Structural and Combinatorial Aspects of Graded Rings with Applications

by

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Declaration

I hereby declare that this thesis contains no material which has been accepted for the award of any other higher degree or graduate diploma in any tertiary institution, and that this thesis includes joint work with co-authors of our refereed research papers and, to the best of my knowledge, contains no material previously published or written by another person, except where due reference is made in the text.

Allison Plant

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Abstract

Algebraic structure is at the heart of mathematics and graded ring structures arise in many natural applications and contexts. Particular examples of graded rings actively investigated in recent decades include generalized matrix rings, the Morita rings associated with Morita contexts, polynomial rings and the ring of symmetric functions. We describe a graded construction for all these rings. In order to extend our investigation to as wide a class of graded rings as possible, we consider rings graded by partial groupoids. We present homogeneous sums as graded by induced partial groupoids.

Homogeneity of ideals and radicals of graded rings is one the most interesting and fundamental ideas in graded ring theory. We introduce a consistent definition for the graded Jacobson radical for group graded rings without unity. We compare the graded Jacobson radical for both rings with unity, and those without. We find that for group graded rings, the descriptions are equivalent.

We provide some necessary lemmas for rings without unity which have appeared in the case the ring is afforded unity. These lay the relevant foundation for our investigations. For example, we show that $\mathcal{J}(R) \cap R_e \subseteq \mathcal{J}(R_e)$ (where e is idempotent) for all groupoid graded rings without unity.

We give a generalization of Bergman's 'folklore' lemma for group graded rings with unity to partial groupoid graded rings without unity. Since homogeneous sums and generalized matrix rings are both graded by induced partial groupoids, our generalization of Bergman's lemma applies to these graded rings as well. Our results also yield three corollaries on the Jacobson radical of graded F-algebras (where F is a field).

In 1985 Anderson, Divinsky, and Sulinski defined an *invariant ideal* I_0 in which $R_1 I_0 R_1 \subseteq I_0$ for any $\mathbb{Z}_2$-graded ring R and found that the Jacobson radical was 'invariant' in all $\mathbb{Z}_2$-graded rings. We define a new concept of S-invariance and it turns out that the results of several previous authors fit our definition. For example a 1989 result of Jespers and Wauters is equivalent to saying that the Jacobson radical is S-invariant in all generalized matrix rings. We specify, with necessary and sufficient conditions on S, exactly for which graded rings the Jacobson radical is S-invariant.

The ring of symmetric functions is a graded ring with important applications in mathematical physics. Structural aspects of this graded ring are described. Using the transition matrices of the symmetric functions we are able to write the spin characters of the symmetric group in terms of the ordinary ones. This leads us to describe a new algorithm for the spin characters. We also present simpler algorithms in two special cases.

We include the ring of Hirota derivatives as a practical example of a graded ring without unity. The BKP equations are one example of its homogeneous elements. Setting up this example leads us to introduce the generalized Q-operators and we describe some connections between them and the BKP equations. By associating the generalized Q-operators with shifted Young diagrams, we generate the lower weight portion of the BKP hierarchy.

Motivation for these studies is directed mostly by the investigations of other authors and driven mostly by curiosity.

Acknowledgements

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The School of Mathematics and Physics at the University of Tasmania is a humble academic world filled with, I think, many outstanding scholars. I feel pleased and blessed to have been welcomed into that world and given the chance to complete this thesis.

I express my warmest wishes to Dr M.A. Salam, who introduced me to symmetric functions and guided me toward a future in mathematics. It was Salam who introduced me to the problem of developing the BKP hierarchy and to the spin characters and the world of projective representations.

The School of ITMS at the University of Ballarat provided many resources during the writing up stage, and is worthy of a special mention here. Firstly, I would like to thank Professor Sid Morris. His belief in, passion for, and support of, mathematics in Australia provided me with enough spark to re-ignite my interest in this thing when I'd all but chucked it in. Also from ITMS, I am grateful for the friendships of Dr Prahbu Manyem and Dr David Yost.

I have left until almost last, Dr Karl Hoffman from University of New Orleans. I will always be grateful of the experience of an impromptu lecture provided by Dr Hoffman on projective representations. When I say a lecture, I mean that upon being asked if he knew anything of the topic, he indicated I should sit down and proceeded to provide the most insightful account, from any source, I have come across. Every question asked was elaborated upon in great detail until the point came that I believed that his knowledge is perhaps quite limitless.

'Til last I have left the uppermost thanks of all, to those who make up the class of the most important people in my life:

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for my childhood memories - my brother Mark
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Introduction

Mathematics is a universal language able to be shared between people from nations all over the world. Not only is mathematics a beautiful abstract expression of patterns, it is also used to describe the world around us, be it in physics or engineering, chemistry and biology. When one starts out on a PhD, the young mathematician has already begun to feel that all areas of mathematics are, in some sense, connected. Of course, since Gauss, it has become impossible to gain a grasp on all areas; yet one then realizes that a solid foundation in any one of the areas of algebra, analysis, topology or number theory for example; will provide some understanding of the others. For me, algebraic structure is the attraction.

Graded rings provide an elegant way to describe many situations. In this thesis, we give some new results from the abstract area of the Jacobson radical of graded rings. As a 'real-world' example of a graded ring without unity, we include the ring of Hirota derivatives. Also, via symmetric functions we provide a new combinatorial recipe for calculating the spin characters of the symmetric group.

Every effort has been made to present these topics in such a way as to be easily understood.

Semigroup graded rings were first mentioned by Schiffels in 1960 [64] with a focus on rings graded by cancellative semigroups. Indeed, rings graded by cancellative semigroups or commutative semigroups have been of particular interest (see [2], [9], [11], [12], [13], [21], [33] and [53], for example) and results for group graded rings are plentiful (see [3], [8], [17], [49], [50] and [54] for example). Throughout, we try to extend the definitions and results to as wide a class of graded rings as possible. By considering rings graded by partial groupoids, we achieve this.

Groupoid graded rings have only been considered more recently and the first positive results on groupoid graded rings were obtained by Kelarev in [38]. Indeed, one can view generalized matrix rings as graded by partial groupoids; but it is not associativity which the structure grading the ring lacks, it's completeness of the operation. It makes sense in this case to complete the operation by adjoining a zero. In doing so, the grading structure becomes the Brandt semigroup of matrix units (with a zero). And so our main consideration for groupoid graded rings is to give as generalized an account as possible.

Amitsur's comprehensive account of Morita contexts [2] appeared in 1971. In this thesis, we approach Morita contexts as rings graded by the Brandt semigroup of order 2. We provide a definition of an S-invariant radical, and Amitsur's paper includes a result which says, using this definition, that the Jacobson radical is S-invariant. Anderson, Divinsky and Sulinski wrote an impressive paper [3] in 1985 in which a definition of an invariant radical appeared.

Subsequent to Amitsur's paper is the work of Jespers and Wauters [69] on generalized matrix rings which appeared in 1989. One of their results tells of the invariance of the Jacobson radical. Others have published on the concept of S-invariance, such as [30], [31], [62], and [63]. In some senses, the previous results in this area have forced our definition of S-invariance, in order, at least, to sort out the different semigroups which satisfy the property.

We first examine the results of these previous authors in the context of our new definition of S-invariance. We are able to provide necessary and sufficient conditions on S to ensure that for any ring graded by S, its Jacobson radical is S-invariant. Our main theorem and lemmas from Chapter 3 appear in [39]. Gardner and Kelarev [22] have subsequently generalized our results and found that the concept of S-invariance extends to other radical classes.

Graded rings with finite support have been actively investigated recently (see, for example, [7], [13], [19]). Evidently, every group graded ring with finite support is a cancellative homogeneous sum. We show that homogeneous sums are graded by an induced partial groupoid. It's convenient to adjoin a zero to partial groupoids, and this is the approach we take early on with the Brandt groupoid. In the case a zero is adjoined, the partial groupoid becomes a semigroup, and so we describe both generalized matrix rings and Morita contexts as Brandt semigroup graded rings.

Homogeneity of the radical has received a lot of attention over the past few decades (for example in [5], [6], [8], and [50]). The consistent approach has been to investigate connections between the graded Jacobson radical and the radical itself.

In the book of Năstăsescu and Van Oystaeyen [49] on group graded rings, two equivalent descriptions of the graded Jacobson radical appeared, for rings with unity. Two years later Năstăsescu [50] found that $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ for a finite group G of order $n \in \mathbb{N}$ where R is a G-graded ring with unity and $\mathcal{J}_{gr}$ is the

G-graded Jacobson radical. It seems an equivalent lemma in the case for rings without unity had appeared in a preprint of Bergman [8]. No confirmation of this 'folklore' lemma has yet appeared. The lemma in itself is okay; but what is not clear is whether or not the property relates to rings with unity or to those without. To this end, we provide a carefully considered definition of the graded Jacobson radical for rings without unity. Using this description of the graded Jacobson radical we are able to circumvent any potential problem by showing that if the lemma is true for the case of rings with unity, then it is also true for the case of rings without unity.

In Chapter 4, we describe homogeneity situations using a partial groupoid grading. We are able to generalize Bergman's 'folklore' lemma for finite group graded rings to rings graded by partial groupoids. Our main theorem, along with several corollaries and examples from Chapter 4 appears in [39] and [41]; noteworthy is that at that time we had not made the necessary precursive connection to rings without unity that we do indeed give here.

So, why all the fuss about unity? For those of us who prefer not to afford the ring a unity, we are from time to time pestered by others who can not see the point. Indeed what is often demanded is: "Can you describe to me a practical example of a ring without unity that has natural applications?" We attempt to provide such a ring here - a graded one of course - and in order to provide a suitably *practical* example we conduct a thorough search through the ring of symmetric functions. It turns out that this graded ring is naturally combinatoric in behaviour and so you will find from time to time some discussion that may seem a little out of place. If you make it to the end of the text, then hopefully the reasons for their inclusion are put into context.

A further motivation for including a somewhat comprehensive account of the ring of symmetric functions is that it is a highly applicable graded ring, prominently used by mathematical physicists in a diverse range of settings.

Soliton equations are a non-dispersing shallow water wave equation admitting abundant exact solutions (see [26] for example). In 1971, Hirota [25] developed a method for writing these non-linear P.D.E.s as linear combinations of differential operators, called Hirota derivatives. We describe the ring of Hirota derivatives as a graded ring without unity.

In 1983 Jimbo and Miwa [34] released a landmark paper on soliton equations. Included in the paper is a description of the KP and BKP equations in Hirota form. The equations come in a hierarchy. The BKP equations, for example, occur at every even weight larger than 6. In 1988 Nimmo [51] developed a connection between the Schur S-functions and the KP hierarchy using Young diagrams, and left the problem of doing the same thing for the BKP equation open. We utilize the ideas developed by Nimmo [51] to give the Hirota derivatives a relevant meaning in the theory of symmetric functions. This leads us to introduce generalized Q-operators. We make some connections between the BKP hierarchy and the generalized Q-operators, and also some conjectures, leaving many aspects of this study open for further investigation. By representing the generalized Q-operators as shift Young diagrams, we generate the lower weight portion of the BKP hierarchy. Initial conjectures were published with Salam [55], although the required foundation was not included. All the required details are included here to ensure that the construction is valid and does indeed imply symmetric function behaviour. The approach given here is different from that in [32] and [52], in that we do not use supersymmetric functions.

In 1911, Schur [65] described various symmetric functions and introduced the spin characters of the symmetric group in his definitive paper on the projective representation of the symmetric group. The spin characters were not paid much regard until the 1960s when Morris wrote a very comprehensive account [46] and gave some recursive formulae for them. More recently, spin characters have been of interest. For example in 1995, Morris [48] gives further improvements and results subsequent to his previous work. Also, it is with the spin characters that we are able to describe the generalized Q-functions we use to (partly) develop the BKP hierarchy.

By examining the various transition matrices between the different bases of the ring of symmetric functions, we are able to write the spin characters of the symmetric group in terms of the ordinary characters of the symmetric group. This approach allows us to describe a new, non-recursive, combinatorial algorithm for the spin characters.

An index is included at the back of the thesis followed by a glossary of notation. Entries in the glossary of notation with a $\star$ in place of a page reference are not explicitly introduced in the text of the thesis. They are used in this thesis, and so are included in the list to provide clarity and completeness.

The Bibliography includes some entries (specifically [57] and [58]) which are not directly referred to elsewhere in the text. We include them in the Bibliography as they are important papers in semigroup theory which have been influential to the investigations here.

We hope you find this thesis self-contained, enjoyable, comprehensive and easy to read.

Chapter 1

Preliminaries

This chapter introduces background information and prerequisites for understanding the thesis. It contains the standard facts that we need, well known in the theories of semigroups, rings and radicals, partitions, and Young diagrams. Our new results and related terminology are presented in the later chapters.

1.1 Groupoids and groups

A *groupoid* $(S, \circ)$ is a non-empty set S together with a binary operation $\circ$. The binary operation on the set is not necessarily associative. The properties of the binary operation are often used to describe the groupoid. For example, a set S together with a commutative binary operation is called a commutative groupoid. It is worth pointing out that in many texts, and indeed in this thesis, we denote the groupoid $(S, \circ)$ more simply as S. Of course, the operation on S must always be made clear.

If a groupoid is associative, then it is called a *semigroup*. Properties and standard results from semigroup theory will be discussed in Section 1.2.

A subset T of a groupoid S is *closed under the operation* $\circ$ on S if $s \circ t \in T$ for any $s, t \in T$. The binary operation just defined on T is said to be the operation *induced on* T *from* S. A non-empty subset T of S forms a *subgroupoid* if it is closed under the operation induced from S.

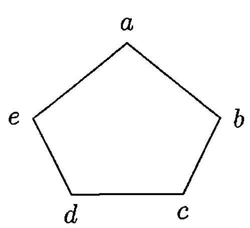
A subgroupoid T of a groupoid S is a *left* (*right*) *ideal* of S if $ST \subseteq T$ (respectively, $TS \subseteq T$). If T is both a left and right ideal of S then T is called a two-sided ideal, or more simply, an *ideal* of S.

If A is a non-empty subset of a groupoid S, then the intersection of all subgroupoids of S containing A is also a subgroupoid of S denoted by $\langle A \rangle$. An element a of a groupoid S *generates* S and is a *generator* of S if $\langle a \rangle = S$. If A is a non-empty subset of S and $\langle A \rangle = S$, then A is called a *set of generators* of S.

A groupoid S is *left* (*right*) *cancellative* if the equality $zx = zy$ (respectively, $xz = yz$) implies $x = y$ for any $x, y, z \in S$. A groupoid S is *cancellative* if it is both left and right cancellative.

A *partial* binary operation $\circ$ on a set S will assign to *some* of the ordered pairs, a value in S. In other words, a partial binary operation on S is a mapping from a non-empty proper subset of $S \times S$ into S. When discussing the properties of partial groupoids we use the standard terminology of groupoid and semigroup theory assuming that the corresponding products used in the definition are defined. For example, a partial groupoid is *left* (*right*) *cancellative* if the equality $zx = zy$ (respectively, $xz = yz$) implies $x = y$ for any $x, y, z \in S$ whenever the products zx and zy (respectively, xz and yz) are defined.

Example 1.1 Let n be an odd integer. Consider a set S with n elements and let each element of the set S be represented by a vertex of an n-sided regular polygon. We define a partial binary operation on S by setting $a \circ b$ equal to the element represented by the vertex of the polygon opposite the side connecting a and b when a and b are next to each other; and undefined otherwise. The operation is cancellative and commutative; but not associative. To see this, let the set $S = \{a, b, c, d, e\}$ be represented by a pentagon and consider the product $a \circ b \circ c$.



$$\begin{aligned} (a \circ b) \circ c &= d \circ c = a \\ a \circ (b \circ c) &= a \circ e = c \quad \neq (a \circ b) \circ c. \end{aligned}$$

The set S under this operation is a finite cancellative partial groupoid.

If a groupoid S satisfies the identity $xy = x$ (respectively, $xy = y$) for any $x, y \in S$, then associativity is easily verified and S is called a *left* (*right*) *zero semigroup*.

A semigroup with identity is called a *monoid*. Suppose $(G, \circ)$ is a monoid with identity element e and that for each element $a \in G$, there exist an *inverse element* $b \in G$ such that $a \circ b = b \circ a = e$, then G is said to form a *group* under the operation $\circ$. If the operation is also commutative, that is if $a \circ b = b \circ a$ for all $a, b \in G$, then the group G is called *Abelian*.

In this thesis, we are not concerned directly with groups; however we do make use of two results from finite group theory. The first is a consequence of Lagrange's Theorem, and the other a consequence of Sylow's First Theorem.

Theorem 1.2 (Lagrange's Theorem, [29], Corollary I.4.6) *If H is a subgroup of a finite group G, then the order $|H|$ of the subgroup H divides the order $|G|$ of the group G. Indeed, the number of distinct left (right) cosets of H in G is $|G|/|H|$.*

In fact, the converse to Lagrange's Theorem is *not* true. When a group G has order n, then for a divisor m of n, there may not exists a subgroup of order m. Sylow's First Theorem tells us which subgroups of certain order must exist.

Theorem 1.3 (Sylow's First Theorem, [29], Theorem II.5.7) *Suppose G is a group of order $p^n m$ where $n, m \in \mathbb{N}$, p is a prime with p and m relatively prime. Then G contains a subgroup of order p^i for each $1 \leq i \leq n$.*

The following corollary is an immediate consequence of Lagrange's Theorem and Sylow's First Theorem.

Corollary 1.4 *For a finite group G, the least common multiple of the order of all subgroups of G equals the order $|G|$ of the group G itself.*

Proof. Lagrange's theorem tells us that the order of any subgroup of G is a divisor of the order $|G|$ of G. So we don't have subgroups of orders which aren't multiples of the prime factors of $|G|$. This means that the least common multiple of the order of all subgroups is bounded above by $|G|$. Sylow's Theorem ensures that for each prime factor p_j of $|G|$ diving into $|G|$ say n_j times, there is a subgroup of order $p_j^{n_j}$ of G. This then means that the least common multiple of the order of all subgroups of G is exactly the order of the group G itself. □

Theorem 1.5 ([29], Corollary II.4.4) *Let G be a group with g any element in G and Z_g the centralizer of g in G. Then the number of elements in the conjugacy class H_g of g is determined by*

$$|H_g| = \frac{|G|}{|Z_g|}.$$

We illustrate this theorem in the context of the symmetric group in Example 1.30.

1.2 Semigroups

The terms introduced in Section 1.1 for groupoids also apply to semigroups.

An element a of a semigroup S is *regular* if there exists $b \in S$ such that $aba = a$. A semigroup in which every element is regular is called a *regular semigroup.* Two elements a and b of a semigroup S are *inverses* of each other if $aba = a$ and $bab = b$. A semigroup that contains a unique inverse element for every element in the semigroup is called an *inverse semigroup.*

Example 1.6 Suppose a is a regular element of a semigroup S, say $axa = a$ with $x \in S$. Consider the element $b = xax$. Then

$$aba = a(xax)a = ax(axa) = axa = a$$

and

$$bab = (xax)a(xax) = x(axa)(xax) = xa(xax) = x(axa)x = xax = b.$$

Hence b is an inverse of a.

An element e of a semigroup is *idempotent* if $e^2 = e$. A semigroup S is a *band* if all elements of S are idempotent. A commutative band is called a *semilattice*.

Let $E(S)$ denote the set of idempotents of a semigroup S. For idempotents $e, f \in E(S)$, we define a partial order on $E(S)$ by writing $e \leq f$ if $ef = fe = e$ and we say that "e is less than or equal to f". If $e \leq f$ and $e \neq f$ then we write $e < f$ and say that e is less than f. If the semigroup S contains a zero element 0, then 0 is less than any other idempotent element of S. A nonzero idempotent e of a semigroup S is said to be *primitive* if 0 is the only idempotent of S less than e. A semigroup with 0 is *null* if $ab = 0$ for all $a, b \in S$.

The definition of a subsemigroup is analogous to that of a subgroupoid. That is, any non-empty subset T of a semigroup S is a *subsemigroup* of S if T is closed under the operation $\circ$ on S (and T is itself a semigroup under the binary operation induced on T from S). Associativity in T is guaranteed since $\circ$ is an associative binary operation on S and T is a subset of S.

A proper two-sided (left, right) ideal T of a semigroup S is (left, right) *maximal* if it is not contained in any other proper two-sided (left, right) ideal of S. A two-sided (left, right) ideal M of a semigroup S is said to be (left, right) *minimal* if it does not properly contain any other nonzero two-sided (left, right) ideal of S. If a semigroup S has a minimal two-sided ideal K, then K is unique and is contained in every other two-sided ideal of S. This minimal ideal K is called the *kernel* of S. A semigroup is said to be *simple* if it has no proper two-sided ideals. For example, the kernel of a semigroup, if it exists, is itself a simple semigroup. A semigroup S with zero is said to be 0-*simple* if $S^2 \neq 0$ and the only proper two-sided ideal of S is 0. With $S^2 \neq 0$ we exclude the null semigroup.

By a *completely* 0-*simple semigroup*, we mean a 0-simple semigroup containing a primitive idempotent.

Example 1.7 Let I be a set and put $S = (I \times I) \cup \{0\}$ where the element 0 is distinct from any element in I. For $i, j, k, l \in I$ define a binary operation $\cdot$ by

$$\begin{aligned}(i,j)\cdot(k,l) &= \begin{cases} (i,l) & \text{if } j = k \\ 0 & \text{otherwise}\end{cases} \\ 0\cdot(i,j) &= (i,j)\cdot 0 = 0\cdot 0 = 0.\end{aligned}$$

Then S is closed under this operation and forms a semigroup. The elements $(i, j) \in S$ are called $I \times I$ *matrix units.* Consider any element of the form $(i, i) \in S$. Then $(i, i)^2 = (i, i) \cdot (i, i) = (i, i)$ and we see that these elements are the only nonzero idempotents of S. Since S has nonzero idempotents, then $S^2 \neq 0$. In fact, S is completely 0-simple. To see this, suppose that S has an ideal A that contains a nonzero element $(i, j) \in A$. Then for any element $(k, l) \in S$, there is $(k, i), (j, l) \in S$ so that $(k, i) \cdot (i, j) \cdot (j, l) = (k, l)$. This means that $S = S(i, j)S$. Since A is an ideal of S and $(i, j) \in A$, we see that $S(i, j)S \subseteq SAS \subseteq A$. So $A = S$, and S is 0-simple. Now, for any two distinct nonzero idempotent elements $(i, i), (j, j) \in S$, we see that $(i, i) \cdot (j, j) = 0$. So the idempotents are primitive, and S is completely 0-simple.

Lemma 1.8 ([14], Theorem 2.51) *If a semigroup is completely* 0*-simple, then it is regular.*

We denote by S^0 the semigroup S with a zero element adjoined and by S^1 the semigroup S with an identity element adjoined. So $S^0 = S \cup \{0\}$, and $S^1 = S \cup \{1\}$. For all purposes of our proofs, in the thesis, we may assume that 0 and 1 are external to S in the notation just given.

Let a be an element of a semigroup S. A *principal ideal* is an ideal generated by a single element, and we denote by:

$L(a)$ the principal left ideal $S^1a = \{xa \mid x \in S^1\}$ of S generated by a;

$R(a)$ the principal right ideal $aS^1 = \{ax \mid x \in S^1\}$ of S generated by a;

$J(a)$ the principal (two-sided) ideal $S^1aS^1 = \{xay \mid x, y \in S^1\}$ of S generated by a.

The set of all generators of S^1a is denoted by L_a; the set of all generators of aS^1 is denoted by R_a; and the set of all generators of S^1aS^1 is denoted by J_a.

Let a and b be elements of a semigroup S. The *Green equivalence relations*, $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ are defined by:

$a\mathcal{L}b$ if and only if $L(a) = L(b)$

and we say that a and b are in the same $\mathcal{L}$-class;

$a\mathcal{R}b$ if and only if $R(a) = R(b)$

and we say that a and b are in the same $\mathcal{R}$-class;

$a\mathcal{J}b$ if and only if $J(a) = J(b)$

and we say that a and b are in the same $\mathcal{J}$-class.

The following results can be found in Clifford and Preston [14] or Howie [28] and are stated here without proof.

Lemma 1.9 ([14], Lemma 1.13) *An element a of a semigroup S is regular if and only if $a\mathcal{L}e$ or $a\mathcal{R}e$ for some idempotent element $e \in S$.*

In other words, an element a of a semigroup S is regular if and only if the principal left (right) ideal of S generated by a has an idempotent generator.

Theorem 1.10 ([14], Theorem 1.17) *A semigroup S is inverse if and only if every principal left ideal and every principal right ideal of S has a unique idempotent generator.*

Lemma 1.11 ([14], Lemma 2.14) *Any idempotent element e of a semigroup S is a right identity element of L_e and a left identity element of R_e.*

Let I be an ideal of a semigroup S. The *Rees factor semigroup* S/I is described as the semigroup formed by taking every element of $S\backslash I$ (under the operation induced from S), adjoining a zero element, and identifying every element of I with the zero element. Sometimes we say that I is *factored out* of S.

Let S be a semigroup. Consider the principal two-sided ideal $J(a)$ generated by an element a of S. Denote by $I(a)$ the set consisting of all those elements of $J(a)$ which do not generate $J(a)$. That is, $I(a) = J(a)\backslash J_a$. It is easy to see that if $I(a)$ is non-empty, it is an ideal of S. Indeed, look at any $x \in I(a)$ and $y \in S$. Since $x \in I(a) \subset J(a)$ and $J(a)$ is an ideal, it follows that $xy \in J(a)$. Also $J(xy) \subseteq J(x) \subset J(a)$ so xy does not generate $J(a)$. Hence $xy \in I(a)$. Similarly $yx \in I(a)$. Since $I(a)$ is an ideal of S contained in $J(a)$, it is an ideal of $J(a)$. We call the Rees factor semigroup $F_a = J(a)/I(a)$ a *principal factor* of S.

Lemma 1.12 ([14], Lemma 2.39) *Each principal factor of a semigroup S is either 0-simple, or simple, or null. Only if S has a kernel is there a simple principal factor, and in this case, the kernel is the only principal factor.*

Corollary 1.13 *Let e be a primitive idempotent of a semigroup S. Then the principal factor $F_e = J(e)/I(e)$ is completely 0-simple.*

Suppose S is any semigroup and let I and J be indexing sets with $i \in I$ and $j \in J$. We adjoin a zero element 0 to S that is not contained in either of the indexing sets I or J. By an *$I \times J$ matrix over S^0*, we mean a mapping A from $I \times J$ into S^0. The assignment of the element (i,j) to $a \in S^0$ is denoted by a_{ij}. If $|I| = n$ and $|J| = m$, the $I \times J$ matrix over S^0 is an array of elements of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

where each $a_{ij} \in S^0$. We write $A = (a_{ij})$. We use $(a)_{ij}$ to mean the matrix over S^0 having $a \in S^0$ in the i^{th} row and j^{th} column and zeros everywhere else and call $(a)_{ij}$ a *Rees $I \times J$ matrix over S^0*. It is important to notice the positioning of the brackets, since this distinguishes an $I \times J$ matrix over S^0 from a Rees $I \times J$ matrix over S^0. In the special case of an $I \times I$ matrix over S^0 where $S = \{e\}$, the matrices $(e)_{ij}$ with $i, j \in I$ are isomorphic with the matrix units (i,j) introduced in Example 1.7. Sometimes the matrix units $(e)_{ij}$ are denoted more simply by e_{ij} when, in context, it is clear that we are referring to matrices rather than to a matrix entry.

Let I and Λ be sets with $i, j, k \in I$ and $\lambda, \mu, \nu \in \Lambda$. For any semigroup S, we can define a binary operation $\circ$ on the set of Rees $I \times \Lambda$ matrices over S^0. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over S^0 and for any $(a)_{j\mu}$, $(b)_{k\nu}$ put

$$(a)_{j\mu} \circ (b)_{k\nu} = (ap_{\lambda i}b)_{j\nu}$$

where $a, b, p_{\lambda i} \in S^0$. The set of all Rees $I \times \Lambda$ matrices over S^0 forms a semigroup under this operation. We call this semigroup the *Rees $I \times \Lambda$ matrix semigroup over the semigroup with zero S^0 with sandwich matrix P* and denote it by $\mathcal{M}^0(S; I; \Lambda; P)$.

Theorem 1.14 (Rees' Theorem, [14], Theorem 3.5) *A semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero.*

Example 1.15 Let I be an indexing set with $i, j \in I$ and consider the one-element group $G = \{e\}$. We adjoin a zero to G and describe the $I \times I$ *identity matrix* Δ over G^0 as having elements (δ_{ij}) determined by

$$\delta_{ij} = \begin{cases} e & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

If we use the identity matrix Δ as the sandwich matrix, then the set of Rees $I \times I$ matrices $(e)_{ij}$ over G^0 forms a Rees $I \times I$ matrix semigroup over $\{e, 0\}$. In fact, the semigroup so formed is clearly isomorphic with the completely 0-simple semigroup of matrix units from Example 1.7. It follows from Rees' Theorem that $\mathcal{M}^0(\{e, 0\}; I; I; \Delta)$ is a regular semigroup. In fact, we shall see in the next Theorem that $\mathcal{M}^0(\{e, 0\}; I; I; \Delta)$ is an inverse semigroup.

The Brandt groupoid was introduced by Brandt in 1927 [10] as a particulary elegant example of a partial groupoid satisfying a set of 4 axioms. We shall restrict our interest in the Brandt groupoid to the case when the operation is complete. This is done by simply sending any undefined pairs to an adjoined zero. In doing this we form a semigroup. Also, we shall not make use of the axiomatic definition here, but rather, speak of the Brandt semigroups as the Rees matrix semigroups that they are isomorphic to. For further details on the axiomatic approach and the Brandt groupoid, the reader is referred to [14].

The Rees matrix semigroup $\mathcal{M}^0(G^0; I, I; \Delta)$ over a group G^0 with zero, indexing sets I with $|I| = n$ and sandwich matrix given by the $n \times n$ identity matrix Δ, is called the *Brandt semigroup* and is denoted by $B(G, n)$.

We sometimes write the elements of $B(G,n)$ as triples by putting $B(G,n) = (I \times G \times I) \cup \{0\}$ with multiplication of elements given by

$$\begin{aligned}(i,g,j)\cdot(k,h,l) &= \begin{cases}(i,gh,l) & \text{when} \quad gh \neq 0\\ 0 & \text{otherwise}\end{cases}\\ (i,g,j)\cdot 0 &= 0\cdot(i,g,j) = 0\cdot 0 = 0\end{aligned}$$

In the case when $G^0 = \{e, 0\}$, the Brandt semigroup is just the semigroup of $n \times n$ matrix units, and so we denote the semigroup of $n \times n$ matrix units more simply by B_n.

Theorem 1.16 ([14], Theorem 3.9) *A semigroup S with zero is a Brandt semigroup if and only if S is a completely 0-simple inverse semigroup. In fact, the following three conditions on a semigroup S with zero are equivalent.*

(i) S is a Brandt semigroup.

(ii) S is a completely 0*-simple inverse semigroup.*

(iii) S is isomorphic with a Rees $I \times I$ matrix semigroup over a group with 0 *with the $I \times I$ identity matrix Δ as the sandwich matrix.*

Example 1.17 Let A be a set and suppose $\{S_\alpha \mid \alpha \in A\}$ is a family of semigroups each with a zero element. We can identify all zeros of each these semigroups S_α with, and denote them by, the same symbol 0.

Let $S_\alpha^* = S_\alpha/0$ mean the set S_α with zero disjoined. Now let S consist of the zero element together with the disjoint union of all sets S_α^*. Define the product of two elements x and y of S to be their product in S_α if they are from the same S_α; or zero otherwise. That is, $S_\alpha S_\beta = 0$ if $\alpha \neq \beta$. Then S together with the product just described, forms a semigroup called the 0-*direct union* of the semigroups S_α.

Clearly, S is a regular semigroup if each of the semigroups S_α is regular, and every primitive idempotent in S_α is also primitive in S. Since completely 0-simple semigroups are regular (Lemma 1.8) and, by definition, contain primitive idempotents, it follows that the 0-direct union of completely 0-simple semigroups is regular and contains a primitive idempotent.

Theorem 1.18 ([15], Theorem 6.39 and Exercise 6.5-5) *A semigroup S is a 0-direct union of completely 0-simple inverse semigroups if and only if S is an inverse semigroup in which every nonzero idempotent is primitive.*

1.3 Rings and radicals

We refer to the ring $(R, +, \times)$ more simply as R, and write the multiplication by juxtaposing the elements. If the multiplicative operation in a ring R is commutative, we call R a *commutative ring.* If R is not the ring 0 consisting of just the zero element and if a multiplicative identity exists then it is written as 1 and the ring is said to be a ring *with unity* or a ring with an identity element.

Throughout, we assume that all rings are associative, but it is neither assumed that they are commutative or with unity.

The notation $I \lhd R$ will mean "I is an ideal of R". A ring R is said to be *simple* if $R^2 \neq 0$ and R has no proper two-sided ideals.

If A is any subset of a ring R, we denote by $\langle A \rangle$ the smallest ideal of R containing A, and this is equal to the intersection of all ideals of R which contain A. We call $\langle A \rangle$ the *ideal of R generated by A.*

Suppose I is an ideal of a ring R, then the *residue classes modulo I* form a ring called the *quotient ring*, denoted by R/I. Sometimes we say that I is *factored out* of R. If R is commutative or with unity, then the same is true for R/I. Every ideal of a ring is the kernel of some ring homomorphism. In fact, any homomorphic image of a ring R is isomorphic to some quotient ring R/I of R. This property is the First Isomorphism Theorem ([29], Corollary III.2.10).

Theorem 1.19 (Second Isomorphism Theorem, [29], Theorem III.2.12) *Suppose I and J are ideals of a ring R. Then $I/(I \cap J)$ is isomorphic to $(I + J)/J$.*

Let R be a ring. A *left R-module* consists of an additive Abelian group A and a ring R together with a mapping $R \times A \to A$. This mapping acts as a multiplication on each element of A by each element of R *on the left*, and so we denote the image of (r, a) in A by $ra \in A$. For all $r, s \in R$ and $a, b \in A$ the following conditions must be met:

(i) $r(a + b) = ra + rb$;

(ii) $(r + s)a = ra + sa$;

(iii) $r(sa) = (rs)a$.

Properties (i) and (ii) mean that the mapping is *linear* and property (iii) means that the mapping is *associative*. A right R-module is defined analogously.

Let R and S be rings. An Abelian group A is an *R–S bimodule* provided that A is both a left R-module and right S-module and

(iv) $r(as) = (ra)s$

for all $a \in A$, $r \in R$ and $s \in S$.

A *class* is a collection of objects which satisfy certain defining properties. Classes are more fundamental than sets. Hungerford [29] provides and elegant example which distinguishes the subtle difference between a class and a set.

Example 1.20 ([29], pp.2) Consider the class $M = \{X \mid X \text{ is a set and } X \notin X\}$. M is a class; but not a set. To see this, suppose M is a set. Then either $M \in M$ or $M \notin M$. However, if $M \in M$ then $M \notin M$, and similarly if $M \notin M$ then $M \in M$. These contradictions mean that M cannot be a set.

A class $\mathcal{X}$ of rings is called a *radical class* if $\mathcal{X}$ satisfies the following axioms:

(i) $\mathcal{X}$ is closed under homomorphisms.
That is, if $R \in \mathcal{X}$ and $I \lhd R$ then $R/I \in \mathcal{X}$;

(ii) Every ring R contains an $\mathcal{X}$-ideal, $\mathcal{X}(R)$, which contains every other $\mathcal{X}$-ideal of R;

(iii) (a) $\mathcal{X}$ is closed under extension.
That is, for $I \lhd R$ if $I \in \mathcal{X}$ and $R/I \in \mathcal{X}$ then $R \in \mathcal{X}$;

or

(b) $\mathcal{X}\left(R/\mathcal{X}(R)\right) = 0$.

For any ring R, the $\mathcal{X}$-ideal described in property (ii) is the $\mathcal{X}$-radical of R. If a ring R is in the radical class $\mathcal{X}$, then R is an $\mathcal{X}$-radical ring. For any radical class $\mathcal{X}$, we consider the class $\mathcal{S}$ of all rings R for which $\mathcal{X}(R) = 0$, and call $\mathcal{S}$ the *semisimple class* corresponding to $\mathcal{X}$. We call a ring R in the semisimple class $\mathcal{S}$ corresponding to $\mathcal{X}$, an $\mathcal{X}$-semisimple ring.

A radical class $\mathcal{X}$ is *hereditary* if every ideal of an $\mathcal{X}$-radical ring is itself an $\mathcal{X}$-radical ring. This means that for any ring R and ideal I of R, the equality $\mathcal{X}(I) = I \cap \mathcal{X}(R)$ holds.

An element a of a ring R is *quasiregular* if there is some b in R such that $a \circ b = a + b - ab = 0$. From this point on, we reserve the use of the symbol $\circ$ to be the binary operation we have just described.

A *quasiregular ring* is one in which all elements are quasiregular. In this case, for each $a \in R$ there exists a unique $b \in R$ with $a \circ b = b \circ a = 0$. We say that b is the *quasi-inverse* of a.

The class $\mathcal{J}$ of all quasiregular rings is an hereditary radical class called the *Jacobson radical class.* If R is a ring, then there is a quasiregular ideal $\mathcal{J}(R) \in \mathcal{J}$ of R which contains every other quasiregular ideal of R. The ideal $\mathcal{J}(R)$ is the *Jacobson radical* of the ring R. Throughout this thesis we reserve use of the symbol $\mathcal{J}(R)$ for the Jacobson radical of the ring R, and when we say a ring is semisimple we mean that it is $\mathcal{J}$-semisimple. Also, we might sometimes call the Jacobson radical of the ring, more simply, the radical of the ring.

When we speak of idempotent elements in a ring R, we simply mean those elements $e \in R$ which are idempotent in the semigroup formed by R under multiplication.

Example 1.21 Let R be a ring and suppose the Jacobson radical $\mathcal{J}(R)$ contains some idempotent $e \in R$. Since all elements of $\mathcal{J}(R)$ are quasiregular, there is some $a \in \mathcal{J}(R)$ with $a \circ e = a + e - ae = 0$. Multiplying by e on the right gives $ae + e - ae = 0$ yielding $e = 0$. So for any ring R, the Jacobson radical $\mathcal{J}(R)$ contains no nonzero idempotents.

Since fields contain a multiplicative identity, it follows from Example 1.21 that all fields are semisimple.

We use $\mathbb{C}$ to denote the field of complex numbers; $\mathbb{R}$ to denote the field of real numbers; $\mathbb{Z}$ to denote the ring of integers; and $\mathbb{N}$ or $\mathbb{Z}^+$ to denote the set of natural numbers.

Theorem 1.22 ([29], Theorem ix.2.3) *If R is a ring, the following properties hold for the Jacobson radical $\mathcal{J}(R)$ of R:*

(i) $\mathcal{J}(R)$ is the intersection of all the left annihilators of simple left R-modules;

(ii) $\mathcal{J}(R)$ is the unique largest quasiregular ideal of R.

Example 1.23 Suppose R is a nil ring. Then for any $r \in R$ there is a positive integer $n \in \mathbb{N}$ such that $r^n = 0$. It follows that

$$r \circ \left(-r - r^2 - r^3 - r^4 + \ldots + -r^{n-1}\right) = 0.$$

So every nilpotent element of a ring has a quasi-inverse. In fact any nil ideal is a quasiregular ideal and so the Jacobson radical contains all nil ideals.

Suppose $(R, +, \times)$ is a ring and let I be an indexing set of order $|I| = n$, with $i, j \in I$. An $n \times n$ matrix over R is a mapping M from $I \times I$ into R. The assignment of the element (i, j) to $r \in R$ is written r_{ij}. We put $M = (r_{ij})$ in the same way we did for a matrix over a semigroup (pp. 10) , and addition $+_m$ in M is given by $(r_{ij}) +_m (t_{ij}) = ((r + t)_{ij})$ for $r, t \in R$. Since $(R, +)$ is an Abelian group $(M, +_m)$ forms an Abelian group. Multiplication $\times_m$ in M is described by $(r_{ij}) \times_m (t_{ij}) = ((\sum_{k=1}^{n} r_{ik} t_{kj})_{ij})$, and $(M, \times_m)$ forms a semigroup. So $(M, +_m, \times_m)$ is a ring called the $n \times n$ matrix ring over the ring R.

An $n \times n$ matrix over a ring R is often called a *square matrix*, and is denoted by $M_n(R)$. For $i, j \in I$, the entries $r_{ij} \in R$ with $i = j$ are called *diagonal entries.* If $r_{ij} = 0$ whenever $j > i$ we say that the "entries above the diagonal" are zero, and call the matrix *lower triangular.* Similarly, if $r_{ij} = 0$ whenever $i > j$ we say the "entries below the diagonal" are zero and the matrix is called *upper triangular.* These definitions only make sense for the class of square matrices.

We conclude this section with a property of the Jacobson radical of square matrices over a ring. For proof or more detail the reader is referred to Karpilovsky ([37]).

Proposition 1.24 ([23], Corollary 4.9.7 with Theorem 4.9.3) *For any ring R and any positive integer n,*

$$\mathcal{J}(M_n(R)) = M_n(\mathcal{J}(R)).$$

Corollary 1.25 *The ring of $n \times n$ matrices over any field is semisimple.*

1.4 Partitions

This section introduces the terminology of partitions and the symmetric group, along with the preliminaries required for Chapter 5, where we investigate some of the relationships in the ring of symmetric functions.

Each group of finite order is isomorphic either to a symmetric group or to a subgroup of a symmetric group, and so the symmetric group may be considered as one of the most elegant and useful examples in elementary group theory.

A *permutation of n objects* is a bijection from the set $\{1, 2, \cdots, n\} \subseteq \mathbb{N}$ onto itself. The *symmetric group S_n* is the set of all permutations of n objects under composition. Composition of permutations is performed from right to left. So $\pi\rho$ is the permutation obtained by first applying ρ, followed by π. The order of the symmetric group is $n!$.

If π is a permutation then the *two-line notation* for π is the array

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

which one might read as "1 goes to $\pi(1)$, 2 goes to $\pi(2)$, etc." The two-line notation can become a little cumbersome. Another standard way of writing a permutation is as a product of cycles.

A *cycle of length k*, $(i_1, i_2, \cdots, i_k)$, is an ordered k-tuple of elements from a subset of the numbers $\{1, 2, ..., n\}$ (with $n \geq k$), which are exchanged amongst themselves. More specifically, the cycle $(i_1, i_2, \cdots, i_k)$ is the permutation

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_2 & i_3 & \cdots & i_1 \end{pmatrix}.$$

Example 1.26 Consider the permutations

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

Then $a = (142)(3)$ and $b = (1423)$. We can read the cycle b as "1 goes to 4, 4 goes to 2, 2 goes to 3, and 3 goes to 1", cycling through the numbers from left to right. The identity permutation is $e = (1)(2)(3)(4)$.

Every permutation can be written as a product of *independent cycles*. Independent cycles are ones in which each number appears in, at most, one cycle.

The *cycle structure* of a permutation is the number of cycles of each length appearing in it when it is written as a product of independent cycles. We write the cycle structure as

$$(1^{m_1}, 2^{m_2}, \cdots, n^{m_n})$$

where each m_k is the number of independent cycles of length k. In Example 1.26, the permutation a has cycle structure $(1^1, 2^0, 3^1, 4^0)$ and the permutation b has cycle structure $(1^0, 2^0, 3^0, 4^1)$.

All permutations of the symmetric group $\mathcal{S}_n$ with the same cycle structure belong to the same conjugacy class and all permutations in a conjugacy class of $\mathcal{S}_n$ have the same cycle structure. Another way to describe the cycle structure is to use the concept of a partition.

A *partition* λ is a finite list of positive integers $\lambda = (\lambda_1 \lambda_2 \cdots \lambda_d)$ arranged in weakly descending (meaning non-increasing) order so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0.$$

The components λ_i of the partition λ are called *parts*, and the number of parts in a partition λ is called the *length* of the partition and is denoted by $l(\lambda)$. The sum of the parts $\lambda_1 + \lambda_2 + \cdots + \lambda_d = n$ is called the *weight* of the partition and is denoted by $|\lambda|$. We write $\lambda \vdash n$ to mean λ is a partition of weight n and often say, more simply, that λ is a partition of n. We reserve $\mathcal{P}_n$ to mean the set of all partitions of weight n. The number of occurrences of a part λ_i in a partition λ is called the *multiplicity of* λ_i *in* λ and is denoted by $m_{\lambda_i}(\lambda)$ or in context, more simply by m_{λ_i}. We usually write the multiplicity of each part as a superscript with

$$\lambda = \left(\lambda_1^{m_{\lambda_1}} \lambda_2^{m_{\lambda_2}} \cdots \lambda_j^{m_{\lambda_j}}\right).$$

The *reverse lexicographical ordering* $\mathcal{L}_n$ on the set $\mathcal{P}_n$ of all partitions of $n \in N$ is the subset of $\mathcal{P}_n \times \mathcal{P}_n$ consisting of all ordered pairs (μ, λ) such that either

$\mu = \lambda$ or else the first non-vanishing difference $\mu_i - \lambda_i$ is positive. $\mathcal{L}_n$ is a total ordering, which means that every element of $\mathcal{P}_n$ can be put in reverse lexicographical order with any other element of $\mathcal{P}_n$, and if $(\mu, \lambda) \in \mathcal{L}_n$ we write $\mu \geq \lambda$. If $\mu \neq \lambda$ and $\mu \geq \lambda$, we write $\mu > \lambda$ and say that μ is larger than λ in the reverse lexicographical ordering.

Example 1.27 When $n = 5$, $\mathcal{L}_5$ arranges $\mathcal{P}_5$ in the sequence

$$(5), (41), (32), (31^2), (2^21), (21^3), (1^5).$$

Using partitions, we have a neat way of describing the conjugacy classes of the symmetric group. Let's see what we mean here, by example. Notice that we usually call the conjugacy class of a permutation more simply the *class* of the permutation.

Example 1.28 Consider the symmetric group $\mathcal{S}_3$.

Element :	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$;	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$;	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$;	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$;
Cycles :	(1)(2)(3)	(12)(3)	(13)(2)	(1)(23)
Class :	(1^3)	(21)	(21)	(21)

Element :		$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$;	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	
Cycles :		(123)	(132)	
Class :		(3)	(3)	.

The identity element of any symmetric group $\mathcal{S}_k$ is always in a conjugacy class by itself; its cycle structure is (1^k).

Suppose $\pi \in \mathcal{S}_n$ is a permutation with conjugacy class described by $\lambda \in \mathcal{P}_n$ a partition of weight $n \in \mathbb{N}$. Whenever we talk of the conjugacy class H_π of π we use H_λ. The number of elements in the class H_λ is denoted by h_λ. Likewise, Z_λ is used for the centralizer Z_π of π and z_λ denotes the number of elements in the centralizer Z_λ.

Lemma 1.29 ([61], Proposition 1.1.1) For a partition $\lambda \vdash n$ of weight n, the number of elements in the centralizer z_λ is determined by Frobenius' formula

$$z_\lambda = \prod_i \lambda_i^{m_{\lambda_i}} m_{\lambda_i}!$$

where m_{λ_i} is the multiplicity of the i^{th} part of the partition λ and the product is taken over all i for which λ_i is a part of λ.

Let $\pi \in \mathcal{S}_n$ be a permutation with class $\lambda \in \mathcal{P}_n$. Theorem 1.5 tells us that we can determine the size h_λ of each conjugacy class H_λ by the size z_λ of the centralizer. That is, $h_\lambda = |H_\lambda| = \frac{|\mathcal{S}_n|}{z_\lambda}$. For the symmetric group $\mathcal{S}_n$ the size of each conjugacy class is

$$h_\lambda = \frac{n!}{z_\lambda} = \frac{n!}{\prod_i \lambda_i^{m_{\lambda_i}} m_{\lambda_i}!}.$$

Example 1.30 Consider the symmetric group $\mathcal{S}_4$. The number of elements of $\mathcal{S}_4$ is $4! = 24$. The number of partitions of weight 4 is 5. Written in order, they are $(4), (31), (2^2), (21^2)$, and the identity (1^4), and each of these partitions corresponds to a class of $\mathcal{S}_4$. The number of elements in each class of $\mathcal{S}_4$ is:

$h_{(1^4)} = \frac{24}{1^4 \times 4!} = 1;$ $\quad h_{(21^2)} = \frac{24}{2 \times 1^2 \times 2!} = 6;$ $\quad h_{(2^2)} = \frac{24}{2^2 \times 2!} = 3;$

$h_{(31)} = \frac{24}{3 \times 1} = 8;$ $\quad h_{(4)} = \frac{24}{4} = 6.$

Indeed, $1 + 6 + 3 + 8 + 6 = 24$.

Every partition $\lambda \vdash n$ of weight n can be associated with a *Young diagram* Y^{λ} involving n boxes (cells, circles, dots, etc.,) with the i^{th} row containing λ_i boxes.

Example 1.31 Suppose $\lambda_1 = (44321)$ and $\lambda_2 = (63^21)$. Then $\lambda_1 \vdash 14$ and $l(\lambda_1) = 5$ while $\lambda_2 \vdash 13$ and $l(\lambda_2) = 4$. The Young diagrams for λ_1 and λ_2 are

$$Y^{\lambda_1} = \begin{ytableau} \; & \; & \; & \star \\ \; & \; & \star & \star \\ \; & \star & \star \\ \star & \star \\ \star \end{ytableau} \qquad Y^{\lambda_2} = \begin{ytableau} \; & \; & \star & \star & \star & \star \\ \; & \; & \star \\ \star & \star & \star \\ \star \end{ytableau}$$

A *Young tableau* τ for a partition $\lambda \vdash n$ is an assignment of n numbers (not necessarily all different) to the n boxes of the Young diagram Y^{λ}. *Standard numbering* means that the assignment of the numbers $1, 2, \cdots, d \leq n$ is such that the numbers are strictly increasing from left to right across each row and strictly increasing down each column. When the numbering is standard we call the tableau τ *standard.*

There are several methods of *semi-standard numbering.* One of them is *unitary numbering* in which the assignment of the numbers $1, 2, \cdots, d \leq n$ is such that the numbers are weakly increasing (meaning non-decreasing) from left to right across each row and strictly increasing down each column.

The *staircase* of a Young diagram consists of all the boxes in a continuous outer ribbon going from the lower left to the upper right (or vice versa). The staircases of the Young diagrams in Example 1.31 are marked with $\star$'s.

Another type of semi-standard numbering is *regular numbering* in which the assignment of the numbers $1, 2, \cdots, d \leq n$ is such that the numbers are:

(i) weakly increasing from left to right across each row ;

(ii) weakly increasing down each column ;

and

(iii) like digits form a continuous staircase of some subdiagram.

When the numbering is unitary we call the tableau τ *unitary* and when the numbering is regular we call the tableau τ *regular.*

Example 1.32 Suppose $\lambda = (321^2)$. Then $\lambda \vdash 7$, $l(\lambda) = 4$ and the Young diagram for λ is

$$Y^\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{2}{c}{} \\ \cline{1-1} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

For this Young diagram, some examples of unitary tableaux are τ_1 and τ_2

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

τ_1 τ_2

unitary unitary

Some examples of regular tableaux are τ_3 and τ_4

$$\begin{array}{ccc}
\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 1&3\\\cline{1-2} 4\\\cline{1-1} 4\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3\\\cline{1-2} 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \\
\tau_3 & \tau_4 & \tau_5 \\
\text{regular} & \text{regular} & \text{not regular}
\end{array}$$

The tableau τ_5 is not regular since the 2s are not arranged in a continuous staircase of any subdiagram. Actually, τ_5 is not unitary either.

For each Young tableau we define a *word* by reading the numbers in successive rows from right to left, starting at the top row. The numbers which make up the word are called the *elements* of the word. A *standard word* is one in which the numbers $1, 2, \cdots, n$ each occur only once. The *indices* of the elements of the word are defined recursively by :

(i) the number 1 has index 0 ;

(ii) if r has index i, the number $r+1$ has :

  (a) index i if it is to the right of r **or** ;

  (b) index $i+1$ if it is to the left of r.

For each standard word w, the *charge of the word*, $c(w)$, is the sum of the indices of each element of w. The *charge of a nonstandard word* is the sum of the charges of the standard subwords. We extract the subwords recursively by :

(i) Check if the word is standard. If it is, we are done, otherwise - proceed ;

(ii) Starting from the left, mark the first 1 that occurs in the word ;

(iii) Call the largest marked number in the word k, and search for the first occurrence of $k+1$ reading across from k to the end of the word, cycling through to the start of the word if required, always from left to right :

   (a) if $k+1$ is found, mark it and go back to the start of step (iii) to search for, and mark, the appropriate $k+2$.

   (b) if no $k+1$ is found, the marked numbers form a standard subword. Extract the marked standard subword and go back to step (i) to begin the extraction of the next standard subword.

Example 1.33 Consider the following tableau τ which is both unitary and regular:

$$\tau = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} .$$

The word associated with this tableau is $w = 221143$. We extract the standard words:

$$22\underline{1}143$$
$$\underline{2}2\underline{1}143$$
$$\underline{2}2\underline{1}14\underline{3}$$
$$\underline{2}2\underline{1}1\underline{43}$$

and removing the first standard word $w_1 = 2143$ leaves $w_2 = 21$, which is standard. So we have the set W of standard subwords of τ

$$W = \{2143; 21\}.$$

The indices, which are written as subscripts, are

$$W = \{2_1 1_0 4_2 3_1, 2_1 1_0\}.$$

Summing all these indices tells us that the charge of the word $w = 221143$ is $c(w) = 5$.

Let λ be a partition with Young diagram Y^λ. Suppose $\rho = (\rho_1, \rho_2, \cdots, \rho_d)$ is a partition of length d with $\rho \vdash |\lambda|$ the same weight as λ. We can *inject the partition ρ into the diagram Y^λ* by writing ρ_1 1's; ρ_2 2's; $\cdots$; ρ_d d's into the boxes of Y^λ to form a Young tableau.

Example 1.34 Consider $\lambda = (42)$ and $\rho = (321)$. Then <u>*all*</u> the possible unitary tableaux formed by injecting ρ into λ are:

$$\underset{\tau_6}{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}} \qquad \underset{\tau_7}{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}} \quad .$$

<u>*All*</u> the possible *regular* tableaux formed by injecting ρ into λ are:

$$\underset{\tau_8}{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}} \qquad \underset{\tau_9}{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}} \quad .$$

There are no possible standard tableaux formed by injecting ρ into λ.

A *negative application* is an injection of numbers giving a regular tableau in which like digits occupy an even number of rows. Since like digits must form a staircase (of some subdiagram) in regular tableaux, we can think of a negative application as one which gives a staircase (of some subdiagram) of even height. By definition, negative applications only have meaning when regular tableaux are formed. In this example τ_8 has 1 negative application and τ_9 has 0 negative applications. In Example 1.32 τ_3 has 2 negative applications and τ_4 has 1 negative application.

Chapter 2

Graded Rings

Semigroup graded rings have been of interest for several decades since their first appearance in 1960 in [64], mainly devoted to rings graded by cancellative semigroups. Rings graded by cancellative semigroups or commutative semigroups have been of particular interest (see [2], [9], [11], [12], [13], [53], and [64], for example) and results for group graded rings are plentiful (see [3], [8], [17], [54], and [62] for example).

Groupoid graded rings have only been considered more recently and the first positive results on groupoid graded rings were obtained by Kelarev in [38]. We begin with a definition of a groupoid graded ring, and then extend this to define a partial groupoid graded ring.

Throughout, we try to extend the definitions and results to as wide a class of graded rings as possible. By considering rings graded by partial groupoids (where possible), we achieve this.

2.1 Examples

Let S be a groupoid (semigroup, group) and let R be a ring (not necessarily with unity) which can be expressed as a direct sum $R = \bigoplus_{s\in S} R_s$ of additive subgroups R_s of R with $s \in S$. If $R_sR_t \subseteq R_{st}$ for all $s,t \in S$ then we say that R is a *groupoid-graded* (*semigroup graded, group graded*) *ring.* We refer to $R = \bigoplus_{s\in S} R_s$ as an S-grading of R and the subgroups R_s as the s-components of R. If we have the stronger condition that $R_sR_t = R_{st}$ for all $s,t \in S$, then we say that the ring R is *strongly* graded by S. Any element r_s in R_s (where $s \in S$) is said to be *homogeneous of degree s*. Each element $r \in R$ can be expressed as a unique sum $r = \sum_{s\in S} r_s$ of homogeneous elements $r_s \in R_s$. We define the *support* of r to be the set $\operatorname{supp}(r) = \{s \in S \mid r_s \neq 0\}$. We can extend this definition to $\operatorname{supp}(R) = \bigcup \operatorname{supp}(r) = \{s \in S \mid R_s \neq 0\}$. If $\operatorname{supp}(R)$ is a finite set then we say that the ring R has *finite support.*

For any subset $G \subseteq S$ we define $R_G = \sum_{g\in G} R_g$. Similarly put $r_G = \sum_{g\in G} r_g$. If G is a subsemigroup of S then R_G is a subring of R. If G is a left (right, two-sided) ideal of S then R_G is a left (right, two-sided) ideal of R.

If S is a partial groupoid, then we say that $R = \bigoplus_{s\in S} R_s$ is *S-graded* when:

(i) $R_sR_t \neq 0$ implies that the product st is defined;

(ii) $R_sR_t \subseteq R_{st}$ whenever st is defined.

If st is defined for all pairs $s,t \in S$, then condition (i) is redundant, and the definition coincides with that of a groupoid graded ring.

There are various classes of rings which can be presented as graded rings. The following examples are of interest to situations considered in later chapters.

Example 2.1 (Homogeneous Sums.) Homogeneous sums are equivalent to rings graded by partial groupoids.

Let R be a ring not necessarily with unity, S a finite set, and let R be the direct sum of additive subgroups R_s where $s \in S$. Denote by $H(R) = \cup_{s \in S} R_s$ the set of all homogeneous elements of R. Then $R = \bigoplus_{s \in S} R_s$ is said to be a *homogeneous sum* of additive subgroups R_s whenever $H(R)$ is closed under multiplication. This idea was first mentioned by Kelarev in [38].

Suppose $R = \bigoplus_{s \in S} R_s$ is a homogeneous sum. If $R_s R_t \neq 0$, then there exists a unique element u in S such that $R_s R_t \subseteq R_u$. To see this, we will set up a contradiction. Firstly, since $R_s R_t \neq 0$, there exists $r_s \in R_s$ and $r_t \in R_t$ with $r_s r_t = x \neq 0$. Since $H(R)$ is closed under multiplication, $x \in R_u$ for some $u \in S$. Next, suppose that $R_s R_t \not\subseteq R_u$. Then there exists some $a \in R_s$ and $b \in R_t$ with $ab = c \in R_v$ with $u \neq v \in S$. Now, put $a' = (r_s + a) \in R_s$ and $b' = (r_t + b) \in R_t$. Then $a'b' = d \in R_y$ for some $y \in S$ and also

$$R_y \ni a'b' = (r_s + a)(r_t + b) = r_s r_t + r_s b + a r_t + ab.$$

But R_y is an additive subgroup of R and recall that $r_s r_t \in R_u$ and $ab \in R_v$ with $u \neq v$. This contradicts the homogeneity of the sum. Indeed, $u = v = y$.

This means we can introduce a partial operation on S by putting $st = u$ for all triples $s, t, u \in S$ such that $0 \neq R_s R_t \subseteq R_u$. Then S becomes a partial groupoid, which we call the *partial groupoid induced by R.*

Example 2.2 (Ring of 2×2 matrices over $\mathbb{R}$.) Let $M = M_2(\mathbb{R})$ be the ring of 2×2 matrices with entries from the field of reals. Consider the elements

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly $M = e_{11}\mathbb{R} + e_{12}\mathbb{R} + e_{21}\mathbb{R} + e_{22}\mathbb{R}$, and in fact M is graded. We put B_2 as the set of these four e_{ij}, $(i, j \leq 2)$ above with a zero 0 adjoined. Under normal matrix multiplication (as described on page 17), B_2 forms the semigroup which grades M. To write this formally we just need to put $R_{e_{ij}} = e_{ij}\mathbb{R}$. Because B_2 also has a zero we also need to put $R_0 = 0$ and then $M = \bigoplus_{s \in B_2} R_s$. So in fact M is a contracted B_2-graded ring.

The semigroup B_2 which grades the ring here is just the Brandt semigroup described in Example 1.15.

If e is an idempotent element of a groupoid (or semigroup) S and R is an S-graded ring, then the e component of R is a subring of R. This is easy to see. Since R_e is an additive subgroup, we need only consider the product $R_e^2 = R_e R_e \subseteq R_{e^2}$. This means that R_e is also closed under multiplication and so is indeed a subring of R.

Example 2.3 Consider the subring R of $M_2(\mathbb{R}) \times M_2(\mathbb{R})$ given by $R = R_e + R_g$ where

$$R_e = \left\{ \left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \right) \mid r \in \mathbb{R} \right\}$$
$$R_g = \left\{ \left(0, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \mid a, b, c, d \in \mathbb{R} \right\}.$$

Since $R_e R_g \subseteq R_g$, $R_g R_e \subseteq R_g$, $R_g R_g \subseteq R_g$ and $R_e R_e \subseteq R_e$ we see that R is graded by the two-element semilattice $Y_2 = \{e, g\}$ with identity element e. It follows that

$$R_e = \left\{ \left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \right) \mid r \in \mathbb{R} \right\}$$

is a subring of R. In fact, since R is graded by a semilattice, R_g is also a subring of R.

If S is a semigroup with zero 0 and R an S-graded ring with $R_0 = 0$ then we say R is a *contracted* S-graded ring. In Example 2.3 the semilattice Y_2 graded ring R is not contracted since g is the zero of the semilattice but $R_g \neq 0$.

Example 2.4 (Generalized Matrix Rings.) A ring $R = \bigoplus_{i,j=1}^{n} R_{ij}$ is said to be a *generalized matrix ring* if

$$R_{ij}R_{k\ell} \subseteq \begin{cases} R_{i\ell} & \text{if} \quad j = k; \\ 0 & \text{otherwise.} \end{cases}$$

Following in the same fashion as Example 2.2 with a relaxing of the size of the matrix from 2 to any $n \in \mathbb{N}$, we consider the Brandt semigroup B_n made up of a zero element 0 and the matrix units elements e_{ij} for $i, j \leq n$.

The Brandt semigroup B_n grades R. We write this formally as $R = \bigoplus_{s \in B_n} R_s$.

We have to put $R_0 = 0$, and so generalized matrix rings are contracted B_n-graded rings.

Example 2.5 (Semigroup Rings.) Let A be a ring, S a finite or infinite semigroup (respectively, group). The the *semigroup (group) ring* $A[S]$ *of* S *over* A is the associative ring consisting of all finite formal sums

$$\alpha = \sum_{s \in S} a_s s$$

where each coefficient $a_s \in A$. The word 'formal' here means that, in general, we only make use of the form of each term without calculating any exact values.

Because the sums are finite, only finitely many of the coefficients a_s can be nonzero.

Consider another element $\beta = \sum_{s\in S} b_s s$ of $A[S]$. Then addition in $A[S]$ is defined component-wise by

$$\alpha + \beta = \sum_{s\in S} a_s s + \sum_{s\in S} b_s s = \sum_{s\in S}(a_s + b_s)s.$$

Multiplication in $A[S]$ is given by

$$\begin{aligned}\alpha\beta &= \left(\sum_{s\in S} a_s s\right)\left(\sum_{s\in S} b_s s\right)\\ &= \sum_{s\in S}\left(\sum_{uv=s} a_u b_v\right) s\end{aligned}$$

where $(a_u u)(b_v v) = (a_u b_v)(uv)$ for $u, v \in S$. Now for any semigroup (group) ring $R = A[S]$, if we put $R_s = As$ for each $s \in S$ then

$$R = AS = \bigoplus_{s\in S} As = \bigoplus_{s\in S} R_s$$

is an S-graded ring. We cannot generalize this construction to make 'groupoid' rings, because we could loose associativity of the multiplication in the ring.

Example 2.6 (Morita Rings associated with Morita contexts.) Morita contexts where introduced in 1958 by Morita in his paper on "Duality for Modules". In 1971, Amitsur wrote a comprehensive and interesting paper [2] on "Rings of Quotients and Morita Contexts".

A *Morita context* is a set $M = (R, V, W, S)$ where R and S are rings, V is an R-S bimodule, W is an S-R bimodule and the products $V \times W$ to R and $W \times V$ to S are associative bilinear mappings. This means that the following conditions are satisfied:

(i) $(v_1 + v_2)w = v_1 w + v_2 w$

(ii) $v(w_1 + w_2) = vw_1 + vw_2$

(iii) $(rv)w = r(vw)$

(iv) $s(wv) = (sw)v$

where $v_1, v_2, v \in V$, $w_1, w_2, w \in W$, $r \in R$, and $s \in S$.

Associated with any Morita context $M = (R, V, W, S)$ is a set of matrices

$$N = \left\{ \begin{bmatrix} r & v \\ w & s \end{bmatrix} \mid r \in R,\ v \in V,\ w \in W,\ s \in S \right\}.$$

which forms a ring under normal matrix addition and multiplication. We call the ring N the *Morita ring associated with the Morita context M*. To ensure a well-defined grading we need to put $N_0 = 0$, and so the Morita ring associated with a Morita context is a contracted B_2-graded ring.

Example 2.7 (Ring of Polynomials in n commuting indeterminates over R.) Let R be a ring. Elements of the formal finite power series

$$R[x_1, x_2, \cdots, x_n] = \left\{ \sum_{(\lambda_1, \lambda_2, \cdots \lambda_n) \in \mathbb{N}^n} a_{\lambda_1, \lambda_2, \cdots \lambda_n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \mid a_{\lambda_1, \lambda_2, \cdots \lambda_n} \in R \right\}$$

form a graded ring called the *ring of polynomials over the ring R in n commuting indeterminates.*

Suppose R does not have a multiplicative identity. To describe the construction of the ring of polynomials over the ring R, we need to introduce a element 1, external to R and put $1 \times a = a \times 1 = a$ for any $a \in R$. We shall not adjoin 1 to R. We do not need to consider this element 1 under addition with any elements of R as this case will never arise. Also, this external element 1 is not in $R[x_1, x_2, \cdots, x_n]$. If R is with unity, then we shall use 1 to mean the multiplicative identity of R. Now, whenever $\lambda_i = 0$ we put $x^{\lambda_i} = 1$. The *coefficients*

$a_{\lambda_1,\lambda_2,\cdots\lambda_n}$ are members of the ring R and whenever $a_{\lambda_1,\lambda_2,\cdots\lambda_n} = b_{\lambda_1,\lambda_2,\cdots\lambda_n}$ for all n-tuples $(\lambda_1, \lambda_2, \cdots \lambda_n)$ the two elements

$$\sum_{(\lambda_1,\lambda_2,\cdots\lambda_n)\in\mathbb{N}^n} a_{\lambda_1,\lambda_2,\cdots\lambda_n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

and

$$\sum_{(\lambda_1,\lambda_2,\cdots\lambda_n)\in\mathbb{N}^n} b_{\lambda_1,\lambda_2,\cdots\lambda_n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

are considered equal. The $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ are called *monomials* and a *term* is a coefficient multiplied by a monomial. The monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_n}^{\lambda_n}$ is said to have *degree* k if $k = \sum_j^n \lambda_j$.

The indeterminates are not unknown or variable elements from the ring R, but the idea of them goes hand in hand with the idea of grading of the ring. In a sense, the indeterminates are indexing or separating (as described formally above) elements of the ring.

Whenever the sum of the powers $\lambda_1, \lambda_2, \cdots \lambda_n$ equals $m \in \mathbb{N}$ for each nonzero term of the polynomial, we say that the polynomial is *homogeneous of degree* m. Let $R^m[x_1, x_2, \cdots x_n]$ consist of all homogenous polynomials of degree m. Then $R^m[x_1, x_2, \cdots x_n]$ forms a group under the addition induced from $R[x_1, x_2, \cdots x_n]$. Put $R^0[x_1, x_2, \cdots x_n] = R$. Since

$$R^m[x_1, x_2, \cdots, x_n] R^l[x_1, x_2, \cdots x_n] = R^{m+l}[x_1, x_2, \cdots x_n]$$

for any $m, l \in \mathbb{N}$ we see that the ring of polynomials over the ring R in n indeterminates is a strongly graded ring. To verify this, we need to give the operations in $R[x_1, x_2, \cdots x_n]$.

Addition is just component-wise so that

$$\sum_{(\lambda_1,\lambda_2,\cdots\lambda_n)\in\mathbb{N}^n} a_{\lambda_1,\lambda_2,\cdots\lambda_n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} + \sum_{(\lambda_1,\lambda_2,\cdots\lambda_n)\in\mathbb{N}^n} b_{\lambda_1,\lambda_2,\cdots\lambda_n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

$$= \sum_{(\lambda_1,\lambda_2,\cdots\lambda_n)\in\mathbb{N}^n} (a_{\lambda_1,\lambda_2,\cdots\lambda_n} +_R b_{\lambda_1,\lambda_2,\cdots\lambda_n})\, x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

where $+_R$ is the addition given in R. We see that the $R^m[x_1, x_2, \cdots x_n]$ do indeed form an additive subgroup under this operation.

Multiplication is not so easy to just jot down here and we will not attempt it since it is not crucial to the thesis. We will describe the multiplication in enough detail to justify our remarks above about the grading of the ring. The multiplication of elements in $R[x_1, x_2, \cdots, x_n]$ is what you would expect intuitively in that we distribute across the brackets. In other words, we multiply each and every term in one polynomial by each and every term in the other. Multiplying coefficients is easy since this is just multiplication in the ring R. We must be careful of the order of the operation, since if our ring R is non-commutative, then so is the polynomial ring. To multiply the indeterminates we use the rule that $x_i^j x_i^k = x_i^{j+k}$ and $x_{i_1}^j x_{i_2}^k = x_{i_1}^j x_{i_2}^k$ whenever $i_1 \neq i_2$. This description of the multiplication, although not exactly explicit, is enough to see that the multiplication of the homogeneous subgroups does indeed give the grading described above.

2.2 The ring of symmetric functions

Symmetric polynomials form a graded subring of the ring of polynomials in n indeterminates over $\mathbb{Z}$. Because this subring is graded, we can set up an inverse system of natural projections enabling an inverse limit of the additive subgroups. We describe this process in Section 2.2.1. By taking the inverse limit, we are essentially allowing the symmetric 'polynomials' to have an infinite number of indeterminates. After taking the inverse limit to allow an infinite number of indeterminates, the members of the graded ring are called, by convention, symmetric functions. This is the approach taken by MacDonald in his definitive text [44] on symmetric functions, and our discussion here follows this style of construction. An alternative approach (see [61]) is to define the ring of symmetric functions as the vector space spanned by all monomial symmetric functions.

The Schur S-functions are an important base for the ring of symmetric functions. They establish a strong connection between the theory of symmetric functions and the combinatorial theory of the Young diagrams.

2.2.1 From symmetric polynomials to symmetric functions

A polynomial from the ring $\mathbb{Z}[x_1, x_2, \cdots, x_n]$ of polynomials in n indeterminates is a *symmetric polynomial* if it is invariant under the action of the symmetric group. The symmetric group acts on polynomials by permuting or interchanging the variables, and so symmetric polynomials are just those which don't change when we permute or interchange the n indeterminates.

The set of symmetric polynomials in n indeterminates

$$\Lambda_n = \mathbb{Z}[x_1, x_2, \cdots, x_n]^{S_n}$$

forms a subring of the ring $\mathbb{Z}[x_1, x_2, \cdots, x_n]$ of polynomials in n indeterminates.

For $k \geq 0$, let $\Lambda_n^{(k)}$ consist of the homogeneous symmetric polynomials of degree k, and include the zero polynomial in each $\Lambda_n^{(k)}$ for all $k \geq 0$. Including the zero means that each $\Lambda_n^{(k)}$ is a group under addition. Also, since $\Lambda_n^{(k)}\Lambda_n^{(j)} \subseteq \Lambda_n^{(k+j)}$, the ring

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^{(k)}$$

forms a graded ring of symmetric polynomials in n indeterminates.

Define the *monomial symmetric polynomial* $m_\lambda(x_1, x_2, \cdots, x_n)$ in n indeterminates corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_j)$ with $j \leq n$ parts and of weight k, by using λ^0 to mean the partition λ with $n - j$ zeros adjoined, so that $\lambda^0 = (\lambda_1, \lambda_2, \cdots, \lambda_j, 0_{j+1}, \cdots, 0_n)$ is a partition of length n. Then we sum over all the distinct monomials in n indeterminates $\{x_1, x_2, \cdots, x_n\}$ with the parts of λ^0 as exponents. The idea is that whenever $j < n$, in each monomial $j - n$ of the indeterminates have the form $x_i^{0_i}$, and vanish, with j indeterminates surviving (with a nonzero exponent). That is

$$m_\lambda(x_1, x_2, \cdots, x_n) = \sum x_1^{\lambda_1^0} x_2^{\lambda_2^0} \cdots x_n^{\lambda_n^0}$$

where the sum is over all distinct permutations of λ^0 putting any $x_i^{0_i} = 1$. Since $\lambda \vdash k$, then $m_\lambda(x_1, x_2, \cdots, x_n)$ is homogeneous of degree k.

Example 2.8 Suppose $\lambda = (21)$ with $x = (x_1, x_2, x_3, x_4)$. Then

$$\begin{aligned} m_{(21)}(x_1, x_2, x_3, x_4) &= x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_1 + x_2^2x_3 + x_2^2x_4 \\ &\quad + x_3^2x_1 + x_3^2x_2 + x_3^2x_4 + x_4^2x_1 + x_4^2x_2 + x_4^2x_3 \end{aligned}$$

is a symmetric polynomial of degree 3.

In the theory of symmetric functions it is the convention to work in infinitely many variables. To construct the graded ring of symmetric functions, we use the natural projection $\rho_{n+1,n} : \Lambda_{n+1} \to \Lambda_n$ making any x_{n+1} terms in Λ_{n+1} equal to zero. Clearly $\rho_{n+1,n}$ is a surjective ring homomorphism. Next, we restrict $\rho_{n+1,n}$ to act on polynomials of degree $k \leq n$ by putting

$$\rho_{n+1,n}^k : \Lambda_{n+1}^{(k)} \to \Lambda_n^{(k)}$$

so that $\rho^k_{n+1,n}$ is also injective. This means that $\rho^k_{n+1,n}(\Lambda^{(k)}_{n+1}) = \Lambda^{(k)}_n$. Taking the limit of this inverse system

$$\Lambda^{(k)} = \lim_{\leftarrow n} \Lambda^{(k)}_n$$

gives $\Lambda^{(k)}$, the set of homogenous symmetric functions of degree k, with zero.

The $\Lambda^{(k)}$ are additive groups, and we use these additive groups to construct the graded *ring of symmetric functions* by putting

$$\Lambda = \bigoplus_k \Lambda^{(k)}.$$

Taking the inverse limit of symmetric polynomials of degree k in n indeterminates is a neat and subtle way of supposing an infinite number of indeterminates without changing the nature of the elements of the homogeneous subgroups. If we were to simply just extend the number of indeterminates infinitely this would allow the product $\prod_i(1+x_i)$, for example, into the ring. Since this product does not have finite support, it cannot be an element in any graded ring.

Next, we use the natural projection $\rho^k_n : \Lambda^{(k)} \longrightarrow \Lambda^{(k)}_n$ mapping symmetric functions of degree k in n indeterminates to symmetric polynomials of degree k to describe certain classical symmetric functions. Notice that the projection ρ^k_n is also an isomorphism for all $n \geq k$.

For any partition $\lambda \vdash k$, the *monomial symmetric function* $m_\lambda(x)$ must satisfy the projection to n indeterminates

$$\rho^k_n(m_\lambda(x)) = m_\lambda(x_1, \cdots, x_n)$$

for every $n \geq k$, where $m_\lambda(x_1, \cdots, x_n)$ is the symmetric polynomial in n indeterminates. This style of constructing the monomial symmetric function is the approach taken by MacDonald [44]. The monomial symmetric functions are also described as elements from the formal power series ring (see [61], for example).

The space spanned by all monomial symmetric functions of degree k is $\Lambda^{(k)}$, the additive subgroup of degree k symmetric polynomials ([61], Proposition 4.3.3).

This means that the monomial symmetric functions form an integer basis for the ring of symmetric functions Λ, and so any symmetric function from Λ can be written, in a unique way, as a finite linear combinations of monomials with integer coefficients. Whence $\Lambda = \mathbb{Z}[m_\lambda]$.

2.2.2 Power-sum symmetric functions and the Schur S-functions

There are several other bases for $\Lambda^{(k)}$. We are specifically interested in two of them: the power-sum symmetric functions, and the Schur S-functions.

For any $r \in \mathbb{N}$, the r^{th} power-sum symmetric function is $p_r(x) = \sum_{i\geq 1} x_i^r$. The power-sum symmetric functions are multiplicative and so for any partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ we write $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$. The power-sum symmetric functions p_λ are well known as a $\mathbb{Q}$-basis of the ring of symmetric functions (see [44], page 16 for example).

A popular way to describe symmetric functions is via their generating functions. The *generating function* for the power-sum symmetric functions is

$$P(x) = \sum_{i\geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}.$$

This is because

$$\sum_{i\geq 1} \frac{d}{dt} \log (1 - x_i t)^{-1} \quad = \quad \sum_{i\geq 1} \frac{x_i}{1 - x_i t}$$

$$
\begin{aligned}
&= \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} && (2.1)\\
&= \sum_{r\geq 1} p_r t^{r-1}
\end{aligned}
$$

and the power-sum p_r is described as the coefficients of t^{r-1} (as in Equation 5.2).

Of more use to us in the coming chapters (specifically see Section 5.3) is the approach taken by Littlewood in his book [43] on "Group characters and matrix representations of groups". Littlewood talks about functions which are *associated* with power-sum symmetric functions, and we will call these functions *Littlewood-associated functions.* For example, for the power-sum symmetric functions the Littlewood-associated function is

$$G(\alpha; t) = \prod_{i\geq 1} \frac{1}{(1-\alpha_i t)}$$

with connection to the generating function made by observing that

$$\frac{G'(\alpha;t)}{G(\alpha;t)} = \sum_{i\geq 1} \frac{d}{dt} \log \frac{1}{(1-\alpha_i t)} \quad . \tag{2.2}$$

An important family of symmetric functions are the Schur S-functions. Frobenius was able to describe them via a characteristic mapping. We give a brief overview here, sufficient to lay the groundwork required, and suggest either [43] for a comprehensive treatment or [44] (Section I.7) for a more concise account relevant to this setting.

A characteristic mapping is an isomorphism from the ring generated by the characters of the symmetric group into the ring of symmetric functions. The structure preserving properties are a key feature of the mapping, especially in the context of making analogies at later stages of this thesis. Frobenius' theorem says that there is a characteristic mapping which maps the group characters χ^λ

of the symmetric group to the symmetric function $s_\lambda(x)$. Explicitly we have

$$s_\lambda(x) = \sum_{\rho \vdash |\lambda|} z_\rho^{-1} \chi_\rho^\lambda p_\rho(x)$$

where χ^λ is the character on the class ρ, which has centralizer of size z_ρ^{-1}; and $p_\rho(x)$ is a power-sum symmetric function. The power-sum symmetric function occurs in the description here definitively. Indeed the right hand side of this expression is just the characteristic mapping of χ^λ, the left hand side being its image in the ring of symmetric functions. The symmetric functions $s_\lambda(x)$ are called *Schur S-functions.*

The Schur S-functions form a $\mathbb{Z}$-basis of the ring of symmetric functions ([24] or [44], I.3.3) and provide a connection between the theory of symmetric functions and the combinatorial theory of Young diagrams. The foundations of the combinatorial approach to Schur S-functions are outlined in Appendix C.

2.2.3 Transition matrices

If $u_\lambda(x)$ and $v_\mu(x)$ are any two bases of the homogeneous subgroup of symmetric functions $\Lambda^{(k)}$ of degree k, each indexed by partitions of weight k, then there is a non-singular (invertible) matrix $M(u_\lambda, v_\mu)$ with integer coefficients $M_{\lambda\mu}$ so that

$$u_\lambda(x) = \sum_{\mu \vdash k} M_{\lambda\mu} v_\mu(x).$$

The matrix $M(u_\lambda, v_\mu)$ of coefficients $M_{\lambda\mu}$ is called the *transition matrix from $u_\lambda(x)$ to $v_\mu(x)$.*

The transition matrix K from the Schur S-functions $s_\lambda(x)$ to the monomial symmetric functions $m_\mu(x)$ has coefficients $K_{\lambda\mu}$ in the equation

$$s_\lambda(x) = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu} m_\mu(x).$$

The numbers $K_{\lambda\mu}$ are called *Kostka numbers* and the matrix K is called the *Kostka matrix.* Actually, it turns out that there is a broader definition of Kostka numbers and the Kostka matrix, which we meet in Chapter 6. When we do meet the Kostka numbers and Kostka matrix again, we see that there is a t dependence that has not come into play yet. It turns out the the case described here is for when $t = 0$.

The transition matrix from the power-sum symmetric functions to the Schur S-functions is just the character table of the symmetric group S_n. This is because of the orthogonality of the characters. This means that

$$p_\rho(x) \quad = \quad \sum_{\lambda \vdash |\rho|} \chi^\lambda_\rho s_\lambda(x) \tag{2.3}$$

where χ^λ_ρ is the character χ^λ on the class ρ.

In Appendix A we give Schensted's algorithm for calculating the ordinary characters of the symmetric group which we utilize in Chapter 6 within our new algorithm for calculating the spin characters of the symmetric group.

2.3 Graded Ideals

In line with our previously mentioned motivation of extending the definitions and results to as wide a class of graded rings as possible, we start of with a definition of a partial groupoid graded ideal. Of course, if our ideal or ring is graded by a stronger structure, all of these definitions and results will still hold.

Let S be a partial groupoid and suppose I is an ideal (left, right or two-sided) of an S-graded ring A. Then I is said to be an *S-graded ideal* if

$$I = \bigoplus_{s \in S} (I \cap A_s) = \bigoplus_{s \in S} I_s.$$

Suppose that B is another S-graded ring. A homomorphism $f : A \to B$ of rings is said to be *graded* if $f(A_s) \subseteq B_s$ for all $s \in S$. If our graded ring homomorphism f is surjective, then clearly $f(A_s) = B_s$. Further, if f is bijective then we say that f is an *S-graded isomorphism.*

Karpilovsky [37] gives a detailed account of the behaviour of the Jacobson radical of graded rings with unity. We require two analogous propositions based on Karpilovsky's comprehensive account, but in the context of partial groupoid graded rings without unity. For these next two propositions, we include a reference to the analogous results in Karpilovsky, and supply proofs here whenever appropriate.

In some parts, the proofs in Karpilovsky still hold in this context. However, because for example, our ring doesn't contain unity, in other instances we require, and have supplied, the appropriate proofs.

The following proposition occurs analogously with Karpilovsky's [37] Proposition 22.6; in which A and B are graded algebras. The proofs here are based wherever possible on those given in Karpilovsky.

Proposition 2.9 ([37], Proposition 22.6) *Let S be a partial groupoid and A and B be S-graded rings.*

(i) If I is an S-graded ideal of A, then A/I is an S-graded ring by setting:

$$(A/I)_s = (A_s + I)/I \qquad \text{for all } s \in S.$$

(ii) If $f : A \to B$ is an S-graded homomorphism, then $f(A)$ is an S-graded subring of B and $\ker f$ *is an S-graded ideal of A. The map $g : A/\ker f \to f(A)$ given by $(a + \ker f) \mapsto f(a)$ for $a \in A$ is an S-graded isomorphism.*

(iii) An ideal I of A is S-graded if and only if I is the kernel of some S-graded homomorphism $f : A \to B$ of S-graded rings.

Proof.

(i) Since A is an S-graded ring and I is an S-graded ideal of A, we have

$$A = \bigoplus_{s \in S} A_s \quad \text{and} \quad I = \bigoplus_{s \in S} I_s = \bigoplus_{s \in S} (I \cap A_s).$$

Consider the factor ring A/I. For each $s \in S$ we see that $A_s/I_s = A_s/(I \cap A_s) = (A_s + I)/I$ by the Second Isomorphism Theorem (Theorem 1.19).

Now, we suppose st is defined and by putting $(A/I)_s = A_s/I_s$, we consider the product of $(A/I)_s$ and $(A/I)_t$ for any $s, t \in S$ to show that A/I is graded. We see that

$$\begin{aligned}
(A/I)_s(A/I)_t &= [(A_s + I)/I]\,[(A_t + I)/I] \\
&= (A_s + I)(A_t + I)/I \\
&\subseteq (A_sA_t + I)/I \\
&= (A/I)_{st}
\end{aligned}$$

since st is defined.

Next, suppose $(A/I)_s(A/I)_t \neq 0$. We must show that st is defined. Since $0 \neq (A/I)_s(A/I)_t \subseteq (A_sA_t + I)/I$ we see that $A_sA_t \neq 0$. Since A is S-graded then st must be defined, as required.

Hence $(A/I) = \bigoplus_{s\in S}(A/I)_s$.

(ii) Since $f(A)$ is a subring of B and f is an S-graded homomorphism, $f(A_s) \subseteq B_s$ for all $s \in S$, and so

$$f(A_s) \subseteq f(A) \cap B_s \text{ for all } s \in S.$$

Since B is S-graded and f is structure preserving, we can take a graded sum of these sets giving

$$f(A) = \bigoplus_{s\in S} f(A_s) = \bigoplus_{s\in S} (f(A) \cap B_s).$$

This means that $f(A)$ is an S-graded subring of B.

Next, $f : A \to B$ is a ring homomorphism and so $\ker f$ is an ideal of A. Now for any $a \in \ker f$ we have $a = \sum_{s\in S} a_s$ and $f(a) = \sum_{s\in S} f(a_s)$. Since f is an S-graded homomorphism, we know that $f(a_s) \in B_s$ for each $s \in S$. Because B is S-graded, this means that each a_s is in the kernel of f. So indeed, $\ker f = \bigoplus_{s\in S} (\ker f \cap A_s)$ is a graded ideal.

Finally, we consider the mapping $g : A/\ker f \to f(A)$ given by $g(a + \ker f) \mapsto f(a)$. To show that g is an S-graded isomorphism is straightforward knowing, as we now do, that the kernel is a graded ideal. For any $s \in S$ we see that

$$\begin{aligned} g((A/\ker f)_s) &= g((A_s + \ker f)/\ker f) \\ &= f(A_s) \end{aligned}$$

as desired.

(iii) We have already shown that the kernel is an ideal. (The remainder of this proof coincides with that given in Karpilovsky). Now suppose that I is some S-graded ideal of our S-graded ring A. Then the natural homomorphism $A \mapsto A/I$ is a graded homomorphism with kernel I.

$\square$

Proposition 2.10 ([37], Proposition 6.18) *Let S be a semigroup, R is an S-graded ring with $\mathcal{J}(R)$ the Jacobson radical of R and e an idempotent element of a semigroup S. Then*

$$R_e \cap \mathcal{J}(R) \subseteq \mathcal{J}(R_e).$$

Proof. Put $A = R_e \cap \mathcal{J}(R)$ with $a \in A$ and let $r \in R_e$. Then $ra \in R_eR_e \subseteq R_e$ since e is idempotent and $ra \in \mathcal{J}(R)$ because $\mathcal{J}(R)$ is an ideal of R. So $ra \in R_e \cap \mathcal{J}(R)$ and similarly $ar \in R_e \cap \mathcal{J}(R)$. Hence $R_e \cap \mathcal{J}(R)$ is a two-sided ideal of R_e.

Since $\mathcal{J}(R_e)$ contains all quasiregular ideals of R_e, we will show that each $x_e \in R_e \cap \mathcal{J}(R)$ has a quasi-inverse in the e component of R.

Take $x_e \in R_e \cap \mathcal{J}(R)$. Since $x \in \mathcal{J}(R)$, there is a $y \in \mathcal{J}(R)$ so that $x \circ y = 0$. The support of y is the set $\operatorname{supp}(y) = \{s \in S \mid y_s \neq 0\}$. The support of y is finite and

$$\begin{aligned} x_e \circ y &= x_e + y - xy \\ &= x_e + y_e + \sum_{s \neq e} y_s - x_e y_e - \sum_{\substack{es = e \\ s \neq e}} x_e y_s - \sum_{es \neq e} x_e y_s \qquad (2.4) \\ &= 0. \end{aligned}$$

We identify four of the terms in the sum just given as belonging to R_e

$$\left(x_e + y_e - x_e y_e - \sum_{\substack{es = e \\ s \neq e}} x_e y_s \right) \in R_e$$

with the remaining terms coming from homogeneous components of the ring other than R_e. In other words supp $\left(\sum_{s \neq e} y_s - \sum_{es \neq e} x_e y_s\right) \subseteq S \backslash \{e\}$.

Since the entire sum (Equation 2.4) equals 0, we can component-wise add the terms and reconcile the sum in each component. So for the e component of R we have

$$x_e + y_e - x_e y_e - \sum_{\substack{es = e \\ s \neq e}} x_e y_s = 0.$$

We see that if the end most term here equals zero then $x_e \circ y_e = 0$ and we are done. We proceed now to show that this is indeed the case.

Since we are reconciling Equation 2.4 component-wise, we can look to the terms of the sum in the homogenous components of R other than R_e. The component-wise reconciliation requires that $\left(\sum_{s \neq e} y_s - \sum_{es \neq e} x_e y_s\right) = 0$ and so

$$\sum_{s \neq e} y_s = \sum_{es \neq e} x_e y_s.$$

Because of the component-wise reconciliation, $|\text{supp } (y) \backslash \{e\}|$ must equal the cardinality of $\{es \mid es \neq e, s \in \text{ supp } (y) \backslash \{e\}\}$.

Now, let's consider the mapping θ_e between these two sets which sends each $s \in \{\text{supp}\,(y) \backslash \{e\}\}$ to $es \in \{es \mid es \neq e, s \in \text{ supp } (y) \backslash \{e\}\}$. The sets are the same size and so θ_e is bijective mapping onto $\{es \in \text{ supp}\,(y)/\{e\} \mid es \neq e\}$ for all $s \neq e$. It follows that $es = e$ only when $s = e$. This means that

$$\sum_{\substack{es = e \\ s \neq e}} x_e y_s = 0$$

because no such s exists. And so the terms in the R_e component of our original sum are just $0 = x_e + y_e - x_e y_e = x_e \circ y_e$. Since $y_e \in \mathcal{J}(R) \cap R_e$, then this ideal of R_e is indeed quasiregular and contained in $\mathcal{J}(R_e)$. $\square$

We describe the graded Jacobson radical here for group graded rings *with unity*. In Section 2.4 we give an equivalent description of the graded Jacobson radical for rings without unity. In Balaba's recent paper [5] both descriptions are used and we also discuss her conditions on their equivalence in Section 2.4.

Let G be a group with identity element e and R a G-graded ring with unity. A G-graded left (right,two-sided) ideal T of R is an *G-graded-maximal* left (right, two-sided) ideal if $T \neq R$ and T is not contained in any other proper G-graded left (right, two-sided) ideals of R. In this case the *graded Jacobson radical* $\mathcal{J}_{gr}(R)$ of R is defined to be the intersection of all G-graded-maximal left ideals of R. The grading of the radical is by the same structure G which grades the ring R itself. We state Bergman's Lemma for finite group G-graded rings with unity.

Theorem 2.11 ([8]), ([50], Theorem 5.4) *Let G be a finite group of order $n \in \mathbb{N}$ and let R be a G-graded ring with unity. Then $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ where $\mathcal{J}_{gr}$ is the G-graded Jacobson radical.*

In Section 2.4 (Theorem 2.13) we extend this result to rings without unity and in Chapter 4 (Theorem 4.2) we give a generalization Bergman's Lemma by relaxing to a partial groupoid grading.

2.4 Unital extensions of graded ideals and rings

Several investigations of the graded Jacobson radical of rings have appeared over the last two decades (see [1], [5], [6], [16], [37] or [50] for example) often for rings which are afforded unity. It suits us here to explore the graded Jacobson radical of rings without unity, and our aim is to provide the necessary connections to rings with unity.

By a unital extension of a ring, we mean an embedding of a ring R without unity into a ring R^u with unity. We do this in the standard way (see [29] for example) by describing a monomorphism from R to R^u.

The ring R^u is made up of the additive group $R \bigoplus \mathbb{Z}$, where R is a ring without unity and $\mathbb{Z}$ is the ring of integers. Elements in $R \bigoplus \mathbb{Z}$ are denoted by ordered pairs $\{(r, n) \mid r \in R, n \in Z\}$.

Addition in R^u is component-wise and multiplication is given by

$$(r, n)(s, m) = (rs + mr + ns, nm)$$

where $r, s \in R$ with $n, m \in \mathbb{Z}$ and the product mr (and analogously ns) means the sum of the ring element r with itself m times in the case of a non-negative integer m, or the sum of the additive inverse $-r$ of the ring element with itself $|m|$ times in the case of a negative integer m. The product rs is determined by the ring multiplication, and the product nm is just the usual product of integers. The set R^u together with component-wise addition and the multiplication just described, forms a ring with an identity element $(0, 1)$. The essential idea here is that the embedding is described by a monomorphism in order to preserve the structure of our original ring R.

For the remainder of this section, we require specifically that R be group graded. In order to consider unital extensions of graded rings, this restriction makes sense. It allows us to place the 'unity' carefully into our graded ring without causing major offence to the structure of our ring.

So we begin with a ring R graded by a group G with group identity e. Any element $r \in R$ can be written uniquely as $r = \sum_{g \in G} r_g$. We embed our G-graded ring R into R^u in a similar manner described for embedding rings in the previous paragraphs, maintaining all the notation introduced there. We identify R with its copy in R^u and since $(0,1)R_g \subseteq R_g$ and $R_g(0,1) \subseteq R_g$ for all $g \in G$, we can grade R^u by putting the identity element $(0,1)$ in the e component, whence:

$$R^u = R^u_e \oplus \bigoplus_{g \in G \setminus \{e\}} R_g.$$

For any $r \in R$ we have

$$(r, n) = (r_e, n) + \sum_{\substack{g \in G \\ g \neq e}} (r_g, 0).$$

Recalling that R_e is a subring of R, we can see that the e component in R^u is just given by the standard unital extension of R_e in R. We shall reserve the use of R^u to always mean the unital extension of R.

Let S be a groupoid (or semigroup or group). A left module T over an S-graded ring R is an *S-graded left module* if there exists additive subgroups T_s of T with

$$T = \bigoplus_{s \in S} T_s$$

and $R_u T_v \subseteq T_{uv}$ for all $u, v \in S$. The graded right module is defined analogously with the ring interacting from the right. Throughout this section we will use module to mean left module and graded module to mean graded left module. We omit the entirely analogous discourse for right modules.

Let S be a groupoid (or semigroup or group). An S-graded module T over an S-graded ring R is a *graded-simple module* 0 and T are its only graded submodules.

The left annihilator of any S-graded module T is

$$\mathcal{A}(T) = \{a \in R \mid at = 0 \text{ for all } t \in T\}.$$

Annihilators of modules are ideals.

We reverted briefly to our more general S-graded rings for the previous two definitions only, and return now, and until the conclusion of this section, to rings graded by groups with group identity e.

For a ring R graded by a group G, the *graded Jacobson radical* $\mathcal{J}_{gr}(R)$ of R is defined to be the intersection of all left annihilators of all G-graded-simple R-modules. We use the $_{gr}$ here to indicate a graded structure. Indeed, for a group graded ring R, the radical is graded by the group G which grades the ring itself. If the ring is not graded by a group, then this description of the graded Jacobson radical may turn out not to be graded!

For rings with unity, the definition of the graded Jacobson radical $\mathcal{J}_{gr}(R)$ just given is equivalent with the definition given on page 50 ([49]). Since this definition is sensible for both rings with unity and those without, we will use $\mathcal{J}_{gr}(R)$ to mean the graded Jacobson radical as just described.

Aside 2.12 After an exhaustive search of the literature by others and me it was found that no actual definition of the graded Jacobson radical for rings without unity has yet appeared. In [5] a comprehensive account of special radicals of graded rings without unity was presented. Unfortunately the descriptions given in the section for the Jacobson radical came (in the most part) from [49], on group graded rings with unity. The error is only noticeable to the acute observer in that the word *regular* is omitted in the description of the radical involving maximal ideals. The definition given here is the obvious one, but was only decided upon after careful consideration. To date, it is only okay for group graded rings. In [1] the idea of describing graded Jacobson radicals (for rings with unity) in a more relaxed grading is discussed. Included in the paper by Balaba [5] is an interesting example of when this description yields an ideal which is not graded. We present her example at the end of this chapter.

We get the following generalization of Theorem 2.11 for rings with unity to rings without unity.

Theorem 2.13 *Let G be a finite group of order $n \in \mathbb{N}$ and let R be a G-graded ring with or without unity. Then $n\mathcal{J}(R) \subseteq \mathcal{J}_{gr}(R)$ where $\mathcal{J}_{gr}$ is the G-graded Jacobson radical. In other words, for any r in $\mathcal{J}(R)$ all homogenous components of nr belong to $\mathcal{J}(R)$.*

A proof in the special case of a ring with identity was given in 1984 by Năstăsescu [50]. An unpublished 1973 paper of Bergman [8] is widely quoted as the original source of this result. We have not had access to the latter paper, and have been unable to find out whether it deals with rings generally or only unital ones. We therefore prove the result assuming the special case applying to rings with unity, in view of the inaccessibility of [8], and for the sake of completeness.

Proof. We shall use module throughout to mean left module. As previously stated, the argument with right modules is symmetrically analogous to the one with left modules.

Suppose M is a module over a ring R and denote by R^u the standard unital extension of R. The module M becomes a unital R^u module by putting

$$(r, n)m = rm + nm$$

for $(r, n) \in R^u$ and $m \in M$.

Let Y be a G-graded R-module with $Y = \oplus_g Y_g$ and $R_h Y_g \subseteq Y_{hg}$. For $(r_e, n) \in R_e^u$ and any $y_g \in Y_g$ we have $(r_e, n)y_g = r_e y_g + n y_g \in Y_g$ and so Y is a G-graded unital R^u-module. Any G-graded unital R^u-module is a G-graded R-module

Similarly, if K is an G-graded R-submodule of a G-graded R-module M, then K is a unital G-graded R^u-submodule the G-graded unital R^u-module M, and vice versa.

So any G-graded-simple R-module is also a unital G-graded-simple R^u-module, and vice versa.

Let G be a group and suppose M is any G-graded-simple R-module over a G-graded ring R with left annihilator $\mathcal{A}(M)$. Take any $a \in \mathcal{A}(M)$. Then $a = \sum_{g \in G} a_g$. For any $h \in G$, pick an $m \in M_h$ (since M is graded). Then

$$0 = \left(\sum_{g \in G} a_g \right) m = \sum_{g \in G} a_g m$$

where each of the $a_g m \in M_{gh}$ for each $g \in G$. Since the sum runs over distinct gh (here G is a group), we have $a_g m = 0$ for all $g \in G$. And so all the homogeneous components $a_g \in \mathcal{A}(M)$ are in the annihilator of M and the annihilator is a G-graded ideal

$$\mathcal{A}(M) = \bigoplus_{g\in G} (R_g \cap \mathcal{A}(M)) = \bigoplus_{g\in G} \mathcal{A}(M)_g.$$

Consider the group graded ring R with unital extension R^u with M again a G-graded-simple R-module. We describe the set

$$\mathcal{A}(M)^u = \{r \in R^u \mid rm = 0 \ \ \forall\, m \in M\}$$

containing the elements in R^u which 'kill' the elements in M. This set forms a graded ideal of R^u.

For any $a \in \mathcal{A}(M)$, the element $(a, 0) \in R^u$ is in $\mathcal{A}(M)^u$ since $(a, 0)m = am = 0$. And so, $\mathcal{A}(M) \subseteq \mathcal{A}(M)^u$. It is clear to see that the elements $(a, 0) \in R^u$ behave exactly as the elements $a \in R$. So to compare $\mathcal{A}(M)$ with its unital extension, we need only consider the the e-component.

Suppose there is an $(r_e, n) \in \mathcal{J}_{gr}(R^u)\backslash\mathcal{J}_{gr}(R)$. Then $(r_e, n) \in \mathcal{A}(M)^u_e = \mathcal{A}(M)^u \cap R^u_e$ with $n \neq 0$. So for all graded simple R-modules M we have

$$(r_e, n)m = r_e m + nm = 0 \qquad n \neq 0.$$

This means that multiplication of an element in any simple module by r_e has the same effect as multiplying by $-n \in \mathbb{Z}$. For every prime p, $\mathbb{Z}_p$ can be considered a graded-simple R-module with trivial ring multiplication so that $(r_e, n)x = nx$ for any $x \in \mathbb{Z}_p$. It is graded by letting $\mathbb{Z}_p$ be the e component, with all other components being equal to zero. This implies that $nx = 0$ where $n \in \mathbb{Z}$ and so p must divide n.

Since $\mathbb{Z}_p$ is a graded-simple R-module for any p, then for all $(r_e, n) \in \mathcal{J}_{gr}(R^u) \backslash \mathcal{J}_{gr}(R)$ we require that p must divide n for all p. This implies that $n = 0$ and we have our contradiction.

This means that no extra killers are admitted by unital extension. Hence, if R is a ring without unity, then $\mathcal{J}_{gr}(R) = \mathcal{J}_{gr}(R^u)$. Applying Theorem 2.11 yields

$$n\mathcal{J}(R) = n\mathcal{J}(R^u) \subseteq \mathcal{J}_{gr}(R^u) = \mathcal{J}_{gr}(R)$$

which completes the proof. $\square$

In Chapter 4 we generalize Theorem 2.13 for the case of rings graded by cancellative partial groupoids.

We conclude this chapter with Balaba's example, as mentioned earlier.

Example 2.14 ([5], Example 6) Consider the set $S = \{(1,1), (1,2), (2,1), (2,2)\}$ under

$$(r, s) \cdot (t, u) = (r, u) \qquad (r, s, t, u \in \{1, 2\}).$$

Then $(S, \cdot)$ forms a rectangular band. The semigroup ring $A = kS$ with coefficients in a field k is S-graded in the usual way. That is

$$A = kS = k(1,1) \oplus k(1,2) \oplus k(2,1) \oplus k(2,2).$$

Let M be any S-graded-simple A-module. Then $(1,1)M = (1,2)M$ and the element $(1,1) - (1,2)$ annihilates M, for any simple A-module M. This puts $(1,1) - (1,2)$ in $\mathcal{J}_{gr}(A)$.

Now if we put $N = A(1,1)$, then N is a graded-simple A-module but $(1,1)$ doesn't annihilate N. The consequence is that $(1,1) \notin \mathcal{J}_{gr}(A)$.

This means that $\mathcal{J}_{gr}(A)$ is actually an *ungraded* ideal of A.

Chapter 3

Rings with Invariant Radicals

Let S be a semigroup and $R = \bigoplus_{s \in S} R_s$ an S-graded ring. If e is an idempotent element of S, then the R_e component of R is a subring of R (see pp. 32). The results of several authors have included investigations of the relationships between the Jacobson radicals $\mathcal{J}(R_e)$ and $\mathcal{J}(R_f)$ of the subrings R_e and R_f for idempotents e, f in S. For example, Amitsur considered invariant radicals in his lengthy paper on Morita contexts [2] and Jespers, Wauters gave comprehensive results for generalized matrix rings [69]. These authors found that the Jacobson radical was invariant for the graded structures of their respective interests.

3.1 Simultaneous generalizations using S-invariance

We now introduce a new concept of S-invariance which enables us to obtain simultaneous generalizations of several previous results known by other authors.

We say that the Jacobson radical is *S-invariant* if, for every S-graded ring

$R = \bigoplus_{s \in S} R_s$, and for all idempotents $e, f \in E(S)$ and all $x, y \in S$ such that $xey = f$, the following inclusion always holds

$$R_x \mathcal{J}(R_e) R_y \subseteq \mathcal{J}(R_f).$$

This definition is motivated by the results of Andrunkanivic [4], Anderson, Divinski and Sulinski [3], Amitsur [2] and Wauters, Jespers [69]. We also find necessary and sufficient conditions on S to ensure that the Jacobson radical is S-invariant.

Amitsur [2] considered radical classes invariant in Morita contexts.

Theorem 3.1 ([2], Theorem 20) *For all Morita contexts $M = (R, V, W, S)$, we have $V\mathcal{J}(S)W \subseteq \mathcal{J}(R)$ where $\mathcal{J}$ is the Jacobson radical.*

With the new concept of S-invariance introduced at the start of this section, we obtain the following generalization of Amitsur's result. This generalization allows us to use Amitsur's result whenever a ring has a B_2-grading.

Lemma 3.2 Amitsur's Theorem is equivalent to saying that the Jacobson radical is B_2-invariant.

Proof. Recall that a Morita context is a set $M = (R, V, W, S)$ where R and S are rings, V is an R-S bimodule, W is an S-R bimodule and the products $V \times W$ to R and $W \times V$ to S are associative bilinear mappings.

Consider the element $vsw \in V\mathcal{J}(S)W$ where $v \in V$, $s \in \mathcal{J}(S)$ and $w \in W$. Since $s \in \mathcal{J}(S)$ the element $swv \in \mathcal{J}(S)$ has a quasi-inverse, s' say. Thus

$swv + s' - s'swv = 0$ and so $s' = twv$ for some $t \in \mathcal{J}(S)$. This yields

$$swv + twv - twvswv = (s + t - twvs)(wv) = 0.$$

Consider the element $r = vtw + vsw - vtwvsw$ and now consider the product

$$\begin{aligned} rv &= (vsw + vtw - vtwvsw)v \\ &= v(sw + tw - twvsw)v \\ &= v(s + t - twvs)wv \\ &= 0. \end{aligned}$$

So

$$r^2 = r(vsw + vtw - vtwvsw) = rv(sw + tw - twvsw) = 0.$$

Hence $(-r) \circ (r) = 0$ and $(-r) \circ (vtw) \circ (vsw) = (-r) \circ (vtw + vsw - vtwvsw) = (-r) \circ (r) = 0$. So any vsw from $V\mathcal{J}(S)W$ is quasiregular with a quasi-inverse in $V\mathcal{J}(S)W$, and so $V\mathcal{J}(S)W$ is a quasiregular ideal in R.

Since every B_2-graded ring is a Morita context (Example 2.6), Amitsur's result says that the Jacobson radical class is B_2-invariant. □

Several authors have considered radicals invariant in group graded rings; including Andrunkanivic [4], Anderson, Divinski, Sulinski [3], Jaegermann [31] and Sands [63]. Here we provide a simultaneous generalization of these results.

Lemma 3.3 For every group G, the Jacobson radical is G-invariant ([4], [63]).

Proof. Let R be any G-graded ring, where G is a group. For any $g \in G$ we have a Morita context given by $\begin{bmatrix} R_1 & R_g \\ R_{g^{-1}} & R_1 \end{bmatrix}$ where 1 is the identity element of the group, and $g^{-1} \in G$ is the inverse of g. From Amitsur's result (Theorem 3.1) we get $R_g \mathcal{J}(R_1) R_{g^{-1}} \subseteq \mathcal{J}(R_1)$. Since $1 \in G$ is the only idempotent, we need only consider this product. So the Jacobson radical class is G-invariant. □

Wauters, Jespers [69] considered generalized matrix rings. One of their results is equivalent to saying that the Jacobson radical is B_n-invariant for all n.

Lemma 3.4 The Jacobson radical is B_n-invariant for all n.

Proof. Suppose R is a B_n-graded ring. Take any nonzero elements $x, y \in B_n$ and nonzero idempotents $e, f \in B_n$ with $xey = f$.

Recall that the Brandt semigroup B_n is the semigroup of $n \times n$ matrix units over an indexing set I with $|I| = n$, as given in Example 1.7. We can express x as (i, j) where $i, j \in I$. Similarly, we can express y as (k, l) with $k, l \in I$. Also, we can express any nonzero idempotents as (m, m) and (n, n) with $m, n \in I$.

Because the product $xey = (i, j) \cdot (m, m) \cdot (k, l)$ is defined and nonzero, it follows that $j = m$ and so we must have $k = m = j$. Next, since $(i, j) \cdot (j, j) \cdot (j, l) = xey = f = (n, n)$ we see that $n = i$ and so must $l = n = i$. Therefore the set (x, e, y, f) is just a Brandt semigroup B_2 over an indexing set of order 2.

Put $R' = R_x + R_e + R_f + R_y$. Then R' is a B_2-graded subring of R. Lemma 3.2 yields that $R_x \mathcal{J}(R_e) R_y \subseteq \mathcal{J}(R_f)$ where $\mathcal{J}$ is the Jacobson radical class. Since R is an arbitrary B_n-graded ring, it follows that the Jacobson radical is B_n-invariant.

□

3.2 S-invariance of the Jacobson radical

We start by giving necessary and sufficient conditions on the semigroup S for the Jacobson radical class to be S-invariant. These conditions generalize some previous results by other authors, as mentioned in Section 3.1.

Theorem 3.5 *Let S be a semigroup, $P(S)$ the ideal generated by all idempotents in S and, $L(S)$ the union of all ideals of S contained in $P(S)$ which do not contain idempotents. Let $K = P(S)/L(S)$. Then the following conditions are equivalent:*

(i) the Jacobson radical class is S-invariant;

(ii) K is a 0-direct union of Brandt semigroups;

(iii) if $a \in K \backslash 0$, then there exists a unique element $x \in K$ such that $axa = a$;

(iv) K is an inverse semigroup in which every nonzero idempotent is primitive;

(v) K is an inverse semigroup which is the union of its 0-minimal right ideals.

Remark 3.6 Amitsur's result on B_2-invariant radicals holds for other radical classes and the same can be said for S-invariance. Specifically, the Baer, Levitzki and Nil radicals are also S-invariant for the semigroup described in our main theorem. It makes sense that the concept of S-invariance can be applied to other radical classes. Indeed, subsequent to our paper [40], Gardner and Kelarev, in [22], provide a large selection of radical classes which are S-invariant for the S-graded rings described our Theorem 3.5.

Before we give the proof of this theorem, we will construct some examples of graded rings for which the Jacobson radical is *not* invariant.

Lemma 3.7 *Let S be a semigroup and $e, f \in S$ be nonzero idempotents such that $SeS \supset SfS$. Then there exists an S-graded ring R such that the Jacobson radical class is not S-invariant.*

Proof. Let $U = SeS$ and $V = SfS$. Since $f \in U$, then there exist $x, y \in S$ such that $f = xey$. We may assume that x and y belong to V because otherwise we could replace x and y by $fx \in V$ and $yf \in V$ using the equality $f = f^3 = (fx)e(yf)$. Let $M = M_2(\mathbb{R})$, the ring of 2×2 matrices with entries from the field of real numbers $\mathbb{R}$, and let T be the subring $e_{12}\mathbb{R}$. Consider the semigroup ring $M[S]$. Clearly $M[V]$ is an ideal of MS and $T[U]$ is a subring of $M[S]$. Hence the sum $R = T[U] + M[V]$ is a subring of $M[S]$. For any $s \in S$ we put

$$R_s = \begin{cases} Ms & \text{if} \quad s \in V \\ Ts & \text{if} \quad s \in U \backslash V \\ 0 & \text{if} \quad s \notin U. \end{cases}$$

Then $R_x = Mx$, $R_y = My$, $R_f = Mf$ and $R_e = Te$. Since $R_e^2 = T^2e^2 = 0$ it follows that R_e is quasiregular. Thus $0 \neq e_{12}f = (e_{11}e_{12}e_{22})(xey) = (e_{11}x)(e_{12}e)(e_{22}y) \in R_x\mathcal{J}(R_e)R_y$. It follows that the Jacobson radical is not invariant, because obviously $\mathcal{J}(R_f) = 0$ cannot contain $e_{12}f$. $\square$

Example 3.8 Consider the subring R of $M_2(\mathbb{R}) \times M_2(\mathbb{R})$ given by $R = R_e + R_g$ where

$$R_e = \left\{ \left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \right) \mid r \in \mathbb{R} \right\}$$

$$R_g = \left\{ \left(0, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \mid a, b, c, d \in \mathbb{R} \right\}.$$

The ring R is graded by the two-element semilattice $Y_2 = \{e, g\}$ with identity e. Since $R_e^2 = 0$ it follows that $\mathcal{J}(R_e) = R_e$. Suppose that the Jacobson radical is invariant. Then

$$\mathcal{J}(R_g) \supseteq R_g \mathcal{J}(R_e) R_g = R_g R_e R_g \neq 0.$$

However this contradicts Corollary 1.25 since $R_g \simeq M_2(\mathbb{R})$ is semisimple.

Lemma 3.9 *Let S be a semigroup. If S contains a subsemigroup isomorphic to the two-element semilattice, then there exists an S-graded ring R such that $\mathcal{J}(R)$ is not S-invariant.*

Proof. Let S be a semigroup. Consider the ring R graded by the two-element semilattice Y_2 constructed in Example 3.8. Suppose that $Y_2 \subseteq S$. Let $P = M_2(\mathbb{R}) \times M_2(\mathbb{R})$, $D_g = R_g$, $D_e = R_e$ and $D_s = 0$ for any $s \in S \backslash Y_2$. The subring $D = \bigoplus_{s \in S} D_s$ of P is a contracted S-graded ring and the Jacobson radical is not invariant. $\square$

Example 3.10 Consider the subring M of $M_2(\mathbb{R})$ given by $M = M_x + M_y$ where

$$M_x = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

$$M_y = \left\{ \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \mid c, d \in \mathbb{R} \right\}.$$

The ring M is graded by the two-element left zero band $X_2 = \{x, y\}$. Consider the ideal of M_x given by

$$I_x = \left\{ \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \mid r \in \mathbb{R} \right\}$$

Since $I_x^2 = 0$, I_x is a quasiregular ideal of M_x. Suppose there exists a quasiregular ideal K of M_x with $I_x \subset K$. Then $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K$ and K cannot be quasiregular. It follows that $\mathcal{J}(M_x) = I_x$. Similarly $\mathcal{J}(M_y) = \left\{ \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix} \mid r \in R \right\}$.

Now, suppose that the Jacobson radical is invariant. Then $M_y\mathcal{J}(M_x)M_y \subseteq \mathcal{J}(M_y)$. Take any $u, v \in M_y$ and any $w \in \mathcal{J}(M_x)$, then

$$\begin{aligned} M_y\mathcal{J}(M_x)M_y \ni uwv &= \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & cr \end{bmatrix}\begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ crs & crt \end{bmatrix} \\ &\notin \mathcal{J}(M_y). \end{aligned}$$

This is a contradiction and the Jacobson radical is not invariant.

Similarly, the subring N of $M_2(\mathbb{R})$ given by $N = N_x + N_y$ where

$$N_x = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$
$$N_y = \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \mid c, d \in \mathbb{R} \right\}.$$

graded by the two-element right zero band $Z_2 = \{x, y\}$, does not have an invariant Jacobson radical.

Lemma 3.11 *Let S be a semigroup. If S contains a subsemigroup isomorphic to the two-element left or right zero band then there exists an S-graded ring R such that $\mathcal{J}(R)$ is not S-invariant.*

Proof. Consider the rings M and N graded respectively by the two-element left zero ideal X_2 and two-element right zero ideal Z_2 as constructed in Example 3.10. Suppose that either $X_2 \subseteq S$ or $Y_2 \subseteq S$. Then, as in the proof of

Lemma 3.9 we can construct a contracted S-graded ring whose Jacobson radical is not invariant. $\square$

3.3 Proofs and remarks

Remark 3.12 *Actually the equivalence of points (ii), (iii), (iv) and (v) in our Theorem 3.5 is given in Clifford and Preston's second volume on the algebraic properties of semigroups. For further details, see Exercise 6, Section 6.5 in [15].*

We now give the proof for Theorem 3.5.

Proof. (i) $\Rightarrow$ (ii) : Suppose that the Jacobson radical is S-invariant. Denote by $E(S)$ the set of all idempotents in S and let $P(S)$ be the ideal generated by $E(S)$. If $P(S) = \emptyset$, the assertion is trivial, and so we may assume $P(S) \neq \emptyset$. Let $L(S)$ be the union of all ideals of S contained in $P(S)$ which do not contain idempotents. Clearly $P(S) \neq L(S)$.

We shall use the same letters to denote the elements in S and their images in the quotient semigroup $P(S)/L(S)$.

For any nonzero element $a \in P(S)/L(S)$, denote by $\langle a \rangle$ the ideal generated by a, by $I(a)$ the ideal of non-generating elements and by F_a the principal factor $\langle a \rangle / I(a)$ containing a.

First, suppose that $P(S)/L(S)$ has a nonzero idempotent e which is not primitive. Then $P(S)/L(S)$ contains a nonzero idempotent $f \neq e$ such that $ef = fe = f$. Therefore $SfS \subset SeS$. This contradicts Lemma 3.7 and it follows that all nonzero idempotents of $P(S)/L(S)$ are primitive.

Next, take any nonzero $a \in P(S)/L(S)$. We shall show that a has an inverse element. To do this, it is enough to show that the principal factor F_a is an inverse semigroup.

Since $a \in P(S)$, there exists a nonzero idempotent $e \in E(S)$, with $a \in SeS$. Since $a \notin L(S)$ there exists an idempotent $f \in \langle a \rangle$. So $SfS \subseteq \langle a \rangle \subseteq SeS$. Lemma 3.7 shows that $SfS = \langle a \rangle = SeS$. This means that $e \in P(S)$. Consider the principal factor F_a. Since F_a contains a primitive idempotent (namely e), it follows from Corollary 1.13 that F_a is completely 0-simple.

Since F_a is completely 0-simple, from Lemma 1.8 it follows that F_a is regular. By Lemma 1.9, each $\mathcal{L}$ and each $\mathcal{R}$ class contains an idempotent. Let $f \in F_a$ be an idempotent. From Lemma 1.11 this idempotent is a right identity of L_f. Suppose L_f contains another idempotent, h say. Since f is a right identity, $hf = h$. Also f is primitive, so it follows that if h is less than or equal to f, then either $h = 0$ or $h = f$. Now, by Lemma 3.11 F_a cannot contain a subsemigroup isomorphic to the two-element left or right zero band. Hence $f = h$ and each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains exactly one idempotent.

From Theorem 1.10 it follows that F_a is an inverse semigroup and so a has an inverse element. Since a was an arbitrary element, it follows that $P(S)/L(S)$ is an inverse semigroup in which every nonzero idempotent is primitive. Using Theorem 1.18 together with Theorem 1.16 we see that $P(s)/L(s)$ is a 0-direct union of Brandt semigroups.

(ii) $\Rightarrow$ (i) : Let S be a semigroup, R an S-graded ring, G a group and I a non-empty indexing set. Suppose that the quotient semigroup $P(S)/L(S)$ is a 0-direct union of Brandt semigroups $\{(i, g, j) \mid i, j \in I, g \in G^0\}$.

Let $e, f \in S$ be idempotents and take any $x, y \in S$ such that $e = xfy$. Then the idempotents e and f must belong to the same 0-direct component of $P(S)/L(S)$.

First, consider the case when x and y are also in the same 0-direct component as e and f. Pass to the principal factor F of $P(S)/L(S)$ containing e, f, x, y. Again we shall use the same letters e, f, x, y to denote the images of e, f, x, y in F.

Let $1 \in G$ denote the identity element of G. Since $e, f \notin L(S)$, then clearly e, f are nonzero idempotents of F. It is easily seen that $e = (i, 1, i)$ and $f = (j, 1, j)$ for some $i, j \in I$. It follows from the equality $xfy = e$ that if $x = (i, g, j)$ for some $g \in G^0$, then $y = (j, g^{-1}, i)$, where $g^{-1} \in G$ is the inverse of g.

For each $g \in G^0$, denote by $\mathcal{R}_g$ the sum of all $R_{(i,g,j)}$ where $i, j \in I$. For any $g, h \in G^0$ and any $i, j, k, l \in I$ we get $(i, g, j)(k, h, l) = (i, gh, l)$ if $j = k$, and $(i, g, j)(k, h, l) = 0$ otherwise. In both cases $R_{(i,g,j)}R_{(k,h,l)} \subseteq \mathcal{R}_{gh}$. Therefore $\mathcal{R} = \bigoplus_{g \in G^0} \mathcal{R}_g$ is G^0-graded. The identity component, $\mathcal{R}_1 = \bigoplus_{i,j} R_{(i,1,j)}$ is a generalized matrix ring. Therefore $\mathcal{J}(\mathcal{R}_1) \cap R_e = \mathcal{J}(R_e)$ for any idempotent $e \in S$ (Proposition 2.10). Since $\mathcal{R}$ is group graded and $\mathcal{J}(\mathcal{R})$ is G-invariant (Example 3.3) it follows that $\mathcal{R}_g \mathcal{J}(\mathcal{R}_1) \mathcal{R}_{g^{-1}} \subseteq \mathcal{J}(\mathcal{R}_1)$. Also $R_x \subseteq \mathcal{R}_g$ and $R_y \subseteq \mathcal{R}_{g^{-1}}$ so we get

$$R_x \mathcal{J}(R_f) R_y \subseteq \mathcal{R}_g \mathcal{J}(\mathcal{R}_1) \mathcal{R}_{g^{-1}} \bigcap R_x R_f R_y \subseteq \mathcal{J}(\mathcal{R}_1) \cap R_e = \mathcal{J}(R_e),$$

as required.

Second, consider arbitrary $x, y \in S$. Put $M = R_x \mathcal{J}(R_f) R_y$ and let $T = R_e^1 M R_e^1$ be the ideal generated by M in R_e. In order to prove $M \subseteq \mathcal{J}(R_e)$ we shall show that $T \subseteq \mathcal{J}(R_e)$.

Let F be the principal factor of S containing e and f. Then ex and ye are nonzero elements of F, because otherwise we would get $e = exfye = 0$. For such elements we have proved that $\mathcal{J}(R_e) \supseteq R_{ex}\mathcal{J}(R_f)R_{ye}$.

Consider the ideal $T^3 \subseteq TR_e^1MR_e^1T \subseteq R_eR_x\mathcal{J}(R_f)R_yR_e \subseteq R_{ex}\mathcal{J}(R_f)R_{ye} \subseteq \mathcal{J}(R_e)$. Hence $T \subseteq \mathcal{J}(R_e)$, as required.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): For the equivalence of (ii), (iii) and (iv) see Venkatesan [68]. For the equivalence of (ii), (iii) and (v) see Preston ([56] and [57]). Alternatively, the equivalence of (ii), (iii), (iv) and (v) are stated in the exercise set concluding Section 6.5 of Clifford and Preston's second volume [15]. $\Box$

Remark 3.13 Actually it is of interest now to consider when $R_xJ(R_e)R_y \supseteq J(R_f)$ because this will give us some ideas on the stronger property

$$R_xJ(R_e)R_y = J(R_f).$$

This happens only in the trivial case when S is without idempotents, so that there are no e, f, x, y such that $xey = f$. To this end, take a ring R with zero multiplication, and consider the semigroup ring RS. Then all products on the left hand side are zero, and so don't contain the right hand side.

Chapter 4

Rings with Homogeneous Radicals

By defining a homogeneous radical we give a generalization of Bergman's Lemma (Theorem 2.13) on finite group graded rings to rings graded by partial groupoids.

Graded rings with finite support have been actively investigated recently (see, for example, [7], [13], [19]). Evidently, every group graded ring with finite support is a cancellative homogeneous sum. We saw in Examples 2.1 and 2.4 that homogeneous sums and generalized matrix rings are both graded by induced partial groupoids. Therefore our main theorem and corollaries apply to these graded rings, as well.

4.1 Homogeneous sums

Let S be a non-empty partial groupoid and recall that each element r of an S-graded ring $R = \bigoplus_{s\in S} R_s$ can be expressed as a unique sum $r = \sum_{s\in S} r_s$ of homogeneous elements $r_s \in R_s$ for $s \in S$. The support of each $r \in R$ is given by $\operatorname{supp}(r) = \{s \in S \mid r_s \neq 0\}$. We have seen (Example 2.1) that the set of homogeneous elements $H(R) = \bigcup_{s\in S} R_s$ is the union of all components R_s for $s \in S$. Similarly we put

$$H(r) = \{r_s \mid s \in \operatorname{supp}(r)\},$$

and assume $H(0) = 0$.

If I is a subset of R, then we put $H(I) = \bigcup_{r\in I} H(r)$ and $\operatorname{supp}(I) = \bigcup_{r\in I} \operatorname{supp}(r)$. A subset I of R is said to be *homogeneous* if $H(I) \subseteq I$. If I is a homogeneous subring of R, then it is clear that $I = \bigoplus_{s\in S} I_s$ is a homogeneous sum, where $I_s = I \cap R_s$.

Let $R = \bigoplus_{s\in S} R_s$ be a homogeneous sum and $H(R) = \bigcup_{s\in S} R_s$ be the set of homogeneous elements of R. We say that R is a *cancellative* homogeneous sum if, given any homogeneous elements $x \in R_s$, $y \in R_t$, $z \in H(R)$ and any $u \in S$, each of the conditions $0 \neq xz, yz \in R_u$ or $0 \neq zx, zy \in R_u$ implies that $s = t$. This is equivalent to saying that the partial groupoid induced by R is cancellative.

For a partial groupoid S, denote by $m(S)$ the least common multiple of the orders of all subgroups of S. Evidently, $m(S) \leq |S|!$. However, it seems the value of $m(S)$ must be much less than $|S|!$. So let's give a better upper bound for $m(S)$ in terms of $|S|$.

Remark 4.1 Take a positive integer n and list all primes $p_1, p_2, \ldots, p_k$ which are less than n. Consider the set M_n of products of the form $p_1^{a_1} \cdots p_k^{a_k}$, where $p_1^{a_1} + \cdots + p_k^{a_k} - k < n$ and define $m(n)$ as the maximum product in M_n. Then for every partial groupoid S,

$$m(S) \leq m(|S|).$$

Proof. Put $n = |S|$ and $m = m(S)$. Consider the prime decomposition of $m = p_1^{a_1} \cdots p_k^{a_k}$. Given that m is the least common multiple of the orders of all subgroups of S, for every $1 \leq i \leq k$, we see that S has a subgroup T_i such that $p_i^{a_i}$ is a factor of $|T_i|$. By Sylow's First Theorem (Theorem 1.3), T_i contains a subgroup H_i of order precisely $p_i^{a_i}$.

For any $i \neq j$, the intersection of H_i and H_j will either be empty or contain only the identity element. Since the union of all the sets H_i is contained in S, we have $p_1^{a_1}+\cdots+p_k^{a_k}-x = |\bigcup_{i=1}^{k} H_i| \leq |S| = n$, where $0 \leq x \leq k-1$ is included in this sum to cover any cases of non-empty intersections in the subgroups because of potential shared identities. This means that $p_1^{a_1} + \cdots + p_k^{a_k} - (k-1) \leq p_1^{a_1} + \cdots + p_k^{a_k} - x \leq |S|$ and hence $p_1^{a_1} + \cdots + p_k^{a_k} - k \leq |S| - 1 < |S|$, so that that m belongs to M_n. Therefore m is less than or equal to the maximum number $m(n)$ in M_n. $\square$

When dealing with homogeneous sums $R = \bigoplus_{s \in S} R_s$ with S a finite set, we consider S as a partial groupoid induced by R and denote the least common multiple of the orders of all subgroups of S by $m(R)$. So, if $R = \bigoplus_{s \in S} R_s$ is a homogeneous sum then, considering S as a partial groupoid induced by R, we get $m(R) = m(S)$.

4.2 Homogeneous components of the radical

Theorem 4.2 *Let R be a cancellative homogeneous sum, and let $m = m(R)$. Then, for every $r \in \mathcal{J}(R)$, all homogeneous components of mr belong to $\mathcal{J}(R)$.*

Before we give the proof, we show that the number $m(R)$ in the main theorem cannot be replaced by smaller numbers.

Lemma 4.3 *Let S be a finite partial groupoid and put $m = m(S)$. Then there exists an S-graded ring R so that for any number $\ell < m$*

$$\ell H(\mathcal{J}(R)) \not\subseteq \mathcal{J}(R).$$

Proof. Consider the prime decomposition $m = p_1^{a_1} \cdots p_k^{a_k}$ of m. As in the proof for Remark 4.1, denote by H_i a subgroup of order $p_i^{a_i}$ in S. Let F_i be the ring of residues modulo $p_i^{a_i}$. Denote by R the direct sum of group rings $F_i[H_i]$, for $i = 1, \ldots, k$. For $s \in S$, put $R_s = 0$ if s does not belong to the union of all subgroups of S. If s is contained in some subgroup of S, then denote by R_s the direct sum of all sets $F_i s$ such that $s \in H_i$ for appropriate $i = 1, \ldots, k$. It is easily seen that $R = \bigoplus_{i=1}^{k} F_i[H_i] = \bigoplus_{s \in S} R_s$ is S-graded.

Next, take any $i = 1, \cdots, k$ and consider the natural mapping $\phi_i : F_i H_i \to F_i$ induced by collapsing H_i to the identity element 1. Specifically, ϕ_i sends $\sum_{j=1}^{p_1^{a_1}} f_j h_j \to \sum_{j=1}^{n} f_j$ where the f_j's are in F_i and the h_j's in H_i. Then ϕ_1 is a surjective ring homomorphism called the *augmentation map*. The kernel of the augmentation map is given by

$$\ker \phi_i = \left\{ \sum_{j=1}^{p_1^{a_i}} f_j h_j \,\middle|\, f_j \in F_i, h_j \in H_i, \sum_{j=1}^{p_i^{a_i}} f_j = 0 \right\}$$

and this is the *augmentation ideal.* Karpilovsky ([37] Theorem 43.6) tells us that the Jacobson radical of $F_i[H_i]$ is the augmentation ideal plus $\mathcal{J}(F_i)$. Actually $\mathcal{J}(F_i) = \langle p_i \rangle \lhd Z_{p_i^{a_i}}$. So $\mathcal{J}(F_i[H_i]) \subseteq \ker \phi$.

It follows that the Jacobson radical of each F_iH_i is equal to the augmentation ideal of F_iH_i, i.e., to the set

$$\left\{ \sum_{j=1}^{n} f_j h_j \;\middle|\; f_j \in F_i, h_j \in H_i, \sum_{j=1}^{n} f_j = 0 \right\}.$$

This means that $\mathcal{J}(R)$ is the direct sum of all these augmentation ideals.

Look at any number ℓ less than m. There exists i such that $p_i^{a_i}$ does not divide ℓ. Take any $g \in H_i \backslash \{e\}$, where e is the identity of H_i. The element $1-g$ belongs to $\mathcal{J}(R)$. However, $\ell \neq 0$ in F_i, and so $0 \neq \ell g \notin \mathcal{J}(R)$. □

4.3 Proof

Proof of Theorem 4.2. Suppose the contrary. Then we can find a minimal counter-example to the theorem, that is, there exists a cancellative partial groupoid S with minimal $|S|$ and an S-graded ring R such that $m(R)H(\mathcal{J}(R)) \not\subseteq \mathcal{J}(R)$. Let $k = |S|$ and $n = m(R)$.

First, consider the case where $|\operatorname{supp}(R_sR)| < k$ for some $s \in S$. We will denote R_s by W. (This notation is needed so that we may refer back to this part of the proof later when we shall use the same reasoning with another set W. Our argument is valid for any set W contained in one homogeneous component of R such that $|\operatorname{supp}(WR)| < k$). Let $K = R^1WR^1$. Since K is a homogeneous ideal of R, evidently R/K is a homogeneous sum, too. We are going to show that R/K is also a counter-example to the theorem. This will allow us to factor out K and assume that $W = 0$, which will lead to a contradiction.

Denote by $E(R)$ the additive subgroup generated by the set $nH(\mathcal{J}(R))$ of homogeneous components of elements of $n\mathcal{J}(R)$. Take any $x \in \mathcal{J}(R)$ and $y \in H(R)$. For any such y, there exists $r \in R$ and $b \in S$ such that $y = r_b$. Given that S is cancellative, all $x_a y$ belong to distinct homogeneous components for different $a \in S$. Therefore $(nx_a)y = n(xr)_{ab} \in nH(\mathcal{J}(R))$. Thus $nH(\mathcal{J}(R))$ is an ideal of the multiplicative semigroup $H(R)$. Hence $E(R)$ is an ideal of R.

Recall that $K = R^1WR^1$ and let us introduce $I = R^1WR$, $L = E(R) \cap K$, $F = E(R) \cap I$ and $P = IE(R)$. Then $P \subseteq F \subseteq L \subseteq E(R)$ are ideals of R and we will show that they are all quasiregular.

We begin with P. Pick any $e \in nH(\mathcal{J}(R))$. There exist $r \in \mathcal{J}(R)$ and $g \in S$ such that $e = nr_g$. Consider an arbitrary t in S. Given that W is contained in one homogeneous component of R, obviously R_tWR is a homogeneous right ideal of R. Since quasiregularity is inherited by right ideals, we get

$$R_tWRr \subseteq \mathcal{J}(R) \cap R_tWR \subseteq \mathcal{J}(R_tWR).$$

Denote by T the partial groupoid induced on $\operatorname{supp}(R_tWR)$ by R_tWR. Let $\ell = m(T)$. Since $|T| < k$, the minimality of k implies that $\ell H(\mathcal{J}(R_tWR)) \subseteq \mathcal{J}(R_tWR)$. If a product uv is defined in T for some $u, v \in T$, then the product uv is also defined in S. Therefore every subgroup of T is also a subgroup of S. It follows that ℓ divides n. Therefore $nH(\mathcal{J}(R_tWR)) \subseteq \mathcal{J}(R_tWR)$. Since $R_tWRr \subseteq \mathcal{J}(R_tWR)$ and $e = nr_g$ we get

$$R_tWRe \subseteq nH(\mathcal{J}(R_tWR)) \subseteq \mathcal{J}(R_tWR).$$

Given that e was an arbitrary generating element of $E(R)$, it follows that

$$R_tWRE(R) \subseteq \mathcal{J}(R_tWR).$$

Since $E(R)$ is an ideal of $H(R)$, we see that $R_tWRE(R)$ is a quasiregular right ideal of R. Therefore $P = IE(R) = \sum_{t \in S} R_tWRE(R)$ is a sum of quasiregular right ideals and is hence quasiregular.

Further, $P = IE(R) \supseteq FE(R) \supseteq F^2$ implies $(F/P)^2 = 0$ and so F/P is quasiregular. Since P is quasiregular and F/P is quasiregular, it now follows that F is also quasiregular.

In order to show that L is quasiregular, consider the homogeneous ideal $K = R^1WR^1$. Clearly, $K^2 = R^1WR^1K \subseteq I$. Therefore, we obtain

$$L^2 = (E(R) \cap K)^2 \subseteq (E(R) \cap I) = F.$$

Since F is quasiregular, L is also quasiregular.

Next, let $\overline{R}$ denote the S-graded quotient ring R/K. For $X \subseteq R$ and $r \in R$ denote by $\overline{X}$ and $\overline{r}$ the respective images of X and r in $\overline{R}$. Denote by V the partial groupoid induced on $\operatorname{supp}(\overline{R})$ by $\overline{R}$ and let $m(V) = u$. Then $E(\overline{R})$ is the additive subgroup generated by $uH(\mathcal{J}(\overline{R}))$. All products defined in V are also defined in S, because $\overline{R}_s\overline{R}_t \neq 0$ implies $R_sR_t \neq 0$. Hence all subgroups of V are also subgroups of S. It follows that u divides n. This and $\overline{\mathcal{J}(R)} \subseteq \mathcal{J}(\overline{R})$ give us $\overline{E(R)} \subseteq E(\overline{R})$.

Suppose that $\overline{R}$ is not a counter-example to our theorem. Then $E(\overline{R}) \subseteq \mathcal{J}(\overline{R})$, and so $\overline{E(R)} \subseteq E(\overline{R}) \subseteq \mathcal{J}(\overline{R})$. Hence $\overline{E(R)}$ is quasiregular and therefore $E(R)/L \cong \overline{E(R)}$ is quasiregular. Since L is quasiregular, this implies that $E(R)$ is quasiregular. This contradiction shows that $\overline{R}$ is also a counter-example. Therefore without loss of generality we may assume that $W = 0$ from the very beginning.

Finally, since $R_s = W = 0$, we see that R is a homogeneous sum of R_t where t runs over $S\backslash\{s\}$. This contradicts the minimality of k and so the case when $|\operatorname{supp}(R_sR)| < k$ for some $s \in S$ is impossible.

Now, consider that case when $|\operatorname{supp}(R_s R)| = k$ for all $s \in S$. Then the products st are defined for all pairs $s, t \in S$.

First, suppose that S is not associative. Then there exist $a, b, c \in S$ with $(ab)c \neq a(bc)$. Thus $R_a R_b R_c \subseteq R_{(ab)c} \cap R_{a(bc)} = \emptyset$. Therefore $|\operatorname{supp}(R_a R_b R)| < k$. As for the case when $|\operatorname{supp}(R_s R)| < k$, we can set $R_s R_t = W$ and we get $W = 0$. This means that $R_s R_t = 0$. As we have shown above this case is impossible.

Second, suppose that S is associative. Then S is a semigroup. Since every finite cancellative semigroup is a group, our counter-example is graded by a finite group. However, for rings graded by finite groups the theorem is true by Theorems 2.11 and 2.13. This contradiction completes the proof. □

The following example shows that Theorem 4.2 cannot be generalized to homogeneous sums which are not cancellative.

Example 4.4 Let $R = R_0 + R_1$ be the subring of $M_2(\mathbb{R}) \times M_2(\mathbb{R})$ where

$$R_0 = \left\{ \left(0, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \mid a, b, c, d \in \mathbb{R} \right\}$$

$$R_1 = \left\{ \left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \right) \mid r \in \mathbb{R} \right\}.$$

The set $S = \{0, 1\}$ with respect to ordinary multiplication of integers is an idempotent semigroup. Clearly R is S-graded and S is not cancellative. Consider the ideal

$$I = \left\{ \left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, 0 \right) \mid r \in \mathbb{R} \right\}.$$

Since $I^2 = 0$, I is a quasiregular ideal and since $R/I \simeq M_2(\mathbb{R})$ is semisimple, it follows that $I = \mathcal{J}(R)$. Take any $x \in I$, say

$$x = \underbrace{\left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \right)}_{x_1} + \underbrace{\left(0, \begin{bmatrix} 0 & -r \\ 0 & 0 \end{bmatrix} \right)}_{x_0}.$$

where $r \in \mathbb{R}$. Evidently, R_0 is an ideal of R and so $R_0 \cap \mathcal{J}(R) = \mathcal{J}(R_0) = 0$. However

$$0 \neq nx_0 \notin \mathcal{J}(R_0)$$

for any $n \in \mathbb{N}$. Hence the analogue of our main theorem for homogeneous sums which are not cancellative does not hold.

4.4 Corollaries

If G is a finite group then Lagrange's Theorem along with Sylow's First Theorem (Corollary 1.4) says that the least common multiple of the order of all subgroups of G equals $|G|$. Then by definition $m(R)$ equals $|G|$ for every G-graded ring R. Thus, in the case of a ring graded by a finite group Theorem 4.2 tells us that $|G|H(\mathcal{J}(R)) \subseteq \mathcal{J}(R)$. This is exactly Bergman's lemma (Theorem 2.13).

Recall from Example 2.4 that a ring $R = \bigoplus_{i,j=1}^{m} R_{ij}$ is said to be a generalized matrix ring if

$$R_{ij}R_{k\ell} \subseteq \begin{cases} R_{i\ell} & \text{if} \quad j = k; \\ 0 & \text{otherwise.} \end{cases}$$

Considering the induced partial groupoid on the indexing set

$$S = \{(i,j) \mid i,j = 1, \dots, m\},$$

we see that S is cancellative and has only subgroups of order one. Therefore $m(R) = 1$. So Theorem 4.2 shows that the radical of R is homogeneous. That is

$H(\mathcal{J}(R)) \subseteq \mathcal{J}(R)$ for every generalized matrix ring R. This was earlier proved by Bergman in [9] and Wauters, Jespers in [69].

Let an F-algebra R be a cancellative homogeneous sum over a field F of characteristic zero or prime characteristic greater than $|S|$ where S is the partial groupoid induced by R. Then $m(R)$ is invertible in F, and we immediately get the following.

Corollary 4.5 *Let an F-algebra R be a cancellative homogeneous sum, S the partial groupoid induced by R and F a field with* $\operatorname{char} F = 0$ *or* $\operatorname{char} F > |S|$. *Then the Jacobson radical of R is homogeneous.*

Our results can be used to deduce various corollaries concerning semisimplicity of graded rings. For example, by analogy with group graded terminology we say that homogeneous sum $R = \bigoplus_{s \in S} R_s$ is *non-degenerate* if and only if each of the equalities $(rR)_e = 0$ or $(Rr)_e = 0$ implies $r = 0$, where $e \in E(S)$ is any idempotent. We now have the following corollary.

Corollary 4.6 *Let S be a finite cancellative partial groupoid with identity e, let F be a field with* $\operatorname{char} F = 0$ *or* $\operatorname{char} F > |S|$, *and let $R = \bigoplus_{s \in S} R_s$ be a non-degenerate S-graded F-algebra. If R_e is semisimple, then R is semisimple.*

Proof. Suppose that R is not semisimple. Corollary 4.5 shows that there exists a nonzero homogeneous element $x \in \mathcal{J}(R)$. By non-degeneracy $(xR)_e \neq 0$. Hence there exists a homogeneous element y such that $0 \neq xy \in R_e \cap \mathcal{J}(R)$. Since e is the identity of S, clearly R_e is a direct summand of the right R_e-module R. Therefore Proposition 2.10 tells us that $R_e \cap \mathcal{J}(R) \subseteq \mathcal{J}(R_e)$. Thus $\mathcal{J}(R_e) \neq 0$. This contradiction shows that R is semisimple. □

Say that an S-graded ring R is *faithful* if and only if $rR_t = 0$ implies $R_sR_t = 0$, and $R_tr = 0$ implies $R_tR_s = 0$, for any $s, t \in S$ and $0 \neq r \in R_s$.

Corollary 4.7 *Let S be a finite cancellative partial groupoid, F a field with $\operatorname{char} F = 0$ or $\operatorname{char} F > |S|$, and let $R = \bigoplus_{s\in S} R_s$ be a faithful S-graded F-algebra. If all subrings among the homogeneous components R_s are semisimple, then R is semisimple.*

Proof. Suppose that R is not semisimple. To get a contradiction we shall prove that, for some idempotent e, the homogeneous component R_e is a ring with nonzero radical.

Put $I = \mathcal{J}(R)$. Corollary 4.5 says that I is homogeneous and that $I = \bigoplus_{s\in S} I_s$, where $I_s = I \cap R_s$. Let T be the support of I, that is, the set of all s such that $I_s \neq 0$. Faithfulness easily yields that $R_T = \bigoplus_{t\in T} R_t$ is a subring of R. Obviously R_T is not semisimple because $\mathcal{J}(R_T) \supseteq I$. To simplify the notation we assume that $R = R_T$ and that all components R_s are nonzero.

If all rings among the components R_s are nilpotent (in particular, if there are no rings among these components), then R is nilpotent by [38], Theorem 1. Look at any component R_s. Recursively define $t_1 = s$, $Q_1 = R_s$, and $t_{i+1} = t_i^2$, $Q_{i+1} = Q_iR_{t_i}$, for $i \geq 1$. Given that R is nilpotent, there exists a positive integer n such that $Q_n \neq 0$ and $Q_{n+1} = 0$. Put $t = t_n$. The faithfulness and $Q_nR_t = Q_{n+1} = 0$ yield $R_t^2 = 0$. Thus $R_t \in \mathcal{J}$ is a radical ring, a contradiction.

Next, consider the case where a component R_e is a ring but is not nilpotent. Then $R_e^2 \subseteq R_e$ gives us $e^2 = e$.

Choose a nonzero homogeneous element r from $R_e \cap \mathcal{J}(R)$. This time we denote by I the quasiregular homogeneous ideal generated in R by r, and let T be the support of I. As above, we may assume that $R = R_T$ and all the components R_s are nonzero.

We claim that R_e is a direct summand of R as a left R_e-module.

Indeed, pick any $s \in S$. There exist $t \in S$ and $y \in R_t$ such that $0 \neq ry \in R_s$. If $R_e ry = 0$, then $R_e R_s = 0$ by faithfulness. On the other hand, if $R_e ry \neq 0$, then choose an element $x \in R_e$ such that $xry \neq 0$. We get $(xr)y \in R_{et} = R_s$ and $x(ry) \in R_{es}$; whence $es = s$. Therefore $R_e R_s \subseteq R_s$. It follows that $\bigoplus_{e \neq s \in S} R_s$ is a left R_e-module.

By Proposition 2.10 $\mathcal{J}(R_e) \supseteq R_e \cap \mathcal{J}(R)$. Therefore $0 \neq r \in \mathcal{J}(R_e)$, a contradiction. $\square$

Chapter 5

The Ring of Hirota Derivatives

Just as Frobenius had shown that the ordinary group characters of the symmetric group mapped to S-functions, Schur called the characteristic mapping of the irreducible characters of the projective representation (we call these the spin characters), *Q-functions.* These Q-functions have come to be called Schur Q-functions. In this chapter, we just call them more simply, Q-functions. In Chapter 6 we describe a more generalized version of Q-functions through the Hall-Littlewood polynomials.

The aim of this chapter is to present a graded ring without unity which has physical world applications. Graded rings, of course, arise in many natural contexts, and so we chose a graded ring from one of many. The title of this chapter reveals the choice of ring, and our investigation leads us to a hierarchy of partial differential equations (PDEs) called BKP equations. It turns out that the solutions to these equations (see [32]) are the Schur Q-functions.

Our main idea is to make use of the characteristic mapping used by Schur in order to introduce *generalized Q-operators.* We describe the algebraic structure of the generalized Q-operators as additive subgroups of the ring of Hirota derivatives, and use shifted Young diagrams to represent them. In the final sections of this chapter, we make some connections between these generalized Q-operators and the BKP equations. For example, we use a Pieri-type formula for Q-functions as a raising operator for the shifted diagrams and show that this action generates a lower weight part of the BKP hierarchy. The fact that this action generates the lower part of the hieracrhy was published by us in a previous paper [55] with co-author Salam. At that time, however, we had not made the necessary connections to the characteristic mapping, nor to the generalized power-sum symmetric functions. In the first three sections of this chapter, we now make those connections. We also conjecture that certain elements of the subgroups of generalized Q-operators are the so-called seeds of the BKP hierarchy. In the final section of the next chapter we show that equations in Hirota form can be written in terms of the generalized Q-operators.

We use $\mathbb{C}$ to denote the complex numbers, and $GL(k)$ to denote the general linear group of $k \times k$ invertible matrices.

5.1 The Q-functions

The Q-functions were first introduced by Schur as a projective representations analogy of Frobenius' theorem for ordinary characters. A *projective representation* of the symmetric group $\mathcal{S}_n$ is a mapping M from $\mathcal{S}_n$ into $GL(k)$ so that for $x, y \in \mathcal{S}_n$

$$M(x)M(y) = \tau(x,y)M(xy)$$

for some $\tau(x,y) \in \mathbb{C}^* = \mathbb{C}/\{0\}$. Because the linear transformations are invertible the mapping τ is a 2-cocycle. The set of cohomology classes of 2-cocyles forms an Abelian group called the *Schur multiplier.* We refer to Section 1 of [67] for further details, or to [27] for a comprehensive account. In our case, since we are mapping from the symmetric group, the nature of the equivalence classes of these 2-cocycles is available. Indeed, they have a 2-element classification for $n \geq 4$ as $\mathbb{Z}_2$. This equivalency is determined by the Schur multiplier. (It is interesting to note that the account of Schur multipliers recommended in [27] is written by Karpilovsky [36], author of [37] on the Jacobson radical of graded rings.) Representations M of $\mathcal{S}_n$ for which $\tau(x,y) \equiv 1$ correspond with the ordinary linear representations; otherwise there are group elements so that $\tau(x,y) \equiv -1$ and these correspond to a double cover $\widetilde{\mathcal{S}_n}$ of $\mathcal{S}_n$. It is the characters of this double cover that we mean when we talk about the irreducible characters of the projective representation. The double cover is sometimes called the *spin representation* and its characters are called *spin characters.* We use this terminology, and in Chapter 6 we write the spin characters in terms of the ordinary ones. We also use this relationship to describe a new combinatorial algorithm to determine spin character values.

Some specifics on spin characters, in the context of Q-functions, are noted here. Denote by ζ^λ_μ the spin character ζ^λ on the class μ of the symmetric group $\mathcal{S}_n$. Use $\mathcal{OP}$ to mean the class of partitions with all parts odd integers, and call the members of $\mathcal{OP}$ *odd part partitions.* Use $\mathcal{DP}$ to mean the class of partitions with all parts distinct integers (so that the parts are written in strict descending order) and call the members of $\mathcal{DP}$ *distinct part partitions.* Only spin characters ζ^λ with $\lambda \in \mathcal{DP}$ on the class $\rho \in \mathcal{OP}$ are relevant here, consistent with the same restrictions required on the spin characteristic mapping, as in [65], and [27].

For any $\lambda \in \mathcal{DP}$, the *Q-function* Q_λ is determined by the mapping of spin characters ζ^λ so that

$$Q_\lambda(x) = \sum_{\substack{\rho \vdash |\lambda| \\ \rho \in \mathcal{OP}}} 2^{\frac{1}{2}[l(\lambda)+l(\rho)+\epsilon]} z_\mu^{-1} \zeta_\rho^\lambda p_\rho(x) \tag{5.1}$$

where: $l(\lambda)$ means the length of the partition λ, as discussed in Section 1.4; ϵ is 0 or 1 as required; ζ_ρ^λ are the spin characters on the class $\rho \in \mathcal{OP}$; $p_\rho(x)$ are the power-sum symmetric functions; and, z_μ is the size of the centralizer, determined by Frobenius' formula (see Lemma 1.29). The right hand side of this expression is just the characteristic mapping of ζ^λ, analogous with the mapping discussed on page 43.

This formulation of the Q-functions is due to Schur and first appeared in [65]. The Q-functions appear in their more generalized form as Hall-Littlewood functions in [27] (see also Section 6.1). The Q-functions we just described here are also referred to as Schur Q-functions; but as mentioned in the introduction to this chapter, in this thesis, we usually call them, more simply, Q-functions.

5.2 The ring of Hirota derivatives

Recall that the main aim of this chapter is to provide examples of graded rings with applications. The first of these examples will be the ring of Hirota derivatives, whose elements play a crucial role in the describing certain soliton equations. One of the first descriptions of soliton wave behaviour was made by Scottish engineer, mathematician, and physicist John Scott Russell [59] who observed (from horseback) a non-dispersing shallow water wave traveling along a canal in Scotland. For an accessible and insightful introduction to solitons, see [45].

The *Hirota derivatives* D_x are differential operators acting on a pair of functions:

$$D_x^n f(x)\cdot g(x) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2) \quad |_{x_1=x_2=x}$$

where ∂_{x_1} is the standard partial differential operator with respect to x_1.

Example 5.1

$$\begin{aligned} D_x f(x)\cdot g(x) &= (\partial_{x_1} - \partial_{x_2}) f(x_1)g(x_2) && |_{x_1=x_2=x} \\ &= f_{x_1} g(x_2) - f(x_1) g_{x_2} && |_{x_1=x_2=x} \\ &= f_x g - f g_x. \end{aligned}$$

The minus sign is the thing to notice here. In this sense, the Hirota derivatives are an anti-symmetrical version of the classical Leibniz rule for differentiating products of functions.

If the Hirota derivatives act on a pair of the same function (i.e. on $f \cdot f$) more than once, then the 'splitting' of the variable into dummy variables with re-evaluation at the end allows the resultant to survive the first act:

$$\begin{aligned} D_x^2 f(x)\cdot f(x) &= D_x (f_{x_1} f - f f_{x_2}) && |_{x_1=x_2=x} \\ &= (f_{x_1x_1} f - f_{x_1} f_{x_2} - f_{x_1} f_{x_2} + f f_{x_2x_2}) && |_{x_1=x_2=x} \\ &= 2\left(f_{xx}\cdot f - (f_x)^2\right) \end{aligned}$$

An interesting aspect of the previous expression is that

$$\begin{aligned} \partial_x^2 \log f &= \partial_x \left(\frac{1}{f}\cdot f_x\right) \\ &= \frac{1}{f} f_{xx} - \frac{1}{f^2}\cdot f_x \cdot f_x \\ &= \frac{1}{f^2}\left(f\cdot f_{xx} - f_x f_x\right) \\ &= \frac{1}{2f^2} D_x^2 \left(f(x)\cdot f(x)\right). \end{aligned}$$

This relationship is key to the way these differential operators play a crucial role in describing soliton equations. This is demonstrated in Example 5.3 of Section 5.4.

Use $\mathbb{Q}[D_x]$ to denote polynomials of Hirota derivatives with rational coefficients with the additive subgroups of degree k monomials of D_x denoted by $\mathbb{Q}[D_x]^{(k)}$. Then the Hirota derivatives form the basis of a graded ring

$$\mathbb{Q}[D_x] = \bigoplus_{k \in \mathbb{N}} \mathbb{Q}[D_x]^{(k)}$$

since $\mathbb{Q}[D_x]^{(m)}\mathbb{Q}[D_x]^{(n)} \subseteq \mathbb{Q}[D_x]^{(m+n)}$. The structure grading the ring is the natural numbers under addition $(\mathbb{N}, +)$, a semigroup without unity. This means that the ring $\mathbb{Q}[D_x]$ is a graded ring without unity. It seems mention-worthy that the ring $\mathbb{Q}[D_x]$ does not contain constants.

The Hirota derivatives are multiplicative and extend into many variables by introducing a partition notation by first putting, for example $x = x_1$, and writing D_x more simply as D_1. Then $D_\lambda = D_{\lambda_1} D_{\lambda_2} \cdots D_{\lambda_k}$ for any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. For partitions written using multiplicity of the parts $\lambda = (\lambda_1^{m_{\lambda_1}}, \cdots, \lambda_j^{m_{\lambda_j}})$, this means that $D_\lambda = D_{\lambda_1}^{m_{\lambda_1}} \cdots D_{\lambda_j}^{m_{\lambda_j}}$.

5.3 Littlewood-associated functions and the generalized power-sums

The function Littlewood associated with the power-sum symmetric functions (see Section 2.2) is

$$G(\alpha; t) = \prod_{i \geq 1} (1 - \alpha_i t)^{-1}$$

which is connected to the generating function $P(\alpha)$ by observing that $P(\alpha) = G'(t) \cdot G(t)^{-1}$, as shown in Section 2.2.

If the Littlewood-associated function for a power-sum symmetric function is generalized in a meaningful way to a symmetric function of several sets of variables, then the symmetric functions are called *generalized power-sum symmetric functions*, with corresponding *generalized Schur functions* obtained via the group characteristic mapping. These are introduced in Section 6.4 of Littlewood's book [43] on group characters and matrix representations of the symmetric group, where two other Littlewood-associated functions are described for which properties of the power-sum symmetric functions hold: the reciprocal, and the rational. The rational function gives rise to generalized power-sum symmetric functions in two sets of indeterminates α and β. The Littlewood-associated function in rational form is

$$F(\alpha, \beta; t) = \prod_{i \geq 1} \frac{(1 - \beta_i t)}{(1 - \alpha_i t)}.$$

with connection to the generating function given by

$$\frac{F'(\alpha, \beta; t)}{F(\alpha, \beta; t)} = \sum_i \frac{d}{dt} \log \frac{1 - \beta_i t}{1 - \alpha_i t}$$

made by analogy with Equation 2.2.

In this case

$$\begin{aligned}
\sum_{i \geq 1} \frac{d}{dt} \log \frac{1 - \beta_i t}{1 - \alpha_i t} &= \sum_{i \geq 1} \frac{\alpha_i}{1 - \alpha_i t} - \sum_{i \geq 1} \frac{\beta_i}{1 - \beta_i t} \\
&= \sum_{i \geq 1} \sum_{r \geq 1} \alpha_i^r t^{r-1} - \sum_{i \geq 1} \sum_{r \geq 1} \beta_i^r t^{r-1} \qquad (5.2) \\
&= \sum_{r \geq 1} \overline{p}_r(\alpha, \beta) t^{r-1}
\end{aligned}$$

and so the *generalized power-sum symmetric function* is (see [42], p.109) is

$$\overline{p}_r(\alpha,\beta) = \sum_{i\geq 1} \alpha_i^r - \sum_{i\geq 1} \beta_i^r.$$

Because we are using Littlewood-associated rational functions, this ensures a meaningful analogy between the power-sum symmetric functions and the generalized power-sum symmetric functions, and so the generalized power-sum symmetric functions are multiplicative with $\overline{p}_\lambda = \overline{p}_{\lambda_1}\overline{p}_{\lambda_2}\cdots\overline{p}_{\lambda_k}$ for any partition $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_k)$.

For each $n \in \mathbb{N}$ assign the power-sum symmetric function $p_n(x)$ to the partial differential operator ∂_{x_n}. Then we can write $\psi(\overline{p}_n(\alpha,\beta)) = D_n$ as the natural assignment of the generalized power-sum symmetric function $\overline{p}_n$ to the Hirota derivative D_n. For any $\lambda = (\lambda_1, \cdots \lambda_k)$ we see that $\psi(\overline{p}_\lambda) = \psi(\overline{p}_{\lambda_1}\cdots\overline{p}_{\lambda_k}) = D_{\lambda_1}\cdots D_{\lambda_k} = D_\lambda$. The idea of making a connection between the generalized power-sum symmetric functions and the Hirota derivatives was first mentioned by Nimmo in [51].

Since the ordinary characteristic mapping of generalized power-sum symmetric functions defines generalized Schur S-functions, we use the spin characteristic mapping introduced by Schur to analogously define generalized Q-functions. So for any $\lambda \in \mathcal{DP}$ the *generalized Q-functions* are given by

$$\overline{Q}_\lambda(x) = \sum_{\rho\in\mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\mu^{-1} \zeta_\rho^\lambda \overline{p}_\rho(\alpha,\beta)$$

where $\overline{p}_\rho(\alpha,\beta)$ are the generalized power-sum symmetric functions in two variables with rational Littlewood-associated function. The coefficients are exactly as in Equation 5.1.

We introduce the idea of a natural assignment of generalized Q-functions to a a linear combination Hirota derivatives by noticing that

$$\begin{aligned}\psi(\overline{Q}_\lambda(x)) &= \psi\left(\sum_{\rho\in\mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\mu^{-1}\zeta_\rho^\lambda \overline{p}_\rho(\alpha,\beta)\right)\\ &= \sum_{\rho\in\mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\mu^{-1}\zeta_\rho^\lambda \psi(\overline{p}_\rho(\alpha,\beta))\\ &= \sum_{\rho\in\mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\mu^{-1}\zeta_\rho^\lambda D_\rho.\end{aligned}$$

We denote $\psi(\overline{Q}_\lambda)$ by $\widetilde{Q}_\lambda$ and call it a *generalized Q-operator.*

Example 5.2 By definition, the generalized Q-operators $\widetilde{Q}_\lambda$ are only meaningful for distinct part partitions λ with weight at least 4. The first suitable partition (lowest in the reverse lexicographical ordering) is $\lambda = (31)$. In this case the sum is over all $\mu \in \mathcal{OP}$ of weight 4. This description gives rise to two summands. One for $\rho = (1^4)$ and one for $\rho = (31)$. Using the spin character tables given in Appendix B, we readily obtain

$$\widetilde{Q}_{31} = \frac{4}{3}D_{(31)} + \frac{4}{3}D_{(1^4)}.$$

5.4 Hirota's form and the BKP equations

Often the PDEs that arise in connection with some physical ('real-world') problems are not easily handled by direct analysis. A good example of this is a family of evolution equations called the BKP equations. The BKP equations are non-linear PDEs that occur in a hierarchy. In 1983 Jimbo and Miwa [34] gave a detailed analysis of the BKP hierarchy of equations using vertex operators to describe the connection between members lower in the hierarchy to those of higher degree. With a view to exploring the structure of the BKP hierarchy in a graded ring context, we will begin with the lowest order equation in Hirota form.

Writing a PDE in Hirota form means expressing it as a polynomial of Hirota derivatives acting on a pair of polynomials τ satisfying a certain bilinear conditions ([34], Theorem 2.1 describes the bilinear identity). When this condition is met, the polynomials are called τ-functions.

Example 5.3 The *Kadomtsev-Petviashvilli* equation, more easily called the KP equation $(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$ is changed into Hirota form by substituting

$$u = 2\frac{\delta^2}{\delta x^2}\log\tau = 2\frac{D_{x^2}}{f_x}(\tau\cdot\tau)$$

and then integrating with respect to t to give

$$\left(D_x^4 + D_x D_t + 3D_y^2\right)\tau\cdot\tau = 0.$$

We index the Hirota derivatives in partition notation with $x = x_1, y = x_2$ and $t = -\frac{1}{4}x_3$ by convention, yielding

$$\left(D_{(1^4)} + 3D_{(2^2)} - 4D_{(31)}\right)\tau\cdot\tau = 0.$$

The solutions to the KP equation turn out to be Schur S-functions (see [32] or [70]).

The *bilinear Kadomstev-Petviashvili equation*, called the BKP equation, first appeared in [20] and [34]. The Hirota form of the first non-trivial BKP equation is

$$(D_{(1^6)} - 5D_{(31^3)} - 5D_{(3^2)} + 9D_{(51)})\tau\cdot\tau = 0.$$

This equation is the first in a hierarchy of equations. Only equations of even weight occur in the hierarchy. Also, the number of equations of any particular weight n can be determined. Let any partition with all parts even be called an even part partition. Then the number of equations of weight n in the BKP hierarchy is given by the number of odd part partitions of weight n minus the number of even part partitions of weight n ([34], pp. 999).

Nimmo [51] put forward the idea of diagram KP and BKP hierarchies with the KP and BKP equations being represented by Young diagrams. The idea of the BKP diagram hierarchy is illustrated by Figure 5.1. The first BKP equation (weight 6) is symbolized at the top the figure by $BKP_{(\,)}$. Nimmo called this equation a *seed*, because when we apply a rasing operator to it (this action is illustrated by the downward arrows, R, in the figure) higher order equations in the hierarchy are generated. Nimmo also determined the existence of new seeds in the hierarchy occurring whenever the weight of the equation is an even triangular number. Two of the weight 10 BKP equations arise under the action of the rasing operator, and the third weight ten equation is a new seed.

For the KP hierarchy, Nimmo was successful in producing a shift s-operator, represented by Young diagrams, and found a suitable raising operator to generate the hierarchy. For the BKP hierarchy, Nimmo was unable to to find a suitable shift operator. Nimmo proposed that the same raising operator used for the KP hierarchy would also generate the BKP hierarchy but also says that "this is the consequence of a fundamental algebraic property that is not yet apparent to the author and is the subject of further investigation".

In Section 5.6, we provide a diagram BKP hierarchy with generalized Q-operators being represented by *shifted diagrams*. We also make use of a Pieri type formula as the raising operator. This raising operator was conjectured by us in a previous paper [55] with Salam, as mentioned in the introduction to this chapter. But at that time we had not made the necessary connections to the characteristic mapping and the generalized Q-functions. The improvements are significant;

Figure 5.1: Nimmo's idea for a diagram BKP hierarchy

$$
\begin{tikzcd}
\text{Weight :} & & & & \\
6 & & BKP_{(\cdot)} \arrow[dl, "R_2"'] \arrow[ddr, "R_4"] & & \\
8 & BKP_{(2)} \arrow[d, "R_2"'] & & & \\
10 & BKP_{(2^2)} & & BKP_{(4)} & BKP_{(\cdot)}
\end{tikzcd}
$$

but there is still more to be done.

5.5 Generalized Q-operators

Recall that for $\lambda \in \mathcal{DP}$ the generalized Q-operators are determined by

$$\widetilde{Q}_\lambda = \sum_{\rho \in \mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\mu^{-1} \zeta_\rho^\lambda D_\rho.$$

Since the generalized Q-operators are only written in terms of Hirota derivatives parameterized by odd part partitions, we put $H = \mathbb{Q}\,[D_1, D_3, \cdots, D_{2n-1}]$, the ring of polynomials of Hirota derivatives in odd parts with rational coefficients. Then $D_1, D_3, \cdots, D_{2n-1}$ are linearly independent and form a basis for H. Next, we consider the class of distinct part partitions λ with length $l(\lambda) = k$ and weight $|\lambda| = 2n \geq 4$. Denote by $\widehat{H}_k$ the additive subgroup of H made up of linear combinations of generalized Q-operators $\widetilde{Q}_\lambda$ with integer coefficients. The hat notation is used to mean that the weight of the partitions is even, with the index k being the length of the partitions.

Elements from the group $\widehat{H}_k$ have a Young-diagram type of representation as *shifted Young diagrams*, which are used for strictly distinct part partitions. The defining characteristic of the shifted Young diagrams is that the first box on any row must start along the main diagonal, thus ensuring that the parts of the partition are strictly decreasing (whereas the row lengths in ordinary Young diagrams are weakly decreasing). We denote shifted Young diagrams by $\mathcal{Y}^{\lambda}$ and throughout this chapter we just call them diagrams.

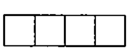
Proposition 5.4 Every group $\widehat{H}_k$ has a *core element* and a *core*. The core is a diagram which is contained in the diagram of every other element in the group. In particular, whenever $k \equiv (-1, 0) \pmod 4$ the core is called a *triangular core*. In the case that $k \equiv (1, 2) \pmod 4$, the core is not triangular, and we say it contains *a thorn*.

Proof. Put $k \equiv 0 \bmod 4$. This means $k = 4m$ for some $m \in \mathbb{N}$. Consider the distinct part partition λ with $4m$ parts so that $\lambda_1 = 4m, \lambda_2 = 4m-1, \cdots, \lambda_{4m} = 1$. Then λ has even weight and the core element of $\widehat{H}_k$ is just $\widetilde{Q}_{\lambda}$. This is because the diagram of $\widetilde{Q}_{\lambda}$ is contained in the diagram of all other elements of $\widehat{H}_k$. That is, λ is the lowest weight partition in the reverse lexicographical ordering of length k with even weight. The core is clearly triangular.

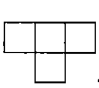
Fix the case $k \equiv -1 \bmod 4$ relative to the previous one by putting, without loss of generality, $k = 4m - 1$ with m the same as before. The core in this case is obtained by deleting the $4m$ boxes of the top row of core $\mathcal{Y}^{\lambda}$. Clearly the diagram is still triangular, and since we are ignoring an even number of boxes, the corresponding generalized $\widetilde{Q}$-operator is indeed in $\widehat{H}_{k=4m-1}$.

In the final two cases, the thorns occur because of the restriction here to partitions of even weight. Start with the case $k \equiv 1 \bmod 4$. Again, fix this case to the previous two and add an extra row of $4m+1$ boxes. In doing this, we have made a triangular diagram of odd weight. This forces the addition of the extra box (which we call the thorn) to the top row. (This is illustrated in the Example 5.5). The diagram made in this way is the core of $\widehat{H}_{k=4m+1}$. A similar argument occurs for the case $k \equiv 2 \bmod 4$. Notice that the triangular numbers for $k \equiv (1,2) \bmod 4$ are always odd, forcing the addition of the thorn. $\square$

Example 5.5 Elements in the group $\widehat{H}_1$ are indexed by length 1 partitions (necessarily distinct part) with weight even and greater than or equal to four. The core element is $\widetilde{Q}_{(4)}$ and the core is

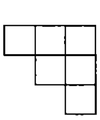
 .

The group $\widehat{H}_2$ consist of length 2 partitions with even weight $w \geq 4$. The parts of the length 2 partitions in $\widehat{H}_2$ are just $\lambda_1 \geq 3$ and $\lambda_2 = \lambda_1 - 2n$, where $2n < w$. The core element is $\widetilde{Q}_{(31)}$ and the core of $\widehat{H}_2$ is

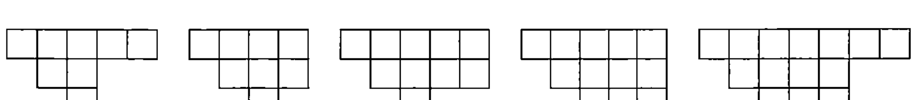
 .

The box located at the right-most end of the top row of $\mathcal{Y}^{(31)}$ is the thorn.

The group $\widehat{H}_3$ has the first triangular core

 .

The diagrams of some lower weight elements of $\widehat{H}_3$ are shown here. Notice that all the diagrams of elements in $\widehat{H}_3$ contain the core.



We conjecture a connection between the triangular core elements of $\widehat{H}_k$ and the seeds in the BKP hierarchy.

Conjecture 5.6 *The seeds of the* BKP *hierarchy are just the core elements of $\widehat{H}_k$ which have triangular cores. The weight of the seeds is $\frac{1}{2}k(k+1)$.*

Example 5.7 The first triangular core occurs in $\widehat{H}_3$ and the core element is $\widetilde{Q}_{(321)}$. Using Equation 5.1 we find that

$$\widetilde{Q}_{(321)} = \frac{8}{45}\left(D_{(1^6)} - 5D_{(31^3)} + 9D_{(51)} - 5D_{(3^2)}\right).$$

This is the first non-trivial equation of the BKP hierarchy. The weight of the equation is 6.

The next triangular core occurs in $\widehat{H}_4$ and the core element is $\widetilde{Q}_{(4321)}$. In this case we get $\widetilde{Q}_{(4321)} = \frac{16}{28350}(6S_1 - 90S_2)$ where

$$\begin{aligned} S_1 &= \left[D_{(1^{10})} + 63D_{(51^5)} - 225D_{(71^3)} - 175D_{(3^31)} + 525D_{(91)} - 189D_{(5^2)}\right] \qquad \text{and} \\ S_2 &= \left[6D_{(51^5)} - 5D_{(3^21^4)} - 15D_{(71^3)} + 15D_{(531^2)} - 5D_{(3^31)} + 10D_{(91)} - 15D_{(73)} + 9D_{(5^2)}\right] \end{aligned}$$

are two of the three weight 10 equations given in Jimbo and Miwa [34], with the third one being

$$S_3 = \left[D_{(31^7)} - 21D_{(531^2)} + 35D_{(91)} - 15D_{(73)}\right].$$

In the next section we apply a raising operator to the first triangular core and we obtain two other linearly independent weight 10 equations.

The next triangular core occurs in $\widehat{H}_7$ and the weight of the equation is 28. Our investigation of the BKP seeds has not extended this far yet.

5.6 The BKP hierarchy

In order to explore the structure of the BKP hierarchy and the role of the subgroups $\widehat{H}_k$ of generalized Q-operators, we turn first to the idea of describing a meaningful notion of 'multiplication' for the generalized Q-operators. Since the generalized Q-operators are elements in the ring of Hirota Derivatives, then the obvious choice is multiplication by the base elements.

Recall that the the relationship between the D_r and the $\widetilde{Q}_\lambda$ is analogous to that between the $\bar{p}_r$ and $\bar{Q}_\lambda$ which is analogous to that between the p_r and Q_λ. (We have used only the notation here rather than verbal description for ease of reading). This means that the action of $D_r\widetilde{Q}_\lambda$ is analogous to p_rQ_λ. To do this, we make use of a Pieri-type formula for the Q-functions in which strips and double strips come into play.

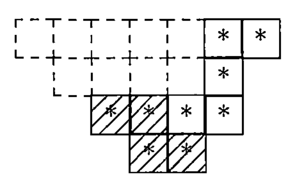
If λ and μ are distinct part partitions with $\mu \subseteq \lambda$ (meaning that the shifted diagram of μ is contained in that of λ), then the difference λ/μ forms a *shifted skew diagram* $\mathcal{Y}^{\lambda/\mu}$. A shifted skew diagram $\mathcal{Y}^{\lambda/\mu}$ is said to be a *strip* if it is connected and contains at most one box on every diagonal coordinate $x \in \mathcal{Y}^\lambda$ of the shifted diagram $\mathcal{Y}^\lambda$. The *height* of a strip is the number of rows it occupies.

A shifted skew diagram is called a *double strip* if it is the union of two strips which both start on the first diagonal. To calculate the depth of a double strip $\mathcal{Y}^{\lambda/\mu}$, we split it into two strips, y_1 and y_2, using the rule that a box lying on the j^{th} diagonal of $\mathcal{Y}^{\lambda/\mu}$ belongs to y_k $(k = 1, 2)$ if the intersection of $\mathcal{Y}^{\lambda/\mu}$ and the j^{th} diagonal has cardinality k. Then the *depth* of the double strip $\mathcal{Y}^{\lambda/\mu}$ is

$$d(\lambda/\mu) = \frac{|y_2|}{2} + ht(y_1)$$

where: $ht(y_1)$ is the height of the strip y_1; and $|y_2|$ means the number of boxes in the the strip y_2.

Example 5.8 Suppose $\mu = (54)$ and $\lambda = (7542)$. Then the skew shifted diagram $\mathcal{Y}^{\lambda/\mu}$ is a double strip.



The shaded boxes represent y_2. The depth of $\mathcal{Y}^{\lambda/\mu}$ is 4.

Theorem 5.9 ([46]) and [35] *For $\mu \in \mathcal{DP}$ we have*

$$p_r Q_\mu = \sum_\lambda 2^{l(\lambda)-l(\mu)} h^\lambda_{\mu(r)} Q_\lambda \tag{5.3}$$

summing over all distinct part partitions λ such that $|\lambda| = |\mu| + r$; and $\mathcal{Y}^{\lambda/\mu}$ is a strip or double strip with

$$h^\lambda_{\mu(r)} = \begin{cases} (-1)^{ht(\lambda/\mu)-1} & \text{if } \mathcal{Y}^{\lambda/\mu} \text{ is a strip;} \\ 2(-1)^{d(\lambda/\mu)-1} & \text{if } \mathcal{Y}^{\lambda/\mu} \text{ is a double strip} \end{cases}$$

where: $ht(\lambda/\mu)$ is the height of the strip $\mathcal{Y}^{\lambda/\mu}$; and $d(\lambda/\mu)$ is the depth of the double strip $\mathcal{Y}^{\lambda/\mu}$.

Remark 5.10 *The coefficients $2^{l(\lambda)-l(\mu)}$ will only occur non-trivially whenever the action of the raising operator increases the height of the diagrams. i.e. when $D_r : \widehat{H}_k \mapsto \widehat{H}_k \oplus \widehat{H}_{k+1}$. The implications of this action on the* BKP *hierarchy are yet to be investigated. Further, the application of double strips has not yet* *appeared in this investigation as double strips will not occur until higher weight equations are investigated.*

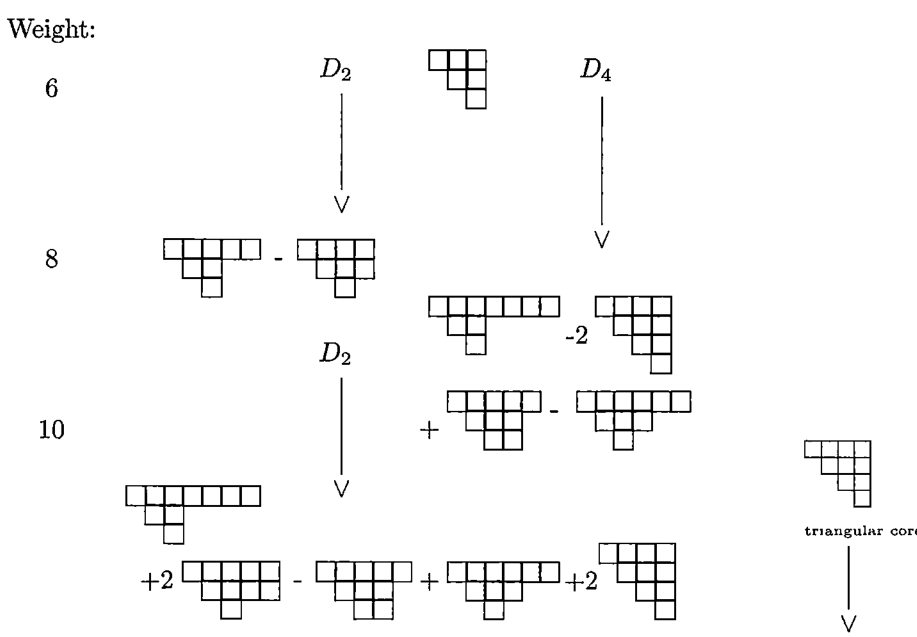
Corollary 5.11 *For $\mu \in \mathcal{DP}$ we have*

$$D_r \widetilde{Q}_\mu = \sum_\lambda 2^{l(\lambda)-l(\mu)} h^\lambda_{\mu(r)} \widetilde{Q}_\lambda \tag{5.4}$$

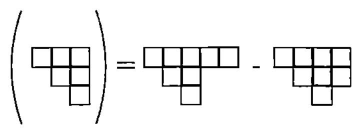
with all the same conditions and notations as stated in Theorem 5.9.

Conjecture 5.12 *Applying the action of Theorem 5.9 to the core elements with triangular cores in $\widehat{H}_k$ generates the BKP hierarchy.*

Figure 5.2: Construction of a lower weight portion BKP Hierarchy



Example 5.13 The first core element with a triangular core is $\widetilde{Q}_{(321)}$. Applying the action described in Theorem 5.9 produces

$D_2[\tilde{Q}_{(321)}] = D_2$ 

This yields a weight 8 equation:

$$\widetilde{Q}_{(521)}-\widetilde{Q}_{(431)} = \frac{8}{315}\left(D_{(1^8)} + 7D_{(31^5)} - 21D_{(51^3)} - 35D_{(3^21^2)} + 90D_{(71)} + -42D_{(53)}\right)$$

which is a scalar multiple of the one given in [34].

Figure 5.2 illustrates the weight 6 to weight 10 portion of the diagram BKP hierarchy. For weight 10 we find three linearly independent equations (one of these is the new seed described in Example 5.7):

$\widetilde{Q}_{(721)} + 2\widetilde{Q}_{(541)} - \widetilde{Q}_{(532)} - \widetilde{Q}_{(631)} + 2\widetilde{Q}_{(4321)} = \frac{16}{28350}\left(16S_1 - 30S_2 - 105S_3\right)$

$\widetilde{Q}_{(721)} + \widetilde{Q}_{(532)} - \widetilde{Q}_{(631)} - 2\widetilde{Q}_{(4321)} = \frac{16}{28350}\left(-10S_1 - 210S_2 + 105S_3\right)$

$\widetilde{Q}_{(4321)} = \frac{16}{28350}\left(6S_1 - 90S_2\right)$

where S_1, S_2 and S_3 are the three weight 10 equations given in [34], and also previously on page 96.

5.7 Conclusions

The Hirota form of the equations in the BKP hierarchy are elements in a graded ring without unity. The ring consists of polynomials of Hirota derivatives. The even integers under addition grade the ring and we have

$$H = \bigoplus_{2n \in 2\mathbb{N}} H^{(2n)}$$

where $H^{(2n)}$ is the homogeneous subgroup of degree $2n$. The members of the BKP hierarchy are homogeneous elements of this ring. The ring also has an interesting collection of subgroups $\widehat{H}_k$ of generalized Q-operators. Some connections between these subgroups and the members of the BKP hierarchy are being realized. We have shown that the diagrams of generalized Q-functions can be used to develop a low weight portion of the BKP hierarchy, and in Section 6.4 (of the next chapter) we write the Hirota derivatives in terms of generalized Q-functions. Keeping in mind that our initial motivation was to develop an example of a graded ring without unity that has a strong connection to real-world applications, we conclude this chapter.

Chapter 6

Spin Characters of the Symmetric Group

In Chapter 5, in order to make the connection between the Q-functions and the Hirota derivatives, we made use of the spin characters. In this chapter we relate the spin characters of the symmetric group to the ordinary ones. The connection revolves around the different transition matrices between the symmetric functions. Because we write the spin characters in this way, we are also able to describe a new combinatorial algorithm for the spin characters. The algorithm we write is just an amalgamation of two existing ones.

Our algorithm yields two simple special cases. One of these special cases links directly back to Schur's work in 1911, in that a corollary to our algorithm was known to Schur [65] (according to [48]).

6.1 The Hall-Littlewood functions

The Hall-Littlewood functions are a generalized type of symmetric function and their introduction is made easier by the fact that have already met Schur's Q-functions. Indeed the Schur Q-functions can be described as a special case of the Hall-Littlewood Q-functions, as we see by the end of this section. We only give the details we require to describe our main result on the spin characters. The field is quite young; but a thorough treatment was given by Hoffman and Humphries [27] in 1992.

The Hall-Littlewood polynomials are symmetric in the n indeterminates $x_1, \cdots, x_n$ with coefficients in $\mathbb{Z}[t]$, making them elements in the ring $\Lambda_n[t]$ of symmetric polynomials with coefficients in $\mathbb{Z}[t]$. We approach the Hall-Littewood functions in the same manner that we approached the symmetric functions in Section 2.2. That is, we first describe the Hall-Littlewood polynomials and then pass to the inverse limit to give the Hall-Littlewood functions using natural projections. We sometimes write HL as an abbreviation of "Hall-Littlewood".

The *Hall-Littlewood P-polynomials* are given by

$$P_\lambda(x_1 \cdots x_n; t) = \sum_{\omega \in \mathcal{S}_n / \mathcal{S}_n^\lambda} \omega \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

where $\mathcal{S}_n^\lambda$ is the subgroup of permutations $\omega \in \mathcal{S}_n$ such that $\lambda_{\omega(i)} = \lambda_i$. We pass to the inverse limit requiring that, for every $\lambda \vdash k$, the image of the *Hall-Littlewood functions* $P_\lambda(x; t)$ from the subgroup $\Lambda^{(k)}[t]$ be the Hall-Littlewood polynomials $P_\lambda(x_1, \cdots, x_n; t)$ in $\Lambda_n^{(k)}[t]$ for each $n \geq k$. The HL P-functions are algebraically independent over $\mathbb{Z}[t]$ and form a $\mathbb{Z}[t]$-basis of the ring $\Lambda[t]$ ([44], Proposition III.2.7).

The *Hall-Littlewood Q-functions* are scalar multiples of HL P-functions given by

$$Q_\lambda(x;t) \quad = \quad b_\lambda(t)P_\lambda(x;t) \tag{6.1}$$

where

$$b_\lambda(t) = \prod_{i\geq 1} \varphi_{m_{\lambda_i}}(t)$$

and

$$\varphi_r(t) = (1-t)(1-t^2)\cdots(1-t^r), \tag{6.2}$$

with m_{λ_i} being the multiplicity of the part λ_i in λ.

When $t = -1$, Equation 6.2 vanishes for any $r \geq 2$. This means that the multiplicity m_{λ_i} of any part λ_i must equal 1. In other words, the HL Q-function $Q_\lambda(x;-1)$ is non-zero only for distinct part partitions. Indeed, the HL Q-functions $Q_\lambda(x;-1)$ are exactly the Q-functions introduced by Schur and the subject of Chapter 5. This provides an important link to the spin characters because Schur's Q-functions are given definitively as a characteristic mapping.

Notice also that when $t = -1$ (and necessarily $\lambda \in \mathcal{DP}$)

$$\begin{aligned} b_\lambda(-1) &= \prod_{i\geq 1} \varphi_{m_{\lambda_i}}(-1) \\ &= \prod_{i\geq 1}(1--1) \\ &= 2^{l(\lambda)}, \end{aligned}$$

and Equation 6.1 becomes

$$Q_\lambda(x;-1) = 2^{l(\lambda)}P_\lambda(x;-1).$$

The transition matrix $X(t)$ between the power-sum symmetric functions and the Hall-Littlewood P-functions has coefficients $X^\lambda_\rho(t)$ determined by

$$p_\rho(x) = \sum_\lambda X^\lambda_\rho(t) P_\lambda(x;t). \tag{6.3}$$

When $t = 1$ the HL P-functions are just the monomial symmetric functions: $P_\lambda(x;1) = m_\lambda(x)$.

When $t = 0$ the HL P-functions are the Schur S-functions: $P_\lambda(x;0) = s_\lambda(x)$, and so the entries in the transition matrix $X(0)$ are the ordinary group characters of the symmetric group (see Equation 2.3)

$$X^\lambda_\rho(0) = \chi^\lambda_\rho.$$

To determine the orthogonality relationships for the Hall-Littlewood functions, MacDonald [44] compares equivalent series expansions of

$$\prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j}.$$

The expansions we give here all come from section III.7 of MacDonald's book [44]. Firstly,

$$\prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} = \sum_\lambda z_\lambda(t)^{-1} p_\lambda(x) p_\lambda(y)$$

where $z_\lambda(t)$ is a generalized form of Frobenius' formula for the size of the centralizer, and is given by

$$z_\lambda(t) = z_\lambda \prod_{i \geq 1} \left(1 - t^{\lambda_i}\right)^{-1}. \tag{6.4}$$

Also

$$\begin{aligned} \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} &= \sum_\lambda P_\lambda(x;t) Q_\lambda(y;t) \\ &= \sum_\lambda b_\lambda(t) P_\lambda(x;t) P_\lambda(y;t). \end{aligned}$$

And so

$$\sum_\lambda z_\lambda(t)^{-1}p_\lambda(x)p_\lambda(y) = \sum_\lambda b_\lambda(t)P_\lambda(x;t)P_\lambda(y;t).$$

Comparing coefficients, via Equation 6.3 we get

$$X'(t)z(t)^{-1}X(t) = b(t)$$

and

$$X(t)b(t)^{-1}X'(t) = z(t)$$

where: $X'(t)$ is the transpose of $X(t)$; and $b(t)$ (respectively $z(t)$) is used to denote the matrix with the entries $b_\lambda(t)$ (respectively $z_\lambda(t)$) along the diagonal, and zeros elsewhere. From these come the *orthogonality relations*:

$$\begin{aligned}\sum_{|\rho|=n} z_\rho(t)^{-1}X^\lambda_\rho(t)X^\lambda_\mu(t) &= \delta_{\lambda\mu}b_\lambda(t),\\ \sum_{|\lambda|=n} b_\lambda(t)^{-1}X^\lambda_\rho(t)X^\lambda_\sigma(t) &= \delta_{\rho\sigma}z_\rho(t).\end{aligned}$$

Using

$$X(t)^{-1} = b(t)^{-1}X'(t)z(t)^{-1}$$

gives

$$Q_\lambda(x;t) = \sum_\rho z_\rho(t)^{-1}X^\lambda_\rho(t)p_\rho(x).$$

Lemma 6.1 *When $t = -1$ and ρ is an odd part partition, the entries in the transition matrix $X(t)$ between the power-sum symmetric functions and the Hall-Littlewood P-functions are scalar multiples of the spin characters of the symmetric group. Specifically, for $\rho \in \mathcal{OP}$*

$$X^\lambda_\rho(-1) = 2^{\frac{1}{2}[l(\lambda)-l(\rho)+\epsilon]}\zeta^\lambda_\rho \tag{6.5}$$

$$\zeta^\lambda_\rho = 2^{\frac{1}{2}[l(\rho)-l(\lambda)-\epsilon]}X^\lambda_\rho(-1). \tag{6.6}$$

Proof. Form the orthogonality relations, we have

$$Q_\lambda(x;t) = \sum_\rho z_p(t)^{-1} X_\rho^\lambda(t) p_\rho(x). \tag{6.7}$$

When ρ is an odd part partition and $t=-1$, Equation 6.4 yields

$$z_\rho(-1) = z_\rho 2^{-l(\rho)}.$$

Using this, we evaluate Equation 6.7 at $t=-1$ to get

$$Q_\lambda(x;-1) = \sum_\rho z_p^{-1} 2^{l(\rho)} X_\rho^\lambda(-1) p_\rho(x). \tag{6.8}$$

Since $Q_\lambda(x;-1)$ is Schur's Q-function, we can compare this equation to Schur's original equation introducing the Q-functions (Equation 5.1):

$$Q_\lambda(x) = \sum_{\rho\in\mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\rho^{-1} \zeta_\rho^\lambda p_\rho(x).$$

Comparing coefficients gives Equation 6.5. Re-arranging to make the spin characters the object gives Equation 6.6. □

6.2 A new recipe for the spin characters

A rich and well-established connection between the theory of symmetric functions and the combinatorial properties of Young diagrams and tableaux enables us a write a new combinatorial algorithm for calculating the spin characters. The algorithm we describe is just an amalgamation of two existing theorems/algorithms: the first is due to Lascoux and Schützenber and appears shortly (Theorem 6.2); the second is Schensted's algorithm for the ordinary characters of the symmetric group. Appendix A gives the details of Schensted's algorithm.

The *Kostka matrix* $K(t)$ is the transition matrix between the Schur S-functions and the HL P-functions and has coefficients $K_{\lambda\mu}(t)$ in the equation

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t) P_\mu(x;t).$$

The numbers $K_{\lambda\mu}(t)$ are called *Kostka numbers.*

When $t = 1$, recall that the HL P-functions are just the monomial symmetric functions. In this case, the Kostka numbers and Kostka matrix just described are the same as those given in Section 2.2.3.

We are interested in the case $t = -1$. Lascoux and Schützenberger found a combinatorial formula for the Kostka numbers for any value of t. Their formula depends on the charges of words formed by injected partitions to form unitary tableaux. We refer to Section 1.4 for all the required definitions.

Theorem 6.2 (Theorem of Lascoux and Schützenberger, [44], Theorem III.6.5) *The elements of the Kostka Matrix are given by*

$$K_{\lambda\rho}(t) = \sum_\tau t^{c(\tau)}$$

where the sum is over all possible unitary tableaux τ formed by injecting ρ into the Young diagram Y^λ; and $c(\tau)$ is the charge of the word associated with the tableau τ.

Example 6.3 Let $t = -1$ and consider $K_{(42)(2^21^2)}(-1)$. The possible unitary tableaux and their associated words are:

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 3&4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\ \hline 2&3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\ \hline 2&2 \\ \cline{1-2}\end{array} \; .$$

$w_1 = \{221143\}$ $w_2 = \{321142\}$ $w_3 = \{421132\}$ $w_4 = \{431122\}$

For the word w_1, the set of standard subwords is $\{2143, 21\}$ as described in Example 1.33. The indices are also determined in the example just mentioned ($w_1 : \{2_1 1_0 4_2 3_1, 2_1 1_0\}$) and so the charge of the first tableau is $c(\tau_1) = 5$. Similarly we obtain

$$\begin{aligned} w_2 &: \{3_1 1_0 4_1 2_0, 2_1 1_0\} \quad \text{and so} \quad c(\tau_2) = 3 \\ w_3 &: \{4_2 1_0 3_1 2_0, 2_1 1_0\} \quad \text{and so} \quad c(\tau_3) = 4 \\ w_4 &: \{4_2 3_1 1_0 2_0, 1_0 2_0\} \quad \text{and so} \quad c(\tau_4) = 3. \end{aligned}$$

Hence $K_{(42)(2^2 1^2)}(-1) = (-1)^5 + 2 \times (-1)^3 + (-1)^4 = -2.$

The Kostka matrix $K(t)$ can be used to connect the transition matrix $X(t)$ and the ordinary characters. Specifically ([44], Equation III.7.6')

$$X^{\lambda}_{\rho}(t) = \sum_{\mu \geq \lambda} \chi^{\mu}_{\rho} K_{\mu\lambda}(t) \tag{6.9}$$

where the sum is over all partitions $\mu \geq \lambda$ in the reverse lexicographical ordering.

Theorem 6.4 *Suppose $\lambda \in \mathcal{DP}$ is a distinct part partition of weight n with length $l(\lambda)$, and $\rho \in \mathcal{OP}$ is an odd part partition of weight n and length $l(\rho)$. The spin character ζ^{λ} on the class ρ is*

$$\zeta^{\lambda}_{\rho} \quad = \quad 2^{\frac{1}{2}[l(\rho) - l(\lambda) - \epsilon]} \sum_{\mu \geq \lambda} K_{\mu\lambda}(-1) \chi^{\mu}_{\rho} \tag{6.10}$$

where the sum is over all partitions μ greater than or equal to λ in the reverse lexicographical ordering; χ^{μ} is the ordinary character of the symmetric group S_n on the class ρ; $K_{\mu\lambda}(-1)$ are the Kostka numbers with $t = -1$, and ϵ is appropriately 0 *or* 1.

Proof. Using Equation 6.6 from Lemma 6.1 with Equation 6.9 is all that is required.□

Remark 6.5 Theorem 6.4 allows us to determine the spin characters using known combinatorial methods for calculating the ordinary characters. We give a recipe using a "build-up" method. This means that for larger order characters, we do not rely on needing to know the spin characters of lower orders. Of course, it may be of interest to investigate a recipe using a staircase stripping method. We give the build-up method here because it fits well with Lascoux and Schützenberger's algorithm for the Kostka numbers.

Algorithm 6.6 *Suppose $\lambda \in \mathcal{DP}$ is a distinct part partition of weight n and $\rho \in \mathcal{OP}$ is an odd part partition of weight n. To calculate the spin character ζ_ρ^λ of the symmetric group S_n, we must consider partitions $\mu \vdash |\lambda|$ where $\mu \geq \lambda$ (in the reverse lexicographic ordering). For each of these μ:*

(i) Calculate the charge $c(\tau)$ of each of the unitary *tableaux τ formed by injecting λ into μ. Compute the sum $\sum_\tau (-1)^{c(\tau)}$;*

(ii) Calculate the number of negative applications of each of the regular *tableaux σ formed by injecting ρ into μ. Denote by $n_e(\sigma_\mu)$ the number of tableaux σ which involve an even number of negative applications and by $n_o(\sigma_\mu)$ the number of tableaux σ which involve an odd number of negative applications. Find the difference $n_e(\sigma_\mu) - n_o(\sigma_\mu)$, and call this difference $\Delta n(\sigma_\mu)$.*

Then

$$\zeta_\rho^\lambda \quad = \quad 2^{\frac{1}{2}[l(\rho)-l(\lambda)-\epsilon]} \sum_{\mu \geq \lambda} \left[\Delta n\left(\sigma_\mu\right) \cdot \left(\sum_\tau (-1)^{c(\tau)} \right) \right] \tag{6.11}$$

where: τ has shape μ; $l(\rho)$ and $l(\lambda)$ denote the lengths of the partitions ρ and λ respectively; and ϵ is 1 if $l(\rho) - l(\lambda)$ is odd, and 0 otherwise.

Proof. Part (i) of our algorithm is exactly Lascoux and Schützenberger's algorithm for the Kostka numbers $K_{\mu\lambda}(-1)$. Part (ii) of our algorithm is exactly Schensted's build-up staircase recipe for the ordinary characters which is detailed in Appendix A. Lascoux and Schützenberger Theorem (Theorem 6.2) require tableaux of shape μ. Since Schensted's build-up staircase recipe also requires tableaux of shape μ, we naturally merge the two algorithms. Indeed, Equation 6.11 is just Equation 6.10 with $K_{\mu\lambda}(-1)$ replaced by $\sum_\tau(-1)^{c(\tau)}$ using Lascoux and Schützenberger's theorem (Theorem 6.2) and with χ^μ_ρ replaced by $\Delta n\,(\sigma_\mu)$ using Schensted's recipe (Algorithm A.1). □

Example 6.7 Suppose we want to calculate the spin character $\zeta^{(42)}$ on the class $\rho = (31^3)$. Then we must consider all partitions μ of weight 6 such that $\mu \geq \lambda = (42)$. So we put $\mu_1 = (6)$; $\mu_2 = (51)$; $\mu_3 = (42)$ and sum over these μ_i. Figure 6.1 contains a table in which each row is indexed by each one of the μ_i just listed. The left hand column in the table shows the sum $\sum_\tau(-1)^{c(\tau)}$ from part (i) of Algorithm 6.6. The middle column shows the calculation of the differences $\Delta(\sigma_\mu)$ described in part (ii) of Algorithm 6.6. In the right-most column multiples of the two are summed, producing a total to be multiplied by $2^{\frac{1}{2}(l(\rho)-l(\lambda)-\epsilon)}$. In this case we calculate $\zeta^{(42)}_{(31^3)} = -2$.

Spin character tables are provided in Appendix B. Referring to the table for characters of degree 6, we look for the column headed (31^3) and the row indexed by (42). The entry there is the value of $\zeta^{(42)}_{(31^3)}$ and this value is -2.

Figure 6.1: Diagrams and calculations for Example 6.7

	Unitary Tableaux Inject $\lambda = (42)$	Regular Tableaux Inject $\rho = (31^3)$	
$\mu = (6)$	$\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 \end{ytableau}$ $w = \{221111\}$ $w = \{2_1 1_0; 2_1 1_0; 1_0; 1_0\}$ $c(w) = 2$ $(-1)^2 = 1$	$\begin{ytableau} 1 & 1 & 1 & 2 & 3 & 4 \end{ytableau}$ $\#$ neg $= 0$ $n_e(\sigma_\mu) = 1$ $\quad n_o(\sigma_\mu) = 0$ $\Delta n(\sigma_\mu) = 1 - 0 = 1$	$1 \times 1 = 1$
$\mu = (51)$	$\begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ 2 \end{ytableau}$ $w = \{211112\}$ $w = \{1_0 2_0; 2_1 1_0; 1_0; 1_0\}$ $c(w) = 1$ $(-1)^1 = -1$	$\begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 4 \end{ytableau}$ $\quad \begin{ytableau} 1 & 1 & 1 & 2 & 4 \\ 3 \end{ytableau}$ $\#$ neg $= 0$ $\quad \#$ neg $= 0$ $\begin{ytableau} 1 & 1 & 1 & 3 & 4 \\ 2 \end{ytableau}$ $\quad \begin{ytableau} 1 & 1 & 2 & 3 & 4 \\ 1 \end{ytableau}$ $\#$ neg $= 0$ $\quad \#$ neg $= 1$ $n_e(\sigma_\mu) = 3$ $\quad n_o(\sigma_\mu) = 1$ $\Delta n(\sigma_\mu) = 3 - 1 = 2$	$-1 \times 2 = -2$
$\mu = (42)$	$\begin{ytableau} 1 & 1 & 1 & 1 \\ 2 & 2 \end{ytableau}$ $w = \{111122\}$ $w = \{1_0 2_0; 1_0 2_0; 1_0; 1_0\}$ $c(w) = 0$ $(-1)^0 = 1$	$\begin{ytableau} 1 & 1 & 1 & 2 \\ 3 & 4 \end{ytableau}$ $\quad \begin{ytableau} 1 & 1 & 2 & 3 \\ 1 & 4 \end{ytableau}$ $\#$ neg $= 0$ $\quad \#$ neg $= 1$ $\begin{ytableau} 1 & 1 & 1 & 3 \\ 2 & 4 \end{ytableau}$ $\quad \begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 2 \end{ytableau}$ $\#$ neg $= 0$ $\quad \#$ neg $= 1$ $\begin{ytableau} 1 & 1 & 1 & 4 \\ 2 & 3 \end{ytableau}$ $\quad \begin{ytableau} 1 & 1 & 2 & 4 \\ 1 & 3 \end{ytableau}$ $\#$ neg $= 0$ $\quad \#$ neg $= 1$ $n_e(\sigma_\mu) = 3$ $\quad n_o(\sigma_\mu) = 3$ $\Delta n(\sigma_\mu) = 3 - 3 = 0$	$1 \times 0 = 0$
			$1 + -2 + 0$ $= -1$

Also $l(\rho) = 4$ and $l(\lambda) = 2$. Hence

$$\zeta^{(42)}_{(31^3)} = 2^{\frac{4-2}{2}} \times -1 = -2.$$

Key for Notation

$\#$ neg the number of negative applications

6.3 Special cases

6.3.1 Length one partitions

In the special case when λ is a length one partition of weight n, i.e. so that $\lambda = (n)$, our algorithm for the value of the spin character $\zeta^{\lambda=(n)}$ on any class ρ simplifies.

Theorem 6.8 *Suppose $\lambda = (n)$ is a length one partition of weight n. The spin character $\zeta_\rho^{\lambda=(n)}$ on the class ρ is*

$$\zeta_\rho^{\lambda=(n)} = 2^{\frac{1}{2}[l(\rho)-\delta]} \tag{6.12}$$

where

$$\delta = \begin{cases} 1 & \text{whenever } n \text{ is odd} \\ 2 & \text{whenever } n \text{ is even.} \end{cases}$$

Proof. Consider the special case that $\lambda = (n)$ is a length one partition of weight n. Using our recipe (Algorithm 6.6) we need consider only $\mu \geq \lambda$. Since $\lambda = (n)$, the only μ satisfying this requirement is $\mu = (n)$. This means that the sum over μ in Equation 6.12 reduces to the product of $\sum_\tau (-1)^{c(\tau)}$ determined in step (i) and $\Delta n(\sigma_\mu)$ determined in step (ii) of Algorithm 6.6, both evaluated at $\mu = (n)$.

Whenever $\mu = (n)$ and $\lambda = (n)$, the only unitary tableaux formed by the injection of λ into μ is the trivial one row tableau with 1's everywhere. So the extracted word $w = (111 \cdots 1)$ of length n is a standard word with charge 0. Hence we always have $\sum_\tau (-1)^{c(\tau)} = (-1)^0 = 1$. This reduces the spin character calculation to the product of $\Delta n(\sigma_\mu)$ and a power of two.

Next, $\Delta n(\sigma_\mu)$ in step (ii) of Algorithm 6.6 depends on the number of negative applications involved in injecting the class ρ into Y^μ. This calculation becomes trivial in this case since the number of rows in Y^μ is one, meaning the number of negative applications must always be 0, since we will never have an even number of rows. There is always one and only one way to inject any partition ρ into a one row diagram to give a regular tableau. Indeed, we always have only one trivial even negative application in this case. And so we always obtain 1 even and 0 odd negative applications for any ρ, whence $\Delta n(\sigma_\mu) = n_e(\sigma_\mu) - n_o(\sigma_\mu) = 1$.

Hence, in the special case that $\lambda = (n)$, we always get $\Delta n(\sigma_\mu) \cdot \left(\sum_\tau (-1)^{c(\tau)}\right) = 1$ summed only once in the case $\mu = (n)$; and so the spin character is just $2^{\frac{1}{2}(l(\rho)-l(\lambda)-\epsilon)}$. Since $l(\lambda) = 1$, this expression reduces further to

$$\zeta_\rho^n = 2^{\frac{1}{2}(l(\rho)-1-\epsilon)}.$$

Next, combine the constants 1 and ϵ (where $\epsilon = 0$ or 1 appropriately), by putting $-1-\epsilon = -\delta$ where

$$\delta = \begin{cases} 1 & \text{whenever } \mathrm{l}(\rho) \text{ is odd} \\ 2 & \text{whenever } \mathrm{l}(\rho) \text{ is even.} \end{cases}$$

Finally, notice that since $\rho \vdash n$ is an odd part partition, if n is even, then ρ must have an odd number of parts. Likewise, when n is odd, ρ must have an even number of parts. This means we can describe δ in terms of n and remove the dependance on $l(\rho)$. $\square$

Example 6.9 The first row of the character table corresponds to the irreducible character $\zeta^{\lambda=(n)}$. Using Corollary 6.8 we can easily give the first row of the spin character table for different n. Let's consider the case when, say, $n = 10$. Since n is even, the value of δ here is $\delta = 2$.

$$\zeta_\rho^{(10)} = 2^{\frac{1}{2}(l(\rho)-2)}$$

class :	(91)	(73)	(71^3)	(5^2)	(531^2)	(51^5)	(3^31)	(31^7)	(1^{10})
$l(\rho)$	2	2	4	2	4	6	4	8	10
$\zeta_\rho^{(10)}$	1	1	2	1	2	4	2	8	16

Next, we provide the first row in a spin character table of odd degree, say, $n = 11$. This makes it easy to see how the δ in Corollary 6.8 is written without dependence on $l(\rho)$; but rather by the parity of n. In this case, n is odd and so the value of δ here is $\delta = 1$.

$$\zeta_\rho^{(11)} = 2^{\frac{1}{2}(l(\rho)-1)}$$

class :	(11)	(91^2)	(731)	(71^4)	(5^21)	(53^2)	(531^3)
$l(\rho)$	1	3	3	5	3	3	5
$\zeta_\rho^{(11)}$	1	2	2	4	2	2	4

class :	(51^6)	(3^31^2)	(3^21^5)	(31^8)	(1^{11})
$l(\rho)$	7	5	7	9	11
$\zeta_\rho^{(11)}$	8	4	8	16	32.

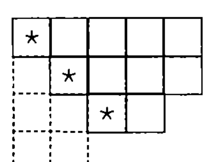
Looking at the spin character tables given in Appendix B, we see that all the characters calculated are in fact true and correct.

Remark 6.10 *A similar form of Equation 6.12 from Theorem 6.8 is mentioned by Morris [46] as having appeared in Schur's original paper [65]. We have, of course, obtained this result in a completely different way to that of Schur.*

6.3.2 When the class is the identity element

For distinct part partitions $\lambda = (\lambda_1, \cdots, \lambda_k)$ of length k, shifted diagrams can be represented in another type of diagram $\widetilde{\mathcal{Y}}^{\lambda}$ with a main diagonal of k boxes marked with a $\star$ say. Put λ_i boxes in the i^{th} row to the right of the marked box for each $1 \leq i \leq k$. Then put $\lambda_i - 1$ dashed-boxes in the i^{th} column below the marked box for each $1 < i \leq k$. The diagram $\widetilde{\mathcal{Y}}^{\lambda}$ constructed in this way is called the *shifted symmetric diagram.*

Example 6.11 Consider $\lambda = (431)$. The shifted symmetric diagram $\widetilde{\mathcal{Y}}^{(431)}$ is



.

Graphically, the *hook-length* of a partition ρ at coordinate x in the Young diagram Y^{ρ} is the number of boxes along the row to the right of x plus the number of boxes down the column below x plus 1 (for the box x itself). For the shifted diagram $\mathcal{Y}^{\lambda}$, the hook-length $h(x)$ for each $x \in \mathcal{Y}^{\lambda}$ is defined to be the hook-length at x in the shifted symmetric diagram $\widetilde{\mathcal{Y}}^{\lambda}$. We write $h(x)$ as $\widetilde{h}(x)$ to clarify this point.

Example 6.12 Consider $\lambda = (431)$. Then for the shifted diagram $\mathcal{Y}^{\lambda}$, the hook-lengths have been calculated and injected into the diagram $\widetilde{\mathcal{Y}}^{\lambda}$ at the corresponding boxes for all coordinates $x \in \mathcal{Y}^{\lambda}$.

$\star$	7	5	4	2
	$\star$	4	3	1
		$\star$	1	

Proposition 6.13 ([44], p. 134) *When the class ρ is restricted to the identity element (1^n) of the symmetric group S_n, the coefficients in transition matrix $X^{\lambda}_{(1^n)}$ from the power-sum symmetric functions $p_{\rho=(1^n)}$ to the HL P-functions $P_{\lambda\vdash n}(x;t=-1)$ can be calculated using the hook-lengths of the shifted diagram. Explicitly*

$$X^{\lambda}_{(1^n)}(-1) = \frac{n!}{\prod_{x\in\mathcal{Y}^\lambda}\widetilde{h}(x)}$$

where $\widetilde{h}(x)$ is the hook length at x in the shifted diagram $\mathcal{Y}^\lambda$ of $\lambda\vdash n$.

Using this hook-length formula and making use of Equation 6.6 yields the following corollary to Theorem 6.4.

Corollary 6.14 *Let $\rho\vdash n$ and $\lambda\vdash n$ be partitions of weight n with $\rho=(1^n)$. Then the spin character $\zeta^{\lambda}_{1^n}$ on class $\rho=(1^n)$ is given by*

$$\zeta^{\lambda}_{(1^n)} = 2^{\frac{1}{2}[n-l(\lambda)-\epsilon]}\frac{n!}{\prod_{x\in\mathcal{Y}^\lambda}\widetilde{h}(x)}$$

where $\mathcal{Y}^\lambda$ is the shifted diagram of λ; $\widetilde{h}(x)$ is the hook length of $x\in\mathcal{Y}^\lambda$, and ϵ is 0 or 1 accordingly.

Proof. Combining Equation 6.6 from Lemma 6.1 with the hook-length formula given in Proposition 6.13 completes the proof. □

Example 6.15 Consider the class (1^6) of $\mathcal{S}_6$. For $\lambda = (42)$ we have $l(\lambda) = 2$. The diagram $\widetilde{\mathcal{Y}}^\lambda$ of λ with the hook lengths for each coordinate $x \in \mathcal{Y}^\lambda$ injected at each $x \in \mathcal{Y}^\lambda$ is

$$\begin{array}{ccccc}
\star & 6 & 4 & 3 & 1 \\
 & \star & 2 & 1 & \\
 & & & & \\
 & & & &
\end{array}$$

and so

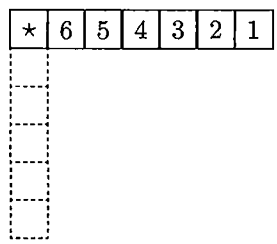
$$\begin{aligned}
\zeta_{(1^6)}^{\lambda=(42)} &= 2^{\frac{1}{2}(n-l(\lambda)-\epsilon)} \frac{n!}{\prod_{x \in \mathcal{Y}(\lambda)} \widetilde{h}(x)} \\
&= 2^{\frac{1}{2}(6-2-\epsilon)} \frac{6!}{6 \times 4!} \\
&= 2^{\frac{1}{2}(4)} 5 \\
&= 20
\end{aligned}$$

where the appropriate value for ϵ is $\epsilon = 0$.

Example 6.16 Consider the class (1^6) of $\mathcal{S}_6$. For $\lambda = (6)$ we have $l(\lambda) = 1$. In this case we can use Theorem 6.8 to determine

$$\zeta_{(1^6)}^{(6)} = 2^{\frac{1}{2}(6-2)} = 2^2 = 4.$$

For the purpose of illustration, we apply Corollary 6.14 here and leave it to the reader to see, by example, how this hook-length formula can be used to prove Theorem 6.8. The shifted symmetric diagram of $\widetilde{\mathcal{Y}}^\lambda$ of λ with the hook lengths for each coordinate $x \in \mathcal{Y}^\lambda$ injected at each $x \in \mathcal{Y}^\lambda$ is



and so

$$\begin{aligned}\zeta^{\lambda=(6)}_{(1^6)} &= 2^{\frac{1}{2}(n-l(\lambda)-\epsilon)}\frac{n!}{\prod_{x\in\mathcal{Y}(\lambda)}\widetilde{h}(x)}\\ &= 2^{\frac{1}{2}(6-1-\epsilon)}\frac{6!}{6!}\\ &= 2^{\frac{1}{2}(4)} = 4\end{aligned}$$

where the appropriate value for ϵ here is 1.

6.4 Generalized Q-operators revisited

To conclude this chapter, we make one final proposition about the BKP hierarchy. Its appearance here is somewhat out of place, in that it relates to the contents of the previous chapter; but this result relies upon Lemma 6.1 developed earlier in this chapter.

Recall that we use $\widehat{H}_k$ to be the additive subgroup of the ring of Hirota derivatives whose elements are the generalized Q-operators with $l(\lambda) = k$ and $|\lambda| = 2n \geq 4$. Put $\widehat{H} = \cup_k \widehat{H}_k$ as the direct union of additive subgroups of the ring.

Proposition 6.17 *The members of the* BKP *hierarchy are elements of the subgroup $\widehat{H}$ of generalized Q-operators. Indeed, any Hirota derivative is expressible as an integer sum of generalized Q-operators. Explicitly*

$$D_\rho(x) = \sum_{\mathcal{DP}\ni\lambda\vdash|\rho|} 2^{-\frac{1}{2}(l(\lambda)+l(\rho)-\epsilon)}\zeta^\lambda_\rho \widetilde{Q}_\lambda(x)$$

where ϵ is 0 or 1 accordingly.

Proof. This result is a consequence of Lemma 6.1, keeping in mind that the generalized Q-operators are developed via Littlewood-associated functions and the spin characteristic mapping introduced by Schur (Equation 5.1).

To begin, recall that the transition matrix between the HL P-functions $P_\lambda(x;t)$ and the power-sum symmetric functions $p_\rho(x)$ has coefficients $X_\lambda^\rho(t)$ in the equation

$$p_\rho(x) = \sum_\lambda X_\rho^\lambda(t) P_\lambda(x;t).$$

Now, using Equation 6.5 from Lemma 6.1 we readily obtain

$$p_\rho(x) = \sum_\lambda 2^{\frac{1}{2}(l(\lambda)-l(\rho)+\epsilon)} \zeta_\rho^\lambda P_\lambda(x;t). \tag{6.13}$$

Next, we evaluate Equation 6.13 at $t = -1$ and replace the HL P-functions $P_\lambda(x;-1)$ with HL Q-functions $Q_\lambda(x;-1)$ via Equation 6.3 to give

$$p_\rho(x) = \sum_{\mathcal{DP} \ni \lambda \vdash |\rho|} 2^{-\frac{1}{2}(l(\lambda)+l(\rho)-\epsilon)} \zeta_\rho^\lambda Q_\lambda(x;-1).$$

Since the $Q_\lambda(x;-1)$ are just the Schur Q-functions, relying upon the analogue between the power-sum symmetric functions and the Hirota derivatives set up in Section 5.3 completes the proof. □

Appendix A

Ordinary Characters of the Symmetric Group

Schenstead [66] gives an excellent description of a recursive staircase algorithm for calculating the ordinary characters of the symmetric group "based upon a famous formula due to Frobenius" ([66], p.142). From this she goes on to give a build-up staircase form of this algorithm. We employ Schensted's "build-up" algorithm in part (ii) of our Algorithm 6.6.

Algorithm A.1 (Schensted's build-up staircase algorithm, [66], 3.5.3) To calculate the character χ^{μ}_{ρ} on the class ρ in the irreducible representation μ, first draw the Young diagram Y^{μ}; then inject ρ in a regular manner, with the added restriction that like digits must form a continuous staircase of some subdiagram of Y^{μ}. The value of the character χ^{μ}_{ρ} is equal to the number of ways of doing the above that involve an even number of negative applications minus the number of ways of doing the above that involve an odd number of negative applications.

Example A.2 Suppose $\mu = (2^2)$ and $\rho = (21^2)$. Then all the possible regular tableaux formed by injecting ρ into μ are

$$\underset{\tau_1}{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}} \qquad \underset{\tau_2}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline\end{array}} \; .$$

The tableau τ_1 involves 0 negative applications (an even number of negative applications) and τ_2 involves 1 negative application (an odd number of negative applications). The character $\chi^{(2^2)}_{(21^2)}$ is equal to the number of tableau that involve an even number of negative applications minus the number of tableau that involve an odd number of negative applications. Hence $\chi^{(2^2)}_{(21^2)} = 1 - 1 = 0$.

Appendix B

Spin Character Tables

Spin character tables from [46] and [27] up to degree 10.

The rows are indexed by the characters of the irreducible representations ζ^λ in distinct part partitions; the columns are headed above by the class ρ, an odd part partition. For example, to look up the character $\zeta^{(41)}$ on the class (31^2), we go to the table 'Degree 5', read down the left hand side to the character $\zeta^{(41)}$, written as $\langle 41\rangle$, and then across to the column headed by (31^2). The entry 0 found there is the value of the spin character $\zeta^{(41)}_{31^2}$.

Degree 4

class $\rightarrow$	(1^4)	(31)
$\langle 4\rangle$	2	1
$\langle 31\rangle$	4	-1

Degree 5

class $\rightarrow$	(1^5)	(31^2)	(5)
$\langle 5\rangle$	4	2	1
$\langle 41\rangle$	6	0	-1
$\langle 32\rangle$	4	-1	1

Degree 6

class →	(1^6)	(31^3)	(51)	(3^2)
$\langle 6\rangle$	4	2	1	1
$\langle 51\rangle$	6	2	-1	-2
$\langle 42\rangle$	20	-2	0	2
$\langle 321\rangle$	4	-1	1	-2

Degree 7

class →	(1^7)	(31^4)	(51^2)	(3^21)	(7)
$\langle 7\rangle$	8	4	2	2	1
$\langle 61\rangle$	20	4	0	-1	1
$\langle 52\rangle$	36	0	-1	0	1
$\langle 43\rangle$	20	-2	0	2	-1
$\langle 421\rangle$	28	-4	2	-2	0

Degree 8

class →	(1^8)	(31^5)	(51^3)	(3^21^2)	(71)	(53)
$\langle 8\rangle$	8	4	2	2	1	1
$\langle 71\rangle$	48	12	2	0	-1	-2
$\langle 62\rangle$	112	8	-2	-2	0	2
$\langle 53\rangle$	112	-4	-2	4	0	1
$\langle 521\rangle$	64	-4	1	-2	1	-1
$\langle 431\rangle$	48	-6	2	0	-1	1

Degree 9

class →	(1^9)	(31^6)	(51^4)	(3^21^2)	(71^2)	(531)	(3^3)	(9)
$\langle 9\rangle$	16	8	4	4	2	2	2	1
$\langle 81\rangle$	56	16	4	2	0	-1	-2	-1
$\langle 72\rangle$	160	20	0	-2	-1	0	2	1
$\langle 63\rangle$	224	4	-4	2	0	1	1	-1
$\langle 54\rangle$	112	-4	-2	4	0	-1	-4	1
$\langle 621\rangle$	240	0	0	-6	2	0	-6	0
$\langle 531\rangle$	336	-24	4	0	0	-1	6	0
$\langle 432\rangle$	96	-12	4	0	-2	2	-6	0

Degree 10

class →	(1^{10})	(31^7)	(51^5)	(3^21^4)	(71^3)	(531^2)	(3^31)	(91)	(73)	(5^2)
$\langle 10\rangle$	16	8	4	4	2	2	2	1	1	1
$\langle 91\rangle$	128	40	12	8	2	0	-2	-1	-2	-2
$\langle 82\rangle$	432	72	8	0	-2	-2	0	0	2	2
$\langle 73\rangle$	768	48	-8	0	-2	2	6	0	-1	-2
$\langle 64\rangle$	672	0	-12	12	0	0	-6	0	0	2
$\langle 721\rangle$	400	20	0	-8	1	0	-4	1	-1	0
$\langle 631\rangle$	800	-20	0	-4	2	0	1	-1	1	0
$\langle 541\rangle$	448	-28	2	4	0	-2	2	1	0	-2
$\langle 532\rangle$	432	-36	8	0	-2	1	0	0	-1	2
$\langle 4321\rangle$	96	-12	4	0	-2	2	-6	0	2	-4

Appendix C

A Combinatorial Approach to Symmetric Functions

There exists a rich and well established connection between the ring of symmetric functions and the combinatorial theory of Young diagrams. This connection is highlighted by the fact that one can define both the S-functions and the Q-functions in a purely combinatorial setting. We provide the starting point for the combinatorial approach here. For further details, see [61] for S-functions and either [60] or [67] for more on the Q-functions.

C.1 Schur S-functions

The *Schur polynomials* may be determined combinatorially by

$$s_\lambda(x_1, x_2, \cdots, x_n) = \sum_\tau x^\tau = \sum_\tau \prod_{y \in \tau} x_{\tau_y}$$

where the sum is over all unitary tableau τ of shape λ formed by injecting partitions ρ with $|\lambda|$ parts ρ_i from the set $\{1, 2, \cdots, n\}$ corresponding to the indices in the n-tuple indeterminate.

Example C.1 Suppose $\lambda = (21)$. When $x = (x_1, x_2, x_3)$ there are 8 possible unitary tableaux of shape λ with entries from the set $\{1, 2, 3\}$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}\ ;\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}\ ;\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array}\ ;\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\ ;\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\ ;\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}\ ;\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\ .$$

From these tableaux, we give the Schur polynomial

$$s_{(21)}(x_1x_2x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.$$

The Schur polynomials are symmetric because if we swap the indeterminates with one another, the polynomial remains unchanged. Just as in Section 2.2 we extend the number of indeterminates the Schur polynomials by taking an inverse limit to describe the Schur S-functions. The Schur S-functions provide a $\mathbb{Z}$-basis for the ring of symmetric functions.

For Example C.1, the corresponding Schur S-function is

$$s_{(21)}(x) = \sum_{i,j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k.$$

C.2 Schur Q-functions

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ be an n-tuple indeterminate. The *Q-polynomials* may be determined combinatorially by

$$Q_\lambda(x_1, x_2, \cdots, x_n) = \sum_{\tau} x^\tau = \sum_{\tau} \prod_{y \in \tau} x_{\tau_y}$$

where the sum is over all standard shifted tableau τ of shape λ formed by injecting partitions ρ of length $|\lambda|$, with parts ρ_i from the set made up of the union of the sets of marked and unmarked alphabets $\{1, 2, \cdots, n\} \cup \{1', 2', \cdots, n'\}$ corresponding to the indices in the n-tuple indeterminate. Order the set of alphabets so that $1' < 1 < 2' < 2 < 3' < \cdots$. The rules for standard injection into a shifted diagram are that rows and columns must be weakly increasing with the added restrictions that marked numbers may not have row-wise repetition, and that unmarked numbers may not have column-wise repetition. The marking of the numbers is only relevant for the injection; with the value x_{τ_y} un-marking any marked numbers.

Example C.2 Suppose $\lambda = (21)$. When $x = (x_1, x_2, x_3)$ there are 32 possible standard shifted tableaux of shape λ with column alphabet $\{1, 2, 3\}$, and row alphabet $\{1', 2', 3'\}$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1' & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1' & 1 \\ \hline \multicolumn{1}{c|}{} & 2' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 1' & 1 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1' & 1 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1' & 2' \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1' & 2' \\ \hline \multicolumn{1}{c|}{} & 2' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1 & 3' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 1' & 3' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1 & 3' \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 1' & 3' \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} ; \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 2' & 3' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 2' & 3' \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 2 & 3' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 2 & 3' \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 1' & 2' \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 1' & 2' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 1' & 2 \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 1' & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 3' \\ \cline{2-2}\end{array} \; ; \; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \quad .$$

From these tableaux we can give the Schur Q-polynomial

$$Q_{(21)}(x_1, x_2, x_3) = 4x_1^2x_2 + 4x_1x_2^2 + 4x_1^2x_3 + 4x_1x_3^2 + 4x_2x_3^2 + 4x_3x_2^2 + 8x_1x_2x_3.$$

The Q-function is determined by the inverse limit, and in this case we obtain

$$Q_{(21)}(x) = 4\sum_{i,j} x_i x_j^2 + 8 \sum_{i<j<k} x_i x_j x_k.$$

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Glossary of Notation

Symbol	Concept	Page
$\in$, $s \in S$	membership	$\star$
$\ni$, $S \ni s$	membership	$\star$
$\notin$, $s \notin S$	nonmembership	$\star$
$\lvert S \rvert$	order of a set S	$\star$
$\subset$	proper subset	$\star$
$\subseteq$	subset	$\star$
$\emptyset$	empty set	$\star$
$\cap$	set intersection	$\star$
$\cup$	set union	$\star$
$S \backslash T$	difference of sets S and T	$\star$
$A \times B$	Cartesian product of sets A and B	$\star$
$f : A \to B$	mapping f from set A to set B	$\star$
$f(a) = b$	image of a	$\star$
$a \mapsto f(a)$	explicit rule for mapping element a	$\star$
$\langle a \rangle$	principal ideal (generated by element a)	2
$\langle A \rangle$	subgroupoid generated by A	2
	ideal of a ring generated by set A	13
0	zero element	$\star$
	ideal $\{0\}$ of a groupoid	$\star$
	matrix with zeros everywhere	$\star$
	ideal $\{0\}$ of a ring	$\star$
	ring with just a 0 element	13
a^{-1}	multiplicative inverse of a	$\star$
$-a$	additive inverse of a	$\star$
$\gcd(a, b)$	greatest common divisor of a and b	$\star$
$\operatorname{lcm}(a, b)$	least common multiple of a and b	$\star$
Z_g	centralizer of g	5
H_g	conjugacy class of g	5
$E(S)$	set of idempotents of a semigroup S	6
(i, j)	$I \times I$ matrix units	7

$\star$ indicates that the notation is used in the text ; but is not specifically defined.

Symbol	Concept	Page
S^0	semigroup S with a zero element adjoined	7
S^1	semigroup S with an identity element adjoined	7
$S^* = S\backslash\{0\}$	semigroup S with zero element 0 disjoint	12
$L(a)$	principal left ideal generated by a	8
$R(a)$	principal right ideal generated by a	8
$J(a)$	principal ideal generated by a	8
L_a	set of generators of S^1a	8
R_a	set of generators of aS^1	8
J_a	set of generators of S^1aS^1	8
$\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$	Green equivalence classes	8
S/I	Rees factor semigroup	9
$F(a) = J(a)/I(a)$	principal factor	9
$A = (a_{ij})$	matrix A over S^0	10
a_{ij}	matrix entry in the i^{th} row and j^{th} column	10
$(a)_{ij}$	Rees $I \times J$ matrix	10
e_{ij}	$I \times I$ matrix units	10
$\mathcal{M}^0(S; I; \Lambda; P)$	Rees $I \times \Lambda$ matrix over S^0 with sandwich matrix P	10
Δ	identity matrix	11
$B(G, n)$	Brandt semigroup	11
$B_n = B(\{e, 0\}, n)$	$n \times n$ matrix units	12
$B_n = (I \times G \times I)$	elements of B_n written as ordered triples	12
$I \lhd R$	I is an ideal of R	13
$\langle A \rangle$	ideal of a ring generated by a set A	13
R/I	quotient ring	14
$\ker f$	kernel of ring homomorphism f	
$\mathcal{S}$	semisimple class	15
$a \circ b$	$= a + b - ab$	16
$\mathcal{J}$	Jacobson radical class	16
$\mathcal{J}(R)$	Jacobson radical of R	16
$M_n(R)$	$n \times n$ square matrix over a ring R	18
$\mathcal{S}_n$	Symmetric group of order $n!$	18
$(1^{m_1}, 2^{m_2}, \cdots, n^{m_n})$	cycle structure of a permutation	20
$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$	partition with d parts	20
λ_i	part of a partition	20
$l(\lambda)$	length (number of parts) of a partition	20
$\lvert\lambda\rvert$	weight (sum of parts) of a partition	20
$\lambda \vdash n$	partition of weight n	20
$\mathcal{P}_n$	set of all partitions of weight n	20
$\mathcal{OP}$	the class of odd part partitions	84
$\mathcal{DP}$	the class of distinct part partitions	84
m_{λ_i}	multiplicity of the part λ_i (in λ)	20
$(\lambda_1^{m_{\lambda_1}}, \cdots, \lambda_j^{m_{\lambda_j}})$	partition written using multiplicity of the parts	20

Symbol	Concept	Page
$\mathcal{L}_n$	reverse lexicographical ordering on the set $\mathcal{P}_n$.	20
$\mu \geq \lambda$	$(\mu, \lambda) \in \mathcal{L}_n$	21
(1^n)	identity element of $\mathcal{S}_n$	21
H_λ	conjugacy class of $\mathcal{S}_n$	22
h_λ	size of conjugacy class H_λ	22
Z_λ	centralizer of $\mathcal{S}_n$	22
z_λ	size of centralizer Z_λ	22
Y^λ	Young diagram of the partition λ	23
τ	Young tableau	23
$c(w)$	charge of a word w	25
$R = \bigoplus_{s\in S} R_s$	S-graded ring R	30
R_s	s-component of the S-graded ring R	30
R_e	subring of S-graded ring R with e idempotent	32
$A[S]$	semigroup or group ring of S over A	33
$H(R) = \cup_{s\in S} R_s$	set of all homogeneous elements of R	31
$M = (R, V, W, S)$	Morita context	34, 59
$R[x_1, x_2, \cdots, x_n]$	ring of polynomials in n indeterminates over R	35
Λ_n	ring of symmetric polynomials	38
$\Lambda_n^{(k)}$	homogeneous symmetric polynomials degree k	38
$\Lambda^{(k)} = \lim_{\leftarrow n} \Lambda_n^{(k)}$	subgroup of symmetric functions of degree k ..	40
$\Lambda = \bigoplus_{k\geq 0} \Lambda^{(k)}$	graded ring of symmetric functions	40
$m_\lambda(x_1, \cdots, x_n)$	monomial symmetric polynomial	39
$m_\lambda = m_\lambda(x)$	monomial symmetric function	40
$p_r(x) = \sum_{i\geq 1} x_i^r$	r^{th} power-sum symmetric function	41
$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$	power-sum symmetric function	41
$P(x)$	generating functions for the $p_r(x)$	41
$G(\alpha; t)$	Littlewood-associated function for the $p_r(\alpha)$ ..	42
$s_\lambda(x_1, \cdots, x_n)$	Schur polynomial	125
$s_\lambda = s_\lambda(x)$	Schur S-funtion	43, 126
$M(u_\lambda, v_\mu)$	transition matrix from $u_\lambda(x)$ to $v_\mu(x)$	43
$M_{\lambda\mu}$	entry in the transition matrix $M(u_\lambda, v_\mu)$	43
$K = M(s_\lambda, m_\mu)$	Koskta matrix as a transition matrix	43, 108
$K_{\lambda\mu}$	Kostka number, entry in K	44, 108
$\chi = M(p_\rho, s_\lambda)$	character table as a transition matrix	44
χ_ρ^λ	ordinary character of the symmetric group	44
R^u	unital extension of a ring R without unity	51
(r, n)	element in R^u	51
$(0, 1)$	identity element of R^u	52
R_e^u	unital extension of the e component of R	52
$\mathcal{A}(T)$	annihilator of an S-graded module	53
$J_{gr}(R)$	S-graded Jacobson radical of ring R	50, 53

Symbol	Concept	Page
$H(r)$	homogeneous components of $r \in R$	71
$H(I)$	$= \bigcup_{r \in I} H(r)$	71
$\operatorname{supp}(r)$	support of element r in S-graded ring R	30
$\operatorname{supp}(I)$	$= \bigcup_{r \in I} \operatorname{supp}(r)$	71
$m(S)$	lcm of the orders of all subgroups of S	71
$m(R)$	$= m(S)$ where S is the semigroup induced by R	72
$GL(k)$	General linear group	83
$\mathbb{Z}_2$	Two element group	84
$\widetilde{\mathcal{S}}_n$	double cover of $\mathcal{S}_n$	84
ζ_μ^λ	spin character on the class μ	84
$\mu \in \mathcal{OP}$	μ is an odd part partition	84
$\lambda \in \mathcal{DP}$	λ is a distinct part partitions	84
$Q_\lambda(x)$	Schur Q-function (or more simply, Q-function)	85
D_x^n	Hirota derivative	86
$\mathbb{Q}[D_x]$	graded ring of polynomials of Hirota derivatives in a single variable	87
$D_\lambda = D_{\lambda_1} \cdots D_{\lambda_k}$	partition notation for Hirota derivatives	87
$\overline{p}_n(\alpha, \beta)$	generalized power-sum symmetric function	89
$\overline{p}_\lambda = \overline{p}_{\lambda_1} \overline{p}_{\lambda_2} \cdots \overline{p}_{\lambda_k}$	partition notation for generalized power-sums	89
$\overline{Q}_\lambda(x)$	generalized Q-function	89
$\widetilde{Q}_\lambda(x)$	generalized Q-operator	90
$H = \bigoplus_{2n \in 2\mathbb{N}} H^{(2n)}$	ring of Hirota derivatives in odd parts	93,100
$\widehat{H}_k$	polynomial ring of $\widetilde{Q}_\lambda(x)$ with $l(\lambda) = k$ and $\lvert\lambda\rvert = 2n \geq 4$	93
$\mathcal{Y}^\lambda$	shifted Young diagram	94
$\mathcal{Y}^{\lambda/\mu}$	shifted skew diagram	97
$ht(\lambda/\mu)$	height of the strip λ/μ	97
$d(\lambda/\mu)$	depth of the double strip λ/μ	97
$\Lambda[t]$	ring of symmetric functions over $\mathbb{Q}[t]$	103
$P_\lambda(x_1, \cdots, x_n; t)$	Hall-Littlewood P-polynomial	103
$P_\lambda(x; t)$	Hall-Littlewood P-function	103
$Q_\lambda(x; t)$	Hall-Littlewood Q-function	104
$b_\lambda(t)$	$= \prod_{i \geq 1} \varphi_{m_{\lambda_i}}(t)$	104
$\varphi_r(t)$	$= (1-t)(1-t^2) \cdots (1-t^r)$	104
$Q_\lambda(x; -1) = Q_\lambda(x)$	Schur Q-function	104
$X(t) = M(p_\rho, P_\lambda)$	transition matrix from $p_\rho(x)$ to $P_\lambda(x; t)$	104
$X_\rho^\lambda(t)$	coefficient of $X(t)$	105
$z_\lambda(t)$	generalized form of z_λ	105
$z(t)$	matrix with $z_\lambda(t)$ along diagonal	106
$b(t)$	matrix with $b_\lambda(t)$ along diagonal	106

Symbol	Concept	Page
$K(t) = M(s_\lambda, P_\mu)$	Kostka matrix as a transition matrix	108
$K_{\lambda\mu}(t)$	Kostka number, coefficient of $K(t)$	108
$n_e(\sigma_\mu)$	number of even negative applications	110
$n_o(\sigma_\mu)$	number of odd negative applications	110
$\Delta n(\sigma_\mu)$	$= n_e(\sigma_\mu) - n_o(\sigma_\mu)$	110
$\widetilde{\mathcal{Y}}^\lambda$	shifted symmetric diagram of λ	116
$h(x) = \widetilde{h}(x)$	hook length in $\widetilde{\mathcal{Y}}^\lambda$ for $x \in \mathcal{Y}^\lambda$	116

$\mathbb{C}$	field of complex numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Q}$	field of rational numbers
$\mathbb{Z}$	ring of integers
$\mathbb{N}$, $\mathbb{Z}^+$	set of natural numbers, set of positive integers
$\mathbb{N}^n$	n-tuple of natural numbers
$\mathbb{R}^n$	n-tuple of real numbers