	5.14	
		0 2 1 1 1 1	5.15	

	15	1 1 1 1 1 1	5.16	
		0 2 1 1 1 1	5.17	
		0 1 1 1 1 2	5.18	
	16	1 1 1 1 1 1	5.19	
		0 2 1 1 1 1	5.20	
		0 1 1 1 2 1	5.21	
	17	1 1 1 1 1 1	5.22	
		0 2 1 1 1 1	5.23	72
6	18		6.1	
	to	all are	to	
	43	0 1 1 1 1 1 1	6.26	26
				98

<u>LEMMA 4.4</u> *There are exactly* 2^n *pairwise non-isomorphic members of* $\mathcal{M}$ *on a ground set of size* n.

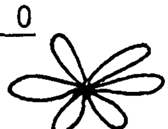
<u>Proof:</u> Without loss of generality we can choose one ordering of the n elements of the ground set E from all the orderings imposed by the various functions μ which induce the matroids of $\mathcal{M}$ on E. Each of the matroids of rank r on E is distinguished by the first elements of E on which the standardised function l takes the values 1, 2, ..., r. There are $\binom{n}{r}$ ways of choosing those elements in the correct order, i.e. there are $\binom{n}{r}$ matroids of rank r on E. Summing from r = 0 to r = n we have that there are 2^n matroids on E. //

It is interesting to note that 2^n is exactly the lower bound given by Crapo [4] for the number of matroids on a set of size n. However a sharper bound, namely $2^{n^2/12}$ for sufficiently large n has subsequently been obtained [3]. The sharper bound shows that $\mathcal{M}$ is a relatively small sub-class of the class of all matroids.

From the above lemma we see that there are 64 matroids on a ground set of size 6 which are in $\mathcal{M}$, and 34 which are not. Those 34 are distinguished by the excluded minor property of the previous chapter. There are two possibilities for the excluded minor ${}^{k}M(E_1 \dot{\cup} E_2)$, namely $k = 2$ and $k = 3$. The minor given by $k = 2$ is a graphical matroid consisting of two rank 1 circuits, i.e. two sets of 2-multiple edges. The $k = 3$ minor is the matroid whose euclidean representation is two non-intersecting three points lines in the same plane.

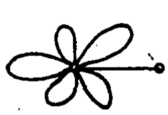
We list all 98 matroids on a set of size 6 and distinguish those which are not members of $\mathcal{M}$ by an asterisk. We use the notation listed in the above table for all of the matroids, and where possible we represent them as graphs.

<u>M(T) = 0</u>

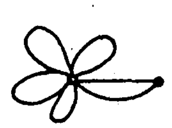


0.1

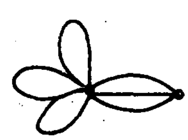
<u>M(T) = 1</u>



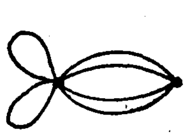
1.1



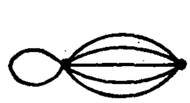
1.2



1.3



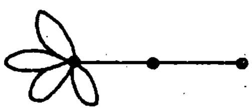
1.4

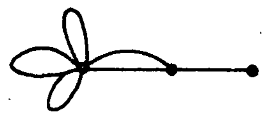


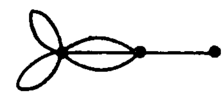
1.5

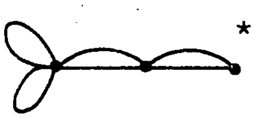
1.6

M(T) = 2

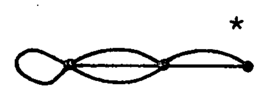

2.1

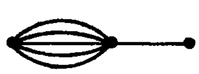

2.2

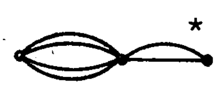

2.3


2.4


2.5

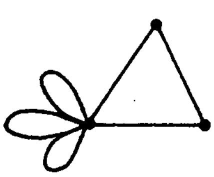

2.6

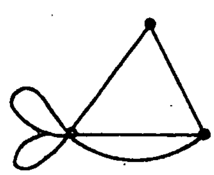

2.7

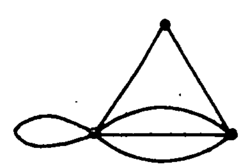

2.8

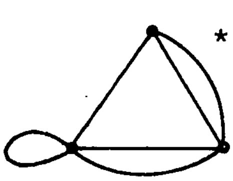

2.9

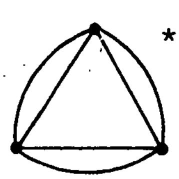
M(T) = 3

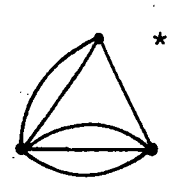

3.1

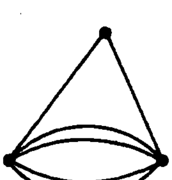

3.2


3.3

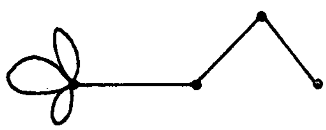

3.4


3.5

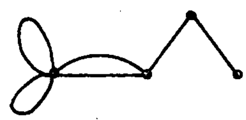

3.6


3.7

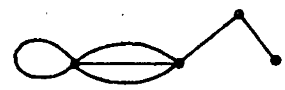
M(T) = 4



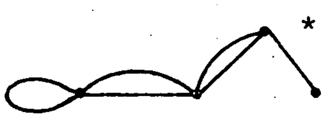
3.8



3.9

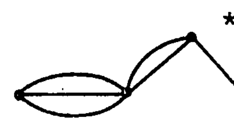


3.10

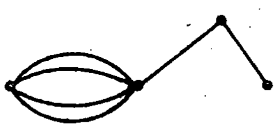


3.11

3.12



3.13



3.14

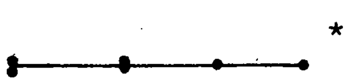
M(T) = 5 These are not graphic, and are represented in euclidean space.

$|\sigma(\phi)| = 2$

4.1

$|\sigma(\phi)| = 1$

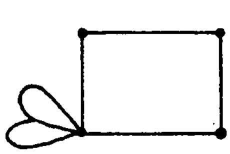
4.2



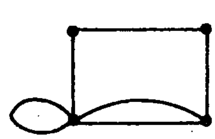
4.3

4.4

M(T) = 6



4.5



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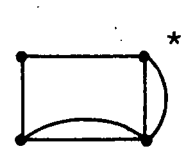
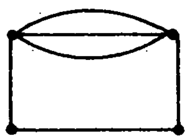
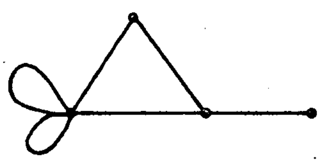


4.7



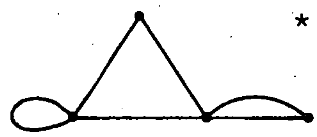
4.8

M(T) = 7

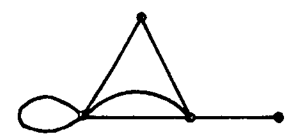


4.9

*

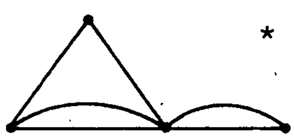


4.10



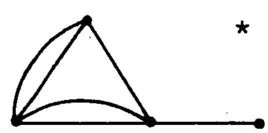
4.11

*

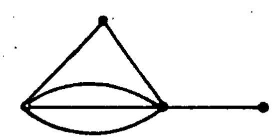


4.12

*



4.13



4.14

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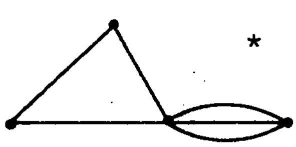
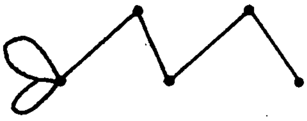


4.15

M(T) = 8

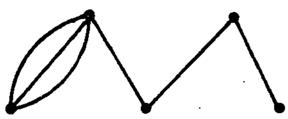


4.16

4.17

*

4.18



4.19

<u>M(T) = 9</u> These are not graphic and are represented in euclidean space.

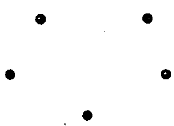
$|\sigma(\phi)| = 1$

5.1

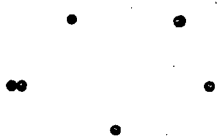
5.2

<u>M(T) = 10</u> These are not graphic and are represented in euclidean space.

$|\sigma(\phi)| = 1$



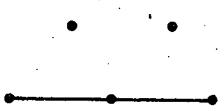
5.3



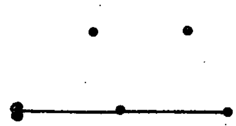
5.4

<u>M(T) = 11</u> These are not graphic and are represented in euclidean space.

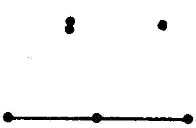
$|\sigma(\phi)| = 1$



5.5



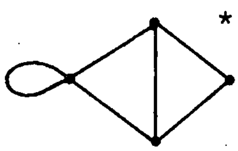
5.6



*

5.7

<u>M(T) = 12</u>



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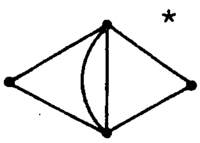
5.8



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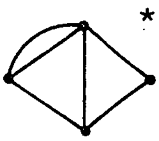
5.9



*

5.10

<u>M(T) = 13</u> These are not graphic and are represented in euclidean space.

$|\sigma(\phi)| = 1$

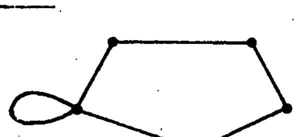
5.11

5.12

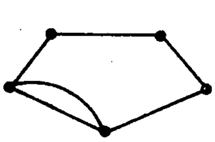
*

5.13

<u>M(T) = 14</u>

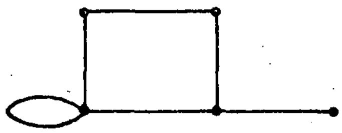


5.14

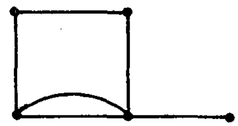


5.15

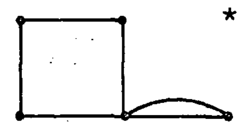
M(T) = 15



5.16

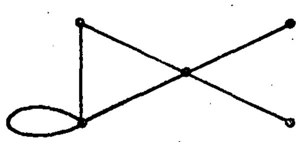


5.17

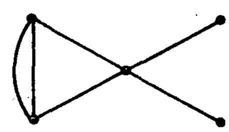


5.18

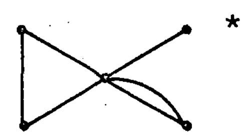
M(T) = 16



5.19

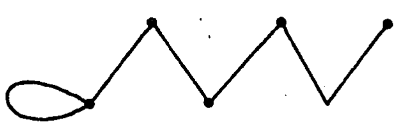


5.20

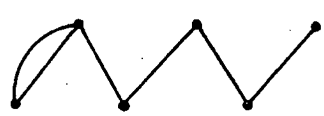


5.21

M(T) = 17



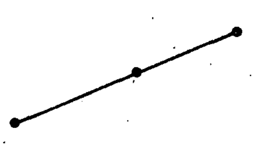
5.22



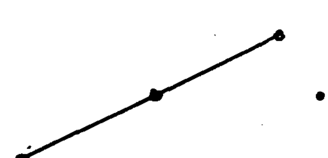
5.23

M(T) = 18 to 43

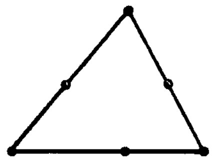
These matroids (6.1-6.26) are precisely those listed above under the heading "Simple Matroids" on $T = \{1,2,3,4,5,6\}$, and so they are not listed again. However those which are not in $\mathbb{M}$ are shown for completeness.



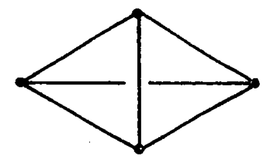
6.4 (Non-graphic)



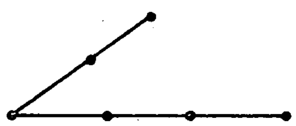
6.5 (Non-graphic)



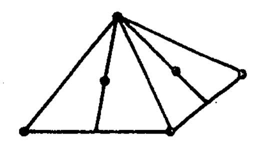
6.6 (Non-graphic)



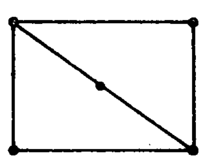
6.7 (Graphic)



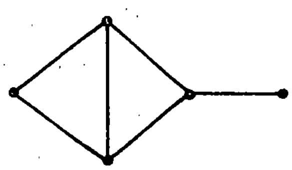
6.9 (Non-graphic)



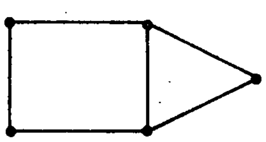
6.14 (Non-graphic)



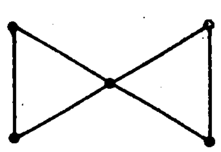
6.15 (Graphic)



6.18 (Graphic)



6.20 (Graphic)



6.21 (Graphic)

As can be seen from the above, there are 68 graphic matroids on a set of size 6, 42 of which are in $\mathcal{M}$ and 26 are not. There are 30 non-graphic matroids, 22 of which are in $\mathcal{M}$ and 8 are not.

The calculation of the number of non-isomorphic matroids on a set of 6 elements seems to be a new result, and so we state it as a theorem.

THEOREM 4.5 *There are* 98 *non-isomorphic matroids on a set of* 6 *elements.* //

CHAPTER 5

In the previous chapter we saw that not all matroids of the class $\mathcal{M}$ are graphic. It is natural to enquire whether they are a subclass of any well known class of matroids, and in this short chapter we answer the question as well as establishing a necessary and sufficient condition for a matroid to belong to $\mathcal{M}$.

<u>LEMMA 5.1</u> *A matroid* $M \in \mathcal{M}$ *on a ground set* E *is a transversal matroid.*

<u>Proof</u>: We construct the family of subsets $\mathcal{U} = (A_i : 1 \le i \le \rho)$, where $A_i = \{a \in E : l(a) \ge i\}$. By Lemma 2.10 a transversal of $\mathcal{U}$ is precisely a basis of M, whence partial transversals are precisely the independent sets of M. //

This is a most interesting result because M^* being also in $\mathcal{M}$ is also transversal. Therefore here we have a subclass of transversal matroids whose dual is also transversal. That not all transversal matroids have duals which are also transversal is shown by the following example. Figure 4 below is a graphic matroid which is transversal, and its dual (Figure 5) is also graphic but is not transversal.

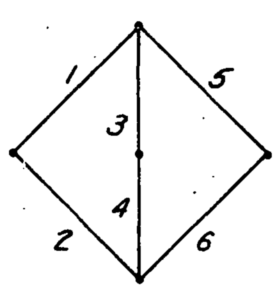


Fig. 4 Graphic Matroid
which is transversal

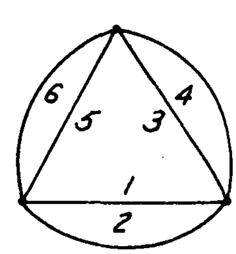


Fig. 5 Graphic Matroid
which is not transversal

The above are the matroids 6.15 and 3.5 of the previous chapter and of course they are not in $\mathcal{M}$.

Not all transversal matroids whose duals are also transversal, belong to $\mathcal{M}$. The following example shows this.

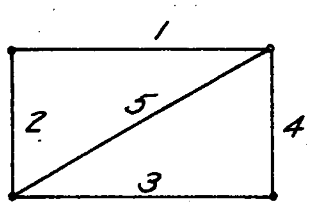


Fig. 6 Graphic Matroid
transversal but $\notin \mathcal{M}$

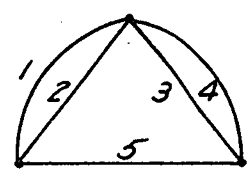


Fig. 7 Transversal Matroid
dual of Fig. 6

The matroid of Fig. 6 is transversal with family $(\{1,2\}, \{3,4\}, \{1,4,5\})$, and the family of the transversal matroid of Fig. 7 is $(\{1,2,5\}, \{3,4,5\})$. They are obviously dual. The excluded minor characterisation shows that they are not in $\mathcal{M}$.

The following theorem shows precisely which transversal matroids are members of $\mathcal{M}$.

<u>THEOREM 5.2</u> *A matroid* M *on a ground set* E *is a member of* $\mathcal{M}$ *if and only if it is transversal having a presentation of a family of nested sets.*

<u>Proof</u>: Given $M(E) \in \mathcal{M}$ we construct a family $\mathcal{U} = (A_i : 1 \le i \le \rho)$, where $E_i = \{a \in E : l(a) \ge i\}$, and ρ is the rank of $M(E)$. The E_i form a chain ordered by strict inclusion.

Conversely let $M(E)$ be a transversal matroid with family of representable sets $\mathcal{U} = (E_i : 1 \le i \le \rho)$ such that $E_1 \supset E_2 \supset \ldots \supset E_\rho$.

For all $a \in E_i \backslash E_{i+1}$, $1 \le i \le \rho - 1$, we assign $l(a) = i$, and for all $a \in E\rho$ we assign $l(a) = \rho$. The function l induces a matroid of the class $\mathbb{M}$ on E and the transversals of $\mathbb{U}$ are precisely the bases of the induced matroid. //

We conclude this chapter with a necessary and sufficient condition for a matroid to belong to the class $\mathbb{M}$.

<u>THEOREM 5.3</u> *Let* $M(E)$ *be a matroid of the class* $\mathbb{M}$ *whose independent sets are the family* $\mathbb{I}$. *Let* $\mathbb{C}$ *be the family of circuits of* $M(E)$. *Let*

$$\mathbb{I}' = (I : I \in \mathbb{I},\ b \notin I \text{ where b is a coloop})$$

and for $I \in \mathbb{I}$, $\mathbb{C}_I = (C_a : a \in C_a \in \mathbb{C},\ a \in I,\ |C_a| \min)$.

(A particular C_a *might not be unique, and the family might have some repetitions of* C_a*'s.) Then* $M(E) \in \mathbb{M}$ *if and only if for any* $\mathbb{C}_I$, *at most* i *circuits have cardinality* $\le i+1$ *for* $1 \le i \le |I|-1$.

<u>Proof</u>: Suppose $M(E) \in \mathbb{M}$ and E is μ-ordered. Consider any $I = \{a_{j1}, \ldots, a_{jm}\} \in \mathbb{I}'$ and let $\mathbb{C}_I = (C_1, \ldots, C_m)$, where $C_i \cap a_{ji} \ne \phi$ and $|C_i|$ is minimum. Then $|C_i| > \mu(a_{ji}) \ge i$ from Lemma 2.3 and at most i circuits have cardinality $\le i+1$. This applies for $1 \le i \le |I|-1$.

Conversely let $M(E)$ be a matroid with circuit structure as described. We define a function μ on the set E as follows:

if a is a loop, let $\mu(a) = 0$;

if a is a coloop, let $\mu(a) = \rho(M)$;

if a is neither a loop nor a coloop, let $\mu(a) = |C_a| - 1$,

where $C_a \cap a \ne \phi$ and $|C_a|$ minimum.

This function μ induces a matroid $M'(E) \in \mathcal{M}$, with the family of independent sets $\mathcal{J}$. We have to show that $\mathcal{J} = \mathcal{I}$. For any $I \in \mathcal{I}$ let $I = \{b_1, \ldots, b_s\} \cup \{b_{s+1}, \ldots, b_t\}$, where the first subset is a member of $\mathcal{I}'$ and the second is not. Because of the assumed circuit structure, $I \in \mathcal{J}$.

For $J \in \mathcal{J}$ we let $J = \{c_1, \ldots, c_m\}$ where $i > j \Rightarrow \mu(c_i) \geq \mu(c_j)$. Suppose for some $i < m$, $\{c_1, \ldots, c_i\}$ is independent in $M(E)$ but $\{c_1, \ldots, c_{i+1}\}$ is dependent. Then the latter contains a circuit of size at most $i+1$ and that circuit must meet c_{i+1}. Therefore $\mu(c_{i+1}) = i$, which is impossible since J is independent in $M'(E)$, so we conclude that the independence of $\{c_1, \ldots, c_i\}$ implies the independence of $\{c_1, \ldots, c_{i+1}\}$. Since at most one of the circuits of $M(E)$ meeting $\{c_1, c_2\}$ has cardinality ≤ 2, $\{c_1, c_2\}$ is independent in $M(E)$, and induction on i gives us that J is independent in $M(E)$. //

CHAPTER 6

In the previous chapters the ground set upon which the matroid is induced is finite. This chapter deals with infinite ground sets, and in this case we use the term pregeometry rather than matroid. From Crapo and Rota [5] we have the following definition:

A *pregeometry* $G(S)$ is a set S endowed with a closure relation σ having the following properties:

(i) the exchange property: if $a \in \sigma(A \cup b)$ and $a \notin \sigma(A)$, then $b \in \sigma(A \cup a)$,

(ii) the finite basis property: any $A \subseteq S$ has a finite subset $A_f \subseteq A$ such that $\sigma(A_f) = \sigma(A)$.

(We recall that a closure relation σ is defined by the properties

(a) $A \subseteq \sigma(A)$, and (b) $A \subseteq \sigma(B) \Rightarrow \sigma(A) \subseteq \sigma(B)$, for all $A,B \subseteq S$.)

As for matroids, a pregeometry has a family of independent sets, and the pregeometry is completely defined by this family. The family $\mathcal{I}$ turns out to have the same properties as the collection of independent sets of a matroid, namely:

(1) $J \subseteq I \in \mathcal{I} \Rightarrow J \in \mathcal{I}$

(2) $I,J \in \mathcal{I}$ and $|I| > |J| \Rightarrow$ there exists an element $x \in I\backslash J$ such that $J \cup x \in \mathcal{I}$.

In addition property (ii) above, the finite basis property, ensures that

(3) all members of $\mathcal{I}$ are finite, and have finitely bounded size.

Therefore a pregeometry $G(S)$ consists of the non-empty set S , together with a non-empty family $\mathcal{I}$ of subsets (called the independent sets) of S, satisfying (1), (2) and (3) above.

We are concerned in this chapter to show that the characterisation of the class $\mathcal{M}$ matroids revealed in Chapter 3 carries over to the class of pregeometries induced in the same manner as $\mathcal{M}$.

A *restriction* $G_T(S)$ of a pregeometry $G(S)$, or a *subgeometry* as Crapo and Rota call it, is the set T endowed with the closure relation σ_{rest} given by

$$\sigma_{rest}(A) = \sigma(A) \cap T.$$

It is easy to show that a restriction is a pregeometry.

<u>LEMMA 6.1</u> *If* $G(S)$ *is a pregeometry on a ground set* S *and* T *is a finite subset of* S, *then* $G_T(S)$ *is a matroid.* //

<u>LEMMA 6.2</u> *Let the function* $\mu: S \to Z$ *be bounded above. Then* μ *defines a pregeometry* $G(S)$ *whose family* $\mathcal{I}$ *of independent sets is given by*

$$\phi \in \mathcal{I}, \textit{ and } I \in \mathcal{I} \textit{ if and only if } \max\{\mu(a) : a \in J\} \geq |J|$$
$$\textit{for all } \phi \neq J \subseteq I .$$

<u>Proof</u>. Since μ is bounded above we are assured of the existence of $\max\{\mu(A) : a \in A\}$ for all $A \subseteq S$. Otherwise the reasoning is the same as in Lemmas 2.1 and 2.2. //

We can, without loss of generality, assume that $0 \leq \mu(a) \leq \rho(G)$ for all $a \in S$, since the function $\nu: S \to Z$ given by $\nu(a) = \min\{\max\{0, \mu(a)\}, \rho(G)\}$ defines the same pregeometry as G. This assumption is made for the rest of the chapter.

We call pregeometries derived in the above manner *pregeometries of the class* $\mathcal{G}$.

The next proof requires *Rado's Selection Principle* which is as follows:

Let $\mathcal{U} = (A_i : i \in I)$ be a family of finite subsets of a set S. Let $\mathcal{J}$ denote the collection of all finite subsets of the index set I and for each $J \in \mathcal{J}$, let θ_J be a choice function of the subfamily $(A_i : i \in J)$. Then there exists a choice function θ of $\mathcal{U}$ with the property that, for each $J \in \mathcal{J}$, there is a K with $J \subseteq K \in \mathcal{J}$ and $\theta_{|J} = \theta_{K|J}$. (For proof see Mirsky [15].)

<u>THEOREM 6.3</u> *The pregeometry* G(S) *is in* $\mathcal{G}$ *exactly when each of its finite restrictions (submatroids) is in* $\mathcal{M}$.

<u>Proof</u>: If $G_\mu \in \mathcal{G}$ then $M = G_\mu | T$ is defined by $\mu_{|T}$, using the same reasoning as in Lemma 2.23.

Conversely if for each $T \subset\subset S$, $G|T = M_{\mu_T}$ for some $\mu_T : T \to Z$, we define a family $(X)_S$ by

$$X_a = \{0, 1, 2, \ldots, \rho(G)\} \text{ for all } a \in S.$$

Then for each $T \subset\subset S$ the function μ_T is a choice function. Rado's Selection Principle ensures the existence of a choice function $\mu : S \to Z$ with $T \subseteq K \subset\subset S \Rightarrow \mu_{|T} = \mu_{K|T} = \mu_T$, and as $X_a = \{0, 1, 2, \ldots \rho(G)\}$ for all $a \in S$, this choice function is bounded. The function μ induces a pregeometry G_μ on Z.

It remains to show that G_μ is identical to G. This will be so if I independent in $G_\mu \Leftrightarrow$ I independent in G.

If I is independent in G_μ then $\max\{\mu(a) : a \in J\} \geq |J|$ for all $J \subseteq I$, whence $\max\{\mu_{|K}(a) : a \in J\} \geq |J|$ for $I \subseteq K \subset\subset S$. Therefore I is independent in $M_{\mu_{|K}} = G|K$, and hence in G.

Conversely if I is independent in G then $|I| < \infty$ and there exists K with $I \subseteq K \subset\subset S$ such that I is independent in $G|K = M_{\mu_{|K}}$.

Therefore $\max\{\mu_{|K}(a): a \in J\} \geq |J|$ for all $J \subseteq I$, whence $\max\{\mu(a): a \in J\} \geq |J|$ for all $J \subseteq I$, and I is independent in G_μ. //

We define a *minor* of a pregeometry to be any contraction of a finite restriction.

THEOREM 6.4 $\mathcal{G}$ *is characterised by the family* ${}^kM$, $k = 2, 3, \ldots$ *of excluded minors.*

Proof: This follows immediately from Theorem 6.3 and Theorem 3.9. //

THEOREM 6.5 *Each* $G_\mu \in \mathcal{G}$ *is characterised by having a finite chain* $\sigma(\phi) = F_0 \subset F_1 \ldots \subset F_k \subseteq S$, *where* F_{i+1} *is the unique minimal non-trivial extension of the flat* F_i, *unless* F_i *has no such extension in which case* $F_{i+1} = E$. *Each flat in* G_μ *is a direct sum of some* F_i *and a free matroid.*

Proof: If a pregeometry G has such a chain then so does any finite restriction. Therefore, by Lemma 3.3., any finite restriction is a matroid in $\mathcal{M}$, whence $G \in \mathcal{G}$.

Conversely let $G \in \mathcal{G}$ and suppose G has two minimal non-trivial extensions F and F' of a flat H. Then there exist circuits $C \subseteq F$ and $C' \subseteq F'$ with $f(C) = f(F)$ and $f(C') = f(F')$. Now in the restriction $G|C \cup C'$, $\sigma_{rest}(F)$ and $\sigma_{rest}(F')$ are minimal non-trivial extensions of $\sigma_{rest}(H)$, but since the restriction is in $\mathcal{M}$ we have $\sigma_{rest}(F) = \sigma_{rest}(F')$, i.e. $\sigma_{rest}(C) = \sigma_{rest}(C')$. It follows from this that $f(C) = f(C')$, whence $F = F'$. We begin the chain with the closure of the empty set and from the above the rest follows.

Any flat is (i) free, or (ii) an F_i, or (iii) a free extension of an F_i. Therefore a flat F is the direct sum of F_i, for some $0 \leq i \leq k$, and the free matroid $M|(F \backslash F_i)$. //

THEOREM 6.6 *If* $G \in \mathcal{G}$ *then it is transversal.*

Proof: Let $S_i = \{a \in S: \mu(a) \geq i\}$ for $i = 1, 2, \ldots, \rho(G)$. Then $\mathcal{U} = (S_i: 1 \leq i \leq \rho(G))$ is a family of subsets of S whose transversals are bases of G. //

THEOREM 6.7 $G \in \mathcal{G}$ *if and only if it is transversal having a presentation of a family of nested sets.*

Proof: Suppose $G \in \mathcal{G}$. Then by Theorem 6.6 it is transversal and its family of representable sets has the desired property.

Conversely let G be a transversal pregeometry with family of subsets $\mathcal{U} = (S_i: 1 \leq i \leq \rho(G))$ having the property $S_\rho \subseteq S_{\rho-1} \cdots \subseteq S_1$. We define a function $\mu: S \to Z$ by

$$\mu(a) = i \quad \text{if} \quad a \in S_i \backslash S_{i+1} \quad \text{for} \quad 1 \leq i \leq \rho-1 ,$$

$$\mu(a) = \rho \quad \text{if} \quad a \in S_\rho$$

$$\mu(a) = 0 \quad \text{if} \quad a \notin S_1 .$$

Then the pregeometry G_μ induced by μ has as its bases sets which can be described by $B = \{a_1, \ldots, a_\rho: \mu(a_i) \geq i$ for $1 \leq i \leq \rho\}$. It is obvious that the transversals of $\mathcal{U}$ and the bases of G_μ are precisely the same, i.e. $G = G_\mu \in \mathcal{G}$. //

The other properties of the class $\mathcal{M}$ carry over to the class $\mathcal{G}$ where appropriate.

CHAPTER 7

In this chapter we examine the matroids induced by $\mu:E^r \to Z$ via the submodular function $f:2^E \to Z$. This somewhat enlarges the class $\mathbb{M}$; for instance some simple graphical matroids were excluded from $\mathbb{M}$ but are induced by the function defined on r-sized subsets. It also provides matroids with a richer structure.

We begin with the function $\mu:E^r \to Z$ and, as in Chapter 2, obtain $f:2^E \to Z$ as follows. Let

$$f(A) = \max\{\mu(a_1,\ldots,a_r): a_1,\ldots,a_r \in A\} \quad \text{for all } \phi \neq A \subseteq E \,,$$

$$f(A) = \min\{\mu(a_1,\ldots,a_r): a_1,\ldots,a_r \in E\} \quad \text{for } A = \phi \,.$$

Functions derived in this manner are said to belong to the *class* F^r.

Functions of this class are always increasing functions, but they are not always submodular. Consider for example $\mu:E^2 \to Z$ defined as follows. Let $\mu(a,b)$ be the integer part of the distance between the points a and b in Euclidean space E. Let $A = \{a,b\}$, $B = \{c,d\}$, and $\mu(a,b) = 5$, $\mu(c,d) = 5$, $\mu(a,c) = 12$, $\mu(b,d) = 12$. Then obviously $f(A) + f(B) < f(A \cup B) + f(A \cap B)$.

There are some functions on r-sized subsets which induce submodular functions in the manner of Chapter 2 but it is not the purpose of this thesis to characterise them, if indeed this is possible. However we construct one such function as follows:

Let $\mu_1, \ldots, \mu_r$ be functions from E into Z. We define a function $\mu:E^r \to Z$ by

$$\mu(a_1,\ldots,a_r) = \mu_1(a_1) + \ldots + \mu_r(a_r) \quad \text{for all } a_i \in E.$$

It follows that $f(A) = \max\{\mu_1(a_1) + \ldots + \mu_r(a_r) : a_1,\ldots,a_r \in A\}$, whence $f(A) = f_1(A) + \ldots + f_r(A)$, for all $A \subseteq E$, where $f_i(A) = \max\{\mu_i(a) : a \in A\}$ for $1 \le i \le r$. In order to distinguish the function f from that of Chapter 2, we designate it f^r and we have $f^r = f_1 + \ldots + f_r$.

<u>LEMMA 7.1</u> *The function* $f^r : 2^E \to Z$ *is submodular.*

<u>Proof</u>. For any $A,B \subseteq E$

$$\begin{aligned} f^r(A \cup B) + f^r(A \cap B) &= f_1(A \cup B) + \ldots + f_r(A \cup B) + f_1(A \cap B) + \ldots \\ &\qquad + f_r(A \cap B) \\ &\le f_1(A) + f_1(B) + \ldots + f_r(A) + f_r(B) \\ &= f^r(A) + f^r(B) . \end{aligned}$$ //

The function μ therefore defines a matroid on E . It is natural to enquire what functions on r-tuples are expressible as sums of functions on singletons. The answer is that there are not very many, as the next lemma shows.

<u>LEMMA 7.2</u> *A function* $\mu : E \times E \to Z$ *can be written* $\mu(a_i,a_j) = \mu_1(a_i) + \mu_2(a_j)$ *for all* $a_i,a_j \in E$ *if and only if*

$$\mu(a_i,a_h) - \mu(a_i,a_k) = \mu(a_j,a_h) - \mu(a_j,a_k) \quad \textit{for all } a_i,a_j,a_h,a_k \in E.$$

<u>Proof</u>. If $\mu(a_i,a_j) = \mu_1(a_i) + \mu_2(a_j)$ for all a_i,a_j then by substitution we have $\mu(a_i,a_h) - \mu(a_i,a_k) = \mu(a_j,a_h) - \mu(a_j,a_k)$.

Conversely suppose we have a function $\mu : E \times E \to Z$ such that $\mu(a_h,a_k) - \mu(a_h,a_j) = \mu(a_i,a_k) - \mu(a_i,a_j)$. Then $\mu(a_i,a_j)$ is determined by the $2n - 1$ terms $\mu(a_1,a_1), \ldots, \mu(a_1,a_n), \mu(a_2,a_1), \ldots, \mu(a_n,a_1)$, where $|E| = n$. We must show that there exist $\mu_1,\mu_2 : E \to Z$ such that the n^2 equations

$\mu_1(a_i) + \mu_2(a_j) = \mu(a_i, a_j)$ are satisfied. There is an infinite number of solutions for $\mu_1(a_1), \ldots, \mu_1(a_n)$, $\mu_2(a_1), \ldots, \mu_2(a_n)$ to the $2n - 1$ equations

$$\begin{aligned}
\mu_1(a_1) + \mu_2(a_1) &= \mu(a_1, a_1) \\
&\vdots \\
\mu_1(a_1) + \mu_2(a_n) &= \mu(a_1, a_n) \\
\mu_1(a_2) + \mu_2(a_1) &= \mu(a_2, a_1) \\
&\vdots \\
\mu_1(a_n) + \mu_2(a_1) &= \mu(a_n, a_1) ,
\end{aligned}$$

and by fixing an integer value of say $\mu_1(a_1)$ we obtain one integer value for each of the others. It remains to show that this solution is consistent with the remaining $n^2 - (2n-1)$ equations. This is so since

$$\begin{aligned}
\mu_1(a_i) + \mu_2(a_j) &= \mu_1(a_i) + \mu_2(a_1) + \mu_1(a_1) + \mu_2(a_j) \\
&\qquad - \mu_1(a_1) - \mu_2(a_1) \\
&= \mu(a_i, a_1) + \mu(a_1, a_j) - \mu(a_1, a_1) \\
&= \mu(a_i, a_j) . \qquad //
\end{aligned}$$

Similar but more complicated results apply for μ defined on larger subsets.

<u>LEMMA 7.3</u> deleted

Since f^r is submodular it defines a matroid on E whose independent sets are given by $\mathcal{I} = \{I: f^r(J) \geq |J| \text{ for all } J \subseteq I\}$. We designate this matroid as M^r , and say that it belongs to the *class* $\mathcal{M}^r$.

We define the *union* of r matroids $M_1, \ldots, M_r$ on E as the matroid whose independent sets are each precisely the union of r subsets of E , each of which is independent in a distinct M_i . The matroid $M_1 \cup \ldots \cup M_r$ is defined by the collection $\mathcal{I}$ of independent sets given by $\mathcal{I} = \{I: I = I_1 \cup \ldots \cup I_r, I_i \in \mathcal{I}_i\}$, where $\mathcal{I}_i$ is the collection of independent sets of the matroid M_i .

<u>LEMMA 7.4</u> *Suppose* $f^2 = f_1 + f_2$ *and that* f^2, f_1 *and* f_2 *induce the matroids* M^2, M_1 *and* M_2 *respectively on a ground set* E . *Then* $M^2 = M_1 \cup M_2$.

<u>Proof</u>. For any I_1, I_2 independent in M_1, M_2 respectively, $f^2(I_1 \cup I_2) = f_1(I_1 \cup I_2) + f_2(I_1 \cup I_2) \geq f_1(I_1) + f_2(I_2) \geq |I_1| + |I_2| \geq |I_1 \cup I_2|$.

Conversely suppose there exists a set I of cardinality $m+1$ which is independent in M^2 but cannot be partitioned into two sets, one of which is independent M_1 and one in M_2 . Further suppose that all sets of size $\leq m$ can be so partitioned. We choose $a \in I$ such that $\mu_1(a) \geq \mu_1(x)$ for all $x \in I\backslash a$, and partition $I\backslash a$ into

I_1 and I_2 which are independent in M_1 and M_2 respectively. Then $f_1(I) = f_1(I_1) = |I_1| = \mu_1(a)$, and furthermore there exists no $b \in I_1$ such that $\mu_2(b) > f_2(I_2)$, otherwise I could be partitioned as required.

The set $I_2 \cup a$ is dependent in M, whence $f_2(J_2 \cup a) = |J_2|$ for some $J_2 \subseteq I_2$. Suppose $J_2 = I_2$; then $f^2(I) = f_1(I) + f_2(I) = |I_1| + |I_2| < |I|$, which contradicts the independence of I in M^2 . Therefore $J_2 \subset I_2$ and $\mu_2(a) \leq f_2(J_2) = |J_2|$. If there exists no $b \in I_1$ such that $\mu_2(b) > f_2(J_2)$ then $f^2(I_1 \cup J_2 \cup a) = |I_1| + |J_2|$, which is impossible since the set $I_1 \cup J_2 \cup a$ is independent in M^2 . Such an element therefore must exist, and by interchanging a and b we obtain $I_1^{(i)}$ and $I_2^{(i)} \cup b$. Again there exists $J_2^{(i)} \subset I_2^{(i)} \cup b$ with $J_2 \subset J_2^{(i)}$ and $c \in (I_2 \cup b)\backslash J_2^{(i)}$ such that $f_2(J_2^{(i)}) = |J_2^{(i)}| = \mu_2(c)$, otherwise I would partition as required. But then $I_1^{(i)} \cup J_2^{(i)} \cup c$ would be dependent in M^2 , unless it is possible again to interchange elements as above. The latter must be true, and in this manner after a finite number s of interchanges we arrive at $I_1^{(s)}$, $I_2^{(s)}$ of the same size of I_1 and I_2 respectively, and an element x not an element of either, such that $f_1(I) = f_1(I_1^{(s)}) = |I_1|$ and $f_2(I) = f_2(I_2^{(s)}) = |I_2| = \mu_2(x)$, which contradicts the independence of I in M^2 . We conclude that if independent sets of size m in M^2 partition as required, then so do those of size $m+1$.

For $|I| = 1$, $f^2(I) \geq 1$ implies that $f_i(I) \geq 1$ for $i = 1$ or 2 or both, whence $I = I_1 \cup I_2$ where I_i is independent in M_i for $i = 1, 2$. //

We now extend the lemma to the general case of the union of r matroids.

<u>THEOREM 7.5</u> *Suppose* $f_1, \ldots, f_r \in \mathsf{F}$ *and define matroids* $M_1, \ldots, M_r \in \mathsf{M}$ *respectively, and that* $f^r = f_1 + \ldots + f_r$ *defines the matroid* $M^r \in \mathsf{M}^r$. *Then* $M^r = M_1 \cup \ldots \cup M_r$.

<u>Proof</u>. Suppose $M^m = M_1 \cup \ldots \cup M_m$ for some $m < r$, and there exists I independent in M^{m+1} which cannot be expressed as the union of $m+1$ sets, each independent in a distinct M_i . We take the union of maximal sets of I , each of which is independent in a distinct M_i ; this is obviously a proper subset of I . Therefore there exists $a \in I$ which when joined to each of these maximal sets forms a set which contains a circuit in the appropriate M_i .

It follows from the above that I is dependent in M^m . Using this and the fact that I is independent in M^{m+1} , we have

$$f^{m+1}(J) = f_1(J) + \ldots + f_{m+1}(J) \geq |J| \text{ for all } J \subseteq I ,$$

but
$$f^m(K) = f_1(K) + \ldots + f_m(K) < |K|$$

for some $K \subseteq I$. If K_i is the subset of K which is independent in M_i , we have $f_i(K) \geq f_i(K_i) \geq |K_i|$, whence $f^m(K) \geq |K| - 1$. But $f^{m+1}(K) \geq |K|$, so it follows that $f_{m+1}(K) \geq 1$. Therefore there exists $b \in K$ such that $\mu_{m+1}(b) \geq 1$. Suppose $b = a$; then a is independent in M_{m+1} , which is contrary to our original supposition. Suppose $b \neq a$; then $b \in I_i$ for some i and $\mu_i(b) \neq 0$. It follows that $(I_i \backslash \{b\}) \cup \{a\}$ is independent in M_i and b is independent in M_{m+1} , which also contradicts our original supposition. Therefore $M^m = M_1 \cup \ldots \cup M_m$ implies that $M^{m+1} = M_1 \cup \ldots \cup M_{m+1}$. Lemma 7.4 shows that the result is true for $m = 2$, so it is true for $m = r$ by induction. //

LEMMA 7.6 deleted

LEMMA 7.7 *Suppose* $\mu_i(a) = 0$ *or* 1 *for all* $a \in E$ *and for* $i = 1, 2, \ldots, r$. *Then a set* $\{a_1, \ldots, a_r\}$ *is independent in* $M^r(E)$ *if and only if the* μ_i*'s can be permuted such that* $\mu_i(a_i) = 1$ *for* $i = 1, 2, \ldots, r$.

Proof. Clearly the set is independent if such a permutation does exist.

Conversely assume $\{a_1, \ldots, a_r\}$ is independent. Suppose the result is true for independent sets of size $m < r$, i.e. there exist $\mu_1, \ldots, \mu_m$ such that $\mu_i(a_i) = 1$ for $i = 1, 2, \ldots, m$. If it is not true for $m+1$ then $\mu_j(a_{m+1}) = 0$ for $j > m$. Also if $\mu_j(a_i) = 1$ for $j > m$ and $i < m$ then $\mu_i(a_{m+1}) = 0$, otherwise by rearrangement of the μ's the theorem is true for $m+1$. But since a_{m+1} is independent some μ_i maps it to 1, so we conclude that $\mu_j(a_i) = 0$ for $j > m$ and $i < m$. This plus $\mu_j(a_{m+1}) = 0$ for $j > m$ gives us that $f(\{a_1, \ldots, a_{m+1}\}) = m$, which is impossible since the set is independent. Therefore if the result is true for m it is true for $m+1$. Clearly the result is true for $m=1$. //

Suppose now that $\mu(a_1, \ldots, a_r) = \mu_1(a_1) + \ldots + \mu_r(a_r)$ and that the maximum value of any μ_i on E is k. Then we can write $\mu_i = \mu_{i1} + \ldots + \mu_{ik}$, where

$$\mu_{ij}(a) = 1 \text{ if } j \leq \mu_i(a)$$

and $\mu_{ij}(a) = 0$ if $j > \mu_i(a)$.

In our usual way we define a function $f_{ij}:2^E \to Z(2)$ as follows:

$$f_{ij}(A) = \max\{\mu_{ij}(a):a \in A\}.$$

Then for any $A \subseteq E$ it is easily verified that

$f_i(A) = f_{i1}(A) + \ldots + f_{ik}(A)$. Therefore $f = f_{11} + \ldots + f_{ik}$.

LEMMA 7.8 $M^r = M_{11} \cup \ldots \cup M_{rk}$ *, where* M_{ij} *is the matroid defined by* μ_{ij}.

Proof. It is only necessary to prove that $M_i = M_{i1} \cup \ldots \cup M_{ik}$.

If I is independent in M_i then $f_i(J) \geq |J|$ for all $J \subseteq |I|$, whence there exist distinct $a_m \in I$, $m = 1,2,\ldots, |I|$, such that $\mu_i(a_m) \geq m$, i.e. $\mu_{im}(a_m) = 1$. Therefore a_m is independent in M_{im} and $M_i \subseteq M_{i1} \cup \ldots \cup M_{ik}$.

Conversely consider a union $I_{i1} \cup \ldots \cup I_{ik}$, where I_{im} is independent in M_{im}. We suppose that all the I_{im} are non-empty since if the inclusion we seek is true for this, it is true for some empty. Now $I_{im} = \{a\}$, where a is such that $\mu_{ij}(a) = 1$ for $j \leq m$, which implies that $\mu_i(a) \geq m$. Therefore $I_{i1} \cup \ldots \cup I_{ik}$ is independent in M_i , whence $M_{i1} \cup \ldots \cup M_{ik} \subseteq M_i$. //

We designate the closure in the matroid $M^r \in \mathbb{M}^r$ by the relation $\sigma^r:2^E \to 2^E$. The following explores the structure of closures in the class $\mathbb{M}^r$ and their relation with closures in matroids of the class $\mathbb{M}$.

LEMMA 7.9. *In the matroid* $M^r(E) \in \mathbb{M}^r$ *, for all* $A \subseteq E$ *,* $\sigma^r(A) = A \cup B$ *, where* B *is maximal with respect to* $f^r(B) = |J|$, *where* $J \subseteq A$ *is maximal with respect to* (i) J *is independent in* $M^r(E)$, *and* (ii) $f^r(J) = |J|$.

Proof. Suppose for any $A \subseteq E$ we have a set $A \cup B$ as described above. Obviously $A \subseteq A \cup B$.

Suppose $A \subseteq C \cup D$, where D is a maximal with respect to $f^r(D) = |K|$, where $K \subseteq C$ is maximal with respect to (i) K is independent and (ii) $f^r(K) = |K|$. Then $f^r(J) = f^r(K)$, whence $B \subseteq D$ and $A \cup B \subseteq C \cup D$. Therefore $\sigma^r(A) = A \cup B$. //

Obviously in determining the structure of the closure we are concerned about the independent set J such that $f^r(J) = |J|$. The next lemma tells us more about this set.

LEMMA 7.10. *Suppose* I *is independent in* $M^r \in \mathcal{M}^r$. *Then* $f(I) = |I|$ *if and only if* $I = I_1 \dot{\cup} \ldots \dot{\cup} I_r$ *where* I_i *is a maximal independent subset of* I *in* $M_i \in \mathcal{M}$, *and* $f_i(I_i) = |I_i|$ *for* $i = 1, 2, \ldots, r$.

Proof. Suppose $f(I) = |I|$. We know that $I = I_1 \cup \ldots \cup I_r$ where I_i is independent in M_i and that $f_i(I) \geq f_i(I_i) \geq |I_i|$. Therefore $|I| \geq f_1(I_1) + \ldots + f_r(I_r) \geq |I_1| + \ldots + |I_r|$, whence $I = I_1 \dot{\cup} \ldots \dot{\cup} I_r$ and $f_i(I_i) = |I_i|$. Further $f_i(I) = f_i(I_i)$, and so I_i is a maximal subset of I independent in M_i .

Conversely suppose $I = I_1 \dot{\cup} \ldots \dot{\cup} I_r$ with $I_1, \ldots, I_r$ having the required properties. Then $f_i(I) = f_i(I_i) = |I_i|$ for $i = 1, 2, \ldots, r$, and so $f(I) = |I_1| + \ldots + |I_r| = |I|$. //

LEMMA 7.11. *If* $\sigma_i : 2^E \to 2^E$ *is the closure relationship in the matroid* $M_i \in \mathcal{M}$ *then* $\sigma^r(A) \subseteq \bigcup_{i=1}^{r} \sigma_i(A)$.

Proof. Suppose $D \subseteq \sigma^r(A)$. We let $D = D_1 \,\dot\cup\, D_2$ with $D_1 \subseteq A \backslash B$ and $D_2 \subseteq B \backslash A$, where B is the set defined in Lemma 7.9. Obviously $D_1 \subseteq \cup\, \sigma_i(A)$. If $J = J_1 \,\dot\cup \ldots \dot\cup\, J_r$ is the maximal independent set contained in A such that $f^r(J) = |J|$, then $J_i \subseteq I_i$ for $i = 1, 2, \ldots, r$, where I_i is the maximal independent set contained in A such that $f_i(I_i) = |I_i|$. Therefore $f^r(D_2) \leq |J_1| + \ldots + |J_r| \leq |I_1| + \ldots + |I_r|$, whence $f_i(D_2) \leq |I_i|$ for at least one i , and so $D_2 \subseteq \sigma_i(A)$. //

We move on now to consideration of circuits in matroids of the class $\mathcal{M}^r$.

LEMMA 7.12. *If* C *is a circuit in the matroid* $M^r \in \mathcal{M}^r$ *then* $C \supseteq C_1 \cup \ldots \cup C_r$ *, where* C_i *is a circuit in* $M_i \in \mathcal{M}$.

Proof. If C is a circuit in M^r then for any $a \in C$, $C = J \cup a$ where J is independent in M^r and $f(C) = f(J) = |J|$. Therefore by Lemma 7.10 $J = J_1 \,\dot\cup \ldots \dot\cup\, J_r$, J_i being a maximal subset of J independent in M_i , and $\mu_i(a) \leq |J_i|$ for $i = 1, 2, \ldots, r$. It follows then that $C_i \subseteq J_i \cup a$ where C_i is a circuit in M_i , and so $C \supseteq C_1 \cup \ldots \cup C_r$. //

LEMMA 7.13. *If* C *is a circuit in* M^r *then for each* i , $1 \leq i \leq r$, *there exist at least two elements* $b,c \in C$ *such that* $\mu_i(b) = \mu_i(c) = |J_i|$.

Proof. If not then we only have say $\mu_i(b) = |J_i|$ and $f(C \backslash b) < |C| - 1$, whence C is not a circuit. //

It is now possible to construct circuits in M^r. We select independent sets $J_1, J_2, \ldots J_r$ from $M_1, M_2, \ldots, M_r$ respectively such that $f_i(J_i) \geq f_i(J_k)$ for all i,k, $f_i(J_i) = |J_i|$ for all i, and the J_i's are disjoint. If there exist, for each i, at least two elements $b,c \in J_i \,\dot{\cup}\, \ldots \,\dot{\cup}\, J_r$ such that $\mu_i(b) = \mu_i(c) = |J_i|$ then we join to the above disjoint union any $a \in E$ such that $0 < \mu_i(a) \leq |J_i|$ for all i, and this gives a circuit in M^r. If for any i, only one element b in $J_1 \,\dot{\cup}\, \ldots \,\dot{\cup}\, J_r$ is such that $\mu_i(b) = |J_i|$ and otherwise $\mu_k(a) \leq |J_k|$, and this joined to the disjoint union of independent sets is a circuit in M^r. If $J = J_1 \,\dot{\cup}\, \ldots \,\dot{\cup}\, J_r$ as above and there exists $a \in E\backslash J$ such that $\mu_i(a) = |J_i|$ for all i then $C = J \cup a$ is a circuit in M^r and we have $C = C_1 \cup \ldots \cup C_r$, where $C_i = J_i \cup a$ is a circuit in M_i.

If $\mathcal{B}_1, \ldots, \mathcal{B}_r$ are the collections of bases of the matroids $M_1, \ldots, M_r \in \mathbb{M}$ then clearly the bases of M^r are the maximal members of the family $(B_1 \cup \ldots \cup B_r \colon B_i \in M_i)$.

We turn now to the consideration of dual matroids of those in $\mathbb{M}^r$. The matroid M^{r*} has as its bases the sets which are the complements of maximal members of the family $(B_1 \cup \ldots \cup B_r \colon B_i \in \mathcal{B}_i)$. More succinctly, the bases are the minimal members of the family $(B_1^* \cap \ldots \cap B_r^* \colon B_i^* \in \mathcal{B}_i^*)$, where $\mathcal{B}_i^*$ is the set of bases of M_i^*. Hence in general M^{r*} does not belong to $\mathbb{M}^r$. However below is an example of a member of $\mathbb{M}^2$ whose dual also belongs to $\mathbb{M}^2$, in fact it is self dual.

Let $E = \{a_1, a_2, a_3, a_4\}$ with $l_1, l_2 \colon E \to Z$ two standardised functions (levels) as shown in figure 8 below.

Fig 8. l_1 and l_2 on E

Then l_1 and l_2 define two matroids M_1 and M_2 having collections of bases $(\{a_3\}, \{a_4\})$ and $(\{a_1\}, \{a_2\})$ respectively. The collection of bases of $M^2 = M_1 \cup M_2$ is $(\{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\})$. The matroids M_1^* and M_2^* have standardised functions or levels as shown in figure 9.

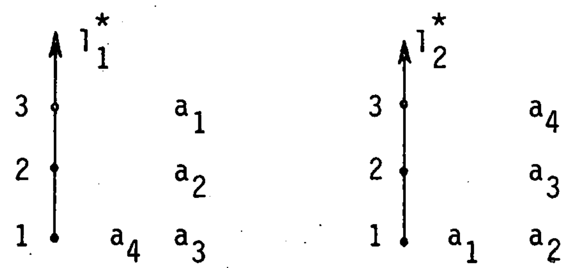
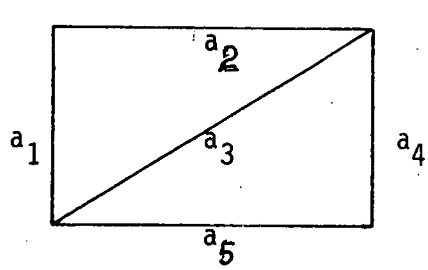


Fig 9. l_1^* and l_2^* on E

The matroids M_1^* and M_2^* have collections of bases $(\{a_4,a_2,a_1\}, \{a_3,a_2,a_1\})$ and $(\{a_1,a_3,a_4\}, \{a_2,a_3,a_4\})$ respectively, whence the collection of bases of M^{2*} is precisely that of M^2 .

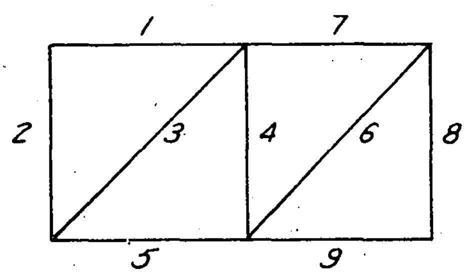
As we remarked in the beginning of this chapter, some very simple graphical matroids, such as that on a quadrilateral with one diagonal, are not in $\mathbb{M}$. However the class $\mathbb{M}^r$, being more complex, does admit some of these, including the example mentioned above. This is shown below, and we chose matroids of rank 1 to build the required matroid.



	a_1	a_2	a_3	a_4	a_5
μ_1	1	1	0	0	0
μ_2	1	0	1	0	1
μ_3	0	0	0	1	1

Fig 10. Quadrilateral with diagonal

A further complication is admissible in $\mathcal{M}^5$, where the M_i's are of rank 1, as shown in figure 11.

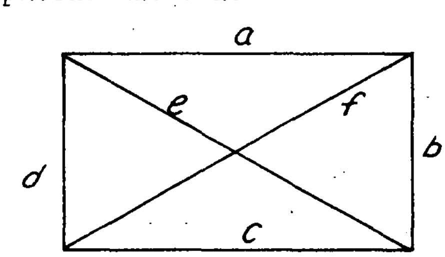


	1	2	3	4	5	6	7	8	9
μ_1	1	1	0	0	0	0	0	0	0
μ_2	1	0	1	0	1	0	0	0	0
μ_3	0	0	0	1	1	0	1	0	0
μ_4	0	0	0	0	0	1	1	0	1
μ_5	0	0	0	0	0	0	0	1	1

Fig 11. $M \in \mathcal{M}^5$ is graphical matroid

It seems that the graphical matroid on a chain of triangles in the manner of figure 11 above could be represented as matroids belonging to the class $\mathcal{M}^r$ for some r. However the limitation of this class for representation of graphical matroids becomes obvious when we consider a quadrilateral with two diagonals, as the following lemma shows.

<u>LEMMA 7.14</u>. *It is not possible to find* $\mu_1,\ldots,\mu_r$ *such that they define a matroid* $M^r \in \mathcal{M}^r$ *on the edges of the graph below which is identical to the graphical matroid.*



<u>Proof</u>. It is possible to find the required functions if and only if it is possible to find a certain number of functions which map the edges to 0 or 1 only, such that these functions define the necessary matroid. We consider then only functions mapping the edges onto 0 or 1.

Suppose it is possible to find μ_1, μ_2, ..., i.e. f_1, f_2, ..., mapping only to 0 and 1, such that they define the required matroid.

Then for $\{a,b,e\}$ there exist, without loss of generality, f_1, f_2 which each map the set to 1. Further, $f_i(\{a,b,e\}) = 0$ for $i \neq 1$ or 2, whence $f_1(x) = f_2(x) = 1$ for at least one $x \in \{a,b,e\}$. Assume $x = e$; then $f_k(\{c,d,e\}) = f_j(\{c,d,e\}) = 1$ for some k,j, and $f_i(\{c,d,e\}) = 0$ for $i \neq k,j$. It follows then that k and j are 1 and 2, whence $f(\{a,b,c,d\}) = 2$ which is impossible. Therefore we can assume, without loss of generality, that $f_1(a) = f_2(a) = 1$. If $f_i(\{d,f\}) = 1$ for $i \neq 1$ or 2 then $f(\{a,d,f\}) \geq 3$ which is impossible. Therefore $f(\{a,b,d,f\}) = 2$ which is also impossible and we conclude that it is impossible to find the required f_1, f_2, ... //

We saw that matroids of the class $\mathbb{M}$ are transversal. This result is now extended to the class $\mathbb{M}^r$, and strengthened.

THEOREM 7.15. $M^r \in \mathbb{M}^r$ *is transversal.*

Proof. From Lemma 5.1 $M_1, \ldots, M_r$ are transversal matroids, whence there exist families $(X)_{I_1}, \ldots, (X)_{I_r}$ of sets of E such that the partial transversals of $(X)_{I_i}$ for $i = 1, 2, \ldots, r$ are precisely the independent sets of $M_1, \ldots, M_r$ respectively. Therefore the independent sets of M^r are precisely the partial transversals of $(X)_I$ where $I = I_1 \cup \ldots \cup I_r$. //

The following theorem shows that the reverse is also true.

THEOREM 7.16. *Let* $M(E)$ *be a transversal matroid of rank* r. *Then* $M(E) \in \mathbb{M}^r$.

Proof. Let $\mathbb{U}$ be a presentation of $M(E)$ and let $(E_1, \ldots, E_r)$ be a subfamily of $\mathbb{U}$ such that its transversals are bases of $M(E)$. We define functions $\mu_1, \ldots, \mu_r : E \to Z$ as follows. Let $\mu_i(a) = 1$ for

$a \in E_i$ and $\mu_i(a) = 0$ for $a \notin E_i$.

Suppose $I = \{a_1, \ldots, a_m\}$ is independent in $M(E)$. Then there exists a subfamily $(E_{i1}, \ldots, E_{im})$ of $\mathfrak{U}$ with $a_j \in E_{ij}$ and $\mu_j(a_j) = 1$ for $1 \leq j \leq m$. Therefore I is independent in the matroid M^r defined by $\mu_1, \ldots, \mu_r$.

Conversely suppose $I = \{a_1, \ldots, a_m\}$ is independent in M^r defined by $\mu_1, \ldots, \mu_r$. Then there exist $\mu_{i1}, \ldots, \mu_{im}$ such that $\mu_{ij}(a_j) = 1$ for $1 \leq j \leq m$, where $a_j \in E_{ij}$ and it follows that I is independent in $M(E)$. //

We now define $\mathbb{M}^f$ to be the class of matroids consisting precisely of all subclasses $\mathbb{M}^r$, r finite. Then we have the following exact description of the matroids of this thesis.

<u>THEOREM 7.17</u>. *The class of all finite transversal matroids is exactly* $\mathbb{M}^f$. //

A FAILED CONJECTURE

We remarked in the introduction that one motivation for this study was the hope of building up from functions defined on singletons to functions defined on subsets of size r, in order thereby to obtain a function $f:2^E \to Z$ which is identical to a well known submodular function or perhaps even to the rank function of a well known matroid. Another approach is to begin with a submodular function $f:2^E \to Z$ and define $f^r:E^r \to Z$ by $f^r(A) = \max\{f(B): B \subseteq A, |B| \leq r\}$. We know that f^1 is submodular and we conjecture that if f^r is submodular then f^{r+1} is also. From this we would have submodular functions f^1, f^2, to f^ρ, where ρ is the rank of the matroid defined by f, and furthermore f^ρ and f define the same matroid.

However f^r being submodular does not imply that f^{r+1} is submodular as the following example shows.

Let $E = \{a,b,c,d\}$ with $f:2^E \to Z$ given by

$$f(a) = 2 \quad , \quad f(b) = f(c) = f(d) = 1$$

$$f(ab) = f(bc) = f(cd) = f(ac) = f(bd) = 2 \ ,$$

$$f(ad) = 3$$

$$f(abc) = 2, \ f(bcd) = f(cda) = f(dab) = 3$$

$$f(abcd) = 3 \ .$$

Then f is increasing and submodular. We know from Chapter 2 that f^1 is submodular, but f^2 is not, as can be seen by considering the sets $A = \{a,b,c\}$ and $B = \{b,c,d\}$. Then $f^2(A) + f^2(B) = 2 + 2$, while $f^2(A \cap B) + f^2(A \cup B) = 2 + 3$.

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INDEX